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Metastable Lifetimes in Random Dynamical Systems

Barbara Gentz

University of Bielefeld, Germany

Florent Barret (Ecole Polytechnique, Palaiseau, France)
Nils Berglund (Université d'Orléans, France)
Bastien Fernandez (CPT Luminy, France, currently at Courant Institute, USA)

Metastability: A common phenomenon

- Observed in the dynamical behaviour of complex systems
- Related to first-order phase transitions in nonlinear dynamics

Characterization of metastability

- Existence of quasi-invariant subspaces Ω_i, $i \in I$
- Multiple timescales
  - A short timescale on which local equilibrium is reached within the Ω_i
  - A longer metastable timescale governing the transitions between the Ω_i

Important feature

- High free-energy barriers to overcome

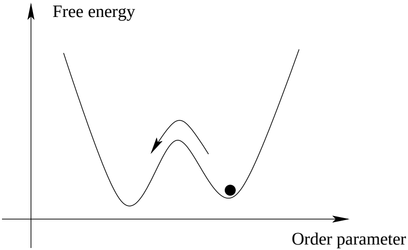
Consequence

- Generally very slow approach to the (global) equilibrium distribution

Metastability in the real world

Examples

- ▹ Supercooled liquid
- ▹ Supersaturated gas
- ▹ Wrongly magnetized ferromagnet

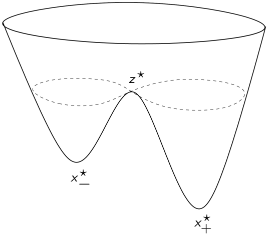


- ▹ Near first-order phase transitions
- ▹ Nucleation implies crossing of energy barrier

Reversible diffusions

Gradient dynamics (ODE)

$$\dot{x}_t^{\text{det}} = -\nabla V(x_t^{\text{det}})$$



Random perturbation by Gaussian white noise (SDE)

$$\mathrm{d}x_t^\varepsilon(\omega) = -\nabla V(x_t^\varepsilon(\omega))\,\mathrm{d}t + \sqrt{2\varepsilon}\,\mathrm{d}B_t(\omega)$$

with

- $V : \mathbb{R}^d \to \mathbb{R}$: confining potential, growth condition at infinity
- $\{B_t(\omega)\}_{t\geq 0}$: d-dimensional Brownian motion

Kolmogorov's forward or Fokker–Planck equation

- Solution $\{x_t^\varepsilon(\omega)\}_t$ is a (time-homogenous) Markov process
- Densities p, $(x,t) \mapsto p(x,t|y,s)$, of the transition probabilities satisfy

$$\frac{\partial}{\partial t} p = \mathcal{L}_\varepsilon p = \nabla \cdot \left[\nabla V(x) p\right] + \varepsilon \Delta p$$

Equilibrium distribution

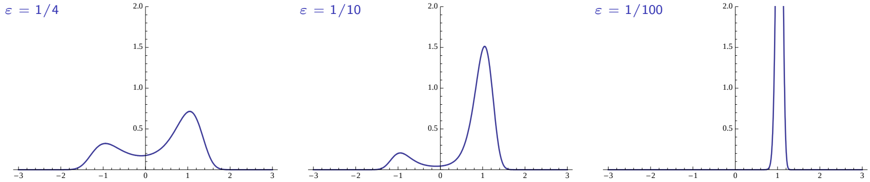
- If $\{x_t^\varepsilon(\omega)\}_t$ admits an invariant density p_0, then $\mathcal{L}_\varepsilon p_0 = 0$
- Easy to verify (for gradient systems)

$$p_0(x) = \frac{1}{Z_\varepsilon} \mathrm{e}^{-V(x)/\varepsilon} \qquad \text{with} \qquad Z_\varepsilon = \int_{\mathbb{R}^d} \mathrm{e}^{-V(x)/\varepsilon} \, \mathrm{d}x$$

- Invariant measure or equilibrium distribution

$$\mu_\varepsilon(dx) = \frac{1}{Z_\varepsilon} \mathrm{e}^{-V(x)/\varepsilon} \, dx$$

- μ_ε concentrates in the minima of V



Transition times between potential wells

First-hitting time of a small ball $B_\delta(x_+^\star)$ around minimum $x_+^\star$

$$\tau_+ = \tau^\varepsilon_{x_+^\star}(\omega) = \inf\{t \geq 0 \colon x_t^\varepsilon(\omega) \in B_\delta(x_+^\star)\}$$

Eyring–Kramers Law [Eyring 35, Kramers 40]

- $d = 1$: $\quad \mathbb{E}_{x_-^\star} \tau_+ \simeq \dfrac{2\pi}{\sqrt{V''(x_-^\star)|V''(z^\star)|}} \, \mathrm{e}^{[V(z^\star)-V(x_-^\star)]/\varepsilon}$
- $d \geq 2$: $\quad \mathbb{E}_{x_-^\star} \tau_+ \simeq \dfrac{2\pi}{|\lambda_1(z^\star)|} \sqrt{\dfrac{|\det \nabla^2 V(z^\star)|}{\det \nabla^2 V(x_-^\star)}} \, \mathrm{e}^{[V(z^\star)-V(x_-^\star)]/\varepsilon}$

where $\lambda_1(z^\star)$ is the unique negative eigenvalue of $\nabla^2 V$ at saddle $z^\star$

Proving Kramers Law

- Exponential asymptotics and optimal transition paths via large deviations approach [Wentzell & Freidlin 69–72]

$$\lim_{\varepsilon \to 0} \varepsilon \log \mathbb{E}_{x_-^\star} \tau_+ = V(z^\star) - V(x_-^\star)$$

  Only 1-saddles are relevant for transitions between wells

- Low-lying spectrum of generator of the diffusion (analytic approach) [Helffer & Sjöstrand 85, Miclo 95, Mathieu 95, Kolokoltsov 96, ...]
- Potential theoretic approach [Bovier, Eckhoff, Gayrard & Klein 04]

$$\mathbb{E}_{x_-^\star} \tau_+ = \frac{2\pi}{|\lambda_1(z^\star)|} \sqrt{\frac{|\det \nabla^2 V(z^\star)|}{\det \nabla^2 V(x_-^\star)}}\, \mathrm{e}^{[V(z^\star) - V(x_-^\star)]/\varepsilon} [1 + \mathcal{O}((\varepsilon|\log \varepsilon|)^{1/2})]$$

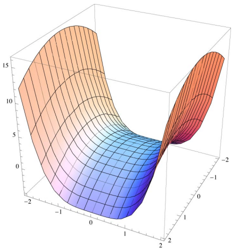
- Full asymptotic expansion of prefactor [Helffer, Klein & Nier 04]
- Asymptotic distribution of τ_+ [Day 83, Bovier, Gayrard & Klein 05]

$$\lim_{\varepsilon \to 0} \mathbb{P}_{x_-^\star} \{\tau_+ > t \cdot \mathbb{E}_{x_-^\star} \tau_+\} = \mathrm{e}^{-t}$$

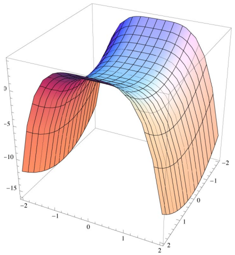
Non-quadratic saddles

What happens if $\det \nabla^2 V(z^\star) = 0$?

$$\det \nabla^2 V(z^\star) = 0 \quad \Rightarrow \quad \text{At least one vanishing eigenvalue at saddle } z^\star$$
$$\Rightarrow \quad \text{Saddle has at least one non-quadratic direction}$$
$$\Rightarrow \quad \text{Kramers Law not applicable}$$



Quartic unstable direction



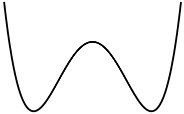
Quartic stable direction

Why do we care about this non-generic situation?

Parameter-dependent systems may undergo bifurcations

Example: Two harmonically coupled particles

$$V_\gamma(x_1, x_2) = U(x_1) + U(x_2) + \tfrac{\gamma}{2}(x_1 - x_2)^2$$

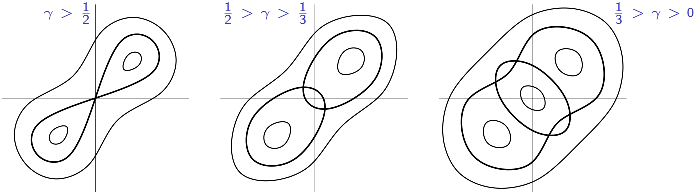


$U(x) = \frac{x^4}{4} - \frac{x^2}{2}$

Change of variable: Rotation by $\pi/4$ yields

$$\widehat{V}_\gamma(y_1, y_2) = -\frac{1}{2}y_1^2 - \frac{1-2\gamma}{2}y_2^2 + \frac{1}{8}\left(y_1^4 + 6y_1^2y_2^2 + y_2^4\right)$$

Note: $\det \nabla^2 \widehat{V}_\gamma(0,0) = 1 - 2\gamma \;\Rightarrow\;$ Pitchfork bifurcation at $\gamma = 1/2$



Further examples: More particles

- N particles with nearest-neighbour coupling: $i \in \Lambda = \mathbb{Z}/N\mathbb{Z}$

$$V_\gamma(x) = \sum_{i\in\Lambda} U(x_i) + \frac{\gamma}{4} \sum_{i\in\Lambda} (x_{i+1} - x_i)^2$$

Results [Berglund, Fernandez & G. 07]
  - Bifurcation diagram, showing a series of symmetry-breaking bifurcations
  - Optimal transition paths
  - Exponential asymptotics of transition times

- Ginzburg–Landau SPDE on a compact interval: $x \in [0, L]$, various b.c.

$$\partial_t \phi(x,t) = \partial_{xx}\phi(x,t) + \phi(x,t) - \phi(x,t)^3 \;\; (+ \text{ weak spatio–temporal noise})$$

Energy functional $V_L(\phi) = \int_0^L [U(\phi(x)) + \frac{1}{2}\phi'(x)^2] \, \mathrm{d}x$

Results [Maier & Stein 01; Berglund & G. 09; Barret, Berglund & G. in prep.]
  - Pitchfork bif. at $L = 2\pi$ (periodic b.c.) or $L = \pi$ (Neumann b.c.)
  - Subexponential asymptotics of transition times at critical L

Degenerate saddles: An example

Assume $z^\star = 0$ and eigenvalues $\lambda_1 < 0 = \lambda_2 < \lambda_3 \leq \dots \leq \lambda_d$ of $\nabla^2 V(0)$

$$V(x) = -\frac{1}{2}|\lambda_1|x_1^2 + \frac{1}{2}\sum_{j=3}^{d} \lambda_j x_j^2 + \sum_{1\leq i\leq j\leq k\leq d} V_{ijk} x_i x_j x_k + \dots$$

Normal form: There exists a polynomial $g(y) = \mathcal{O}(\|y\|^2)$ s.t.

$$V(y+g(y)) = -\frac{1}{2}|\lambda_1|y_1^2 + C_3 y_2^3 + C_4 y_2^4 + \frac{1}{2}\sum_{j=3}^{d} \lambda_j y_j^2 + \text{higher-order terms}$$

Simplest case: $z^\star = 0$ a saddle with $C_3 = 0$ and $C_4 > 0$. In this case,

$$V(y+g(y)) = -\frac{1}{2}|\lambda_1|y_1^2 + C_4 y_2^4 + \frac{1}{2}\sum_{j=3}^{d} \lambda_j y_j^2 + \text{higher-order terms}$$

$$= -u_1(y_1) + u_2(y_2) + \frac{1}{2}\sum_{j=3}^{d} \lambda_j y_j^2 + \text{higher-order terms}$$

Main result

- Assume $x_-^\star$ is a quadratic local minimum of V (non-quadratic minima can be dealt with)
- Assume $x_+^\star$ is another local minimum of V
- Assume $z^\star = 0$ is the relevant saddle for passage from $x_-^\star$ to $x_+^\star$
- Normal form near saddle

$$V(y) = -u_1(y_1) + u_2(y_2) + \frac{1}{2}\sum_{j=3}^{d} \lambda_j y_j^2 + \dots$$

- Assume growth conditions on u_1, u_2

Theorem [Berglund & G., to appear in MPRF 2010]

$$\mathbb{E}_{x_-^\star} \tau_+ = \frac{(2\pi\varepsilon)^{d/2}\,\mathrm{e}^{-V(x_-^\star)/\varepsilon}}{\sqrt{\det \nabla^2 V(x_-^\star)}} \Bigg/ \varepsilon \frac{\int_{-\infty}^{\infty} \mathrm{e}^{-u_2(y_2)/\varepsilon}\,\mathrm{d}y_2}{\int_{-\infty}^{\infty} \mathrm{e}^{-u_1(y_1)/\varepsilon}\,\mathrm{d}y_1} \prod_{j=3}^{d} \sqrt{\frac{2\pi\varepsilon}{\lambda_j}}$$

$$\times \left[1 + \mathcal{O}((\varepsilon|\log\varepsilon|)^\alpha)\right]$$

where $\alpha > 0$ is explicitly known, depends on the growth conditions on u_1, u_2

Corollaries: From quadratic saddles to saddles with a quartic direction

- Quadratic saddle: $V(y) = -\frac{1}{2}|\lambda_1|y_1^2 + \frac{1}{2}\sum_{j=2}^{d}\lambda_j y_j^2 + \dots$

$$\mathbb{E}_{x_-^\star}\tau_+ = 2\pi\sqrt{\frac{\lambda_2\dots\lambda_d}{|\lambda_1|\det\nabla^2 V(x_-^\star)}}\,e^{[V(z^\star)-V(x_-^\star)]/\varepsilon}[1+\mathcal{O}((\varepsilon|\log\varepsilon|)^{1/2})]$$

- Quartic stable direction: $V(y) = -\frac{1}{2}|\lambda_1|y_1^2 + C_4 y_2^4 + \frac{1}{2}\sum_{j=3}^{d}\lambda_j y_j^2 + \dots$

$$\mathbb{E}_{x_-^\star}\tau_+ = \frac{2C_4^{1/4}\varepsilon^{1/4}}{\Gamma(1/4)}\sqrt{\frac{(2\pi)^3\lambda_3\dots\lambda_d}{|\lambda_1|\det\nabla^2 V(x_-^\star)}}\,e^{[V(z^\star)-V(x_-^\star)]/\varepsilon}[1+\mathcal{O}((\varepsilon|\log\varepsilon|)^{1/4})]$$

Corollaries:
From quadratic saddles to saddles with a quartic direction

- Quadratic saddle: $V(y) = -\frac{1}{2}|\lambda_1|y_1^2 + \frac{1}{2}\sum_{j=2}^{d}\lambda_j y_j^2 + \dots$

$$\mathbb{E}_{x_-^\star}\tau_+ = 2\pi\sqrt{\frac{\lambda_2\dots\lambda_d}{|\lambda_1|\det\nabla^2 V(x_-^\star)}}\,\mathrm{e}^{[V(z^\star)-V(x_-^\star)]/\varepsilon}[1+\mathcal{O}((\varepsilon|\log\varepsilon|)^{1/2})]$$

- Quartic unstable direction: $V(y) = -C_4 y_1^4 + \frac{1}{2}\sum_{j=2}^{d}\lambda_j y_j^2 + \dots$

$$\mathbb{E}_{x_-^\star}\tau_+ = \frac{\Gamma(1/4)}{2C_4^{1/4}\varepsilon^{1/4}}\sqrt{\frac{(2\pi)^1\lambda_2\dots\lambda_d}{\det\nabla^2 V(x_-^\star)}}\,\mathrm{e}^{[V(z^\star)-V(x_-^\star)]/\varepsilon}[1+\mathcal{O}((\varepsilon|\log\varepsilon|)^{1/4})]$$

Corollaries: Worse than quartic . . .

▷ Quartic unstable direction: $V(y) = -C_4 y_1^4 + \frac{1}{2}\sum_{j=2}^{d} \lambda_j y_j^2 + \dots$

$$\mathbb{E}_{x_-^\star} \tau_+ = \frac{\Gamma(1/4)}{2C_4^{1/4}\varepsilon^{1/4}} \sqrt{\frac{2\pi\lambda_2 \dots \lambda_d}{\det \nabla^2 V(x_-^\star)}}\, e^{[V(z^\star)-V(x_-^\star)]/\varepsilon} [1 + \mathcal{O}((\varepsilon|\log\varepsilon|)^{1/4})]$$

▷ Degenerate unstable direction: $V(y) = -C_{2p} y_1^{2p} + \frac{1}{2}\sum_{j=2}^{d} \lambda_j y_j^2 + \dots$

$$\mathbb{E}_{x_-^\star} \tau_+ = \frac{\Gamma(1/2p)}{pC_{2p}^{1/2p}\varepsilon^{1/2(1-1/p)}} \sqrt{\frac{2\pi\lambda_2 \dots \lambda_d}{\det \nabla^2 V(x_-^\star)}}\, e^{[V(z^\star)-V(x_-^\star)]/\varepsilon} [1 + \mathcal{O}((\dots)^{1/2p})]$$

Corollaries: Pitchfork bifurcation

Pitchfork bif.: $V(y) = -\frac{1}{2}|\lambda_1|y_1^2 + \frac{1}{2}\lambda_2 y_2^2 + C_4 y_2^4 + \frac{1}{2}\sum_{j=3}^{d}\lambda_j y_j^2 + \dots$

- For $\lambda_2 > 0$ (possibly small wrt. ε):

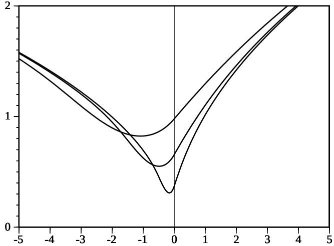
$$\mathbb{E}_{x_-^\star}\tau_+ = 2\pi\sqrt{\frac{(\lambda_2 + \sqrt{2\varepsilon C_4})\lambda_3\dots\lambda_d}{|\lambda_1|\det\nabla^2 V(x_-^\star)}}\,\frac{e^{[V(z^\star)-V(x_-^\star)]/\varepsilon}}{\Psi_+(\lambda_2/\sqrt{2\varepsilon C_4})}[1+R(\varepsilon)]$$

where

$$\Psi_+(\alpha) = \sqrt{\frac{\alpha(1+\alpha)}{8\pi}}\,e^{\alpha^2/16}\,K_{1/4}\Big(\frac{\alpha^2}{16}\Big)$$

$$\lim_{\alpha\to\infty}\Psi_+(\alpha) = 1$$

$K_{1/4} =$ modified Bessel fct. of 2*nd* kind



$\lambda_2 \mapsto$ prefactor

$\varepsilon = 0.5,\ \varepsilon = 0.1,\ \varepsilon = 0.01$

- For $\lambda_2 < 0$: Similar
  (involving eigenvalues at new saddles and $I_{\pm 1/4}$)

Ginzburg–Landau SPDE: Stationary states

$$\partial_t u(x,t) = \partial_{xx} u(x,t) + u(x,t) - u(x,t)^3 + \sqrt{2\varepsilon}\xi(t,x)$$

- $x \in [0,L]$ and $u(x,t) \in \mathbb{R}$, weak space–time white noise $\sqrt{2\varepsilon}\xi$
- Neumann b.c.: $\partial_x u(0,t) = \partial_x u(L,t) = 0$
- Energy functional $V(u) = V_L(u) = \int_0^L [U(u(x)) + \frac{1}{2}u'(x)^2]\,\mathrm{d}x$
- Stationary states for $L < \pi$:
  - $u_\pm(x) \equiv \pm 1$ (uniform and stable; global minima)
  - $u_0(x) \equiv 0$ (uniform and unstable; transition state)
  - Activation energy $V(u_0) - V(u_\pm) = L/4$
- Stationary states for $L > \pi$:
  - $u_\pm(x) \equiv \pm 1$ (uniform and stable; remain global minima)
  - $u_0(x) \equiv 0$ (uniform and unstable; no longer transition state)
  - $u_{\text{inst},\pm}(x)$ of instanton shape (pair of unstable states; transition states)
  - Additional stationary states as L increases; not transition states
- As $L \nearrow \pi$: Pitchfork bifurcation

Ginzburg–Landau SPDE: Classical results and reduction to finite-dimensional systems

Classical results [Faris & Jona-Lasinio 82]

- Existence and uniqueness of mild solution
- Large deviation principle

Reduction to finite-dimensional systems: Galerkin approximation

- Truncate Fourier series

$$u_d(x,t) = \frac{1}{\sqrt{L}} y_0(t) + \frac{2}{\sqrt{L}} \sum_{k=1}^{d} y_k(t) \cos(\pi k x / L)$$

- Rewrite potential in Fourier variables; retain only modes with $k \leq d$

$$V^{(d)}(y) = \frac{1}{2} \sum_{k=0}^{d} \lambda_k y_k^2 + V_4^{(d)}(y) \ , \quad \lambda_k = -1 + (\pi k / L)^2$$

- Apply results for finite-dimensional systems

Ginzburg–Landau SPDE: Uniform control of error terms

▷ Result for the Galerkin approximation

$$\varepsilon^{\gamma} C(d)\, \mathrm{e}^{\Delta W^{(d)}/\varepsilon}[1 - R_d^-(\varepsilon)] \leq \mathbb{E}_{u_-^{(d)}} \tau_{u_+^{(d)}} \leq \varepsilon^{\gamma} C(d)\, \mathrm{e}^{\Delta W^{(d)}/\varepsilon}[1 + R_d^+(\varepsilon)]$$

(The contribution ε^{γ} is only present at bifurcation points / non-quadratic saddles)

▷ The following limits exist and are finite

$$\lim_{d\to\infty} C(d) =: C(\infty) \qquad \text{and} \qquad \lim_{d\to\infty} \Delta W^{(d)} =: \Delta W^{(\infty)}$$

▷ Important: Uniform control of error terms (uniform in d):

$$R^{\pm}(\varepsilon) := \sup_d R_d^{\pm}(\varepsilon) \to 0 \quad \text{as} \quad \varepsilon \to 0$$

Away from bifurcation points, c.f. [Barret, Bovier & Méleard 09]

Ginzburg–Landau SPDE: Taking the limit $d \to \infty$

- For any ε, the distance between $u(x,t)$ and Galerkin approximation $u^{(d)}(x,t)$ becomes small on any finite time interval $[0,T]$
  [Liu '03, Blömker & Jentzen '09]
- Uniform error bounds and large deviation results allow to decouple limits of small ε and large d
- Yielding

$$\varepsilon^{\gamma} C(\infty)\, e^{\Delta W^{(\infty)}/\varepsilon}[1 - R^{-}(\varepsilon)] \leq \mathbb{E}_{u_-} \tau_{u_+} \leq \varepsilon^{\gamma} C(\infty)\, e^{\Delta W^{(\infty)}/\varepsilon}[1 + R^{+}(\varepsilon)]$$

Ginzburg–Landau SPDE

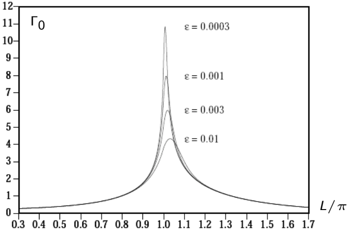
Theorem [Barret, Berglund & G., in preparation]

For the Ginzburg–Landau SPDE with Neumann b.c., $L < \pi$

(Similar expression for $L > \pi$)

$$\mathbb{E}_{u_-} \tau_{u_+} = \frac{1}{\Gamma_0(L)} \, e^{L/4\varepsilon}$$

where the rate prefactor satisfies
(recall: $\lambda_1 = -1 + (\pi/L)^2$)



$$\Gamma_0(L) = \frac{1}{2^{3/4}\pi} \sqrt{\frac{\sinh(\sqrt{2}L)}{\sin L}} \sqrt{\frac{\lambda_1}{\lambda_1 + \sqrt{3\varepsilon/4L}}} \Psi_+\left(\frac{\lambda_1}{\sqrt{3\varepsilon/4L}}\right)[1 + \mathcal{O}(\dots)]$$

$$\longrightarrow \frac{\Gamma(1/4)}{2(3\pi^7)^{1/4}} \sqrt{\sinh(\sqrt{2}\pi)} \, \varepsilon^{-1/4} [1 + \mathcal{O}(\dots)] \qquad \text{as } L \nearrow \pi$$

Thank you for your attention!

Ginzburg–Landau equation

$$\partial_t u(x,t) = \partial_{xx} u(x,t) + u(x,t) - u(x,t)^3 + \text{noise}$$

- On finite interval $x \in [0, L]$
- $u(x,t) \in \mathbb{R}$ (one-dimensional, representing e.g. magnetization)
- Boundary conditions
  - Periodic b.c. $u(0,t) = u(L,t)$ and $\partial_x u(0,t) = \partial_x u(L,t)$
  - Neumann b.c. with zero flux $\partial_x u(0,t) = \partial_x u(L,t) = 0$
- Weak space–time white noise

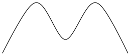
Deterministic dynamics minimizes energy functional

$$V(u) = \int_0^L \left[\tfrac{1}{2} u'(x)^2 - \tfrac{1}{2} u(x)^2 + \tfrac{1}{4} u(x)^4\right] \mathrm{d}x$$

as

$$\partial_{xx} u(x,t) + u(x,t) - u(x,t)^3 = -\frac{\delta V}{\delta u}$$

Stationary states for the deterministic system

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} u(x) = -u(x) + u(x)^3 = -\frac{\mathrm{d}}{\mathrm{d}u}\Bigg[$$  $$\Bigg]$$

- Uniform stationary states
  - $u_\pm(x) \equiv \pm 1$ (stable; global minima of V)
  - $u_0(x) \equiv 0$ (unstable – when is u_0 a transition state?)
- Periodic b.c.: For $k = 1, 2, \dots$ and $L > 2\pi k$
  - Continuous one-parameter family of stationary states

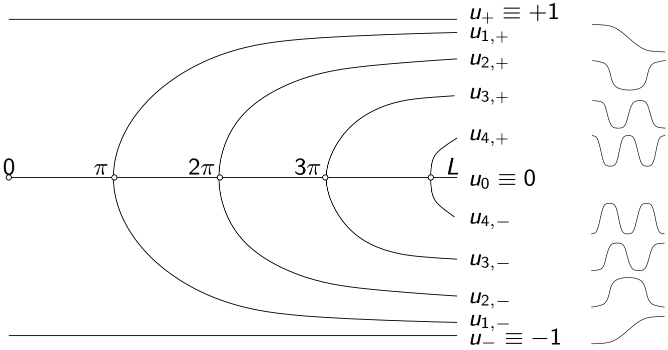
    $$u_{k,\varphi}(x) = \sqrt{\frac{2m}{m+1}}\,\mathrm{sn}\Big(\frac{kx}{\sqrt{m+1}} + \varphi, m\Big) \quad \text{where} \quad 4k\sqrt{m+1}\,\mathrm{K}(m) = L$$
- Neumann b.c.: For $k = 1, 2, \dots$ and $L > \pi k$
  - Two stationary states

    $$u_{k,\pm}(x) = \pm\sqrt{\frac{2m}{m+1}}\,\mathrm{sn}\Big(\frac{kx}{\sqrt{m+1}} + \mathrm{K}(m), m\Big) \quad \text{where} \quad 2k\sqrt{m+1}\,\mathrm{K}(m) = L$$

Stationary states: Neumann b.c.

For $k = 1, 2, \ldots$ and $L > \pi k$:

$$u_{k,\pm}(x) = \pm\sqrt{\frac{2m}{m+1}}\,\mathrm{sn}\left(\frac{kx}{\sqrt{m+1}} + \mathrm{K}(m), m\right) \quad \text{where} \quad 2k\sqrt{m+1}\,\mathrm{K}(m) = L$$



Stability of the stationary states: Neumann b.c.

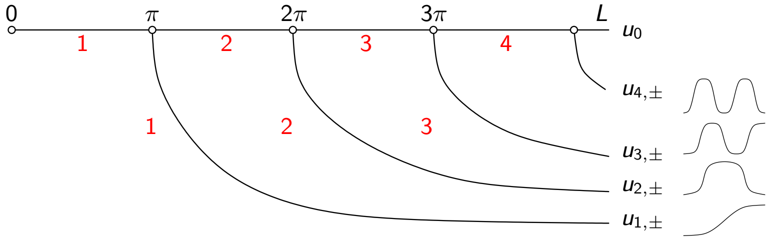
Consider linearization of GL equation at stationary solution $u : [0, L] \to \mathbb{R}$

$$\partial_t v = A[u]v \quad \text{where} \quad A[u] = \frac{d^2}{dx^2} + 1 - 3u^2$$

Stability is determined by the eigenvalues of $A[u]$

- $u_\pm(x) \equiv \pm 1$: $A[u_\pm]$ has eigenvalues $-(2 + (\pi k/L)^2)$, $k = 0, 1, 2, \ldots$
- $u_0(x) \equiv 0$: $A[u_0]$ has eigenvalues $1 - (\pi k/L)^2$, $k = 0, 1, 2, \ldots$

Counting the number of positive eigenvalues: None for $u_\pm$ and ...



Stability of the stationary states: Neumann b.c.

- For $L < \pi$:
  - $u_{\pm}(x) \equiv \pm 1$ are stable; global minima
  - $u_0(x) \equiv 0$ is unstable; transition state
  - Activation energy $V(u_0) - V(u_{\pm}) = L/4$

- For $L > \pi$:
  - $u_{\pm}(x) \equiv \pm 1$ remain stable; global minima
  - $u_0(x) \equiv 0$ remains unstable; but no longer forms the transition state
  - $u_{1,\pm}(x)$ are the new transition states (of instanton shape)

- Pitchfork bifurcation as L increases through π:
  Uniform transition state u_0 bifurcates into pair of instanton states $u_{1,\pm}$

- Subsequent bifurcations at $L = k\pi$ for $k = 2, 3, \ldots$ do not affect transition states

Ginzburg–Landau equation with noise

$$\begin{cases} \partial_t u(x,t) &= \partial_{xx} u(x,t) + u(x,t) - u(x,t)^3 + \sqrt{2\varepsilon}\xi(t,x) \\ u(\cdot,0) &= \varphi(\cdot) \\ \partial_x u(0,t) &= \partial_x u(L,t) = 0 \qquad \text{(Neumann b.c.)} \end{cases}$$

- Space–time white noise $\xi(t,x)$ as formal derivative of Brownian sheet
- Mild / evolution formulation, following [Walsh '86]:

$$u(x,t) = \int_0^L G_t(x,z)\varphi(z)\,\mathrm{d}z + \int_0^t \int_0^L G_{t-s}(x,z)\left[u(s,z) - u(s,z)^3\right]\mathrm{d}z\,\mathrm{d}s$$
$$+ \sqrt{2\varepsilon}\int_0^t \int_0^L G_{t-s}(x,z)W(\mathrm{d}s,\mathrm{d}z)$$

where

- G is the heat kernel
- W is the Brownian sheet

Existence and a.s. uniqueness [Faris & Jona-Lasinio 82]

Question

How long does a noise-induced transition from the global minimum $u_-(x) \equiv -1$ to (a neighbourhood of) $u_+(x) \equiv 1$ take?

$$\tau_{u_+} = \text{first hitting time of such a neighbourhood}$$

Metastability: We expect $\mathbb{E}_{u_-} \tau_{u_+} \sim \mathrm{e}^{\text{const}/\varepsilon}$

We seek

- Activation energy ΔW
- Transition rate prefactor Γ_0^{-1}
- Exponent α of error term

such that

$$\mathbb{E}_{u_-} \tau_{u_+} = \Gamma_0^{-1} \, \mathrm{e}^{\Delta W/\varepsilon} [1 + \mathcal{O}(\varepsilon^\alpha)]$$

Large deviations for the Ginzburg–Landau equation

Large deviation principle [Faris & Jona–Lasinio '82]:

- For $L \leq \pi$:

$$\Delta W = V(u_0) - V(u_-) = L/4$$

- For $L > \pi$:

$$\Delta W = V(u_{1,\pm}) - V(u_-) = \frac{1}{3\sqrt{1+m}}\left[8\mathrm{E}(m) - \frac{(1-m)(3m+5)}{1+m}\mathrm{K}(m)\right]$$

Formal computation of the prefactor for the GL equation

Consider $L < \pi$

- Transition state: $u_0(x) \equiv 0$, $V[u_0] = 0$
- Activation energy: $\Delta W = V[u_0] - V[u_-] = L/4$
- Eigenvalues at stable state $u_-(x) \equiv -1$: $\mu_k = 2 + (\pi k/L)^2$
- Eigenvalues at transition state $u_0 \equiv 0$: $\lambda_k = -1 + (\pi k/L)^2$

Thus formally [Maier & Stein '01, '03]

$$\Gamma_0 \simeq \frac{|\lambda_0|}{2\pi} \sqrt{\prod_{k=0}^{\infty} \frac{\mu_k}{|\lambda_k|}} = \frac{1}{2^{3/4}\pi} \sqrt{\frac{\sinh(\sqrt{2}L)}{\sin L}}$$

For $L > \pi$: Spectral determinant computed by Gelfand's method

Problems

- What happens when $L \nearrow \pi$? (Approaching bifurcation)
- Is the formal computation correct in infinite dimension?

Ginzburg–Landau equation: Introducing Fourier variables

- Fourier series

$$u(x,t) = \frac{1}{\sqrt{L}} y_0(t) + \frac{2}{\sqrt{L}} \sum_{k=1}^{\infty} y_k(t) \cos(\pi k x/L) = \frac{1}{\sqrt{L}} \sum_{k\in\mathbb{Z}} \tilde{y}_k(t)\, e^{i\, k\pi x/L}$$

- Rewrite energy functional V in Fourier variables

$$V(y) = \frac{1}{2} \sum_{k=0}^{\infty} \lambda_k y_k^2 + V_4(y)\,, \qquad \lambda_k = -1 + (\pi k/L)^2$$

where

$$V_4(y) = \frac{1}{4L} \sum_{k_1+k_2+k_3+k_4=0} \tilde{y}_{k_1} \tilde{y}_{k_2} \tilde{y}_{k_3} \tilde{y}_{k_4}$$

- Resulting system of SDEs

$$\dot{y}_k = -\lambda_k y_k - \frac{1}{L} \sum_{k_1+k_2+k_3=k} \tilde{y}_{k_1} \tilde{y}_{k_2} \tilde{y}_{k_3} + \sqrt{2\varepsilon}\, \dot{W}_t^{(k)}$$

with i.i.d. Brownian motions $W_t^{(k)}$

Truncating the Fourier series

- Truncate Fourier series (projected equation)

$$u_d(x,t) = \frac{1}{\sqrt{L}} y_0(t) + \frac{2}{\sqrt{L}} \sum_{k=1}^{d} y_k(t) \cos(\pi k x / L)$$

- Retain only modes $k \leq d$ in the energy functional V

$$V^{(d)}(y) = \frac{1}{2} \sum_{k=0}^{d} \lambda_k y_k^2 + V_4^{(d)}(y)$$

where

$$V_4^{(d)}(y) = \frac{1}{4L} \sum_{\substack{k_1+k_2+k_3+k_4=0 \\ k_i \in \{-d,\dots,0,\dots,+d\}}} \tilde{y}_{k_1} \tilde{y}_{k_2} \tilde{y}_{k_3} \tilde{y}_{k_4}$$

- Resulting d-dimensional system of SDEs

$$\dot{y}_k = -\lambda_k y_k - \frac{1}{L} \sum_{\substack{k_1+k_2+k_3=k \\ k_i \in \{-d,\dots,0,\dots,+d\}}} \tilde{y}_{k_1} \tilde{y}_{k_2} \tilde{y}_{k_3} + \sqrt{2\varepsilon}\, \dot{W}_t^{(k)}$$

Reduction to finite-dimensional system

- Show the following result for the projected finite-dimensional systems

$$\varepsilon^\gamma C(d)\, e^{\Delta W^{(d)}/\varepsilon}[1 - R_d^-(\varepsilon)] \le \mathbb{E}_{u_-^{(d)}} \tau_{u_+^{(d)}} \le \varepsilon^\gamma C(d)\, e^{\Delta W^{(d)}/\varepsilon}[1 + R_d^+(\varepsilon)]$$

(The contribution ε^γ is only present at bifurcation points / non-quadratic saddles)

- The following limits exist and are finite

$$\lim_{d\to\infty} C(d) =: C(\infty) \qquad \text{and} \qquad \lim_{d\to\infty} \Delta W^{(d)} =: \Delta W^{(\infty)}$$

- Important: Uniform control of error terms (uniform in d):

$$R^\pm(\varepsilon) := \sup_d R_d^\pm(\varepsilon) \to 0 \quad \text{as} \quad \varepsilon \to 0$$

Away from bifurcation points, c.f. [Barret, Bovier & Méleard 09]

Taking the limit $d \to \infty$

- For any ε, distance between $u(x,t)$ and solution $u^{(d)}(x,t)$ of the projected equation becomes small [Liu '03] on any finite time interval $[0,T]$
- Uniform error bounds and large deviation results allow to decouple limits of small ε and large d
- Yielding

$$\varepsilon^\gamma C(\infty)\, e^{\Delta W^{(\infty)}/\varepsilon}[1 - R^-(\varepsilon)] \leq \mathbb{E}_{u_-}\tau_{u_+} \leq \varepsilon^\gamma C(\infty)\, e^{\Delta W^{(\infty)}/\varepsilon}[1 + R^+(\varepsilon)]$$

Result for the Ginzburg–Landau equation

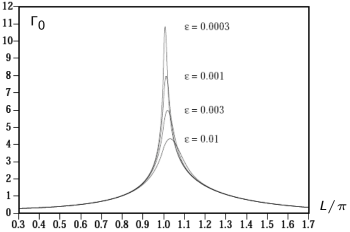
Theorem [Barret, Berglund & G., in preparation]

For the Ginzburg–Landau equation with Neumann b.c., $L < \pi$

(Similar expression for $L > \pi$)

$$\mathbb{E}_{u_-}\tau_{u_+} = \frac{1}{\Gamma_0(L)}\,\mathrm{e}^{L/4\varepsilon}[1+\mathcal{O}((\varepsilon|\log\varepsilon|)^{1/4})]$$

where the rate prefactor satisfies
(recall: $\lambda_1 = -1 + (\pi/L)^2$)



$$\Gamma_0(L) = \frac{1}{2^{3/4}\pi}\sqrt{\frac{\sinh(\sqrt{2}L)}{\sin L}}\sqrt{\frac{\lambda_1}{\lambda_1+\sqrt{3\varepsilon/4L}}}\Psi_+\Big(\frac{\lambda_1}{\sqrt{3\varepsilon/4L}}\Big)$$

$$\longrightarrow \frac{\Gamma(1/4)}{2(3\pi^7)^{1/4}}\sqrt{\sinh(\sqrt{2}\pi)}\,\varepsilon^{-1/4} \qquad \text{as } L \nearrow \pi$$

Towards a proof in the finite-dimensional case: Potential theory for Brownian motion I

First-hitting time $\tau_A = \inf\{t > 0 \colon B_t \in A\}$ of $A \subset \mathbb{R}^d$

Fact I: The expected first-hitting time $w_A(x) = \mathbb{E}_x \tau_A$ is a solution to the Dirichlet problem

$$\begin{cases} \Delta w_A(x) = 1 & \text{for } x \in A^c \\ w_A(x) = 0 & \text{for } x \in A \end{cases}$$

and can be expressed with the help of the Green function $G_{A^c}(x, y)$ as

$$w_A(x) = \int_{A^c} G_{A^c}(x, y)\, \mathrm{d}y$$

Potential theory for Brownian motion II

The equilibrium potential (or capacitor) $h_{A,B}$ is a solution to the Dirichlet problem

$$\begin{cases} \Delta h_{A,B}(x) = 0 & \text{for } x \in (A \cup B)^c \\ h_{A,B}(x) = 1 & \text{for } x \in A \\ h_{A,B}(x) = 0 & \text{for } x \in B \end{cases}$$

Fact II: $h_{A,B}(x) = \mathbb{P}_x[\tau_A < \tau_B]$

The equilibrium measure (or surface charge density) is the unique measure $\rho_{A,B}$ on ∂A s.t.

$$h_{A,B}(x) = \int_{\partial A} G_{B^c}(x, y)\, \rho_{A,B}(\mathrm{d}y)$$

Capacities

Key observation: For a small ball $C = B_\delta(x)$,

$$\int_{A^c} h_{C,A}(y)\,\mathrm{d}y = \int_{A^c}\int_{\partial C} G_{A^c}(y,z)\,\rho_{C,A}(\mathrm{d}z)\,\mathrm{d}y$$

$$= \int_{\partial C} w_A(z)\,\rho_{C,A}(\mathrm{d}z) \simeq w_A(x)\,\mathrm{cap}_C(A)$$

where $\mathrm{cap}_C(A) = \int_{\partial C} \rho_{C,A}(\mathrm{d}y)$ denotes the capacity

$$\Rightarrow \quad \mathbb{E}_x \tau_A = w_A(x) \simeq \frac{1}{\mathrm{cap}_{B_\delta(x)}(A)} \int_{A^c} h_{B_\delta(x),A}(y)\,\mathrm{d}y$$

Variational representation via Dirichlet form

$$\mathrm{cap}_C(A) = \int_{(C\cup A)^c} \|\nabla h_{C,A}(x)\|^2\,\mathrm{d}x = \inf_{h\in\mathcal{H}_{C,A}} \int_{(C\cup A)^c} \|\nabla h(x)\|^2\,\mathrm{d}x$$

where $\mathcal{H}_{C,A}$ = set of sufficiently smooth functions h satisfying b.c.

General case

$$dx_t^\varepsilon = -\nabla V(x_t^\varepsilon)\, dt + \sqrt{2\varepsilon}\, dB_t$$

What changes as the generator Δ is replaced by $\varepsilon\Delta - \nabla V \cdot \nabla$?

$$\mathrm{cap}_C(A) = \varepsilon \inf_{h \in \mathcal{H}_{C,A}} \int_{(C\cup A)^c} \|\nabla h(x)\|^2 \, \mathrm{e}^{-V(x)/\varepsilon} \, \mathrm{d}x$$

$$\mathbb{E}_x \tau_A = w_A(x) \simeq \frac{1}{\mathrm{cap}_{B_\delta(x)}(A)} \int_{A^c} h_{B_\delta(x),A}(y) \, \mathrm{e}^{-V(y)/\varepsilon} \, \mathrm{d}y$$

It remains to investigate capacity and integral.

Assume, $x = x_-^\star$ is a quadratic minimum. Use rough *a priori* bounds on h

$$\int_{A^c} h_{B_\delta(x_-^\star),A}(y) \, \mathrm{e}^{-V(y)/\varepsilon} \, \mathrm{d}y \simeq \frac{(2\pi\varepsilon)^{d/2} \, \mathrm{e}^{-V(x_-^\star)/\varepsilon}}{\sqrt{\det \nabla^2 V(x_-^\star)}}$$

Estimating the capacity

For the truncated energy functional

$$V^{(d)}(y) = \frac{1}{2}\sum_{k=0}^{d} \lambda_k y_k^2 + V_4^{(d)}(y) = -\frac{1}{2}y_0^2 + u_1(y_1) + \frac{1}{2}\sum_{k=2}^{d} \lambda_k y_k^2 + \dots$$

where

$$u_1(y_1) = \frac{1}{2}\lambda_1 y_1^2 + \frac{3}{8}y_1^4$$

To show

$$\operatorname{cap}_C(A) = \varepsilon \frac{\int_{-\infty}^{\infty} e^{-u_1(y_1)/\varepsilon}\, dy_2}{\sqrt{2\pi\varepsilon}} \prod_{j=2}^{d} \sqrt{\frac{2\pi\varepsilon}{\lambda_j}} \left[1 + \mathcal{O}(R(\varepsilon))\right]$$

where $R(\varepsilon) = \mathcal{O}((\varepsilon|\log\varepsilon|)^{1/4})$ is uniformly bounded in d

Sketch of the proof

Proof follows along the lines of [Bovier, Eckhoff, Gayrard & Klein 04]

- Upper bound: Use Dirichlet form representation of capacity

$$\text{cap} = \inf_h \Phi(h) \leqslant \Phi(h_+) = \Phi(h_+) = \varepsilon \int \|\nabla h_+(y)\|^2 \, e^{-V(y)/\varepsilon} \, dy$$

  Choose $\delta = \sqrt{c\varepsilon|\log \varepsilon|}$ and

$$h_+(z) = \begin{cases} 1 & \text{for } y_0 < -\delta \\ f(y_0) & \text{for } -\delta < y_0 < \delta \\ 0 & \text{for } y_0 > \delta \end{cases}$$

  where $\varepsilon f''(y_0) + \partial_{y_0} V(y_0, 0) f' y_0) = 0$ with b.c. $f(\pm\delta) = 0$ or 1, resp.
- Lower bound: Bound Dirichlet form for capacity from below by
  - restricting domain
  - taking only 1*st* component of ∇h
  - using b.c. derived from *a priori* bound on $h_{C,A}$