Analysis 1

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Contents

1 Real numbers

1.1 Axioms of real numbers and their immediate consequences

Set $\mathbb{R}$ is called the set of *real numbers* and its elements are called real numbers (*reelle Zahlen*) if the following axioms are satisfied:

Axiom 1.1 (Axioms of addition) *For any two elements $x, y \in \mathbb{R}$ the expression $x + y$ is defined as a real number with the following properties:*

1. *(Neutral element) There exists a zero element $0 \in \mathbb{R}$ such that*
$$x + 0 = 0 + x = x \text{ for any } x \in \mathbb{R}.$$

2. *(Negative element) For any $x \in \mathbb{R}$ there exists $-x \in \mathbb{R}$ such that*
$$x + (-x) = (-x) + x = 0.$$

3. *(Associative law - Assoziativgesetz) For all $x, y, z \in \mathbb{R}$,*
$$(x + y) + z = x + (y + z).$$

4. *(Commutative law - Kommutativgesetz) For all $x, y \in \mathbb{R}$,*
$$x + y = y + x.$$

Axiom 1.2 (Axioms of multiplication) *For any two elements $x, y \in \mathbb{R}$ the expression $x \cdot y$ (or just xy) is defined as a real number with the following properties:*

1. *(Neutral element) There exists a unit element $1 \in \mathbb{R} \setminus \{0\}$ such that*
$$x \cdot 1 = 1 \cdot x = x \text{ for any } x \in \mathbb{R}.$$

2. *(The inverse element) For any $x \in \mathbb{R} \setminus \{0\}$ there exists $x^{-1} \in \mathbb{R}$ such that*
$$x \cdot x^{-1} = x^{-1} \cdot x = 1.$$

3. *(Associative law) For all $x, y, z \in \mathbb{R}$,*
$$(x \cdot y) \cdot z = x \cdot (y \cdot z).$$

4. *(Commutative law) For all $x, y \in \mathbb{R}$,*
$$x \cdot y = y \cdot x.$$

5. *(Distributive law - Distributivgesetz) For all $x, y, z \in \mathbb{R}$,*
$$(x + y) \cdot z = x \cdot z + y \cdot z.$$

Any set that satisfies the axioms of addition and multiplication is called a *field* (*Körper*), and the two sets of axioms are called the axioms of a field (*Körperaxiome*).

Axiom 1.3 (Axioms of order - *Anordnungsaxiome*) *For any two elements $x, y \in \mathbb{R}$ the relation $x \leq y$ is defined, that is, either $x \leq y$ is true or not, with the following properties:*

1. *(reflexivity) $x \leq x$ for any $x \in \mathbb{R}$.*
2. *$x \leq y$ and $y \leq x$ imply $x = y$.*
3. *(transitivity) $x \leq y$ and $y \leq z$ imply $x \leq z$*
4. *for any two elements $x, y \in \mathbb{R}$, either $x \leq y$ or $y \leq x$ is true.*
5. *if $x \leq y$ then $x + z \leq y + z$ for any $z \in \mathbb{R}$*
6. *if $0 \leq x$ and $0 \leq y$ then $0 \leq x \cdot y$.*

The relation $x \leq y$ is spelt out as "x is at most y" or "x is smaller than or equal to y". Any set with a relation $\leq$ that satisfies the axioms 1-4 is called an *ordered set*. The axioms 5,6 provide the link between the arithmetic operations and the order.

Axiom 1.4 *(Axiom of completeness - Vollständigkeitaxiom) If X, Y are non-empty subsets of $\mathbb{R}$ such that $x \in X$ and $y \in Y$ imply $x \leq y$ then there exists $c \in \mathbb{R}$ such that $x \leq c \leq y$ for all $x \in X$ and $y \in Y$.*

It is useful to represent the real numbers as points on a straight line (although for a rigorous introduction of geometry one needs the theory of real numbers!). Then the meaning of the axiom of completeness is that if a set X is on the left of the set Y then there is a point c that separates these two sets. In other words, the straight line contains no "holes" or "punctures".

Consequences of the addition axioms.

- *Zero element is unique.*

Indeed, if $0'$ is another zero then $0' + 0$ must be equal to both $0'$ and 0 whence

$$0' = 0' + 0 = 0.$$

- *The negative of any $x \in \mathbb{R}$ is unique.*

If y and z are two negatives to x then

$$y = y + 0 = y + (x + z) = (y + x) + z = 0 + z = z.$$

In particular, it follows that $-(-x) = x$ because $x + (-x) = 0$ and, hence, x satisfies the definition of the negative to $-x$. Also, we see that $-0 = 0$ because $0 + 0 = 0$ and, hence, 0 satisfies the definition of the negative to 0.

- *The equation $x + a = b$ has a unique solution $x = b + (-a)$.*

Indeed, for this value of x, we have

$$(b + (-a)) + a = b + ((-a) + a) = b + 0 = b.$$

On the other hand, the equation $x + a = b$ implies

$$(x + a) + (-a) = b + (-a)$$

whence

$$x + (a + (-a)) = b + (-a)$$

and $x = b + (-a)$.

The sum $b + (-a)$ is denoted shortly $b - a$ and is called the *difference* of b and a. The operation of taking difference is called *subtraction.*

Consequences of the multiplication axioms.

- *The unit element is unique.*
- *The inverse of any $x \in \mathbb{R} \setminus \{0\}$ is unique.*
- *The equation $x \cdot a = b$ has a unique solution $x = b \cdot a^{-1}$ provided $a \neq 0$.*

The proofs are the same as for the case of addition and are omitted. The expression $b \cdot a^{-1}$ is called the *quotient* of b and a and is denoted by b/a or $\frac{b}{a}$. The corresponding operation is called the *division.*

Consequences of the distributive law.

- $x \cdot 0 = 0 \cdot x = 0$ *for any* $x \in \mathbb{R}$

Indeed, we have

$$x \cdot 0 = x \cdot (0 + 0) = x \cdot 0 + x \cdot 0$$

which implies that

$$x \cdot 0 = x \cdot 0 - x \cdot 0 = 0.$$

- $x \cdot y = 0$ *implies* $x = 0$ *or* $y = 0$.

Indeed, if $y \neq 0$ then solving the equation $x \cdot y = 0$ with respect to x we obtain $x = 0/y = 0 \cdot y^{-1} = 0$.

- $-x = (-1) \cdot x$ *for any* $x \in \mathbb{R}$

Indeed, $x + (-1) \cdot x = 1 \cdot x + (-1) \cdot x = (1 + (-1)) \cdot x = 0 \cdot x = 0$ whence it follows that $(-1) \cdot x$ is the negative to x.

It follows, that $-(x + y) = -x - y$ because

$$-(x + y) = (-1) \cdot (x + y) = (-1) \cdot x + (-1) \cdot y = -x - y.$$

- $(-1) \cdot (-x) = x$

Applying the previous claim to $-x$, we obtain

$$-(-x) = (-1) \cdot (-x).$$

It is clear from the definition of the negative that $-(-x) = x$, which finishes the proof.

- $x \cdot (-y) = -(x \cdot y)$ *and* $(-x) \cdot (-y) = x \cdot y$.

We have

$$x \cdot (-y) = x \cdot ((-1) \cdot y) = (-1) \cdot (x \cdot y) = -(x \cdot y)$$

and, applying this to $-x$ instead of x,

$$(-x) \cdot (-y) = -((-x) \cdot y) = -(-(x \cdot y)) = x \cdot y.$$

In particular, we have $(-1) \cdot (-1) = 1$.

Consequences of the axioms of order.

The relation $x \leq y$ will be equivalently written as $y \geq x$. If in addition $x \neq y$ then we write $x < y$ or $y > x$ (strict inequality).

• *If $x < y$ and $y \leq z$ then $x < z$. Similarly, if $x \leq y$ and $y < z$ then $x < z$.*

By the transitivity, we have $x \leq z$. Let us show that $x \neq z$. Assuming from the contrary that $x = z$, we have $x < y$ and $y \leq x$, which implies by the axioms of order that $x = y$, which is impossible by $x < y$. The second claim is proved similarly.

• *For any two numbers $x, y \in \mathbb{R}$, exactly one of the relations holds:*

$$x < y, \ x = y, \ \ x > y.$$

If $x = y$ then only the middle relation holds. Assume $x \neq y$. By the axioms of order, we have either $x \leq y$ or $x \geq y$ which implies $x < y$ or $x > y$. If these two relations hold simultaneously then by the axiom we have $x = y$. Hence, exactly one of these two relations holds.

• *$a \leq b$ and $x \leq y$ imply $a + x \leq b + y$.*

By the order axiom, we have

$$a + x \leq b + x$$

and

$$b + x \leq b + y.$$

Combining these together, we obtain $a + x \leq b + y$.

• *The following inequalities are equivalent:*

$$x < y \ \ \Leftrightarrow \ \ \ y - x > 0 \ \ \ \Leftrightarrow \ -x > -y \ .$$

It suffices to prove the three implications:

$$x < y \ \Rightarrow \ \ y - x > 0 \ \Rightarrow \ -y < -x \ \Rightarrow \ x < y. \tag{1.1}$$

If $x < y$ then adding to the both sides $(-x)$ we obtain $0 < y - x$. If the latter holds then adding to the both sides $(-y)$ we obtain $-y < -x$. Hence, the first two implications in (1.1) give us that

$$x < y \ \ \Rightarrow \ -y < -x.$$

Changing here $-y$ to x and $-x$ to y, we obtain

$$-y < -x \ \ \Rightarrow \ x < y,$$

which is exactly the last implication in (1.1).

A number $x \in \mathbb{R}$ is called *positive* if $x > 0$ and *negative* if $x < 0$. Applying (1.1) with $y = 0$ we obtain that x is negative if and only if $-x$ is positive. Similarly, x is positive if and only if $-x$ is negative.

• *If x and y are both positive or negative then $x \cdot y$ is positive (in particular, $x \cdot x > 0$ provided $x \neq 0$). If one of x, y is positive and the other is negative then $x \cdot y$ is negative.*

If $x, y > 0$ then, by the axiom, we have $x \cdot y \geq 0$. If $x \cdot y = 0$ then one of x, y must be 0, which is impossible. Hence, $x \cdot y > 0$. If $x, y < 0$ then $-x, -y$ are positive, and

$$x \cdot y = (-x) \cdot (-y) > 0.$$

If $x > 0$ and $y < 0$ then $-y > 0$ and

$$-(x \cdot y) = x \cdot (-y) > 0$$

which implies that $x \cdot y < 0$.

- $1 > 0$ *and* $(-1) < 0$.

Since by the axiom $1 \neq 0$ and $1 = 1 \cdot 1$, and by the previous claim $1 \cdot 1 > 0$, we conclude $1 > 0$. It follows that $(-1) < 0$.

- *Let* $x < y$. *If* $a > 0$ *then* $a \cdot x < a \cdot y$. *If* $a < 0$ *then* $a \cdot x > a \cdot y$.

Note that $x < y$ implies $y - x > 0$. Therefore,

$$a \cdot y - a \cdot x = a \cdot (y - x) > 0,$$

which implies the claim in the case $a > 0$. The case $a < 0$ is treated similarly.

- *If* $x > 0$ *then* $x^{-1} > 0$. *If* $x > y > 0$ *then* $x^{-1} < y^{-1}$.

Indeed, if $x^{-1} < 0$ then

$$1 = x \cdot x^{-1} < 0,$$

which contradicts $1 > 0$. To prove the second claim, observe that $y < x$ implies

$$x^{-1} \cdot y < x^{-1} \cdot x = 1$$

and

$$x^{-1} = x^{-1} \cdot \left(y \cdot y^{-1}\right) = \left(x^{-1} \cdot y\right) \cdot y^{-1} < 1 \cdot y^{-1} = y^{-1},$$

which was to be proved.

Consequences of the axiom of completeness.

Let S be a non-empty subset of $\mathbb{R}$. A real number a is called an *upper bound* for S if $x \leq a$ for any $x \in S$. Similarly, a is a *lower bound* of S if $x \geq a$ for any $x \in S$.

Definition. The supremum $\sup S$ of a set $S \subset \mathbb{R}$ is the smallest upper bound of S. The infimum of a set $S \subset \mathbb{R}$ is the largest lower bound of S.

Clearly, for any $x \in S$ we have

$$\inf S \leq x \leq \sup S.$$

A number $a \in S$ is called the *maximum* of S if $a \in S$ and $x \leq a$ for any $x \in S$. If the maximum of S exists then it is denoted by $\max S$. In this case, we have $\max S = \sup S$. Indeed, $\max S$ is an upper bound for S. If $a < \max S$ then a cannot be an upper bound because $\max S \in S$ which means, that $\max S$ is the smallest upper bound.

Similarly one defines the minimum $\min S$. If it exists then it coincides with the infimum of S. However, in general $\sup S$ and $\inf S$ exist under milder conditions than $\max S$ and $\min S$, as we'll see below.

A set S is called *bounded from above* if it has an upper bound, and *bounded from below* if it has a lower bound.

Theorem 1.1 *Any non-empty subset $S \subset \mathbb{R}$ bounded from above has the supremum. Similarly, any non-empty subset $S \subset \mathbb{R}$ bounded from below has the infimum.*

Proof. Denote by U the set of upper bounds for S. Since S is bounded, the set U is not empty. For any $x \in S$ and $y \in U$, we have by construction $x \leq y$. Hence, by the completeness axiom, there exists $c \in \mathbb{R}$ that separates S and U, that is, $x \leq c \leq y$ for all $x \in S$ and $y \in U$. This means that

- c is an upper bound for S and, hence, $c \in U$;
- c is the smallest number in U.

Hence, c is the smallest upper bound of S, that is, the supremum of S, which proves the existence of $\sup S$. The infimum is handled in the same way. ∎

For any two reals $a < b$, define the following intervals:

$$\begin{aligned} (a, b) &= \{x \in \mathbb{R} : a < x < b\} && \text{open} \\ (a, b] &= \{x \in \mathbb{R} : a < x \leq b\} && \text{semi-open from left} \\ [a, b) &= \{x \in \mathbb{R} : a \leq x < b\} && \text{semi-open from right} \\ [a, b] &= \{x \in \mathbb{R} : a \leq x \leq b\} && \text{closed.} \end{aligned}$$

Proposition 1.2 *Let S be any of these intervals. Then S is non-empty,* $\inf S = a$, *and* $\sup S = b$.

Remark. It is easy to show that the set $S = (a, b)$ has neither maximum nor minimum. Indeed, it $\max S$ exists then $\max S = \sup S = b$ whereas $b \notin S$ and, hence, b cannot be the maximum of S.

Proof. Note that $2 := 1 + 1 > 1$ because $1 > 0$. Then $0 < 2^{-1} < 1^{-1}$, which can be written as $\frac{1}{2} \in (0, 1)$. More generally, let us show that $c := \frac{1}{2}(a + b)$ belongs to (a, b) (which will prove, in particular, that $S \neq \emptyset$). Indeed, we have $a + b < b + b = 2b$, which implies $\frac{1}{2}(a + b) < b$, that is $c < b$. Similarly, one proves that $c > a$.

Set $x = \inf S$ and prove that $x = a$. Clearly, a is a lower bound for S, whence $x \geq a$. Assume that $x > a$ and bring this assumption to contradiction. Since x is a lower bound for S and $c \in S$, we have $x \leq c$, which implies $x < b$ and, hence, $(a, x) \subset (a, b)$. Set $y = \frac{1}{2}(a + x)$ and note that, by the above argument,

$$y \in (a, x) \subset (a, b) \subset S,$$

which implies $y \geq \inf S = x$. We have obtained a contradiction with the condition $y < x$, which finishes the proof. ∎

By Theorem 1.1, $\sup S$ is defined for any bounded from above non-empty set of reals. Let us define $\sup S$ for *all* sets $S \subset \mathbb{R}$ using the notations $+\infty$ and $-\infty$ for the positive and negative infinity. These are additional ideal points with the properties that $+\infty$ is larger than any real, and $-\infty$ is smaller than any real. The set $\mathbb{R} \cup \{+\infty, -\infty\}$ is called the *extended* real line and is denoted by $\overline{\mathbb{R}}$, and its elements are called extended reals.

If S is non-empty but unbounded from above then define $\sup S$ by

$$\sup S = +\infty.$$

If S is empty then set $\sup S = -\infty$. Similarly, if S is non-empty and unbounded from below then set $\inf S = -\infty$. If S is empty then set $\inf S = +\infty$.

Theorem 1.1 and the above conventions imply the following.

Corollary. *Any subset of* $\mathbb{R}$ *has the supremum and the infimum with values in* $\overline{\mathbb{R}}$.

The notion of an interval is easily extended to the case when $a, b \in \overline{\mathbb{R}}$ to include $+\infty$ and $-\infty$. For example, if $a \in \mathbb{R}$ then $[a, +\infty)$ denotes the set of all $x \in \mathbb{R}$ such that $x \geq a$, and $(a, +\infty)$ is the set of $x \in \mathbb{R}$ such that $x > a$. The statement of Proposition 1.2 remains true in this case as well.

1.2 Natural numbers and mathematical induction

The purpose of this section is to introduce the notion of a natural number (*natürliche Zahl*). We expect that the set $\mathbb{N}$ of natural numbers is a subsets of $\mathbb{R}$ and should satisfy the following properties:

- $1 \in \mathbb{N}$
- if $x \in \mathbb{N}$ then also $x + 1 \in \mathbb{N}$.

However, these two properties are still not enough to specify $\mathbb{N}$ – for example, the full set $\mathbb{R}$ also satisfies these properties. In order to give a complete definition of $\mathbb{N}$, let us introduce the following notion.

Definition. A set $S \subset \mathbb{R}$ is called *inductive* if $x \in S$ implies $x + 1 \in S$.

For example, the set $\mathbb{R}$ itself is inductive. Also, intervals of the form $(a, +\infty)$ and $[a, +\infty)$ are inductive sets, whereas any bounded interval is not inductive.

Definition. The set $\mathbb{N}$ of natural numbers is the intersection of all inductive sets containing 1.

Theorem 1.3 *The set* $\mathbb{N}$ *is the smallest inductive set containing* 1.

Proof. Denote by $\mathcal{F}$ the family of all inductive sets containing 1. Then the statement of Theorem 1.3 contains two claims:

- $\mathbb{N} \in \mathcal{F}$ (that is, $\mathbb{N}$ is an inductive set containing 1),
- $\mathbb{N}$ is the *smallest* set in $\mathcal{F}$, that is, $\mathbb{N} \subset S$ for any $S \in \mathcal{F}$.

The family $\mathcal{F}$ is non-empty, for example, $\mathbb{R} \in \mathcal{F}$. By definition, we have

$$\mathbb{N} = \bigcap_{S \in \mathcal{F}} S. \tag{1.2}$$

Since each $S \in \mathcal{F}$ contains 1, it follows that $\mathbb{N}$ also contains 1. Since each S is inductive, we have

$$x \in \mathbb{N} \implies x \in S \ (\forall S \in \mathcal{F}) \implies x + 1 \in S \implies x + 1 \in \mathbb{N},$$

that is, $\mathbb{N}$ is inductive. Hence, $\mathbb{N} \in \mathcal{F}$. By (1.2) we have $\mathbb{N} \subset S$ for any $S \in \mathcal{F}$, which finishes the proof. ■

For example, the interval $[1, +\infty)$ is an inductive set and contains 1, which implies $\mathbb{N} \subset [1, +\infty)$. Hence, 1 is the smallest natural number. Since $\mathbb{N}$ contains 1, it must contain also $2 = 1 + 1$, then it must contain $3 = 2 + 1$, etc.

Suppose that we need to prove a statement $P(n)$ (*Aussage*) depending on a natural parameter n. Theorem 1.3 provides the following method of proving $P(n)$ for all $n \in \mathbb{N}$, which consists of two steps:

(1) prove that $P(1)$ is true,

(2) prove that if $P(n)$ is true then $P(n+1)$ is also true.

These two conditions imply that $P(n)$ is true for all $n \in \mathbb{N}$. Indeed, let S be the set of those n, for which $P(n)$ is true. Then, by step 1, S contains 1 and, by step 2, S is inductive. Hence, by Theorem 1.3, S contains all $\mathbb{N}$, which means that $P(n)$ holds for all $n \in \mathbb{N}$.

This method of the proof is called the *mathematical induction* (*Vollständige Induktion*). The step 1 is called the *inductive basis* (*Induktionanfang*), and the step 2 is called the *inductive step* (*Induktionsschritt*). The assumption $P(n)$ in the inductive step is called the *inductive hypothesis* (*Induktionsvoraussetzung*).

Let us illustrate this method in the proof of the following theorem.

Theorem 1.4 (*a*) *The sum $n + m$ and the product nm of two natural numbers nm are again natural numbers.*

(*b*) *If n, m are two natural numbers and $n > m$ then $n - m \in \mathbb{N}$.*

Proof. (*a*) Let us prove that $n + m \in \mathbb{N}$ using induction in n.

- Inductive basis. If $n = 1$ then $1 + m \in \mathbb{N}$ because $m \in \mathbb{N}$ and $\mathbb{N}$ is an inductive set.

- Inductive step. The inductive hypothesis $P(n)$ is as follows: if $m \in \mathbb{N}$ then $n + m \in \mathbb{N}$. Assuming that $P(n)$ holds for some n, let us prove $P(n+1)$, that is, $m \in \mathbb{N}$ implies $(n+1) + m \in \mathbb{N}$. Indeed, we have the identity

$$(n + 1) + m = (n + m) + 1.$$

  By the inductive hypothesis we have $n + m \in \mathbb{N}$, and by Theorem 1.3 we conclude that $(n + m) + 1 \in \mathbb{N}$.

Let us now prove that $nm \in \mathbb{N}$ using again induction in n.

- Inductive basis. If $n = 1$ then $nm = m \in \mathbb{N}$.

- Inductive step. If it is already known that $nm \in \mathbb{N}$ then

$$(n + 1) m = nm + m \in \mathbb{N}$$

  since both numbers nm and m are natural, and we have already proved that the sum of natural numbers is natural.

(b) Let us prove the claim by induction in m. The inductive basis is stated as follows: if $n \in \mathbb{N}$, $n > 1$, then $n - 1 \in \mathbb{N}$. It suffices to prove that the set

$$S = \{n \in \mathbb{N} : n = 1 \text{ or } n - 1 \in \mathbb{N}\}$$

contains $\mathbb{N}$. Clearly, S contains 1. If $n \in S$ then $n \in \mathbb{N}$ and, hence, $n + 1 \in \mathbb{N}$. Since $(n + 1) - 1 = n \in \mathbb{N}$, we conclude that $n + 1 \in S$, which implies that S is an inductive set. By Theorem 1.3, S contains $\mathbb{N}$. [1]

The inductive step. The inductive hypothesis $P(m)$ is as follows: if $n \in \mathbb{N}$ and $n > m$ then $n - m \in \mathbb{N}$. Assuming that $P(m)$ holds for some m, let us prove $P(m + 1)$, that is, if $n > m+1$ then $n - (m + 1) \in \mathbb{N}$. Using the properties of addition and subtraction, one easily proves that

$$n - (m + 1) = (n - m) - 1.$$

We have $n > m$, which implies by the inductive hypothesis that $n - m \in \mathbb{N}$. Since $n - m > 1$, by the inductive basis we conclude that $(n - m) - 1 \in \mathbb{N}$, which was to be proved. ■

Denote by $\mathbb{Z}$ the union of $\{0\}$, $\mathbb{N}$, and the negatives of $\mathbb{N}$, which can be written as follows:

$$\mathbb{Z} := \{0\} \cup \mathbb{N} \cap (-\mathbb{N}).$$

The elements of $\mathbb{Z}$ are called *integers* (*ganze Zahlen*) and $\mathbb{Z}$ is called the set of integers. Obviously, a positive integer is the same as a natural number.

Corollary. *If $x, y \in \mathbb{Z}$ then also $x + y$, $x - y$, xy belong to $\mathbb{Z}$.*

Proof. If $x = 0$ or $y = 0$ then the claims are trivial. Otherwise, there are natural numbers n, m such that $x = n$ or $x = -n$ and $y = m$ or $y = -m$. Considering various cases and using Theorem 1.4, we obtain the claim. For example, in the case $x = n$ and $y = -m$ we have $x + y = n - m$. If $n = m$ then $x + y = 0 \in \mathbb{Z}$, if $n > m$ then $x + y \in \mathbb{N}$ by Theorem 1.4, and if $n < m$ then by Theorem 1.4 $m - n \in \mathbb{N}$ and, hence, $n - m = -(m - n) \in \mathbb{Z}$. ■

Corollary. *If $x, y \in \mathbb{Z}$ then the condition $x > y$ is equivalent to $x \geq y + 1$.*

Proof. Indeed, $x > y$ implies that $x - y > 0$. Since $x - y$ is an integer, this means that $x - y \in \mathbb{N}$. Since 1 is the smallest natural number, it follows that $x - y \geq 1$ and, hence, $x \geq y + 1$. The opposite implication is obvious: $x \geq y + 1$ implies $x > y$ because $y + 1 > y$. ■

Claim (Extension of the principle of mathematical induction) *Let n_0 be an integer and $P(n)$ be a statement that depends on an integer parameter $n \geq n_0$. If $P(n_0)$ is true and $P(n)$ implies $P(n + 1)$ for any $n \geq n_0$ then $P(n)$ is true for all $n \geq n_0$.*

(See Problem no.8).

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[1] Alternatively, this argument can be stated as a proof by induction in n. Namely, let us prove that, for any $n \in \mathbb{N}$,

$$\text{either } n = 1 \text{ or } n - 1 \in \mathbb{N}. \qquad (*)$$

Inductive basis: if $n = 1$ then $(*)$ is trivially satisfied.

Inductive step: assuming that a number $n \in \mathbb{N}$ satisfies $(*)$, let us prove that $n + 1$ does too. Indeed, we have $n + 1 \in \mathbb{N}$ and $(n + 1) - 1 = n \in \mathbb{N}$, which means that $n + 1$ satisfies $(*)$

1.3 Finite sequences and finite sets

For any two integers $n \leq m$, denote by $\{n, ..., m\}$ the set of all integers k such that $n \leq k \leq m$.

Definition. A finite sequence (*endliche Folge*) $\{a_k\}_{k=n}^m$ is a mapping (*Abbildung*) $a : \{n, ..., m\} \to \mathbb{R}$ with notation $a(k) = a_k$.

The sum $\sum_{k=n}^m a_k$ of a finite sequence $\{a_k\}_{k=n}^m$ can be defined by induction in m as follows:

- if $m = n$ then $\sum_{k=n}^m a_k = a_n$.
- if $\sum_{k=n}^m a_k$ is already defined then set

$$\sum_{k=n}^{m+1} a_k = \left(\sum_{k=n}^{m} a_k\right) + a_{m+1}. \tag{1.3}$$

By the principle of mathematical induction, we conclude that the sum $\sum_{k=n}^m a_k$ is defined for all integers $m \geq n$. Frequently we use a less formal notation

$$\sum_{k=n}^{m} a_k = a_n + a_{n+1} + ... + a_m.$$

Similarly, one defines the product

$$\prod_{k=n}^{m} a_k = a_n \cdot ... \cdot a_m.$$

In particular, when all $a_k = a$ then we obtain the *powers* of a:

$$a^n = \prod_{k=1}^{n} a = \underbrace{a \cdot a \cdot ... \cdot a}_{n \text{ times}},$$

for any $n \in \mathbb{N}$. Equivalently, the powers a^n can be defined inductively by $a^1 = a$ and $a^{n+1} = a^n \cdot a$. Using induction, one easily proves the following identities:

$$a^n \cdot a^m = a^{n+m} \text{ and } (a^n)^m = a^{n \cdot m}. \tag{1.4}$$

If $a \neq 0$ then the powers a^n are defined also for non-positive integers n as follows: $a^0 := 1$ and, for a negative integer n, set $a^n := (a^{-1})^{-n}$. The identities (1.4) remain true for all integers n, m.

The next statement plays an important role in the study of finite sets.

Theorem 1.5 *Let S be a non-empty subset of $\mathbb{Z}$. If S is bounded from above then* $\max S$ *exists. If S is bounded from below then* $\min S$ *exists. In particular, if $S \subset \mathbb{N}$ then* $\min S$ *always exists.*

Proof. Let S be bounded from above. Then, by Theorem 1.1, $\sup S$ exists. Set $a = \sup S$. By definition, a is the smallest upper bound for S. Let us prove that $a \in S$, which will imply $a = \max S$. Assume from the contrary that $a \notin S$. Since $a - 1$ is not an upper bound for S, there exists $b \in S$ such that $b > a - 1$. Since a is an upper bound for S, we have also $b \leq a$. Since $a \notin S$ and $b \in S$, $a \neq b$ and, hence, $b < a$. Hence, we have $b \in (a-1, a)$. Similarly, there exists $c \in S$ such that $c \in (b, a)$. The numbers c and b are both in S and, hence, c and b are integers. Since $c - b > 0$, $c - b$ is a positive integer and, hence, $c - b \geq 1$, whereas by construction $c - b < a - (a - 1) = 1$. This contradiction finishes the proof. The case of $\inf S$ is treated in the same way. ■

Theorem 1.6 (The Archimedes principle) *For any real x there exists a unique integer n such that $n \leq x < n + 1$.*

Such a number n is called the *integer part* of x and is denoted by $[x]$. For example, $\left[\frac{1}{2}\right] = 0$ and $\left[-\frac{1}{2}\right] = -1$.

Proof. Fix $x \in \mathbb{R}$ and consider the set

$$S = \{k \in \mathbb{Z} : k > x\}.$$

Let us first show that this set is non-empty. Assuming the contrary we obtain that x is an upper bound for $\mathbb{Z}$. Hence, $\mathbb{Z}$ is bounded from above and, by Theorem 1.5, $\mathbb{Z}$ has the maximum, say m. But then $m + 1$ is also an integer and must be in $\mathbb{Z}$, which contradicts to $m + 1 > \max \mathbb{Z}$.

Since the set S is non-empty and is bounded from below, we obtain by Theorem 1.5 that S has the minimum; let $l = \min S$. Then consider $n = l - 1$. By construction, $n \notin S$, which implies $n \leq x$. On the other hand, $n + 1 = l \in S$, that is, $n + 1 > x$. Hence, this n satisfies the required conditions

$$n \leq x < n + 1.$$

Assume that there is another integer n' also satisfying

$$n' \leq x < n' + 1.$$

The comparison the two lines shows that $n' < n + 1$ and, hence, $n' \leq n$. In the same way $n \leq n'$, which implies $n = n'$. ■

Let X, Y be two arbitrary sets and $f : X \to Y$ be a mapping (*Abbildung*) from X to Y, that is, a rule that associates to any element $x \in X$ an element $y \in Y$, which is denoted by $f(x)$. The element $y = f(x)$ is called the *image* (*Bild*) of x, and x is called a *preimage* or *inverse image* (*Urbild*) of y. A mapping $f : X \to Y$ is called *injective* if every point $y \in Y$ has at most one preimage (that is, different points in X have different images). A mapping f is called *surjective* if every point $y \in Y$ has at least one preimage. Finally, f is called *bijective* if every $y \in Y$ has exactly one preimage. Equivalently, f is bijective if f is injective and surjective.

For a bijective mapping f, the inverse mapping $f^{-1} : Y \to X$ is defined as follows: $f^{-1}(y)$ is equal to the unique element $x \in X$ such that $f(x) = y$. Clearly, f^{-1} is also a bijection.

Definition. Two sets X, Y are called *equivalent* (*Gleichmächtig*) if there is a bijection $f : X \to Y$.

Notation: $X \sim Y$. The following are simple properties of the equivalence:

- $X \sim X$,
- $X \sim Y$ implies $Y \sim X$
- $X \sim Y$, $Y \sim Z$ imply $X \sim Z$.

For example, let us prove the last property. For any two mappings $f : X \to Y$ and $g : Y \to Z$ consider the through mapping $h : X \to Z$ defined by $h(x) = g(f(x))$ (the mapping h is called the composition of f and g and is denoted by $g \circ f$). If f and g are bijections then h is also a bijections, whence $X \sim Z$.

For any natural number n, denote

$$F_n := \{1, ..., n\} = \{k \in \mathbb{N} : 1 \leq k \leq n\}.$$

Definition. A set S is called *finite* if S is either empty or S is equivalent to F_n for some $n \in \mathbb{N}$.

If $S \sim F_n$ then one says that the number of elements in S is n or that the *cardinality* of S is n. This can also be written as follows: $\operatorname{card} S = n$.

Theorem 1.7 (a) *A subset of a finite set is finite.*

(b) *The union of two finite sets is finite.*

Proof. (a) It suffices to prove that any subset of F_n is finite. Inductive basis: If $n = 1$ then any subset of $F_1 = \{1\}$ is either F_1 or $\emptyset$ and there is nothing to prove. Inductive step. Assuming that any subset of F_n is finite, let us prove that any subset of F_{n+1} is finite. Consider a subset $S \subset F_{n+1}$. If $S = F_{n+1}$ then there is nothing to prove. Otherwise, there is $a \in F_{n+1} \setminus S$. Without loss of generality, we can assume that $a = n + 1$ so that

$$S \subset F_{n+1} \setminus \{n + 1\} = F_n.$$

However, by the inductive hypothesis, any subset of F_n is finite and, hence, S is finite.

(b) For any two sets A, B, the union $A \cup B$ is a disjoint union of $A \setminus B$, $B \setminus A$, $A \cap B$. By part (a), all these sets are finite. Hence, it suffices to prove that the union of two disjoint finite sets is finite. Hence, let A, B be two disjoint finite sets, $\operatorname{card} A = n$ and $\operatorname{card} B = m$. Then $A \sim F_n$ and $B \sim F_m$. Consider the set

$$n + F_m = \{n + x : x \in F_m\}.$$

Obviously, $n + F_m \sim F_m$ and, hence, $B \sim n + F_m$. The sets F_n and $n + F_m$ are disjoint, which implies that $A \cup B$ is equivalent to the union of F_n and $n + F_m$. Finally, it is easy to see that

$$F_n \cup (n + F_m) = \{1 \leq k \leq n\} \cup \{n + 1 \leq k \leq n + m\} = F_{n+m},$$

which implies that $A \cup B \sim F_{n+m}$ and $A \cup B$ is finite. ∎

Remark. As it follows from the proof, if A and B are disjoint finite sets then

$$\operatorname{card}(A \cup B) = \operatorname{card} A + \operatorname{card} B.$$

Theorem 1.8 (The Dirichlet principle) *Let n, m be natural numbers and $n > m$. Then there is no injective mapping from F_n to F_m.*

Remark. A folklore version of this theorem is called the *pigeonhole principle*: if n letters are put in m pigeonholes and $n > m$ then there is a pigeonhole with more than 1 letters in it.

Proof. The condition $n > m$ means $n \geq m + 1$ and, hence $F_{m+1} \subset F_n$. If a mapping $f : F_n \to F_m$ is injective then its restriction $f|_{F_{m+1}}$ is also injective. Hence, it suffices to consider the case $n = m + 1$. Let us prove by induction in $n > 1$ that any mapping $f : F_n \to F_{n-1}$ is not injective.

Inductive basis. The minimal value of n is 2. Consider a mapping $f : F_2 \to F_1$. Since $F_1 = \{1\}$ and $F_2 = \{1, 2\}$, it follows that $f(1) = f(2) = 1$ and, hence, f is not injective.

Inductive step. Assuming that the inductive hypothesis is true for some n, let us prove that any mapping $f : F_{n+1} \to F_n$ is not injective. Assume from the contrary that a mapping $f : F_{n+1} \to F_n$ is injective. Without loss of generality we can assume that $f(n+1) = n$. The injectivity of f implies that, for any $x \in F_n$, $f(x) < n$ and, hence, $f(x) \leq n - 1$. Let $g = f|_{F_n}$ be the restriction of f to F_n. Then the image of g lies in F_{n-1} so that we can consider g as a mapping from F_n to F_{n-1}. By the inductive hypothesis, g is not injective, which clearly contradicts the injectivity of f. ∎

Corollary. *If $n \neq m$ then F_n and F_m are not equivalent.*

Proof. Assume that $n > m$. If $F_n \sim F_m$ then there is a bijection $f : F_n \to F_m$. But then f is an injection, which contradicts Theorem 1.8. ∎

Hence, a set S may be equivalent to at most one of the sets F_n, which makes the notion $\operatorname{card} S$ well-defined.

1.4 Some consequences of the completeness axiom

Definition. An infinite sequence $\{a_k\}_{k=1}^{\infty}$ is a mapping $a : \mathbb{N} \to \mathbb{R}$ with the notation $a(k) = a_k$. More generally, if n is an integer then an infinite sequence $\{a_k\}_{k=n}^{\infty}$ is a mapping $a : \{k \in \mathbb{Z} : k \geq n\} \to \mathbb{R}$.

Similarly one defines sequences of elements from a set S rather than numbers - just replace in the above definition $\mathbb{R}$ by S. In particular, consider a sequence of bounded intervals $\{I_k\}_{k=1}^{\infty}$, that is, I_k is one of the intervals of the form (a_k, b_k), $[a_k, b_k)$, $(a_k, b_k]$, $[a_k, b_k]$ where $a_k, b_k \in \mathbb{R}$, $a_k < b_k$. We say that the sequence $\{I_k\}_{k=1}^{\infty}$ is *nested* if $I_{k+1} \subset I_k$ for any $k \geq 1$. In this case, we have $a_k \leq a_{k+1}$ and $b_{k+1} \leq b_k$.

Theorem 1.9 (The principle of nested intervals / the Cauchy-Cantor lemma) *Let $\{I_k\}_{k=1}^{\infty}$ be a nested sequence of closed bounded intervals in $\mathbb{R}$. Then the intervals $\{I_k\}_{k=1}^{\infty}$ have a common point (that is, there exists a point x such that $x \in I_k$ for all k).*

Remark. It is essential that the intervals are closed. Indeed, consider the following nested sequence of semi-open intervals: $I_k = (0, \frac{1}{k}]$. We claim that these intervals have empty intersection. Indeed, if $x \in I_k$ for all $k \in \mathbb{N}$ then $0 < x < \frac{1}{k}$ and, hence, $x^{-1} \geq k$. However, this contradicts the Archimedes principle. Similarly, the boundedness of the intervals is essential: the unbounded intervals $(-\infty, -k]$ have no common point.

Proof. Let us use the above notation for the intervals. The condition $a_k \leq a_{k+1}$ implies by induction that $a_k \leq a_l$ whenever $k \leq l$ (use induction in l). Similarly, the condition $b_{k+1} \leq b_k$ implies that $b_k \leq b_l$ whenever $k \geq l$. Let us show that $a_k < b_l$ for any pair of natural numbers k, l. Indeed, if $k \leq l$ then

$$a_k \leq a_l < b_l,$$

and if $k > l$ then

$$a_k < b_k \leq b_l,$$

so that in the both cases we have $a_k < b_l$.

Let A be the set of all numbers a_k and B - the set of all numbers b_k. Then $a \in A$ and $b \in B$ imply $a \leq b$. By the axiom of completeness we conclude that there is a point $c \in \mathbb{R}$ such that $a \leq c \leq b$ for all $a \in A$ and $b \in B$. This exactly means that $c \in [a_k, b_k]$ for all k. ∎

In the next statement we'll consider a family $\{I_\alpha\}_{\alpha \in S}$ of intervals parametrized by an index α that may vary in an arbitrary index set S. This means that I is a mapping from the set S to the set of all intervals, and $I(\alpha)$ is denoted by I_α. If $T \subset S$ then the family $\{I_\alpha\}_{\alpha \in T}$ is called a *subfamily* of $\{I_\alpha\}_{\alpha \in S}$. The subfamily $\{I_\alpha\}_{\alpha \in T}$ is finite if the set T is finite.

Theorem 1.10 (The compactness principle / the Borel-Lebesgue lemma) *Let a bounded closed interval $[a, b]$ be covered by a family $\{I_\alpha\}_{\alpha \in S}$ of open intervals, where S is an arbitrary index set. Then there exists a finite subfamily $\{I_\alpha\}_{\alpha \in T}$ that also covers $[a, b]$.*

Remark. This property of the interval $[a, b]$ is called *compactness*. In more details this notion will be studied in Analysis 2. It is essential that the interval $[a, b]$ is closed and bounded. For example, consider a semi-open interval $(0, 1]$ and the following open intervals $I_k = (\frac{1}{k}, 2)$. Then the family $\{I_k\}_{k=1}^{\infty}$ covers $(0, 1]$ while no finite subfamily does. Indeed, let T be a finite set of indices k and let us show that $\bigcup_{k \in T} I_k$ does not cover $(0, 1]$. The finiteness of T implies that T is bounded as a subset of $\mathbb{R}$ (see Exercise 14) Let m be an upper bound for T. Then $k \leq m$ for all $k \in T$, which implies $\frac{1}{k} \geq \frac{1}{m}$. Hence, $\frac{1}{m} \notin I_k$. However, $\frac{1}{m} \in (0, 1]$, which means that the finite family $\{I_k\}_{k \in S}$ does not cover $(0, 1]$.

Proof. Assume from the contrary that no finite subfamily of $\{I_\alpha\}_{\alpha \in S}$ covers $[a, b]$. Set $c = \frac{a+b}{2}$ and consider two intervals $[a, c]$ and $[c, b]$. If both of them can be covered by finite subfamilies of $\{I_\alpha\}_{\alpha \in S}$ then so can be the union (since the union of finite sets is finite by Theorem 1.7). Therefore, one of these intervals cannot be covered by a finite subfamily of $\{I_\alpha\}_{\alpha \in S}$. Denote this interval by $[a_1, b_1]$. Considering now $c_1 = \frac{a_1+b_1}{2}$ we obtain that one of the intervals $[a_1, c_1]$ and $[c_1, b_1]$ cannot be covered by a finite subfamily of $\{I_\alpha\}_{\alpha \in S}$; denote this interval by $[a_2, b_2]$, etc. More precisely, we construct a nested sequence $\{[a_n, b_n]\}_{n=0}^{\infty}$ of intervals such that each $[a_n, b_n]$ cannot be covered by a finite subfamily of $\{I_\alpha\}_{\alpha \in S}$. Namely, define the sequence by induction in n as follows.

- Inductive basis for $n = 0$. Set $a_0 = a$ and $b_0 = b$.
- Inductive step. If $[a_n, b_n]$ is already constructed for some n then construct $[a_{n+1}, b_{n+1}]$ as follows. Set $c_n = \frac{a_n+b_n}{2}$. Then one of the intervals $[a_n, c_n]$, $[c_n, b_n]$ cannot be covered by a finite subfamily of $\{I_\alpha\}_{\alpha \in S}$. Choose this interval and denote it by $[a_{n+1}, b_{n+1}]$.

By the principle of mathematical induction, $[a_n, b_n]$ is defined for all non-negative integers n. By construction, the sequence $\{[a_n, b_n]\}_{n=0}^{\infty}$ is nested. By Theorem 1.9, there is a common point x for all intervals $[a_n, b_n]$. Since $x \in [a, b]$, the point x belongs to some interval $I_\alpha = (c, d)$ so that $c < x < d$. We claim that there exists $n \in \mathbb{N}$ such that

$$[a_n, b_n] \subset (c, d) . \tag{1.5}$$

which will be a contradiction to the assumption that $[a_n, b_n]$ cannot be covered by a finite subfamily of $\{I_\alpha\}_{\alpha \in S}$. By construction we have

$$b_{n+1} - a_{n+1} = \frac{b_n - a_n}{2},$$

which implies by induction that

$$b_n - a_n = \frac{b - a}{2^n}.$$

We need to find n large enough to satisfy (1.5). For that we use the following version of the Archimedes principle (Theorem 1.6):

Claim *If l, m are positive numbers then there exists a natural number n such that $\frac{l}{n} < m$.*

Indeed, by Theorem 1.6 there is $n \in \mathbb{N}$ such that $n > \frac{l}{m}$, which is equivalent to $\frac{l}{n} < m$.

Let m be the smallest of the numbers $x - c$ and $d - x$. Since $c < x < d$, the number m is positive. Applying the above claim with $l = b - a$, we conclude that there exists $n \in \mathbb{N}$ such that $\frac{b-a}{n} < m$. By Bernoulli's inequality (see Exercise 9), we have

$$2^n \geq 1 + n > n,$$

which implies

$$b_n - a_n = \frac{b - a}{2^n} < \frac{b - a}{n} < m.$$

Using $b_n - a_n < m$ and $m \leq d - x$, we obtain

$$b_n = (b_n - x) + x \leq (b_n - a_n) + x < m + (d - m) = d$$

and using $m \leq x - c$, we obtain

$$a_n = x - (x - a_n) \geq x - (b_n - a_n) > (m + c) - m \geq c.$$

Hence, we have proved that $c < a_n < b_n < d$, whence (1.5) follows. ■

1.5 Numeral systems

For the next theorem, we need the following modification of the principle of the mathematical induction.

Claim *Let $P(n)$ be a statement depending on the natural parameter n. Assume that the following two conditions take place:*

- (Inductive basis) $P(1)$ *is true*

- (Inductive step) *For any* $n \in \mathbb{N}$, *if* $P(k)$ *is true for all* $k \in \{1, ..., n\}$ *then also* $P(n+1)$ *is true.*

  Then $P(n)$ *is true for all* $n \in \mathbb{N}$.

Proof. Let $Q(n)$ be the statement that $P(k)$ is true for all $k \in \{1, ..., n\}$. Let us prove by induction that $Q(n)$ holds for all $n \in \mathbb{N}$, which of course implies that also $P(n)$ holds.

Inductive basis. $Q(1)$ is true because $Q(1)$ is equivalent to $P(1)$.

Inductive step. Assuming that $Q(n)$ holds for some n let us prove that $Q(n+1)$ holds. Indeed, $Q(n)$ means that $P(k)$ is true for all $k \in \{1, .., n\}$. By the hypothesis, this implies that $P(n+1)$ also true. Hence, $P(k)$ is true for all $k \in \{1, ..., n+1\}$, which means hat $Q(n+1)$ is true. ∎

Theorem 1.11 *Let* $q > 1$ *be a natural number. For any* $x \in \mathbb{N}$, *there exists a unique non-negative integer* n *and a unique sequence* $\{a_k\}_{k=0}^{n}$ *of integers such that* $0 \leq a_k < q$, $a_n \neq 0$ *and*

$$x = \sum_{k=0}^{n} a_k q^k = a_n q^n + a_{n-1} q^{n-1} + ... + a_1 q + a_0. \tag{1.6}$$

The identity (1.6) is called the representation of x in the q-ic *numeral system* (= *the positional system with the base* q). The familiar symbolic way of writing down the identity (1.6) is as follows:

$$x = a_n a_{n-1} ... a_0$$

(*do not mix up with the product*). Each number a_k in this context is called a *digit* (*eine* $Ziffer$). The number n is called the *order* of x.

Most frequently used positional systems are the *decimal* system (with base $q =$ ten), the *binary* system (with base $q = 2$), and the *hexadecimal* system (with base $q =$ sixteen). The binary system is remarkable by the simplicity of operations with the digits since there are only two digits: 0 and 1. The numbers $1, 2, 3, 4, 5, 6, 7, 8, 9$ look in the binary system as follows:

$$1,\ 10,\ 11,\ 100,\ 101,\ 110,\ 111,\ 1000,\ 1001.$$

In the hexadecimal system, alongside the usual digits $0, 1, ..., 9$ one uses the digits A, B, C, D, E, F to represent the numbers from ten to fifteen. For example, the number FF in the hexadecimal system is equal to

$$\underbrace{FF = F \cdot q + F}_{\text{hex}} = \underbrace{15 \cdot 16 + 15 = 255}_{\text{decimal}}.$$

Proof of Theorem 1.11. Let us prove the statement by induction in x using the above modification of the principle of the mathematical induction.

Inductive basis. If $x = 1$ then (1.6) holds with $n = 0$ and $a_0 = 1$. Moreover, with any other choice of n and a_k the right hand side of (1.6) is larger than 1 so that the representation of 1 in the q-ic system is unique.

Inductive step. Assume that any natural number $y < x$ admits a unique representation in q-ic system and prove the same for x. Set $y = \left[\frac{x}{q}\right]$, that is, y is the integer part of $\frac{x}{q}$. This means, that y is the unique integer such that

$$y \leq \frac{x}{q} < y + 1. \tag{1.7}$$

Obviously, $y \geq 0$. Set $r = x - qy$. Obviously, r is an integer and

$$x = qy + r. \tag{1.8}$$

It follows from (1.7) that $0 \leq r < q$, that is, r is a q-ic digit. If $y = 0$ then $x < q$ and the q-ic representation of x is given by $n = 0$ and $a_0 = r$. Assume $y > 0$. Then $y \in \{1, ..., x - 1\}$ and, by the inductive hypothesis, y can be represented in q-ic system, say, as follows

$$y = b_m q^m + ... + b_0.$$

Then by (1.8)

$$x = b_m q^{m+1} + ... + b_0 q + r,$$

which is a q-ic representation of x because r and all b_k are q-ic digits.

Now let us prove the uniqueness of the representation of x in the form (1.6). Observe that

$$x = a_n q^n + ... + a_1 q_1 + a_0 = q\left(a_n q^{n-1} + ... + a_1\right) + a_0 = qy + a_0,$$

where

$$y = a_n q^{n-1} + ... + a_1. \tag{1.9}$$

Dividing the identity $x = qy + a_0$ by q and using $0 \leq a_0 < q$, we obtain

$$y \leq \frac{x}{q} < y + 1.$$

By Theorem 1.6, y is uniquely determined by these conditions (and $y = \left[\frac{x}{q}\right]$). Therefore, $a_0 = x - qy$ is also uniquely determined. Since $y < x$, the representation (1.9) of y in the q-ic system is unique by the inductive hypothesis, which implies that all the digits $a_1, ..., a_n$ are uniquely determined. Hence, all the digits $a_0, ..., a_n$ are uniquely determined, which finishes the proof. ∎

Now we consider the representation in the q-ic system of real numbers. Consider first an infinite sequence $\{a_k\}_{k=1}^{\infty}$ of q-ic digits (that is, integers between 0 and $q - 1$) and the infinite sum

$$\sum_{k=1}^{\infty} a_k q^{-k} = a_1 q^{-1} + a_2 q^{-2} + ... \ , \tag{1.10}$$

so far as a formal expression. The expression $\sum_{k=1}^{\infty} a_k q^{-k}$ is called a *series* (*eine Reihe*). The purpose of the next statement is to define the notion of the sum of the series[2].

Lemma 1.12 *For any sequence $\{a_k\}_{k=1}^{\infty}$ of q-ic digits, there exists a unique $x \in \mathbb{R}$ such that*

$$\sum_{k=1}^{n} a_k q^{-k} \leq x \leq \sum_{k=1}^{n} a_k q^{-k} + q^{-n}, \tag{1.11}$$

for any $n \in \mathbb{N}$. Moreover, $x \in [0, 1]$.

[2]Later in this course the sum of a series will be defined for a more general series than (1.10).

Definition. The unique number x satisfying (1.11) is called the sum of the series (1.10), and one writes

$$x = \sum_{k=1}^{\infty} a_k q^{-k}.$$

The sum $\sum_{k=1}^{n} a_k q^{-k}$ is called the n-th *partial sum* of the series (1.10). The inequality (1.11) means that the n-th partial sum can be considered as an approximation to the sum of the series with error at most q^{-n}.

A series of the form (1.10) is also called an (infinite) q-ic *fraction.* Hence, any q-ic fraction represents a real number $x \in [0, 1]$. The usual symbolic way of writing down a q-ic fraction is as follows:

$$0, a_1 a_2 a_3 ...$$

(or using '.' after 0 instead of ',').

Proof of Lemma 1.12. Set

$$x_n = \sum_{k=1}^{n} a_k q^{-k}$$

and

$$y_n = \sum_{k=1}^{n} a_k q^{-k} + q^{-n} = x_n + q^{-n}.$$

Let us show that the sequence of intervals $[x_n, y_n]$ is nested. It is obvious that $x_n < y_n$ and $x_{n+1} \geq x_n$. We only need to prove that $y_{n+1} \leq y_n$. Indeed, using $a_{n+1} \leq q - 1$, we obtain

$$\begin{aligned} y_{n+1} &= \sum_{k=1}^{n+1} a_k q^{-k} + q^{-(n+1)} \\ &= \sum_{k=1}^{n} a_k q^{-k} + a_{n+1} q^{-(n+1)} + q^{-(n+1)} \\ &= \left(y_n - q^{-n}\right) + (a_{n+1} + 1)\, q^{-(n+1)} \\ &\leq y_n - q^{-n} + q \cdot q^{-(n+1)} \\ &= y_n. \end{aligned}$$

By the principle of the nested intervals (Theorem 1.9), we conclude the existence of a common point for all intervals $[x_k, y_k]$.

Assume that there exists two distinct common points x and x', and let $x > x'$. Consider the difference $l := x - x' > 0$. The assumption that both x and x' belong to $[x_n, y_n]$ implies that, for any $n \in \mathbb{N}$,

$$l \leq y_n - x_n = q^{-n} \leq 2^{-n} < \frac{1}{n}.$$

On the other hand, by the Archimedes principle, there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < l$. This contradiction finishes the proof of the uniqueness.

For the proof of the fact that $x \in [0, 1]$ see Exercise 17. ■

Example. Setting $a_k = 1$ for all $k \in \mathbb{N}$, we obtain the series

$$\sum_{k=1}^{\infty} q^{-k} = q^{-1} + q^{-2} + ..., \tag{1.12}$$

which is called the *infinite geometric series*. Let us prove that the sum of the infinite geometric series is given by the formula

$$\sum_{k=1}^{\infty} q^{-k} = \frac{1}{q-1}, \tag{1.13}$$

that is,

$$\underbrace{0,111...}_{q-\text{ic}} = \frac{1}{q-1}.$$

Recall that, by Exercise 10, the following identity holds for any real $r \neq 1$:

$$1 + r + r^2 + ... + r^n = \frac{r^{n+1}-1}{r-1}.$$

Setting $r = q^{-1}$ and using the notation x_n, y_n as in the above proof, we obtain, for any $n \in \mathbb{N}$,

$$\begin{aligned} x_n &= q^{-1} + q^{-2} + ... + q^{-n} \\ &= \frac{q^{-(n+1)} - 1}{q^{-1} - 1} - 1 = \frac{q^{-(n+1)} - q^{-1}}{q^{-1} - 1} \\ &= \frac{q^{-1}\left(q^{-n} - 1\right)}{q^{-1}\left(1 - q\right)} = \frac{1 - q^{-n}}{q-1} < \frac{1}{q-1} \end{aligned}$$

and

$$\begin{aligned} y_n &= x_n + q^{-n} = \frac{1 - q^{-n}}{q-1} + q^{-n} \\ &= \frac{1 - 2q^{-n} + q^{-(n-1)}}{q-1} \geq \frac{1}{q-1} \end{aligned}$$

where in the last line we have used $q \geq 2$ and

$$1 - 2q^{-n} + q^{-(n-1)} \geq 1 - qq^{-n} + q^{-(n-1)} = 1.$$

Hence, $\frac{1}{q-1} \in [x_n, y_n]$, which was to be proved.

Example. Let us give an example showing that the representation of a number x as a q-ic fraction may be not unique. Indeed, if $a_1 = 0$ and $a_k = q - 1$ for all $k \geq 2$ then by (1.13)

$$\begin{aligned} x &= \underbrace{0,0\left(q-1\right)\left(q-1\right)...}_{q-\text{ic}} \\ &= \left(q-1\right)q^{-2} + \left(q-1\right)q^{-3} + ... \\ &= q^{-1}\left(q-1\right)\left(q^{-1} + q^{-2} + ...\right) \\ &= q^{-1} \\ &= \underbrace{0,100...}_{q-\text{ic}} \end{aligned}$$

Hence, this number x has two representations as an infinite q-ic fraction.

To provide a unique representation of x in the form of a q-ic fraction, one needs to put an additional restriction on a_k.

Definition. A q-ic fraction $0, a_1a_2a_3...$ is called *proper* if $a_k < q-1$ occurs infinitely often[3].

Theorem 1.13 *Let $q > 1$ be a natural number. For any $x \in [0,1)$, there exists a unique representation of x as a proper q-ic fraction.*

Let us use Theorems 1.11 and 1.13 to represent any non-negative real number x in the q-ic system. Any $x \geq 0$ can be uniquely represented in the form $x = y + z$ where y is a non-negative integer and $z \in [0,1)$ (indeed, y is the integer part of x and z is the *fractional* part of x defined by $z = x - y$). If $y = 0$ then $x \in [0,1)$ and its q-ic representation is given by Theorem 1.13. Assume $y > 0$. Then, by Theorem 1.11, y can be represented in the form

$$y = \sum_{k=0}^{n} a_k q^k = \underbrace{a_n a_{n-1} .... a_0}_{q-\text{ic}}$$

where a_k are q-ic digits and $a_n \neq 0$. By Theorem 1.13, z can be represented in the form

$$z = \sum_{k=1}^{\infty} b_k q^{-k} = \underbrace{0, b_1 b_2 b_3 ...}_{q-\text{ic}} ,$$

where b_k are q-ic digits. Hence, we obtain the following representation of x in the q-ic system

$$x = \sum_{k=0}^{n} a_k q^k + \sum_{k=1}^{\infty} b_k q^{-k},$$

which is symbolically written in the form

$$x = \underbrace{a_n a_{n-1} ... a_0, b_1 b_2 b_3 ...}_{q-\text{ic}} .$$

The expression $a_n a_{n-1} ... a_0, b_1 b_2 b_3 ...$ (where a_k and b_k are q-ic digits) is called a q-ic *numeral* (or a q-ic number). Hence, any non-negative real number can be represented by a q-ic numeral.

Proof of Theorem 1.13. By definition, the identity $x = 0, a_1 a_2 ...$ means that, for any $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} a_k q^{-k} \leq x \leq \sum_{k=1}^{n} a_k q^{-k} + q^{-n}. \tag{1.14}$$

Also, the fraction$0, a_1 a_2 ....$ is proper if the following set

$$S = \{k \in \mathbb{N} : a_k < q - 1\}$$

[3]That is, the set

$$\{k \in \mathbb{N} : a_k < q - 1\}$$

is infinite.

is infinite (recall that $0 \leq a_k \leq q-1$).

Let us first prove the following property of the properness.

Claim *A* q-ic *fraction* $x = 0, a_1 a_2 \ldots$ *is proper if and only if, for any* $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} a_k q^{-k} \leq x < \sum_{k=1}^{n} a_k q^{-k} + q^{-n}. \tag{1.15}$$

The distinction between (1.14) and (1.15) is that the right inequality in (1.15) is *strict*[4].

Let us prove that if (1.15) holds then the fraction $0, a_1 a_2 \ldots.$ is proper. Assume from the contrary, that the fraction $0, a_1 a_2 \ldots$ is not proper and prove that the right inequality in (1.15) fails for some n. The fact that the fraction is not proper means that the set S is finite and, hence, has an upper bound (see Exercise 14). This means that there exists $n \in \mathbb{N}$ such that $a_k = q-1$ for all $k \geq n$. Setting

$$y = \sum_{k=1}^{n-1} a_k q^{-k}$$

we obtain

$$\begin{aligned}
x &= \sum_{k=1}^{\infty} a_k q^{-k} = \sum_{k=1}^{n-1} a_k q^{-k} + \sum_{k=n}^{\infty} a_k q^{-k} \\
&= y + \sum_{k=n}^{\infty} (q-1)\, q^{-k} \\
&= y + (q-1)\, q^{-(n-1)} \left(q^{-1} + q^{-2} + \ldots\right) \\
&= y + (q-1)\, q^{-(n-1)} \frac{1}{q-1} \\
&= y + q^{-(n-1)},
\end{aligned}$$

where we have used the formula (1.13) for the sum of the infinite geometric series.

On the other hand, using $a_n = q-1$, we obtain

$$\begin{aligned}
\sum_{k=1}^{n} a_k q^{-k} + q^{-n} &= \sum_{k=1}^{n-1} a_k q^{-k} + a_n q^{-n} + q^{-n} \\
&= y + (a_n + 1)\, q^{-n} \\
&= y + q^{-(n-1)},
\end{aligned}$$

[4] For example, consider in the decimal system the fraction

$$x = 0, 19999\ldots$$

For any $n \geq 2$, the right hand side of (1.15) is equal to

$$0, \underbrace{199\ldots9}_{n} + 10^{-n} = 0, 2 = x.$$

Hence, the right inequality in (1.15) fails.

whence

$$x = \sum_{k=1}^{n} a_k q^{-k} + q^{-n},$$

which means that the strict inequality in (1.15) does not take place.

Assuming now that the fraction $0, a_1 a_2 ...$ is proper, let us prove that (1.15) holds for any $n \in \mathbb{N}$. We need only to prove that, for any $n \in \mathbb{N}$

$$x < \sum_{k=1}^{n} a_k q^{-k} + q^{-n}.$$

By definition of a proper fraction, the set

$$S := \{k \in \mathbb{N} : a_k < q - 1\}$$

is infinite, which implies that it has no upper bound (see Exercise 14). Hence, for any $n \in \mathbb{N}$ there exists $m > n$ such that $m \in S$, that is, $a_m < q - 1$. Applying (1.14) with n replaced by m, we obtain

$$\begin{aligned} x &\leq \sum_{k=1}^{m} a_k q^{-k} + q^{-m} \\ &= \sum_{k=1}^{n} a_k q^{-k} + \sum_{k=n+1}^{m} a_k q^{-k} + q^{-m} \\ &< \sum_{k=1}^{n} a_k q^{-k} + (q-1) \sum_{k=n+1}^{m} q^{-k} + q^{-m}, \end{aligned}$$

where we have applied $a_k \leq q - 1$ and $a_m < q - 1$. Using the formula

$$1 + r + r^2 + ... + r^{k-1} = \frac{r^k - 1}{r - 1} = \frac{1 - r^k}{1 - r}$$

for the sum of a finite geometric series (see Exercise 10), we obtain

$$\begin{aligned} \sum_{k=n+1}^{m} q^{-k} &= q^{-(n+1)} + q^{-(n+2)} + ... + q^{-m} \\ &= q^{-(n+1)} \left(1 + q^{-1} + ... + q^{-(m-n-1)}\right) \\ &= q^{-(n+1)} \frac{1 - q^{-(m-n)}}{1 - q^{-1}} \\ &= q^{-(n+1)} q \frac{1 - q^{-(m-n)}}{q - 1} = \frac{q^{-n} - q^{-m}}{q - 1}, \end{aligned}$$

whence

$$\begin{aligned} x &< \sum_{k=1}^{n} a_k q^{-k} + (q-1) \frac{q^{-n} - q^{-m}}{q - 1} + q^{-m} \\ &= \sum_{k=1}^{n} a_k q^{-k} + q^{-n}, \end{aligned}$$

which was to be proved.

In the view of the above Claim, Theorem 1.13 can be restated as follows: for any $x \in [0,1)$ there is a unique sequence $\{a_k\}_{k=1}^{\infty}$ of q-ic digits such that (1.15) holds for all $n \in \mathbb{N}$. Let us construct a_n by induction in n.

Inductive basis for $n = 1$. The relations (1.15) for $n = 1$ look as follows:

$$a_1 q^{-1} \leq x < a_1 q^{-1} + q^{-1}, \tag{1.16}$$

which is equivalent to

$$a_1 \leq qx < a_1 + 1. \tag{1.17}$$

By the Archimedes principle (see Theorem 1.6), such integer a_1 exists and is unique; in fact, (1.17) means that $a_1 = [qx]$, which is the integer part of qx. In particular, a_1 is uniquely defined. Let us show that a_1 is a digit. Indeed, the hypothesis $0 \leq x < 1$ implies $0 \leq qx < q$ and, hence, $0 \leq [qx] < q$, which means that $a_1 = [qx]$ is a q-ic digit.

Inductive step. Assuming that $a_1, ..., a_n$ are already defined, let us show that there exists a unique digit a_{n+1} such that

$$\sum_{k=1}^{n+1} a_k q^{-k} \leq x < \sum_{k=1}^{n+1} a_k q^{-k} + q^{-(n+1)}. \tag{1.18}$$

Setting $y = \sum_{k=1}^{n} a_k q^{-k}$, let us rewrite (1.18) in the form

$$y + a_{n+1} q^{-(n+1)} \leq x < y + a_{n+1} q^{-(n+1)} + q^{-(n+1)},$$

or

$$a_{n+1} \leq q^{n+1}(x - y) < a_{n+1} + 1,$$

which is equivalent to

$$a_{n+1} = \left[q^{n+1}(x - y)\right].$$

In particular, a_{n+1} is unique. Let us show that a_{n+1} defined by this formula, is a q-ic digit, that is $0 \leq a_{n+1} < q$. Indeed, by the inductive hypothesis (1.15), we have

$$y \leq x < y + q^{-n},$$

which implies

$$\begin{aligned} 0 &\leq x - y < q^{-n}, \\ 0 &\leq q^{n+1}(x - y) < q, \\ 0 &\leq \left[q^{n+1}(x - y)\right] < q \end{aligned}$$

which was to be proved. ■

Remark. A positional numeral system can be used to prove the existence of a set $\mathbb{R}$ satisfying the axioms of real numbers. For example, one can define non-negative real numbers as binary numerals

$$a_n a_{n-1} ... a_0, b_1 b_2 ... ,$$

where each a_k, b_l is a binary digit, that is, 0 or 1. Then one defines addition, multiplication, and the order of binary numerals using operations with the digits, and proves all the axioms of reals. However, the proofs are quite long and technical, and will not be considered here.

1.6 Cardinal numbers

The purpose of this section is to introduce the tools that will allow to compare two sets in order to determine which set is "larger". Recall the equivalence relation between arbitrary sets: $X \sim Y$ if there is a bijection between X and Y.

Definition. The family of all sets that are equivalent to X is called the *cardinal number* of X (or the *cardinality* of X) and is denoted by $|X|$.

Although $|X|$ is not a real number but a family of sets, $|X|$ possesses various properties reminiscent of those of real numbers, which justify this notation and the term "the cardinal number".

Claim $|X| = |Y|$ *if and only if* $X \sim Y$*.*

Proof. Indeed, assuming $X \sim Y$, we obtain

$$Z \in |X| \iff Z \sim X \iff Z \sim Y \iff Z \in |Y|$$

Hence, if $X \sim Y$ then $|X| = |Y|$. Conversely, if $|X| = |Y|$ then $X \in |Y|$ (because $X \in |X|$) whence $X \sim Y$. ■

One can define inequality between cardinal numbers as follows.

Definition. We write $|X| \leq |Y|$ if X is equivalent to a subset of Y.

Claim $|X| \leq |Y|$ *if and only if there exists an injective mapping* $f : X \to Y$*.*

Proof. If X is equivalent to a subsets $Y' \subset Y$ then there is a bijection $g : X \to Y'$. Defining $f : X \to Y$ by $f(x) = g(x)$ we obtain an injection. Conversely, if there exists an injection $f : X \to Y$ then restricting the target Y to $f(X)$, we obtain a bijection between X and $f(X)$, which means that $X \sim f(X)$. Hence, X is equivalent to a subset of Y, which means $|X| \leq |Y|$. ■

Let us list some further properties of the inequality between cardinal numbers:

1. If $|X| \leq |Y|$ and $|Y| \leq |Z|$ then $|X| \leq |Z|$ (see Exercise 18).
2. If $|X| \leq |Y|$ and $|Y| \leq |X|$ then $|X| = |Y|$.
3. for any two sets X, Y, either $|X| \leq |Y|$ or $|Y| \leq |X|$ is true.

The properties 2 and 3 are, in fact, deep theorems that we do not prove here (and which will not be used).

Example. Let $Y = \mathbb{N}$ and X be the set of even natural numbers, that is,

$$X = \{2n : n \in \mathbb{N}\} = \{2, 4, 6, ...\}.$$

Since $X \subset Y$, we obtain $|X| \leq |Y|$. However, in fact $|X| = |Y|$ because there is a bijection $f : X \to Y$ given by $f(x) = x/2$.

This example shows that a proper subset of a set Y can be equivalent to Y. By Theorem 1.8, this can occur only if Y is infinite.

Definition. A set X is said to be countable (*abzählbar*) if $|X| = |\mathbb{N}|$.

For example, the set of all even numbers is countable. If X is a countable set then there is a bijection $f : \mathbb{N} \to X$. Denoting $f(n) = x_n$, we see that X can be identified with the sequence $\{x_n\}_{n=1}^{\infty}$. We write this as follows: $X = \{x_1, x_2, ...\}$ and say that the set X is *enumerated* by natural numbers.

Theorem 1.14 *(a) A subset of a countable set is either finite or countable.*

(b) The direct product of two countable sets is countable.

(c) If $\{X_n\}_{n=1}^{\infty}$ is a sequence of countable sets then their union $X = \bigcup_{n=1}^{\infty} X_n$ is also countable.

Proof. (a) It suffices to prove that any subset X of $\mathbb{N}$ is either finite or countable. Assume that X is infinite, and construct a bijection $f : X \to \mathbb{N}$, which will prove that X is countable. Let us construct by induction in n a sequence $\{X_n\}_{n=1}^{\infty}$ of infinite subsets of X as follows.

Inductive basis $n = 1$: define $X_1 = X$.

Inductive step. If X_n is already defined and $a_n = \min X_n$ then set $X_{n+1} = X_n \setminus \{a_n\}$. Note that X_{n+1} is also infinite because if X_{n+1} is finite then $X_n = X_{n+1} \cup \{a_n\}$ is also finite by Theorem 1.7.

By construction, the sequence $\{X_n\}_{n=1}^{\infty}$ is nested, that is, $X_{n+1} \subset X_n$, and $X_n \setminus X_{n+1} = \{a_n\}$. Since a_n is a lower bound for X_{n+1} and $a_n \notin X_{n+1}$, we obtain that

$$a_{n+1} = \min X_{n+1} > a_n.$$

The sequence $\{a_n\}_{n=1}^{\infty}$ belongs to X. Let us show that X coincides with this sequence, which will prove that $|X| = |\mathbb{N}|$. Consider an arbitrary $x \in X$ and define the set

$$S = \{n \in \mathbb{N} : x \in X_n\}.$$

This set is non-empty because $1 \in S$. Let us show that S is bounded. The inequalities $a_1 \geq 1$ and $a_{n+1} > a_n$ imply by induction that $a_n \geq n$. Therefore, for any $n > x$ we have $a_n > x$ and, hence, $x \notin X_n$ and $n \notin S$. This implies that x is an upper bound for S. Since S is a bounded subset of $\mathbb{N}$, S has the maximum; let $m = \max S$. This means that $x \in X_m$ but $x \notin X_{m+1}$, that is,

$$x \in X_m \setminus X_{m+1} = \{a_m\}.$$

We conclude that $x = a_m$, which finishes the proof.

(b) Let X and Y be two countable sets. The direct product $X \times Y$ is the set of pairs (x, y) where $x \in X$ and $y \in Y$, that is,

$$X \times Y = \{(x, y) : x \in X, y \in Y\}.$$

Without loss of generality, we can assume that $X = Y = \mathbb{N}$ and prove that the set $\mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$ is countable. For any $k \in \mathbb{N}$, consider the set

$$D_k = \{(n, m) : n, m \in \mathbb{N},\ n + m = k\}.$$

Clearly, each set D_k, is finite and $\mathbb{N}^2$ is a disjoint union of all the sets D_k, $k \in \mathbb{N}$. On the diagram below, the elements of $\mathbb{N}^2$ are arranged in a table and the sets D_k are the diagonals of this table. Clearly, $\mathbb{N}^2$ is a disjoint union of all the sets D_k. Then one can enumerate $\mathbb{N}^2$ by enumerating successively each diagonal D_k – the number of each element

(n, m) is shown as a subscript:

$$
\begin{array}{ccccccccccc}
 & m=1 & & m=2 & & m=3 & & m=4 & & m=5 & ... \\
n=1 & (1,1)_1 & & (1,2)_3 & & (1,3)_6 & & (1,4)_{10} & & (1,5)_{15} & ... \\
 & & \nearrow & & \nearrow & & \nearrow & & \nearrow & & \\
n=2 & (2,1)_2 & & (2,2)_5 & & (2,3)_9 & & (2,4)_{14} & & ... & \\
 & & \nearrow & & \nearrow & & \nearrow & & & & \\
n=3 & (3,1)_4 & & (3,2)_8 & & (3,3)_{13} & & ... & & & \\
 & & \nearrow & & \nearrow & & & & & & \\
n=4 & (4,1)_7 & & (4,2)_{12} & & ... & & & & & \\
 & & \nearrow & & & & & & & & \\
n=5 & (5,1)_{11} & & ... & & & & & & & \\
... & ... & & & & & & & & &
\end{array}
$$

To describe this more precisely, denote by a_k be the total number of elements in all sets $D_1, D_2, ..., D_k$. Since $\operatorname{card} D_k = k - 1$, we obtain

$$a_k = \sum_{l=0}^{k-1} l = \frac{(k-1)\,k}{2}$$

(the right equality can be easily verified by induction in k). Then define a mapping $f : \mathbb{N}^2 \to \mathbb{N}$ as following: if $(n, m) \in D_k$ then set

$$f(n, m) = a_{k-1} + m.$$

Since m runs from 1 to $k - 1$ in D_k, we obtain

$$
\begin{aligned}
f(D_k) &= \{a_{k-1} + 1, ..., a_{k-1} + (k-1)\} \\
&= \{a_{k-1} + 1, ..., a_k\}
\end{aligned}
$$

where we have used the fact that $a_k = a_{k-1} + (k - 1)$. In other words, we have

$$f(D_k) = \{x \in \mathbb{N}\colon\ a_{k-1} < x \le a_k\},$$

whence it follows that $\mathbb{N}$ is a disjoint union of the sets $f(D_k)$. Hence, f is a bijection between $\mathbb{N}^2$ and $\mathbb{N}$, which finishes the proof.

(c) Let each set X_n be enumerated as follows: $X_n = \{x_{nm}\}_{m=1}^{\infty}$. Define a mapping $f : \mathbb{N}^2 \to X$ by setting

$$f(n, m) = x_{nm}$$

(that is, $f(n,m)$ is the m-th element of X_n). Next, use the following result: if $f : Y \to X$ is a surjection then $|X| \leq |Y|$ (see Exercise 18). In our case, f is a surjection, and we conclude that that $|X| \leq |\mathbb{N}^2|$. Using $|\mathbb{N}^2| = |\mathbb{N}|$, we obtain $|X| \leq |\mathbb{N}|$. Since set X is infinite as a union of infinite sets, we conclude that $|X| = |\mathbb{N}|$. ■

Definition. A real number x is called *rational* if it can be represented in the form $x = \frac{n}{m}$ where n, m are integers, and *irrational* otherwise.

The set of all rational numbers is denoted by $\mathbb{Q}$, so that the following inclusions take place $\mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$. It follows from Exercise 2 that the set $\mathbb{Q}$ is closed under the arithmetic operations of addition, subtraction, multiplication, and division.

Proposition *The set* $\mathbb{Q}$ *is countable, that is* $|\mathbb{Q}| = |\mathbb{N}|$.

Proof. Let us construct a mapping $f : \mathbb{N}^2 \to \mathbb{Q}$ as follows: for any couple $(n,m) \in \mathbb{N}^2$, set

$$f(n,m) = \begin{cases} \frac{n}{m}, & m \neq 0 \\ 0, & m = 0. \end{cases}$$

Then f is a surjection which implies $|\mathbb{Q}| \leq |\mathbb{N}^2|$. By Theorem 1.14 we have $|\mathbb{N}^2| = |\mathbb{N}|$, and by the same theorem $|\mathbb{Q}| \leq |\mathbb{N}|$ implies that $\mathbb{Q}$ is countable (because $\mathbb{Q}$ is infinite). ■

For any two sets X, Y, we write $|X| < |Y|$ if $|X| \leq |Y|$ but $|X| \neq |Y|$.

Definition. A set X is called *uncountable* if $|\mathbb{N}| < |X|$.

The next theorem provides an important example of an uncountable set.

Theorem 1.15 *The set* $\mathbb{R}$ *is uncountable, that is,* $|\mathbb{N}| < |\mathbb{R}|$.

The cardinal number $|\mathbb{R}|$ is called *continuum*.

Proof. Assume that $\mathbb{R}$ is countable so that $\mathbb{R} = \{x_1, x_2, ...\}$. Let us construct a nested sequence of closed bounded intervals $\{I_n\}_{n=1}^{\infty}$ as follows. Choose I_1 arbitrarily that I_1 does not contain x_1, for example, $I_1 = [x_1 + 1, x_2 + 2]$. If $I_n = [a_n, b_n]$ has been defined, then choose $I_{n+1} \subset I_n$ so that I_{n+1} does not contain x_{n+1}. Indeed, if $x_{n+1} \notin I_n$ then set $I_{n+1} = I_n$. If $x_{n+1} \in I_n$ then ether $x_{n+1} < b_n$ or $x_{n+1} > a_n$. In the first case, set $I_n = \left[\frac{x_{n+1}+b_n}{2}, b_n\right]$ and in the second case set $I_n = \left[a_n, \frac{a_n+x_{n+1}}{2}\right]$. By construction, we obtain a nested sequence of closed bounded intervals $\{I_n\}_{n=1}^{\infty}$ such that $x_n \notin I_n$ for any $n \geq 1$. By the principle of nested intervals, there is a point x that belongs to all I_n. Then $x \neq x_n$ for any $n \geq 1$ because $x_n \notin I_n$. This means that the point x is not in the sequence $\{x_n\}$ and, hence, $\mathbb{R}$ cannot be enumerated. ■

Corollary. *There are irrational numbers.*

Indeed, we have by the previous theorems, $|\mathbb{Q}| < |\mathbb{R}|$ which implies that $\mathbb{R} \setminus \mathbb{Q}$ is non-empty.

Theorem 1.16 *If* $|X| = |\mathbb{R}|$ *and* $|Y| \leq |\mathbb{N}|$ *then* $|X \cup Y| = |\mathbb{R}|$ *and* $|X \setminus Y| = |\mathbb{R}|$.

In the next argument, we use the following notation: $A \sqcup B$ denotes the union $A \cup B$ provided the sets A and B are disjoint, that is, $A \cap B = \emptyset$.

Proof. Observe that

$$X \cup Y = X \sqcup (Y \setminus X)$$

and $|Y \setminus X| \leq |Y| \leq |\mathbb{N}|$. Hence, renaming $Y \setminus X$ into Y, we can assume that X and Y are disjoint.

Let us use the following fact: any infinite set contains a countable subset (see Exercise 19). Let X_0 be any countable subset of X and set $X_1 = X \setminus X_0$. Then

$$X = X_0 \sqcup X_1$$

and, hence,

$$X \cup Y = X_0 \sqcup X_1 \sqcup Y = (X_0 \sqcup Y) \sqcup X_1. \tag{1.19}$$

However, since X_0 us countable and Y is either countable or finite we conclude by Theorem 1.14 that $X_0 \cup Y$ is also countable (see also Exercise 20). In particular, we have

$$X_0 \sqcup Y \sim X_0,$$

which together with (1.19) implies

$$X \cup Y \sim X_0 \sqcup X_1 = X,$$

which was to be proved.

Remark. As we see from the above argument, the assumption $|X| = |\mathbb{R}|$ was not used at all. All we need is the fact that set X is infinite. Hence, this part of Theorem 1.16 can be stated as follows: If set X is infinite and $|Y| \leq |\mathbb{N}|$ then $|X \cup Y| = |X|$.

Let us prove now the second claim of Theorem 1.16 in the following stronger form: if X is uncountable and $|Y| \leq |\mathbb{N}|$ then $|X \setminus Y| = |X|$. Since $X \setminus Y = X \setminus (X \cap Y)$ and $|X \cap Y| \leq |Y| \leq |\mathbb{N}|$, we can rename $X \cap Y$ by Y and assume in the sequel that $Y \subset X$. Then we have

$$X = (X \setminus Y) \cup Y. \tag{1.20}$$

Applying the above Remark to the difference $X \setminus Y$ and noticing that $X \setminus Y$ is infinite (indeed, if $X \setminus Y$ is finite then by (1.20) X is at most countable, which contradicts to the assumption that X is uncountable), we obtain

$$|(X \setminus Y) \cup Y| = |X \setminus Y|.$$

Combining this with (1.20), we conclude that $|X| = |X \setminus Y|$, which was to be proved. ∎

Using Theorem 1.16, one can prove that any of the intervals $(a, b), [a, b], [a, b), (a, b]$ where $a < b$, is equivalent to $\mathbb{R}$ (see Exercise 24).

Algebraic numbers. We know already the following classes of real numbers:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$$

and that their cardinal numbers satisfy the relations:

$$|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}| < |\mathbb{R}|.$$

Let us introduce yet another class of real numbers.

Definition. A real number x is called *algebraic*, if x satisfies an equation of the form

$$x^n + a_1 x^{n-1} + a_2 x^{n-2} + ... + a_n = 0,$$

where n is a natural number and all the coefficients a_k are rational numbers. In other words, x is a root of a polynomial with rational coefficients.

For example, any rational number x is algebraic since it satisfies the equation $x+a_1 = 0$ with $a_1 = -x \in \mathbb{Q}$ (here $n = 1$). Also, the number $x = \sqrt{2}$ is an algebraic number because it satisfies the equation $x^2 - 2 = 0$ (with $n = 2$).

It is possible to prove that the sum, difference, product, and ratio of algebraic numbers is again algebraic.

Denote by $\mathbb{A}$ the set of algebraic real numbers. Then $\mathbb{Q} \subset \mathbb{A} \subset \mathbb{R}$ and $\mathbb{Q} \neq \mathbb{A}$.

Theorem 1.17 $|\mathbb{A}| = |\mathbb{N}|$.

The real numbers that are not algebraic are called *transcendental.*

Corollary. *The set of transcendental numbers has the cardinality of* $|\mathbb{R}|$. *In particular, there exists at least one transcendental number.*

Proof. Indeed, the set of transcendental numbers is $\mathbb{R} \setminus \mathbb{A}$, and by Theorems 1.17 and 1.16, we obtain that

$$|\mathbb{R} \setminus \mathbb{A}| = |\mathbb{R} \setminus \mathbb{N}| = |\mathbb{R}| .$$

■

Proof of Theorem 1.17. For any $n \in \mathbb{N}$, let P_n be the set of the polynomials of the form $x^n + a_1 x^{n-1} + ... + a_n$ with $a_k \in \mathbb{Q}$. Let R_n be the set of all roots of the polynomials from P_n. Let us prove that $|R_n| = |\mathbb{N}|$. We use without proof the fact that each polynomial has a finite number of roots (in fact, at most n). Note that P_n can be identified with the set of all sequences $\{a_k\}_{k=1}^n$ of n rational numbers a_k. This implies that $P_{n+1} = P_n \times \mathbb{Q}$ because

$$\{a_1, a_2, ..., a_{n+1}\} = (\{a_1, ..., a_n\}, a_{n+1}) \in P_n \times \mathbb{Q}.$$

Using $P_1 = \mathbb{Q}$ and $|\mathbb{Q}| = |\mathbb{N}|$ as well as $|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$ (by Theorem 1.14), let us prove by induction that $|P_n| = |\mathbb{N}|$.

Inductive basis: $|P_1| = |\mathbb{Q}| = |\mathbb{N}|$.

Inductive step: $|P_{n+1}| = |P_n \times \mathbb{Q}| = |P_n \times \mathbb{N}| = |\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$.

Then R_n is the union of the sets of roots of all polynomials from P_n, that is, R_n is the union of a sequence of finite sets, which is countable by Theorem 1.14. Hence[5], $|R_n| = |\mathbb{N}|$.

Finally, we have $\mathbb{A} = \bigcup_{n=1}^{\infty} R_n$, and by Theorem 1.14 we conclude that $|\mathbb{A}| = |\mathbb{N}|$. ■

All the infinite sets considered above had the cardinal numbers $|\mathbb{N}|$ or $|\mathbb{R}|$. It is natural to ask whether there exist other infinite cardinal numbers. The question whether there exists a cardinal number strictly between $|\mathbb{N}|$ and $|\mathbb{R}|$ is known as the continuum hypothesis. It happens to be extremely difficult and to have an amazing answer: this question cannot be resolved within the conventional axiomatic system of the set theory and the existence or non-existence of such a set could be regarded as an independent axiom. On the contrary, the question of the existence of a cardinal number strictly larger than $|\mathbb{R}|$ is solved easily by Theorem 1.18 below.

---

[5] Strictly speaking, this argument only shows that $|R_n| \leq |\mathbb{N}|$, which is enough for the proof of Theorem 1.17. The identity $|R_n| = |\mathbb{N}|$ follows from the observation that R_n is infinite because R_n contains all the numbers of the form r^n where $r \in \mathbb{Q}$.

Definition. For any set X, denote by 2^X the set of all subsets of X (including $\emptyset$ and X).

The notation 2^X is motivated by the following observation.

Claim *If X is finite and* $\operatorname{card} X = n$ *then* $\operatorname{card} 2^X = 2^n$ *(in other words, there exist 2^n subsets of X).*

See Exercise 23.

Theorem 1.18 *For any set X, we have $|X| < \left|2^X\right|$.*

In particular, we have $\left|2^{\mathbb{R}}\right| > |\mathbb{R}|$. Another consequence of this theorem is that there is no largest cardinal number.

Proof. By definition, $|X| < |Y|$ if $|X| \leq |Y|$ and $|X| \neq |Y|$. Recall also that $|X| \leq |Y|$ if there exists a injection $f : X \to Y$, and $|X| = |Y|$ if there is a bijection $f : X \to Y$. So, in order to prove that $|X| < |Y|$, we need to show that:

- there is an injection $f : X \to Y$,
- there is no bijection $f : X \to Y$.

It is easy to construct an injection $f : X \to 2^X$, for example, as follows: $f(x) = \{x\}$, where $\{x\}$ is the subset of X that consists only of x.

Let $f : X \to 2^X$ be any mapping, and let us prove that f is not even surjective (then f is not a bijection). Consider the following set

$$S = \{x \in X : x \notin f(x)\} \tag{1.21}$$

which is a subset of X and, hence, is an element of 2^X (note that $f(x)$ is a subset of X so it makes sense to ask whether $x \in f(x)$ or not). Let us show that S has no preimage, which will prove the above claim. Assume that $f(y) = S$ for some $y \in X$, and consider two cases:

1. If $y \in S$ then by (1.21) $y \notin f(y) = S$ so we obtain a contradiction.
2. If $y \notin S$ then by (1.21) $y \in f(y) = S$ – again a contradiction.

Hence, $f(y) \neq S$, which was to be proved. ■

1.7 Complex numbers

Definition. The set $\mathbb{C}$ of *complex numbers* is the set of all couples (x, y) of real numbers x, y, with the following operations of addition and multiplication:

- $(x, y) + (x', y') = (x + x', y + y')$
- $(x, y) \cdot (x', y') = (xx' - yy', xy' + yx')$.

One can also say that $\mathbb{C}$ is $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ endowed with the above two operations.

Consider the complex numbers of the form $(x, 0)$. It follows from the definition that

$$\begin{aligned} (x,0) + (y,0) &= (x+y, 0) \\ (x,0) \cdot (y,0) &= (xy, 0). \end{aligned}$$

Hence, the addition and multiplication of complex numbers of the form $(x, 0)$ matches those for real numbers. For that reason, the set of complex numbers of the form $(x, 0)$ is identified with $\mathbb{R}$ by the rule $(x, 0) \mapsto x$. By this identification, we can assume that $\mathbb{R} \subset \mathbb{C}$ and write $(x, 0) = x$.

The complex number $(0, 1)$ plays a particularly important role and is denoted by i. It is called the *imaginary unit*. By definition, we have

$$i^2 = (0,1) \cdot (0,1) = (-1, 0) = -1$$

so that $i^2 = -1$.

Note also, that for all $a \in \mathbb{R}$ and $(x, y) \in \mathbb{C}$,

$$a \cdot (x,y) = (a,0) \cdot (x,y) = (ax, ay).$$

This implies that

$$(x,y) = (x,0) + (0,1)\,(y,0) = x + y \cdot (0,1) = x + yi.$$

Hence, any complex number can be written in the form $x+yi$ (or $x+iy$) and this notation is usually preferred to (x, y).

Normally, we denote complex number by a single letter, say $z = x + iy$. The number x is called the *real part* of z and y is called the *imaginary part* of z, and they are denoted by $x = \operatorname{Re} z$, $y = \operatorname{Im} z$. In particular, we have

$$z = \operatorname{Re} z + i \operatorname{Im} z.$$

Clearly, $z \in \mathbb{R}$ if and only if $\operatorname{Im} z = 0$.

For what follows, we need the following lemma.

Lemma 1.19 *For any non-negative real number a there exists a unique non-negative real number x such that $x^2 = a$.*

Definition. The unique number x as above is called the square root of a and is denoted by $\sqrt{a}$.

For the proof of existence of the square root see Exercise 21 where the case $a = 2$ was considered. The general case is treated exactly in the same way. The uniqueness of the square root is trivial: for any two distinct positive real numbers x, y we have either $x > y$ or $x < y$ which imply, respectively, that either $x^2 < y^2$ or $x^2 > y^2$ so that $x^2 = y^2$ is impossible. This argument not only proves the uniqueness of $\sqrt{a}$ but also yields that if $0 \le a \le b$ then $\sqrt{a} \le \sqrt{b}$.

Definition. For any complex number $z = x+iy$, its *modulus* or *absolute value* (*der Betrag*) $|z|$ is defined by

$$|z| = \sqrt{x^2 + y^2} = \sqrt{(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2}.$$

It follows that $|z|$ is a non-negative real and $|z| = 0$ if and only if $z = 0$.

If $z \in \mathbb{R}$ that is, $z = x$ then
$$|z| = \sqrt{x^2} = |x| .$$
Hence, in this case the modulus of z as a complex number coincides with the modulus of z as a real number.

Consider one more operation on complex numbers: *conjugation*. For any $z = x+iy \in \mathbb{C}$ define its *conjugate* $\bar{z}$ by
$$\bar{z} = x - iy = \operatorname{Re} z - i \operatorname{Im} z.$$
Clearly, $\bar{\bar{z}} = z$. In the next statement, we collect useful properties of the operations on complex numbers.

Theorem 1.20 (*a*) *The following relations hold for complex numbers:*
$$z_1 + z_2 = z_2 + z_1, \quad (z_1 + z_2) + z_3 = z_1 + (z_2 + z_3) \tag{1.22}$$
$$z_1 z_2 = z_2 z_1, \quad (z_1 z_2) z_3 = z_1 (z_2 z_3) \tag{1.23}$$
$$(z_1 + z_2) z_3 = z_1 z_3 + z_2 z_3 \tag{1.24}$$
$$z + \bar{z} = 2 \operatorname{Re} z, \quad z\bar{z} = |z|^2 \tag{1.25}$$
$$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2, \quad \overline{z_1 z_2} = \bar{z}_1 \bar{z}_2 \tag{1.26}$$
$$|z_1 z_2| = |z_1| |z_2| \quad \textit{and} \quad |z_1 + z_2| \le |z_1| + |z_2| . \tag{1.27}$$

(*b*) *For any $z_1, z_2 \in \mathbb{C}$ such that $z_2 \neq 0$, there exists a unique complex number w (denoted also as $\frac{z_1}{z_2}$) such that $z_2 w = z_1$. Also we have the identities*
$$\frac{z_1}{z_2} = |z_2|^{-2} z_1 \bar{z}_2 \quad \textit{and} \quad \left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}. \tag{1.28}$$

Proof. (*a*) The identities (1.22), (1.24), and the first identity in (1.23) (the commutative law for multiplication) are trivially verified by direct computation. Let us prove the second identity in (1.23) (the associative law for multiplication). Setting $z_k = x_k + iy_k$ we obtain
$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i (x_1 y_2 + y_1 x_2)$$
and
$$\begin{aligned} (z_1 z_2) z_3 &= ((x_1 x_2 - y_1 y_2) + i (x_1 y_2 + y_1 x_2)) (x_3 + i y_3) \\ &= (x_1 x_2 - y_1 y_2) x_3 - y_3 (x_1 y_2 + y_1 x_2) + i (x_1 x_2 - y_1 y_2) y_3 + (x_1 y_2 + y_1 x_2) x_3 \\ &= x_1 x_2 x_3 - y_1 y_2 x_3 - x_1 y_2 y_3 - y_1 x_2 y_3 + i (x_1 x_2 y_3 - y_1 y_2 y_3 + x_1 y_2 x_3 + y_1 x_2 x_3) . \end{aligned}$$
Obviously, the above expression is invariant under the cyclic permutation of indices:
$$1 \mapsto 2, \quad 2 \mapsto 3, \quad 3 \mapsto 1$$
(for example, the term $x_1 x_2 x_3$ does not change, the term $y_1 y_2 x_3$ goes to $y_2 y_3 x_1 = x_1 y_2 y_3$, the term $x_1 y_2 y_3$ goes to $x_2 y_3 y_1 = y_1 x_2 y_3$, etc). This implies that
$$(z_1 z_2) z_3 = (z_2 z_3) z_1 = z_1 (z_2 z_3)$$

where in the last identity we have used the commutative law.

To prove (1.25), set $z = x + iy$. Then $\bar{z} = x - iy$ and

$$z\bar{z} = (x + iy)(x - iy) = \left(x^2 + y^2\right) + i(xy - xy) = x^2 + y^2 = |z|^2$$

and $z + \bar{z} = 2x = 2\operatorname{Re} z$.

The first identity in (1.26) is trivial, the second is proved as follows:

$$\overline{z_1 z_2} = \overline{(x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2)} = (x_1x_2 - y_1y_2) - i(x_1y_2 + y_1x_2)$$

and

$$\bar{z}_1\bar{z}_2 = (x_1 - iy_1)(x_2 - iy_2) = (x_1x_2 - y_1y_2) - i(x_1y_2 + y_1x_2)$$

whence $\overline{z_1 z_2} = \bar{z}_1\bar{z}_2$.

The first identity in (1.27) is proved as follows:

$$\begin{aligned} |z_1z_2|^2 &= (x_1x_2 - y_1y_2)^2 + (x_1y_2 + y_1x_2)^2 \\ &= x_1^2x_2^2 - 2x_1x_2y_1y_2 + y_1^2y_2^2 + x_1^2y_2^2 + 2x_1x_2y_1y_2 + y_1^2x_2^2 \\ &= x_1^2x_2^2 + y_1^2y_2^2 + x_1^2y_2^2 + y_1^2x_2^2 \\ &= \left(x_1^2 + y_1^2\right)\left(x_2^2 + y_2^2\right) = |z_1|^2|z_2|^2. \end{aligned}$$

Before we prove the second relation in (1.27), observe that

$$\begin{aligned} |z_1 + z_2|^2 &= (z_1 + z_2)(\bar{z}_1 + \bar{z}_2) = z_1\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_1 + z_2\bar{z}_2 \\ &= |z_1|^2 + 2\operatorname{Re}(z_1\bar{z}_2) + |z_2|^2, \end{aligned}$$

where we have used

$$z_1\bar{z}_2 + z_2\bar{z}_1 = z_1\bar{z}_2 + \overline{z_1\bar{z}_2} = 2\operatorname{Re}(z_1\bar{z}_2).$$

Since $|\operatorname{Re} z| \leq |z|$, we obtain

$$|z_1 + z_2|^2 \leq |z_1|^2 + 2|z_1\bar{z}_2| + |z_2|^2 = |z_1|^2 + 2|z_1||z_2| + |z_2|^2 = (|z_1| + |z_2|)^2,$$

whence $|z_1 + z_2| \leq |z_1| + |z_2|$. This inequality is referred to as the *triangle inequality*.

(*b*) Multiplying the equation $z_2w = z_1$ by $\bar{z}_2$ and using (1.25), we obtain $|z_2|^2 w = z_1\bar{z}_2$. Noticing that $|z_2|$ is a positive real and multiplying by $|z_2|^{-2}$, we obtain

$$w = |z_2|^{-2} z_1\bar{z}_2. \tag{1.29}$$

Conversely, defining w by (1.29), we obtain

$$z_2w = |z_2|^{-2} z_1\bar{z}_2z_2 = |z_2|^{-2} z_1 |z_2|^2 = z_1.$$

This proves the existence and uniqueness of w as well as the first identity in (1.28). Taking the modulus of the both sides of (1.29), we obtain

$$\left|\frac{z_1}{z_2}\right| = |w| = |z_2|^{-2}|z_1||\bar{z}_2| = |z_2|^{-1}|z_1| = \frac{|z_1|}{|z_2|}.$$

■

2 Limits of sequences

2.1 Convergent and divergent sequences

Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of reals. We will be interested what happens with x_n when n becomes very large. Different sequences may exhibit different behavior. For example, in the sequence $a_n = (-1)^n$ the common term x_n jumps between 1 and -1. In the sequence $b_n = \frac{1}{n}$ the common term becomes smaller and very close to zero when n increases and becomes very large. It is natural to say that the sequence $\{b_n\}$ has the limit 0 whereas $\{a_n\}$ has no limit at all.

Definition. We say that a sequence $\{x_n\}_{n=1}^{\infty}$ of reals *converges* to $a \in \mathbb{R}$ if for any real $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_n - a| < \varepsilon$ for any natural $n \geq \mathbb{N}$. Shortly, this is written as follows:

$$\boxed{\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \text{ such that } |x_n - a| < \varepsilon \quad \forall n \geq N.} \tag{2.1}$$

The number a is called the *limit* (*der Grenzwert*) of $\{x_n\}$, and this is denoted by

$$x_n \to a$$

(x_n converges to a, or x_n tends to a, or x_n goes to a) or by

$$\lim_{n\to\infty} x_n = a$$

(the limit of x_n is a). A sequence is called *convergent* if it converges to some $a \in \mathbb{R}$. A sequence is called *divergent* if it is not convergent.

The definition (2.1) of the limit can be stated also as follows:

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \text{ such that } n \geq N \implies |x_n - a| < \varepsilon.$$

Or, one can say that, for any $\varepsilon > 0$, the condition $|x_n - a| < \varepsilon$ holds for all large enough n.

One can regard the convergence $x_n \to a$ as an approximation of a by the elements of the sequence $\{x_n\}$. Let $\varepsilon > 0$ be a prescribed bound for the error of the approximation. Then the convergence $x_n \to a$ means that, whatever is $\varepsilon > 0$, all the terms x_n with large enough n are approximations of a with error $< \varepsilon$. It is important that ε must take arbitrary positive values to ensure that the approximation of a by x_n occurs with an arbitrary precision.

Definition. For any $a \in \mathbb{R}$ and $\varepsilon > 0$, the interval $U_\varepsilon(a) = (a - \varepsilon, a + \varepsilon)$ is called the *ε-neighborhood* of a (*die ε-Umgebung*).

The condition $|x_n - a| < \varepsilon$ means that $x_n \in U_\varepsilon(a)$. Hence, the definition of convergence $x_n \to a$ can be stated as follows:

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \text{ such that } x_n \in U_\varepsilon(a) \text{ for all } n \geq N.$$

Or one can say that $x_n \to a$ if, for any $\varepsilon > 0$, $x_n \in U_\varepsilon(a)$ for all large enough n.

Theorem 2.1 *(a) $x_n \to a$ if and only if the set*

$$S_\varepsilon = \{n \in \mathbb{N} : x_n \notin U_\varepsilon(a)\}$$

is finite for any $\varepsilon > 0$.

(b) If the limit exists then it is unique.

(c) If the limit exists then the sequence is bounded from above and below.

(d) $x_n \to a$ if and only if $|x_n - a| \to 0$.

(e) If $x_n \to a$, $y_n \to b$ and $x_n \le y_n$ then $a \le b$.

(f) If $\{x_n\}, \{y_n\}, \{z_n\}$ are three sequences such that $x_n \to a$, $z_n \to a$ for some a and $x_n \le y_n \le z_n$ for all large enough n then also $y_n \to a$.

Proof. (a) If $x_n \to a$ then for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $x_n \in U_\varepsilon(a)$ for all $n \ge N$ which means that $n \notin S_\varepsilon$ for any $n \ge N$. Therefore, $n \in S_\varepsilon$ implies $n < N$ so that S_ε is bounded and, hence, finite.

If S_ε is finite then S_ε has an upper bound, say N. Then $n > N$ implies $n \notin S_\varepsilon$ that is, $x_n \in U_\varepsilon$, which means that $x_n \to a$.

(b) Let a and b be two distinct limits of the same sequence $\{x_n\}$, and let $a > b$. Set $\varepsilon = \frac{a-b}{2}$ and observe that the intervals $U_\varepsilon(a)$ and $U_\varepsilon(b)$ are disjoint. By part (a), outside $U_\varepsilon(a)$ there can be only finitely many terms x_n. But if $x_n \in U_\varepsilon(b)$ then $x_n \notin U_\varepsilon(a)$ which means that only finitely many terms x_n belong to$U_\varepsilon(b)$, which contradicts the condition $x_n \to b$.

(c) Let $x_n \to a$. Fix some $\varepsilon > 0$ say $\varepsilon = 1$. The part of the sequence x_n that is contained in $U_\varepsilon(a)$ is bounded because $U_\varepsilon(a)$ is bounded. Outside $U_\varepsilon(a)$ there are only finitely many terms, and any finite set of reals is bounded (see Exercise 14). Hence, the sequence is bounded as the union of two bounded sets.

(d) Set $y_n = |x_n - a|$. By definition, $y_n \to 0$ if, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|y_n - 0| < \varepsilon$ for all $n \ge N$. But this exactly means $|x_n - a| < \varepsilon$ that is, $x_n \to a$.

(e) Assume $a > b$ and set $\varepsilon = \frac{a-b}{2} > 0$ so that the neighborhoods $U_\varepsilon(a)$ and $U_\varepsilon(b)$ are disjoint. By definition, there exists $N' \in \mathbb{N}$ such that $x_n \in U_\varepsilon(a)$ for all $n \ge N'$, and $N'' \in \mathbb{N}$ such that $y_n \in U_\varepsilon(b)$ for all $n \ge N''$. Hence, for all $n \ge N = \max(N', N'')$, we have both conditions $x_n \in U_\varepsilon(a)$ and $y_n \in U_\varepsilon(b)$ which implies $x_n > y_n$ and which contradicts the hypothesis.

(f) For any $\varepsilon > 0$ there exists $N' \in \mathbb{N}$ such that $x_n \in U_\varepsilon(a)$ for all $n \ge N'$ and there exists $N'' \in \mathbb{N}$ such that $z_n \in U_\varepsilon(a)$ for all $n \ge N''$. Then, for all $n \ge \max(N', N'')$, both x_n and z_n are in $U_\varepsilon(a)$, which implies that also $y_n \in U_\varepsilon(a)$. Hence, we obtain $y_n \to a$. ∎

Example.

1. Let us show that the sequence $x_n = \frac{1}{n}$ converges to 0. Indeed, by the Archimedes principle, for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$. Then, for any $n \ge N$, $\frac{1}{n} < \varepsilon$ which can be written as $\left|\frac{1}{n} - 0\right| < \varepsilon$. Hence, $\frac{1}{n} \to 0$.

2. Let us show that the sequence $x_n = (-1)^n$ does not converge. The term x_n takes two values 1 and -1. If $x \ne 1$ and $x \ne -1$ then x cannot be the limit because there is a small neighborhood $U_\varepsilon(x)$ that contains no elements x_n. If $x = 1$ then outside $U_1(x)$ there are infinitely many terms that are equal to -1, and if $x = -1$ then outside $U_1(x)$ there are infinitely many terms that are equal to 1. Hence, there is no limit. This example also shows that a bounded sequence need not be convergent.

3. Consider a sequence $\{x_n\}$ such that $x_n = a$ for all $n \geq n_0$. Then $x_n \to a$. Indeed, for any $\varepsilon > 0$, $|x_n - a| = 0 < \varepsilon$ for all $n \geq n_0$.

4. Consider a sequence $\{x_n\} = n$. We claim that it is divergent. Indeed, if x is any real and $\varepsilon > 0$ then there is N such that $N > x + \varepsilon$. Hence, $x_n \notin U_\varepsilon(x)$ for any $n > N$ which implies the divergence.

5. Consider the sequence $x_n = a^n$ for a real a and prove that $\{x_n\}$ converges if and only if $a \in (-1, 1]$.

   Let $a > 1$, say $a = 1 + c$ where $c > 0$. Then, by Bernoulli's inequality,

   $$a^n = (1+c)^n \geq 1 + nc,$$

   which implies that a^n is unbounded and, hence, cannot converge. If $a < -1$ then $|x_n| = |a|^n$ and by the same argument $\{|x_n|\}$ is unbounded. Hence, $\{x_n\}$ is unbounded and cannot converge.

   If $a = -1$ then the sequence $x_n = (-1)^n$ has been already considered and it is divergent.

   If $a = 1$ then $\{x_n\}$ is a constant sequence $x_n = 1$ that converges to 1.

   If $a = 0$ then $x_n = 0$ and $x_n \to 0$.

   Let $0 < a < 1$. Then set $b = \frac{1}{a} > 1$ so that $a^n = \frac{1}{b^n}$. Writing $b = 1 + c$ where $c > 0$ we obtain $b^n \geq 1 + nc > nc$ and, hence,

   $$0 < a^n < \frac{1}{nc}.$$

   Similarly to one of the previous examples, we see that $\frac{1}{nc} \to 0$. Hence, by Theorem 2.1(f) we conclude $a_n \to 0$.

   If $-1 < a < 0$ then $0 < |a| < 1$ and $|a^n| = |a|^n \to 0$ by the previous argument, which implies that also $a^n \to 0$.

2.2 General properties of limit

Theorem 2.2 *If $x_n \to a$ and $y_n \to b$ then $x_n + y_n \to a + b$ and $x_n y_n \to ab$. If $y_n \neq 0$ and $b \neq 0$ then also $\frac{x_n}{y_n} \to \frac{a}{b}$.*

In particular, if $y_n = b$ is a constant sequence then we obtain $bx_n \to ab$. This implies that $-x_n \to -a$ and, hence, $y_n - x_n \to b - a$.

Proof. We have

$$|x_n + y_n - (a+b)| = |x_n - a + y_n - b| \leq |x_n - a| + |y_n - b|.$$

For any $\varepsilon > 0$ $\exists N' \in \mathbb{N}$ such that $|x_n - a| < \varepsilon$ for all $n \geq N'$, and $\exists N'' \in \mathbb{N}$ such that $|y_n - b| < \varepsilon$ for all $n \geq N''$. Hence, for any $n \geq N = \max(N', N'')$, we obtain

$$|x_n + y_n - (a+b)| < \varepsilon + \varepsilon = 2\varepsilon.$$

Since 2ε takes arbitrary positive values when ε varies in $(0, +\infty)$, we can rename 2ε to ε and conclude that for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$|x_n + y_n - (a + b)| < \varepsilon,$$

whence $x_n + y_n \to a + b$.

To prove the second claim of the theorem, estimate the difference $x_n y_n - ab$ as follows:

$$\begin{aligned} |x_n y_n - ab| &= |x_n y_n - x_n b + x_n b - ab| \\ &\leq |x_n (y_n - b)| + |(x_n - a) b| \\ &\leq \sup \{|x_n|\} |y_n - b| + |b| |x_n - a| . \end{aligned} \tag{2.2}$$

Before we can proceed, let us prove the following claim.

Claim *If $z_n \to 0$ then also $c z_n \to 0$ for any $c \in \mathbb{R}$.*

If $c = 0$ then there is nothing to prove. Assume $c \neq 0$. The hypothesis $z_n \to 0$ means that, for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|z_n| < \varepsilon$ for all $n \geq N$. The latter implies $|c z_n| < |c| \varepsilon$ for all $n \geq N$. Since $|c| \varepsilon$ takes arbitrary positive values, renaming $|c| \varepsilon$ to ε we obtain $c z_n \to 0$.

Returning to (2.2), observe that the sequence $\{x_n\}$ is bounded and, hence, $\sup \{|x_n|\}$ exists as a real number (see Theorem 2.1). Since $|y_n - b| \to 0$ and $|x_n - a| \to 0$, we obtain by the above claim that also

$$|b| |x_n - a| \to 0$$

and

$$\sup \{|x_n|\} |y_n - b| \to 0.$$

By the first part of Theorem 2.2, we obtain that

$$z_n := \sup \{|x_n|\} |y_n - b| + |b| |x_n - a| \to 0.$$

Since

$$0 \leq |x_n y_n - ab| \leq z_n,$$

we conclude by Theorem 2.1 that also $|x_n y_n - ab| \to 0$ and, hence, $x_n y_n \to ab$.

For the third part of this theorem, it suffices to prove that $\frac{1}{y_n} \to \frac{1}{b}$ since then we can multiply this convergence by $x_n \to a$ to obtain $\frac{x_n}{y_n} \to \frac{a}{b}$. Assume for simplicity that $b > 0$ (the case $b < 0$ is similar). There exists $N \in \mathbb{N}$ such that $y_n \in U_{b/2}(b) = (b/2, 3b/2)$ for all $n \geq N$. In particular, $y_n > b/2$ for all $n \geq N$. For such n, we have

$$\left| \frac{1}{y_n} - \frac{1}{b} \right| = \left| \frac{b - y_n}{b y_n} \right| \leq \frac{|b - y_n|}{b^2/2}.$$

Since $|b - y_n| \to 0$, it follows that also $\left| \frac{1}{y_n} - \frac{1}{b} \right| \to 0$ and, hence, $\frac{1}{y_n} \to \frac{1}{b}$. ■

Example. Consider the sequence

$$x_n = \frac{a n^2 + b n + c}{a' n^2 + b' n + c}$$

where $a' \neq 0$. Then

$$x_n = \frac{n^2 \left(a + \frac{b}{n} + \frac{c}{n^2} \right)}{n^2 \left(a' + \frac{b'}{n} + \frac{c}{n^2} \right)} = \frac{a + \frac{b}{n} + \frac{c}{n^2}}{a' + \frac{b'}{n} + \frac{c}{n^2}}.$$

As we have seen above, $\frac{1}{n} \to 0$. Hence, also $\frac{1}{n^2} \to 0$ and $\frac{b}{n} \to 0$, $\frac{c}{n^2} \to 0$ etc. Applying all the parts of Theorem 2.2, we conclude that $x_n \to \frac{a}{a'}$.

2.3 Existence of limit

Subsequence. We start with the following definition.

Definition. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of reals and $\{n_k\}_{k=1}^{\infty}$ be a sequence of natural numbers such that $n_k < n_{k+1}$ for any $k \in \mathbb{N}$. Then the sequence $\{x_{n_k}\}_{k=1}^{\infty}$ is called a *subsequence* of $\{x_n\}_{n=1}^{\infty}$.

For example, if $n_k = 2k$ then $\{x_{n_k}\} = \{x_2, x_4, ...\}$ consists of all even elements of the sequence $\{x_n\}$.

Claim *If $x_n \to a$ then also $x_{n_k} \to a$ for any subsequence $\{n_k\}$.*

Proof. By definition of the convergence $x_n \to a$,

$$\text{for any } \varepsilon > 0 \ \text{ there is } N \in \mathbb{N} \text{ such that } |x_n - a| < \varepsilon \text{ for all } n \geq N.$$

The conditions $n_1 \geq 1$ and $n_{k+1} > n_k$ imply by induction $n_k \geq k$ for all natural k. Hence, if $k \geq N$ then $n_k \geq N$, which implies $|x_{n_k} - a| < \varepsilon$. We obtain that

$$\text{for any } \varepsilon > 0 \text{ there is } N \in \mathbb{N} \text{ such that } |x_{n_k} - a| < \varepsilon \text{ for all } k \geq N.$$

which exactly means that $x_{n_k} \to a$. ∎

Of course, the fact that a subsequence converges does not imply in general that the sequence converges. For example, the sequence $x_n = (-1)^n$ does not converge while its subsequence $x_{2k} = 1$ does.

Theorem 2.3 (Theorem of Bolzano-Weierstrass) *Any bounded sequence has a convergent subsequence.*

Proof. By hypothesis, all terms of the sequence $\{x_n\}$ are located in some interval $[a, b]$, $a < b$. Set $c = \frac{a+b}{2}$. At least one of the interval $[a, c]$ and $[c, b]$ contains infinitely many terms of the sequence $\{x_n\}$. Denote this interval by $[a_1, b_1]$ so that $[a_1, b_1] \subset [a, b]$,

$$b_1 - a_1 = \frac{b-a}{2},$$

and $[a_1, b_1]$ contains infinitely many terms of $\{x_n\}$. Let $c_1 = \frac{a_1+b_1}{2}$ and select one of the intervals $[a_1, c_1]$, $[c_1, b_1]$ that contains infinitely many terms of $\{x_n\}$; denote this interval by $[a_2, b_2]$, etc. By induction, we obtain a sequence $\{[a_k, b_k]\}_{k=1}^{\infty}$ such that $[a_k, b_k] \subset [a_{k-1}, b_{k-1}]$,

$$b_k - a_k = \frac{b_{k-1} - a_{k-1}}{2} = \frac{b-a}{2^k},$$

and $[a_k, b_k]$ contains infinitely may terms of $\{x_n\}$. The sequence $\{[a_k, b_k]\}$ is nested and, hence, it has a common point, say x.

Let us construct a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to x$. Choose n_1 so that $x_{n_1} \in [a_1, b_1]$. Then choose n_2 so that

$$n_2 > n_1 \text{ and } \ x_{n_2} \in [a_2, b_2].$$

Such n_2 exists because $[a_2, b_2]$ contains infinitely many terms of the sequence $\{x_n\}$. Then choose n_3 so that

$$n_3 > n_2 \text{ and } x_{n_3} \in [a_3, b_3],$$

etc. Each time we can select $n_k > n_{k-1}$ so that $x_{n_k} \in [a_k, b_k]$ because $[a_k, b_k]$ contains infinitely many terms of $\{x_n\}$. Let us prove that the subsequence $\{x_{n_k}\}$ constructed in this way converges to x. Indeed, since both x_{n_k} and x are located in $[a_k, b_k]$, we have

$$|x_{n_k} - x| \le b_k - a_k \to 0 \text{ as } k \to \infty,$$

whence $x_{n_k} \to x$. ∎

Cauchy sequences. Let us define the notion of a Cauchy sequence.

Definition. A sequence $\{x_n\}$ is called *Cauchy* (or a *Cauchy sequence)*

$$\text{for any } \varepsilon > 0 \text{ there is } N \in \mathbb{N} \text{ such that } |x_n - x_m| < \varepsilon \text{ for all } n, m \ge N. \tag{2.3}$$

Shortly this can be stated as follows: $x_n - x_m \to 0$ as $n, m \to \infty$.

Theorem 2.4 *A sequence of reals converges if and only if it is a Cauchy sequence.*

Proof. Convergent $\Longrightarrow$ Cauchy. If $x_n \to a$ then

$$\text{for any } \varepsilon > 0 \text{ there is } N \in \mathbb{N} \text{ such that } |x_n - a| < \varepsilon \text{ for all } n \ge N.$$

Then we have, for all $n, m \ge N$,

$$|x_n - x_m| = |(x_n - a) - (x_m - a)| \le |x_n - a| + |x_m - a| < 2\varepsilon.$$

Renaming 2ε to ε we obtain that $\{x_n\}$ satisfies the definition of a Cauchy sequence.

Cauchy $\Longrightarrow$ Convergent. Let $\{x_n\}$ be a Cauchy sequence. Let us first show that this sequence is bounded. Indeed, by definition there exists N such that $|x_n - x_N| \le 1$ for any $n \ge N$. Hence, all sequence $\{x_n\}_{n=N}^{\infty}$ is contained in the interval $[-x_N, x_N]$ and, hence, is bounded. The remaining part $\{x_n\}_{n=1}^{N-1}$ is bounded as a finite sequence. Hence, the whole sequence $\{x_n\}_{n=1}^{\infty}$ is bounded as well.

By Theorem 2.3 there exists a convergent subsequence $\{x_{n_k}\}$. Assume that $x_{n_k} \to a$ and prove that, in fact, all sequence $\{x_n\}$ goes to a. The convergence $x_{n_k} \to a$ means that

$$\text{for any } \varepsilon > 0 \text{ there is } K \in \mathbb{N} \text{ such that } |x_{n_k} - a| < \varepsilon \text{ for all } k \ge K.$$

Choose k so big that $k \ge K$ and $n_k \ge N$ (this is the case if $k \ge \max(K, N)$). Then, for any $n \ge N$, we obtain by (2.3) (just put $m = n_k$ there) that

$$|x_n - x_{n_k}| < \varepsilon.$$

Together with $|x_{n_k} - a| < \varepsilon$, this implies

$$|x_n - a| = |x_n - x_{n_k} + x_{n_k} - a| \le |x_n - x_{n_k}| + |x_{n_k} - a| < 2\varepsilon.$$

Hence, $|x_n - a| < 2\varepsilon$ for any $n \ge N$, which means that $x_n \to a$. ∎

Monotone sequences. The last type of sequences we consider here are the monotone sequences.

Definition. A sequence $\{x_n\}_{n=1}^{\infty}$ of reals is called *monotone increasing* if $x_{n+1} \geq x_n$ for any $n \in \mathbb{N}$, and *monotone decreasing* if $x_{n+1} \leq x_n$ for any $n \in \mathbb{N}$. A sequence is called monotone if it is either monotone increasing or monotone decreasing.

Example. The sequence $x_n = n$ is monotone increasing, the sequence $x_n = \frac{1}{n}$ is monotone decreasing, the constant sequence $x_n = a$ is both monotone increasing and decreasing, the sequence $x_n = (-1)^n$ is not monotone.

If $\{x_n\}$ is monotone increasing then x_n is bounded from below, for example, x_1 is a lower bound for $\{x_n\}$. If $\{x_n\}$ is monotone decreasing then x_1 is an upper bound for $\{x_n\}$ and, hence, this sequence is bounded from above.

Theorem 2.5 *Any bounded monotone sequence converges.*

Recall that, by Theorem 2.1, any convergent sequence is bounded. Hence, a monotone sequence is convergent if and only if it is bounded.

Proof. Let $\{x_n\}_{n=1}^{\infty}$ be a bounded monotone increasing sequence. The boundedness implies that the supremum $a = \sup\{x_n\}$ exists and is a real number (see Theorem 1.1). Let us prove that $x_n \to a$. For any $\varepsilon > 0$, the number $a - \varepsilon$ is not an upper bound for $\{x_n\}$. Hence, there exists $N \in \mathbb{N}$ such that $x_N > a - \varepsilon$. Since the sequence $\{x_n\}$ is monotone increasing, we have that also $x_n > a - \varepsilon$ for any $n \geq N$. On the other hand, by the definition of a, $x_n \leq a$ whence $x_n \in (a - \varepsilon, a]$ and $|x_n - a| < \varepsilon$, for any $n \geq N$. We conclude that $x_n \to a$.

In the same way one proves that a bounded monotone decreasing sequence converges to its infimum. ■

2.4 Limits $+\infty$ and $-\infty$

We'll define here what it means that $x_n \to +\infty$ or $-\infty$. Recall that the definition of the convergence $x_n \to a$ can be stated as follows:

$$\text{for any } \varepsilon > 0 \text{ there exists } N \in \mathbb{N} \text{ such that } x_n \in U_\varepsilon(a) \text{ for all } n \geq N.$$

To adapt this definition to the case $a = \pm\infty$, let us define the neighborhoods of ∞ as follows: for any $E \in \mathbb{R}$, set

$$U_E(+\infty) = (E, +\infty) = \{x \in \mathbb{R} : x > E\}$$

and

$$U_E(-\infty) = (-\infty, E) = \{x \in \mathbb{R} : x < E\}.$$

Definition. We say that a sequence $\{x_n\}$ has limit $+\infty$ (or tends to $+\infty$, or goes to $+\infty$, or *diverges* to $+\infty$) and write $\lim_{n\to\infty} x_n = +\infty$ or $x_n \to +\infty$ if

$$\text{for any } E \in \mathbb{R} \text{ there exists } N \in \mathbb{N} \text{ such that } x_n \in U_E(+\infty) \text{ for all } n \geq N. \tag{2.4}$$

Similarly, $x_n \to -\infty$ if

$$\text{for any } E \in \mathbb{R} \text{ there exists } N \in \mathbb{N} \text{ such that } x_n \in U_E(-\infty) \text{ for all } n \geq N. \tag{2.5}$$

Note the terminology: if $x_n \to a$ where $a \in \mathbb{R}$ then we say that x_n *converges* to a, while if $a = +\infty$ or $-\infty$ then we say that x_n *diverges* to a, although in the both cases $\lim x_n$ exists as an element of the extended real line $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$.

Example. 1. The sequence $x_n = n$ tends to $+\infty$ because for any $E \in \mathbb{R}$ there exists $N \in \mathbb{N}$ such that $N > E$. Hence, for any $n \geq N$, we have $x_n > E$, that is, $x_n \in U_E(+\infty)$, which means $x_n \to +\infty$. Similarly, the sequence $x_n = -n$ diverges to $-\infty$. A small modification of this argument shows that the sequence $x_n = cn$ diverges to $+\infty$ if $c > 0$ and to $-\infty$ if $c < 0$.

2. Consider the sequence $x_n = a^n$ and show that if $a > 1$ then $a^n \to +\infty$ (recall that if $-1 < a \leq 1$ then the sequence $\{a^n\}$ is convergent). Indeed, writing $a = 1 + c$ where $c > 0$ and using Bernoulli's inequality, we obtain

$$a^n = (1+c)^n > cn.$$

Since $cn \to +\infty$, we obtain that also $a^n \to +\infty$.

If $a < -1$ then the sequence $x_n = a^n$ switches the sign ("+" if n is even and "−" if n is odd) so that it does not satisfy (2.4) or (2.5) with $E = 0$. Hence, in this case $\{x_n\}$ diverges without having limit.

Operations with $+\infty$ and $-\infty$. For any $a \in \overline{\mathbb{R}}$ define

$$(+\infty) + a = a + (+\infty) = \begin{cases} +\infty, & -\infty < a \leq +\infty \\ \text{undefined}, & a = -\infty, \end{cases}$$

and a similar rule holds for $(-\infty) + a$. The multiplication is defined by

$$(+\infty) \cdot a = a \cdot (+\infty) = \begin{cases} +\infty, & 0 < a \leq +\infty \\ -\infty, & -\infty \leq a < 0 \\ \text{undefined}, & a = 0. \end{cases}$$

and a similar rule holds for $-\infty \cdot a$. Finally, division by ∞ is defined by

$$\frac{a}{+\infty} = \begin{cases} 0, & a \in \mathbb{R} \\ \text{undefined}, & a = +\infty \text{ or } a = -\infty. \end{cases}$$

Hence, the following operations are undefined: $\infty - \infty$, $\infty \cdot 0$, and $\frac{\infty}{\infty}$. For completeness, recall that $\frac{a}{0}$ is not defined either.

In the next statement, we collect extensions of Theorems 2.2, 2.3, and 2.5 in the case when the limit may be $\pm\infty$.

Theorem 2.6 *(a) If $\{x_n\}$ and $\{y_n\}$ are sequences of reals such that $x_n \to a$ and $y_n \to b$ where $a, b \in \overline{\mathbb{R}}$ then $x_n + y_n \to a + b$, $x_n y_n \to ab$, and $x_n / y_n \to a/b$ provided the expressions $a + b$, ab, a/b are defined (and $y_n \neq 0$ in the latter case).*

(b) Any sequence of reals has a subsequence whose limit exists in $\overline{\mathbb{R}}$.

(c) If $\{x_n\}$ is a monotone increasing sequence of reals then

$$\lim_{n\to\infty} x_n = \sup\{x_n\}, \tag{2.6}$$

and if $\{x_n\}$ is a monotone decreasing sequence of reals then

$$\lim_{n\to\infty} x_n = \inf\{x_n\}.$$

Proof. (a) See Theorem 2.2 and Exercise 37.

(b) See Theorem 2.3 and Exercise 36.

(c) If $\sup\{x_n\}$ is finite then (2.6) was proved in Theorem 2.5. In the case when $\sup\{x_n\} = +\infty$, one needs to prove that if $\{x_n\}$ is an unbounded monotone increasing sequence then

$$\lim_{n\to\infty} x_n = +\infty,$$

which is done using the same argument as in the proof of Theorem 2.5. ∎

2.5 Limit points

Definition. A number $a \in \overline{\mathbb{R}}$ is called a limit point of a sequence $\{x_n\}$ if a is the limit of a subsequence of $\{x_n\}$.

Of course, if a sequence $\{x_n\}$ has limit a, then a is a limit point of $\{x_n\}$. The converse is not true. For example, if $x_n = (-1)^n$ then both 1 and -1 are limit points of $\{x_n\}$, while this sequence has no limit.

Denote by L the set of all limit points of a sequence $\{x_n\}$. By Theorem 2.6, the set L is non-empty.

Definition. For a sequence $\{x_n\}$, its *limit superior* is defined by

$$\limsup_{n\to\infty} x_n := \sup L.$$

(limsup of x_n as $n \to \infty$), the *limit inferior* is defined by

$$\liminf_{n\to\infty} x_n := \inf L.$$

(liminf of x_n as $n \to \infty$).

Alternative notation:

$$\limsup_{n\to\infty} x_n = \varlimsup_{n\to\infty} x_n \quad \text{and} \quad \liminf_{n\to\infty} x_n = \varliminf_{n\to\infty} x_n.$$

Theorem 2.7 (a) *The number $a \in \overline{\mathbb{R}}$ is a limit point of $\{x_n\}$ if and only if any neighborhood of a contains infinitely many terms of $\{x_n\}$.*

(b) *The following identities take place:*

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \sup_{k\ge n}\{x_k\} \quad \text{and} \quad \liminf_{n\to\infty} x_n = \lim_{n\to\infty} \inf_{k\ge n}\{x_k\}. \tag{2.7}$$

(c) *Both* $\limsup x_n$ *and* $\liminf x_n$ *are limit points of the sequence* $\{x_n\}$.

Hence, $\limsup x_n$ is the maximal limit point and $\liminf x_n$ is the minimal limit point.

Proof. (a) Assume for simplicity that $a \in \mathbb{R}$ (the cases $a = +\infty$ or $-\infty$ are handled similarly). Assume that, for any $\varepsilon > 0$, the ε-neighborhood $U_\varepsilon(a)$ contains infinitely many terms of $\{x_n\}$, and prove that a is a limit point. For that, we need to construct a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to a$. Select n_1 so that $x_{n_1} \in U_1(a)$. If $x_{n_{k-1}}$ is already defined then choose $n_k > n_{k-1}$ so that $x_{n_k} \in U_{1/k}(a)$ (this is possible because $U_{1/k}(a)$ contains infinitely many terms, in particular those with the index $n > n_{k-1}$). Then $|x_{n_k} - a| < \frac{1}{k}$ whence $|x_{n_k} - a| \to 0$ and $x_{n_k} \to a$.

Assume now that a is a limit point and prove that $U_\varepsilon(a)$ contains infinitely many terms of $\{x_n\}$. Indeed, let a be the limit of a subsequence $\{x_{n_k}\}$. Then by Theorem 2.1, $U_\varepsilon(a)$ contains all the terms of $\{x_{n_k}\}$ except for finitely many terms; hence, $U_\varepsilon(a)$ contains infinitely many terms of $\{x_n\}$.

$(b)+(c)$ Denote

$$y_n = \sup_{k\geq n}\{x_k\} = \sup\{x_n, x_{n+1}, ...\}.$$

Comparing with $y_{n+1} = \sup\{x_{n+1}, x_{n+2}, ...\}$, we see that the supremum in the definition of y_n is taken over a larger set, which implies $y_n \geq y_{n+1}$. Hence, the sequence $\{y_n\}$ is decreasing and, by Theorem 2.6, has a limit $a \in \overline{\mathbb{R}}$. Note that

$$a = \lim_{n\to\infty} y_n = \lim_{n\to\infty}\sup_{k\geq n}\{x_k\}.$$

We need to prove that $a = \sup L$. Let us first prove that $a \in L$. Assume for simplicity that $a \in \mathbb{R}$ and show that, for any $\varepsilon > 0$ and for any $N \in \mathbb{R}$ there is $n \geq N$ such that $x_n \in U_\varepsilon(a)$ – this will imply that $U_\varepsilon(a)$ contains infinitely many terms of $\{x_n\}$ and, hence, a is a limit point. Indeed, $U_\varepsilon(a)$ contains all y_n with large enough n; in particular, there exists $m \geq N$ such that $y_m \in U_\varepsilon(a)$. Find $\delta > 0$ so small that $U_\delta(y_m) \subset U_\varepsilon(a)$. Since $y_m = \sup_{n\geq m}\{x_n\}$, there exists $n \geq m$ such that $x_n \in U_\delta(y)$. Hence, for this n we have $n \geq N$ and $x_n \in U_\varepsilon(a)$, which was to be proved.

Now let us prove that a is an upper bound for L. Indeed, if $b \in L$ then b is a limit point of $\{x_n\}$ and, hence, b is also a limit point of any sequence $\{x_n, x_{n+1}, ...\}$. Clearly, a limit point of a sequence is bounded by its supremum. This implies

$$b \leq \sup\{x_n, x_{n+1}, ...\} = y_n.$$

Passing in the inequality $b \leq y_n$ to the limit as $n \to \infty$, we obtain $b \leq a$. Therefore, a is an upper bound for L.

Since $a \in L$ and a is an upper bound for L, we conclude that $a = \sup L$. This proves the first identity in (2.7) and at the same time the fact that $\limsup x_n$ belongs to L and, hence, is a limit point of $\{x_n\}$.

In the same way one treats $\liminf$. ■

2.6 Series

2.6.1 Definitions and examples

A *series* (*die Reihe*) is an infinite sum of the form $\sum_{k=1}^\infty a_k$ where $\{a_k\}_{k=1}^\infty$ is a sequence of reals. We have seen series of specific form before in the context of q-ic numeral systems. Here we define the notion of the sum of a general series as follows. First, define the n-th *partial sum* of the series $\sum_{k=1}^\infty a_k$ by

$$S_n = \sum_{k=1}^n a_k.$$

Definition. If $\lim_{n\to\infty} S_n$ exists (either finite or infinite) then the value of $\sum_{k=1}^\infty a_k$ is defined to be $\lim_{n\to\infty} S_n$, that is,

$$\sum_{k=1}^\infty a_k = \lim_{n\to\infty} S_n.$$

Hence, there are three possibilities:

1. $\lim_{n\to\infty} S_n$ is finite, that is, the sequence $\{S_n\}$ converges. Then one says that the series $\sum_{k=1}^{\infty} a_k$ also converges, and the value of $\sum_{k=1}^{\infty} a_k$ is a real number.

2. $\lim_{n\to\infty} S_n = +\infty$ (or $-\infty$). Then one says that the series $\sum_{k=1}^{\infty} a_k$ diverges to $+\infty$ (or $-\infty$), and the value of $\sum_{k=1}^{\infty} a_k$ is $+\infty$ (or $-\infty$).

3. $\lim_{n\to\infty} S_n$ does not exist. Then one says that the series $\sum_{k=1}^{\infty} a_k$ diverges, and in this case the value of $\sum_{k=1}^{\infty} a_k$ is undefined.

Example. *(The geometric series)* Let $a_k = x^k$ where $-1 < x < 1$. By Exercise 10,

$$S_n = x + x^2 + ... + x^n = \frac{1 - x^{n+1}}{1 - x} - 1 = \frac{x - x^{n+1}}{1 - x}.$$

Since $|x| < 1$, we have $x^{n+1} \to 0$ as $n \to \infty$, which implies $S_n \to \frac{x}{1-x}$. Hence,

$$\sum_{k=1}^{\infty} x^n = \frac{x}{1 - x}.$$

Example. Consider the series

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + ... + \frac{1}{k \cdot (k+1)} + ... = \sum_{k=1}^{\infty} \frac{1}{k\,(k+1)}.$$

To evaluate the partial sum S_n, use the following identity

$$\frac{1}{k \cdot (k+1)} = \frac{1}{k} - \frac{1}{k+1},$$

which implies that

$$S_n = \sum_{k=1}^{n} \frac{1}{k\,(k+1)} = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + ... + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}.$$

Alternatively, one can prove by induction that $S_n = 1 - \frac{1}{n+1}$ using the identity

$$S_{n+1} = S_n + \frac{1}{(n+1)\,(n+2)} = S_n + \frac{1}{n+1} - \frac{1}{n+2}.$$

Since $\frac{1}{n+1} \to 0$ as $n \to \infty$, we obtain that $S_n \to 1$ and, hence,

$$\sum_{k=1}^{\infty} \frac{1}{k\,(k+1)} = 1.$$

2.6.2 Non-negative series

Definition. A series $\sum_{k=1}^{\infty} a_k$ is called *non-negative* if all the terms a_k are non-negative reals.

For a non-negative series, the sequence $\{S_n\}$ of the partial sums is **monotone increasing** and, hence, $\lim S_n$ exists in $\overline{\mathbb{R}}$. Therefore, the sum $\sum_{k=1}^{\infty} a_k$ of a non-negative series is **always** defined and, moreover,

$$\sum_{k=1}^{\infty} a_k \in [0, +\infty] .$$

Hence, for a non-negative series only two possibilities occur: either $\sum_{k=1}^{\infty} a_k < +\infty$ and the series $\sum_{k=1}^{\infty} a_k$ is convergent, or $\sum_{k=1}^{\infty} a_k = +\infty$ and the series is divergent to $+\infty$.

Example. *(The harmonic series)* Consider the series

$$1 + \frac{1}{2} + \frac{1}{3} + ... = \sum_{k=1}^{\infty} \frac{1}{k}$$

and prove that it diverges to $+\infty$. The partial sum of this series is

$$S_n = 1 + \frac{1}{2} + ... + \frac{1}{n}.$$

Let us show that the sequence $\{S_n\}$ is not Cauchy. Indeed, we have

$$S_{2n} - S_n = \underbrace{\frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{2n}}_{n \text{ terms}} \geq n \frac{1}{2n} = \frac{1}{2},$$

whereas if $\{S_n\}$ is Cauchy then $S_{2n} - S_n \to 0$ as $n \to \infty$. Hence, $\{S_n\}$ is not convergent, whence it follows that $S_n \to +\infty$ and $\sum_{k=1}^{\infty} \frac{1}{k} = \infty$.

Note that the sequence $\{S_n\}$ of partial sum of this series increases with n very slowly. For example,

$$S_{1000} = \sum_{k=1}^{1000} \frac{1}{k} \approx 7,4854709 \quad \text{and} \quad S_{10000} = \sum_{k=1}^{10000} \frac{1}{k} \approx 9,787\,606.$$

2.6.3 General tests for convergence

Theorem 2.8 (The tail test) *The series $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=m}^{\infty} a_k$ converge or diverge simultaneously, for any $m \in \mathbb{N}$.*

The series $\sum_{k=m}^{\infty} a_k$ is called a *tail* of the series $\sum_{k=1}^{\infty} a_k$.

Proof. Let $S_n = \sum_{k=1}^{n} a_k$ and $T_n = \sum_{k=m}^{n} a_k$ where $n > m$. Then

$$S_n - T_n = \sum_{k=1}^{m-1} a_k =: C,$$

where C does not depend on n. Therefore, $T_n = S_n - C$ and, hence, if the sequence $\{S_n\}$ converges then so does $\{T_n\}$ and vice versa. ■

Example. The series $\sum_{k=1}^{\infty} \frac{1}{k+2006}$ diverges because

$$\sum_{k=1}^{\infty} \frac{1}{k+2006} = \frac{1}{2007} + \frac{1}{2008} + ... = \sum_{k=2007}^{\infty} \frac{1}{k},$$

which is a tail of the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$.

Theorem 2.9 (The divergence test) *If $\sum_{k=1}^{\infty} a_k$ converges then $\lim_{k\to\infty} a_k \to 0$. In other words, if a_k does not tend to 0 then the series $\sum_{k=1}^{\infty} a_k$ diverges.*

Proof. We have

$$S_n - S_{n-1} = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n-1} a_k = a_n.$$

If the series $\sum a_k$ converges then the sequence $\{S_n\}$ converges, say, $S_n \to S$. Then also $S_{n-1} \to S$, which implies $a_n = S_n - S_{n-1} \to 0$. ■

Example. Consider again the geometric series $\sum_{k=1}^{\infty} x^k$ where $|x| \geq 1$. In this case, x^k does not tend to 0 (because $\left|x^k\right| = |x|^k \geq 1$) and we conclude that the series $\sum_{k=1}^{\infty} x^k$ diverges. If $x \geq 1$ then this series is non-negative and, hence, $\sum_{k=1}^{\infty} x^k = +\infty$. If $x \leq -1$ then the sum $\sum_{k=1}^{\infty} x^k$ is not defined. For example, the sum $\sum_{k=1}^{\infty} (-1)^k$ is not defined.

Theorem 2.10 *Let $\sum_{k=1}^{\infty} a_k = A$ and $\sum_{k=1}^{\infty} b_k = B$ where $A, B \in \overline{\mathbb{R}}$.*

(a) If $A + B$ is defined then $\sum_{k=1}^{\infty} (a_k + b_k) = A + B$.

(b) If $\lambda \in \mathbb{R}$ and λA is defined then $\sum_{k=1}^{\infty} \lambda a_k = \lambda A$.

(c) If $a_k \leq b_k$ for all $k \in \mathbb{N}$ then $A \leq B$.

Proof. Parts (a) and (b) follow immediately from Theorem 2.6, and part (c) – from Theorem 2.1. ■

Corollary. (The comparison test for non-negative series) *If $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ are non-negative series such that $a_k \leq b_k$ then*

$$\sum_{k=1}^{\infty} a_k \leq \sum_{k=1}^{\infty} b_k. \tag{2.8}$$

In particular, if $\sum_{k=1}^{\infty} b_k$ converges then so does $\sum_{k=1}^{\infty} a_k$, and if $\sum_{k=1}^{\infty} a_k$ diverges then so does $\sum_{k=1}^{\infty} b_k$.

Proof. This follows from Theorem 2.10(c) and the fact that the values of $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ are always defined. ■

Example. Consider the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ and let us prove that this series converges by comparing it with the convergent series $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$. The obvious inequality $\frac{1}{k^2} > \frac{1}{k(k+1)}$

does not help here because we need the estimate in the opposite direction. Using $k \geq \frac{1}{2}(k+1)$, we obtain

$$\frac{1}{k^2} \leq \frac{2}{k(k+1)}.$$

Hence, (2.8) yields

$$\sum_{k=1}^{\infty} \frac{1}{k^2} \leq 2 \sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 2.$$

In particular, the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges. In fact, we have[6]

$$\sum_{k=1}^{\infty} \frac{1}{k^2} \approx 1,644\,9341.$$

For another proof of the convergence of $\sum_{k=1}^{\infty} \frac{1}{k^2}$ see Exercise 41.

2.6.4 Complex-valued sequences and series

One can consider sequences $\{z_n\}_{n=1}^{\infty}$ of complex numbers and define the notion of the limit exactly as in the real case.

Definition. A sequence $\{z_n\}$ of complex numbers converges to $a \in \mathbb{C}$ if $|z_n - a| \to 0$ as $n \to \infty$.

Theorem 2.11 *$z_n \to a$ if and only if $\operatorname{Re} z_n \to \operatorname{Re} a$ and $\operatorname{Im} z_n \to \operatorname{Im} a$.*

Proof. Let $z_n = x_n + iy_n$ and $a = x + iy$ where $x_n, y_n, x, y \in \mathbb{R}$. Then $z_n - a = (x_n - x) + i(y_n - y)$ and

$$|z_n - a|^2 = |x_n - x|^2 + |y_n - y|^2,$$

which implies that $|z_n - a| \to 0$ if and only if $|x_n - x| \to 0$ and $|y_n - y| \to 0$. ■

By means of Theorem 2.11, many results on real-valued series can be transferred to those with complex terms.

Definition. A sequence $\{z_n\}$ of complex numbers is called Cauchy if $|z_n - z_m| \to 0$ as $n, m \to \infty$. That is,

for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|z_n - z_m| < \varepsilon$ for all $n, m \geq N$.

Similarly to Theorem 2.11, one proves that a sequence $\{z_n\}$ is Cauchy if and only if the both sequences $\{\operatorname{Re} z_n\}$ and $\{\operatorname{Im} z_n\}$ are Cauchy.

Corollary. *A sequence $\{z_n\}$ of complex numbers converges if and only if it is Cauchy.*

[6] It is possible to show that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{1}{6}\pi^2.$$

Proof. Let $x_n = \operatorname{Re} z_n$ and $y_n = \operatorname{Im} z_n$. Then we have the following equivalences:

$$\begin{aligned}\{z_n\} \text{ converges} &\iff \{x_n\} \text{ and } \{y_n\} \text{ converge}\\ &\iff \{x_n\} \text{ and } \{y_n\} \text{ are Cauchy}\\ &\iff \{z_n\} \text{ is Cauchy.}\end{aligned}$$

■

Definition. A series $\sum_{k=1}^{\infty} a_k$ of complex numbers converges to $a \in \mathbb{C}$ if the sequence $\{S_n\}$ of its partial sums converges to a.

Definition. A series $\sum_{k=1}^{\infty} a_k$ with of complex numbers is called *absolutely convergent* if $\sum_{k=1}^{\infty} |a_k| < \infty$.

The next theorem extend the comparison test of the previous section to complex-valued series.

Theorem 2.12 (The comparison test) *Assume that $|a_k| \le b_k$ for all $k \ge 1$ where $a_k \in \mathbb{C}$ and $\sum_{k=1}^{\infty} b_k$ is a non-negative convergent series. Then also the series $\sum_{k=1}^{\infty} a_k$ converges and*

$$\left|\sum_{k=1}^{\infty} a_k\right| \le \sum_{k=1}^{\infty} b_k. \tag{2.9}$$

If the condition $|a_k| \le b_k$ holds for all $k \ge 1$ then one says that the series $\sum a_k$ is *dominated* by $\sum b_k$.

Corollary. *If a series $\sum a_k$ converges absolutely then it converges.*

Proof. Indeed, set $b_k = |a_k|$. Then the series $\sum b_k$ converges and $|a_k| \le b_k$. By Theorem 2.12 we conclude that $\sum a_k$ is also convergent. ■

Proof of Theorem 2.12. Let $A_n = \sum_{k=1}^{n} a_k$ and $B = \sum_{k=1}^{n} b_k$. Then, for all indices $n > m$, we have

$$|A_n - A_m| = \left|\sum_{k=m+1}^{n} a_k\right| \le \sum_{k=m+1}^{n} |a_k| \le \sum_{k=m+1}^{n} b_k = B_n - B_m. \tag{2.10}$$

Since the sequence $\{B_n\}$ converges, it is Cauchy, that is, $B_n - B_m \to 0$ as $n, m \to \infty$. Hence, also $|A_n - A_m| \to 0$, the sequence $\{A_n\}$ is Cauchy, and the series $\sum a_k$ converges. The inequality (2.9) follows from $|A_n| \le B_n$ which is proved similarly to (2.10). ■

Example. Consider the series $\sum_{k=1}^{\infty} \frac{c_k}{k^2}$ where $\{c_k\}$ is any bounded sequence of complex numbers (for example, $c_k = (-1)^k$). We claim that this series converges absolutely. Indeed, let C be an upper bound for $\{|c_k|\}$. Then

$$\left|\frac{c_k}{k^2}\right| \le \frac{C}{k^2}$$

and since $\sum \frac{C}{k^2} = C \sum \frac{1}{k^2}$ is a non-negative convergent series, we conclude by Theorem 2.12 that $\sum \frac{c_k}{k^2}$ converges absolutely.

2.6.5 Specific tests for convergence

Theorem 2.13 (The ratio test) *If* $\{a_n\}_{n=1}^{\infty}$*is a sequence of non-zero complex numbers such that*

$$\limsup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|<1$$

then the series $\sum_{n=1}^{\infty}a_n$ *converges absolutely.*

Proof. Denote for simplicity $r_n=\left|\frac{a_{k+1}}{a_k}\right|$ and let $r=\limsup_{k\to\infty}r_n$. Note that $r<1$ and choose a number q such that $r<q<1$. We claim that $r_n<q$ for all large enough n, that is, the following set

$$S=\{n\in\mathbb{N}:r_n\geq q\}$$

is finite. If S is infinite then we obtain a subsequence of $\{r_n\}$ such that all its terms are in $[q,+\infty)$. This subsequence has a limit point, say R, and it follows that $R\in[q,+\infty]$. In particular, $R>r$. However, R is a limit point also for $\{r_n\}$, which contradicts to $R>r$ because the maximal limit point of $\{r_n\}$ is r.

Hence, we have proved that the set S is finite, that is, there exists N such that $r_n<q$ for all $n\geq N$. It follows that

$$|a_{n+1}|\leq q\,|a_n|\ \text{ for all } n\geq N.$$

By induction, we obtain

$$|a_n|\leq q^{n-N}\,|a_N|\,,$$

whence by $0<q<1$

$$\sum_{n=N}^{\infty}|a_n|\leq|a_N|\sum_{n=N}^{\infty}q^{n-N}=|a_N|\sum_{n=0}^{\infty}q^n<\infty.$$

Hence, the series $\sum_{n=1}^{\infty}|a_n|$ converges, which was to be proved. ∎

Example. Consider the series $\sum_{n=0}^{\infty}\frac{x^n}{n!}$ and prove that it converges absolutely for any $x\in\mathbb{C}$. Setting $a_n=\frac{x^n}{n!}$, we obtain

$$\left|\frac{a_{n+1}}{a_n}\right|=\left|\frac{x^{n+1}n!}{(n+1)!x^n}\right|=\left|\frac{x}{n+1}\right|=\frac{|x|}{n+1}\to 0.$$

Hence, $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=0$ and the series converges absolutely by the ratio test.

Definition. The series $\sum_{n=0}^{\infty}\frac{x^n}{n!}$ is called the *exponential* series and its sum is called the exponential function of x and is denoted by

$$\exp(x)=\sum_{n=0}^{\infty}\frac{x^n}{n!}. \tag{2.11}$$

The number $\exp(1)$ is also denoted by e so that

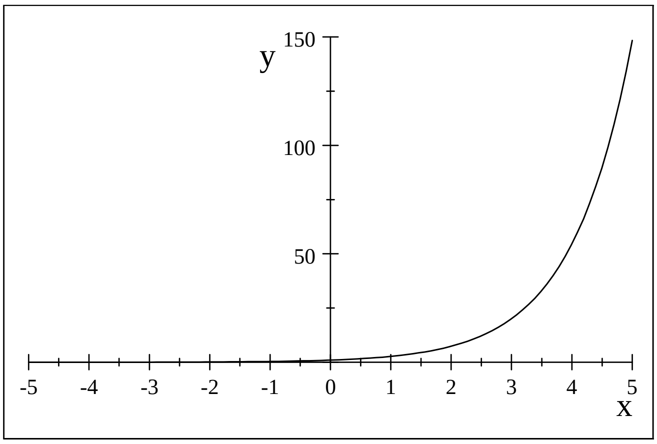
$$e=\sum_{n=0}^{\infty}\frac{1}{n!}=1+\frac{1}{1!}+\frac{1}{2!}+...+\frac{1}{n!}+....$$

The approximate value of e is

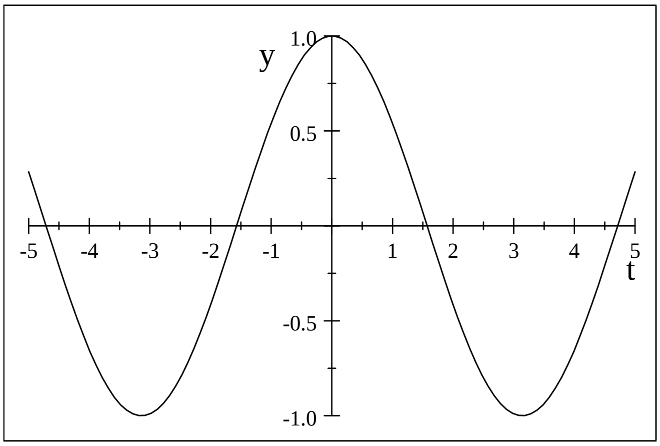
$$e = 2,718281828459045...$$

and e is known to be a transcendental number.

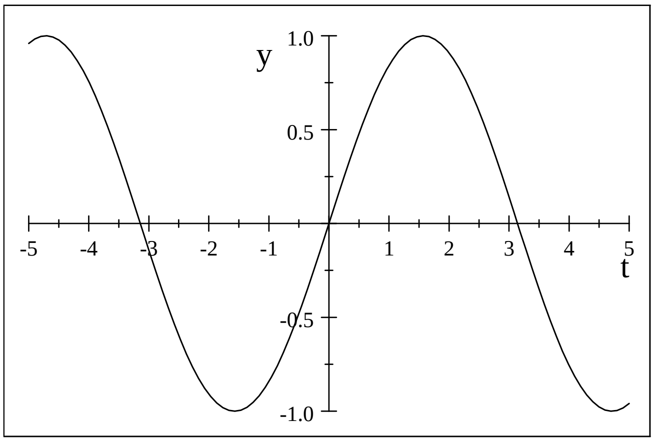
The graph of function $y = \exp(x)$ for $x \in \mathbb{R}$ looks as follows:



If x is complex, for example, $x = it$ where $t \in \mathbb{R}$ then $\exp(it)$ is a complex number, and the graph of its real part $y = \operatorname{Re}\exp(it)$ is as follows:



The graph of the function $y = \operatorname{Im}\exp(it)$ is as follows:



2.6.6 Product series

Let $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$ be two series of complex numbers. The *Cauchy product* of these two series is the series $\sum_{n=0}^{\infty} c_n$ where

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

To understand the meaning of c_n, consider the following infinite table containing all possible terms $a_k b_l$ with non-negative integers k, l:

$$\begin{array}{ccccccccc}
a_0b_0 & a_0b_1 & a_0b_2 & \dots & a_0b_{n-k} & \dots & a_0b_{n-1} & \boxed{a_0b_n} & \dots \\
a_1b_0 & a_1b_1 & a_1b_2 & \dots & \dots & \dots & \boxed{a_1b_{n-1}} & \dots & \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & & \\
a_kb_0 & \dots & \dots & \dots & \boxed{a_kb_{n-k}} & \dots & & & \\
\dots & \dots & \dots & \dots & \dots & & & & \\
\dots & \dots & \dots & \dots & & & & & \\
a_{n-1}b_0 & \boxed{a_{n-1}b_1} & \dots & & & & & \dots & \dots \\
\boxed{a_nb_0} & \dots & & & & & \dots & a_nb_n & \dots \\
\dots & & & & & & \dots & \dots & \dots
\end{array}$$

The terms of the form $a_k b_{n-k}$ with fixed n lie on the n-th diagonal of this table (see the boxed terms). Hence, c_n is the sum of all the terms on the n-th diagonal. Then the sum $\sum_{n=0}^{\infty} c_n$ "contains" all the terms $a_k b_l$ in this table, and we expect that

$$\sum_{n=0}^{\infty} c_n = \sum_{k,l} a_k b_l = \sum_{k=0}^{\infty} a_k \sum_{l=0}^{\infty} b_l. \tag{2.12}$$

However, this argument is non-rigorous since it requires change of the summation order in series, which does not always preserve the sum. The next statement provides the conditions under which (2.12) is true.

Theorem 2.14 (The Cauchy product theorem) *If $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are absolutely convergent series of complex numbers then their Cauchy product $\sum_{n=0}^{\infty} c_n$ converges absolutely and*

$$\sum_{n=0}^{\infty} c_n = \left(\sum_{k=0}^{\infty} a_k\right)\left(\sum_{l=0}^{\infty} b_l\right).$$

Proof. Set

$$A_n = \sum_{k=0}^{n} a_k, \quad B_n = \sum_{l=0}^{n} b_l, \quad C_n = \sum_{m=0}^{n} c_m$$

and notice that $\lim_{n\to\infty} C_n = \sum_{m=0}^{\infty} c_m$ and

$$\lim_{n\to\infty} A_n B_n = \left(\sum_{k=0}^{\infty} a_k\right)\left(\sum_{l=0}^{\infty} b_l\right).$$

We need to prove that the sequence $\{C_n\}$ converges and

$$\lim_{n\to\infty} C_n = \lim_{n\to\infty} A_n B_n \ . \tag{2.13}$$

Observe that

$$C_n = \sum_{m=0}^{n} c_m = \sum_{m=0}^{n}\sum_{k=0}^{m} a_k b_{m-k} = \sum_{\{k,l:k+l\leq n\}} a_k b_l$$

That is, C_n is the sum of all terms $a_k b_l$ in the triangle $\{k,l : k+l \leq n\}$. For $A_n B_n$ we have

$$A_n B_n = \sum_{k=0}^{n} a_k \sum_{l=0}^{n} b_l = (a_0 + ... + a_n)(b_0 + ... + b_n) = \sum_{\{k,l:k\leq n,l\leq n\}} a_k b_l$$

Hence, $A_n B_n$ is the sum of all $a_k b_l$ in the rectangle $\{k,l : k \leq n, l \leq n\}$.

Consider first the case when all terms a_k and b_k are non-negative real numbers. Obviously, we have the inclusions

$$\{k+l \leq n\} \subset \{k \leq n, l \leq n\}$$

and, for $m = [n/2]$,

$$\{k \leq m, l \leq m\} \subset \{k+l \leq n\}$$

(that is, the triangle $\{k+l \leq n\}$ is "squeezed" between two rectangles), which implies

$$A_m B_m \leq C_n \leq A_n B_n.$$

Since the both sequences $\{A_n B_n\}$ and $\left\{A_{[n/2]} B_{[n/2]}\right\}$ have the same limit as $n \to \infty$, we obtain that $\{C_n\}$ has the same limit as $\{A_n B_n\}$. In particular, the sequence $\{C_n\}$ converges, which means that the series $\sum_{n=0}^{\infty} c_n$ also converges. Since $c_n \geq 0$, this series converges absolutely.

Consider now the general case when a_n and b_n are complex. By hypothesis, the series $\sum |a_k|$ and $\sum |b_k|$ converge. Set

$$c_n^* = \sum_{k=0}^{n} |a_k| \, |b_{n-k}| \, ,$$

that is, $\sum c_n^*$ is the Cauchy product of $\sum |a_k|$ and $\sum |b_k|$. By the first part of the proof, the series $\sum c_n^*$ is convergent. Then we have

$$|c_n| = \left| \sum_{k=0}^{n} a_k b_{n-k} \right| \leq \sum_{k=0}^{n} |a_k| \, |b_{n-k}| = c_n^*,$$

which implies by the comparison test that $\sum c_n$ converges absolutely.

We are left to show that $A_n B_n - C_n \to 0$, which will imply (2.13). We have

$$A_n B_n - C_n = \sum_{\{k,l:k\leq n,l\leq n\}} a_k b_l - \sum_{\{k,l:k+l\leq n\}} a_k b_l = \sum_{\{k,l:k\leq n.l\leq n.k+l>n\}} a_k b_l.$$

It follows by the triangle inequality that

$$|A_nB_n - C_n| \le \sum_{\{k,l:k\le n.l\le n.k+l>n\}} |a_k|\,|b_l|\,.$$

Denote by A_n^*, B_n^*, C_n^* the partial sums of the series $\sum|a_k|$, $\sum|b_k|$, and $\sum c_k^*$, respectively. Then we have

$$A_n^*B_n^* - C_n^* = \sum_{\{k,l:k\le n.l\le n.k+l>n\}} |a_k|\,|b_l|$$

whence it follows that

$$|A_nB_n - C_n| \le |A_n^*B_n^* - C_n^*|\,.$$

Since by the first part of the proof $|A_n^*B_n^* - C_n^*| \to 0$, we conclude that also $|A_nB_n - C_n| \to 0$, which was to be proved. ■

Example. Let us prove some properties of the exponential function.

1. For all $x, y \in \mathbb{C}$,

$$\exp(x+y) = \exp(x)\exp(y)\,. \tag{2.14}$$

Consider the series

$$\exp(x) = \sum_{k=0}^{\infty}\frac{x^k}{k!} \quad\text{and}\quad \exp(y) = \sum_{k=0}^{\infty}\frac{y^k}{k!} \tag{2.15}$$

and let $\sum_{n=0}^{\infty} c_n$ be the Cauchy product of these series, that is,

$$c_n = \sum_{k=0}^{n}\frac{x^k}{k!}\frac{y^{n-k}}{(n-k)!}.$$

By the binomial formula, we have

$$(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^ky^{n-k} = \sum_{k=0}^{n}\frac{n!}{k!\,(n-k)!}x^ky^{n-k} = n!c_n,$$

whence

$$c_n = \frac{(x+y)^n}{n!}.$$

Hence, we obtain

$$\sum_{n=0}^{\infty}c_n = \sum_{n=0}^{\infty}\frac{(x+y)^n}{n!} = \exp(x+y)\,.$$

By Theorem 2.14, we conclude that

$$\exp(x+y) = \sum_{n=0}^{\infty}c_n = \left(\sum_{k=0}^{\infty}\frac{x^k}{k!}\right)\left(\sum_{k=0}^{\infty}\frac{y^k}{k!}\right) = \exp(x)\exp(y)\,,$$

which finishes the proof.

2. Recall the notation $\exp(1) = e$. Let us show that, for any integer k,

$$\exp(k) = e^k. \tag{2.16}$$

Let us first prove by induction that (2.16) holds for all $k \in \mathbb{N}$. If $k = 1$ then (2.16) amounts to the definition of e. Inductive step: if (2.16) holds for some k then by (2.14)

$$\exp(k+1) = \exp(k)\exp(1) = e^k e = e^{k+1}.$$

If $k = 0$ then $\exp(k) = 1$ by (2.15), which matches $e^0 = 1$. Before we consider the case $k < 0$, observe that, by (2.14) with $y = -x$,

$$\exp(x)\exp(-x) = \exp(0) = 1. \tag{2.17}$$

Hence, if $k < 0$ then

$$\exp(k) = \frac{1}{\exp(-k)} = \frac{1}{e^{-k}} = e^k.$$

3. For any $x \in \mathbb{C}$, $\exp(x) \neq 0$ and, for any $x \in \mathbb{R}$, $\exp(x)$ is real and positive.

Indeed, it follows from (2.17) that $\exp(x) \neq 0$. If $x \in \mathbb{R}$ then $\exp(x) \in \mathbb{R}$ just by the definition

$$\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

If $x \geq 0$ then $\exp(x) > 0$ because all the terms of the series are non-negative and the term with $k = 0$ is 1. If $x < 0$ then we obtain by (2.17)

$$\exp(x) = \frac{1}{\exp(-x)} > 0.$$

The identity (2.16) motivates the following definition of e^x for any complex x:

$$e^x = \exp(x).$$

For example, we obtain $e^{1/2} = \sqrt{e}$ because $e^{1/2}e^{1/2} = e^{1/2+1/2} = e$ and, hence, $e^{1/2}$ satisfies the definition of $\sqrt{e}$.

Naturally, the question arrises how to define a^x for any positive a and any real (or complex) x. This question will be addressed later on in the next Chapter.

2.6.7 Conditionally convergent series

Definition. A series $\sum_{k=1}^{\infty} a_k$ is said to *converge conditionally* if it converges but does not converge absolutely, that is,

$$\sum_{k=1}^{\infty} |a_k| = +\infty.$$

Example. Consider the *alternating harmonic series*

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \approx 0,69314718. \tag{2.18}$$

By Exercise 42 this series converges. However, it does not converge absolutely because

$$\sum_{k=1}^{\infty} \frac{1}{k} = +\infty.$$

One should be careful with conditionally convergent series: they do not satisfy the commutative and associative laws! For example, by changing the order of terms in such a series one can obtain a divergent series (or even make its sum to be equal to any prescribed number). Let us illustrate this phenomenon on the example of the series (2.18). Observe first that

$$\sum_{k=1}^{\infty}\frac{1}{2k}=\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k}=+\infty$$

and

$$\sum_{k=1}^{\infty}\frac{1}{2k-1}=+\infty$$

because $\frac{1}{2k-1}\geq\frac{1}{2k}$. Therefore, there exists $N_1\in\mathbb{N}$ such that

$$1+\frac{1}{3}+...+\frac{1}{2N_1-1}>1.$$

Consider now the tail of the series

$$\sum_{k=N_1+1}^{\infty}\frac{1}{2k-1},$$

which also diverges to $+\infty$. Therefore, there exists $N_2>N_1$ such that

$$\frac{1}{2N_1+1}+\frac{1}{2N_1+3}...+\frac{1}{2N_2-1}>1.$$

Then there exists N_3 such that

$$\frac{1}{2N_2+1}+\frac{1}{2N_2+3}+...+\frac{1}{2N_3-1}>1,$$

etc. Now consider the series, which is obtain by changing the order of the alternating harmonic series:

$$\begin{aligned}&\left(1+\frac{1}{3}+...+\frac{1}{2N_1-1}\right)-\frac{1}{2}\\&+\left(\frac{1}{2N_1+1}+\frac{1}{2N_1+3}...+\frac{1}{2N_2-1}\right)-\frac{1}{4}\\&+\left(\frac{1}{2N_2+1}+\frac{1}{2N_2+3}+...+\frac{1}{2N_3-1}\right)-\frac{1}{6}+...\end{aligned}$$

Then sum in each row is at least $1-\frac{1}{2}=\frac{1}{2}$, hence, the series is divergent.

Let us state without proof the following theorem.

Theorem *(a) If $\sum a_k$ is an absolutely convergent series of complex numbers and $\sum b_k$ is a series that is obtained from $\sum a_k$ by changing the order of summation and/or by grouping the terms, then $\sum b_k$ also converges absolutely and*

$$\sum b_k=\sum a_k.$$

(b) If $\sum a_k$ is a conditionally convergent series of real numbers, then, for any $c\in\overline{\mathbb{R}}$ there exists a series $\sum b_k$, which is obtain from $\sum a_k$ by changing the order of terms and such that $\sum b_k=c$.

3 Continuous functions

3.1 Limit of a function

We introduce the notion of the limit of a function, which is similar to the notion of the limit of a sequence. Let us start with a particular case.

Definition. Let f be a real-valued function defined in some interval $I = (r, +\infty)$ where $r \in \mathbb{R}$, and let $b \in \mathbb{R}$. We write that

$$\lim_{x \to +\infty} f(x) = b \tag{3.1}$$

if, for any $\varepsilon > 0$, there exists $N \in \mathbb{R}$ such that $|f(x) - b| < \varepsilon$ for all $x > N$ (more precisely, for all $x > N$ and $x \in I$ so that $f(x)$ is defined).

Recall the definition of a neighborhood: for any real x, a neighborhood of x is an interval

$$U_\varepsilon(x) = (x - \varepsilon, x + \varepsilon),$$

where $\varepsilon > 0$, a neighborhood of $+\infty$ is an interval

$$U_E(+\infty) = (E, +\infty),$$

a neighborhood of $-\infty$ is an interval

$$U_E(-\infty) = (-\infty, E),$$

where $E \in \mathbb{R}$.

Then the above Definition can be stated as follows: $\lim_{x \to +\infty} f(x) = b$ if, for any $\varepsilon > 0$, there exists $N \in \mathbb{R}$ such that

$$x \in U_N(+\infty) \cap I \implies f(x) \in U_\varepsilon(b). \tag{3.2}$$

Suppressing the subscripts in the notation of neighborhood, we state this Definition in yet another form: $\lim_{x \to +\infty} f(x) = b$ if, for any neighborhood $U(b)$ of b there is a neighborhood $U(+\infty)$ of $+\infty$ such that

$$x \in U(+\infty) \cap I \implies f(x) \in U(b).$$

Terminology: as in the case of sequences, we also say that $f(x)$ *converges* (tends, goes) to b as x goes to $+\infty$ and write $f(x) \to b$ as $x \to +\infty$.

In the last form, the definition of the limit can be extended as follows.

Definition. Let I be an open interval in $\mathbb{R}$ and $a \in \overline{\mathbb{R}}$ be either a point in I or one of the endpoints of I. Set $J = I \setminus \{a\}$ and let $f : J \to \mathbb{R}$ be a real-valued function on J. Let $b \in \overline{\mathbb{R}}$. We write

$$\lim_{x \to a} f(x) = b$$

if, for any neighborhood $U(b)$ of b, there exists a neighborhood $U(a)$ of a such that

$$x \in U(a) \cap J \implies f(x) \in U(b).$$

For simplicity of terminology, for any interval $I = (x, y)$ denote by $\overline{I}$ the closed interval $[x, y]$. The interval $\overline{I}$ is called the *closure* of I in $\overline{\mathbb{R}}$. Hence, the condition that either $a \in I$ or a is an endpoint of I can be equivalently stated that $a \in \overline{I}$. The above definition can be stated as follows:

Definition. Let I be an open interval in $\mathbb{R}$ and let $a \in \overline{I}$. Set $J = I \setminus \{a\}$ and let $f : J \to \mathbb{R}$ be a real-valued function on J. We write

$$\lim_{x \to a} f(x) = b, \tag{3.3}$$

where $b \in \overline{\mathbb{R}}$, if, for any neighborhood $U(b)$ of b, there exists a neighborhood $U(a)$ of a such that

$$x \in U(a) \cap J \implies f(x) \in U(b).$$

A more precise notation would be

$$\lim_{\substack{x \to a \\ x \in J}} f(x) = b$$

which indicated the domain J of f. Note that by definition $a \notin J$. We will normally used a simplified notation (3.3). We also write $f(x) \to b$ as $x \to a$ and say that $f(x)$ goes (tends) to b as x goes to a (if $b \in \mathbb{R}$ then one says that $f(x)$ converges to b, and if $b = +\infty$ or $-\infty$ then $f(x)$ diverges to b).

The case $a = +\infty$ considered above fits also into this definition since in this case $I = J = (r, +\infty)$.

Consider the case when $a, b \in \mathbb{R}$, denote the neighborhood of b by $U_\varepsilon(b)$ and the neighborhood of a by $U_\delta(a)$. Then this definition reads as follows: $\lim_{x \to a} f(x) = b$ if, for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|x - a| < \delta \text{ and } x \in J \implies |f(x) - b| < \varepsilon.$$

In this form the definition of the limit is stated in most textbooks, and it is convenient for applications.

Example. Let $f(x) = x^2$, $x \in \mathbb{R}$, and let us show that $f(x) \to a^2$ as $x \to a \in \mathbb{R}$. Indeed, for any $x \in J = \mathbb{R} \setminus \{a\}$, we have

$$\left|f(x) - a^2\right| = \left|x^2 - a^2\right| = |x - a|\,|x + a|.$$

If $|x - a| < \delta$ then $x \in (a - \delta, a + \delta)$ whence $|x| \le |a| + |\delta|$ and $|x + a| \le 2|a| + |\delta|$. Since δ can be chosen, we can assume that $\delta \le 1$, whence also $|x + a| \le 2|a| + 1$. This implies that under condition $|x - a| < \delta$, we have

$$\left|f(x) - a^2\right| < \delta(2|a| + 1).$$

It follows that

$$\left|f(x) - a^2\right| < \varepsilon,$$

where $\varepsilon > 0$ is given, provided

$$\delta \le \frac{\varepsilon}{2|a| + 1}.$$

Hence, we can take $\delta = \min\left(1, \frac{\varepsilon}{2|a|+1}\right)$.

Example. Let $f(x) = \exp(x)$ and let us show that $f(x) \to 1$ as $x \to 0$. For that, write

$$|\exp(x) - 1| = \left|\sum_{k=1}^{\infty} \frac{x^k}{k!}\right| \le \sum_{k=1}^{\infty} |x|^k = \frac{|x|}{1-|x|}$$

assuming $|x| < 1$. Given $\varepsilon > 0$ let us find $0 < \delta \le 1$ such that

$$|x| < \delta \implies \frac{|x|}{1-|x|} < \varepsilon. \tag{3.4}$$

Indeed, using $\frac{|x|}{1-|x|} = \frac{1}{1-|x|} - 1$ we rewrite the last inequality in the form

$$\frac{1}{1-|x|} < 1 + \varepsilon,$$

which is equivalent to $1 - |x| > \frac{1}{1+\varepsilon}$ and to $|x| < \frac{\varepsilon}{1+\varepsilon}$. Hence, taking $\delta = \frac{\varepsilon}{1+\varepsilon}$ (and noticing that $\delta < 1$) we obtain from (3.4)

$$|x| < \delta \implies |\exp(x) - 1| < \varepsilon,$$

which proves that $f(x) \to 1$ as $x \to 0$.

Example. Consider $f(x) = \frac{1}{|x-1|}$ and show that $f(x) \to +\infty$ as $x \to 1$. For that we need to show that, for any $E \in \mathbb{R}$, there exists $\delta > 0$ such that

$$0 < |x-1| < \delta \implies \frac{1}{|x-1|} > E.$$

Indeed, just take $\delta = \frac{1}{E}$ is $E > 0$ (and δ is any if $E < 0$).

Theorem 3.1 *Let I be an open interval in $\mathbb{R}$ and let $a \in \overline{I}$. Let f be a real-value function defined on $J = I \setminus \{a\}$. Then the condition*

$$\lim_{x \to a} f(x) = b,$$

(where $b \in \overline{\mathbb{R}}$) is equivalent to the following: for any sequence $\{x_n\} \subset J$,

$$\lim_{n\to\infty} x_n = a \implies \lim_{n\to\infty} f(x_n) = b.$$

Proof. Consider the case when both a and b are real numbers. If $f(x) \to b$ as $x \to a$ then, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|x - a| < \delta \text{ and } x \in J \implies |f(x) - b| < \varepsilon.$$

If $\{x_n\} \subset J$ is a sequence such that $x_n \to a$ then there exists $N \in \mathbb{N}$ such that

$$n \ge N \implies |x_n - a| < \delta.$$

Hence, we conclude by the previous lines that

$$n \geq N \implies |f(x_n) - b| < \varepsilon$$

which means that $f(x_n) \to b$ as $n \to \infty$.

Conversely, assume that for any sequence $\{x_n\} \subset J$, the condition $x_n \to a$ implies $f(x_n) \to b$, and prove that $\lim_{x \to a} f(x) = b$. Assume from the contrary that the latter is not true. Then this means that there exists $\varepsilon > 0$ such that for any $\delta > 0$ for some $x \in J$ holds $|x - a| < \delta$ while $|f(x) - b| \geq \varepsilon$. Fix $\varepsilon > 0$ that exists by this condition and set $\delta = \frac{1}{k}$, $k \in \mathbb{N}$. Then there is $x_k \in J$ such that $|x_k - a| < 1/k$ and $|f(x_k) - b| \geq \varepsilon$. Hence, we obtain a sequence $\{x_k\} \subset J$ such that $x_k \to a$ while $f(x_k) \not\to b$.

In the same way one treats the cases when a and/or b are infinite. ∎

Theorem 3.2 *Let I be an open interval in $\mathbb{R}$ and let $c \in \overline{I}$. Let f, g be a real-valued functions defined in $J = I \setminus \{c\}$ such that*

$$\lim_{x \to c} f(x) = a \quad \text{and} \quad \lim_{x \to c} g(x) = b,$$

where $a, b \in \overline{\mathbb{R}}$.

(a) Then

$$\lim_{x \to c} (f + g) = a + b, \quad \lim_{x \to c} fg = ab, \quad \lim_{x \to c} \frac{f}{g} = \frac{a}{b}$$

provided the expressions $a + b$, ab, $\frac{a}{b}$ are defined (and $g \neq 0$ in the case of $\frac{f}{g}$).

(b) Let $f(x) \leq g(x)$ for all $x \in J$. Then $a \leq b$.

(c) If f, g, h are three functions on J such that $f \leq h \leq g$ and

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = a$$

then also

$$\lim_{x \to c} h(x) = a.$$

Proof. (a) Indeed, for any sequence $\{x_n\}_{n=1}^{\infty} \subset J$ such that $x_n \to c$, we have by Theorem 3.1 $f(x_n) \to a$ and $g(x_n) \to b$. By Theorem 2.6, we obtain $(f + g)(x_n) \to a + b$ provided $a + b$ is defined. Hence, applying once again Theorem 3.1, we conclude $(f + g)(x) \to a + b$ as $x \to c$. In the same way one treats other cases.

(b) Taking any sequence $x_n \to c$, we obtain by Theorem 3.1 that $f(x_n) \to a$, $g(x_n) \to b$. Since $f(x_n) \leq g(x_n)$, Theorem 2.1 implies that $a \leq b$.

(c) Taking any sequence $x_n \to c$ and applying Theorem 2.1, we obtain the claim. ∎

3.1.1 Composite functions

Let A, B, C be three sets and consider two mappings $f : A \to B$ and $g : B \to C$. Then $g(f(x))$ is defined as an element of C for any $x \in A$. The mapping $g(f(x)) : A \to C$ is called the *composition* of f and g or a *composite mapping* and is denoted by $g \circ f$:

$$g \circ f(x) = g(f(x)).$$

If A, B, C are subsets of $\mathbb{R}$ then we use the term function instead of mapping. Let us emphasize that the composition $g \circ f$ is defined whenever the image of f is contained in the domain of g.

Let us further extend the definition of the limit of a function to the case when the domain of the function is not necessarily an interval.

Definition. For any set $A \in \mathbb{R}$, its closure $\overline{A}$ in $\overline{\mathbb{R}}$ is defined by

$$\overline{A} = \left\{a \in \overline{\mathbb{R}} : \text{for any neighborhood } U(a), U(a) \cap A \neq \emptyset\right\}.$$

Clearly, we always have $A \subset \overline{A}$.

Example. Let I be any interval with endpoints $a, b \in \overline{\mathbb{R}}$. Then $\overline{I} = [a, b]$, which matches the previous definition of $\overline{I}$. Furthermore, let $J = I \setminus \{c\}$ where $c \in \overline{\mathbb{R}}$. Then $\overline{J} = [a, b]$.

Example. $\overline{\mathbb{Q}} = \overline{\mathbb{R}}$ because for any $a \in \overline{\mathbb{R}}$ and any neighborhood $U(a)$ there are rational numbers in $U(a)$.

Definition. Let $J \subset \mathbb{R}$ be a non-empty set and $f : J \to \mathbb{R}$ be a function on J. Let $a \in \overline{J}$ and $b \in \overline{\mathbb{R}}$.Then we write

$$\lim_{\substack{x \to a \\ x \in J}} f(x) = b$$

if, for any neighborhood $U(b)$ of b, there is a neighborhood $U(a)$ of a such that

$$x \in U(a) \cap J \implies f(x) \in U(b).$$

In the previous lecture, we have considered the case when $J = I \setminus \{a\}$ where I is an open interval and $a \in \overline{I}$. This case is a particular case of the present definition because $\overline{J} = \overline{I}$. Theorems 3.1 and 3.2 remain true for an arbitrary set J.

Theorem 3.3 (Limit of a composite function) *Let A, B be subsets of $\mathbb{R}$ and f and g be real-valued functions on A and B, respectively. Assume that, for some $a \in \overline{A}$, $b \in \overline{B}$, $c \in \overline{\mathbb{R}}$,*

$$\lim_{\substack{x \to a \\ x \in A}} f(x) = b$$

and

$$\lim_{\substack{y \to b \\ y \in B}} g(y) = c.$$

If the composite function $g(f(x))$ is defined on A, that is, if $f(A) \subset B$, then

$$\lim_{\substack{x \to a \\ x \in A}} g(f(x)) = c.$$

Proof. For any neighborhood $U(c)$ of c, there is a neighborhood $U(b)$ of b such that

$$y \in U(b) \cap B \implies g(y) \in U(c). \tag{3.5}$$

Given the neighborhood $U(b)$, there exists a neighborhood $U(a)$ such that

$$x \in U(a) \cap A \implies f(x) \in U(b).$$

By the condition $f(A) \subset B$, we have also $f(x) \in B$ for any $x \in A$, whence we see that

$$x \in U(a) \cap A \implies f(x) \in U(b) \cap B.$$

Applying (3.5) with $y = f(x)$, we obtain that, for any neighborhood $U(c)$ of c, there exists a neighborhood $U(a)$ of a such that

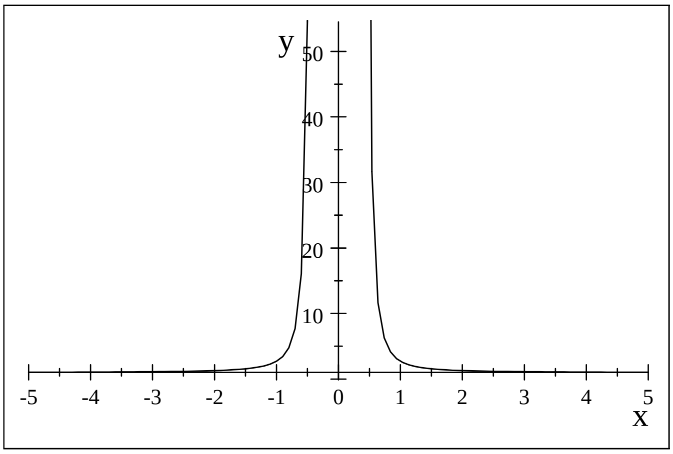
$$x \in U(a) \cap A \implies g(f(x)) \in U(c), \tag{3.6}$$

which means that

$$\lim_{\substack{x \to a \\ x \in A}} g(f(x)) = c.$$

■

Example. Let us evaluate $\lim_{x\to 0} \exp\left(\frac{1}{x^2}\right)$. Note that the domain of the function $\exp\left(\frac{1}{x^2}\right)$ is $\mathbb{R} \setminus \{0\}$.



The graph of function $\exp\left(\frac{1}{x^2}\right)$

We have

$$\lim_{x\to 0} \frac{1}{x^2} = +\infty,$$

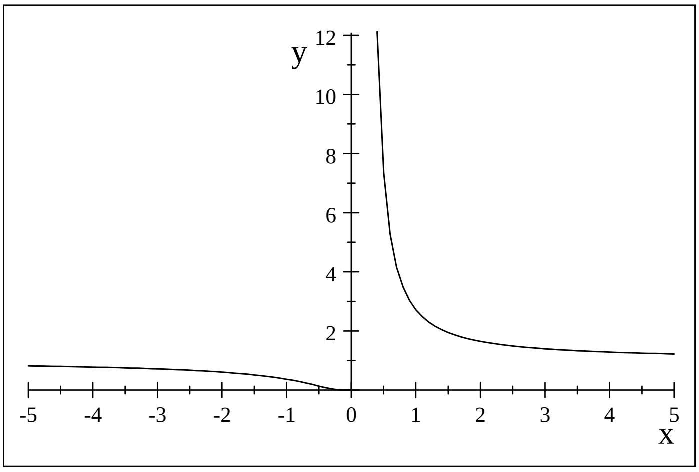
which follows from $\frac{1}{|x|} \to +\infty$. Since $\exp(y) > y$ (which follows from the definition of exp), we obtain

$$\lim_{y\to+\infty} \exp(y) = +\infty.$$

By Theorem 3.3, we obtain

$$\lim_{x\to 0} \exp\left(\frac{1}{x^2}\right) = \lim_{y\to+\infty} \exp(y) = +\infty. \tag{3.7}$$

Example. Consider now the function $f(x) = \exp\left(\frac{1}{x}\right)$ and find its limits as $x \to 0$ in the two domains: $x > 0$ and $x < 0$.



The graph of function $\exp\left(\frac{1}{x}\right)$

Using the fact that

$$\lim_{\substack{x\to 0\\ x>0}} \frac{1}{x} = +\infty,$$

we obtain

$$\lim_{\substack{x\to 0\\ x>0}} \exp\left(\frac{1}{x}\right) = \lim_{y\to+\infty} \exp(y) = +\infty.$$

Using that

$$\lim_{\substack{x\to 0\\ x<0}} \frac{1}{x} = -\infty,$$

we obtain

$$\lim_{\substack{x\to 0\\ x<0}} \exp\left(\frac{1}{x}\right) = \lim_{y\to-\infty} \exp(y) = \lim_{y\to-\infty} \frac{1}{\exp(-y)} = \lim_{z\to+\infty} \frac{1}{\exp(z)} = \frac{1}{+\infty} = 0.$$

3.2 Continuous functions

Definition. Let f be a real-valued function on a set $J \subset \mathbb{R}$. We say that f is *continuous* (*stetig*) at a point $a \in J$ if

$$\lim_{\substack{x\to a\\ x\in J}} f(x) = f(a).$$

If f is not continuous at a then we say that f is *discontinuous* at a.

If f is continuous at all points of J then f is called *continuous on J* or just *continuous.*

Example. Trivial examples of continuous functions are $f(x) = \text{const}$ and $f(x) = x$.

Let us show that function $f(x) = \exp(x)$ is continuous at any point $a \in \mathbb{R}$. It has been shown already that

$$\lim_{x\to 0} \exp(x) = 1 = \exp(0),$$

that is, $\exp(x)$ is continuous at $x = 0$. For an arbitrary $a \in \mathbb{R}$, we have

$$\exp(x) = \exp(x - a + a) = \exp(a)\exp(x - a).$$

Since

$$\lim_{x \to a} \exp(x-a) = \lim_{y \to 0} \exp(y) = 1,$$

it follows that

$$\lim_{x \to a} \exp(x) = \exp(a).$$

Theorem 3.4 *Let f, g be two functions defined on a set $J \subset \mathbb{R}$. If f and g are continuous at a point $a \in J$ then also the functions $f+g$, fg, f/g are continuous at a (for the case f/g assume that $g \neq 0$).*

If f, g are continuous on J then also $f+g$, fg, f/g are continuous on J (for the case f/g assume that $g \neq 0$).

Proof. The first claim follows immediately from Theorem 3.2. For example, for the sum $f+g$ we have

$$\lim_{x \to a} (f+g)(x) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = f(a) + g(a) = (f+g)(a),$$

whence the continuity of $f+g$ at a follows. Similarly one treats the functions fg and f/g.

The second claim follows obviously from the first one. ∎

Example. Since the functions $f(x) = x$ and $g(x) = \text{const}$ are continuous on $\mathbb{R}$, it follows that also any function $f(x) = cx^n$ is continuous on $\mathbb{R}$ for any $n \in \mathbb{R}$ and $c \in \mathbb{R}$. Furthermore, any polynomial

$$f(x) = c_0 + c_1 x + c_2 x^2 + ... + c_n x^n$$

is continuous on $\mathbb{R}$.

Consider a *rational* function $f(x) = \frac{g(x)}{h(x)}$ where g and h are two polynomials, which is defined in the set $\{h \neq 0\}$. Then f is continuous in the domain $\{h \neq 0\}$.

Theorem 3.5 *Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ are two functions, where $A, B \subset \mathbb{R}$ and let the composition $g \circ f$ be defined (that is, $f(A) \subset B$). If f is continuous at $a \in A$ and g is continuous at $b = f(a)$ then $g \circ f$ is continuous at a.*

If f is continuous on A and g is continuous on B then $g \circ f$ is continuous on A.

Proof. By Theorem 3.3, we have

$$\lim_{x \to a} g(f(x)) = \lim_{y \to f(a)} g(y) = \lim_{y \to b} g(y) = g(b) = g(f(a)),$$

that is, $g(f(x))$ is continuous at a.

If f is continuous at any point $a \in A$ and g is continuous at any $b \in B$ then g is continuous at $b = f(a) \in B$ (by the assumption that $f(A) \subset B$) and, hence, $g \circ f$ is continuous at a. Therefore, $g \circ f$ is continuous on A. ∎

Example. Let $f(x)$ be a rational function. Then $\exp(f(x))$ is continuous on the domain of $f(x)$.

Example. Consider the function

$$f(x) = \begin{cases} \exp\left(-\frac{1}{x^2}\right), & x \neq 0, \\ 0, & x = 0, \end{cases}$$

and prove that $f(x)$ is continuous on $\mathbb{R}$. The function $\exp\left(-\frac{1}{x^2}\right)$ is continuous in the domain $\{x \neq 0\}$ by Theorem 3.5. Hence, we are left to prove that f is continuous at 0, that is,

$$\lim_{\substack{x\to 0\\ x\in\mathbb{R}}} f(x) = 0. \tag{3.8}$$

As it follows from one of the above examples,

$$\lim_{\substack{x\to 0\\ x\neq 0}} \exp\left(-\frac{1}{x^2}\right) = 0,$$

that is, for any $\varepsilon > 0$ there is $\delta > 0$ such that

$$|x| < \delta, \quad x \neq 0 \implies \left|\exp\left(-\frac{1}{x^2}\right)\right| < \varepsilon.$$

Allowing now $x = 0$ and using the fact that $f(0) = 0$, we obtain that

$$|x| < \delta \implies |f(x)| < \varepsilon,$$

whence (3.8) follows.

3.3 Global properties of continuous functions

3.3.1 The intermediate value theorem

Theorem 3.6 *(*The intermediate value theorem - *Zwischenwertsatz) Let $f(x)$ be a continuous function defined on a closed interval $[a,b]$. If $f(a) < 0$ and $f(b) > 0$, then there exists $c \in (a,b)$ such that $f(c) = 0$.*

Proof. Consider the set

$$S = \{x \in [a,b] : f(x) \geq 0\}.$$

This set is bounded and non-empty (indeed, $b \in S$). Hence, it has the infimum $c = \inf S \in [a,b]$. Let us show that $f(c) = 0$.

Let us first show that $f(c) \geq 0$. If $c = b$ then there is nothing to prove. If $c < b$ then $c + \frac{1}{n} < b$ for all large enough $n \in \mathbb{N}$. Since $c + \frac{1}{n}$ is not a lower bound for S (because c is the greatest lower bound of S), there is $x_n \in S$ such that $x_n < c + \frac{1}{n}$. On the other hand, $x_n \geq c$ because c is a lower bound for S. Hence, $c \leq x_n < c + \frac{1}{n}$ and $f(x_n) \geq 0$. Taking limit as $n \to \infty$, we obtain $x_n \to c$ and, hence, $f(x_n) \to f(c)$. Since $f(x_n) \geq 0$, we obtain $f(c) \geq 0$.

Let us prove that $f(c) \leq 0$. Note that $c \neq a$ because $f(c) \geq 0$ while $f(a) < 0$. Hence, $x_n = c - \frac{1}{n} > a$ for all large enough $n \in \mathbb{N}$. Since c is a lower bound for S, $x_n \notin S$ whence $f(x_n) \leq 0$. Since $x_n \to c$ as $n \to \infty$, we obtain that $f(c) \leq 0$, which finishes the proof.
■

Example. Let $f(x)$ be a polynomial of an *odd* degree n with real coefficients; that is,

$$f(x) = x^n + c_1 x^{n-1} + ... + c_n,$$

where $c_k \in \mathbb{R}$ and n is odd. Let us prove that there is at least one real root, that is, there is $x \in \mathbb{R}$ such that $f(x) = 0$. By Exercise 47, we have

$$\lim_{x \to +\infty} f(x) = +\infty,$$

which implies that there is $b \in \mathbb{R}$ with $f(b) > 0$. Similarly, using the fact that n is odd and $\lim_{x \to -\infty} x^n = -\infty$, we obtain that

$$\lim_{x \to -\infty} f(x) = -\infty.$$

It follows that there is $a \in \mathbb{R}$ such that $f(a) < 0$. By Theorem 3.6, there is $x \in \mathbb{R}$ such that $f(x) = 0$.

Recall the following notation. If f is real-valued function defined on a set A then the image of f is the set

$$f(A) = \{f(x) : x \in A\}.$$

Then define

$$\sup_A f = \sup f(A)$$

and

$$\inf_A f = \inf f(A).$$

Theorem 3.7 *Let f be a continuous function on an interval I. Then the image $J = f(I)$ is an interval. Moreover, the endpoints of J are $\inf_I f$ and $\sup_I f$.*

Note that the types of the intervals I, J are not predefined: they may be open, semi-open, or closed, as well as bounded or unbounded.

Proof. It suffices to show that if

$$\inf f < c < \sup f$$

then $c \in f(I)$. By definition of sup and inf, there are numbers $a, b \in f(I)$ such that

$$\inf f < a < c < b < \sup f.$$

Let $a = f(s)$ and $b = f(t)$ where $s, t \in [a, b]$. Assume for simplicity that $s < t$. Consider on $[s, t] \subset I$ the function

$$g(x) = f(x) - c$$

and observe that $g(s) = a - c < 0$ and $g(t) = b - c > 0$. Hence, by Theorem 3.6, the function g vanishes at a point $x \in (s, t)$ which implies $f(x) = c$ and $c \in f(I)$. $\blacksquare$

Example. Let us show that $\exp(\mathbb{R}) = (0, +\infty)$. Since $\exp(x) > 0$ for any $x \in \mathbb{R}$, we have $\exp(\mathbb{R}) \subset (0, +\infty)$. Observing that

$$\lim_{x \to +\infty} \exp(x) = +\infty$$

and

$$\lim_{x \to -\infty} \exp(x) = \lim_{y \to +\infty} \exp(-y) = \lim_{y \to +\infty} \frac{1}{\exp(y)} = \frac{1}{+\infty} = 0,$$

we obtain that $\sup \exp(x) = +\infty$ and $\inf \exp(x) = 0$. Hence, by Theorem 3.7, $\exp(\mathbb{R})$ is an interval with the endpoints 0 and $+\infty$, which can be only $(0, +\infty)$.

3.3.2 The maximal value theorem

If f is a real-valued function defined on a set A then the *maximal value* (maximum) of f is the maximum of $f(A)$ if it exists. Notation:

$$\max_A f = \max f(A).$$

In general, $\max_A f$ may not exist, but if it exists then it is equal to $\sup_A f$. Similarly one defined the minimal value $\min_A f$.

Theorem 3.8 (The maximal value theorem) *Let f be a continuous function on a closed bounded interval I. Then both $\max_I f$ and $\min_I f$ exist.*

Proof. Let $I = [a, b]$. Let us first show that $f(I)$ is bounded from above, that is, there is $C \in \mathbb{R}$ such that $f(x) \le C$ for all $x \in [a, b]$. If such C does not exists then, for any $n \in \mathbb{N}$, there is $x_n \in [a, b]$ such that $f(x_n) > n$. The latter implies $f(x_n) \to +\infty$ as $n \to \infty$. The sequence $\{x_n\}$ is bounded and, hence, it contains a convergent subsequence, say $x_{n_k} \to x$, where $x \in [a, b]$. By the continuity of f, we have

$$f(x_{n_k}) \to f(x) \text{ as } k \to \infty,$$

which contradicts $f(x_{n_k}) \to +\infty$.

Since $f(I)$ is a bounded set, it has a finite supremum $M = \sup_I f$. Let us show that function f takes the value M, that is, there is $x_0 \in [a, b]$ where $f(x_0) = M$, which will imply that $M = \max_I f$. By the definition of the supremum, $M - \varepsilon$ is not an upper bound for any $\varepsilon > 0$. Hence, for any $\varepsilon > 0$ there is $x \in [a, b]$ such that $f(x) > M - \varepsilon$. Taking $\varepsilon = \frac{1}{n}$ and denoting by x_n this value of x, we obtain a sequence $\{x_n\} \subset [a, b]$ such that $f(x_n) > M - \frac{1}{n}$. Since also $f(x_n) \le M$, we have $f(x_n) \to M$ as $n \to \infty$. The sequence $\{x_n\}$ is bounded and, hence, has a convergent subsequence $\{x_{n_k}\}$, say $x_{n_k} \to x_0$. By the continuity of f, we have $f(x_{n_k}) \to f(x_0)$ whereas by construction $f(x_{n_k}) \to M$. Hence, $f(x_0) = M$, which was to be proved. ■

Corollary. *If f is a continuous function on a closed bounded interval I then the image $f(I)$ is also a closed bounded interval; moreover,*

$$f(I) = \left[\min_I f, \max_I f\right].$$

Proof. By Theorem 3.7 $f(I)$ is an interval with the endpoints A and B, where $A = \inf_I f$ and $B = \sup_I f$. By Theorem 3.8, we have $A = \min_I f$ and $B = \max_I f$, which in particular implies that both A and B belong to $f(I)$. Hence, we conclude that $f(I) = [A, B]$, which proves the both claims. ■

3.4 The inverse function

Recall that a function $f(x)$ on an interval I is called (monotone) increasing if $x \le y$ implies $f(x) \le f(y)$. The function is called *strictly* (monotone) increasing if $x < y$ implies $f(x) < f(y)$. Similarly one defines decreasing functions.

Example. Function $f(x) = \exp(x)$ is strictly increasing on $\mathbb{R}$. Indeed, if $x < y$ then

$$\exp(y) = \exp(y - x + x) = \exp(y - x)\exp(x) > \exp(x)$$

because $z = y - x > 0$ and, hence,

$$\exp(z) = 1 + z + \frac{z^2}{2!} + ... > 1.$$

Example. Function $f(x) = x^n$ is strictly increasing on $[0, +\infty)$ provided $n \in \mathbb{N}$. Indeed, if $y > x$ then setting $z = y - x$ we obtain

$$y^n = (x + z)^n = x^n + \sum_{k=1}^{n-1} \binom{n}{k} x^k z^{n-k} + z^n \geq x^n + z^n > x^n.$$

Example. Function $f(x) = x^n$ is strictly decreasing on $(0, +\infty)$ provided n is a negative integer. This follows from $x^n = \frac{1}{x^{|n|}}$ and the fact that $x^{|n|}$ is strictly increasing.

Definition. Consider a function (a mapping) $f : A \to B$ where A and B are two arbitrary sets. The *inverse function* is a function $f^{-1} : B \to A$ with the following property:

$$y = f(x) \text{ is equivalent to } x = f^{-1}(y)$$

for all $x \in A$ and $y \in B$.

The inverse function may not exist but if it exists then it is unique. Note that if f^{-1} exists then also $(f^{-1})^{-1}$ exists and $(f^{-1})^{-1} = f$.

Note also the following useful identities that follow immediately from the definition:

$$\begin{aligned} f^{-1}(f(x)) &= x \text{ for all } x \in A, \\ f\left(f^{-1}(y)\right) &= y \text{ for all } y \in B. \end{aligned}$$

In other words, $f^{-1} \circ f$ and $f \circ f^{-1}$ are the identity functions.

Example. For the function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$ defined by $f(x) = \frac{1}{x}$ the condition $y = \frac{1}{x}$ is equivalent to $x = \frac{1}{y}$. Hence, f^{-1} exists and coincides with f.

Consider the function $f : [0, +\infty) \to [0, +\infty)$ defined by $f(x) = x^2$. Since the condition $y = x^2$ for non-negative x and y is equivalent to $x = \sqrt{y}$, we see that the inverse function f^{-1} exists and $f^{-1}(y) = \sqrt{y}$.

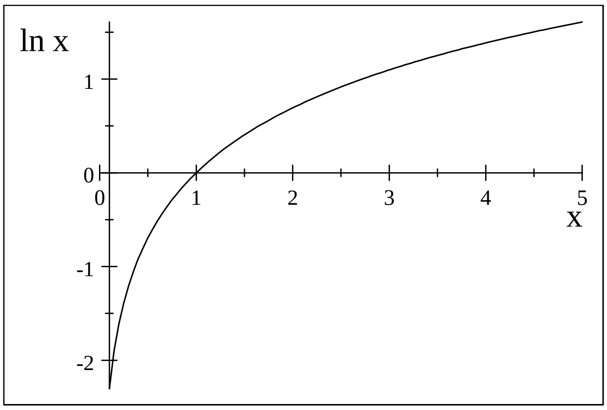
Claim *The inverse function f^{-1} exists if and only if $f : A \to B$ is a bijection.*

Proof. Assume that f^{-1} exists. Then f is a bijection because, for any $y \in B$, the condition $f(x) = y$ is equivalent to $x = f^{-1}(y)$ and, hence, is satisfied for exactly one value of $x \in A$. Conversely, if f is a bijection then for any $y \in B$ there exists a unique $x \in A$ such that $f(x) = y$, which defines the function $x = f^{-1}(y)$. ■

Theorem 3.9 *Let f be a strictly monotone function on an interval I and let $J = f(I)$. Then the inverse function $f^{-1} : J \to I$ exists, is strictly monotone and continuous.*

Example. The function $\exp(x)$ is strictly increasing on $\mathbb{R}$ and $\exp(\mathbb{R}) = (0, +\infty)$. Hence, it has the continuous inverse function defined on $(0, +\infty)$. This function is called the *natural logarithm* and is denoted by ln (sometimes also log). Hence, we have the defining condition for $\ln y$ for any $y > 0$:

$$x = \ln y \iff y = \exp x.$$



The natural logarithm can be used to define a^x for any $a > 0$ and $x \in \mathbb{R}$ (or even $x \in \mathbb{C}$). Indeed, let us set

$$a^x = \exp(x \ln a). \tag{3.9}$$

Function $f(x) = a^x$ is called the *exponential function with the base a*. Let us mention the following properties of this function:

1. $a^1 = a$ because $\exp(\ln a) = a$.

2. $a^{x+y} = a^x a^y$ because

$$a^x a^y = \exp(x \ln a) \exp(y \ln a) = \exp((x + y) \ln a) = a^{x+y}.$$

3. For any $n \in \mathbb{N}$, the present definition of a^n matches the previous inductive definition of a^n because by the above two properties $a^{n+1} = a^n a$.

4. For all $x, y \in \mathbb{R}$,

$$(a^x)^y = a^{xy} \tag{3.10}$$

(see Exercise 51).

Note that $\ln e = 1$ so that by (3.9) $e^x = \exp(x)$ that matches the previous definition of e^x.

Proof of Theorem 3.9. Assume for simplicity that f is strictly increasing. Clearly, f is a bijection from I onto J. Hence, the inverse function exists. To prove the monotonicity fix $y_1 < y_2$ from J and let $x_k = f^{-1}(y_k)$ so that $y_k = f(x_k)$. We claim that $x_1 < x_2$. Indeed, $x_1 = x_2$ implies $y_1 = y_2$ and $x_1 > x_2$ implies $y_1 > y_2$. Since the both outcomes contradict $y_1 < y_2$, the only remaining possibility is $x_1 < x_2$.

Let us prove that f^{-1} is continuous, that is, for any $y \in J$ and any sequence $\{y_n\} \subset J$ such that $y_n \to y$, we have $f^{-1}(y_n) \to f^{-1}(y) = x$. Assume from the contrary that the sequence $x_n = f^{-1}(y_n)$ does not converge to x. Then by Theorem 2.1 there is $\varepsilon > 0$ such that outside $U_\varepsilon(x)$ there are infinitely many terms of the sequence $\{x_n\}$. It follows that one of the intervals $(-\infty, x-\varepsilon]$, $[x+\varepsilon, +\infty)$ contains infinitely many terms of $\{x_n\}$, let it be $(-\infty, x-\varepsilon]$. Renaming the sequence, we can assume $x_n \leq x - \varepsilon$ for all $n \in \mathbb{N}$. Since $x_n \leq x - \varepsilon < x$ and both x_n, x belong to the interval I, it follows that also $x - \varepsilon \in I$. By the monotonicity of f, we obtain that

$$y_n = f(x_n) \leq f(x-\varepsilon),$$

whence

$$y = \lim_{n\to\infty} y_n \leq f(x-\varepsilon) < f(x) = y.$$

This contradiction shows that $f^{-1}(y_n) \to f^{-1}(y)$, which proves the continuity of f^{-1}. ∎

Note that it the continuity of f is not assumed in the statement of Theorem 3.9. If f is still continuous then, by Theorem 3.7, J is an interval. Hence, in this case the domain of the inverse function is also an interval.

Example. The function $y = x^n$ (where $n \in \mathbb{N}$) is strictly increasing on $[0, +\infty)$ and its image is $[0, +\infty)$. Hence, the inverse function exists and is continuous on $[0, +\infty)$ and is denoted by $x = \sqrt[n]{y}$.

Note that, for any $a > 0$, $\sqrt[n]{a} = a^{1/n}$. Indeed, $x = \sqrt[n]{a}$ is the unique positive number that satisfy the equation $x^n = a$. On the other hand, $x = a^{1/n}$ also satisfies this equation by (3.10) since $\left(a^{1/n}\right)^n = a^{\frac{1}{n}n} = a$.

3.5 Trigonometric functions and the number π

Let us define the trigonometric functions $\sin x$ and $\cos x$ by the identities:

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} \quad \text{and} \quad \cos x = \frac{e^{ix} + e^{-ix}}{2}, \tag{3.11}$$

where i is the imaginary unit. Here x can be any complex number but we will use $\sin x$ and $\cos x$ mostly with a real x. Note that in this case the definitions still require the exponential function of a complex argument ix.

It follows from the definition of sin and cos that

$$e^{ix} = \cos x + i \sin x$$

(the Euler formula).

Using the exponential series, we obtain the expansions of $\sin x$ and $\cos x$ into the series as follows:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ... = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \tag{3.12}$$

and

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ... = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} \tag{3.13}$$

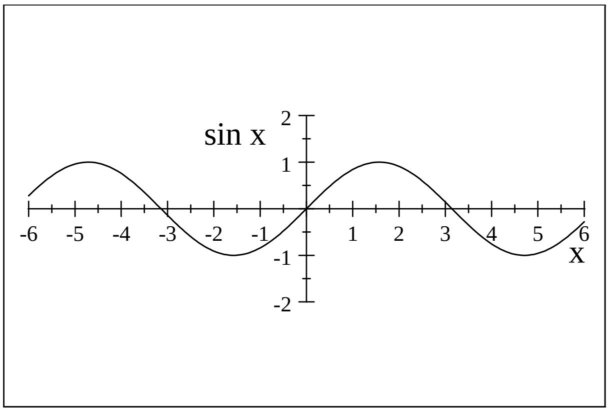
(see Exercise 53). In particular, it follows that $\sin x$ and $\cos x$ are real if $x \in \mathbb{R}$. Also, the expansions (3.12) and (3.13) imply that $\sin x$ is an odd function that is,

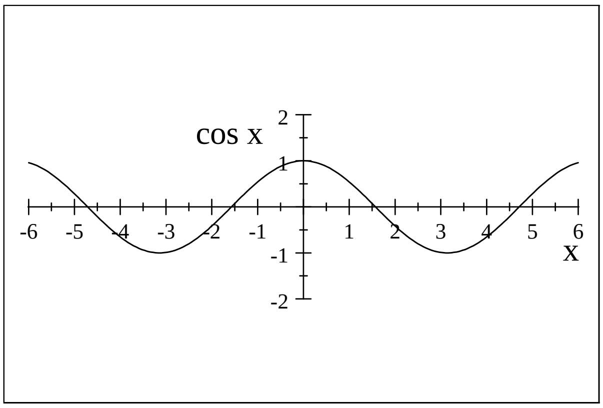
$$\sin(-x) = -\sin x,$$

whereas $\cos x$ is an even function, that is

$$\cos(-x) = \cos x.$$

It follows from (3.11) that the functions $\sin x$ and $\cos x$ are continuous on $\mathbb{R}$ (see Exercise 53). The graphs of $\sin x$ and $\cos x$ are as follows:

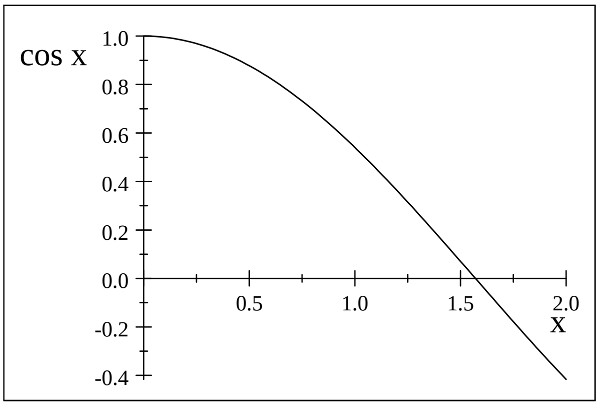


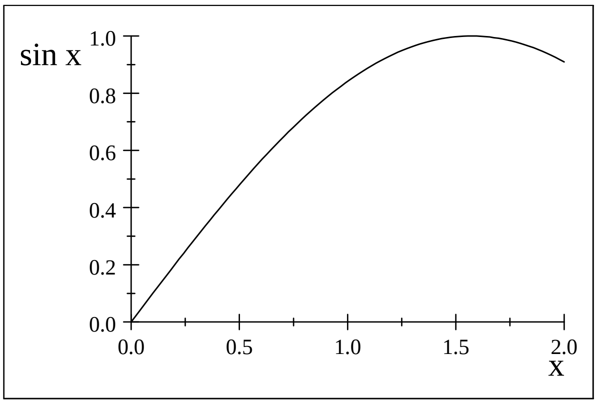


As one can see from the graphs, $\sin x$ and $\cos x$ are *periodic* functions, which is not obvious from either (3.11) or (3.12), (3.13). We'll prove the periodicity in the rest of this section. Let us

Lemma 3.10 (a) *There exists a number* $c \in (0, 2)$ *such that* $\cos c = 0$ *whereas* $\cos x > 0$ *for all* $0 \le x < c$. *In particular,* c *is the smallest positive root of the equation* $\cos x = 0$.

(b) $\sin x > 0$ *for any* $x \in (0, 2)$.





Proof. (a) Note that $\cos 0 = 1 > 0$. Let us show that $\cos 2 < 0$. We have

$$\cos 2 = 1 - \frac{2^2}{2!} + \frac{2^4}{4!} - \frac{2^6}{6!} - ... = a_0 - a_1 + a_2 - a_3 + ...$$

where $a_n = \frac{2^{2n}}{(2n)!}$. The sequence $\{a_n\}$ is strictly decreasing for $n \geq 1$ because

$$\frac{a_{n+1}}{a_n} = \frac{2^2}{(2n+1)(2n+2)} \leq 1.$$

Let $S_n = \sum_{k=0}^{n} (-1)^k a_k$ be the partial sum of this series. We claim that the sequence $\{S_{2m}\}_{m=0}^{\infty}$ of even partial sums is decreasing. For example, have

$$S_2 = a_0 - a_1 + a_2 = a_0 - (a_1 - a_2) \leq a_0 = S_0.$$

Similarly, we obtain in the general case

$$S_{2(m+1)} = S_{2m} - a_{2m+1} + a_{2m+2} \leq S_{2m}.$$

Hence, for any $m \geq 0$,

$$\cos 2 = \lim_{n\to\infty} S_n = \lim_{m\to\infty} S_{2m} \leq S_{2m}.$$

In particular, we have

$$\cos 2 \leq S_2 = 1 - \frac{2^2}{2!} + \frac{2^4}{4!} = -\frac{1}{3} < 0,$$

that is $\cos 2 < 0$. Applying Theorem 3.6 to the interval $[0, 2]$, we conclude that $\cos x = 0$ for some $x \in (0, 2)$.

Consider the set

$$S = \{x \in [0, 2] : \cos x = 0\},$$

which is non-empty by the above argument, and set

$$c = \inf S \in [0, 2].$$

Let us show that $c \in S$. Indeed, there is a sequence $\{x_n\}_{n=1}^{\infty} \subset S$ such that $x_n \to c$ as $n \to \infty$, whence it follows by the continuity of cos

$$\cos c = \lim_{n\to\infty} \cos x_n = 0,$$

whence $c \in S$. Note that $c > 0$ because $\cos 0 > 0$ and $c < 2$ because $\cos 2 < 0$.

Finally, let us show that if $0 < x < c$ then $\cos x > 0$. Since c is a lower bound for S, the condition $x < c$ implies $x \notin S$ whence $\cos x \neq 0$. If $\cos x < 0$ then, applying Theorem 3.6 to the interval $[0, x]$, we obtain that is a point $0 < y < x$ such that $\cos y = 0$ and, hence, $y \in S$, which is impossible because $y < c$. We conclude that $\cos x > 0$, which finishes the proof.

(b) We have

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ... = a_0 - a_1 + a_2 - ...$$

where $a_n = \frac{x^{2n+1}}{(2n+1)!}$. For any $n \geq 0$, we have

$$\frac{a_{n+1}}{a_n} = \frac{x^2}{(2n+2)(2n+3)} < \frac{4}{(2n+2)(2n+3)} < 1,$$

that is, the sequence $\{a_n\}_{n=0}^{\infty}$ is decreasing.

Consider the partial sum $S_n = \sum_{k=0}^{n} (-1)^k a_k$ and prove that the sequence of odd partial sum $\{S_{2m+1}\}_{m=0}^{\infty}$ is increasing. For example,

$$S_3 = a_0 - a_1 + a_2 - a_3 \geq a_0 - a_1 = S_1$$

and similarly

$$S_{2m+1} = S_{2m-1} + a_{2m} - a_{2m+1} \geq S_{2m-1}.$$

Hence

$$\sin x = \lim_{m\to\infty} S_{2m+1} \geq S_{2m+1} \text{ for any } m \geq 0.$$

In particular, we have for any $x \in (0, 2)$

$$\sin x \geq S_1 = x - \frac{x^3}{6} > 0$$

because $x > \frac{x^3}{6}$ is equivalent to $x^2 < 6$, and the latter is true by $x < 2$. ■

Definition. Define the number π by $\pi = 2c$ where c is the smallest positive root of $\cos x$ that exists by Lemma 3.10.

It follows from the above proof that $0 < \pi < 4$. Similarly to the proof of Lemma 3.10, one can show that $c > \frac{3}{2}$ and, hence, $\pi > 3$ (see Exercise 53). A numerical computation shows that

$$\pi = 3,14159265358979...$$

This many decimal digits of π were known as early as in 15^{th} century. Presently π is computed to over 6 billion decimal digits.

Theorem 3.11 (a) *We have the identities*

$$\exp\left(\frac{\pi}{2}i\right) = i, \quad \exp(\pi i) = -1, \qquad \exp(2\pi i) = 1.$$

(b) *The function* $\exp(z)$ *is* $2\pi i$ *periodic, that is,*

$$\exp(z + 2\pi i) = \exp(z) \ \textit{for all } z \in \mathbb{C}. \tag{3.14}$$

(c) *Functions* $\sin x$ *and* $\cos x$ *are* 2π *periodic, that is*

$$\sin(x + 2\pi) = \sin x \ \textit{ and } \ \cos(x + 2\pi) = \cos x \ \textit{ for all } x \in \mathbb{R}.$$

Proof. (a) Let us apply the identity

$$\cos^2 x + \sin^2 x = 1. \tag{3.15}$$

(see Exercise 53). Using as above the notation $c = \pi/2$, we have $\cos c = 0$ whence by (3.15) $|\sin c| = 1$. Since $0 < c < 2$ and, by Lemma 3.10(b), $\sin c > 0$, we obtain $\sin c = 1$. Therefore, by the Euler formula,

$$\exp\left(\frac{\pi}{2}i\right) = \exp(ci) = \cos c + i \sin c = i,$$

whence

$$\begin{aligned} \exp(\pi i) &= \exp(ci)\exp(ci) = i \cdot i = -1, \\ \exp(2\pi i) &= \exp(\pi i)\exp(\pi i) = (-1)^2 = 1. \end{aligned} \tag{3.16}$$

(b) Using (3.16), we obtain

$$\exp(z + 2\pi i) = \exp(z)\exp(2\pi i) = \exp(z).$$

(c) Using (3.12) and (3.14), we have

$$\begin{aligned} \sin(x + 2\pi) &= \frac{1}{2i}\left(\exp(ix + 2\pi i) - \exp(-ix - 2\pi i)\right) \\ &= \frac{1}{2i}\left(\exp(ix) - \exp(-ix)\right) = \sin x, \end{aligned}$$

and a similar argument works for cos. ■

4 Differential calculus

4.1 Definition of the derivative

Definition. Let $f(x)$ be a function defined on an interval I. The *derivative* (*die Ableitung*) of function f at a point $x \in I$ is defined by

$$f'(x) = \lim_{y \to x} \frac{f(y) - f(x)}{y - x},$$

provided the limit exists.

Here the variable y varies in $I \setminus \{x\}$ because this is the domain of the function $y \mapsto \frac{f(y)-f(x)}{y-x}$. The expression $\frac{f(y)-f(x)}{y-x}$ is called the *difference quotient* of function f and its value shows how functions f varies between the points x and y. The derivative $f'(x)$ is the rate of change of function f at the point x.

Setting $h = y - x$, we obtain an equivalent definition:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Example. Let $f(x) = const$. Then $f(y) - f(x) = 0$ whence $f'(x) = 0$. We can write this down as follows:

$$(const)' = 0.$$

Let $f(x) = x$. Then

$$\frac{f(y) - f(x)}{y - x} = \frac{y - x}{y - x} = 1$$

whence $f'(x) = 1$ for all $x \in \mathbb{R}$. Hence,

$$(x)' = 1.$$

Let $f(x) = x^2$. Then

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{2hx + h^2}{h} = 2x.$$

Hence,

$$\left(x^2\right)' = 2x.$$

More generally, one can prove that, for any $n \in \mathbb{N}$,

$$\boxed{(x^n)' = nx^{n-1}}$$

(see Exercise 56).

Let $f(x) = \exp(x)$. By Exercise 43,

$$f'(x) = \lim_{y \to x} \frac{\exp(y) - \exp(x)}{y - x} = \exp(x),$$

whence

$$\boxed{(\exp(x))' = \exp(x).}$$

Let $f(x) = \sin x$. Then

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \\ &= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\ &= \lim_{h\to 0} \sin x \frac{\cos h - 1}{h} + \lim_{h\to 0} \cos x \frac{\sin h}{h}. \end{aligned}$$

By Exercise 54, we have

$$\lim_{h\to 0} \frac{\cos h - 1}{h} = 0 \text{ and } \lim_{h\to 0} \frac{\sin h}{h} = 1$$

whence

$$\boxed{(\sin x)' = \cos x}$$

Similarly one proves that

$$\boxed{(\cos x)' = -\sin x}$$

(see Exercise 56).

Physical meaning of the derivative. Let $f(t)$ be the position function of an object that moves on a straight line. That is, at time t the object is located at the point $f(t) \in \mathbb{R}$. Then

$$\frac{f(t+h) - f(t)}{h} = \frac{\text{displacement of the object}}{\text{time interval}}$$

is the average velocity of the object in the time interval $[t, t+h]$. Taking the limit when $h \to 0$, we obtain the *instantaneous velocity* of the object at time t. Hence, the derivative $f'(t)$ is the instantaneous velocity of the object at time t. For example, if the object is a car then $f'(t)$ is displayed at the speedometer at any time t.

Geometric meaning of the derivative. Consider the graph of the function $y = f(x)$ and consider the a straight line through the points $(x, f(x))$ to $(x+h, f(x+h))$, which is called a *secant line* of the graph. The equation of a straight line that goes through a point (X_0, Y_0) has the form

$$Y = A(X - X_0) + Y_0$$

(we use the capital X and Y to distinguish with the graph of function f), where A is the *slope* (*die Steigung*) of the line. Setting $X_0 = x$, $Y_0 = f(x)$, $X = x+h$, and $Y = f(x+h)$, we obtain the equation for A:

$$f(x+h) = Ah + f(x)$$

whence

$$A = \frac{f(x+h) - f(x)}{h}.$$

Taking $h \to 0$, we obtain $A \to f'(x)$. The limiting position of a secant line when $h \to 0$ is called the *tangent* line of the graph at the point x. The equation of the tangent line is

$$Y = f'(x)(X - x) + f(x). \tag{4.1}$$

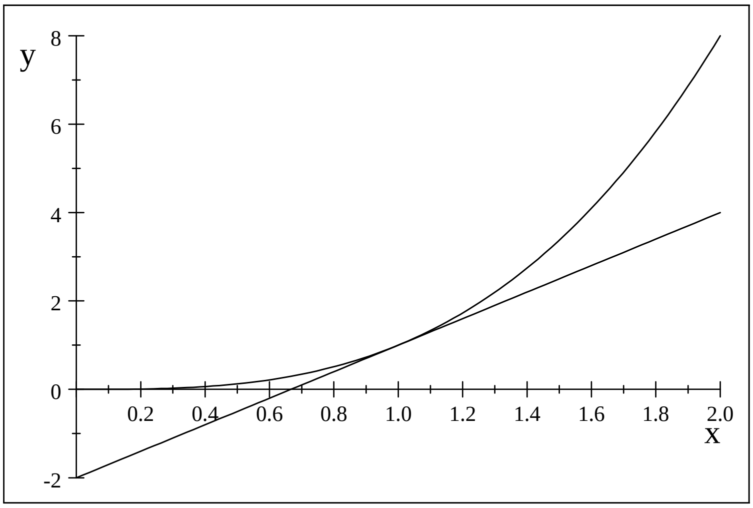
Example. Let $f(x) = x^3$. Then the equation (4.1) becomes

$$Y = 2x^2 (X - x) + x^3.$$

For example, at $x = 1$ we obtain the following tangent line:

$$Y = 3(X - 1) + 1 = 3X - 2.$$

Here are the graphs of the function $f(x)$ and its tangent line at $x = 1$:



The tangent line can be considered as a good approximation of the graph of the function near the point x. Setting $X = x + h$, obtain the following relation

$$f(x + h) \approx f(x) + f'(x) h. \tag{4.2}$$

What is the exact meaning of this relation? To state it let us introduce the following terminology.

Definition. Let $f(x)$ and $g(x)$ be two functions defined on a interval $I \subset \mathbb{R}$ and $a \in \overline{I}$. We write

$$f(x) = o(g(x)) \quad \text{as } x \to a \tag{4.3}$$

if

$$\lim_{x \to a} \frac{f(x)}{g(x)} = 0.$$

One reads out (4.3) as follows : f is little o of g.

For example,

$$x^2 = o(x) \text{ as } x \to 0$$

while

$$x = o\left(x^2\right) \text{ as } x \to +\infty.$$

Lemma 4.1 *If $f'(x)$ exists then*

$$f(x + h) = f(x) + f'(x) h + o(h) \ \ \textit{as } h \to 0. \tag{4.4}$$

The relation (4.4) can be considered as a rigorous version of (4.2).

Proof. We need to prove that

$$f(x+h) - f(x) - f'(x)h = o(h) \text{ as } h \to 0.$$

Indeed, we have

$$\lim_{h\to 0} \frac{f(x+h) - f(x) - f'(x)h}{h} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} - f'(x) = 0,$$

whence (4.4) follows. ■

Example. Let $f(x) = \sin x$. Then by (4.4)

$$\sin(x+h) = \sin x + h\cos x + o(h).$$

For example, if $x = 0$ then we obtain $\sin h = h + o(h)$. One can prove this formula also by using the expansion of $\sin x$ into the series in powers of h.

Differential. The increment h of the variable x is also called the *differential* of x and is denoted by dx (here dx is NOT the product of d and x; dx is just another notation for h, which contains the reference to the variable x). Using this notation, rewrite (4.4) as follows

$$f(x+dx) - f(x) = f'(x)\,dx + o(dx) \text{ as } dx \to 0. \tag{4.5}$$

The left hand side $f(x+dx) - f(x)$ is called the *increment* of the function f. The term $f'(x)\,dx$ on the right hand side is called the *differential* of the function f and is denoted by $df(x)$ so that

$$df(x) = f'(x)\,dx. \tag{4.6}$$

Note that the differential df is a linear function of dx (considering x to be fixed). Since by (4.5)

$$f(x+dx) - f(x) = df(x) + o(dx),$$

we can say that the differential is the main linear part of the increment of function.

Example. Using the derivatives evaluated above, we obtain from (4.6)

$$\begin{aligned} dx^n &= nx^{n-1}dx \\ d\exp(x) &= \exp(x)\,dx \\ d\sin x &= \cos x\ dx \\ d\cos x &= -\sin x\ dx. \end{aligned}$$

Definition. We say that a function f defined on an interval I is *differentiable* at a point $x \in I$ if the derivative $f'(x)$ exists. Function f is differentiable on I if f is differentiable at any point $x \in I$.

Theorem 4.2 *If a function f is differentiable at x then f is continuous at x.*

Proof. We have

$$\lim_{y\to x}\left(f\left(y\right)-f\left(x\right)\right)=\lim_{y\to x}\frac{f\left(y\right)-f\left(x\right)}{y-x}\left(y-x\right)=f'\left(x\right)\lim_{y\to x}\left(y-x\right)=0,$$

whence

$$\lim_{y\to x}f\left(y\right)=x.$$

Hence, f is continuous at x. ■

Example. Consider the function

$$f\left(x\right)=\begin{cases}1, & x>0\\ 0, & x\leq 0\end{cases}$$

Since $f=const$ on $(0,+\infty)$ and on $(-\infty,0)$, we obtain that f is differentiable at $x\neq 0$ and $f'\left(x\right)=0$. Let us show that f is not differentiable at $x=0$. Indeed, at this point the function f is not even continuous since

$$\lim_{\substack{x\to 0\\ x>0}}f\left(x\right)=1\neq 0=f\left(0\right).$$

By Theorem 4.2, function f is not differentiable at 0.

4.2 Rules of differentiation

Theorem 4.3 *Let f and g be two functions on an interval $I\subset\mathbb{R}$, which are differentiable at some point $x\in I$. Then functions $f+g$, fg, $\frac{f}{g}$ are also differentiable at x (in the case of f/g assuming $g\neq 0$) and*

(a)

$$\boxed{\left(f+g\right)'\left(x\right)=f'\left(x\right)+g'\left(x\right).}\tag{4.7}$$

(b) *The product rule (Produktregel)*

$$\boxed{\left(fg\right)'\left(x\right)=f'\left(x\right)g\left(x\right)+f\left(x\right)g'\left(x\right).}\tag{4.8}$$

(c) *The quotient rule (Quotientregel)*

$$\boxed{\left(\frac{f}{g}\right)'\left(x\right)=\frac{f'\left(x\right)g\left(x\right)-f\left(x\right)g'\left(x\right)}{g^2\left(x\right)}.}\tag{4.9}$$

It follows from (4.8) that, for any constant c,

$$\boxed{\left(cf\right)'=cf'.}$$

Proof. (a) Using the definition of the derivative and Theorem 3.2, we obtain

$$\begin{aligned}\left(f+g\right)'\left(x\right) &= \lim_{y\to x}\frac{f\left(y\right)+g\left(y\right)-f\left(x\right)-g\left(x\right)}{y-x}\\ &= \lim_{y\to x}\frac{f\left(y\right)-f\left(x\right)}{y-x}+\lim_{y\to x}\frac{g\left(y\right)-g\left(x\right)}{y-x}\\ &= f'\left(x\right)+g'\left(x\right).\end{aligned}$$

(b) Arguing as above and using the continuity of f, which holds by Theorem 4.2, we obtain

$$\begin{aligned}
(fg)'(x) &= \lim_{y\to x} \frac{f(y)\,g(y) - f(x)\,g(x)}{y-x} \\
&= \lim_{y\to x} \frac{f(y)\,g(y) - f(y)\,g(x)}{y-x} + \lim_{y\to x} \frac{f(y)\,g(x) - f(x)\,g(x)}{y-x} \\
&= \lim_{y\to x} f(y) \lim_{y\to x} \frac{g(y) - g(x)}{y-x} + g(x) \lim_{y\to x} \frac{f(y) - f(x)}{y-x} \\
&= f(x)\,g'(x) + g(x)\,f'(x),
\end{aligned}$$

which was to be proved.

(c) Similarly to (b), we have

$$\begin{aligned}
\left(\frac{1}{g}\right)'(x) &= \lim_{y\to x} \frac{1}{y-x}\left(\frac{1}{g(y)} - \frac{1}{g(x)}\right) \\
&= \lim_{y\to x} \frac{1}{y-x}\left(\frac{g(x) - g(y)}{g(y)\,g(x)}\right) \\
&= \lim_{y\to x} \frac{g(x) - g(y)}{y-x} \lim_{y\to x} \frac{1}{g(y)\,g(x)} \\
&= \frac{-g'(x)}{g^2(x)}.
\end{aligned}$$

Hence, we have

$$\left(\frac{1}{g}\right)'(x) = -\frac{g'(x)}{g^2(x)}, \tag{4.10}$$

which is a particular case of (4.9) for $f = 1$.

For arbitrary f, we obtain using (4.8) and (4.10):

$$\begin{aligned}
\left(\frac{f}{g}\right)' &= \left(f\frac{1}{g}\right)' = f'\left(\frac{1}{g}\right) + f\left(\frac{1}{g}\right)' \\
&= \frac{f'}{g} - \frac{fg'}{g^2} = \frac{f'g - fg'}{g^2}.
\end{aligned}$$

■

Theorem 4.4 (The chain rule - *Kettenregel*) *Let f be a function on an interval A, and g be a function on an interval B such that the composition $g \circ f$ is defined, that is $f(A) \subset B$. If f is differentiable at a point $x \in A$ and g is differentiable at $y = f(x) \in B$, then $g \circ f$ is differentiable at x and*

$$\boxed{(g \circ f)'(x) = g'(y)\,f'(x).} \tag{4.11}$$

One can also write

$$(g \circ f)'(x) = g'(f(x))\,f'(x).$$

Example. Consider a general exponential function $f(x) = a^x$ where $a > 0$ and $x \in \mathbb{R}$. By definition, we have

$$f(x) = \exp(x \ln a),$$

which can be written as the composition of the following two functions:

$$\exp(y) \text{ and } y = x \ln a.$$

Hence, by the chain rule,

$$f'(x) = (\exp(y))' (x \ln a)' = \exp(y) \ln a = a^x \ln a,$$

that is,

$$\boxed{(a^x)' = a^x \ln a.}$$

Example. The function $f(x) = \exp(\cos 2x)$ can written as the composition of three functions:

$$\exp(z), \quad z = \cos y, \quad y = 2x.$$

Applying the chain rule twice, we obtain

$$f'(x) = (\exp(z))' (\cos y)' (2x)' = -2 \exp(z) \sin y = -2 \exp(\cos(2x)) \sin 2x.$$

Before the proof of Theorem 4.4, let us prove the following stronger version of Lemma 4.1.

Lemma 4.5 *Let f be a function on an interval I which is differentiable at a point $x \in I$. Then there exists a function F on I such that*

$$f(X) - f(x) = F(X)(X - x) \tag{4.12}$$

for all $X \in I$ and

$$\lim_{X \to x} F(X) = F(x) = f'(x). \tag{4.13}$$

Remark. This statement covers Lemma 4.1 because by (4.12)

$$\begin{aligned} f(X) - f(x) &= f'(x)(X - x) + (F(X) - f'(x))(X - x) \\ &= f'(x)(X - x) + o(X - x), \end{aligned}$$

where in the last line we have used that $F(X) - f'(x) \to 0$ as $X \to x$.

Proof. Define function F so that (4.12) is satisfied:

$$F(X) = \frac{f(X) - f(x)}{X - x},$$

for $X \neq x$ and set

$$F(x) = f'(x).$$

Clearly, (4.12) holds for all $X \in I$ (if $X = x$ then the both sides of (4.12) vanish), and

$$\lim_{X \to x} F(X) = \lim_{X \to x} \frac{f(X) - f(x)}{X - x} = f'(x).$$

■

Proof of Theorem 4.4. By Lemma 4.5, we have the identities

$$f(X) - f(x) = F(X)(X - x) \quad \text{for all } X \in A$$

and

$$g(Y) - g(y) = G(Y)(Y - y) \quad \text{for all } Y \in B,$$

where

$$\lim_{X \to x} F(X) = F(x) = f'(x)$$

and

$$\lim_{Y \to y} G(Y) = G(y) = g'(y).$$

Then, setting $Y = f(X)$ and noticing that $y = f(x)$, we obtain

$$\begin{aligned} g \circ f(X) - g \circ f(x) &= g(f(X)) - g(f(x)) = G(f(X))(f(X) - f(x)) \\ &= G(f(X))F(X)(X - x), \end{aligned}$$

whence

$$\begin{aligned} (g \circ f)'(x) &= \lim_{X \to x} \frac{g \circ f(X) - g \circ f(x)}{X - x} = \lim_{X \to x} G(F(X))F(X) \\ &= \lim_{X \to x} G(F(X)) \lim_{X \to x} F(X) \\ &= g'(y) f'(x), \end{aligned}$$

which was to be proved. ■

Theorem 4.6 (The derivative of the inverse function) *Let f be a continuous strictly monotone function on an interval I so that the inverse function $f^{-1}(x)$ exists on the interval $J = f(I)$. If f is differentiable at $x \in I$ and $f'(x) \neq 0$ then f^{-1} is differentiable at $y = f(x)$ and*

$$\boxed{(f^{-1})'(y) = \frac{1}{f'(x)}}. \tag{4.14}$$

Remark. The tangent line to the graph of function f at point x has the equation

$$Y - y = A(X - x)$$

where $A = f'(x)$ is the slope. When considering the inverse function, we switch the role of the function and the argument, so that the equation can be written in the form

$$X - x = \frac{1}{A}(Y - y)$$

provided $A \neq 0$. Hence, the slope of the tangent line in this case is $\frac{1}{A}$, which explains the formula (4.14).

Since $y = f(x)$ is equivalent to $x = f^{-1}(y)$, we can rewrite (4.14) as

$$\left(f^{-1}\right)'(y) = \frac{1}{f'(f^{-1}(y))}$$

and
$$\left(f^{-1}\right)'(f(x)) = \frac{1}{f'(x)}.$$

Example. Consider function $f(x) = \exp(x)$ on $I = \mathbb{R}$. We already know that the inverse function to $\exp(x)$ is $\ln y$ defined for $y \in (0, +\infty)$. By (4.14), we obtain
$$(\ln y)' = \frac{1}{(\exp(x))'} = \frac{1}{\exp(x)}$$
provided $y = \exp(x)$, which implies
$$\boxed{(\ln y)' = \tfrac{1}{y}.}$$

Example. Consider a *power function* $f(x) = x^a$ where $x > 0$ and $a \in \mathbb{R}$. By definition of x^a,
$$f(x) = \exp(a \ln x)$$
so that f is the composition of the function $\exp(y)$ and $y = a \ln x$. Using the chain rule and the derivative of $\ln x$, we obtain
$$(x^a)' = (\exp(y))'(a \ln x)' = \exp(y)\frac{a}{x} = x^a \frac{a}{x} = a x^{a-1}.$$
Finally,
$$\boxed{(x^a)' = a x^{a-1}.}$$
In particular,
$$\left(\sqrt{x}\right)' = \left(x^{1/2}\right)' = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}.$$

Proof of Theorem 4.6. Note that $J = f(I)$ is an interval by Theorem 3.7, and the inverse function f^{-1} exists, is continuous, and strictly monotone by Theorem 3.9. By Lemma 4.5, we have the identity
$$f(X) - f(x) = F(X)(X - x) \tag{4.15}$$
where function F satisfies the relations
$$\lim_{X \to x} F(X) = F(x) = f'(x). \tag{4.16}$$

Denoting $Y = f(X)$ and $y = f(x)$, we obtain from (4.15)
$$Y - y = F\left(f^{-1}(Y)\right)\left(f^{-1}(Y) - f^{-1}(y)\right)$$
Setting $G(Y) = F\left(f^{-1}(Y)\right)$, using (4.16) and the continuity of f^{-1} at y, we obtain by Theorem 3.3
$$\lim_{Y \to y} G(Y) = \lim_{Y \to y} F\left(f^{-1}(Y)\right) = \lim_{X \to x} F(X) = f'(x)$$
and
$$G(y) = F\left(f^{-1}(y)\right) = F(x) = f'(x).$$

Since $f'(x) \neq 0$, there is a neighborhood $U(y)$ such that $G(Y)$ does not vanish in $U(y)$. Hence, for any $Y \in U(y) \cap J$, we obtain

$$f^{-1}(Y) - f^{-1}(y) = \frac{1}{G(Y)}(Y - y).$$

By the definition of the derivative, we obtain

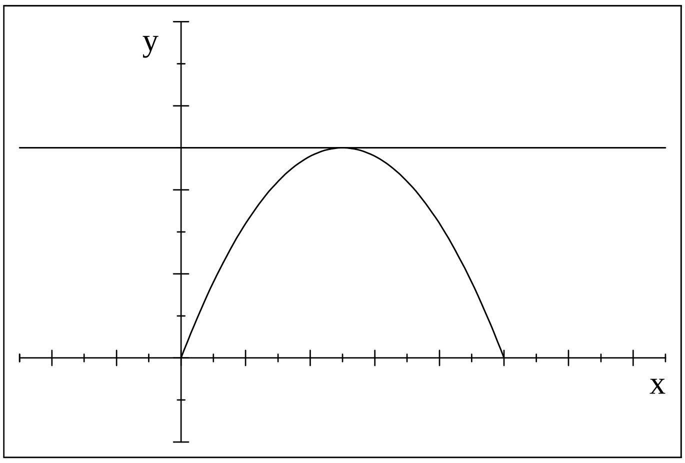
$$\left(f^{-1}\right)'(y) = \lim_{Y \to y} \frac{f^{-1}(Y) - f^{-1}(y)}{Y - y} = \lim_{Y \to y} \frac{1}{G(Y)} = \frac{1}{f'(x)},$$

which was to be proved. ■

4.3 Major theorems of differential calculus

Theorem 4.7 (The Fermat theorem) *Let f be a function defined on an open interval I. Assume that $\max_I f$ exists and let $f(x) = \max_I f$ for some $x \in I$. If f is differentiable at x then $f'(x) = 0$. The same claim holds if $f(x) = \min_I f$.*

The geometric meaning of this theorem is as follows. If f takes its maximal (or minimal) value at a point x inside an open interval then the tangent line at x must be horizontal. The slope of a horizontal line is 0 (see the graph below) which means that $f'(x) = 0$.



Proof. By Lemma 4.5, we have

$$f(X) - f(x) = F(X)(X - x),$$

where

$$\lim_{X \to x} F(X) = F(x) = f'(x).$$

Assume from the contrary that $f'(x) \neq 0$, say, $f'(x) > 0$. Then function $F(X)$ is positive in some neighborhood $U(x)$. Note that $I \cap U(x)$ is an open interval containing x. Taking $X \in I \cap U(x)$ such that $X > x$, we obtain

$$f(X) - f(x) = F(X)(X - x) > 0, \tag{4.17}$$

and, hence, $f(X) > f(x)$, which contradicts the fact that $f(x) = \max_I f$.

Similarly, if $f'(x) < 0$ then there is a neighborhood $U(x)$ where $F(X) < 0$. Choosing $X \in I \cap U(x)$ such that $X < x$, we obtain again (4.17).

The case $f(x) = \min_I f$ is treated in the same way. ∎

Theorem 4.8 (The Rolle Theorem) *Let f be a continuous function on $[a,b]$ differentiable on (a,b). If $f(a) = f(b)$ then there is a point $c \in (a,b)$ such that $f'(c) = 0$.*

Proof. By Theorem 3.8 (the maximal value theorem), function f takes its maximal value at some point $c_1 \in [a,b]$ and its minimal value at some point $c_2 \in [a,b]$. If one of these points is contained in the open interval (a,b) then the value of f' at this point is 0 by Theorem 4.7. Assume that both c_1 and c_2 are the endpoints of this interval. Since f takes the same value at a and b and this value is both maximal and minimal, the function f must be constant on $[a,b]$. Then $f'(c) = 0$ for any $c \in (a,b)$. ∎

Theorem 4.9 (Mean-Value Theorem of Lagrange) *Let f be a continuous function on an interval $[a,b]$ differentiable on (a,b). Then there is a point $c \in (a,b)$ such that*

$$f(b) - f(a) = f'(c)(b-a).$$

Proof. Consider the function

$$h(x) = f(x) - \frac{f(b) - f(a)}{b-a}(x-a)$$

so that $h(b) = h(a) = f(a)$. This function is obviously continuous on $[a,b]$ and differentiable on (a,b). By Theorem 4.8, there is $c \in (a,b)$ such that $h'(c) = 0$. Then we have

$$h' = f' - \frac{f(b) - f(a)}{b-a},$$

whence it follows that

$$f'(c) = \frac{f(b) - f(a)}{b-a},$$

which was to be proved. ∎

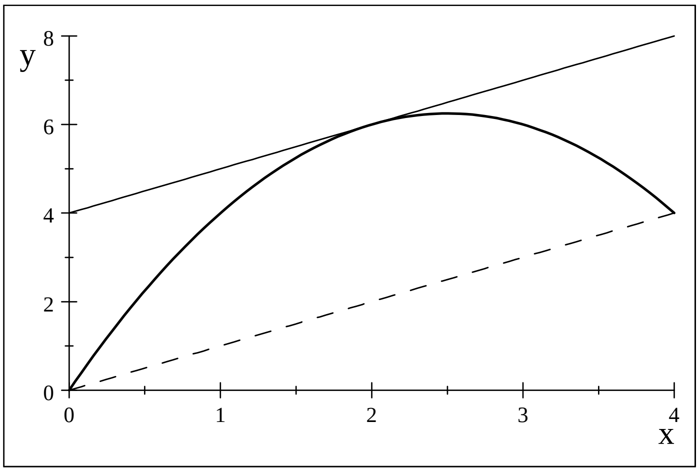
Remark. Let

$$y = A(x-a) + f(a)$$

be the equation of the secant line of the graph of function $f(x)$, which goes through the points $(a, f(a))$ and $(b, f(b))$. Here

$$A = \frac{f(b) - f(a)}{b-a}$$

is the slope of this line. Then Theorem 4.9 says that the slope of a secant line is equal to the slope of the tangent line at some intermediate point $c \in (a,b)$. See the diagram below where $c = 2$.



Theorem 4.10 (The constant test) *Let f be a differentiable function on an interval I. If $f'(x) = 0$ for all $x \in I$ then $f = \text{const}$ on I.*

Note that if $f = \text{const}$ on I then $f'(x) = 0$ for all $x \in I$. Hence, we have the following equivalence: $f' \equiv 0$ on $I \Leftrightarrow f = \text{const}$ on I.

Proof. It suffices to prove that $f(a) = f(b)$ for all distinct $a, b \in I$. Applying Theorem 4.9, we obtain that there exists $c \in (a, b)$ such that

$$f(a) - f(b) = f'(c)(a - b).$$

Since $f'(c) = 0$, we conclude that $f(a) = f(b)$. ■

Example. Consider the following problem: find all functions f such that $f'(x) = x$ on $\mathbb{R}$. First note that if $f(x) = ax^n$ then $f'(x) = anx^{n-1}$. Hence, in order to have $f'(x) = x$, we need to take $n = 2$ and $a = \frac{1}{2}$. Therefore, the function $f(x) = \frac{x^2}{2}$ satisfies the condition $f'(x) = x$. However, the question arises whether there are other functions satisfying it? If f is another function such that $f' = \frac{x^2}{2}$ then $\left(f - \frac{x^2}{2}\right)' = 0$ on $\mathbb{R}$. This implies by Theorem 4.10 that $f - \frac{x^2}{2} = C$ for some constant C and, hence, $f = \frac{x^2}{2} + C$. This is the most general function that satisfies the condition $f' = x$.

Theorem 4.11 (The monotonicity test) *Let f be a continuous function on an interval I differentiable on I_0 where I_0 is the open interval with the same endpoints as I. If $f'(x) \geq 0$ for all $x \in I_0$ then f is a monotone increasing function on I. Furthermore, if $f'(x) > 0$ for all $x \in I_0$ then f is strictly increasing on I.*

Similarly, if $f'(x) \leq 0$ for all $x \in I_0$ then f is a monotone decreasing function on I, and if $f'(x) < 0$ for all $x \in I_0$ then f is strictly decreasing on I.

Proof. Consider two points $a < b$ on I. Applying Theorem 4.9, we obtain that there is $c \in (a, b) \subset I_0$ such that

$$f(b) - f(a) = f'(c)(b - a).$$

Hence, if $f' \geq 0$ then it follows that $f(b) \geq f(a)$, that is, f is monotone increasing. The other claims are proved in the same way. ■

Example. Consider function $\sin x$ on $I = [0, \pi/2]$. By the definition of $\pi/2$, we know that $\cos x > 0$ for $x \in I_0 = (0, \pi/2)$. Since $(\sin x)' = \cos x$, we conclude that $\sin x$ is strictly increasing on $[0, \pi/2]$.

Theorem 4.11 can be used to prove inequalities as follows.

Corollary. (The comparison test)

(a) *Let f and g be two continuous functions on an interval $[a, b)$, $a < b$, that are differentiable in (a, b). Assume that*

1. $f(a) \le g(a)$
2. $f'(x) \le g'(x)$ *for all* $x \in (a, b)$.

Then $f(x) \le g(x)$ for all $x \in (a, b)$. Moreover, if $f'(x) < g'(x)$ in (a, b) then $f(x) < g(x)$ in (a, b).

(b) *Let f and g be two continuous functions on an interval $(a, b]$, $a < b$, that are differentiable in (a, b). Assume that*

1. $f(b) \le g(b)$
2. $f'(x) \ge g'(x)$ *for all* $x \in (a, b)$.
   Then $f(x) \le g(x)$ for all $x \in (a, b)$. Moreover, if $f'(x) > g'(x)$ in (a, b) then $f(x) < g(x)$ in (a, b).

Proof. (a) Set $h = g - f$ and notice that $h(a) \ge 0$ and $h' \ge 0$ in (a, b). By Theorem 4.11, h is monotone increasing $[a, b)$, whence it follows that, for any $x \in (a, b)$,

$$h(x) \ge h(a) \ge 0.$$

Hence, $f(x) \le g(x)$ for all $x \in (a, b)$. If the case of the strict inequality $f'(x) < g'(x)$, we obtain that h is strictly increasing, whence $h(x) > 0$ and $f(x) < g(x)$.

(b) In this case the function h is monotone decreasing and $h(b) \ge 0$, whence

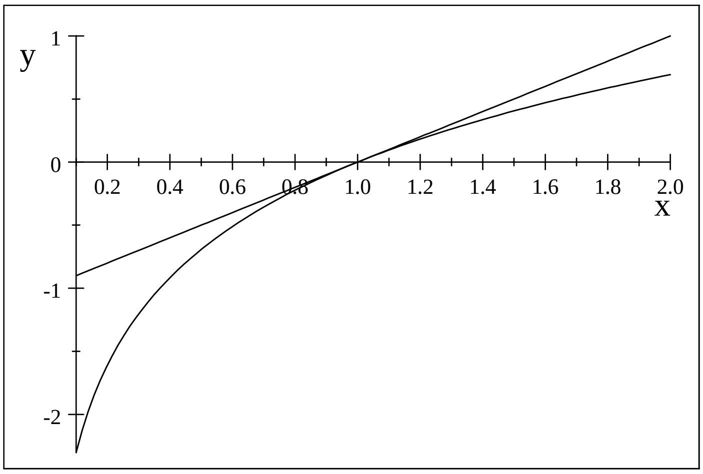
$$h(x) \ge h(b) \ge 0$$

for any $x \in (a, b)$, which implies $f(x) \le g(x)$. In the case of strict inequality $f'(x) < g'(x)$, h is strictly decreasing, whence $h(x) > 0$ and $f(x) < g(x)$. ■

Example. Let us prove that, for all $x > 0$,

$$\ln x \le x - 1. \tag{4.18}$$

See the graphs of these two functions in the next diagram:



If $x = 1$ then the both sides of (4.18) are equal to 0. If $x > 1$ then

$$(\ln x)' = \frac{1}{x} < 1 = (x-1)'.$$

Applying the comparison test (a) in the interval $[1, +\infty)$, we conclude that $\ln x < x - 1$ for all $x > 1$.

If $0 < x < 1$ then

$$(\ln x)' = \frac{1}{x} > 1 = (x-1)'.$$

Applying the comparison test (b) in the interval $(0, 1]$, we obtain $\ln x < x - 1$ in $(0, 1)$. Hence, (4.18) holds for all $x > 0$ and, moreover, the equality in (4.18) is attained only at $x = 1$.

Another consequence of Theorem 4.11 is the following result.

Theorem 4.12 (The inverse function theorem) *Let f be a differentiable function on an interval I such that $f'(x) > 0$ for all $x \in I$ (or $f'(x) < 0$ for all $x \in I$). Then the inverse function $f^{-1}(x)$ exists on the interval $J = f(I)$, is differentiable in J and, for any $y \in J$,*

$$\left(f^{-1}\right)'(y) = \frac{1}{f'(x)}, \tag{4.19}$$

where $x = f^{-1}(y)$ (or x is determined by the condition $y = f(x)$).

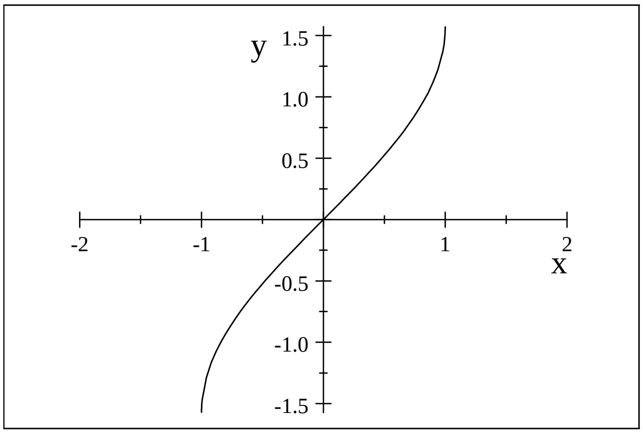
Proof. By Theorem 4.11, f' is strictly monotone on I. Then, by Theorem 3.9, the inverse function f^{-1} exists on J. By Theorem 4.6, f^{-1} is differentiable and its derivative satisfies (4.19). ■

Example. Let $f(x) = \sin x$ on $I = (-\pi/2, \pi/2)$. Since $f'(x) = \cos x$ and $\cos x > 0$ on this interval, we conclude that the inverse function exists on $J = (-1, 1)$ and is differentiable. The inverse function of $\sin x$ is denoted by $\arcsin y$. It follows from (4.14) that

$$(\arcsin y)' = \frac{1}{(\sin x)'} = \frac{1}{\cos x} = \frac{1}{\sqrt{1-y^2}}.$$

In fact, the domain of $\arcsin y$ is $[-1, 1]$ but this function is not differentiable at $y = 1, -1$ (See Exercise 59 for more details).

Here is the graph of $\arcsin x$:



4.4 Higher order derivatives

For a function f defined on an interval I, the derivative $f^{(n)}$ of the order $n \in \mathbb{N}$ (or the n-th derivative) is defined inductively by the following two conditions:

$$f^{(1)} = f' \text{ and } f^{(n)} = \left(f^{(n-1)}\right)' \text{ for any } n > 1,$$

assuming that the above derivatives exist on I. In particular, we have the second derivative

$$f^{(2)} = f'' = (f')',$$

the third derivative

$$f^{(3)} = f''' = (f'')'$$

the forth derivative

$$f^{(4)} = f^{IV} = (f''')',$$

etc.

If $f^{(n)}$ exists on I then we say that function f is n times differentiable on I. Clearly, this also means that $f^{(k)}$ exists on I for all $k \leq n$. We say that f is differentiable ∞ many times if $f^{(n)}$ exists for all $n \in \mathbb{N}$.

Example.

1. Let $f = \exp(x)$. Then $f' = \exp(x)$ and we obtain by induction that

$$(\exp(x))^{(n)} = \exp(x)$$

for any $n \in \mathbb{N}$. In particular, function $\exp(x)$ is differentiable ∞ many times.

2. Let $f = \sin x$. Then

$$f' = \cos x,\ f'' = -\sin x,\ f''' = -\cos x,\ f^{IV} = \sin x.$$

Hence, $f^{(n)}$ repeats periodically as follows:

$$(\sin x)^{(n)} = \begin{cases} \sin x, & n = 4k, \\ \cos x, & n = 4k+1, \\ -\sin x, & n = 4k+2, \\ -\cos x, & n = 4k+3. \end{cases}$$

3. Let $f = x^a$, where $a \in \mathbb{R}$ and $x > 0$. Then

$$f' = ax^{a-1}, \quad f'' = a\left(a-1\right)x^{a-2},$$

etc. By induction, we obtain

$$\left(x^a\right)^{(n)} = a\left(a-1\right)...\left(a-n+1\right)x^{a-n}.$$

4. Let $f = x^k$ where $k \in \mathbb{N}$ and $x \in \mathbb{R}$. Similarly to the previous example, we have

$$\left(x^k\right)^{(n)} = k\left(k-1\right)...\left(k-n+1\right)x^{k-n}.$$

In particular, for $n = k$, we obtain

$$\left(x^k\right)^{(k)} = k! = \text{const}\,.$$

It follows that $\left(x^k\right)^{(k+1)} \equiv 0$ and, moreover, $\left(x^k\right)^{(n)} \equiv 0$ for all $n > k$.

4.4.1 Taylor's formula

Consider a polynomial function

$$f\left(x\right) = c_0 + c_1 x + ... + c_n x^n = \sum_{k=0}^{n} c_k x^k, \tag{4.20}$$

where n is a non-negative integer and all $c_k \in \mathbb{R}$. If $c_n \neq 0$ then the number n is called the *degree* of the polynomial f and is denoted by $\deg f$. If $c_n = 0$ then removing the vanishing term $c_n x^n$ and possibly other terms with $c_k = 0$, we still can represent f in the form (4.20) with a smaller value n such that $c_n \neq 0$. Hence, the degree of the polynomial (4.20) is in general $\leq n$.

We have seen above that $\left(x^k\right)^{(m)} = 0$ whenever $m > k$. Hence, if $m > \deg f$ then we have $f^{(m)} \equiv 0$. It turns out that if $\deg f \leq n$ then f can be recovered by the following values:

$$f\left(a\right),\ f'\left(a\right),\ f''\left(a\right),\ ...,\ f^{(n)}\left(a\right)$$

at some point $a \in \mathbb{R}$.

Lemma 4.13 *For any polynomial f of degree at most n, we have, for all $a, x \in \mathbb{R}$,*

$$f\left(x\right) = f\left(a\right) + \frac{f'\left(a\right)}{1!}\left(x-a\right) + \frac{f''\left(a\right)}{2!}\left(x-a\right)^2 + ... + \frac{f^{(n)}\left(a\right)}{n!}\left(x-a\right)^n. \tag{4.21}$$

Proof. Inductive basis for $n = 0$. If $\deg f = 0$ then $f = \text{const}$, and for this case the identity (4.21) becomes $f\left(x\right) = f\left(a\right)$, which is obviously true.

Inductive step from $n-1$ to n. If $\deg f \leq n$ then $\deg f' \leq n-1$. Hence, by the inductive hypothesis, we obtain the identity

$$f'\left(x\right) = f'\left(a\right) + \frac{f''\left(a\right)}{1!}\left(x-a\right) + \frac{f'''\left(a\right)}{2!}\left(x-a\right)^2 + ... + \frac{f^{(n)}\left(a\right)}{\left(n-1\right)!}\left(x-a\right)^{n-1}. \tag{4.22}$$

Denote by g the right hand side of (4.21). Clearly, we have

$$g'(x) = f'(a) + \frac{f''(a)}{1!}(x-a) + \frac{f'''(a)}{2!}(x-a)^2 + ... + \frac{f^{(n)}(a)}{(n-1)!}(x-a)^{n-1},$$

which together with (4.22) yields the identity $f'(x) = g'(x)$. Hence, $(f-g)' \equiv 0$, whence, by the constant test (Theorem 4.10), $f - g = \text{const}$. Since by (4.21) $g(a) = f(a)$, we conclude that this const is 0, whence $f(x) = g(x)$, which was to be proved. ∎

Denoting $x - a = b$, we can rewrite (4.21) as follows:

$$f(a+b) = f(a) + \frac{f'(a)}{1!}b + \frac{f''(a)}{2!}b^2 + ... + \frac{f^{(n)}(a)}{n!}b^n.$$

In particular, if $f(x) = x^n$, we obtain the binomial formula:

$$(a+b)^n = a^n + \frac{na^{n-1}b}{1!} + \frac{n(n-1)a^{n-2}b^2}{2!} + ... + \frac{n!}{n!}b^n = \sum_{l=0}^{n}\binom{n}{l}a^{n-l}b^l.$$

Theorem 4.14 (Taylor's formula) *Let $f(x)$ be a function on an open interval I such that f is differentiable n times on I. Then, for any $a \in I$*

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + ... + \frac{f^{(n)}(a)}{n!}(x-a)^n + o((x-a)^n) \quad \text{as } x \to a. \tag{4.23}$$

Conversely, if for some real $c_0, c_1, ..., c_n$

$$f(x) = c_0 + c_1(x-a) + ... + c_n(x-a)^n + o((x-a)^n) \quad \text{as } x \to a \tag{4.24}$$

then

$$c_k = \frac{f^{(k)}(a)}{k!}.$$

The meaning of the formula (4.23) is that when x is close to a then $f(x)$ can be approximated by the polynomial

$$T_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + ... + \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + \sum_{k=1}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k,$$

which is called a *Taylor polynomial* of f. In the case when f is a polynomial of the degree at most n, we have by Lemma 4.13 an *exact* identity $f(x) = T_n(x)$.

Proof. Using the notation $T_n(x)$, we need to show that

$$f(x) = T_n(x) + o((x-a)^n) \text{ as } x \to a. \tag{4.25}$$

Let us prove this by induction in n.

Inductive basis. If $n = 1$ then (4.25) becomes

$$f(x) = f(a) + f'(a)(x-a) + o(x-a),$$

which is true by Lemma 4.1.

Inductive step from $n-1$ to n. Note that

$$\begin{aligned} T_n'(x) &= \sum_{k=1}^{n} \frac{f^{(k)}(a)}{k!} k (x-a)^{k-1} = \sum_{k=1}^{n} \frac{f^{(k)}(a)}{(k-1)!} (x-a)^{k-1} \\ &= f'(a) + \sum_{l=1}^{n-1} \frac{(f')^{(l)}(a)}{l!} (x-a)^{l}, \end{aligned}$$

where have changed $l = k-1$. We see from this identity that $T_n'(x)$ is the Taylor polynomial of the order $n-1$ for the function f' . Therefore, by the inductive hypothesis,

$$f'(x) = T_n'(x) + o\left((x-a)^{n-1}\right) \text{ as } x \to a.$$

Set $h = f - T_n$ so that the above relation becomes

$$\lim_{x \to a} \frac{h'(x)}{(x-a)^{n-1}} = 0. \tag{4.26}$$

Applying Theorem 4.9 to function h and noticing that $h(a) = 0$ we obtain, that. for any $x \neq a$, there is $y \in (a, x)$ such that

$$h(x) = h'(y)(x-a).$$

Note that y depends on x so that we can consider y as a function $y(x)$. Also, the condition $y(x) \in (a, x)$ implies that

$$|y-a| \leq |x-a|.$$

Using the above relations, we obtain

$$\left| \frac{h(x)}{(x-a)^n} \right| = \left| \frac{h'(y)}{(x-a)^{n-1}} \right| \leq \left| \frac{h'(y)}{(y-a)^{n-1}} \right|.$$

Since

$$\lim_{x \to a} y(x) = a$$

and, by (4.26),

$$\lim_{x \to a} \left| \frac{h'(y)}{(y-a)^{n-1}} \right| = \lim_{y \to a} \left| \frac{h'(y)}{(y-a)^{n-1}} \right| = 0,$$

we conclude that also

$$\lim_{x \to a} \left| \frac{h(x)}{(x-a)^n} \right| = 0,$$

whence (4.25) follows.

For the second claim, observe that (4.23) and (4.24) imply

$$c_0 + c_1(x-a) + \ldots + c_n (x-a)^n = f(a) + \frac{f'(a)}{1!}(x-a) + \ldots + \frac{f^{(n)}}{n!}(x-a)^n + o\left((x-a)^n\right) \tag{4.27}$$

as $x \to a$. Taking limits of the both sides as $x \to a$, we obtain

$$c_0 = f(a).$$

Subtracting c_0 from the both sides of (4.27) and dividing by $x-a$, we obtain

$$c_1+c_2(x-a)+...+c_n(x-a)^{n-1}=\frac{f'(a)}{1!}+\frac{f''(a)}{2!}(x-a)+...+\frac{f^{(n)}}{n!}(x-a)^{n-1}+o\left((x-a)^{n-1}\right)$$

as $x\to a$. Taking again the limit as $x\to a$, we obtain

$$c_1=\frac{f'(a)}{1!}.$$

Subtracting c_1, dividing by $x-a$, and taking limit as $x\to a$, we obtain

$$c_2=\frac{f''(a)}{2!},$$

etc. ∎

Example. Let $f(x)=\exp(x)$. Since $f^{(n)}(a)=\exp(a)$ for any n, we obtain from (4.23)

$$\exp(x)=\exp(a)\left(1+\frac{(x-a)}{1!}+\frac{(x-a)^2}{2!}+...+\frac{(x-a)^n}{n!}+o((x-a)^n)\right),$$

or, dividing by $\exp(a)$,

$$\exp(x-a)=1+\frac{(x-a)}{1!}+\frac{(x-a)^2}{2!}+...+\frac{(x-a)^n}{n!}+o((x-a)^n).$$

We see that the Taylor polynomial here coincides with the partial sum of the exponential series so that the above formula can be obtained directly from the definition of exp.

Example. Let $f(x)=x^p$, where $x>0$ and $p\in\mathbb{R}$. Then setting $b=x-a$, we obtain from (4.23)

$$(a+b)^p=a^p+\frac{p}{1!}a^{p-1}b+\frac{p(p-1)}{2!}a^{p-2}b^2+...+\frac{p(p-1)...(p-n+1)}{n!}a^{p-n}b^n+o(b^n)$$

as $b\to 0$. Extending the notation for the binomial coefficient $\binom{p}{n}$ to any $p\in\mathbb{R}$ by

$$\binom{p}{n}=\frac{p(p-1)...(p-n+1)}{n!},$$

we obtain an analogue of the binomial formula for the case when the exponent p is an arbitrary real number:

$$(a+b)^p=a^p+\binom{p}{1}a^{p-1}b+\binom{p}{2}a^{p-2}b^2+...+\binom{p}{n}a^{p-n}b^n+o(b^n), \tag{4.28}$$

as $b\to 0$. For example, if $p=\frac{1}{2}$ then we obtain for $n=1$

$$\sqrt{a+b}=\sqrt{a}+\frac{1}{2}\frac{b}{\sqrt{a}}+o(b) \tag{4.29}$$

and for $n=2$

$$\sqrt{a+b}=\sqrt{a}+\frac{1}{2}\frac{b}{\sqrt{a}}-\frac{1}{8}\frac{b^2}{(\sqrt{a})^3}+o(b^2). \tag{4.30}$$

There formulas can be used for approximate evaluation of $\sqrt{a+b}$ if $\sqrt{a}$ is known and b is small compared to a. For example, (4.29) gives

$$\sqrt{26} = \sqrt{25+1} \approx 5 + \frac{1}{2}\frac{1}{5} = 5,1,$$

while by (4.30)

$$\sqrt{26} \approx 5 + \frac{1}{2}\frac{1}{5} - \frac{1}{8}\frac{1}{125} = 5,099$$

For comparison, note that

$$\sqrt{26} = 5,099019\,51359278... \,.$$

Theorem 4.14 (Taylor's formula) can be stated as follows: if $f(x)$ is a function on an open interval I such that f is differentiable n times on I, then, for any $a \in I$,

$$f(x) = T_n(x) + o\left((x-a)^n\right) \quad \text{as } x \to a, \tag{4.31}$$

where $T_n(x)$ is the Taylor polynomial of the function f of order n at point a, defined by

$$T_n(x) = c_0 + c_1(x-a) + ... + c_n(x-a)^n \tag{4.32}$$

and

$$c_k = \frac{f^{(k)}(a)}{k!} \tag{4.33}$$

(using the conventions $f^{(0)} = f$ and $0! = 1$). Furthermore, $T_n(x)$ is the only polynomial of degree at most n that satisfies (4.31).

Example. Let us find the Taylor polynomials $T_n(x)$ for $f(x) = \arcsin x$ at $a = 0$. We already know that this function is differentiable in $(-1,1)$ and

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}.$$

To find higher order derivatives of $\arcsin x$, we need to differentiate the function $g(x) = (1-x^2)^{-1/2}$ (obviously, this function is differentiable ∞ many times). However, since we need the derivatives only at 0, it is easier to find them using (4.28) with $a = 1$ and $b = -x^2$:

$$g(x) = \left(1-x^2\right)^{-1/2} = 1 + \frac{1}{2}x^2 + \frac{1}{2}\frac{3}{2}\frac{x^4}{2!} + \frac{1}{2}\frac{3}{2}\frac{5}{2}\frac{x^6}{3!} + ... + \frac{1}{2}\frac{3}{2}...\frac{2n-1}{2}\frac{x^{2n}}{n!} + o\left(x^{2n}\right).$$

Using the second part of Theorem 4.14, we conclude by (4.33)

$$g^{(2k)}(0) = \frac{1}{2}\frac{3}{2}...\frac{2k-1}{2}\frac{(2k)!}{k!} = \frac{(2k-1)!!\,(2k)!}{2^k k!},$$

where $(2k-1)!! = 1 \cdot 3 \cdot ... \cdot (2k-1)$, and

$$g^{(2k+1)}(0) = 0.$$

Therefore,

$$\begin{aligned} f^{(2k+1)}(0) &= g^{(2k)}(0) = \frac{(2k-1)!!\,(2k)!}{2^k k!} \\ f^{(2k)}(0) &= g^{(2k-1)}(0) = 0. \end{aligned}$$

By (4.33), the Taylor coefficients for $\arcsin x$ are given by $c_{2k} = 0$ and

$$c_{2k+1} = \frac{f^{(2k+1)}(0)}{(2k+1)!} = \frac{(2k-1)!!}{(2k+1)\,2^k k!},$$

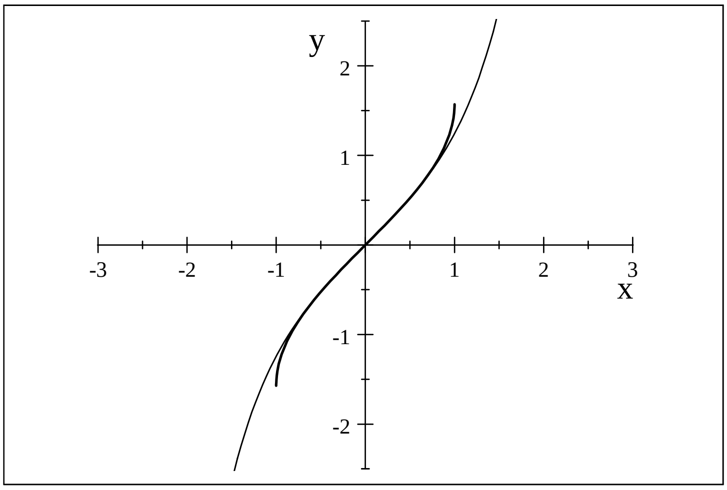
and the Taylor polynomial for $\arcsin x$ is

$$T_{2n+1}(x) = x + \frac{1}{3 \cdot 2 \cdot 1!}x^3 + \frac{1 \cdot 3}{5 \cdot 2^2 \cdot 2!}x^5 + ... + \frac{(2n-1)!!}{(2n+1)\,2^n n!}x^{2n+1}.$$

For example, for $n = 2$ we obtain

$$T_5(x) = x + \frac{1}{2 \cdot 3}x^3 + \frac{1 \cdot 3}{4 \cdot 2 \cdot 5}x^5 = x + \frac{1}{6}x^3 + \frac{3}{40}x^5.$$

The graphs of the function $\arcsin x$ (thick) and its Taylor polynomial $T_5(x)$ are plotted on the next diagram:



We see that $T_5(x)$ provides a good approximation for $\arcsin x$ away from $x \approx 1$. In general, higher order Taylor polynomial provide better approximation for the function.

The Taylor polynomials are widely used in numerical computations for evaluating various functions. However, the formula (4.31) does not give an error estimate for this computation. For that, one needs a more explicit estimate of the difference $f(x) - T_n(x)$, which will be done below. We need first the following result.

Theorem 4.15 *(*Mean-Value Theorem of Cauchy) *Let f, g be continuous functions on an interval $[a, b]$, $a < b$, differentiable on (a, b). Then there is a point $c \in (a, b)$ such that*

$$g'(c)(f(b) - f(a)) = f'(c)(g(b) - g(a)). \tag{4.34}$$

Note that Theorem 4.9 (Lagrange's Mean-Value Theorem) is a particular case of Theorem 4.15 for $g(x) = x$ since in this case (4.34) becomes

$$f(b) - f(a) = f'(c)(b-a).$$

Proof. Consider function

$$h(x) = f(x)(g(b) - g(a)) - g(x)(f(b) - f(a)).$$

Obviously,

$$h(b) - h(a) = (f(b) - f(a))(g(b) - g(a)) - (g(b) - g(a))(f(b) - f(a)) = 0$$

so that $h(a) = h(b)$. By Theorem 4.8 (Rolle's Theorem) there is $c \in (a, b)$ such that $h'(c) = 0$. Since

$$h'(x) = f'(x)(g(b) - g(a)) - g'(x)(f(b) - f(a)),$$

the claim follows. ■

Now we can prove the main theorem in this section.

Theorem 4.16 (Taylor's formula with the remainder term in the Lagrange form) *Let $f(x)$ be a function on an open interval I such that f is differentiable $n+1$ times on I (where $n \geq 0$) Then for all distinct $a, x \in I$, there exists $c \in (a, x)$ such that*

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + ... + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{n+1}(c)}{(n+1)!}(x-a)^{n+1}. \tag{4.35}$$

If $n = 0$ then (4.35) becomes

$$f(x) = f(a) + f'(c)(x-a),$$

which coincides with Theorem 4.9 (Lagrange's Mean Value Theorem).

The difference $f(x) - T_n(x)$ is called the *remainder term.* One can rewrite (4.35) as follows

$$f(x) - T_n(x) = \frac{f^{n+1}(c)}{(n+1)!}(x-a)^{n+1}, \tag{4.36}$$

and this identity is called the *Lagrange form* of the remainder term. Theorem 4.14 gives another expression for the remainder term:

$$f(x) - T_n(x) = o((x-a)^n) \quad \text{as } x \to a, \tag{4.37}$$

which is called the *Peano form* of the remainder term.

If $f^{(n+1)}$ is bounded in a neighborhood of a then (4.36) implies (4.37) because

$$\left|\frac{f^{n+1}(c)}{(n+1)!}(x-a)^{n+1}\right| \leq C|x-a|^{n+1} = o((x-a)^n).$$

However, in general Theorem 4.14 does not imply Theorem 4.16 because in Theorem 4.16 function f must be $n+1$ times differentiable while in Theorem 4.14 – only n times.

Proof. Consider an auxiliary function

$$F(t) = f(x) - \left(f(t) + \frac{f'(t)}{1!}(x-t) + \frac{f''(t)}{2!}(x-t)^2 + ... + \frac{f^{(n)}(t)}{n!}(x-t)^n \right)$$

where the expression in the brackets is the Taylor polynomial of f at the point t (instead of the usual a). We consider now x to be fixed while t varies in $[a, x]$. To find $F'(t)$, evaluate first the derivative of each term separately using the product rule:

$$\begin{aligned} \left(\frac{f^{(k)}(t)}{k!}(x-t)^k \right)' &= -\frac{f^{(k)}(t)}{k!}k(x-t)^{k-1} + \frac{f^{(k+1)}(t)}{k!}(x-t)^k \\ &= -\frac{f^{(k)}(t)}{(k-1)!}(x-t)^{k-1} + \frac{f^{(k+1)}(t)}{k!}(x-t)^k. \end{aligned}$$

Therefore,

$$F'(t) = -f'(t) + \frac{f'(t)}{0!} - \frac{f''(t)}{1!}(x-t) + \frac{f''(t)}{1!}(x-t) - \frac{f'''(t)}{2!}(x-t)^2 - ... - \frac{f^{(n+1)}(t)}{n!}(x-t)^n$$

We see that all the terms cancel out except for the last one, that is

$$F'(t) = -\frac{f^{(n+1)}(t)}{n!}(x-t)^n. \tag{4.38}$$

Now apply Theorem 4.15 with functions $F(t)$ and

$$G(t) = (x-t)^{n+1}$$

on the interval $[a, x]$. We obtain that there is $c \in (a, x)$ such that

$$G'(c)(F(x) - F(a)) = F'(c)(G(x) - G(a)).$$

Observing that

$$\begin{gathered} G(x) = 0, \ \ G(a) = (x-a)^{n+1}, \\ F(x) = 0, \ \ F(a) = f(x) - T_n(x) \\ G'(c) = -(n+1)(x-c)^n \end{gathered}$$

and, by (4.38),

$$F'(c) = -\frac{f^{(n+1)}(c)}{n!}(x-c)^n,$$

we obtain

$$(n+1)(x-c)^n (f(x) - T_n(x)) = \frac{f^{(n+1)}(c)}{n!}(x-c)^n (x-a)^{n+1}.$$

Dividing by $(x-c)^n$ (which does not vanish by $c \in (a, x)$), we obtain (4.35). ■

Example. Consider function $f(x) = \sin x$ and its Taylor polynomials at 0

$$T_3(x) = T_4(x) = x - \frac{x^3}{3!}$$

(this follows either from definition (4.32)-(4.33) of $T_n(x)$ or from the power series (3.12) for $\sin x$ – indeed, $T_n(x)$ is just the n-th partial sum of this series). By Theorem 4.16, we have

$$\sin x - T_4(x) = \frac{f^V(c)}{5!}x^5$$

for some $c \in (0, x)$. Since $f^V(c) = \cos c$ and $|\cos c| \leq 1$, we obtain the following estimate

$$|\sin x - T_4(x)| \leq \frac{x^5}{120},$$

which provides the upper bound for the error when approximating $\sin x$ by $T_4(x)$. For example, setting $x = 0.1$, we obtain

$$\sin 0.1 \approx T_4(0.1) = 0.1 + \frac{0.001}{6} = 0.0998333...$$

and the error of this approximation does not exceed

$$\frac{0.1^5}{120} = 8.333... \times 10^{-8} < 10^{-7}.$$

Hence, in the approximate identity

$$\sin 0.1 \approx 0.0998333...$$

holds with correct 6 decimal places after the point.

4.4.2 Convex functions

Definition. Let f be function defined on an interval $I \subset \mathbb{R}$. Function f is called *convex* if, for all $a, b \in I$ and $t \in (0, 1)$

$$f((1-t)a + tb) \leq (1-t)f(a) + tf(b). \tag{4.39}$$

Function f is called strictly convex if a strict inequality takes place in (4.39).

Function f is called *concave* if, for all $a, b \in I$ and $t \in (0, 1)$,

$$f((1-t)a + tb) \geq (1-t)f(a) + tf(b). \tag{4.40}$$

Function f is called strictly concave if a strict inequality takes place in (4.40).

This definition has the following geometric meaning. Let

$$y = A(x-a) + f(a)$$

be the equation of the secant line through points $(a, f(a))$ and $(b, f(b))$ where

$$A = \frac{f(b) - f(a)}{b - a}$$

is the slope of the line. Let us restrict the variable x to the interval (a, b) and set

$$t = \frac{x-a}{b-a}$$

so that $t \in (0,1)$ and

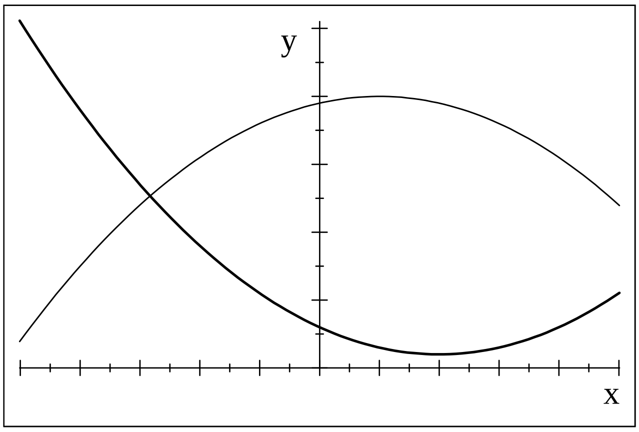
$$x = (b-a)\,t + a = (1-t)\,a + bt$$

With this new parameter, the equation of the secant line is

$$y = (f(b) - f(a))\,t + f(a) = (1-t)\,f(a) + tf(b)\,.$$

Hence, the inequality (4.39) means $f(x) \leq y$ for all $x \in (a,b)$, that is, the graph of the function $f(x)$ on the interval (a,b) lies below the secant line (note that these two line do intersect at the endpoints a and b).

In the same way, function is concave if its graph between any two points lies above the secant line through these points. Typical graphs of convex and concave functions are shown on the next diagram (convex is thick):



Theorem 4.17 (The convexity/concavity test) *Let f be a twice differentiable function on a open interval $I \subset \mathbb{R}$.*

(a) *If $f'' \geq 0$ on I then f is convex on I.*

(b) *If $f'' \leq 0$ in I then f is concave on I.*

Proof. (a) Fix some points $a, b \in I$, $a < b$ and $t \in (0,1)$. Denoting $x = (1-t)\,a + tb$, let us rewrite the definition (4.39) of the convexity as follows

$$\begin{aligned} (1-t+t)\,f(x) &\leq (1-t)\,f(a) + tf(b) \\ (1-t)\,(f(x) - f(a)) &\leq t\,(f(b) - f(x)) \end{aligned}$$

or, using the identity $t = \frac{x-a}{b-a}$,

$$\frac{b-x}{b-a}\,(f(x) - f(a)) \leq \frac{x-a}{b-a}\,(f(b) - f(x))$$

and finally

$$\frac{f(x) - f(a)}{x-a} \leq \frac{f(b) - f(x)}{b-x}. \tag{4.41}$$

Conversely, if (4.41) holds for all $a < x < b$ in I then arguing backwards we obtain (4.39). Hence, it suffices to prove (4.41) for all such triples a, x, b.

By Theorem 4.9 (Mean Value Theorem of Lagrange), we have

$$\frac{f(x) - f(a)}{x - a} = f'(c) \quad \text{and} \quad \frac{f(b) - f(x)}{b - x} = f'(d)$$

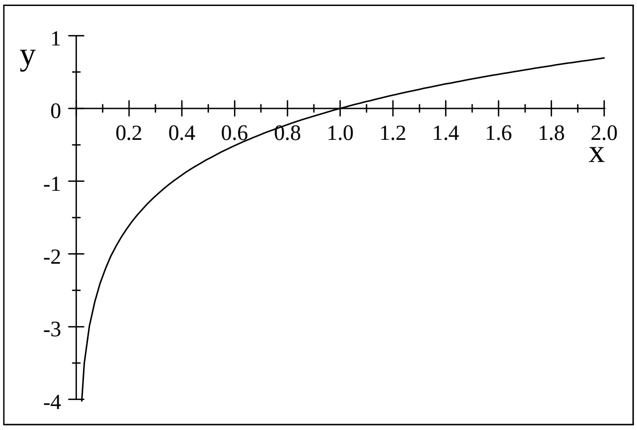
where $c \in (a, x)$ and $d \in (x, b)$. In particular, we have $c < d$. Since $(f')' = f'' \geq 0$, the function f' is monotone increasing by the monotonicity test (Theorem 4.11). Therefore, $f'(c) \leq f'(d)$, whence (4.41) follows.

(b) This part is proved similarly. ■

Example. Consider the function $f(x) = \ln x$, $x > 0$. Since

$$(\ln x)'' = \left(\frac{1}{x}\right)' = -\frac{1}{x^2} < 0,$$

we obtain by Theorem 4.17 that $\ln x$ is concave. The graph of this function is as follows:



Using the definition of the concavity, we have the following inequality

$$\ln((1-t)a + tb) \geq (1-t)\ln a + t \ln b \tag{4.42}$$

for all $a, b > 0$ and $t \in (0, 1)$. Denote $p = \frac{1}{1-t}$ and $q = \frac{1}{t}$ so that $p, q > 1$ and

$$\frac{1}{p} + \frac{1}{q} = 1. \tag{4.43}$$

Then (4.42) implies

$$\ln\left(\frac{a}{p} + \frac{b}{q}\right) \geq \frac{1}{p}\ln a + \frac{1}{q}\ln b = \ln\left(a^{1/p}b^{1/q}\right). \tag{4.44}$$

Setting $x = a^{1/p}$, $y = b^{1/q}$ and applying exp to (4.44), we obtain the *Young inequality*

$$\frac{x^p}{p} + \frac{y^q}{q} \geq xy,$$

which is true for all non-negative x, y and $p, q > 1$ such that (4.43) holds.

A particular case of the Young inequality with $p = q = 2$ is the following familiar inequality:

$$\frac{x^2 + y^2}{2} \geq xy.$$

4.4.3 Local extrema

Definition. Let f be a function defined on an open interval I and $a \in I$. One says that f has at a a *local maximum* if there is a neighborhood $U(a)$ of a such that $f(a)$ is the maximal value of f in $U(a) \cap I$, that is

$$f(a) = \max_{U(a)\cap I} f(x) \ .$$

Similarly one defined a local minimum of f.

One says that f has a local *extremum* at a if f has at a either a local maximum or a local minimum.

Theorem 4.18 (a) (Necessary condition for a local extremum) *Let f be a differentiable function on an open interval. If f has a local extremum at a point $a \in I$ then $f'(a) = 0$.*

(b) (Sufficient condition for a local extremum) *Let f be twice differentiable on an open interval I and let$f'(a) = 0$ for some $a \in I$. If $f''(a) > 0$ then f has a local minimum at a. If $f''(a) < 0$ then f has a local maximum at a.*

Proof. (a) If f has at a a local maximum then f takes at a the maximum value in $U(a) \cap I$. Hence, by Theorem 4.7, $f'(a) = 0$. The same applies to the case of a local minimum.

(b) By the Taylor formula with the remainder term in the Peano form (Theorem 4.14), we have

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + R(x), \tag{4.45}$$

where $R(x) = o\left((x-a)^2\right)$ as $x \to a$, that is,

$$\lim_{x\to a} \frac{R(x)}{(x-a)^2} = 0.$$

If $f''(a) > 0$ then there is a neighborhood $U(a)$ of a such that

$$\left|\frac{R(x)}{(x-a)^2}\right| < \frac{f''(a)}{4} \text{ for all } x \in U(a) \setminus \{a\},$$

which implies that

$$R(x) \geq -\frac{f''(a)}{4}(x-a)^2 \text{ for all } x \in U(a).$$

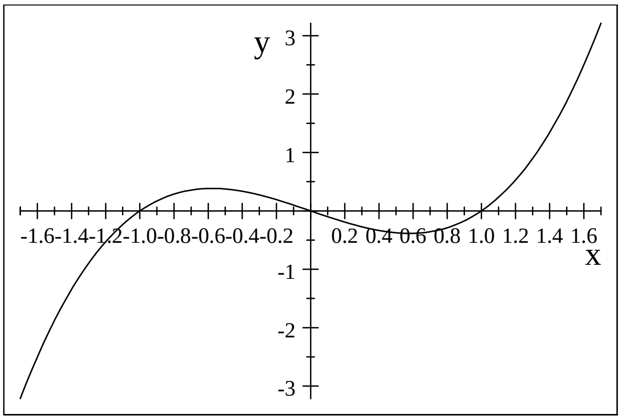
Using also $f'(a) = 0$ we obtain from (4.45) that, for any $x \in U(a)$,

$$f(x) \geq f(a) + \frac{f''(a)}{4}(x-a)^2 \geq f(a),$$

which means that f has at a a local minimum. The case $f''(a) < 0$ is treated similarly. ∎

Example. Consider the function $f(x) = x^3 - x$. Then $f'(x) = 3x^2 - 1$, which has the roots $x = \pm\frac{1}{\sqrt{3}}$. Hence, if f has local extrema then they should be at these two points.

Since $f''(x) = 6x$, we see that $f''(x) > 0$ at $x = \frac{1}{\sqrt{3}}$ and $f''(x) < 0$ at $x = -\frac{1}{\sqrt{3}}$. Hence, f has a local minimum at $\frac{1}{\sqrt{3}}$ and a local maximum at $-\frac{1}{\sqrt{3}}$. The graph of this function is as follows:



4.4.4 l'Hospital's rule

Recall that

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$$

provided both limits in the right hand side exist and their ratio is defined. However, frequently one has to evaluate limits when the ratio on the right hand side is undefined, for example, being of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Such expressions are called *indeterminate forms*. Such limits can be frequently found using the following theorem.

Theorem 4.19 (l'Hospital's rule) *Let f and g be two functions defined and differentiable on an open interval $I \subset \mathbb{R}$, and $a \in \overline{\mathbb{R}}$ be an endpoint of I. Assume that*

(a) either

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0, \tag{4.46}$$

(b) or

$$\lim_{x\to a} g(x) = \pm\infty. \tag{4.47}$$

If

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = b \in \overline{\mathbb{R}}$$

(assuming $g'(x) \neq 0$ on I) then also

$$\lim_{x\to a} \frac{f(x)}{g(x)} = b \tag{4.48}$$

(assuming that $g(x) \neq 0$ on I).

Put simply, the rule is as follows: in order to resolve the indeterminate forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$, replace functions f and g by their derivatives!

Note that in the case (b) typically also the condition

$$\lim_{x \to a} f(x) = \pm\infty \tag{4.49}$$

is satisfied so that one does have an indeterminate form $\frac{\infty}{\infty}$, but (4.49) is not used in the proof and, hence, is not needed for the validity of the statement.

Example. (1) Find $\lim_{x\to 0} \frac{\sin x}{x}$. This is the indeterminate form $\frac{0}{0}$ since both x and $\sin x$ go to 0 as $x \to 0$. Applying Theorem 4.19 in both interval $(0, +\infty)$ and $(-\infty, 0)$, we obtain

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{(\sin x)'}{(x)'} = \lim_{x\to 0} \frac{\cos x}{1} = 1. \tag{4.50}$$

Strictly speaking, the first equality in (4.50) can be justified only after the last equality is obtained. Hence, the rigorous argument runs as follows: since $\lim_{x\to 0} \frac{\sin x}{x}$ is the indeterminate form $\frac{0}{0}$ and since $\lim_{x\to 0} \frac{(\sin x)'}{(x)'}$ exists and is equal to 1, we have by Theorem 4.19 $\lim_{x\to 0} \frac{\sin x}{x} = 1$. The same reasoning applies to all other cases of application of l'Hospital's rule, and will be assumed implicitly.

(2) Note that in order to apply l'Hospital's rule, one must have an indeterminate form. Consider $\lim_{x\to 1} \frac{x^2}{x}$ which is obviously equal to 1. This is not an indeterminate form and l'Hospital's rule is not applicable. Indeed, we see that

$$\lim_{x\to 1} \frac{(x^2)'}{(x)'} = \lim_{x\to 1} \frac{2x}{1} = 2 \neq 1 = \lim_{x\to 1} \frac{x^2}{x}.$$

(3) $\lim_{x\to+\infty} \frac{\exp(x)}{x^2}$ is the indeterminate form $\frac{\infty}{\infty}$ because both $\exp(x)$ and x^2 tend to $+\infty$ as $x \to +\infty$. Write

$$\lim_{x\to+\infty} \frac{\exp(x)}{x^2} = \lim_{x\to+\infty} \frac{(\exp(x))'}{(x^2)'} = \lim_{x\to+\infty} \frac{\exp(x)}{2x}. \tag{4.51}$$

The limit in the right hand side is again an indeterminate form $\frac{\infty}{\infty}$. Differentiating ones again, we write

$$\lim_{x\to+\infty} \frac{\exp(x)}{2x} = \lim_{x\to+\infty} \frac{(\exp(x))'}{(2x)'} = \lim_{x\to+\infty} \frac{\exp(x)}{2} = +\infty. \tag{4.52}$$

Hence, by two applications of l'Hospital's rule, we conclude that

$$\lim_{x\to+\infty} \frac{\exp(x)}{x^2} = +\infty.$$

Applying the same argument, one can prove by induction that

$$\lim_{x\to+\infty} \frac{\exp(x)}{x^n} = +\infty \tag{4.53}$$

for all $n \in \mathbb{N}$.

(4) Find $\lim_{x\to 0} x \ln x$ assuming $x > 0$. Since $x \to 0$ and $\ln x \to -\infty$, we have the indeterminate form $0 \cdot \infty$. To resolve it, represent the limit as follows:

$$\lim_{x\to 0} \frac{\ln x}{1/x}$$

so that it has now the form $\frac{\infty}{\infty}$. By l'Hospital's rule, we obtain

$$\lim_{x\to 0}\frac{\ln x}{1/x}=\lim_{x\to 0}\frac{(\ln x)'}{(1/x)'}=-\lim_{x\to 0}\frac{1/x}{1/x^2}=-\lim_{x\to 0}x=0.$$

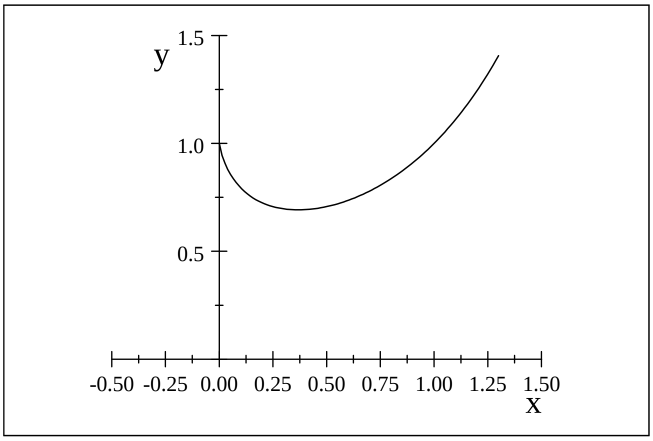
(5) Find $\lim_{x\to 0}x^x$ where $x>0$. This is another kind of an indeterminate form: 0^0. To resolve it, let us take the logarithm of the given function:

$$\lim_{x\to 0}\ln x^x=\lim_{x\to 0}x\ln x=0,$$

by the previous example. Using Theorem 3.3 (the limit of a composite function), we obtain

$$\lim_{x\to 0}x^x=\lim_{x\to 0}\exp\left(\ln x^x\right)=\lim_{y\to 0}\exp\left(y\right)=1.$$

The graph of the function $f(x)=x^x$ is as follows.



(6) Find $\lim_{x\to+1}\frac{\ln x}{x^2-x}$. This is the indeterminate form $\frac{0}{0}$, and we obtain by l'Hospital's rule

$$\lim_{x\to+1}\frac{\ln x}{x^2-x}=\lim_{x\to+1}\frac{(\ln x)'}{(x^2-x)'}=\lim_{x\to+1}\frac{1/x}{2x-1}=1.$$

Proof of Theorem 4.19. (a) For simplicity, let us assume that $b\in\mathbb{R}$ (the case $b=\pm\infty$ is treated similarly). By the definition of the limit, for any $\varepsilon>0$, there is a neighborhood $U(a)$ of a such that

$$\left|\frac{f'(x)}{g'(x)}-b\right|<\varepsilon \text{ for all } x\in U(a)\cap I. \tag{4.54}$$

Let us show that $\frac{f(x)}{g(x)}$ is also closed to b when $x\in U(a)\cap I$. Fix $x\in U(a)\cap I$ and choose some $y\in U(a)\cap I$, $y\neq x$. Applying the Cauchy mean value theorem in the interval $[x,y]$, we obtain that, for some $c\in(x,y)$,

$$f'(c)\left(g(x)-g(y)\right)=g'(c)\left(f(x)-f(y)\right).$$

By hypothesis, $g'(c) \neq 0$. Also, $g(x) \neq g(y)$ by Rolle's theorem (Theorem 4.8) because otherwise the derivative g' would vanish at some point. Hence, we can divide by $g'(c)$ and $g(x) - g(y)$ and obtain

$$\frac{f'(c)}{g'(c)} = \frac{f(x) - f(y)}{g(x) - g(y)} = \frac{\frac{f(x)}{g(x)} - \frac{f(y)}{g(x)}}{1 - \frac{g(y)}{g(x)}},$$

whence

$$\frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)} \left(1 - \frac{g(y)}{g(x)}\right) + \frac{f(y)}{g(x)}$$

and

$$\frac{f(x)}{g(x)} - b = \left(\frac{f'(c)}{g'(c)} - b\right) - \frac{f'(c)}{g'(c)} \frac{g(y)}{g(x)} + \frac{f(y)}{g(x)}. \tag{4.55}$$

Since $c \in U(a) \cap I$, we have by (4.54)

$$\left|\frac{f'(c)}{g'(c)} - b\right| < \varepsilon. \tag{4.56}$$

To make the other terms in the right hand side of (4.55) small enough, we need to choose y so that $\frac{g(y)}{g(x)}$ and $\frac{f(y)}{g(x)}$ are small enough. Since $g(x) \neq 0$, we have by (4.46)

$$\lim_{y \to a} \frac{f(y)}{g(x)} = \lim_{y \to a} \frac{g(y)}{g(x)} = 0.$$

Hence, there is a neighborhood $V(a)$ of a such that

$$\left|\frac{f(y)}{g(x)}\right| < \varepsilon \text{ and } \left|\frac{g(y)}{g(x)}\right| < \varepsilon \text{ for all } y \in V(a) \cap I. \tag{4.57}$$

Hence, choosing some y from $V(a) \cap U(a) \cap I$, we obtain by (4.55), (4.56), and (4.57),

$$\left|\frac{f(x)}{g(x)} - b\right| \leq \varepsilon + (b + \varepsilon)\varepsilon + \varepsilon = (b + 2)\varepsilon + \varepsilon^2. \tag{4.58}$$

Since the right hand side can made smaller than any positive number, we conclude that $\frac{f(x)}{g(x)} \to b$ as $x \to a$.

(b) The proof is similar to (a) but the condition (4.57) is proved differently. First, we chose $U(a)$ to ensure (4.54). Fix some $y \in U(a) \cap I$ and notice that, by (4.47),

$$\lim_{x \to a} \frac{f(y)}{g(x)} = \lim_{x \to a} \frac{g(y)}{g(x)} = 0.$$

Hence, there is a neighborhood $V(a)$ of a such that

$$\left|\frac{f(y)}{g(x)}\right| < \varepsilon \text{ and } \left|\frac{g(y)}{g(x)}\right| < \varepsilon \text{ for all } x \in V(a) \cap I. \tag{4.59}$$

Then, for any $x \in V(a) \cap U(a) \cap I$, we obtain (4.58), which finishes the proof. ■

Waste

Equivalence relation

Let S be an arbitrary set and $\sim$ be a relation on S, that is, for any two elements $x, y \in S$, $x \sim y$ is either true or not ("x is related to y").

Definition. A relation $\sim$ is called an *equivalence relation* if it satisfies the following three conditions:

1. $x \sim x$ for any $x \in S$
2. $x \sim y$ implies $y \sim x$
3. $x \sim y$ and $y \sim z$ imply $x \sim z$

For example, the identity relation $x \sim y$ if $x = y$ satisfies these axioms. On the other hand, if $S = \mathbb{R}$ and $x \sim y$ if $x \leq y$ then the second axiom breaks. Consider another example: $S = \mathbb{R}$ and $x \sim y$ if $|x - y| \leq 1$. Then axioms 1 and 2 are satisfied while 3 does not (for example, $1 \sim 2$ and $2 \sim 3$ while $1 \not\sim 3$).

Proposition 4.20 *If $\sim$ is an equivalence relation on S then S there is a unique family $\mathcal{F}$ of subsets of S such that*

- *each two distinct sets $A, B \in \mathcal{F}$ are disjoint*
- *the union of all the sets from $\mathcal{F}$ is S*
- *$x \sim y$ if and only if x and y belong to the same set from $\mathcal{F}$.*

Shortly, this means that S is split into a disjoint union of subsets such that $x \sim y$ if and only if x and y belong to the same of these subsets. These subsets are called the equivalence classes of the relation $\sim$.

Proof. For any $x \in S$ consider a set A_x of all elements $y \in S$ such that $y \sim x$. Clearly, $x \in A_x$ and, hence, the union of all sets A_x is S. Let us show that, for any two elements $x, y \in S$, either $A_x = A_y$ or $A_x \cap A_y = \emptyset$. Indeed, if $A_x \cap A_y$ is non-empty, say, contains z, then we have $z \sim x$ and $z \sim y$ which implies $x \sim y$. Then for any $t \in A_x$, the relations $t \sim x$ and $x \sim y$ imply $t \sim y$ and, hence, $t \in A_y$. This argument implies $A_x = A_y$. Finally, selecting in the family $\{A_x\}_{x \in S}$ disjoint sets, we obtain the family $\mathcal{F}$ with the required properties. ■

As we see from the proof, the equivalence class containing an element x is A_x, that is, it consists exactly of all elements $y \in S$ such that $y \sim x$.

$\mathbb{N} = \aleph_0$ (aleph – the first letter of the Hebrew alphabet).

p-th means

Consider function $f(x) = x^p$, $x > 0$. Since

$$(x^p)'' = p(p-1)x^{p-2},$$

the function x^p is convex if $p(p-1) \geq 0$, that is, if either $p \leq 0$ or $p \geq 1$, and concave if $p(p-1) \leq 0$, that is, if $0 \leq p \leq 1$. Hence, for all $x, y > 0$ and $t \in (0,1)$

$$\begin{aligned}((1-t)x+ty)^p &\leq (1-t)x^p+ty^p, \text{ if } p \leq 0 \text{ or } p \geq 1, &(4.60)\\ ((1-t)x+ty)^p &\geq (1-t)x^p+ty^p, \text{ if } 0 \leq p \leq 1. &(4.61)\end{aligned}$$

The number

$$M_p = \left(\frac{x^p+y^p}{2}\right)^{1/p}$$

is called the *mean* of the order p of x, y (here we assume $p \neq 0$). For example, if $p = 1$ then

$$M_1 = \frac{x+y}{2}$$

which is the *arithmetic mean* of x, y, and if $p = 2$ then

$$M_2 = \left(\frac{x^2+y^2}{2}\right)^{1/2}$$

is the *quadratic mean* of x, y. It follows from (4.60) and (4.61) that if $p \geq 1$ then

$$M_1 = \frac{x+y}{2} \leq \left(\frac{x^p+y^p}{2}\right)^{1/p} = M_p$$

and if $p \leq 1$ (including $p < 0$) then

$$M_1 \geq M_p.$$

More generally, the following inequality holds: if p, q are non-zero reals such that $p \leq q$ then

$$M_p \leq M_q.$$

Indeed, denoting $X = x^p$ and $Y = y^p$ and $r = \frac{q}{p}$, this inequality amounts to

$$\left(\frac{X+Y}{2}\right)^{1/p} \leq \left(\frac{X^r+Y^r}{2}\right)^{1/q}. \tag{4.62}$$

If $p > 0$ then this is equivalent to

$$\frac{X+Y}{2} \leq \left(\frac{X^r+Y^r}{2}\right)^{1/r},$$

which is true because $r \geq 1$. If $p < 0$ then (4.62) is equivalent to

$$\frac{X+Y}{2} \geq \left(\frac{X^r+Y^r}{2}\right)^{1/r},$$

which is true because $r \leq 1$ (Indeed, if $q > 0$ then $r = \frac{q}{p} < 0$. If $q < 0$ then $p \leq q$ implies $|q| \leq |p|$ and $r = |q| / |p| \leq 1$).

Setting $p = q = 2$ in (4.44), we obtain

$$\ln \left(\frac{x+y}{2} \right) \geq \frac{\ln x + \ln y}{2} = \ln \sqrt{xy},$$

whence it follows that

$$\frac{x+y}{2} \geq \sqrt{xy} \tag{4.63}$$

Of course, this inequality follows also from the identity

$$\frac{x+y}{2} - \sqrt{xy} = \frac{1}{2} \left(\sqrt{x} - \sqrt{y} \right)^2.$$

The expression $\sqrt{xy}$ is called the *geometric mean* of x, y or the mean of the order 0, and is denoted by

$$M_0 = \sqrt{xy}.$$

It follows from (4.63) that $M_0 \leq M_1$, which implies that $M_0 \leq M_p$ for any $p \geq 0$ and $M_0 \geq M_p$ for any $p \leq 0$. Hence, the inequality $M_p \leq M_q$ holds for arbitrary real p, q such that $p \leq q$.

Taylor's formula via l'Hospital's rule

Let us give one more proof of the Taylor formula

$$f(x) = T_n(x) + o\left((x-a)^n \right) \text{ as } x \to a$$

(Theorem 4.14), where f is n times differentiable on an interval I, $a \in I$, and $T_n(x)$ is the Taylor polynomial of f at a of the order n. The proof is by induction in n, and the inductive basis for $n = 1$ is the same as in the main proof.

Inductive step. The derivative $f'(x)$ is $n-1$ times differentiable, and the $(n-1)$-st Taylor polynomial of $f'(x)$ is $T_n'(x)$ where $T_n(x)$ is the n-th Taylor polynomial of f (see the main proof). By the inductive hypothesis,

$$f'(x) - T_n'(x) = o\left((x-a)^{n-1} \right) \text{ as } x \to a,$$

that is,

$$\lim_{x \to a} \frac{f'(x) - T_n'(x)}{(x-a)^{n-1}} = 0.$$

Then, by l'Hospital's rule,

$$\lim_{x \to a} \frac{f(x) - T_n(x)}{(x-a)^n} = \lim_{x \to a} \frac{f'(x) - T_n'(x)}{n(x-a)^{n-1}} = 0,$$

whence the claim follows.