**Numerics and Theory for Stochastic Evolution Equations**

University of Bielefeld, 22–24 November 2006

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# Desynchronisation of coupled bistable oscillators perturbed by additive white noise

Joint work with Nils Berglund & Bastien Fernandez, CPT, Marseille

## Metastability in stochastic lattice models

- ▷ Lattice: $\Lambda \subset \mathbb{Z}^d$
- ▷ Configuration space: $\mathcal{X} = S^\Lambda$, $S$ finite set (e.g. $\{-1, 1\}$)
- ▷ Hamiltonian: $H : \mathcal{X} \to \mathbb{R}$ (e.g. Ising model or lattice gas)
- ▷ Gibbs measure: $\mu_\beta(x) = \mathrm{e}^{-\beta H(x)}/Z_\beta$
- ▷ Dynamics: Markov chain with invariant measure $\mu_\beta$
(e.g. Metropolis such as Glauber or Kawasaki dynamics)

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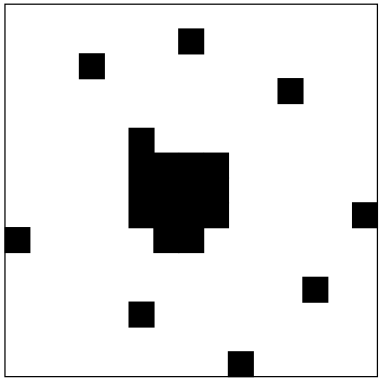
Results (for $\beta \gg 1$) on

- ▷ Transition time between empty and full configuration
- ▷ Transition path
- ▷ Shape of critical droplet




- ▷ Frank den Hollander, *Metastability under stochastic dynamics*, Stochastic Process. Appl. **114** (2004), 1–26
- ▷ Enzo Olivieri and Maria Eulália Vares, *Large deviations and metastability*, Cambridge University Press, Cambridge, 2005

## Metastability in reversible diffusions

$$dx^\sigma(t) = -\nabla V(x^\sigma(t))\,dt + \sigma\,dB(t)$$

- ▷ $V : \mathbb{R}^d \to \mathbb{R}$: potential, growing at infinity
- ▷ $B(t)$: $d$-dimensional Brownian motion

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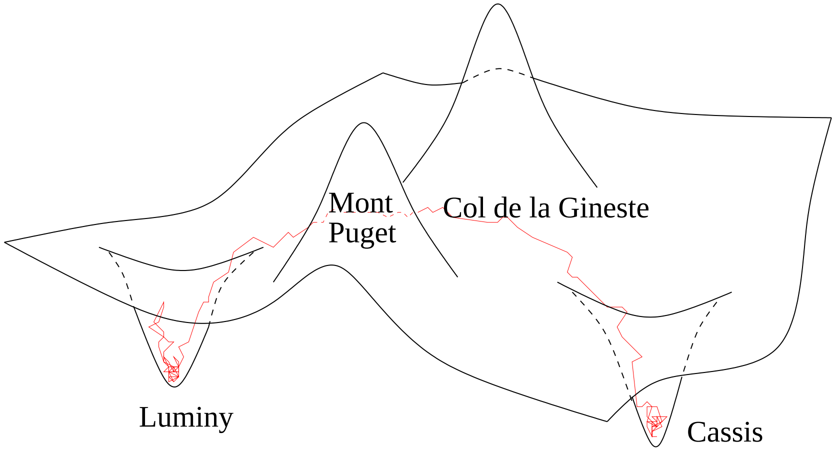


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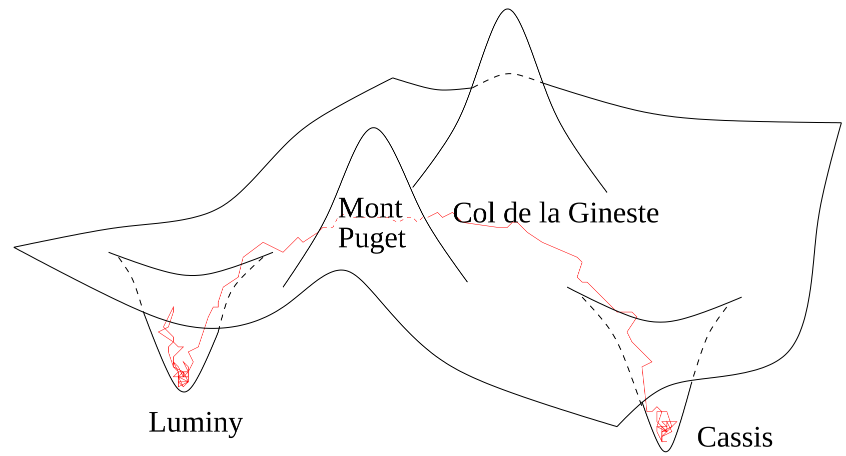
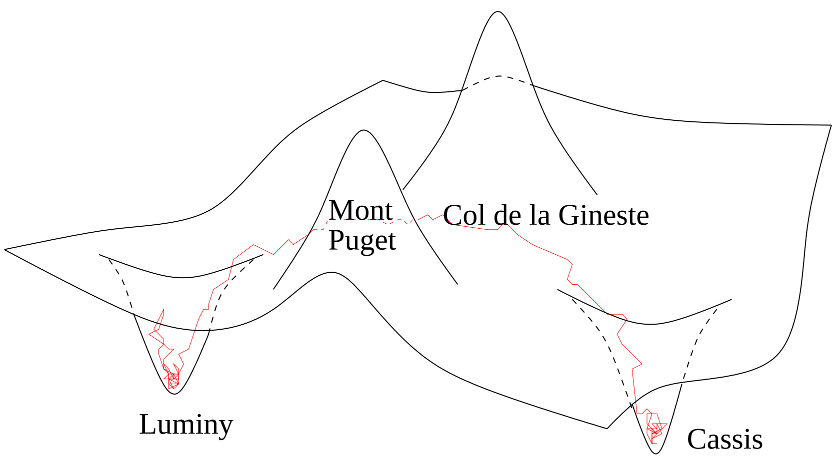



Transition time $\tau$ between potential wells (first-hitting time):

- ▷ Large deviations (Wentzell & Freidlin): $\lim_{\sigma\to 0}\sigma^2 \log \mathbb{E}\tau$
- ▷ Subexponential asymptotics (Bovier, Eckhoff, Gayrard, Klein; Helffer, Nier, Klein)

## Metastability in reversible diffusions

- ▷ Stationary points:
  $\mathcal{S} = \{x \colon \nabla V(x) = 0\}$
- ▷ Saddles of index $k \in \mathbb{N}_0$:
  $\mathcal{S}_k = \{x \in \mathcal{S} \colon \text{Hess}\, V(x) \text{ has } k \text{ negative eigenvalues}\}$



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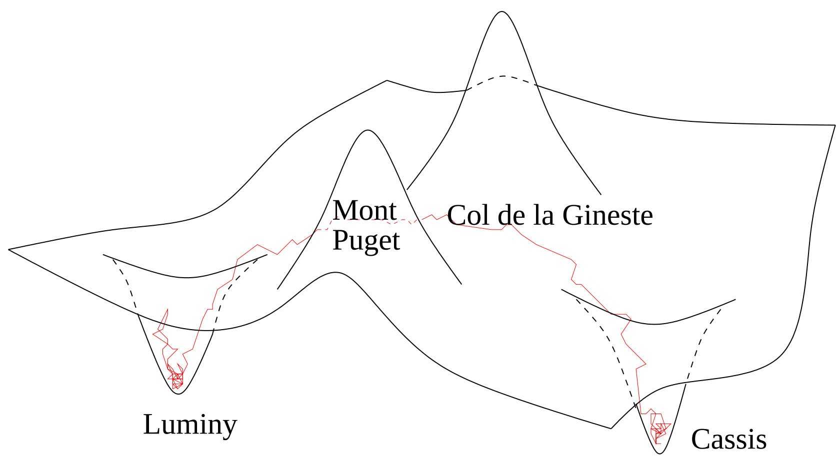
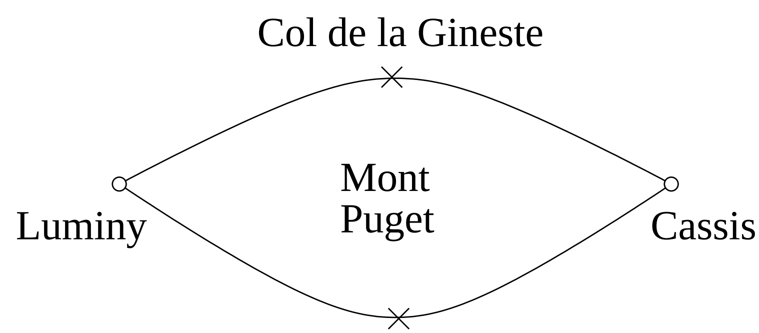
▷ (Multi-)Graph $\mathcal{G} = (\mathcal{S}_0, \mathcal{E})$:

$x \leftrightarrow y$ *iff* $x, y$ belong to unstable manifold of some $s \in \mathcal{S}_1$

▷ $x^\sigma(t)$ resembles Markovian jump process on $\mathcal{G}$



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Gradient system: $dx^\sigma(t) = -\nabla V_\gamma(x^\sigma(t))dt + \sigma\sqrt{N}\,dB(t)$

Global potential: $V_\gamma(x) = \sum_{i\in\Lambda} U(x_i) + \frac{\gamma}{4}\sum_{i\in\Lambda}(x_{i+1} - x_i)^2$

**Weak coupling**

For $\gamma = 0$: $\mathcal{S} = \{-1, 0, 1\}^\Lambda$, $\mathcal{S}_0 = \{-1, 1\}^\Lambda$, $\mathcal{G} =$ hypercube

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**Theorem**

$\forall N \ \exists \gamma^{\star}(N) > 0$ s.t.

- ▷ All $x^{\star}(\gamma) \in S_k(\gamma)$ depend continuously on $\gamma \in [0, \gamma^{\star}(N))$
- ▷ $\frac{1}{4} \leqslant \inf_{N \geqslant 2} \gamma^{\star}(N) \leqslant \gamma^{\star}(3) = \frac{1}{3}\left(\sqrt{3 + 2\sqrt{3}} - \sqrt{3}\right) = 0.2701\ldots$

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For $0 < \gamma \ll 1$:

$$V_\gamma(x^\star(\gamma)) = V_0(x^\star(0)) + \frac{\gamma}{4} \sum_{i \in \Lambda} (x^\star_{i+1}(0) - x^\star_i(0))^2 + \mathcal{O}(\gamma^2)$$

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Dynamics is like in an Ising spin system with Glauber dynamics: Minimize number of interfaces

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| − | − | − | − | − | − | − | − | − | − | − | 0 | + | + | + | + | + |
| − | − | − | − | − | 0 | + | + | + | + | + | + | + | + | + | + | + |
| − | 0 | + | + | + | + | + | + | + | + | + | + | + | + | + | + | + |
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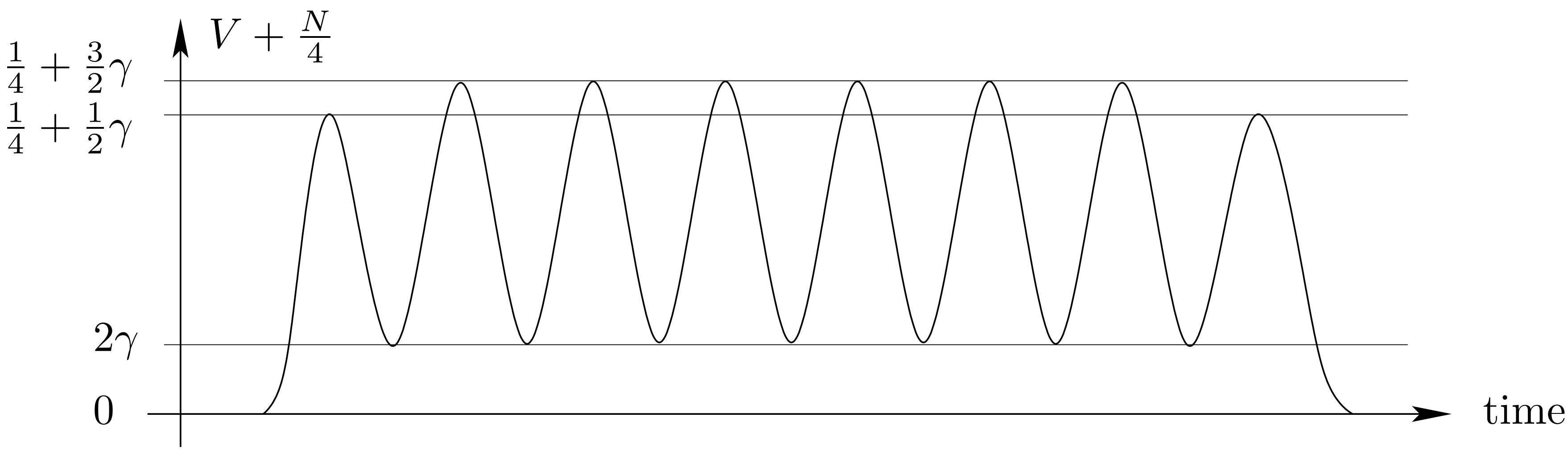


Potential seen along an optimal transition path:
Differences in potential height determine transition times

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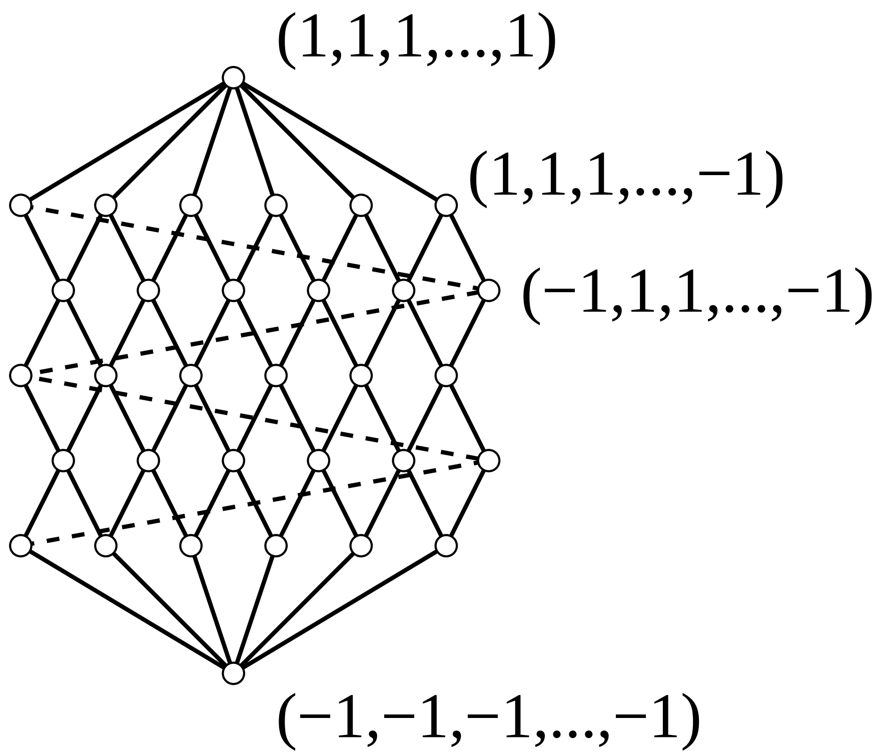
Dynamics is like in an Ising spin system with Glauber dynamics




Partial representation of $\mathcal{G}$ showing only edges contained in optimal transition paths

## Strong coupling: Synchronisation

For all $\gamma \geq 0$: $I^{\pm} = \pm(1, 1, \ldots, 1) \in \mathcal{S}_0$ and $O = (0, 0, \ldots, 0) \in \mathcal{S}$

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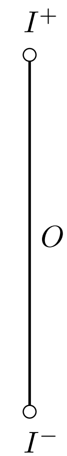


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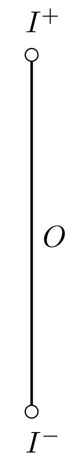


**Theorem**

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**Proof** (using Lyapunov function $W(x)$)

$$\dot{x} = Ax - F(x), \qquad A = \begin{pmatrix} 1-\gamma & \gamma/2 & \dots & \gamma/2 \\ \gamma/2 & & \ddots & \vdots \\ \vdots & \ddots & & \gamma/2 \\ \gamma/2 & \dots & \gamma/2 & 1-\gamma \end{pmatrix}, \qquad F_i(x) = x_i^3$$

$$W(x) = \frac{1}{2}\sum_{i\in\Lambda}(x_i - x_{i+1})^2 = \frac{1}{2}\|x - Rx\|^2 , \qquad Rx = (x_2, \dots, x_N, x_1)$$

$$\frac{\mathrm{d}W(x)}{\mathrm{d}t} = \langle x - Rx, \frac{\mathrm{d}}{\mathrm{d}t}(x - Rx)\rangle \leqslant \langle x - Rx, A(x - Rx)\rangle \leqslant (1 - \frac{\gamma}{\gamma_1})\|x - Rx\|^2$$

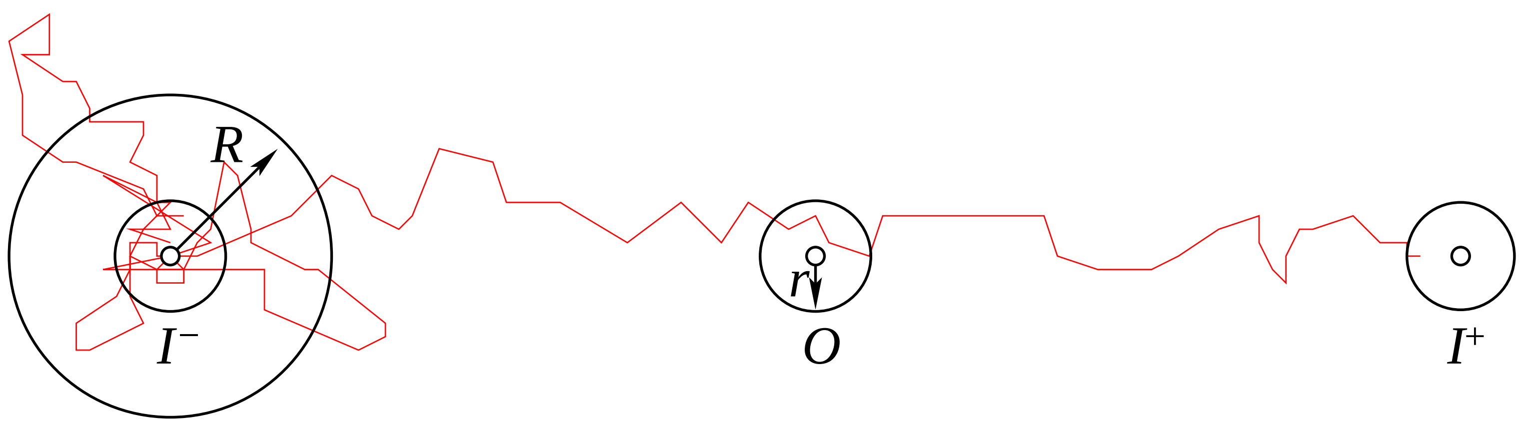
## Strong coupling: Synchronisation

$\tau_+ = \tau^{\text{hit}}(\mathcal{B}(I^+, r))$
$\tau_O = \tau^{\text{hit}}(\mathcal{B}(O, r))$
$\tau_- = \inf\{t > \tau^{\text{exit}}(\mathcal{B}(I^-, R)) \colon x_t \in \mathcal{B}(I^-, r)\}$

**Corollary**

$\forall N \;\; \forall \gamma > \gamma_1(N) \;\; \forall (r, R)$ s.t. $0 < r < R \leqslant \frac{1}{2} \;\; \forall x_0 \in \mathcal{B}(I^-, r)$

- ▷ $\lim_{\sigma \to 0} \mathbb{P}^{x_0}\Big\{ e^{(1/2-\delta)/\sigma^2} \leqslant \tau_+ \leqslant e^{(1/2+\delta)/\sigma^2} \Big\} = 1 \qquad \forall \delta > 0$
- ▷ $\lim_{\sigma \to 0} \sigma^2 \log \mathbb{E}^{x_0}\{\tau_+\} = \dfrac{1}{2}$
- ▷ $\lim_{\sigma \to 0} \mathbb{P}^{x_0}\Big\{ \tau_O < \tau_+ \;\Big|\; \tau_+ < \tau_- \Big\} = 1$



## Intermediate coupling: Reduction via symmetry groups

Global potential $V_\gamma$ is invariant under

- $\triangleright$ $R(x_1, \ldots, x_N) = (x_2, \ldots, x_N, x_1)$
- $\triangleright$ $S(x_1, \ldots, x_N) = (x_N, x_{N-1}, \ldots, x_1)$
- $\triangleright$ $C(x_1, \ldots, x_N) = -(x_1, \ldots, x_N)$

$V_\gamma$ invariant under group $G = D_N \times \mathbb{Z}_2$ generated by $R, S, C$

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$V_\gamma$ invariant under group $G = D_N \times \mathbb{Z}_2$ generated by $R, S, C$

$G$ acts as group of transformations on $\mathcal{X}$, $\mathcal{S}$, $\mathcal{S}_k$ (for all $k$)

**Notions**

- ▷ Orbit of $x \in \mathcal{X}$: $O_x = \{gx \colon g \in G\}$
- ▷ Isotropy group/stabilizer of $x \in \mathcal{X}$: $C_x = \{g \in G \colon gx = x\}$
- ▷ Fixed-point space of a subgroup $H \subset G$:
  $\operatorname{Fix}(H) = \{x \in \mathcal{X} \colon hx = x \;\forall h \in H\}$

**Properties**

- ▷ $|C_x||O_x| = |G|$
- ▷ $C_{gx} = gC_xg^{-1}$
- ▷ $\operatorname{Fix}(gHg^{-1}) = g\operatorname{Fix}(H)$

**Small lattices:** $N = 2$

| $z^\star$ | $O_{z^\star}$ | $C_{z^\star}$ | $\mathsf{Fix}(C_{z^\star})$ |
|---|---|---|---|
| $(0,0)$ | $\{(0,0)\}$ | $G$ | $\{(0,0)\}$ |
| $(1,1)$ | $\{(1,1),(-1,-1)\}$ | $D_2 = \{\mathsf{id}, S\}$ | $\{(x,x)\}_{x\in\mathbb{R}} = \mathcal{D}$ |
| $(1,-1)$ | $\{(1,-1),(-1,1)\}$ | $\{\mathsf{id}, CS\}$ | $\{(x,-x)\}_{x\in\mathbb{R}}$ |
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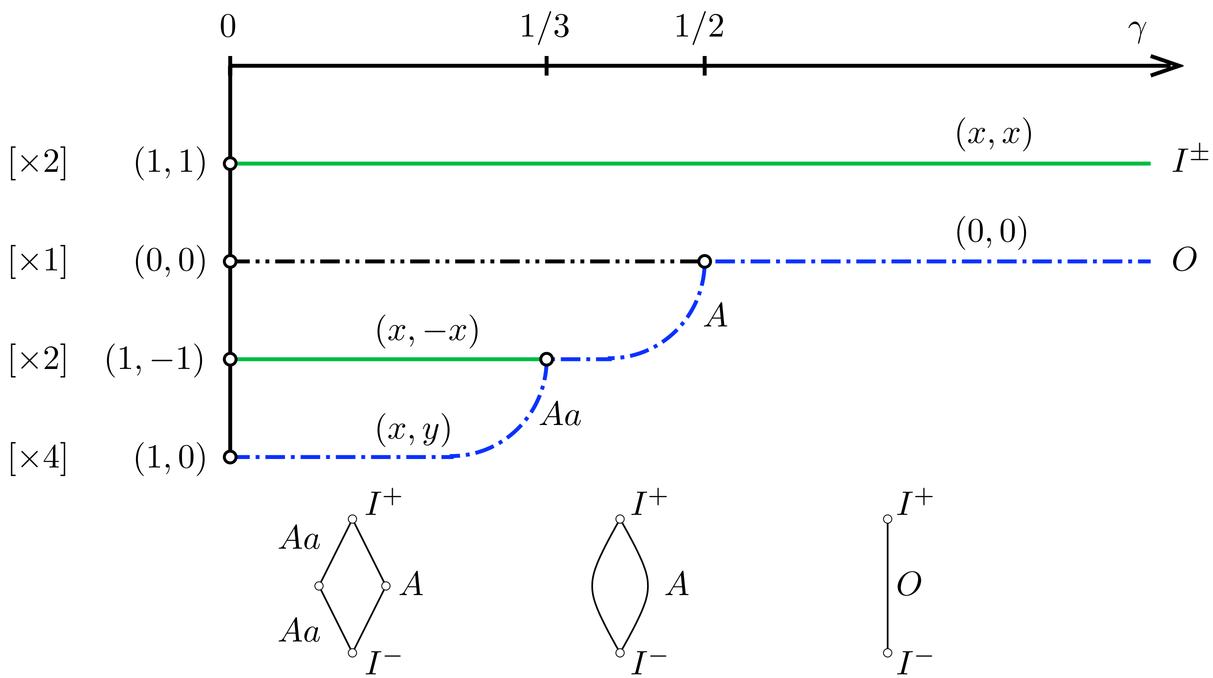
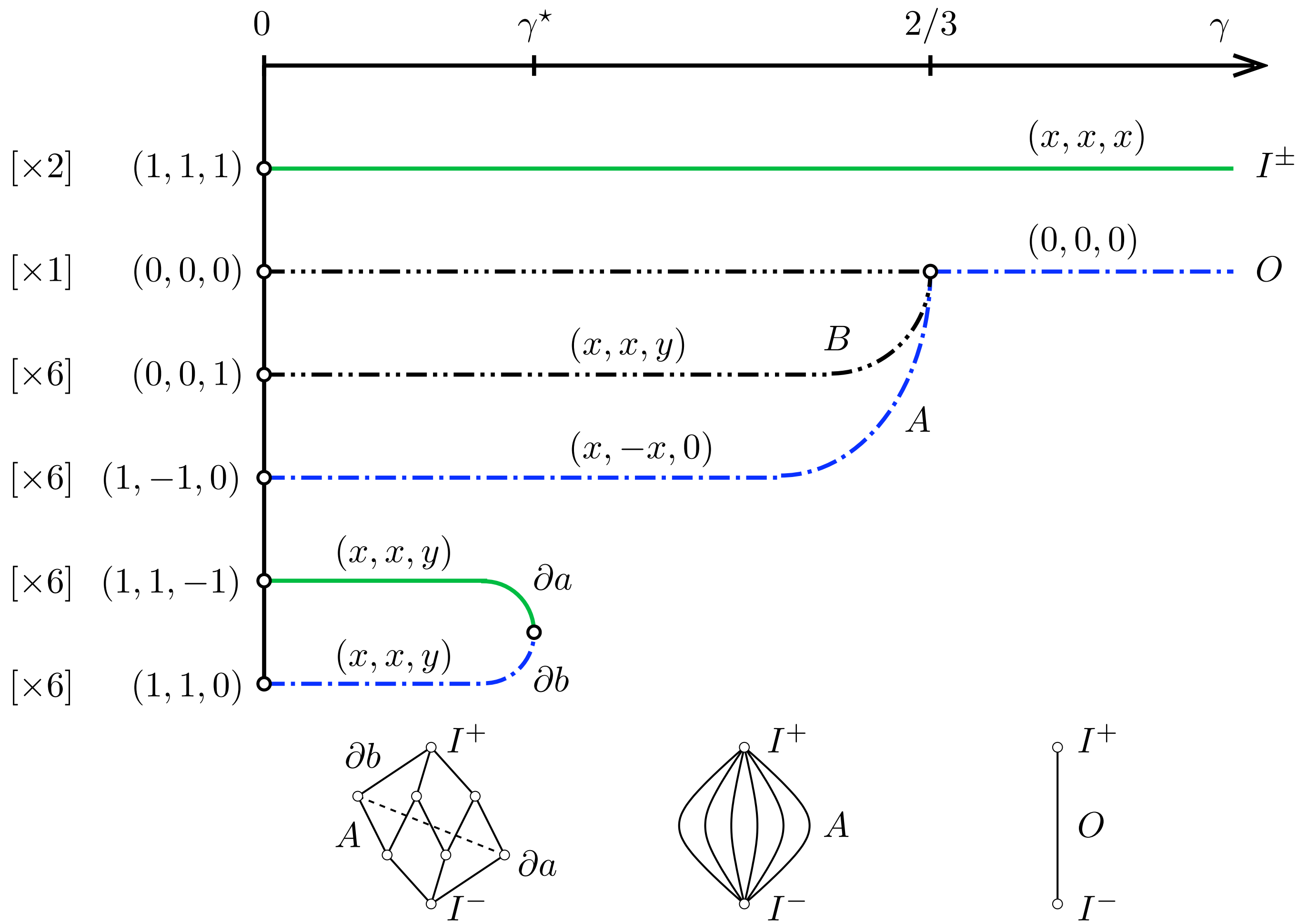
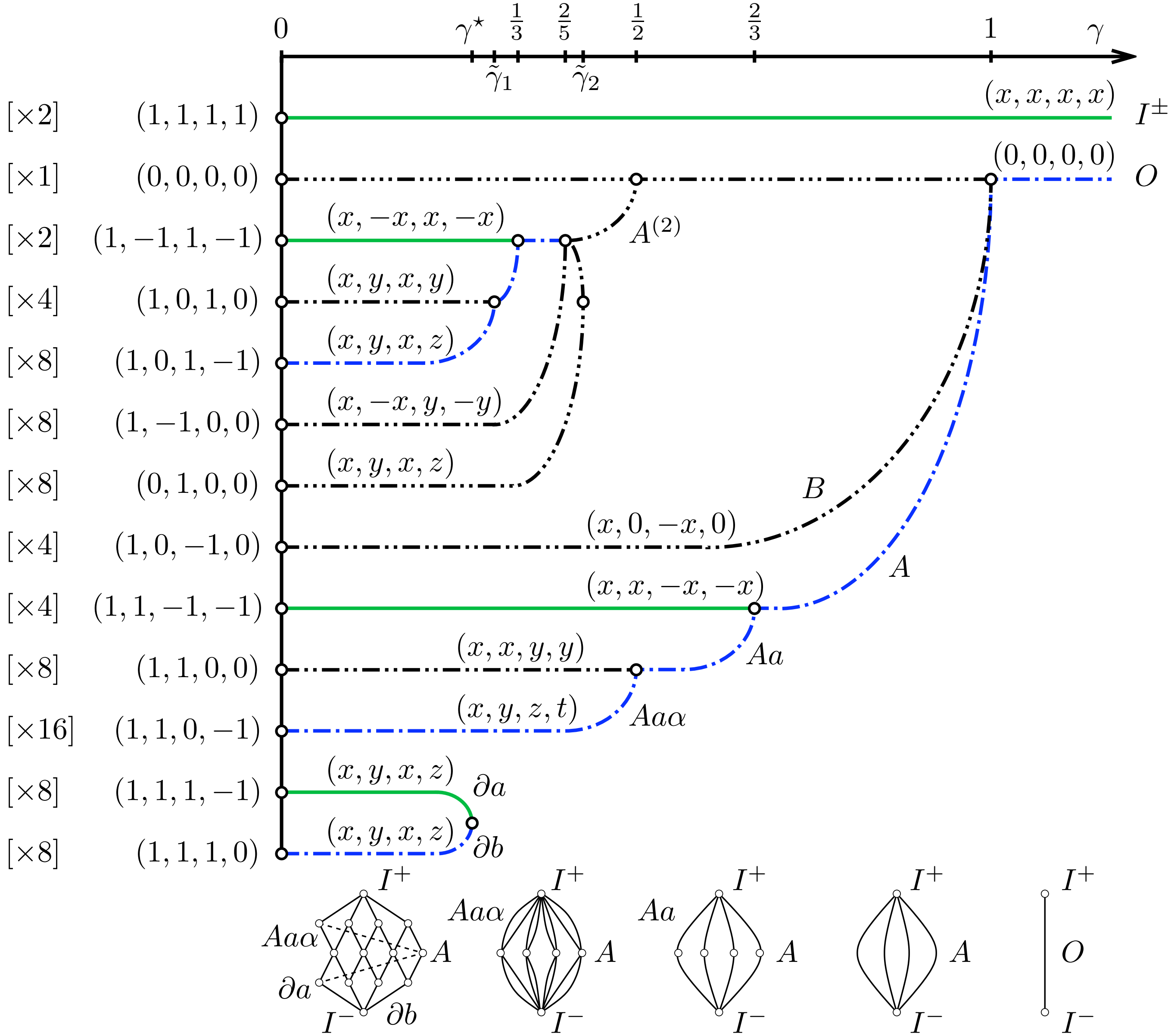


## **Small lattices:** $N = 3$



# Small lattices: $N = 4$



## Desynchronisation transition

### Theorem

$\forall N$ even $\exists \delta(N) > 0$ s.t. for $\gamma_1 - \delta(N) < \gamma < \gamma_1$

- ▷ $|\mathcal{S}| = 2N + 3$
- ▷ $\mathcal{S}$ can be decomposed into

$$\begin{aligned} \mathcal{S}_0 &= O_{I^+} = \{I^+, I^-\} \\ \mathcal{S}_1 &= O_A \;\; = \{A, RA, \dots, R^{N-1}A\} \\ \mathcal{S}_2 &= O_B \;\; = \{B, RB, \dots, R^{N-1}B\} \\ \mathcal{S}_3 &= O_O \;\; = \{O\} \end{aligned}$$

$$A_j = A_j(\gamma) = \frac{2}{\sqrt{3}} \sqrt{1 - \frac{\gamma}{\gamma_1}} \sin\left(\frac{2\pi}{N}\left(j - \frac{1}{2}\right)\right) + \mathcal{O}\left(1 - \frac{\gamma}{\gamma_1}\right)$$

$$V_\gamma(A)/N = -\frac{1}{6}\left(1 - \frac{\gamma}{\gamma_1}\right)^2 + \mathcal{O}\left((1 - \frac{\gamma}{\gamma_1})^3\right)$$

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### Theorem

$\forall N$ even $\exists\delta(N) > 0$ s.t. for $\gamma_1 - \delta(N) < \gamma < \gamma_1$

- $\triangleright$ $|\mathcal{S}| = 2N + 3$
- $\triangleright$ $\mathcal{S}$ can be decomposed into

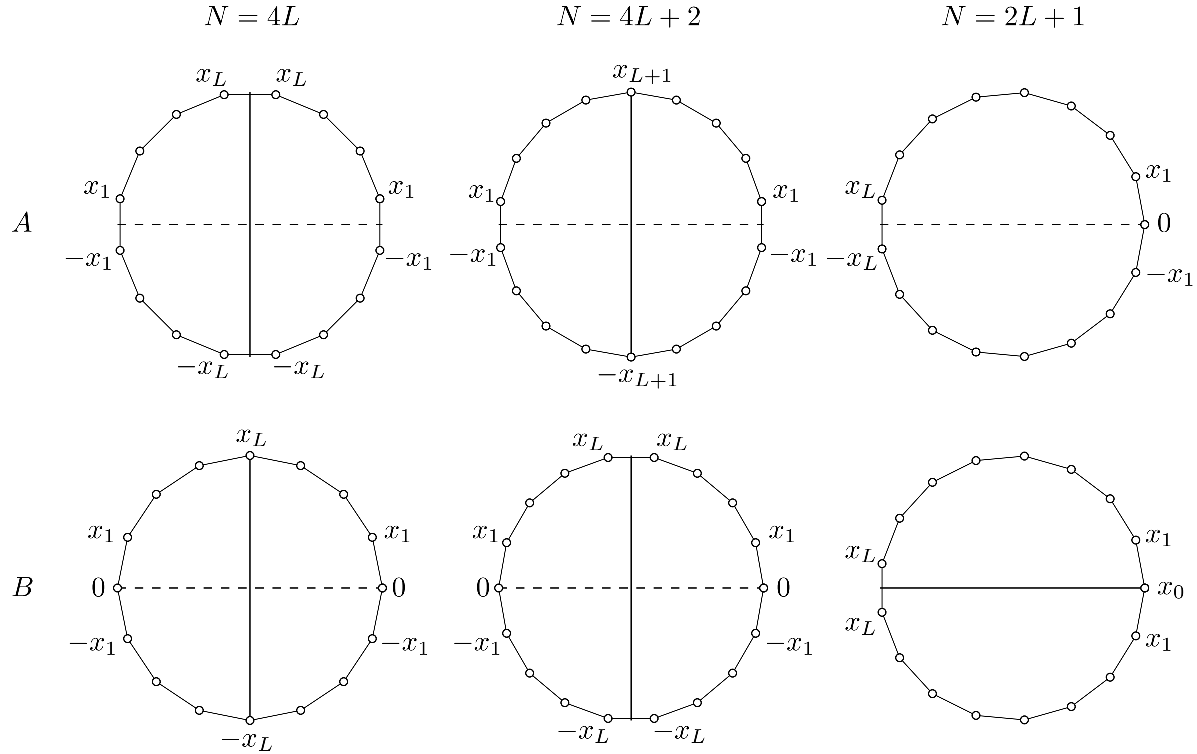
$$\begin{aligned}
\mathcal{S}_0 &= O_{I^+} = \{I^+, I^-\} \\
\mathcal{S}_1 &= O_A \;\; = \{A, RA, \dots, R^{N-1}A\} \\
\mathcal{S}_2 &= O_B \;\; = \{B, RB, \dots, R^{N-1}B\} \\
\mathcal{S}_3 &= O_O \;\; = \{O\}
\end{aligned}$$

$$A_j = A_j(\gamma) = \frac{2}{\sqrt{3}}\sqrt{1 - \frac{\gamma}{\gamma_1}}\sin\Big(\frac{2\pi}{N}\Big(j - \frac{1}{2}\Big)\Big) + \mathcal{O}\Big(1 - \frac{\gamma}{\gamma_1}\Big)$$

$$V_\gamma(A)/N = -\frac{1}{6}\Big(1 - \frac{\gamma}{\gamma_1}\Big)^2 + \mathcal{O}\Big((1 - \frac{\gamma}{\gamma_1})^3\Big)$$

- $\triangleright$ $N$ odd: Similar result, $|\mathcal{S}| \geqslant 4N + 3$
- $\triangleright$ Corollary on $\tau$, with $\tau_0 \mapsto \tau_{\cup gA}$
- $\triangleright$ $A$ and $B$ have particular symmetries (see next slide)

# Symmetries




| $N$ | $x$ | Fix($C_x$) |
|---|---|---|
| $4L$ | $A$ | $(x_1,\ldots,x_L,x_L,\ldots,x_1,-x_1,\ldots,-x_L,-x_L,\ldots,-x_1)$ |
| | $B$ | $(x_1,\ldots,x_L,\ldots,x_1,0,-x_1,\ldots,-x_L,\ldots,-x_1,0)$ |
| $4L+2$ | $A$ | $(x_1,\ldots,x_{L+1},\ldots,x_1,-x_1,\ldots,-x_{L+1},\ldots,-x_1)$ |
| | $B$ | $(x_1,\ldots,x_L,x_L\ldots,x_1,0,-x_1,\ldots,-x_L,-x_L,\ldots,-x_1,0)$ |
| $2L+1$ | $A$ | $(x_1,\ldots,x_L,-x_L,\ldots,-x_1,0)$ |
| | $B$ | $(x_1,\ldots,x_L,x_L,\ldots,x_1,x_0)$ |

## Large N: Sequence of symmetry-breaking bifurcations

Rescaling: $\widetilde{\gamma} = \frac{\gamma}{\gamma_1} = \gamma(1 - \cos(2\pi/N))$,

$$\widetilde{\gamma}_M = \frac{1-\cos(2\pi/N)}{1-\cos(2\pi M/N)} = \frac{1}{M^2}\Big[1 + \mathcal{O}\Big(\frac{M^2}{N^2}\Big)\Big]$$

**Theorem**

$\forall M \geqslant 1\ \exists N_M < \infty$ s.t. for $N \geqslant N_M$ and $\widetilde{\gamma}_{M+1} < \widetilde{\gamma} < \widetilde{\gamma}_M$, $\mathcal{S}$ can be decomposed as

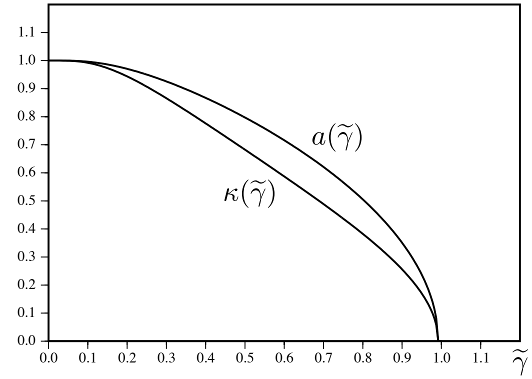
$$\begin{aligned} \mathcal{S}_0 &= O_{I^+} = \{I^+, I^-\} & \\ \mathcal{S}_{2m-1} &= O_{A^{(m)}} & m = 1, \dots, M \\ \mathcal{S}_{2m} &= O_{B^{(m)}} & m = 1, \dots, M \\ \mathcal{S}_{2M+1} &= O_O = \{O\} & \end{aligned}$$

with $A_j^{(m)}(\widetilde{\gamma}) = a(m^2\widetilde{\gamma})\,\mathrm{sn}\Big(\frac{4\,\mathrm{K}(\kappa(m^2\widetilde{\gamma}))}{N} m\Big(j - \frac{1}{2}\Big), \kappa(m^2\widetilde{\gamma})\Big) + \mathcal{O}\Big(\frac{M}{N}\Big)$

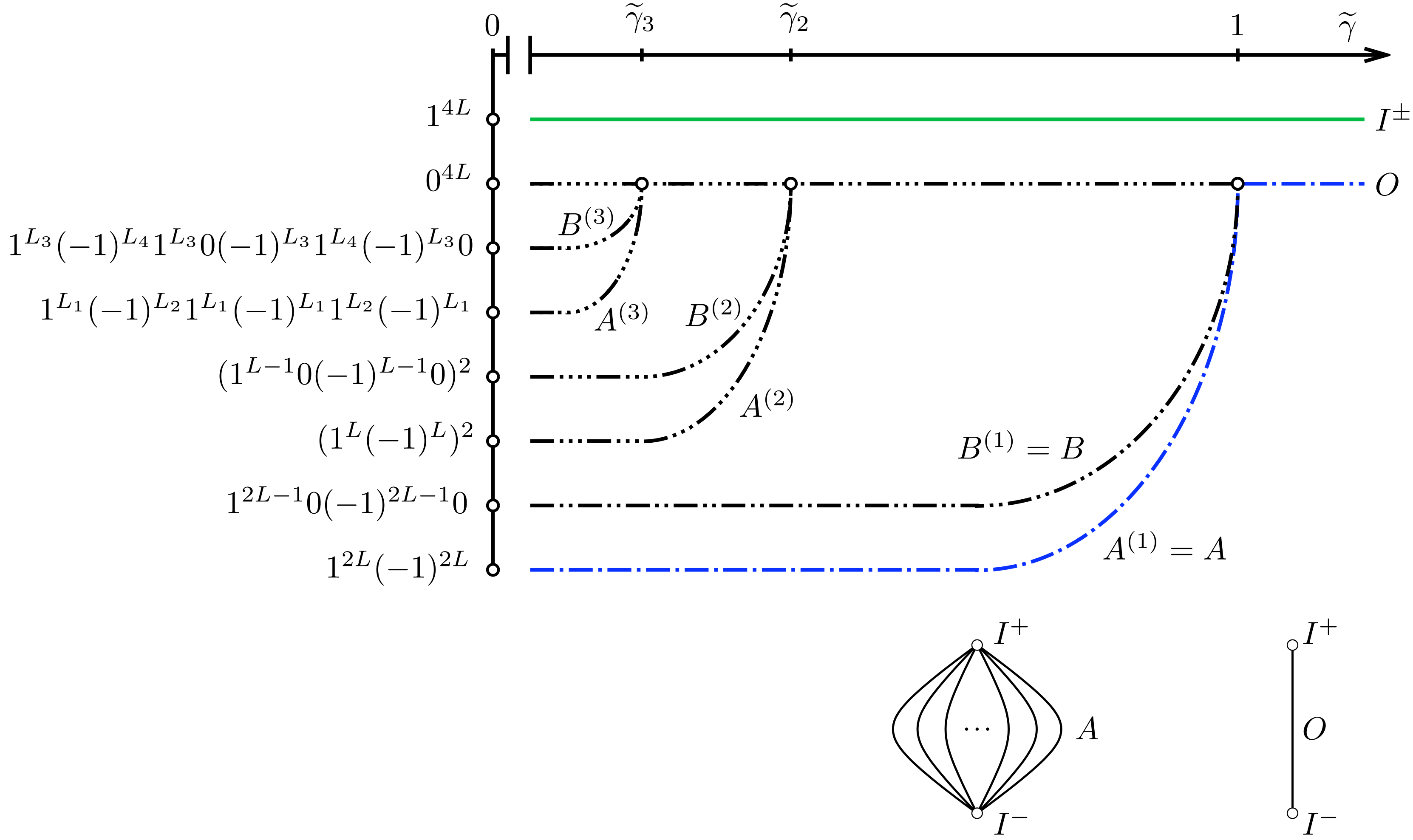
and $\kappa(\widetilde{\gamma})$, $a(\widetilde{\gamma})$ implicitly defined by

$$\widetilde{\gamma} = \frac{\pi^2}{4\,\mathrm{K}(\kappa(\widetilde{\gamma}))^2(1+\kappa(\widetilde{\gamma})^2)}$$

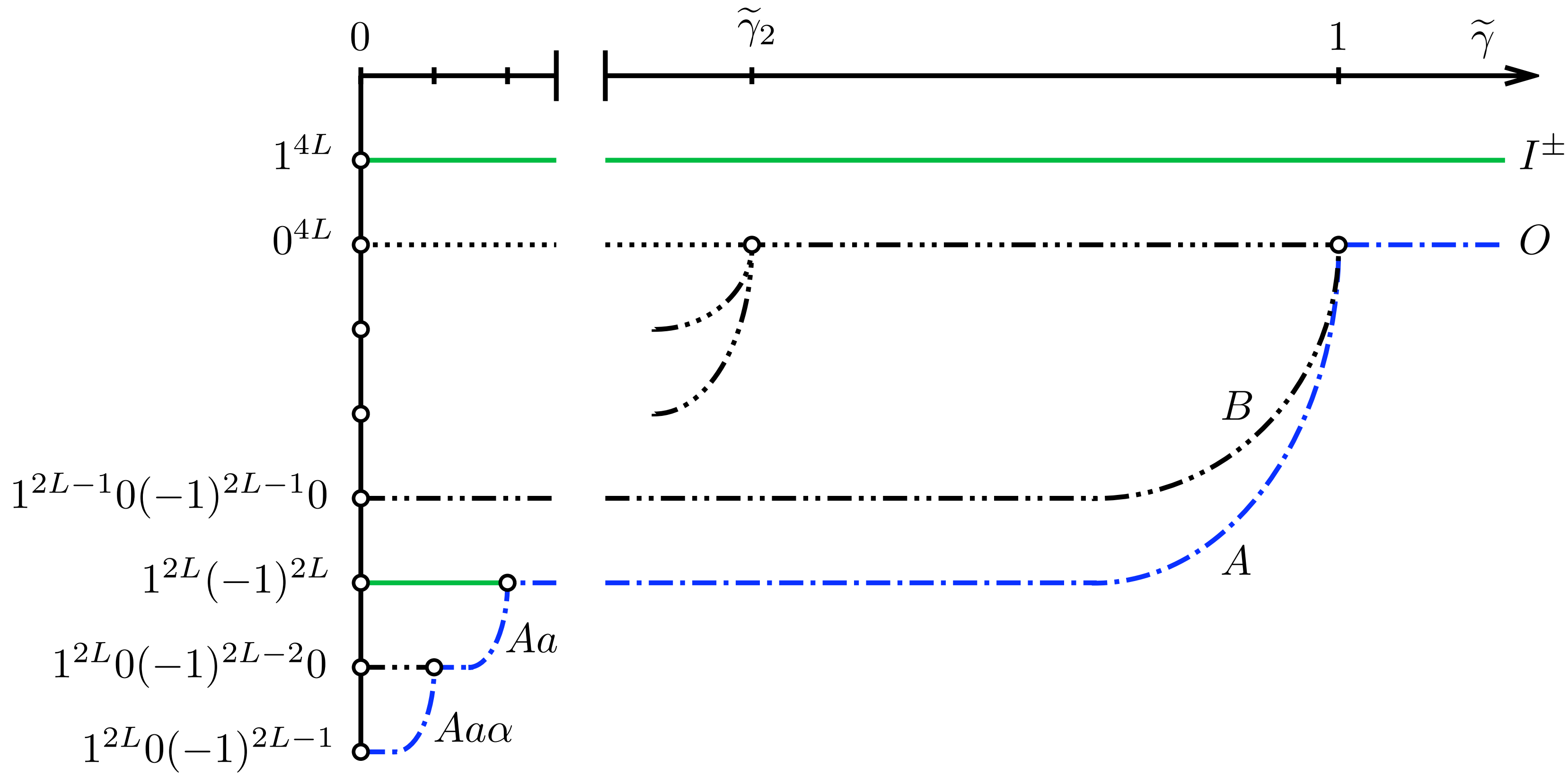
$$a(\widetilde{\gamma})^2 = \frac{2\kappa(\widetilde{\gamma})^2}{1+\kappa(\widetilde{\gamma})^2}$$



# Large N: Bifurcation diagram ($N = 4L$)



## Large N: Bifurcation diagram (N = 4L)

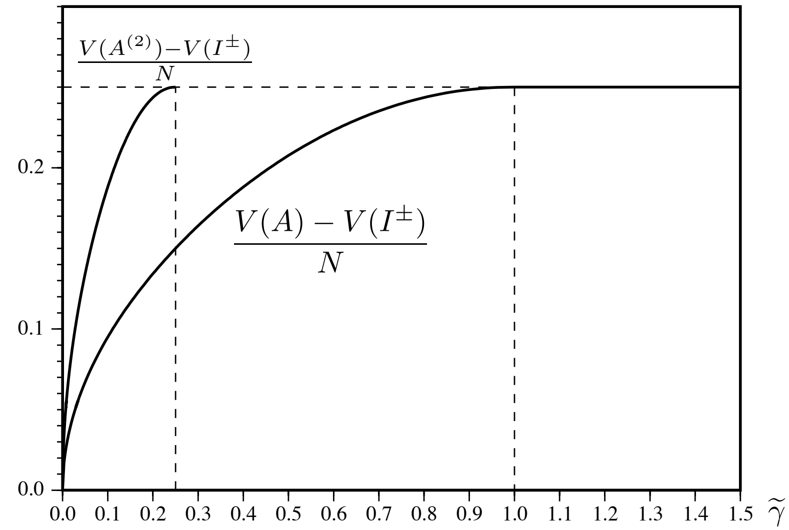



Expected behaviour near zero coupling

## Large N: The transition probabilities

Potential difference $\quad (\kappa = \kappa(\widetilde{\gamma}))$

$$H(\widetilde{\gamma}) = \frac{V(A)-V(I^{\pm})}{N}$$
$$= \frac{1}{4} - \frac{1}{3(1+\kappa^2)}\left[\frac{2+\kappa^2}{1+\kappa^2} - 2\frac{\mathsf{E}(\kappa)}{\mathsf{K}(\kappa)}\right] + \mathcal{O}\left(\frac{\kappa^2}{N}\right)$$



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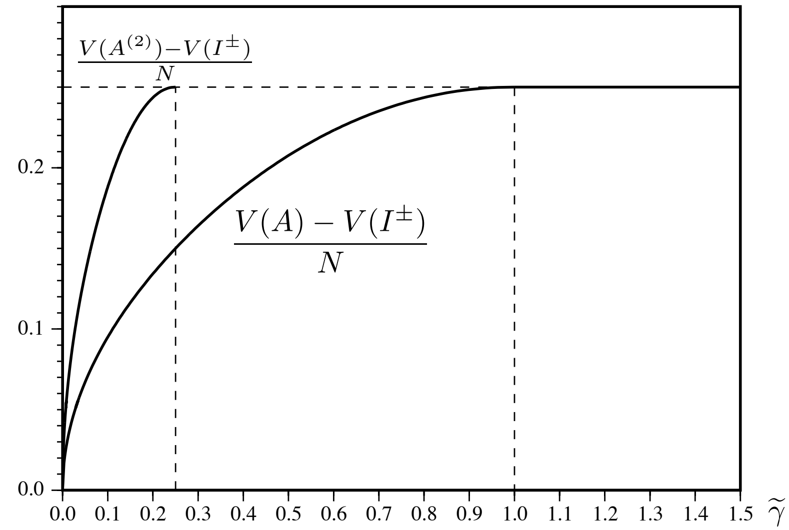


$$\tau_+ = \tau^{\text{hit}}(\mathcal{B}(I^+, r))$$
$$\tau_A = \tau^{\text{hit}}\bigl(\textstyle\bigcup_{g\in G} \mathcal{B}(gA, r)\bigr)$$
$$\tau_- = \inf\{t > \tau^{\text{exit}}(\mathcal{B}(I^-, R)) \colon x_t \in \mathcal{B}(I^-, r)\}$$

### Corollary

$\forall\widetilde{\gamma} \in (0,1]\ \exists N_0(\widetilde{\gamma})\ \forall N \geqslant N_0(\widetilde{\gamma})\ \forall(r,R)$ s.t. $0 < r < R \leqslant \frac{1}{2}\ \forall x_0 \in \mathcal{B}(I^-, r)$

- $\displaystyle\lim_{\sigma\to0} \mathbb{P}^{x_0}\Bigl\{\mathrm{e}^{(2H(\widetilde{\gamma})-\delta)/\sigma^2} \leqslant \tau_+ \leqslant \mathrm{e}^{(2H(\widetilde{\gamma})+\delta)/\sigma^2}\Bigr\} = 1 \qquad \forall\delta > 0$
- $\displaystyle\lim_{\sigma\to0} \sigma^2 \log \mathbb{E}^{x_0}\{\tau_+\} = 2H(\widetilde{\gamma})$
- $\displaystyle\lim_{\sigma\to0} \mathbb{P}^{x_0}\Bigl\{\tau_A < \tau_+ \Bigm| \tau_+ < \tau_-\Bigr\} = 1$

## Ideas of the proof

$$x \in \mathcal{S} \quad \Leftrightarrow \quad f(x_n) + \frac{\gamma}{2}\Big[x_{n+1} - 2x_n + x_{n-1}\Big] = 0$$

$$\Leftrightarrow \quad \begin{cases} x_{n+1} = x_n + \varepsilon w_n - \frac{1}{2}\varepsilon^2 f(x_n) \\ w_{n+1} = w_n - \frac{1}{2}\varepsilon\Big[f(x_n) + f(x_{n+1})\Big] \end{cases}$$

$$\varepsilon = \sqrt{\frac{2}{\gamma}} \simeq \frac{2\pi}{N\sqrt{\tilde{\gamma}}} \ll 1$$

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- ▷ Area-preserving map
- ▷ Discretisation of $\ddot{x} = -f(x)$
- ▷ Almost conserved quantity: $C(x,w) = \frac{1}{2}(x^2 + w^2) - \frac{1}{4}x^4$
  $C(x_{n+1}, w_{n+1}) = C(x_n, w_n) + \mathcal{O}(\varepsilon^3)$

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In action-angle variables $(I, \psi)$:

$$\begin{cases} \psi_{n+1} = \psi_n + \varepsilon\Omega(I_n) + \varepsilon^3 f(\psi_n, I_n, \varepsilon) & \pmod{2\pi} \\ I_{n+1} = I_n + \varepsilon^3 g(\psi_n, I_n, \varepsilon) \end{cases}$$

$I = h(C)$, and $(\psi, C) \mapsto (x, w)$ involves elliptic functions.

**Ideas of the proof**

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$\Omega(I)$ monotonous in $I \Rightarrow$ twist map

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$\Omega(I)$ monotonous in $I \Rightarrow$ twist map

▷ "$\varepsilon^3 = 0$":

$$\begin{cases} \psi_n = \psi_0 + n\varepsilon\Omega(I_0) & \pmod{2\pi} \\ I_n = I_0 \end{cases}$$

Orbit of period $N$ if $N\varepsilon\Omega(I_0) = 2\pi M$, $M \in \{1, 2, \dots\}$

Rotation number $\nu = M/N$; $j \mapsto x_j$ has $2M$ sign changes

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$\Omega(I)$ monotonous in $I$ $\Rightarrow$ twist map.

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Rotation number $\nu = M/N$; $j \mapsto x_j$ has $2M$ sign changes

▷ $\varepsilon > 0$: Poincaré–Birkhoff theorem
$\exists$ at least 2 periodic orbits for each $\nu$ with $2\pi\nu/\varepsilon$ in range of $\Omega$
Problem: Show that there are only 2 such orbits for each $\nu$

**Ideas of the proof**

$$\begin{cases} \psi_{n+1} = \psi_n + \varepsilon\Omega(I_n) + \varepsilon^3 f(\psi_n, I_n, \varepsilon) & \pmod{2\pi} \\ I_{n+1} = I_n + \varepsilon^3 g(\psi_n, I_n, \varepsilon) \end{cases}$$

Generating function: $(\psi_n, \psi_{n+1}) \mapsto G(\psi_n, \psi_{n+1})$ with

$$\partial_1 G(\psi_n, \psi_{n+1}) = -I_n \qquad \partial_2 G(\psi_n, \psi_{n+1}) = I_{n+1}$$

**Ideas of the proof**

$$\begin{cases}\psi_{n+1} = \psi_n + \varepsilon\Omega(I_n) + \varepsilon^3 f(\psi_n, I_n, \varepsilon) & \pmod{2\pi} \\ I_{n+1} = I_n + \varepsilon^3 g(\psi_n, I_n, \varepsilon)\end{cases}$$

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▷ Orbits of period $N$ are stationary points of

$$G_N(\psi_1, \dots, \psi_N) = G(\psi_1, \psi_2) + G(\psi_2, \psi_3) + \dots + G(\psi_N, \psi_1 + 2\pi N\nu)$$

**Ideas of the proof**

$$\begin{cases}\psi_{n+1} = \psi_n + \varepsilon\Omega(I_n) + \varepsilon^3 f(\psi_n, I_n, \varepsilon) & \pmod{2\pi} \\ I_{n+1} = I_n + \varepsilon^3 g(\psi_n, I_n, \varepsilon)\end{cases}$$

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$$G_N(\psi_1, \dots, \psi_N) = G(\psi_1, \psi_2) + G(\psi_2, \psi_3) + \cdots + G(\psi_N, \psi_1 + 2\pi N\nu)$$

In our case, Fourier expansion given by

$$G(\psi_1, \psi_2) = \varepsilon G_0\biggl(\frac{\psi_2 - \psi_1}{\varepsilon}, \varepsilon\biggr) + 2\varepsilon^3 \sum_{p=1}^{\infty} \widehat{G}_p\biggl(\frac{\psi_2 - \psi_1}{\varepsilon}, \varepsilon\biggr) \cos\bigl(p(\psi_1 + \psi_2)\bigr)$$

▷ $N$ particles “connected by springs” in periodic external potential

▷ Analyse stationary points using Fourier variables for $(\psi_1, \dots, \psi_n)$