Mathematics and Statistics Colloquium

Department of Mathematics, University of Texas at Arlington

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The effect of noise on slow-fast systems

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Overview

Introduction: Classical results for autonomous systems

- ▷ Small random perturbations of dynamical systems
- ▷ Exponential asymptotics for first-exit times (Wentzell–Freidlin)
- ▷ Subexponential asymptotics

Slowly time-dependent systems and stochastic resonance

- ▷ The motion of a Brownian particle in a double-well potential
- ▷ Simulations
- ▷ Rigorous results
- ▷ Deterministic dynamics
- ▷ Stochastic dynamics for noise intensities below threshold
- ▷ Stochastic dynamics for noise intensities above threshold

General slow–fast systems

- ▷ Dynamics near slow manifolds
- ▷ Bifurcations and reduced dynamics

Autonomous dynamical systems: ODEs

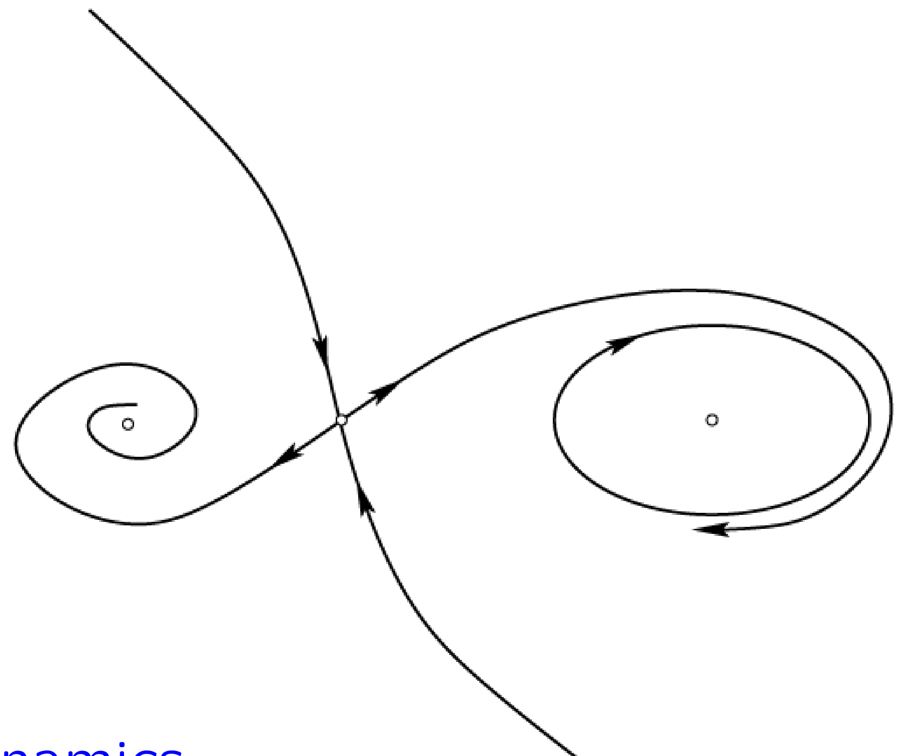
Deterministic ODE

$$\dot{x}_t^{\text{det}} = f(x_t^{\text{det}}) \,, \qquad x_0^{\text{det}} \in \mathbb{R}^d$$

with

$$f : \mathbb{R}^d \to \mathbb{R}^d \,, \quad f \text{ ``well-behaved''}$$

(Existence and uniqueness of solution)



Assumptions on deterministic dynamics

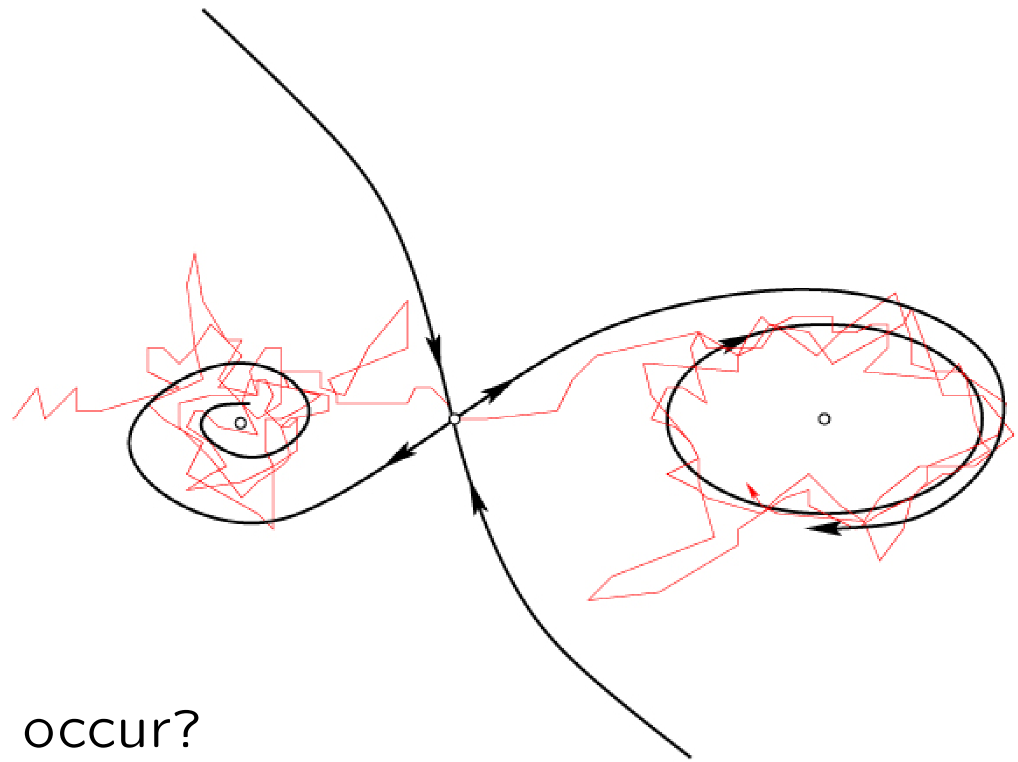
- ▷ Attractors $\mathcal{A}_1, \mathcal{A}_2, \dots$
- ▷ Domains of attraction $\mathcal{B}_1, \mathcal{B}_2, \dots$

Small random perturbations of autonomous systems

$$\mathrm{d}x_t = f(x_t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t\,, \qquad x_0 = x_0^{\mathrm{det}} \in \mathbb{R}^d$$

- ▷ $f : \mathbb{R}^d \to \mathbb{R}^d$ "well-behaved"
- ▷ $\{W_t\}_{t \geq 0}$ d-dim. (standard-) Brownian motion
- ▷ $\sigma > 0$ small

Noise enables transitions
between
domains of attraction



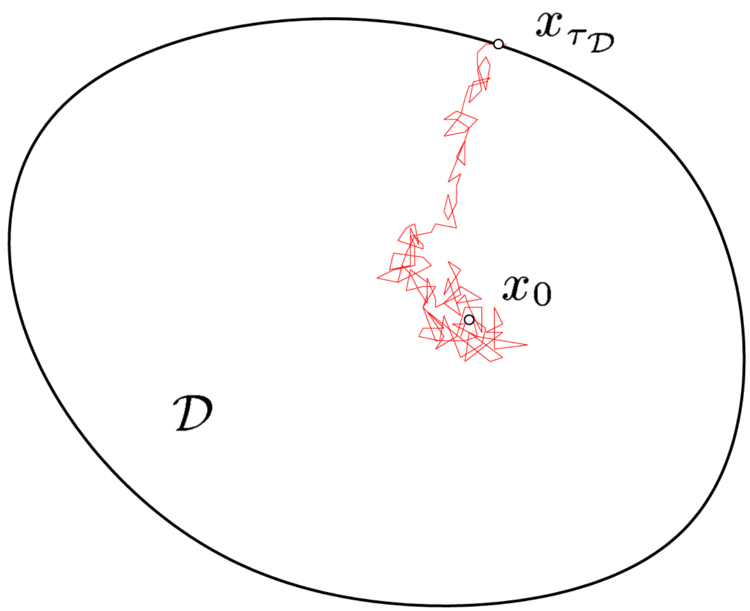
Questions
Transition times?
Transition probabilities?
Where do typical transitions occur?

Diffusion exit from a domain: Exit problem

Bounded domain $\mathcal{D} \ni x_0$ (with smooth boundary)

- ▷ *first-exit time* $\tau = \tau_{\mathcal{D}} = \inf\{t > 0 \colon x_t \notin \mathcal{D}\}$
- ▷ *first-exit location* $x_\tau \in \partial\mathcal{D}$

Distribution of τ and x_τ ?



Interesting case

$\mathcal{D}$ positively invariant
under deterministic flow

Approaches

- ▷ Mean first-exit times and locations via PDEs
- ▷ Exponential asymptotics via Wentzell–Freidlin theory

Exponential asymptotics: Large deviations

Large-deviation rate function

$$I_{[0,t]}(\varphi) = \frac{1}{2}\int_0^t \|\dot\varphi_s - f(\varphi_s)\|^2 \, \mathrm{d}s \qquad \text{for } \varphi \in H_1$$

$$I_{[0,t]}(\varphi) = +\infty \quad \text{otherwise}$$

Large-deviation principle

Probability $\sim \exp\{-I(\varphi)/\sigma^2\}$ to observe sample paths close to φ

Assumptions (for the next two slides)

- $\triangleright$ $\mathcal{D}$ positively invariant
- $\triangleright$ unique, asymptotically stable equilibrium point at $0 \in \mathcal{D}$
- $\triangleright$ $\partial\mathcal{D} \subset$ basin of attraction of 0 (non-characteristic boundary)

Wentzell–Freidlin theory I

Quasipotential

- Quasipotential *with respect to* 0:
  Cost to go against the flow from 0 to z

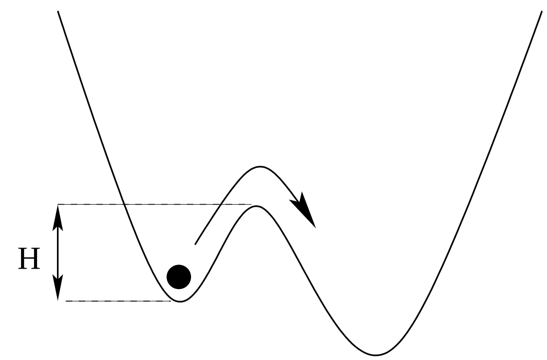
$$V(0,z) = \inf_{t>0} \inf\{I_{[0,t]}(\varphi) \colon \varphi \in \mathcal{C}([0,t],\mathbb{R}^d),\ \varphi_0 = 0,\ \varphi_t = z\}$$

- Minimum of quasipotential on boundary $\partial\mathcal{D}$:

$$\overline{V} := \min_{z\in\partial\mathcal{D}} V(0,z)$$

Gradient case (reversible diffusion)

Drift coefficient deriving from potential:
$f = -\nabla V$



- Cost for leaving potential well: $\overline{V} = 2H$
- Attained for paths against the flow:
  $\dot{\varphi}_t = -f(\varphi_t)$

Wentzell–Freidlin theory II

Theorem [Wentzell & Freidlin $\geqslant$ '70,'84] (under above assumptions)

For arbitrary initial condition $x_0 \in \mathcal{D}$

- Mean first-exit time

  $\mathbb{E}^{x_0}\{\tau\} \sim \mathrm{e}^{\overline{V}/\sigma^2}$ as $\sigma \to 0$

- Concentration of first-exit times

  $\mathbb{P}^{x_0}\Big\{\mathrm{e}^{(\overline{V}-\delta)/\sigma^2} \leqslant \tau \leqslant \mathrm{e}^{(\overline{V}+\delta)/\sigma^2}\Big\} \to 1$ as $\sigma \to 0$ $(\delta > 0)$

- Concentration of exit locations near minima of quasipotential

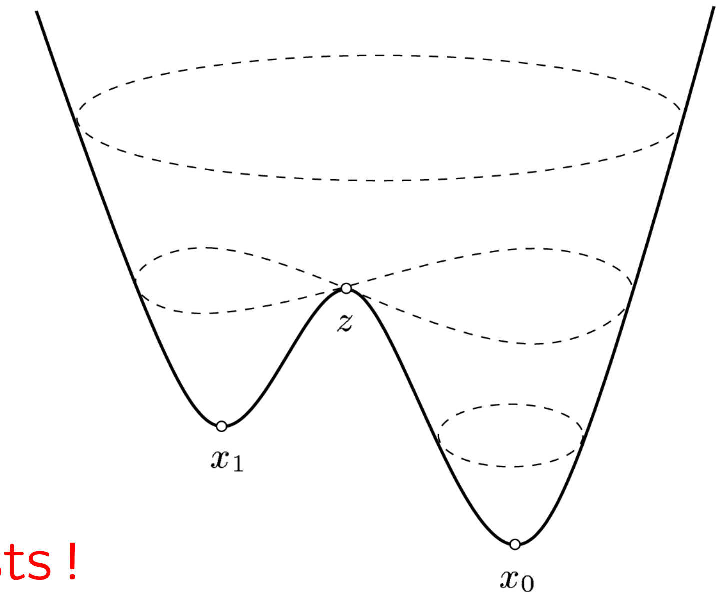
  $\mathbb{P}^{x_0}\Big\{\|x_\tau - z^\star\| < \delta\Big\} \to 1$ as $\sigma \to 0$ $(\delta > 0)$

  ($z^\star$ unique minimum of $z \mapsto V(0,z)$ on $\partial\mathcal{D}$)

Refined results in the gradient case

Simplest case: V double-well potential

First-hitting time τ^{hit} of deeper well



- $\mathbb{E}^{x_1}\{\tau^{\text{hit}}\} = c(\sigma)\, \mathrm{e}^{2\,[V(z)-V(x_1)]\,/\,\sigma^2}$

- $\displaystyle\lim_{\sigma\to 0} c(\sigma) = \frac{2\pi}{|\lambda_1(z)|}\sqrt{\frac{|\det \nabla^2 V(z)|}{\det \nabla^2 V(x_1)}}$ exists !

  $\lambda_1(z)$ unique negative e.v. of $\nabla^2 V(z)$

  ([Eyring '35], [Kramers '40]; [Bovier, Gayrard, Eckhoff, Klein '02–'05], [Helffer, Klein, Nier '04])

- Subexponential asymptotics known; related to geometry at well and saddle / small eigenvalues of the generator

- $\tau^{\text{hit}} \approx$ exp. distributed: $\displaystyle\lim_{\sigma\to 0} \mathbb{P}\Big\{\tau^{\text{hit}} > t\,\mathbb{E}\,\tau^{\text{hit}}\Big\} = \mathrm{e}^{-t}$

  ([Day '82], [Bovier *et al* '02])

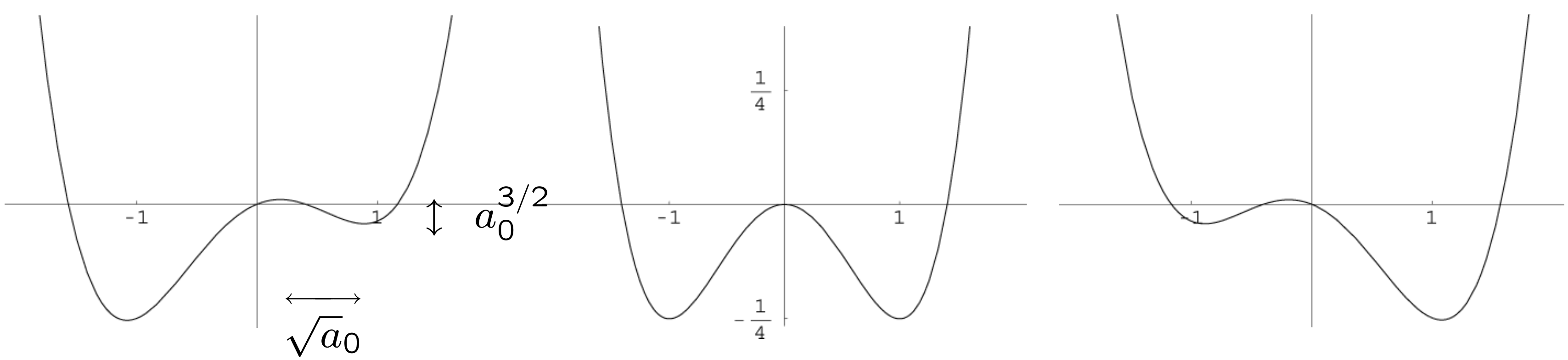
Slowly time-dependent systems

Overdamped motion of a Brownian particle

$$\mathrm{d}x_s = -\frac{\partial}{\partial x} V(x_s, \varepsilon s)\, \mathrm{d}s + \sigma\, \mathrm{d}W_s$$

in a periodically modulated potential

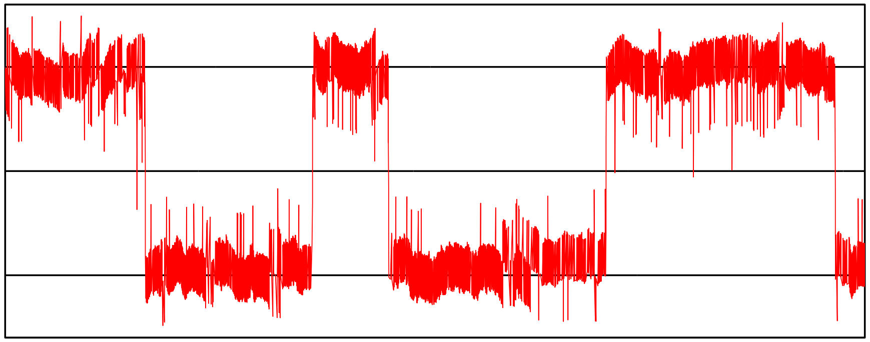
$$V(x, \varepsilon s) = -\frac{1}{2}x^2 + \frac{1}{4}x^4 + (\lambda_c - a_0)\cos(2\pi\varepsilon s)x$$



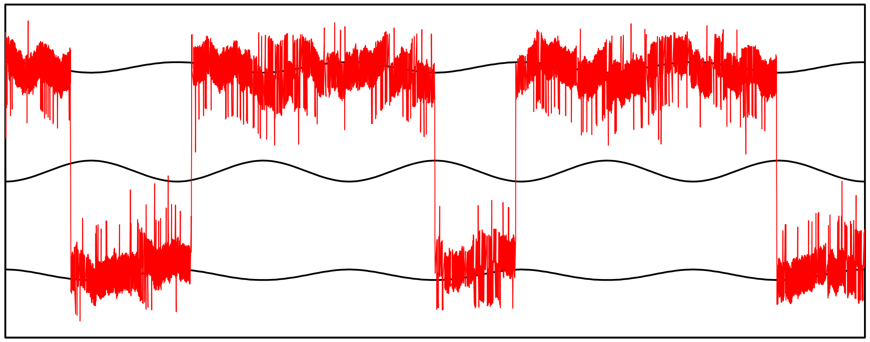
$V(x, 0)$ $\quad$ $V(x, 1/4) = V(x, 3/4)$ $\quad$ $V(x, 1/2)$

Sample paths

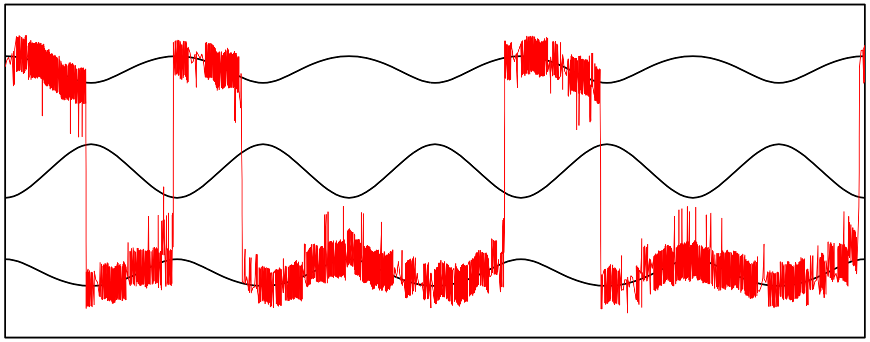
Amplitude of modulation $A = \lambda_c - a_0$
Speed of modulation ε
Noise intensity σ



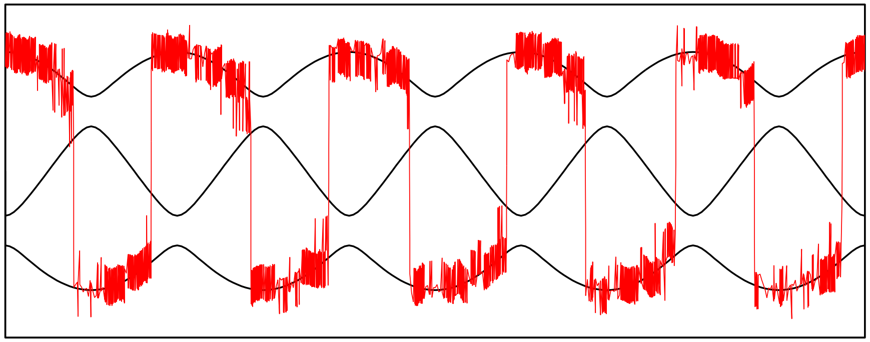
$A = 0.00,\ \sigma = 0.30,\ \varepsilon = 0.001$



$A = 0.10,\ \sigma = 0.27,\ \varepsilon = 0.001$



$A = 0.24,\ \sigma = 0.20,\ \varepsilon = 0.001$



$A = 0.35,\ \sigma = 0.20,\ \varepsilon = 0.001$

Different parameter regimes and stochastic resonance

Synchronisation I

- For matching time scales:
  $2\pi/\varepsilon = T_{\text{forcing}} = 2\,T_{\text{Kramers}} \asymp \mathrm{e}^{2H/\sigma^2}$
- Quasistatic approach: Transitions twice per period likely
  (physics' literature; [Freidlin '00], [Imkeller *et al*, since '02])
- Requires exponentially long forcing periods

Synchronisation II

- For intermediate forcing periods: $T_{\text{relax}} \ll T_{\text{forcing}} \ll T_{\text{Kramers}}$
  and close-to-critical forcing amplitude: $A \approx A_{\text{c}}$
- Transitions twice per period with high probability
- Subtle dynamical effects: Effective barrier heights [Berglund & G '02]

SR outside synchronisation regimes

- Only occasional transitions
- But transition times localised within forcing periods
- Cycling: Distribution of first-exit locations doesn't converge
  ([Day '92], [Maier & Stein '96], [Berglund & G '04], [Berglund & G '05])

Synchronisation regime II

Characterised by 3 small parameters:

$$0 < \sigma \ll 1 \ , \quad 0 < \varepsilon \ll 1 \ , \quad 0 < a_0 \ll 1$$

Recall: Motion of a Brownian particle

$$\mathrm{d}x_s = -\frac{\partial}{\partial x} V(x_s, \varepsilon s) \ \mathrm{d}s + \sigma \, \mathrm{d}W_s$$

$$V(x, \varepsilon s) = -\frac{1}{2}x^2 + \frac{1}{4}x^4 + (\lambda_c - a_0)\cos(2\pi\varepsilon s)x \ , \qquad \lambda_c = \frac{2}{3\sqrt{3}}$$

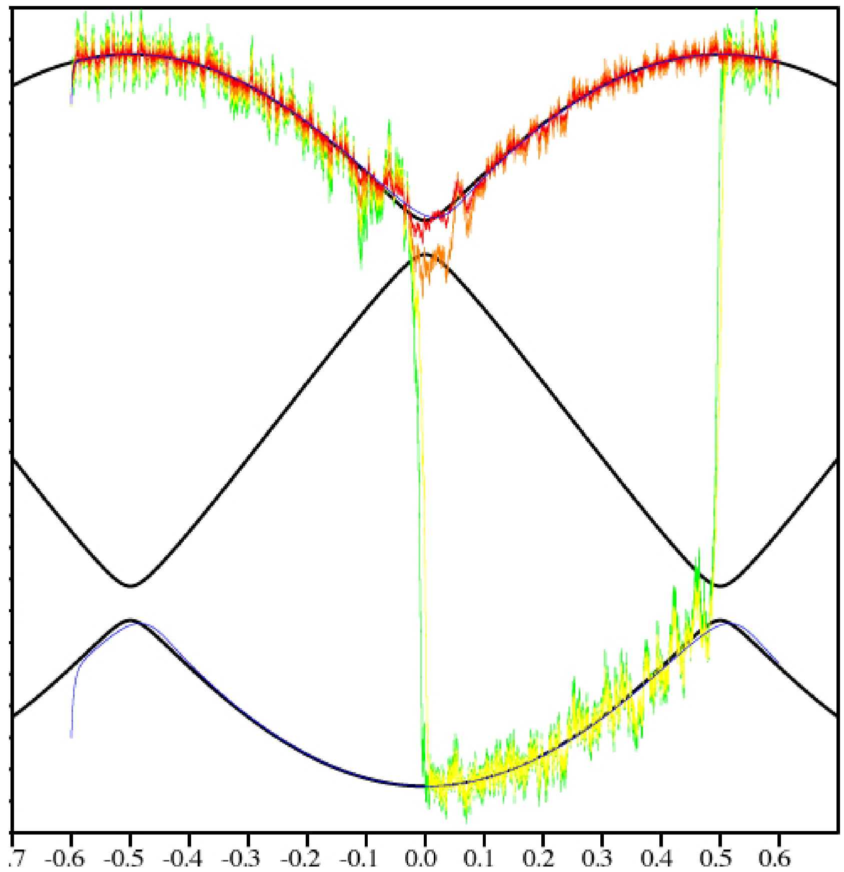
Rewrite in slow time $t = \varepsilon s$:

$$\mathrm{d}x_t = \frac{1}{\varepsilon} f(x_t, t) \ \mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}} \mathrm{d}W_t$$

with drift term

$$f(x, t) = -\frac{\partial}{\partial x} V(x, t) = x - x^3 - (\lambda_c - a_0)\cos(2\pi t)$$

Small-barrier-height regime



System Stochastic resonance

Epsilon	0.005	0.005	0.005	0.005	0.005
Sigma	0	0.03	0.06	0.09	0.12
Gap	0.005	0.005	0.005	0.005	0.005

Time step 0.001
Seeds 0.534154541 0.355564852

Effective barrier heights and scaling of small parameters

Theorem [Berglund & G, Annals of Appl. Probab. '02]
(informal version; exact formulation uses first-exit times from space–time sets)

$$\boxed{\exists \;\text{ threshold value }\; \sigma_{\text{c}} = (a_0 \vee \varepsilon)^{3/4}}$$

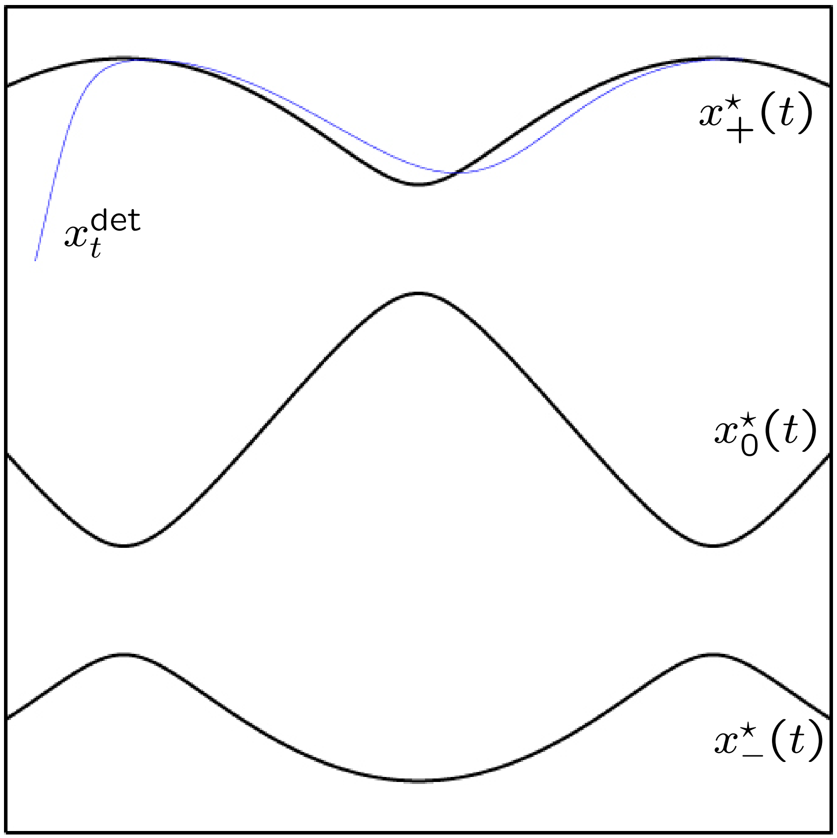
Below: $\sigma \leq \sigma_{\text{c}}$

- ▷ Transitions unlikely
- ▷ Sample paths concentrated in one well
- ▷ Typical spreading $\asymp \dfrac{\sigma}{\big(|t|^2 \vee a_0 \vee \varepsilon\big)^{1/4}} \asymp \dfrac{\sigma}{\big(\text{curvature}\big)^{1/2}}$
- ▷ Probability to observe a transition $\leq \mathrm{e}^{-\mathit{const}\,\sigma_{\text{c}}^2/\sigma^2}$

Above: $\sigma \gg \sigma_{\text{c}}$

- ▷ 2 transitions per period likely (back and forth)
- ▷ with probabilty $\geq 1 - \mathrm{e}^{-\mathit{const}\,\sigma^{4/3}/\varepsilon|\log\sigma|}$
- ▷ Transtions occur near instants of minimal barrier height;
- ▷ Transition window $\asymp \sigma^{2/3}$

Step 1: Deterministic dynamics



- For $t \leq -\mathit{const}$:
  x_t^{det} reaches ε-nbhd of $x_+^\star(t)$
  in time $\asymp \varepsilon|\log \varepsilon|$ (Tihonov '52)
- For $-\mathit{const} \leq t \leq -(a_0 \vee \varepsilon)^{1/2}$:
  $x_t^{\text{det}} - x_+^\star(t) \asymp \varepsilon/|t|$
- For $|t| \leq (a_0 \vee \varepsilon)^{1/2}$:
  $x_t^{\text{det}} - x_0^\star(t) \asymp (a_0 \vee \varepsilon)^{1/2} \geq \sqrt{\varepsilon}$
  (effective barrier height)
- For $(a_0 \vee \varepsilon)^{1/2} \leq t \leq +\mathit{const}$:
  $x_t^{\text{det}} - x_+^\star(t) \asymp -\varepsilon/|t|$
- For $t \geq +\mathit{const}$:
  $|x_t^{\text{det}} - x_+^\star(t)| \asymp \varepsilon$

Step 2: Below threshold $\sigma \leq \sigma_c = (a_0 \vee \varepsilon)^{3/4}$

Behaviour of $y_t = x_t - x_t^{\text{det}}$?

Linearizing the drift coefficent $\longrightarrow$ nonautonomous linear SDE

$$\mathrm{d}y_t^0 = \frac{1}{\varepsilon} a(t) y_t^0 \,\mathrm{d}t + \frac{\sigma}{\sqrt{\varepsilon}} \,\mathrm{d}W_t \,, \qquad y_0 = 0$$

$$a(t) = \partial_x f(x_t^{\text{det}}, t) = \text{curvature} \,; \qquad \alpha(t,s) := \int_s^t a(u)\, du$$

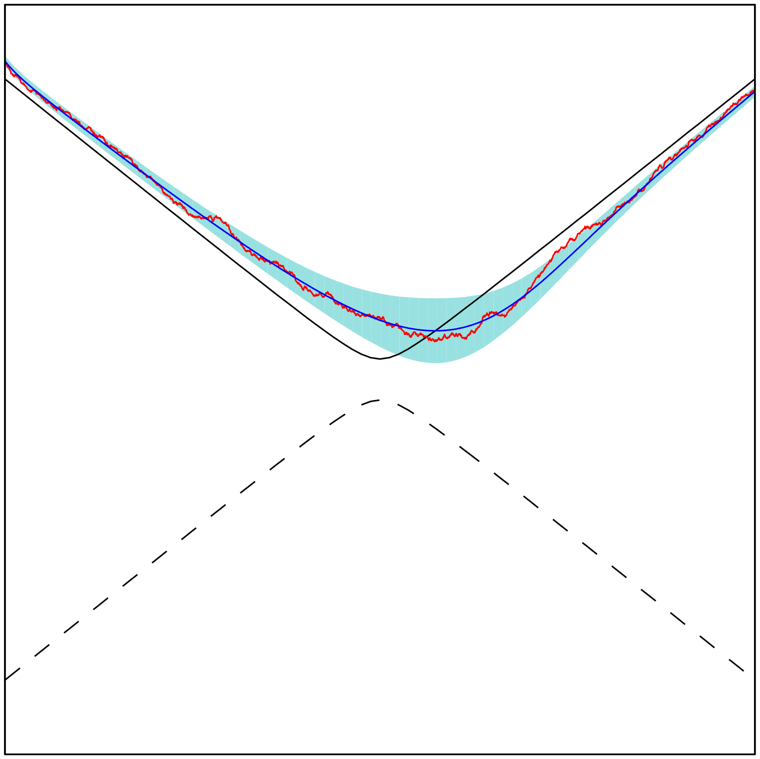
Solution $y_t^0 = \dfrac{\sigma}{\sqrt{\varepsilon}} \displaystyle\int_0^t \mathrm{e}^{\alpha(t,s)/\varepsilon} \,\mathrm{d}W_s$ is a Gaussian process

Variance $v(t) = \dfrac{\sigma^2}{\varepsilon} \displaystyle\int_0^t \mathrm{e}^{2\alpha(t,s)/\varepsilon} \,\mathrm{d}s \sim \dfrac{\sigma^2}{\text{curvature}}$

Concentration result for y_t^0: $\mathbb{P}\{|y_t^0| > \delta\} \leq \mathrm{e}^{-\delta^2/2v(t)}$

Aim: Analogous resultat for the whole sample path $\{y_t\}_{t \geq 0}$

Step 2: Below threshold $\sigma \leq \sigma_c = (a_0 \vee \varepsilon)^{3/4}$



$$v(t) \sim \frac{\sigma^2}{\text{curvature}} \sim \frac{\sigma^2}{(|t|^2 \vee a_0 \vee \varepsilon)^{1/2}}$$

$$\zeta(t) := \frac{v(t)}{\sigma^2}$$

$$\mathcal{B}(h) := \left\{(x,t)\colon |x - x_t^{\text{det}}| < h\sqrt{\zeta(t)}\right\}$$

$\tau_{\mathcal{B}(h)} =$ first-exit time of (x_t, t) from $\mathcal{B}(h)$

Step 2: Below threshold $\sigma \leq \sigma_c = (a_0 \vee \varepsilon)^{3/4}$

Theorem ([Berglund & G '02], [Berglund & G '05])

$\exists\ h_0, c_1, c_2, c_3 > 0 \quad \forall h \leq h_0$

$$C(h/\sigma, t, \varepsilon)\, \mathrm{e}^{-\kappa_- h^2/2\sigma^2} \leq \mathbb{P}\Big\{\tau_{\mathcal{B}(h)} < t\Big\} \leq C(h/\sigma, t, \varepsilon)\, \mathrm{e}^{-\kappa_+ h^2/2\sigma^2}$$

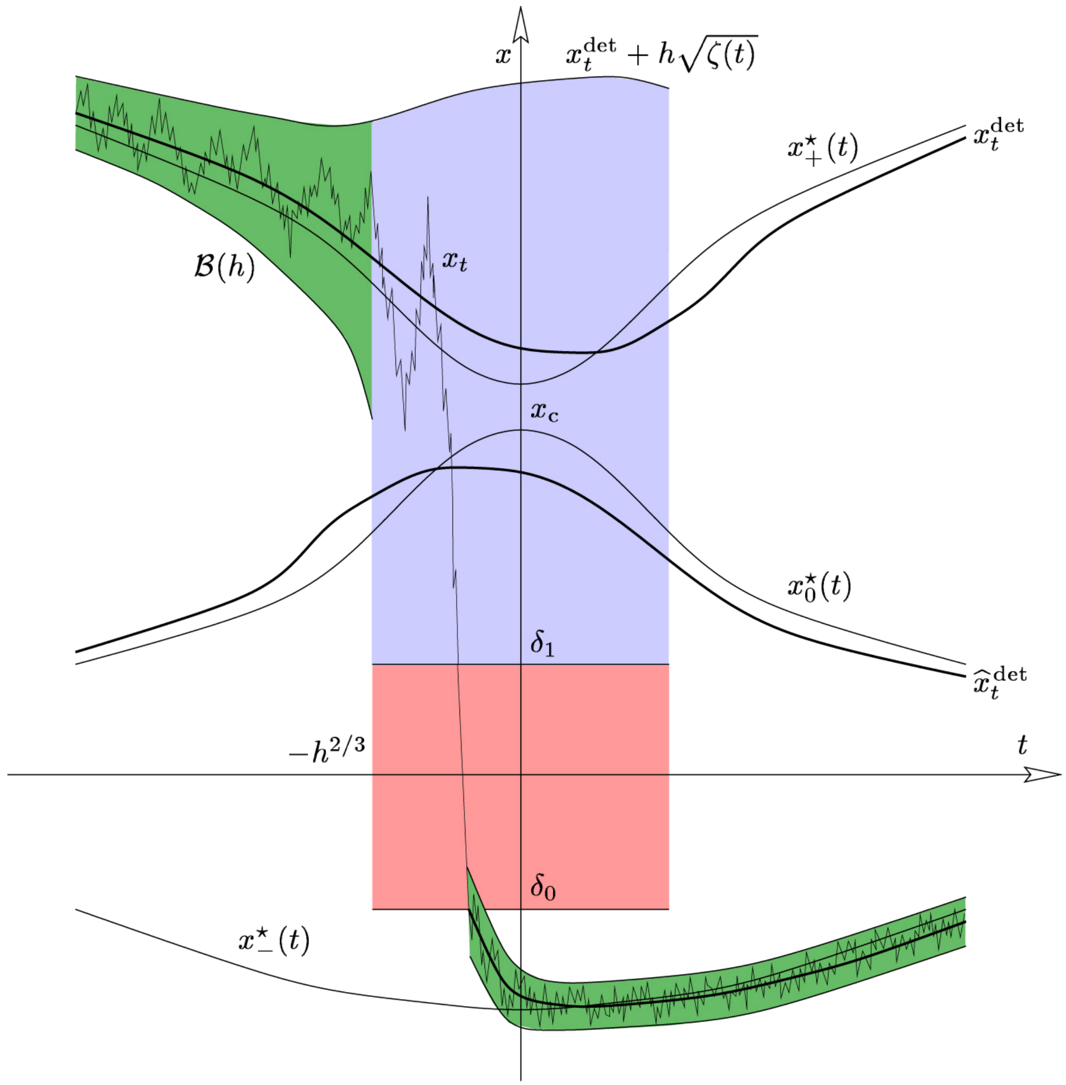
with $\kappa_+ = 1 - c_1 h$, $\kappa_- = 1 + c_1 h + c_1\, \mathrm{e}^{-c_2 t/\varepsilon}$;

$$C(h/\sigma, t, \varepsilon) = \sqrt{\frac{2}{\pi}}\, \frac{|\alpha(t)|}{\varepsilon}\, \frac{h}{\sigma}\left[1 + \mathcal{O}\left(\frac{\sigma}{h}\right) + \frac{t}{\varepsilon}\, \mathrm{e}^{-c_3 h^2/\sigma^2} + \mathrm{e}^{-c_1 t/\varepsilon} + \varepsilon\right]$$

Basic idea

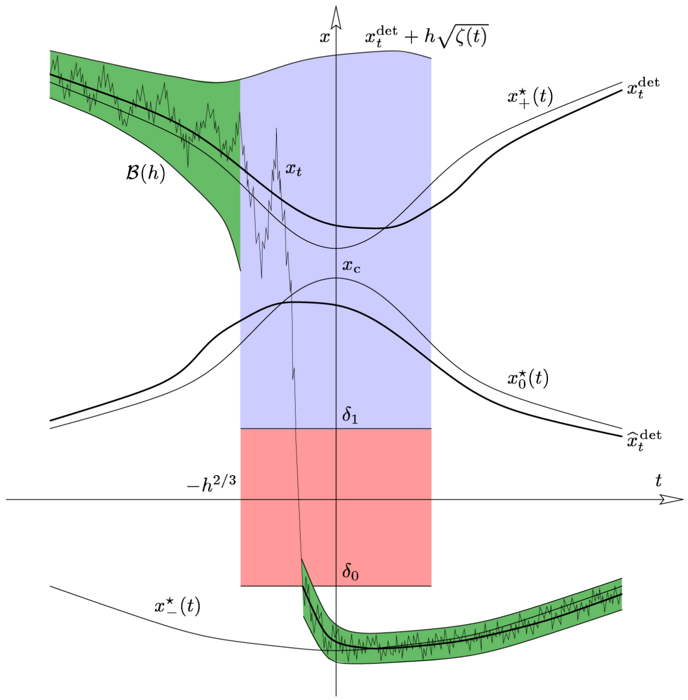
local approximation of y_t by y_t^0; Gaussian process is a rescaled Brownian motion; results on the density of the first-passage time for BM through nonlinear curves

Step 3: Above threshold $\sigma \gg \sigma_c = (a_0 \vee \varepsilon)^{3/4}$



- Typical paths stay below $x_t^{\text{det}} + h\sqrt{\zeta(t)}$
- For $t \ll -\sigma^{2/3}$:
  Transitions unlikely;
  as below threshold
- At time $t = -\sigma^{2/3}$:
  Typical spreading satisfies
  $\sigma^{2/3} \gg x_t^{\text{det}} - x_0^\star(t)$;
  Transitions become likely
- Near saddle:
  Diffusion dominated dynamics
- Levels $\delta_1 > \delta_0$ with $f \asymp -1$;
  δ_0 in domain of attr. of $x_-^\star(t)$;
  Drift dominated dynamics
- Below δ_0: beh. as for small σ

Step 3: Above threshold $\sigma \gg \sigma_c = (a_0 \vee \varepsilon)^{3/4}$



Idea of the proof

With probability $\geq \delta > 0$, in time $\asymp \varepsilon|\log\sigma|/\sigma^{2/3}$, the path reaches

- ▷ x_t^{det} if above
- ▷ then the saddle
- ▷ finally the level δ_1

In time $\sigma^{2/3}$ there are $\dfrac{\sigma^{4/3}}{\varepsilon|\log\sigma|}$ attempts possible

During a subsequent time span of length ε, level δ_0 is reached (with probability $\geq \delta$)

Finally, the path reaches the new well

Result

$$\mathbb{P}\Big\{x_s > \delta_0 \quad \forall s \in [-\sigma^{2/3}, t]\Big\} \leq \mathrm{e}^{-\mathit{const}\ \sigma^{4/3}/\varepsilon|\log\sigma|} \qquad (t \geq -\gamma\sigma^{2/3},\ \gamma \text{ small})$$

General slow–fast systems

Fully coupled SDEs on well-separated time scales

$$\begin{cases} \mathrm{d}x_t = \dfrac{1}{\varepsilon} f(x_t, y_t)\,\mathrm{d}t + \dfrac{\sigma}{\sqrt{\varepsilon}} F(x_t, y_t)\,\mathrm{d}W_t & \text{(fast variables } \in \mathbb{R}^n\text{)} \\ \mathrm{d}y_t = \quad g(x_t, y_t)\,\mathrm{d}t + \sigma' \; G(x_t, y_t)\,\mathrm{d}W_t & \text{(slow variables } \in \mathbb{R}^m\text{)} \end{cases}$$

- ▷ $\{W_t\}_{t \geq 0}$ k-dimensional (standard) Brownian motion
- ▷ $\mathcal{D} \subset \mathbb{R}^n \times \mathbb{R}^m$
- ▷ $f : \mathcal{D} \to \mathbb{R}^n$, $g : \mathcal{D} \to \mathbb{R}^m$ drift coefficients, $\in \mathcal{C}^2$
- ▷ $F : \mathcal{D} \to \mathbb{R}^{n \times k}$, $G : \mathcal{D} \to \mathbb{R}^{m \times k}$ diffusion coefficients, $\in \mathcal{C}^1$

Small parameters

- ▷ $\varepsilon > 0$ adiabatic parameter (*no quasistatic* approach)
- ▷ $\sigma, \sigma' \geq 0$ noise intensities; may depend on ε:
  $\sigma = \sigma(\varepsilon)$, $\sigma' = \sigma'(\varepsilon)$ and $\sigma'(\varepsilon)/\sigma(\varepsilon) = \varrho(\varepsilon) \leq 1$

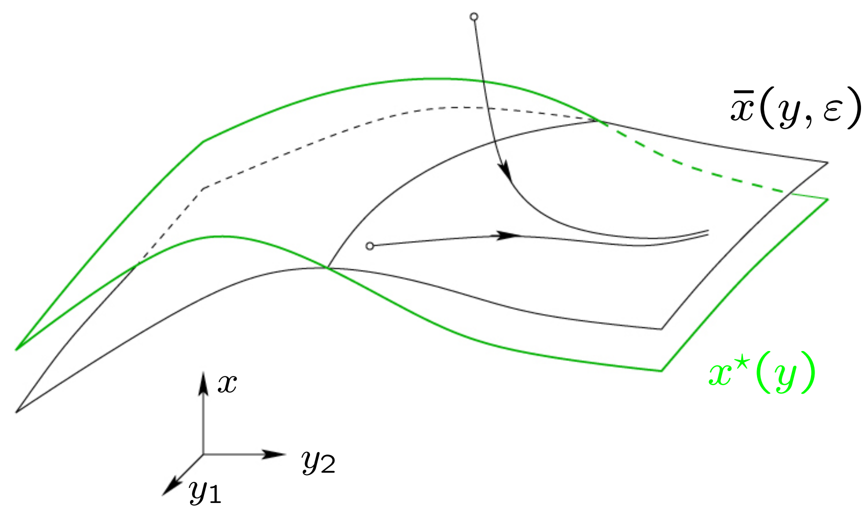
Near slow manifolds: Assumptions on the fast variables

Existence of a slow manifold: $\exists\,\mathcal{D}_0 \subset \mathbb{R}^m \quad \exists\, x^\star : \mathcal{D}_0 \to \mathbb{R}^n$
s.t $\quad (x^\star(y), y) \in \mathcal{D} \quad$ and $\quad f(x^\star(y), y) = 0 \quad$ for $y \in \mathcal{D}_0$

Slow manifold is attracting: Eigenvalues of $A^\star(y) := \partial_x f(x^\star(y), y)$
satisfy $\operatorname{Re} \lambda_i(y) \leq -a_0 < 0$, uniformly in $\mathcal{D}_0$

Theorem ([Tihonov '52], [Fenichel '79])

There exists an *adiabatic manifold*:
$\exists\, \bar{x}(y,\varepsilon)$ s.t.



- $\bar{x}(y,\varepsilon)$ is invariant manifold for deterministic dynamics
- $\bar{x}(y,\varepsilon)$ attracts nearby solutions
- $\bar{x}(y,0) = x^\star(y)$ and $\bar{x}(y,\varepsilon) = x^\star(y) + \mathcal{O}(\varepsilon)$

Consider now *stochastic* system under these assumptions

Typical neighbourhoods of adiabatic manifolds

- ▷ Consider deterministic process $(x_t^{\text{det}} = \bar{x}(y_t^{\text{det}}, \varepsilon), y_t^{\text{det}})$ on (invariant) adiabatic manifold
- ▷ Dev. $\xi_t := x_t - x_t^{\text{det}}$ of fast variables from adiabatic manifold
- ▷ Linearize SDE for ξ_t ; resulting process ξ_t^0 is Gaussian

Key observation

$\dfrac{1}{\sigma^2}\operatorname{Cov}\xi_t^0$ is a particular sol. of the det. slow–fast system

$$\begin{cases} \varepsilon\dot{X}(t) = A(y_t^{\text{det}})X(t) + X(t)A(y^{\text{det}})^{\mathsf{T}} + F_0(y^{\text{det}})F_0(y^{\text{det}})^{\mathsf{T}} \\ \dot{y}_t^{\text{det}} = g(\bar{x}(y_t^{\text{det}}, \varepsilon), y_t^{\text{det}}) \end{cases}$$

with $A(y) = \partial_x f(\bar{x}(y,\varepsilon), y)$ and F_0 0*th*-order approximation to F

- ▷ System admits an adiabatic manifold $\bar{X}(y, \varepsilon)$

Typical neighbourhoods

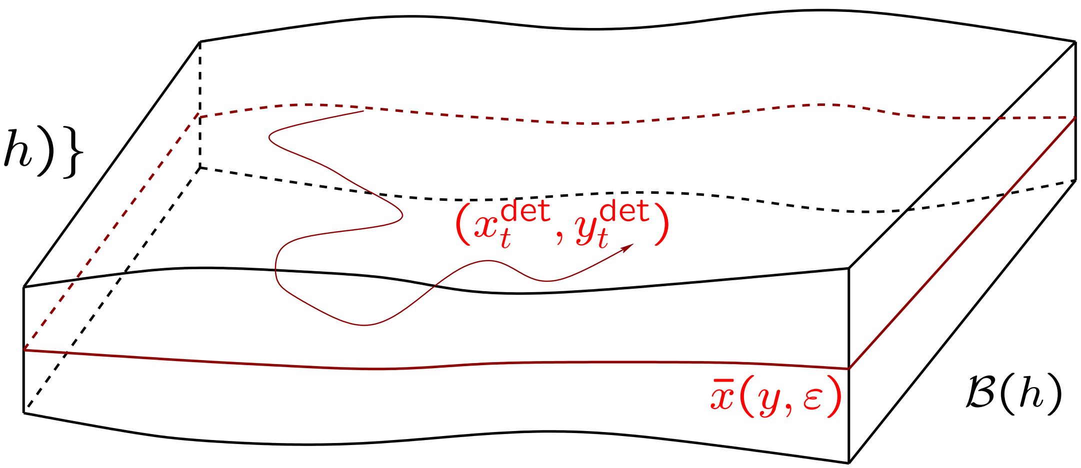
$$\mathcal{B}(h) := \Big\{(x,y)\colon \Big\langle \Big[x - \bar{x}(y,\varepsilon)\Big], \bar{X}(y,\varepsilon)^{-1}\Big[x - \bar{x}(y,\varepsilon)\Big]\Big\rangle < h^2\Big\}$$

Concentration of sample paths near adiabatic manifolds

Define (random) first-exit times

$\tau_{\mathcal{D}_0} := \inf\{s > 0 \colon y_s \notin \mathcal{D}_0\}$

$\tau_{\mathcal{B}(h)} := \inf\{s > 0 \colon (x_s, y_s) \notin \mathcal{B}(h)\}$



Theorem [Berglund & G, J. Differential Equations, 2003]

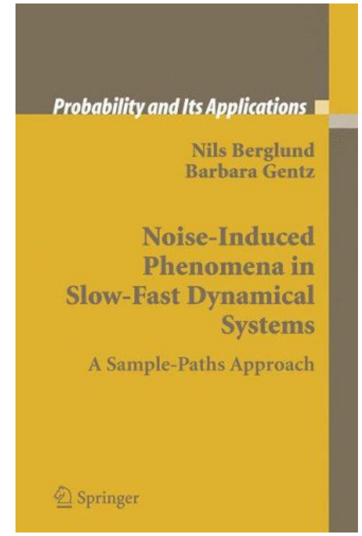
Assume: $\|\overline{X}(y,\varepsilon)\|$, $\|\overline{X}(y,\varepsilon)^{-1}\|$ uniformly bounded in $\mathcal{D}_0$

Then: $\exists\, \varepsilon_0 > 0 \quad \exists\, h_0 > 0 \quad \forall\, \varepsilon \leqslant \varepsilon_0 \quad \forall\, h \leqslant h_0$

$$\mathbb{P}\Big\{\tau_{\mathcal{B}(h)} < \min(t, \tau_{\mathcal{D}_0})\Big\} \leqslant C_{n,m}(t)\, \exp\bigg\{-\frac{h^2}{2\sigma^2}\Big[1 - \mathcal{O}(h) - \mathcal{O}(\varepsilon)\Big]\bigg\}$$

where $C_{n,m}(t) = \Big[C^m + h^{-n}\Big]\bigg(1 + \dfrac{t}{\varepsilon^2}\bigg)$

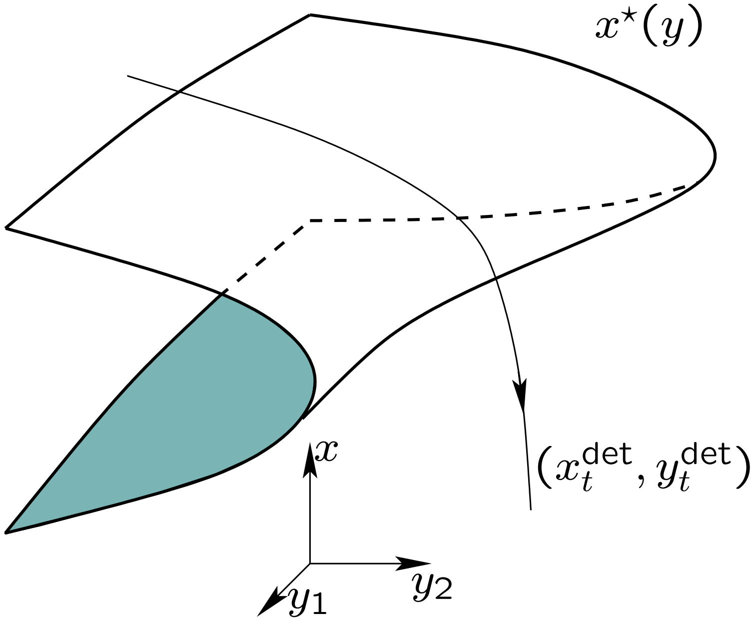
Bifurcations



Question

What happens if (x_t, y_t) approaches a bifurcation point $(\hat{x}, \hat{y})$ for the deterministic dynamics?

Ex.: Saddle–node bifurcation



General approach

- ▷ Apply centre-manifold theorem
- ▷ Concentration results for deviation from centre manifold ([Berglund & G, 2003])
- ▷ Consider reduced dynamics on centre manifold
- ▷ Concentration results for deviation of reduced system from original variables [Berglund & G, 2003]