Blatt 5. Abgabe bis 17.11.23

In all Exercises, Ω is an open subset of $\mathbb{R}^n$.

26. Let u be a non-negative function from $W_0^{1,2}(\Omega)$ and $\{u_k\}$ be a sequence of functions from $W_0^{1,2}(\Omega)$. Prove that if

$$u_k \overset{W^{1,2}}{\longrightarrow} u \text{ and } u_k \overset{\text{a.e.}}{\longrightarrow} u,$$

then also

$$(u_k)_+ \overset{W^{1,2}}{\longrightarrow} u.$$

Hint: Prove that $\left\|\nabla (u_k)_+ - \nabla u\right\|_{L^2}^2 = \int_{\{u_k>0\}} |\nabla u_k - \nabla u|^2 \, dx + \int_{F_k} |\nabla u|^2 \, dx,$

where

$$F_k = \{x \in \Omega : u_k(x) \le 0, u(x) > 0\},$$

and verify that, for almost all $x \in \Omega$,

$$x \notin F_k \text{ for large enough } k,$$

which implies $\mathbf{1}_{F_k} \to 0$ a.e. as $k \to \infty$.

27. Consider in Ω an operator

$$Lu = \sum_{i,j=1}^{n} \partial_i (a_{ij} \partial_j u) + \sum_{i=1}^{n} b_i \partial_i u,$$

where the coefficients a_{ij} and b_i are measurable functions, the matrix (a_{ij}) is uniformly elliptic, and all functions b_i are bounded. Assume that $u \in W^{1,2}(\Omega)$ and $f \in L^2(\Omega)$ satisfy the inequality $Lu \ge f$ weakly in Ω, that is, for any non-negative function $\varphi \in \mathcal{D}(\Omega)$,

$$-\int_\Omega \sum_{i,j=1}^{n} a_{ij} \partial_j u \partial_i \varphi \, dx + \int_\Omega \sum_{i=1}^{n} b_i \partial_i u \, \varphi \, dx \ge \int_\Omega f \varphi dx. \tag{25}$$

The purpose of this question is to prove that then (25) is satisfied for any non-negative function $\varphi \in W_0^{1,2}(\Omega)$.

(a) Prove that if $f \in W_c^{1,2}(\Omega)$ and $f \ge 0$ then there exists a sequence $\{f_k\}$ of functions from $\mathcal{D}(\Omega)$ such that $f_k \ge 0$ and $f_k \overset{W^{1,2}}{\longrightarrow} f$.
Hint: Use the solution of Exercise 8.

(b) Prove the claim of (a) for any non-negative function $f \in W_0^{1,2}(\Omega)$.
Hint: Use Exercise 26 and (a).

(c) Prove that if (25) holds for any non-negative function $\varphi \in \mathcal{D}(\Omega)$ then it holds also for any non-negative function $\varphi \in W_0^{1,2}(\Omega)$.
Hint: Use (b) .

28. Let Ω be a bounded open subset of $\mathbb{R}^n$. Let f be a function on Ω such that $f \in W^{1,2}(\Omega)$ and, for any $x \in \partial\Omega$,

$$\lim_{y \to x,\, y \in \Omega} f(y) = 0. \tag{26}$$

Prove that $f \in W_0^{1,2}(\Omega)$.

Hint: First observe that it suffices to prove this claim assuming that $f \geq 0$. Then, for any $\varepsilon > 0$, show that the function $(f - \varepsilon)_+$ belongs to $W_0^{1,2}(\Omega)$ and that

$$(f - \varepsilon)_+ \overset{W^{1,2}}{\longrightarrow} f \quad \text{as } \varepsilon \to 0.$$

Use Exercises 16 and 8.

29. *(Green's formula for L)* Consider in Ω a uniformly elliptic divergence form operator

$$Lu = \sum \partial_i (a_{ij} \partial_j u)$$

with measurable coefficients a_{ij}. For any $u \in W_{loc}^{1,1}(\Omega)$ we understand Lu in the distributional sense. Let $p, q \in (1, \infty)$ be a pair of Hölder conjugate exponents.

(a) Prove that if $u \in W^{1,p}(\Omega)$ and $Lu \in L^p(\Omega)$ then, for any $v \in W_0^{1,q}(\Omega)$,

$$\int_\Omega Lu\, v\, dx = -\int_\Omega \sum_{i,j=1}^n a_{ij} \partial_j u \partial_i v\, dx. \tag{27}$$

Hint. First prove (27) for $v \in \mathcal{D}(\Omega)$ and then pass to all $v \in W_0^{1,q}(\Omega)$.

(b) Prove (27) if $u \in W_{loc}^{1,p}(\Omega)$, $Lu \in L_{loc}^p(\Omega)$ and $v \in W_c^{1,q}(\Omega)$.
Here $W_c^{1,q}(\Omega)$ is a subspace of $W^{1,q}(\Omega)$ that consists of functions with compact support in Ω.
Hint. You can use without proof that $W_c^{1,q}(\Omega) \subset W_0^{1,q}(\Omega)$ (in the case $q = 2$ this was considered in Exercise 8). Reduce the domain of integration in (27) to a precompact open set $U \subset \Omega$ where $u \in W^{1,p}(U)$, $Lu \in L^p(U)$ and $v \in W_0^{1,q}(U)$, and use (a).