The Effect of Gaussian White Noise on Dynamical Systems

Part III: Bifurcations in Slow–Fast Systems

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19 March 2014

Slowly driven systems in dimension $n = 1$

Slowly driven systems

Recall from Monday's lecture

Parameter dependent ODE, perturbed by small Gaussian white noise

$$\mathrm{d}x_s = \tilde{f}(x_s, \lambda)\,\mathrm{d}s + \sigma\,\mathrm{d}W_s \qquad (x_s \in \mathbb{R}^1)$$

Assume parameter varies slowly in time: $\lambda = \lambda(\varepsilon s)$

$$\mathrm{d}x_s = \tilde{f}(x_s, \lambda(\varepsilon s))\,\mathrm{d}s + \sigma\,\mathrm{d}W_s$$

Rewrite in slow time $t = \varepsilon s$

$$\mathrm{d}x_t = \frac{1}{\varepsilon} f(x_t, t)\,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t$$

Assumptions on Monday

Existence of a uniformly asymptotically stable equilibrium branch $x^\star(t)$

$$\exists!\, x^\star : I \to \mathbb{R} \text{ s.t. } f(x^\star(t), t) = 0$$

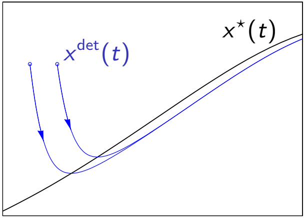
and

$$a^\star(t) = \partial_x f(x^\star(t), t) \leqslant -a_0 < 0$$

[Tihonov 1952]: Then there exists an adiabatic solution $\bar{x}(t, \varepsilon)$

$$\bar{x}(t, \varepsilon) = x^\star(t) + \mathcal{O}(\varepsilon)$$

and $\bar{x}(t, \varepsilon)$ attracts nearby solutions exp. fast



Defining the strip describing the typical spreading

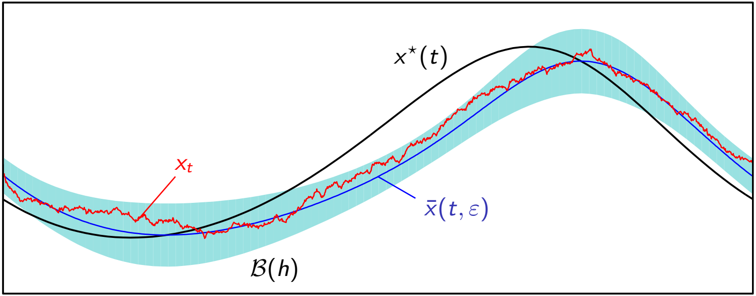
- Let $v(t)$ be the variance of the solution $z(t)$ of the linearized SDE for the deviation $x_t - \bar{x}(t,\varepsilon)$
- $v(t)/\sigma^2$ is solution of a deterministic slowly driven system admitting a uniformly asymptotically stable equilibrium branch
- Let $\zeta(t)$ be the adiabatic solution of this system
- $\zeta(t) \approx 1/|a(t)|$, where $a(t) = \partial_x f(\bar{x}(t,\varepsilon),t) \leq -a_0/2 < 0$

Define a strip $\mathcal{B}(h)$ around $\bar{x}(t,\varepsilon)$ of width $\simeq h\sqrt{\zeta(t)}$ and the first-exit time $\tau_{\mathcal{B}(h)}$

$$\mathcal{B}(h) = \{(x,t)\colon |x - \bar{x}(t,\varepsilon)| < h\sqrt{\zeta(t)}\}$$

$$\tau_{\mathcal{B}(h)} = \inf\{t > 0\colon (x_t,t) \notin \mathcal{B}(h)\}$$

Concentration of sample paths



Theorem [Berglund & G 2002, 2006]

$$\mathbb{P}\bigl\{\tau_{\mathcal{B}(h)} < t\bigr\} \leq \mathit{const}\,\frac{1}{\varepsilon}\biggl|\int_0^t a(s)\,\mathrm{d}s\biggr|\,\frac{h}{\sigma}\;\mathrm{e}^{-h^2[1-\mathcal{O}(\varepsilon)-\mathcal{O}(h)]/2\sigma^2}$$

Next goal

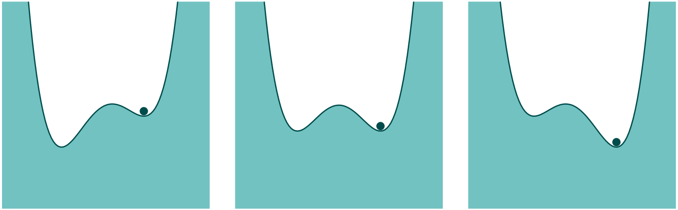

1 km
5088/1.1
Bifurkation
Gesmold
5088/1.3
4,8 km
4,3 km
Melle
MO
Melle
MG
Nieders. Mühlen-Tour
Else-Werre-Tour
OSNABRÜCKER
... die Radler-Region

Avoided bifurcation: Stochastic Resonance

Overdamped motion of a Brownian particle in a periodically modulated potential

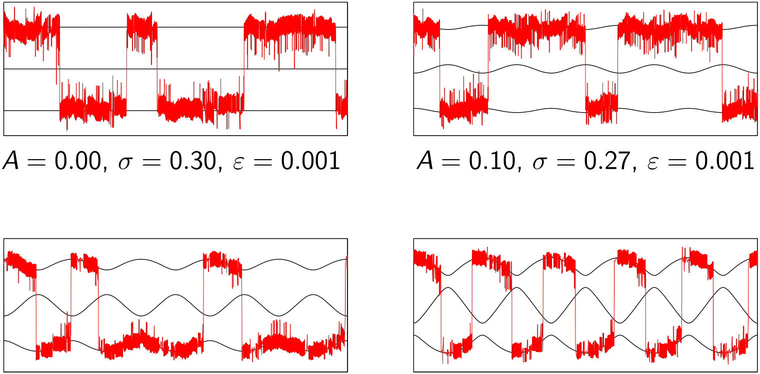
$$\mathrm{d}x_t = -\frac{1}{\varepsilon}\frac{\partial}{\partial x}V(x_t,t)\,\mathrm{d}s + \frac{\sigma}{\sqrt{\varepsilon}}\,\mathrm{d}W_t$$

$$V(x,t) = -\frac{1}{2}x^2 + \frac{1}{4}x^4 + (\lambda_{\mathrm{c}} - a_0)\cos(2\pi t)x$$



Sample paths

Amplitude of modulation $A = \lambda_c - a_0$
Speed of modulation ε
Noise intensity σ



$A = 0.00$, $\sigma = 0.30$, $\varepsilon = 0.001$

$A = 0.10$, $\sigma = 0.27$, $\varepsilon = 0.001$

$A = 0.24$, $\sigma = 0.20$, $\varepsilon = 0.001$

$A = 0.35$, $\sigma = 0.20$, $\varepsilon = 0.001$

Different parameter regimes and stochastic resonance

Synchronisation I

- For matching time scales: $2\pi/\varepsilon = T_{\text{forcing}} = 2\,T_{\text{Kramers}} \asymp e^{2H/\sigma^2}$
- Quasistatic approach: Transitions twice per period likely
  (Physics' literature; [Freidlin 2000], [Imkeller *et al*, since 2002])
- Requires exponentially long forcing periods

Synchronisation II

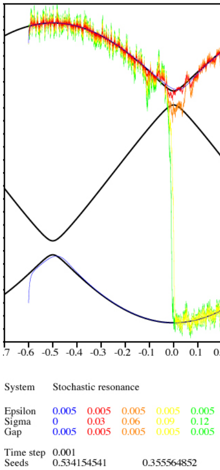
- For intermediate forcing periods: $T_{\text{relax}} \ll T_{\text{forcing}} \ll T_{\text{Kramers}}$
  and close-to-critical forcing amplitude: $A \approx A_c$
- Transitions twice per period with high probability
- Subtle dynamical effects: Effective barrier heights [Berglund & G 2002]

SR outside synchronisation regimes

- Only occasional transitions
- But transition times localised within forcing periods

Synchronisation regime II

Characterised by 3 small parameters: $0 < \sigma \ll 1$, $0 < \varepsilon \ll 1$, $0 < a_0 \ll 1$



Effective barrier heights and scaling of small parameters

Theorem [Berglund & G 2002] (informal version; exact formulation via first-exit times)

$$\exists \text{ threshold value } \sigma_c = (a_0 \vee \varepsilon)^{3/4}$$

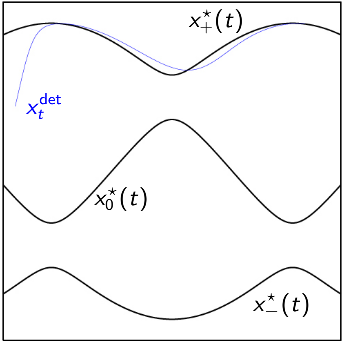
Below: $\sigma \leq \sigma_c$

- Transitions unlikely; sample paths concentrated in one well
- Typical spreading $\asymp \dfrac{\sigma}{\left(|t|^2 \vee a_0 \vee \varepsilon\right)^{1/4}} \asymp \dfrac{\sigma}{\left(\text{curvature}\right)^{1/2}}$
- Probability to observe a transition $\leq e^{-const\, \sigma_c^2/\sigma^2}$

Above: $\sigma \gg \sigma_c$

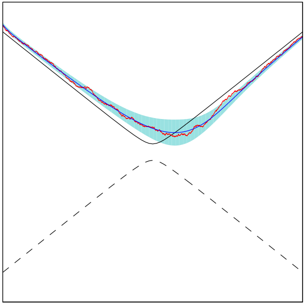
- 2 transitions per period likely (back and forth)
- with probabilty $\geq 1 - e^{-const\, \sigma^{4/3}/\varepsilon|\log \sigma|}$
- Transitions occur near instants of minimal barrier height; window $\asymp \sigma^{2/3}$

Deterministic dynamics



- For $t \leq -const$:
  x_t^{det} reaches ε-nbhd of $x^\star_+(t)$
  in time $\asymp \varepsilon|\log \varepsilon|$ [Tihonov 1952]
- For $-const \leq t \leq -(a_0 \vee \varepsilon)^{1/2}$:
  $x_t^{\text{det}} - x^\star_+(t) \asymp \varepsilon/|t|$
- For $|t| \leq (a_0 \vee \varepsilon)^{1/2}$:
  $x_t^{\text{det}} - x^\star_0(t) \asymp (a_0 \vee \varepsilon)^{1/2} \geq \sqrt{\varepsilon}$
  (effective barrier height)
- For $(a_0 \vee \varepsilon)^{1/2} \leq t \leq +const$:
  $x_t^{\text{det}} - x^\star_+(t) \asymp -\varepsilon/|t|$
- For $t \geq +const$:
  $|x_t^{\text{det}} - x^\star_+(t)| \asymp \varepsilon$

Below threshold: $\sigma \leq \sigma_c = (a_0 \vee \varepsilon)^{3/4}$



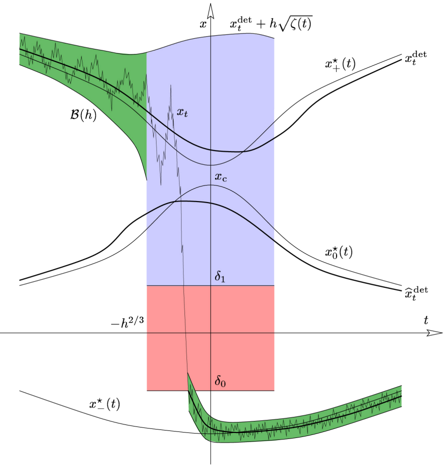
$$v(t) \sim \frac{\sigma^2}{\text{curvature}} \sim \frac{\sigma^2}{(|t|^2 \vee a_0 \vee \varepsilon)^{1/2}}$$

Approach for stable case can still be used

$$C(h/\sigma, t, \varepsilon)\, \mathrm{e}^{-\kappa_- h^2/2\sigma^2} \leq \mathbb{P}\{\tau_{\mathcal{B}(h)} < t\} \leq C(h/\sigma, t, \varepsilon)\, \mathrm{e}^{-\kappa_+ h^2/2\sigma^2}$$

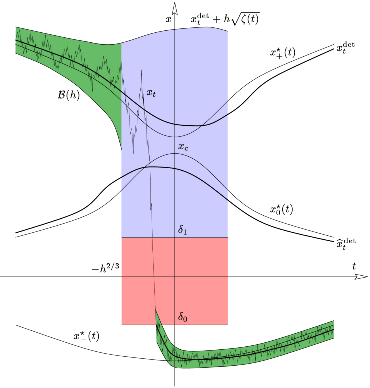
with $\kappa_+ = 1 - \mathcal{O}(\varepsilon) - \mathcal{O}(h)$, $\kappa_- = 1 + \mathcal{O}(\varepsilon) + \mathcal{O}(h) + \mathcal{O}(\mathrm{e}^{-c_2 t/\varepsilon})$

Above threshold: $\sigma \gg \sigma_c = (a_0 \vee \varepsilon)^{3/4}$



- Typical paths stay below $x_t^{\rm det} + h\sqrt{\zeta(t)}$
- For $t \ll -\sigma^{2/3}$:
  Transitions unlikely; as below threshold
- At time $t = -\sigma^{2/3}$:
  Typical spreading is $\sigma^{2/3} \gg x_t^{\rm det} - x_0^\star(t)$
  Transitions become likely
- Near saddle:
  Diffusion dominated dynamics
- $\delta_1 > \delta_0$ with $f \asymp -1$;
  δ_0 in domain of attraction of $x_-^\star(t)$
  Drift dominated dynamics
- Below δ_0: behaviour as for small σ

Above threshold: $\sigma \gg \sigma_c = (a_0 \vee \varepsilon)^{3/4}$



Idea of the proof

With probability $\geq \delta > 0$, in time $\asymp \varepsilon|\log \sigma|/\sigma^{2/3}$, the path reaches

- x_t^{det} if above
- then the saddle
- finally the level δ_1

In time $\sigma^{2/3}$ there are $\dfrac{\sigma^{4/3}}{\varepsilon|\log \sigma|}$ attempts possible

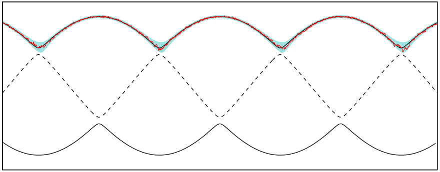
During a subsequent timespan of length ε, level δ_0 is reached (with probability $\geq \delta$)

Finally, the path reaches the new well

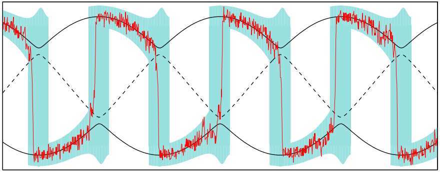
Result

$$\mathbb{P}\{x_s > \delta_0 \quad \forall s \in [-\sigma^{2/3}, t]\} \leq e^{-const\ \sigma^{4/3}/\varepsilon|\log \sigma|} \qquad (t \geq -\gamma\sigma^{2/3},\ \gamma \text{ small})$$

Space–time sets for stochastic resonance

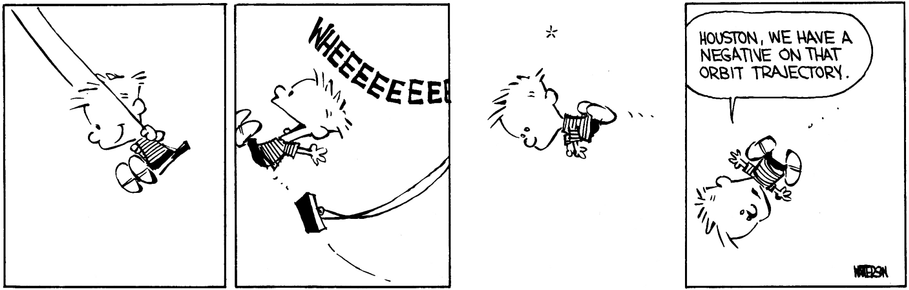


Below threshold



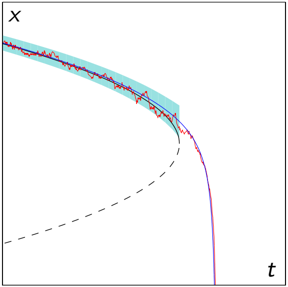
Above threshold

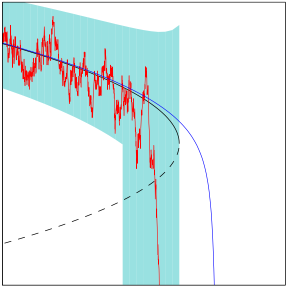
Saddle–node bifurcation



Saddle–node bifurcation (e.g. $f(x,t) = -t - x^2$)

$\sigma \ll \sigma_c = \varepsilon^{1/2}$ $\sigma \gg \sigma_c = \varepsilon^{1/2}$





$\sigma = 0$: Solutions stay at distance $\varepsilon^{1/3}$ above bif. point until time $\varepsilon^{2/3}$ after bif.

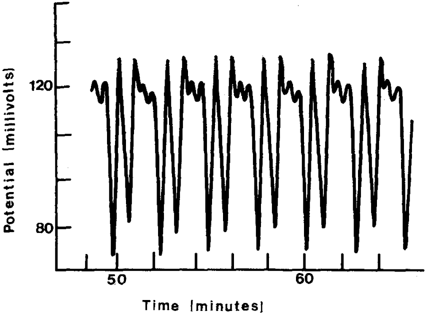
Theorem [Berglund & G 2002]

- If $\sigma \ll \sigma_c$: Paths likely to stay in $\mathcal{B}(h)$ until time $\varepsilon^{2/3}$ after bifurcation; maximal spreading $\sigma/\varepsilon^{1/6}$.
- If $\sigma \gg \sigma_c$: Transition happens typically for $t \asymp -\sigma^{4/3}$ (early transitions); transition probability $\geqslant 1 - e^{-c\sigma^2/\varepsilon|\log \sigma|}$

Mixed-mode oscillations

Mixed-mode oscillations (MMOs)

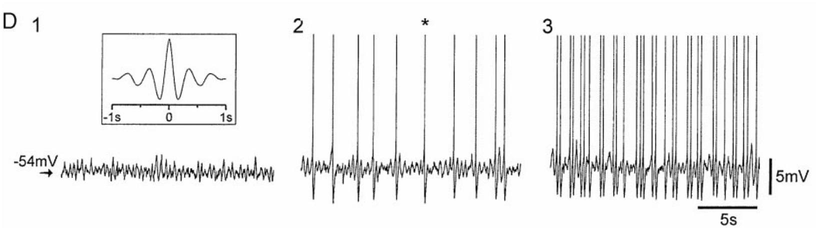
Belousov–Zhabotinsky reaction



Recording from bromide ion electrode; T=25° C; flow rate = 3.99 ml/min; Ce^{+3} catalyst [Hudson, Hart, Marinko '79]

MMOs in Biology

Layer II Stellate Cells



D: subthreshold membrane potential oscillations (1 and 2) and spike clustering (3) develop at increasingly depolarized membrane potential levels positive to about –55 mV. Autocorrelation function (*inset* in 1) demonstrates the rhythmicity of the subthreshold oscillations [Dickson *et al* 2000]

Questions: Origin of small-amplitude oscillations?
Source of irregularity in pattern?

MMOs & slow–fast systems

Observation

MMOs can be observed in slow–fast systems undergoing a folded-node bifurcation (1 fast, 2 slow variables)

Normal form of folded-node [Benoît, Lobry 1982; Szmolyan, Wechselberger 2001]

$$\begin{aligned} \epsilon\dot{x} &= y - x^2 \\ \dot{y} &= -(\mu+1)x - z \\ \dot{z} &= \frac{\mu}{2} \end{aligned}$$

Questions Dynamics for small $\varepsilon > 0$?

MMOs & slow–fast systems

Observation

MMOs can be observed in slow–fast systems undergoing a folded-node bifurcation (1 fast, 2 slow variables)

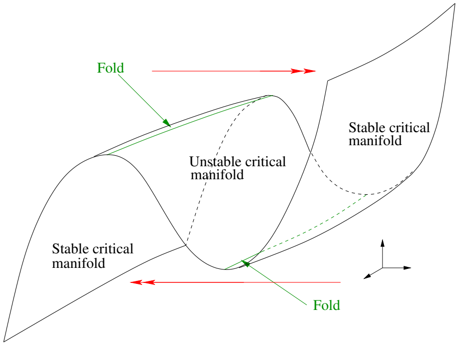
Normal form of folded-node [Benoît, Lobry 1982; Szmolyan, Wechselberger 2001]

$$\epsilon \dot{x} = y - x^2 + \text{noise}$$
$$\dot{y} = -(\mu + 1)x - z + \text{noise}$$
$$\dot{z} = \frac{\mu}{2}$$

Questions Dynamics for small $\varepsilon > 0$?
Effect of noise?

Folded-node bifurcation: Critical manifold and canard solutions

$$\epsilon \dot{x} = y - x^2$$
$$\dot{y} = -(\mu + 1)x - z$$
$$\dot{z} = \frac{\mu}{2}$$



- $\varepsilon = 0$: Critical manifold decomposes into stable and unstable parts + fold line

Folded-node bifurcation: Critical manifold and canard solutions

$$\epsilon \dot{x} = y - x^2$$
$$\dot{y} = -(\mu + 1)x - z$$
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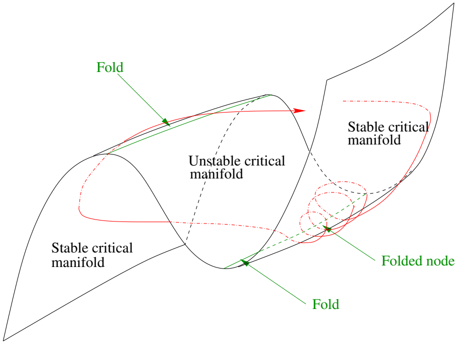
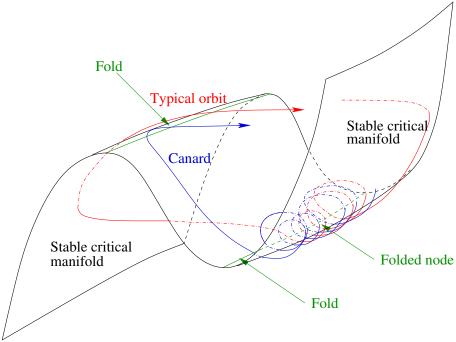


- $\varepsilon = 0$: Critical manifold decomposes into stable and unstable parts + fold line
- Typical solution exhibits small amplitude oscillations

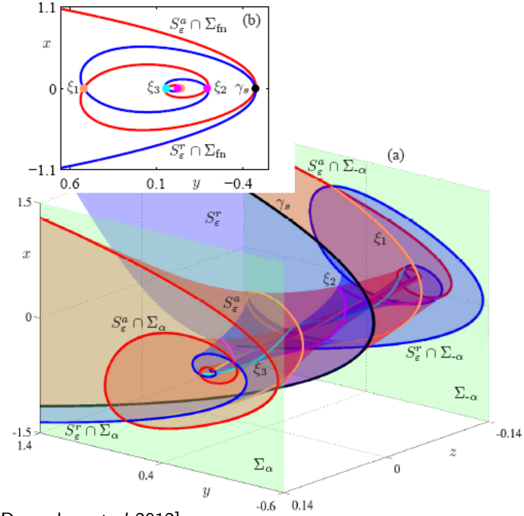
Folded-node bifurcation: Critical manifold and canard solutions

$$\epsilon\dot{x} = y - x^2$$
$$\dot{y} = -(\mu + 1)x - z$$
$$\dot{z} = \frac{\mu}{2}$$



- $\varepsilon = 0$: Critical manifold decomposes into stable and unstable parts + fold line
- Typical solution exhibits small amplitude oscillations
- Existence of canard solutions tracking critical manifold

Folded-node: Adiabatic manifolds and canard solutions



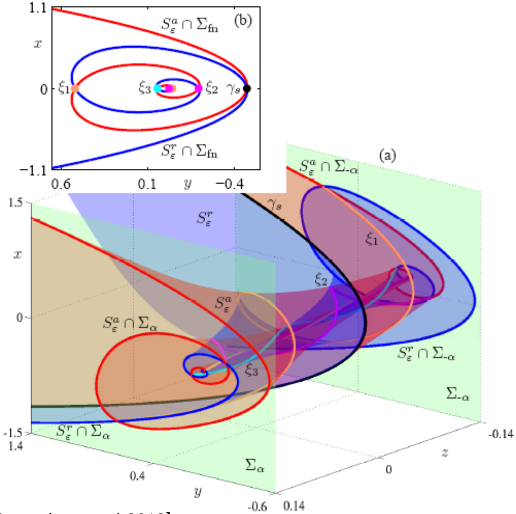
[Desroches *et al* 2012]

Assume

- ε sufficiently small
- $\mu \in (0, 1)$, $\mu^{-1} \notin \mathbb{N}$

Theorem
[Benoît, Lobry 1982;
Szmolyan, Wechselberger 2001;
Wechselberger 2005;
Brøns, Krupa, Wechselberger 2006]

Folded-node: Adiabatic manifolds and canard solutions



[Desroches *et al* 2012]

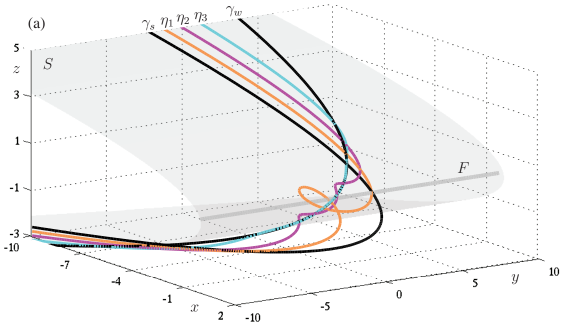
Assume

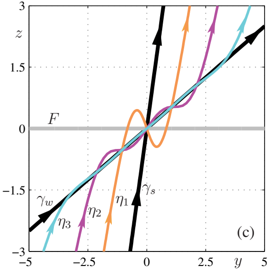
- ε sufficiently small
- $\mu \in (0,1)$, $\mu^{-1} \notin \mathbb{N}$

Theorem

- Existence of *strong* and *weak* (maximal) canard $\gamma_\varepsilon^{\mathrm{s,w}}$
- $2k+1 < \mu^{-1} < 2k+3$: $\exists\ k$ *secondary* canards γ_ε^j
- γ_ε^j makes $(2j+1)/2$ oscillations around $\gamma_\varepsilon^{\mathrm{w}}$

Folded-node: Canard spacing





[Desroches, Krauskopf, Osinga 2008]

Lemma

For $z = 0$: Distance between canards γ_ε^k and γ_ε^{k+1} is $\mathcal{O}(\mathrm{e}^{-c_0(2k+1)^2\mu})$

Stochastic folded nodes: Rescaling

$$
\begin{aligned}
dx_t &= \frac{1}{\varepsilon}(y_t - x_t^2)\,dt + \frac{\sigma}{\sqrt{\varepsilon}}\,dW_t^{(1)} \\
dy_t &= [-(\mu+1)x_t - z_t]\,dt + \sigma'\,dW_t^{(2)} \\
dz_t &= \frac{\mu}{2}\,dt
\end{aligned}
$$

Rescaling (blow-up transformation): $(x, y, z, t) = (\sqrt{\varepsilon}\bar{x}, \varepsilon\bar{y}, \sqrt{\varepsilon}\bar{z}, \sqrt{\varepsilon}\,\bar{t})$

In addition: $(\sigma, \sigma') = (\varepsilon^{3/4}\bar{\sigma}, \varepsilon^{3/4}\bar{\sigma}')$ and consider z as "time"

$$
\begin{aligned}
dx_z &= \frac{2}{\mu}(y_z - x_z^2)\,dz + \frac{\sqrt{2}\sigma}{\sqrt{\mu}}\,dW_z^{(1)} \\
dy_z &= -\frac{2}{\mu}[(\mu+1)x_z + z]\,dz + \frac{\sqrt{2}\sigma'}{\sqrt{\mu}}\,dW_z^{(2)}
\end{aligned}
$$

For small μ: Slowly driven system with two fast variables

Deviation from the adiabatic manifold due to noise

Main idea

- Deterministic reference process $(x_z^{\text{det}}, y_z^{\text{det}})$
- Linearize SDE for $\xi_z := x_z - x_z^{\text{det}}$

Key observation

- Resulting process ξ_z^0 is mean-zero Gaussian
- Covariance matrix $\sigma^2 \overline{X}(z, \varepsilon)$ determines behaviour

We're in business ...

- Calculate asymptotic size of the covariance tube

$$\mathcal{B}(h) = \left\{ (x, y) \colon \left\langle \left[x - \bar{x}(y, \varepsilon) \right], \overline{X}(y, \varepsilon)^{-1} \left[x - \bar{x}(y, \varepsilon) \right] \right\rangle < h^2, \ y \in \mathcal{D}_0 \right\}$$

  using Neishtadt's theorem on delayed Hopf bifurcations
- Use general result on concentration of sample paths for ξ_z in $\mathcal{B}(h)$

Stochastic folded nodes: Concentration of sample paths

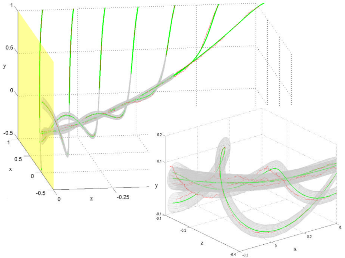
Theorem [Berglund, G & Kuehn 2012]

$$\mathbb{P}\Bigl\{\tau_{\mathcal{B}(h)} < z\Bigr\} \leqslant C(z_0, z) \exp\Bigl\{-\kappa \frac{h^2}{2\sigma^2}\Bigr\} \qquad \forall z \in [z_0, \sqrt{\mu}]$$

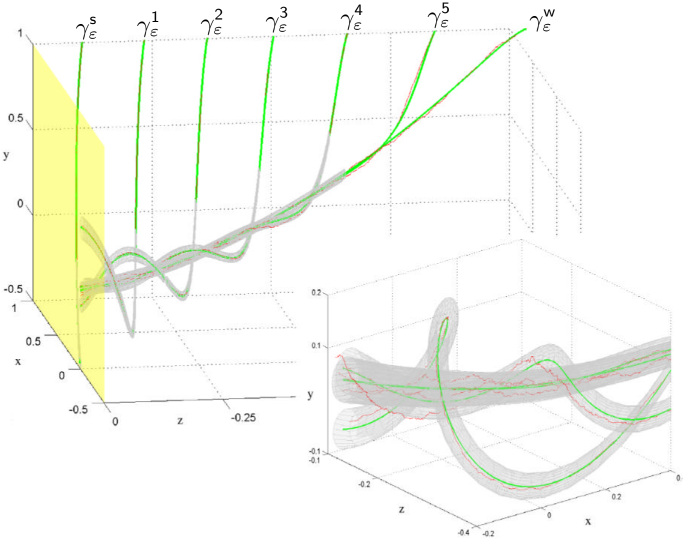
where $\tau_{\mathcal{B}(h)} = \inf\{s > 0 \colon (x_s, y_s) \notin \mathcal{B}(h)\}$

For $z = 0$:

- Distance between canards γ_ε^k and γ_ε^{k+1} is $\mathcal{O}(\mathrm{e}^{-c_0(2k+1)^2\mu})$
- Section of $\mathcal{B}(h)$ is close to circular with radius $\mu^{-1/4}h$
- Noisy canards become indistinguishable when typical radius $\mu^{-1/4}\sigma \approx$ distance



Canards or pasta . . . ?

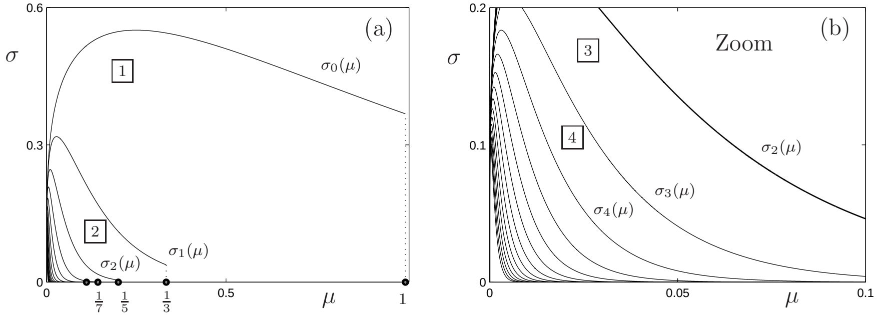


Noisy small-amplitude oscillations

Theorem [Berglund, G & Kuehn 2012]

Canards with $\frac{2k+1}{2}$ oscillations become indistinguishable from noisy fluctuations for

$$\sigma > \sigma_k(\mu) = \mu^{1/4}\,\mathrm{e}^{-(2k+1)^2\mu}$$



Early escape

Model allowing for global returns

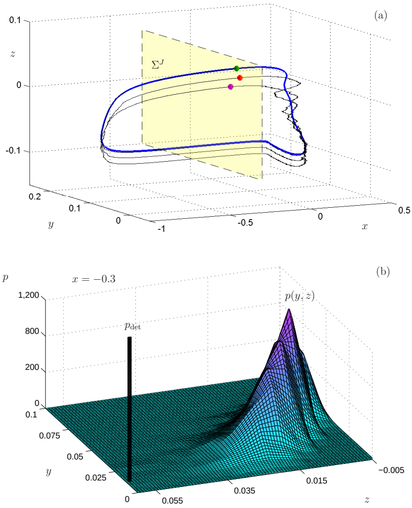
- Consider $z > \sqrt{\mu}$
- $\mathcal{D}_0 =$ neighbourhood of γ^{w}, growing like $\sqrt{z}$

Theorem [Berglund, G & Kuehn 2012]

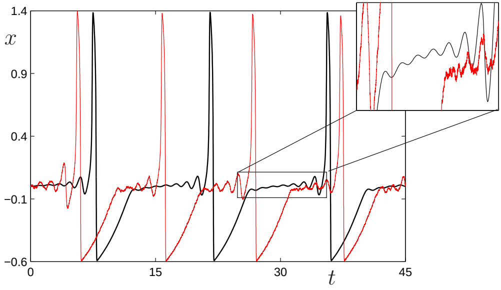
$\exists \kappa, \kappa_1, \kappa_2, C > 0$
s.t.
for $\sigma |\log \sigma|^{\kappa_1} \leqslant \mu^{3/4}$

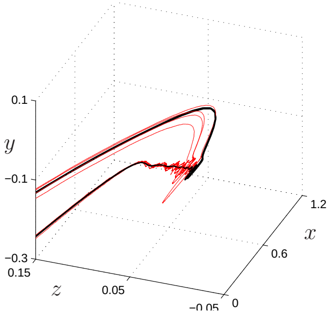
$$\mathbb{P}\{\tau_{\mathcal{D}_0} > z\} \leqslant C|\log \sigma|^{\kappa_2} \mathrm{e}^{-\kappa(z^2-\mu)/(\mu|\log \sigma|)}$$

Note:
r.h.s. small for $z \gg \sqrt{\mu|\log \sigma|/\kappa}$



Mixed-mode oscillations in the presence of noise





Observations

- ▹ Noise smears out small-amplitude oscillations
- ▹ Early transitions modify the mixed-mode pattern
- ▹ Which kind of patterns can arise?

  Partial answer: [Berglund, G & Kuehn, submitted]

References

Sample-paths approach to bifurcations in one-dimensional random slow–fast systems

- N. Berglund and B. Gentz, *A sample-paths approach to noise-induced synchronization: Stochastic resonance in a double-well potential*, Ann. Appl. Probab. 12 (2002), pp. 1419–1470
- N. Berglund and B. Gentz, *The effect of additive noise on dynamical hysteresis*, Nonlinearity 15 (2002), pp. 605–632
- N. Berglund and B. Gentz, *Beyond the Fokker–Planck equation: Pathwise control of noisy bistable systems*, J. Phys. A 35 (2002), pp. 2057–2091
- N. Berglund and B. Gentz, *Metastability in simple climate models: Pathwise analysis of slowly driven Langevin equations*, Stoch. Dyn. 2 (2002), pp. 327–356
- N. Berglund, and B. Gentz, *Noise-induced phenomena in slow–fast dynamical systems. A sample-paths approach*, Springer (2006)

References (cont.)

Early mathematical work on stochastic resonance – quasistatic regime

- M. I. Freidlin, *Quasi-deterministic approximation, metastability and stochastic resonance*, Physica D 137, (2000), pp. 333–352
- S. Herrmann and P. Imkeller, *Barrier crossings characterize stochastic resonance*, Stoch. Dyn. 2 (2002), pp. 413–436
- P. Imkeller and I. Pavlyukevich, *Model reduction and stochastic resonance*, Stoch. Dyn. 2 (2002), pp. 463–506
- M. Fischer and P. Imkeller, *A two state model for noise-induced resonance in bistable systems with delays*, Stoch. Dyn. 5 (2005), pp. 247–270
- S. Herrmann, P. Imkeller, and D. Peithmann, *Transition times and stochastic resonance for multidimensional diffusions with time periodic drift: a large deviations approach*, Ann. Appl. Probab. 16 (2006), 1851–1892

References (cont.)

Mixed-mode oscillations

- J. L. Hudson, M. Hart, and D. Marinko, *An experimental study of multiple peak periodic and nonperiodic oscillations in the Belousov–Zhabotinskii reaction*, J. Chem. Phys. 71 (1979), pp. 1601–1606
- E. Benoît and C. Lobry, *Les canards de* $\mathbb{R}^3$, C.R. Acad. Sc. Paris 294 (1982), pp. 483–488
- C. T. Dickson, J. Magistretti, M. H. Shalisnky, E. Fransen, M. E. Hasselmo, and A. Alonso, *Properties and role of* I_h *in the pacing of subtreshold oscillations in entorhinal cortex layer II neurons*, J. Neurophysiol. 83 (2000), pp. 2562–2579
- P. Szmolyan and M. Wechselberger, *Canards in* $\mathbb{R}^3$, Journal of Differential Equations 177 (2001), pp. 419–453
- M. Wechselberger, *Existence and Bifurcation of Canards in* $\mathbb{R}^3$ *in the Case of a Folded Node*, SIAM J. Applied Dynamical Systems 4 (2005), pp. 101–139
- M. Brøns, M. Krupa, and M. Wechselberger, *Mixed mode oscillations due to the generalized canard phenomenon*, Fields Institute Communications 49 (2006), pp. 39–63
- M. Desroches, J. Guckenheimer, B. Krauskopf, C. Kuehn, H. M. Osinga, and M. Wechselberger, *Mixed-mode oscillations with multiple time scales*, SIAM Review (2012), pp. 211–288

References (cont.)

The effect of noise on canards and mixed-mode oscillations

- ▷ R. B. Sowers, *Random Perturbations of Canards*, Journal of Theoretical Probability 21 (2008), pp. 824–889
- ▷ C. B. Muratov, E. Vanden-Eijnden, *Noise-induced mixed-mode oscillations in a relaxation oscillator near the onset of a limit cycle*, Chaos 18 (2008), p. 015111
- ▷ N. Yu, R. Kuske, Y. X. Li, *Stochastic phase dynamics and noise-induced mixed-mode oscillations in coupled oscillators*, Chaos, 18 (2008), p. 015112
- ▷ N. Berglund, B. Gentz, and C. Kuehn, *Hunting French ducks in a noisy environment*, Journal of Differential Equations 252 (2012), pp. 4786–4841
- ▷ N. Berglund, B. Gentz and C. Kuehn, *From random Poincaré maps to stochastic mixed-mode-oscillation patterns*, arXiv:1312.6353 [math.DS]

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Thank you for your attention !

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