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1. Consider the differential operator L

$$Lu=\sum_{i,j=1}^{n}a_{ij}\left(x\right)\partial_{ij}u+\sum_{k=1}^{n}b_k(x)\partial_k u,$$

where a_{ij} and b_k are continuous functions of x defined in an open subset D of $\mathbb{R}^n$. Assume that the operator L is elliptic at any point, that is, the matrix $\left(a_{ij}\left(x\right)\right)_{i,j=1}^{n}$ is positive definite at any point $x\in D$. Prove the maximum principle: if Ω is a bounded domain such that $\overline{\Omega}\subset D$ and a function $u\in C^2(\Omega)\cap C(\overline{\Omega})$ satisfies in Ω the inequality $Lu\geq 0$ then

$$\max_{\overline{\Omega}}u=\max_{\partial\Omega}u.$$

2. Let L,D,Ω be the same as in Exercise 1. Let u,v be functions of the class $C^2(\Omega)\cap C\left(\overline{\Omega}\right)$.

   (a) *(The comparison principle)* Prove that if $Lu\geq Lv$ in Ω and $u\leq v$ on $\partial\Omega$ then $u\leq v$ in Ω.

   (b) Prove that if

$$\begin{cases}Lu=f & \text{in }\Omega\\ u=g & \text{on }\partial\Omega\end{cases}$$

   then

$$\sup_{\Omega}|u|\leq C\sup_{\Omega}|f|+\sup_{\partial\Omega}|g|,$$

   where C is a constant that depends on Ω and on the coefficients of L in Ω.

   Hint: Compare u with the function $v(x)=-\alpha\exp\left(\gamma x_1\right)+\beta$ with suitable positive constants α,β,γ.

3. Consider a divergence form operator

$$L=\sum_{i,j=1}^{n}\partial_{x_i}\left(a_{ij}\left(x\right)\partial_{x_j}\right)$$

defined in a domain $U\subset\mathbb{R}^n$. Assume that $a_{ij}\in C^1\left(U\right)$. Let V be another domain in $\mathbb{R}^n$ and let $\Phi:U\to V$ be a C^2-diffeomorphism between U and V. We consider $y=\Phi\left(x\right)$ as change of coordinates in U and define for all $l,k=1,...,n$ the following functions:

$$b_{kl}=\sum_{i,j=1}^{n}a_{ij}\frac{\partial y_k}{\partial x_i}\frac{\partial y_l}{\partial x_j}. \tag{1}$$

Set also $D=\det\left(\partial y_k/\partial x_i\right)^{-2}$. Prove that the operator L can be written in the coordinates y as follows:

$$L=\frac{1}{\sqrt{D}}\sum_{i,k=1}^{n}\partial_{y_k}\left(b_{kl}\sqrt{D}\partial_{y_l}\right). \tag{2}$$

Hint. For arbitrary functions $u, v \in C_0^2(U)$ use the Green formula

$$-\int_U Lu\, v\, dx = \int_U \sum_{i,j=1}^n a_{ij}(x)\, \partial_{x_j} u\, \partial_{x_i} v\, dx, \tag{3}$$

express $\partial_{x_j} u$ and $\partial_{x_i} v$ through the derivatives $\partial_{y_l} u$ and $\partial_{y_k} v$, and make change $x = \Phi^{-1}(y)$ in the both integrals.

4. Let φ be a mollifier in $\mathbb{R}^n$ and $f \in L^2(\mathbb{R}^n)$.

   (a) Prove that $f * \varphi \in L^2(\mathbb{R}^n)$ and

   $$\|f * \varphi\|_{L^2} \le \|f\|_{L^2}. \tag{4}$$

   (b) Prove that

   $$f * \varphi_\varepsilon \overset{L^2(\mathbb{R}^n)}{\longrightarrow} f \quad \text{as } \varepsilon \to 0+, \tag{5}$$

   where $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$.
   Hint: Use the following two facts: (i) the set $C_0(\mathbb{R}^n)$ of continuos compactly supported functions is dense in $L^2(\mathbb{R}^n)$ (ii) if $g \in C_0(\mathbb{R}^n)$ then $g * \varphi \in C_0^\infty(\Omega)$ and $g * \varphi_\varepsilon \rightrightarrows g$ as $\varepsilon \to 0$.

   (c) Prove that if $f \in W^{k,2}(\mathbb{R}^n)$ then $f * \varphi \in W^{k,2}(\mathbb{R}^n)$ and

   $$f * \varphi_\varepsilon \overset{W^{k,2}(\mathbb{R}^n)}{\longrightarrow} f \quad \text{as } \varepsilon \to 0.$$

   Hint. Use the following identity for distributional derivatives:

   $$D^\alpha(f * \varphi) = (D^\alpha f) * \varphi. \tag{6}$$