Elliptic partial differential equations

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Chapter 0

Introduction

9.10.23 **Lecture 1**

0.1 Elliptic operators with variable coefficients

In this course we are concerned with partial differential equations in $\mathbb{R}^n$ of the form $Lu = f$ where f is a given function, u is an unknown function, and L is a second order differential operator of one of the two forms:

1. $Lu = \sum_{i,j=1}^{n} \partial_i \left(a_{ij}(x)\partial_j u\right)$ (a divergence form operator)

2. $Lu = \sum_{i,j=1}^{n} a_{ij}(x)\partial_{ij} u$ (a non-divergence form operator).

In the both cases, the matrix (a_{ij}) depends on $x \in \mathbb{R}^n$, is symmetric, that is, $a_{ij} = a_{ji}$, and positive definite. The operators L with positive definite matrices (a_{ij}) are called *elliptic*.

For example, the Laplace operator

$$\Delta = \sum_{i=1}^{n} \partial_{ii} u$$

is both divergence and non-divergence form elliptic operator with the matrix $(a_{ij}) = \mathrm{id}$.

Note that the divergence form operator can be represented in the form

$$Lu = \sum_{i,j=1}^{n} \partial_i \left(a_{ij}(x)\partial_j u\right) = \sum_{i,j=1}^{n} a_{ij}(x)\partial_{ij} u + \left(\partial_i a_{ij}\right)\partial_j u,$$

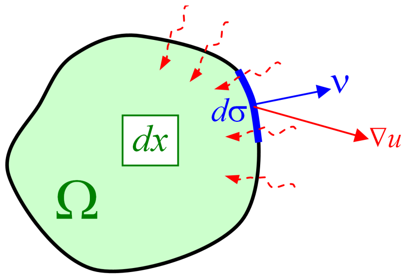
that is the sum of the non-divergence form operator and lower order terms. However, this works only for differentiable coefficients a_{ij}. In fact, the most interesting applications in mathematics requires operators with discontinuous coefficients a_{ij}. Of course, in this case the divergence form operator cannot be understood in the the sense of classical derivatives, and we will define the meaning of Lu in a certain *weak sense*.

0.2 Origin of divergence form operators

One of the origins of divergence form operators is *heat diffusion*. Let $u(x,t)$ denote the temperature in some medium at a point $x \in \mathbb{R}^3$ at time t. Fix a region $\Omega \subset \mathbb{R}^3$. By the Fourier law of thermoconductance, the amount dQ of the heat energy that has flown into Ω through a surface element $d\sigma$ of its boundary $\partial\Omega$ between the time moments t and $t+dt$ is equal to

$$dQ = \sum_{i,j=1}^{3} a_{ij}(x)\nu_i \partial_j u \; d\sigma dt$$

where ν is the outer unit normal vector field to $\partial\Omega$ at a point $x \in d\sigma$ and $a_{ij}(x)$ is the tensor of the thermal conductance of the material of the body.



The dependence of a_{ij} of x means that the conductance may be different at different points, and the dependence on the indices i,j reflects the fact that the conductance may be different in different directions.

The expression

$$\sum_{i,j=1}^{3} a_{ij}(x)\nu_i \partial_j u \tag{0.1}$$

can be regarded as an inner product of the vectors $\nu = (\nu_i)_{i=1}^n$ and $\nabla u = (\partial_j u)_{j=1}^n$ with the coefficients $a_{ij}(x)$ (the symmetry and positive definiteness of this matrix ensure that the expression (0.1) has the properties of an inner product). Hence, the total energy Q that has flown into Ω through its entire boundary between time moments t and $t+h$ is

$$Q = \int_t^{t+h} \int_{\partial\Omega} \sum_{i,j=1}^{3} a_{ij}(x)\nu_i \partial_j u \; d\sigma dt,$$

On the other hand, the amount of heat energy dQ' acquired by a volume element dx of Ω from time t to time $t+h$ is equal to

$$dQ' = (u(x,t+h) - u(x,t))\, c\rho dx,$$

where ρ is the density of the material of the body and c is its heat capacity (both c and ρ are functions of x). Indeed, the volume element dx has the mass ρdx, and increase of its temperature by one degree requires $c\rho dx$ of heat energy. Hence, increase of the temperature from $u(x,t)$ to $u(x,t+dt)$ requires $(u(x,t+h) - u(x,t))\, c\rho dx$ of heat energy. We obtain that the total amount Q' of energy acquired by the entire body Ω from time t to time $t+h$ is equal to

$$Q' = \int_{\Omega} \left(u\left(x, t+h\right) - u\left(x, t\right) \right) c\rho dx.$$

By the law of conservation of energy, in absence of heat sources and sinks, we have $Q = Q'$, that is,

$$\int_t^{t+h} \left(\int_{\partial\Omega} \sum_{i,j=1}^{3} a_{ij}\nu_i \partial_j u \ d\sigma \right) dt = \int_{\Omega} \left(u\left(x, t+h\right) - u\left(x, t\right) \right) c\rho dx.$$

Dividing by h and passing to the limit as $h \to 0$, we obtain

$$\int_{\partial\Omega} \sum_{i,j=1}^{3} a_{ij}\nu_i \partial_j u \, d\sigma = \int_{\Omega} \left(\partial_t u\right) c\rho \, dx. \tag{0.2}$$

Observing that

$$\sum_{i,j=1}^{3} a_{ij}\nu_i \partial_j u = \overrightarrow{F} \cdot \nu$$

where the vector field $\overrightarrow{F}$ has the components

$$F_i = \sum_{j=1}^{3} a_{ij} \partial_j u,$$

and applying the divergence theorem, we obtain

$$\begin{aligned} \int_{\partial\Omega} \sum_{i,j=1}^{3} a_{ij}\nu_i \partial_j u \ d\sigma &= \int_{\partial\Omega} \overrightarrow{F} \cdot \nu \, d\sigma = \int_{\Omega} \operatorname{div} \overrightarrow{F} dx \\ &= \int_{\Omega} \sum_{i=1}^{3} \left(\partial_i F_i\right) dx = \int_{\Omega} \sum_{i,j=1}^{3} \partial_i \left(a_{ij} \partial_j u\right) dx = \int_{\Omega} Lu \, dx, \end{aligned}$$

where

$$Lu = \sum_{i,j=1}^{3} \partial_i \left(a_{ij} \partial_j u\right).$$

is the divergence form operator. This implies together with (0.2) that

$$\int_{\Omega} c\rho \, \partial_t u \, dx = \int_{\Omega} Lu \, dx,$$

Since this identity holds for any region Ω, it follows that the function u satisfies the following *heat equation*

$$c\rho \, \partial_t u = Lu.$$

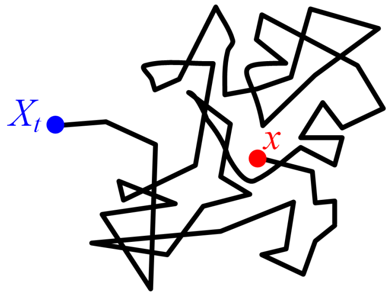
In particular, if u is stationary, that is, does not depend on t, then u satisfies $Lu = 0$.

We have seen that in the above derivation the operator L comes out exactly in the divergence form because of an application of the divergence theorem.

0.3 Origin of non-divergence form operators

The operators in non-divergence form originate from different sources, in particular, from *stochastic diffusion* processes. A stochastic diffusion process in $\mathbb{R}^n$ is mathematical model of Brownian motion in inhomogeneous media. It is described by the family $\{\mathbb{P}_x\}_{x\in\mathbb{R}^n}$ of probability measures, where $\mathbb{P}_x$ is the probability measure on the set Ω_x of all continuous paths $\omega : [0,\infty) \to \mathbb{R}^n$ such that is $\omega(0) = x$.

Define for any $t \geq 0$ a random variable $X(t)$ on Ω_x by $X(t)(\omega) = \omega(t)$. The *random path* $t \mapsto X(t)$ can be viewed as a stochastic movement of a microscopic particle with the initial position $X(0) = x$.



The diffusion process is described by its infinitesimal means

$$\mathbb{E}_x\left(X_i(t+dt) - X_i(t)\right) = b_i dt + o(dt) \quad \text{as } dt \to 0,$$

for any $i = 1, ..., n$, and its infinitesimal covariances

$$\mathbb{E}_x\left(\left(X_i(t+dt) - X_i(t)\right)\left(X_j(t+dt) - X_j(t)\right)\right) = a_{ij}dt + o(dt) \quad \text{as } dt \to 0,$$

for all $i, j = 1, ..., n$, where b_i and a_{ij} are some functions that in general may depend in x and t, but we assume for simplicity that they depend only on x.

By construction, the matrix (a_{ij}) is symmetric and positive definite, as any covariance matrix. The functions a_{ij} and b_i determine the non-divergence form operator with lower order terms:

$$Lu = \sum_{i,j=1}^{n} a_{ij}\partial_{ij}u + \sum_{i=1}^{n} b_i \partial_i u,$$

that has the following relation to the process: for any bounded continuous function f on $\mathbb{R}^n$, the function

$$u(x,t) = \mathbb{E}_x\left(f(X(t))\right)$$

satisfies the heat equation

$$\partial_t u = Lu$$

with the above operator L. This equation is called the *Kolmogorov backward equation*. This operator L is called the *generator* of the diffusion process because it contains all the information about this stochastic process.

Chapter 1

Weak Dirichlet problem for divergence form operators

In this Chapter we deal with the divergence form elliptic operator

$$Lu = \sum_{i,j=1}^{n} \partial_i \left(a_{ij}(x)\partial_j u\right)$$

defined in an open set $\Omega \subset \mathbb{R}^n$. Since the coefficients a_{ij} may be not differentiable, we have to specify exactly how the equation $Lu = f$ is understood.

1.1 Distributions

Let Ω be an open subset of $\mathbb{R}^n$. Denote by $\mathcal{D}(\Omega)$ the linear topological space that as a set coincides with $C_0^\infty(\Omega)$, the linear structure in $\mathcal{D}(\Omega)$ is defined with respect to addition of functions and multiplication by scalars from $\mathbb{R}$, and the topology in $\mathcal{D}(\Omega)$ is defined by means of the following convergence: a sequence $\{\varphi_k\}$ of functions from $\mathcal{D}(\Omega)$ converges to $\varphi \in \mathcal{D}(\Omega)$ in the space $\mathcal{D}(\Omega)$ if the following two conditions are satisfied:

1. $\varphi_k \rightrightarrows \varphi$ in Ω and $D^\alpha \varphi_k \rightrightarrows D^\alpha \varphi$ for any multiindex α of any order;
2. there is a compact set $K \subset \Omega$ such that $\operatorname{supp} \varphi_k \subset K$ for all k.

It is possible to show that this convergence is indeed topological, that is, given by a certain topology.

Any linear topological space $\mathcal{V}$ has a dual linear space $\mathcal{V}'$ that consists of continuous linear functionals on $\mathcal{V}$.

Definition. Any linear continuos functional $f : \mathcal{D}(\Omega) \to \mathbb{R}$ is called a *distribution* in Ω (or generalized functions). The set of all distributions in Ω is denoted by $\mathcal{D}'(\Omega)$. If $f \in \mathcal{D}'(\Omega)$ then the value of f on a *test function* $\varphi \in \mathcal{D}(\Omega)$ is denoted by (f, φ).

Any locally integrable function $f : \Omega \to \mathbb{R}$ can be regarded as a distribution as follows: it acts on any test function $\varphi \in \mathcal{D}(\Omega)$ by the rule

$$(f, \varphi) := \int_\Omega f\varphi \, dx. \tag{1.1}$$

The distributions that are represented in this way by locally integrable functions are called *regular distributions*.

Note that two locally integrable functions f, g determine the same distribution if and only if $f = g$ almost everywhere, that is, if the set

$$\{x \in \Omega : f(x) \neq g(x)\}$$

has measure zero. We write shortly in this case

$$f = g \text{ a.e.} \tag{1.2}$$

Clearly, the relation (1.2) is an equivalence relation, that gives rise to equivalence classes of locally integrable functions. The set of all equivalence classes of locally integrable functions is denoted[1] by $L^1_{loc}(\Omega)$. The identity (1.1) establishes the injective linear mapping $L^1_{loc}(\Omega) \to \mathcal{D}'(\Omega)$ so that $L^1_{loc}(\Omega)$ can be regarded as a subspace of $\mathcal{D}'(\Omega)$.

There are distributions that are not regular, that is, the difference $\mathcal{D}'(\Omega) \setminus L^1_{loc}(\Omega)$ is not empty. For example, define the *delta-function* δ_{x_0} for any $x_0 \in \Omega$ as follows:

$$(\delta_{x_0}, \varphi) = \varphi(x_0).$$

Although historically δ_{x_0} is called delta-*function*, it is a distribution that is not regular and is not determined by any function.

Definition. Let $f \in \mathcal{D}'(\Omega)$. Fix a multiindex α. A *distributional partial derivative* $D^\alpha f$ is a distribution that acts on test functions $\varphi \in \mathcal{D}(\Omega)$ as follows:

$$(D^\alpha f, \varphi) = (-1)^{|\alpha|}(f, D^\alpha \varphi) \quad \forall \varphi \in \mathcal{D}(\Omega), \tag{1.3}$$

where $D^\alpha \varphi$ is the classical (usual) derivative of φ.

Note that the right hand side of (1.3) makes sense because $D^\alpha \varphi \in \mathcal{D}(\Omega)$. Moreover, the right hand side of (1.3) is obviously a linear continuous functions in $\varphi \in \mathcal{D}(\Omega)$, which means that $D^\alpha f$ exists *always* as a distribution.

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Let $f \in L^1_{loc}(\Omega)$. If the distributional derivative $D^\alpha f$ is a regular distribution then the corresponding L^1_{loc}-function is also denoted by $D^\alpha f$ and is called the *weak derivative* of f. An equivalent definition of the weak derivative $D^\alpha f$ is as follows: it is a function from $L^1_{loc}(\Omega)$ such that

$$\int_\Omega D^\alpha f \, \varphi \, dx = (-1)^{|\alpha|} \int_\Omega f \, D^\alpha \varphi \, dx \quad \forall \varphi \in \mathcal{D}(\Omega). \tag{1.4}$$

If $f \in C^k(\Omega)$ then its classical derivative $D^\alpha f$ with $|\alpha| \leq k$ satisfies (1.4) and, hence, is at the same time the weak derivative as well as its distributional derivative.

[1]Sometimes $L^1_{loc}(\Omega)$ is loosely used to denote the set of all locally integrable functions in Ω. However, in a strict sense, the elements of $L^1_{loc}(\Omega)$ are not functions but their equivalence classes.

1.2 Sobolev spaces

As before, let Ω be an open subset of $\mathbb{R}^n$. Fix $p \in [1, \infty)$. A Lebesgue measurable function $f : \Omega \to \mathbb{R}$ is called *p-integrable* if

$$\int_\Omega |f|^p \, dx < \infty.$$

Two measurable functions in Ω (in particular, p-integrable functions) are called equivalent if

$$f = g \text{ a.s.}$$

This is an equivalence relation, and the set of all equivalence classes of p-integrable functions in Ω is denoted by $L^p(\Omega)$. It follows from the Hölder inequality, that $L^p(\Omega) \subset L^1_{loc}(\Omega)$. In particular, all the elements of $L^p(\Omega)$ can be regarded as distributions.

The set $L^p(\Omega)$ is a linear space over $\mathbb{R}$. Moreover, it is a Banach space (=complete normed space) with respect to the norm

$$\|f\|_{L^p} := \left(\int_\Omega |f|^p \, dx \right)^{1/p}.$$

The Banach spaces $L^p(\Omega)$ are called *Lebesgue spaces*.

The case $p = 2$ is of special importance because the space $L^2(\Omega)$ has inner product

$$(f, g)_{L^2} = \int_\Omega fg \, dx,$$

whose norm coincides with $\|f\|_2$ as

$$(f, f)_{L^2}^{1/2} = \left(\int_\Omega f^2 dx \right)^{1/2} = \|f\|_{L^2}.$$

Hence, $L^2(\Omega)$ is a Hilbert space.

Definition. Define the Sobolev space $W^{k,p}$ for arbitrary non-negative integer k and $p \in [1, \infty)$ as follows:

$$W^{k,p}(\Omega) = \{ f \in L^p(\Omega) : D^\alpha f \in L^p(\Omega) \text{ for all } \alpha \text{ with } |\alpha| \le k \}, \tag{1.5}$$

where $D^\alpha f$ is a distributional derivative (that by (1.5) is also a weak derivative).

In words, $W^{k,p}(\Omega)$ is a subspace of $L^p(\Omega)$ that consists of functions whose all weak partial derivatives of the order $\le k$ are also in $L^p(\Omega)$. In particular, $W^{0,p} = L^p$. It is easy to see that $C_0^k(\Omega) \subset W^{k,p}(\Omega)$ for any k and p.

Let us introduce in $W^{k,p}(\Omega)$ the following norm:

$$\|f\|_{W^{k,p}} := \left(\sum_{\alpha : |\alpha| \le k} \int_\Omega |D^\alpha f|^p \, dx \right)^{1/p}.$$

It is possible to show that $\|\cdot\|_{W^{k,p}}$ is indeed a norm, and $W^{k,p}(\Omega)$ is a Banach space with this norm. In the case $p = 2$ this norm is given by the inner product:

$$(f, g)_{W^{k,2}} = \sum_{\alpha : |\alpha| \le k} \int_\Omega D^\alpha f \; D^\alpha g \, dx,$$

so that $W^{k,2}(\Omega)$ is a Hilbert space.

Denote by $L^p_{loc}(\Omega)$ the space of (equivalence classes of) measurable functions f on Ω such that $|f|^p$ is locally integrable. For example, all continuous functions in Ω belong to $L^p_{loc}(\Omega)$. Define the *local Sobolev* space $W^{k,p}_{loc}(\Omega)$ by

$$W^{k,p}_{loc}(\Omega) = \{f \in L^p_{loc}(\Omega) : D^\alpha f \in L^p_{loc}(\Omega) \text{ for all } \alpha \text{ with } |\alpha| \le k\}. \tag{1.6}$$

It is easy to see that $C^k(\Omega) \subset W^{k,p}_{loc}(\Omega)$ for any k and p.

1.3 The weak Dirichlet problem

Let Ω be an open subset of $\mathbb{R}^n$. Consider in Ω an elliptic operator in the divergence form:

$$Lu = \sum_{i,j=1}^n \partial_i (a_{ij}(x)\partial_j u). \tag{1.7}$$

where all functions $a_{ij}(x)$ are measurable in Ω. As before, the matrix (a_{ij}) is symmetric and positive definite. Moreover, here (and everywhere below) we assume that (a_{ij}) is *uniformly elliptic*, that is, for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$,

$$\lambda^{-1}|\xi|^2 \le \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \le \lambda |\xi|^2, \tag{1.8}$$

for some positive constant λ. Equivalently, this means that, for any fixed $x \in \Omega$, all eigenvalues of the matrix $(a_{ij}(x))$ are contained in the interval $[-\lambda^{-1}, \lambda]$.

We define now how to understand the equation $Lu = f$ in the weak sense.

Definition. Let $u \in W^{1,2}_{loc}$ and $f \in L^2_{loc}(\Omega)$. We say that the equation $Lu = f$ is satisfied in a *weak sense* or *weakly* if, for any $\varphi \in \mathcal{D}(\Omega)$,

$$\int_\Omega \sum_{i,j=1}^n a_{ij}\,\partial_j u\,\partial_i\varphi\,dx = -\int_\Omega f\,\varphi\,dx. \tag{1.9}$$

Note that the integral on the right hand side of (1.9) makes sense because the integration can be reduced to a compact set $\operatorname{supp}\varphi$ where φ is bounded and f is integrable. The left hand side makes sense similarly because $\partial_j u \in L^2_{loc}$ and, hence, is integrable on $\operatorname{supp}\varphi$, while $\partial_i\varphi$ and a_{ij} are bounded (the latter follows from (1.8)).

Motivation for this definition is as follows. Assume that $a_{ij} \in C^1$ and $u \in C^2$. Then the equation $Lu = f$ can be understood in the classical sense. Multiplying it by $\varphi \in \mathcal{D}(\Omega)$ and integrating in Ω using integration by parts, we obtain

$$\int_\Omega f\,\varphi\,dx = \sum_{i,j=1}^n \int_\Omega \partial_i(a_{ij}\partial_j u)\,\varphi\,dx = -\sum_{i,j=1}^n \int_\Omega a_{ij}\partial_j u\,\partial_i\varphi\,dx,$$

that is the identity (1.9). Hence, the weak meaning of the equation $Lu = f$ is consistent with the classical one.

Define $W^{1,2}_0(\Omega)$ as the subspace of $W^{1,2}(\Omega)$ that is obtained by taking the closure of $\mathcal{D}(\Omega)$ in $W^{1,2}(\Omega)$, that is

$$W^{1,2}_0(\Omega) = \overline{\mathcal{D}(\Omega)}^{W^{1,2}(\Omega)}.$$

Lemma 1.1 *Let $u \in W^{1,2}(\Omega)$ and $f \in L^2(\Omega)$. Then the equation $Lu = f$ holds weakly if and only if (1.9) holds for all $\varphi \in W_0^{1,2}(\Omega)$.*

Proof. If (1.9) holds for all $\varphi \in W_0^{1,2}(\Omega)$ then, of course, it holds also for all $\varphi \in \mathcal{D}(\Omega)$. Let us prove the converse statement. For any $\varphi \in W_0^{1,2}(\Omega)$ there is a sequence $\{\varphi_k\}$ of functions from $\mathcal{D}(\Omega)$ such that $\varphi_k \to \varphi$ in the norm of $W^{1,2}(\Omega)$. Any φ_k satisfies (1.9), and we would like to pass to the limit as $k \to \infty$. For that, it suffices to verify that the both sides of (1.9) are continuous functionals of $\varphi \in W^{1,2}(\Omega)$. Since they both are linear functionals, it suffices to verify that they are *bounded* linear functionals in $W^{1,2}(\Omega)$.

The functional $\varphi \mapsto \int_\Omega f\varphi dx$ is bounded because

$$\left|\int_\Omega f\varphi\right| \le \|f\|_{L^2}\|\varphi\|_{L^2} \le C\|\varphi\|_{W^{1,2}}$$

where $C = \|f\|_{L^2}$. Let us show that the functional

$$\varphi \mapsto A(\varphi) := \int_\Omega \sum_{i,j=1}^n a_{ij}\partial_j u \partial_i \varphi \, dx$$

is also bounded in $W^{1,2}(\Omega)$, that is,

$$|A(\varphi)| \le C\|\varphi\|_{W^{1,2}}, \tag{1.10}$$

for some constant C and all $\varphi \in W^{1,2}(\Omega)$. Fix $x \in \Omega$ and consider in $\mathbb{R}^n$ the bilinear form

$$(\xi,\eta)_a := \sum_{i,j=1}^n a_{ij}(x)\xi_j\eta_i \text{ for } \xi,\eta \in \mathbb{R}^n.$$

This bilinear form is symmetric and positive definite by the ellipticity of (a_{ij}). Hence, $(\xi,\eta)_a$ is an inner product in $\mathbb{R}^n$. By the Cauchy-Schwarz inequality and (1.8), we obtain

$$|(\xi,\eta)_a| \le \sqrt{(\xi,\xi)_a}\sqrt{(\eta,\eta)_a} \le \lambda|\xi||\eta|.$$

It follows that

$$\begin{aligned} |A(\varphi)| &= \left|\int_\Omega \sum_{i,j=1}^n a_{ij}(x)\partial_j u \partial_i \varphi \, dx\right| \\ &= \left|\int_\Omega (\nabla u, \nabla\varphi)_a \, dx\right| \\ &\le \int_\Omega \lambda|\nabla u||\nabla\varphi| \, dx \\ &\le \lambda\left(\int_\Omega |\nabla u|^2 dx\right)^{1/2}\left(\int_\Omega |\nabla\varphi|^2 dx\right)^{1/2}. \end{aligned}$$

It follows that

$$|A(\varphi)| \le \lambda\|u\|_{W^{1,2}}\|\varphi\|_{W^{1,2}}, \tag{1.11}$$

which proves (1.10) with $C = \lambda \|u\|_{W^{1,2}}$. ■

Definition. We say that a function u solves the Dirichlet problem

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

in the weak sense if

$$\begin{cases} Lu = f \text{ weakly in } \Omega, \\ u \in W_0^{1,2}(\Omega). \end{cases} \tag{D}$$

In other words, the weak meaning of the boundary condition $u|_{\partial\Omega} = 0$ is $u \in W_0^{1,2}(\Omega)$.

Theorem 1.2 *Let Ω be a bounded domain and L be a uniformly elliptic operator in the divergence form in Ω with measurable coefficients. Then the weak Dirichlet problem (D) with the operator (1.7) has exactly one solution for any $f \in L^2(\Omega)$.*

We use in the proof the *Riesz representation theorem*: in any Hilbert space H with inner product $[\cdot,\cdot]$, the equation

$$[u, \varphi] = \ell(\varphi) \quad \forall \varphi \in H$$

has a unique solution $u \in H$ provided ℓ is a bounded linear functional on H.

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Proof. We need to prove that the weak equation $Lu = f$ has a unique solution $u \in W_0^1(\Omega)$ for any $f \in L^2(\Omega)$. Consider in $W_0^{1,2}(\Omega)$ the following bilinear form

$$[u, v]_a := \int_\Omega \sum_{i,j=1}^n a_{ij}(x)\partial_j u(x)\partial_i v(x)dx$$

(the integral converges because a_{ij} are bounded and $\partial_i u, \partial_i v \in L^2(\Omega)$). This form is symmetric by the symmetry of the matrix (a_{ij}).

Applying the uniform ellipticity condition (1.8) with $\xi_j = \partial_j u$ and observing that $|\xi| = |\nabla u|$, we obtain

$$[u, u]_a = \int_\Omega \sum_{i,j=1}^n a_{ij}(x)\partial_j u(x)\partial_i u(x)dx \le \lambda \int_\Omega |\nabla u|^2 dx \le \lambda \|u\|_{W^{1,2}}^2, \tag{1.12}$$

and similarly

$$[u, u]_a \ge \lambda^{-1} \int_\Omega |\nabla u|^2 dx.$$

On the other hand, by the *Friedrichs inequality* we have, for any $u \in W_0^{1,2}(\Omega)$ that

$$\int_\Omega |\nabla u|^2 dx \ge c \int_\Omega u^2 dx,$$

with some positive constant $c = c(\Omega)$. Assuming without loss of generality that $c \le 1$, we obtain

$$\int_\Omega |\nabla u|^2 \, dx \ge \frac{c}{2} \int_\Omega \left(u^2 + |\nabla u|^2\right) dx = \frac{c}{2} \|u\|^2_{W^{1,2}},$$

whence it follows that

$$[u,u]_a \ge \frac{c}{2\lambda} \|u\|^2_{W^{1,2}}. \tag{1.13}$$

In particular, $[u,v]_a$ is positive definite and, hence, is an inner product in $W_0^{1,2}(\Omega)$.

By (1.12) and (1.13), the norm $[u,u]_a^{1/2}$ is equivalent to $\|u\|_{W^{1,2}}$, which implies that $W_0^{1,2}(\Omega)$ with the inner product $[\cdot,\cdot]_a$ is a Hilbert space.

The weak equation $Lu = f$ can be rewritten in the form

$$[u,\varphi]_a = -\int_\Omega f\varphi \, dx \quad \forall \varphi \in W_0^{1,2}(\Omega). \tag{1.14}$$

The right hand side $\ell(\varphi) := -\int_\Omega f\varphi\, dx$ is a bounded linear functional of $\varphi \in W_0^{1,2}(\Omega)$ with respect to the norm of $[\cdot,\cdot]_a$ because

$$|\ell(\varphi)| \le \|f\|_{L^2} \|\varphi\|_{L^2} \le \|f\|_{L^2} \|\varphi\|_{W^{1,2}} \le \text{const}\, [\varphi,\varphi]_a^{1/2}.$$

By the Riesz representation theorem, the equation

$$[u,\varphi]_a = \ell(\varphi) \quad \forall \varphi \in W_0^{1,2}(\Omega),$$

that is equivalent to (1.14), has a unique solution $u \in W_0^{1,2}(\Omega)$, which was to be proved. ∎

1.4 Weak Dirichlet problem with lower order terms

Here we consider a more general operator

$$Lu = \sum_{i,j=1}^n \partial_i \left(a_{ij}\partial_j u\right) + \sum_{i=1}^n b_i(x)\partial_i u \tag{1.15}$$

in an open set $\Omega \subset \mathbb{R}^n$. We assume that the coefficients a_{ij}, b_i are measurable functions of $x \in \Omega$, the second order part $\sum_{i,j=1}^n \partial_i \left(a_{ij}\partial_j u\right)$ is uniformly elliptic, and that all functions b_i are bounded, that is, there is a constant b, such that

$$\sum_{i=1}^m |b_i| \le b \text{ in } \Omega.$$

Definition. Assume that $u \in W^{1,2}(\Omega)$ and $f \in L^2(\Omega)$. We say that the equation $Lu = f$ is satisfied weakly if, for any $\varphi \in W_0^{1,2}(\Omega)$,

$$\int_\Omega \left(\sum_{i,j=1}^n a_{ij}\, \partial_j u\, \partial_i \varphi - \sum_{i=1}^n b_i\, \partial_i u\, \varphi\right) dx = -\int_\Omega f\, \varphi\, dx. \tag{1.16}$$

1.4.1 Uniqueness

Theorem 1.3 (Uniqueness) *Let Ω be a bounded domain and L be the operator (1.15). Then the weak Dirichlet problem*

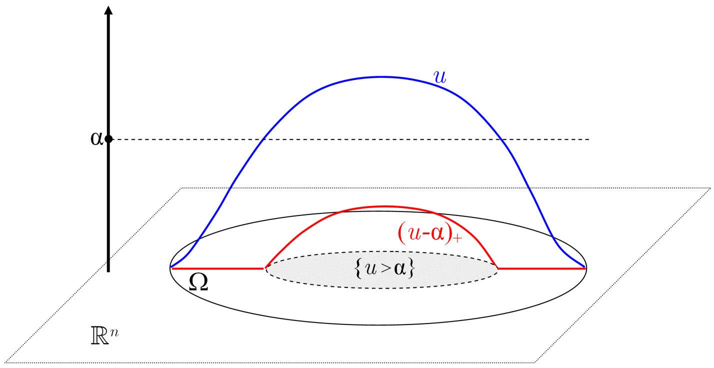
$$\begin{cases} Lu = f & \text{weakly in } \Omega \\ u \in W_0^{1,2}(\Omega) \end{cases}$$

has at most one solution.

For the proof we need some facts about weak derivatives that will be proved later on. Everywhere Ω is an open subset of $\mathbb{R}^n$.

Lemma 1.4 *If $u \in W_0^{1,2}(\Omega)$ then, for any $\alpha \geq 0$, also $(u-\alpha)_+ \in W_0^{1,2}(\Omega)$ and*

$$\nabla (u-\alpha)_+ = \begin{cases} \nabla u & \text{a.e. on the set } \{u > \alpha\} \\ 0 & \text{a.e. on the set } \{u \leq \alpha\} \end{cases} \tag{1.17}$$



Lemma 1.5 *If $u \in W_0^{1,2}(\Omega)$ then, for any $\alpha \in \mathbb{R}$,*

$$\nabla u = 0 \text{ a.e. on the set } \{u = \alpha\}.$$

Besides we are going to use the following inequality that also will be proved later (see Corollary 1.10).

Sobolev inequality. *If $n > 2$ then, for any $\varphi \in W_0^{1,2}(\Omega)$,*

$$\int_\Omega |\nabla \varphi|^2 \, dx \geq c_n \left(\int_\Omega |\varphi|^{\frac{2n}{n-2}} \, dx \right)^{\frac{n-2}{n}},$$

where c_n is a positive constant depending only on n.

If $n = 2$ and Ω is bounded then, for any $q \geq 1$ and for any $\varphi \in W_0^{1,2}(\Omega)$,

$$\int_\Omega |\nabla \varphi|^2 \, dx \geq c \left(\int_\Omega |\varphi|^{2q} \, dx \right)^{1/q},$$

where c is a positive constant depending on q and Ω.

Proof of Theorem 1.3. We need to prove that if $u \in W_0^{1,2}(\Omega)$ and $Lu = 0$ then $u = 0$ a.e. in Ω. It suffices to prove that $u \le 0$ a.e. in Ω since the similar inequality $u \ge 0$ a.e. follows by the same argument applied to $-u$.

We use the notion of the *essential supremum* that is defined by

$$\operatorname*{esssup}_{\Omega} u = \inf \{k \in \mathbb{R} : u \le k \text{ a.e.}\}$$

(note that $u \le k$ a.e. means that the set $\{u > k\}$ has measure 0). Then $u \le 0$ a.e. is equivalent to $\operatorname{esssup} u \le 0$. Let us assume from the contrary that

$$\alpha_0 := \operatorname*{esssup}_{\Omega} u > 0$$

and bring this to contradiction (note that $\alpha_0 = \infty$ is allowed). The weak equation $Lu = 0$ implies that, for any $\varphi \in W_0^{1,2}(\Omega)$,

$$\int_\Omega \sum_{i,j=1}^n a_{ij} \partial_j u \partial_i \varphi \, dx = \int_\Omega \sum_{i=1}^n b_i \partial_i u \, \varphi \, dx. \tag{1.18}$$

The right hand side of (1.18) admits a simple estimate

$$\int_\Omega \sum_{i=1}^n b_i (\partial_i u) \varphi \, dx \le b \int_\Omega |\nabla u| \, |\varphi| \, dx. \tag{1.19}$$

Now we specify function φ as follows: choose α from the interval

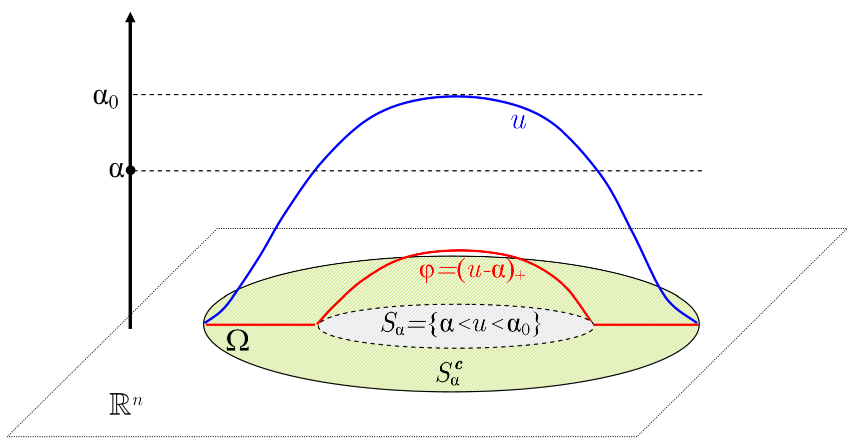
$$0 \le \alpha < \alpha_0$$

and set

$$\varphi = (u - \alpha)_+ .$$

By Lemma 1.4, $\varphi \in W_0^{1,2}(\Omega)$ so that we can use this φ in (1.18). Consider the set

$$S_\alpha := \{x \in \Omega : \alpha < u(x) < \alpha_0\}.$$



Let us verify that

$$\nabla\varphi = \begin{cases} \nabla u & \text{a.e. on } S_\alpha, \\ 0 & \text{a.e. on } S_\alpha^c, \end{cases} \tag{1.20}$$

where $S_\alpha^c = \Omega \setminus S_\alpha$. Indeed, $S_\alpha \subset \{u > \alpha\}$, so that the first line in (1.20) follows from that in (1.17). Note that

$$S_\alpha^c = \{u \le \alpha\} \cup \{u \ge \alpha_0\}.$$

By the second line in (1.17) we have $\nabla\varphi = 0$ a.e. on the set $\{u \le \alpha\}$. On the $\{u \ge \alpha_0\}$ we have by (1.17)

$$\nabla\varphi = \nabla u \text{ a.e.},$$

so it suffices to verify that

$$\nabla u = 0 \text{ a.e. on } \{u \ge \alpha_0\}. \tag{1.21}$$

Indeed, since the set $\{u > \alpha_0\}$ has measure 0 by definition of α_0, we see

$$u = \alpha_0 \text{ a.e. on } \{u \ge \alpha_0\},$$

which implies (1.21) by Lemma 1.5. Thus we finish the proof of (1.20).

Let us now prove that

$$|\nabla u|\,\varphi = \begin{cases} |\nabla\varphi|\,\varphi & \text{a.e. on } S_\alpha, \\ 0, & \text{a.e. on } S_\alpha^c. \end{cases} \tag{1.22}$$

Indeed, the first line in (1.22) follows from that of (1.20). On the set $\{u \le \alpha\}$ we have $\varphi = 0$, while on $\{u \ge \alpha_0\}$ we have by (1.21) $\nabla u = 0$ a.e., which proves the second line in (1.22).

It follows from (1.22) that

$$\int_\Omega |\nabla u|\,\varphi\, dx = \int_{S_\alpha} |\nabla\varphi|\,\varphi dx \le \left(\int_{S_\alpha} \varphi^2 dx\right)^{1/2} \left(\int_{S_\alpha} |\nabla\varphi|^2\, dx\right)^{1/2}.$$

For the left hand side of (1.18) we have by (1.20) and the uniform ellipticity

$$\int_\Omega \sum_{i,j=1}^n a_{ij}\partial_j u \partial_i \varphi\, dx = \int_{S_\alpha} \sum_{i,j=1}^n a_{ij}\partial_j\varphi\partial_i\varphi\, dx \ge \lambda^{-1} \int_{S_\alpha} |\nabla\varphi|^2\, dx.$$

Combining the above two calculations with (1.18), we obtain

$$\lambda^{-1} \int_{S_\alpha} |\nabla\varphi|^2\, dx \le b \left(\int_{S_\alpha} \varphi^2 dx\right)^{1/2} \left(\int_{S_\alpha} |\nabla\varphi|^2\, dx\right)^{1/2}. \tag{1.23}$$

It follows that

$$\int_{S_\alpha} |\nabla\varphi|^2\, dx \le \lambda^2 b^2 \int_{S_\alpha} \varphi^2 dx. \tag{1.24}$$

By the Sobolev inequality we have

$$\int_{S_\alpha} |\nabla\varphi|^2\, dx = \int_\Omega |\nabla\varphi|^2\, dx \ge c\left(\int_\Omega \varphi^{2q} dx\right)^{1/q} \ge c\left(\int_{S_\alpha} \varphi^{2q} dx\right)^{1/q}, \tag{1.25}$$

where

$$\begin{array}{ll} q=\frac{n}{n-2}, & n>2 \\ q>1, & n=2. \end{array}$$

and $c=c\left(q,|\Omega|\right)>0$. On the other hand, we obtain by the Hölder inequality,

$$\int_{S_\alpha}\varphi^2dx=\int_{S_\alpha}1\cdot\varphi^2dx\leq\left(\int_{S_\alpha}1^{q'}dx\right)^{1/q'}\left(\int_{S_\alpha}\varphi^{2q}dx\right)^{1/q}=|S_\alpha|^{1/q'}\left(\int_{S_\alpha}\varphi^{2q}dx\right)^{1/q}$$

where q' is the Hölder conjugate of q, that is, $\frac{1}{q}+\frac{1}{q'}=1$ (so that $q'=\frac{q}{q-1}$), and $|S_\alpha|$ is the Lebesgue measure of the set S_α. Hence,

$$\left(\int_{S_\alpha}\varphi^{2q}dx\right)^{1/q}\geq|S_\alpha|^{-1/q'}\int_{S_\alpha}\varphi^2dx$$

Combining this with (1.24) and (1.25), we obtain

$$c\,|S_\alpha|^{-1/q'}\int_{S_\alpha}\varphi^2dx\leq\lambda^2b^2\int_{S_\alpha}\varphi^2dx.$$

Since $\operatorname{essup}\varphi=\alpha_0-\alpha>0$ and, hence, $\int_\Omega\varphi^2dx>0$, we obtain

$$|S_\alpha|\geq\left(\frac{c}{\lambda^2b^2}\right)^{q'}=:c' \tag{1.26}$$

where the constant c' is positive and does not depend of α.

Now let us bring (1.26) to contradiction. Consider an increasing sequence $\{\alpha_k\}_{k=1}^\infty$ that converges to α_0 as $k\to\infty$. Then the sequence of sets S_{α_k} is decreasing and

$$\bigcap_{k=1}^\infty S_{\alpha_k}=\{x\in\Omega:\forall k\ \ \alpha_k<u(x)<\alpha_0\}=\emptyset.$$

Hence, by the continuity property of the Lebesgue measure,

$$\lim_{k\to\infty}|S_{\alpha_k}|=|\textstyle\bigcap_{k=1}^\infty S_{\alpha_k}|=0,$$

which contradicts (1.26). ■

19.10.23 Lecture 4

1.4.2 Some properties of weak derivatives

Here Ω is an open subset of $\mathbb{R}^n$. Recall that

$$W^{1,2}(\Omega)=\left\{u\in L^2(\Omega):\nabla u\in\vec{L}^2(\Omega)\right\}$$

and

$$\|u\|_{W^{1,2}}^2=\|u\|_{L^2}^2+\||\nabla u|\|_{L^2}^2\,.$$

Recall also that $W_0^{1,2}(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ in $W^{1,2}(\Omega)$,

Lemma 1.6 (Chain rule in $W_0^{1,2}$) *Let ψ be a C^∞-function on $\mathbb{R}$ such that*

$$\psi(0) = 0 \ \ \textit{and} \ \ \sup_{t\in\mathbb{R}} |\psi'(t)| < \infty. \tag{1.27}$$

Then $u \in W_0^{1,2}(\Omega)$ implies $\psi(u) \in W_0^{1,2}(\Omega)$ and

$$\nabla\psi(u) = \psi'(u)\nabla u. \tag{1.28}$$

Proof. Let us first observe that, by (1.27), we have $|\psi(u)| \leq C|u|$ where $C = \sup|\psi'|$. It follows that $\psi(u) \in L^2(\Omega)$. The boundedness of ψ' implies also that $\psi'(u)\nabla u \in \vec{L}^2(\Omega)$.

If $u \in \mathcal{D}(\Omega)$ then obviously $\psi(u)$ is also in $\mathcal{D}(\Omega)$ (in particular, because $\psi(0) = 0$) and, hence, $\psi(u) \in W_0^{1,2}(\Omega)$. In this case the chain rule (1.28) is true because $\nabla\psi(u)$ is the classical derivative.

An arbitrary function $u \in W_0^{1,2}(\Omega)$ can be approximated by a sequence $\{u_k\} \subset \mathcal{D}(\Omega)$ that converges to u in $W^{1,2}(\Omega)$, that is,

$$u_k \stackrel{L^2}{\longrightarrow} u \ \text{ and } \ \ \nabla u_k \stackrel{L^2}{\longrightarrow} \nabla u.$$

By selecting a subsequence, we can assume that also $u_k(x) \to u(x)$ for almost all $x \in \Omega$.

Let us prove that

$$\psi(u_k) \stackrel{L^2}{\longrightarrow} \psi(u) \tag{1.29}$$

$$\nabla\psi(u_k) \stackrel{L^2}{\longrightarrow} \psi'(u)\nabla u. \tag{1.30}$$

The convergence (1.29) follows trivially from $u_k \stackrel{L^2}{\to} u$ and

$$|\psi(u_k) - \psi(u)| \leq C|u_k - u|\ .$$

To prove the convergence (1.30) observe that

$$\begin{aligned} |\nabla\psi(u_k) - \psi'(u)\nabla u| &= |\psi'(u_k)\nabla u_k - \psi'(u)\nabla u| \\ &\leq |\psi'(u_k)(\nabla u_k - \nabla u)| + |(\psi'(u_k) - \psi'(u))\nabla u|, \end{aligned}$$

whence

$$\|\nabla\psi(u_k) - \psi'(u)\nabla u\|_{L^2} \leq C\|\nabla u_k - \nabla u\|_{L^2} + \|(\psi'(u_k) - \psi'(u))\nabla u\|_{L^2}. \tag{1.31}$$

The first term on the right hand side of (1.31) goes to 0 because $\nabla u_k \stackrel{L^2}{\longrightarrow} \nabla u$. It remains to verify that

$$\|(\psi'(u_k) - \psi'(u))\nabla u\|_{L^2}^2 = \int_\Omega |\psi'(u_k) - \psi'(u)|^2 |\nabla u|^2 d\mu \longrightarrow 0 \tag{1.32}$$

as $k \to \infty$. Since $u_k(x) \to u(x)$ a.e. , we have

$$\psi'(u_k) - \psi'(u) \longrightarrow 0 \text{ a.e.}$$

and, hence,

$$\left|\psi'(u_k) - \psi'(u)\right|^2 |\nabla u|^2 \to 0 \text{ a.e..}$$

Since

$$\left|\psi'(u_k) - \psi'(u)\right|^2 |\nabla u|^2 \le 4C^2 |\nabla u|^2$$

and the function $|\nabla u|^2$ is integrable on Ω, we conclude that (1.32) holds by the dominated convergence theorem.

For the next argument we need the convergence in $\mathcal{D}'(\Omega)$: if f and $f_k \in \mathcal{D}'(\Omega)$ then $f_k \overset{\mathcal{D}'}{\to} f$ if

$$(f_k, \phi) \to (f, \phi) \quad \text{for all } \phi \in \mathcal{D}(\Omega) \quad \text{as } k \to \infty.$$

This convergence has the following property: if $f_k \overset{\mathcal{D}'}{\to} f$ then, for any multiindex α,

$$D^\alpha f_k \overset{\mathcal{D}'}{\to} D^\alpha f,$$

because for any $\phi \in \mathcal{D}(\Omega)$

$$(D^\alpha f_k, \phi) = (-1)^{|\alpha|} (f_k, D^\alpha \phi) \to (-1)^{|\alpha|} (f, D^\alpha \phi) = (D^\alpha f, \phi).$$

The convergence (1.29) implies that

$$\nabla \psi(u_k) \overset{\mathcal{D}'}{\to} \nabla \psi(u),$$

which together with (1.30) yields

$$\nabla \psi(u) = \psi'(u) \nabla u. \tag{1.33}$$

It follows that $\psi(u) \in W^{1,2}(\Omega)$. Since $\psi(u_k) \in \mathcal{D}(\Omega)$ and, by (1.29)-(1.30) and (1.33),

$$\psi(u_k) \overset{W^{1,2}}{\longrightarrow} \psi(u),$$

we will conclude that $\psi(u) \in W_0^{1,2}(\Omega)$. ■

Lemma 1.7 *Let $\{\psi_k(t)\}$ be a sequence of C^∞-smooth functions on $\mathbb{R}$ such that*

$$\psi_k(0) = 0 \quad \text{and} \quad \sup_k \sup_{t \in \mathbb{R}} |\psi_k'(t)| < \infty. \tag{1.34}$$

Assume that, for some functions $\psi(t)$ and $\varphi(t)$ on $\mathbb{R}$,

$$\psi_k(t) \to \psi(t) \quad \text{and} \quad \psi_k'(t) \to \varphi(t) \quad \text{for all } t \in \mathbb{R}. \tag{1.35}$$

Then, for any $u \in W_0^{1,2}(\Omega)$, the function $\psi(u)$ is also in $W_0^{1,2}(\Omega)$ and

$$\nabla \psi(u) = \varphi(u) \nabla u.$$

Proof. The function $\psi(u)$ is the pointwise limit of measurable functions $\psi_k(u)$ and, hence, is measurable; by the same argument, $\varphi(u)$ is also measurable. By (1.34), there is a constant C such that

$$|\psi_k(t)| \le C|t|, \tag{1.36}$$

for all k and $t \in \mathbb{R}$, and the same holds for function ψ. Therefore, $|\psi(u)| \leq C|u|$, which implies

$$\psi(u) \in L^2(\Omega).$$

By (1.34), we have also $|\varphi(t)| \leq C$, whence

$$\varphi(u) \nabla u \in \vec{L}^2(\Omega).$$

Since each function ψ_k is smooth and satisfies (1.27), Lemma 1.6 yields that

$$\psi_k(u) \in W_0^{1,2}(\Omega) \quad \text{and} \quad \nabla \psi_k(u) = \psi_k'(u) \nabla u.$$

Let us show that

$$\psi_k(u) \xrightarrow{L^2} \psi(u) \quad \text{and} \quad \nabla \psi_k(u) \xrightarrow{L^2} \varphi(u) \nabla u. \tag{1.37}$$

The dominated convergence theorem implies that

$$\int_\Omega |\psi_k(u) - \psi(u)|^2 d\mu \longrightarrow 0,$$

because the integrand functions tend pointwise to 0 as $k \to \infty$ and, by (1.36),

$$|\psi_k(u) - \psi(u)|^2 \leq 4C^2 u^2,$$

while u^2 is integrable on Ω. Similarly, we have

$$\int_\Omega |\nabla \psi_k(u) - \varphi(u) \nabla u|^2 d\mu = \int_\Omega |\psi_k'(u) - \varphi(u)|^2 |\nabla u|^2 d\mu \longrightarrow 0,$$

because the sequence of functions $|\psi_k'(u) - \varphi(u)|^2 |\nabla u|^2$ tends pointwise to 0 as $k \to \infty$ and is uniformly bounded by the integrable function $4C^2 |\nabla u|^2$.

It follows from $\psi_k(u) \xrightarrow{L^2} \psi(u)$ that

$$\nabla \psi_k(u) \xrightarrow{\mathcal{D}'} \nabla \psi(u),$$

and comparison with (1.37) yields that

$$\nabla \psi(u) = \varphi(u) \nabla u. \tag{1.38}$$

Consequently, $\psi(u) \in W^{1,2}(\Omega)$. It follows from (1.37) and (1.38) that

$$\psi_k(u) \xrightarrow{W^{1,2}} \psi(u).$$

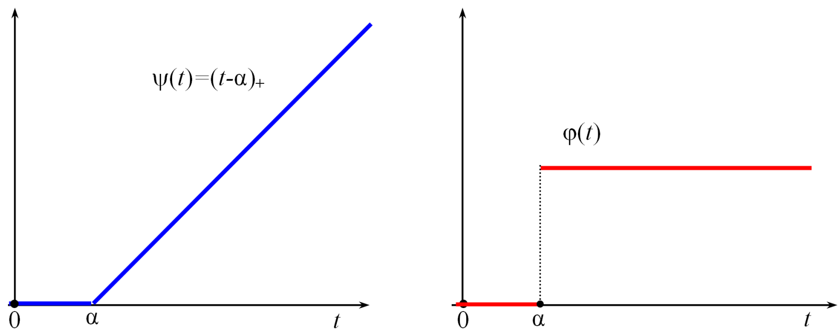
Since $\psi_k(u) \in W_0^{1,2}(\Omega)$ and $W_0^{1,2}(\Omega)$ is a closed subspace of $W^{1,2}(\Omega)$, we conclude that $\psi(u) \in W_0^{1,2}(\Omega)$, which finishes the proof. ■

Proof of Lemma 1.4. We need to prove that if $u \in W_0^{1,2}(\Omega)$ then, for any $\alpha \geq 0$, also $(u - \alpha)_+ \in W_0^{1,2}(\Omega)$ and

$$\nabla (u - \alpha)_+ = \begin{cases} \nabla u & \text{a.e. on the set } \{u > \alpha\} \\ 0 & \text{a.e. on the set } \{u \leq \alpha\} \end{cases}. \tag{1.39}$$

Consider the functions

$$\psi\left(t\right)=\left(t-\alpha\right)_{+}\quad\text{and}\quad\varphi\left(t\right)=\begin{cases}1, & t>\alpha,\\ 0, & t\leq\alpha.\end{cases}$$



Then the claim of Lemma 1.4 can be reformulated as follows: $\psi\left(u\right)\in W_0^{1,2}\left(\Omega\right)$ and

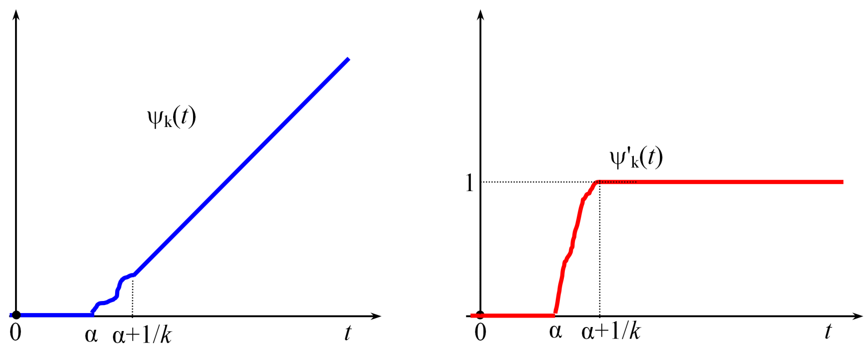
$$\nabla\psi\left(u\right)=\varphi\left(u\right)\nabla u.$$

By Lemma 1.7, it suffices to verify that ψ and φ that can be approximated as in (1.35). For that fix any nonnegative C^∞ function $\eta\left(t\right)$ on $\mathbb{R}$ such that

$$\eta\left(t\right)=\begin{cases}t, & t\geq 1,\\ 0, & t\leq 0.\end{cases}$$

Define ψ_k for any $k\in\mathbb{N}$ by

$$\psi_k\left(t\right)=\frac{1}{k}\eta\left(k\left(t-\alpha\right)\right).$$



If $t\leq\alpha$ then $\psi_k\left(t\right)=0$. If $t>\alpha$ then, for large enough k, we have $k\left(t-\alpha\right)>1$ whence

$$\psi_k\left(t\right)=\frac{1}{k}\left(k\left(t-\alpha\right)\right)=\frac{1}{k}k\left(t-\alpha\right)=t-\alpha\rightarrow\psi\left(t\right)\quad\text{as }k\rightarrow\infty.$$

Hence, $\psi_k(t) \to \psi(t)$ for all $t \in \mathbb{R}$.

Similarly, if $t \leq \alpha$ then $\psi_k'(t) = 0$, and if $t > \alpha$ then, for large enough k, we have $k(t-\alpha) > 1$ whence

$$\psi_k'(t) = \eta'(k(t-\alpha)) = 1 \to \varphi(t) \quad \text{as } k \to \infty.$$

∎

Proof of Lemma 1.5. By Lemma 1.4 with $\alpha = 0$, we have $u_+ \in W_0^{1,2}$ and

$$\nabla u_+ = \begin{cases} \nabla u, & u > 0, \\ 0, & u \leq 0. \end{cases} \tag{1.40}$$

Applying this to function $(-u)$, we obtain that $u_- \in W_0^{1,2}$ and

$$\nabla u_- = \begin{cases} 0, & u \geq 0, \\ -\nabla u, & u < 0. \end{cases} \tag{1.41}$$

Consequently, since $\nabla u_+ = \nabla u_- = 0$ on the set $\{u = 0\}$, we obtain

$$\nabla u = 0 \text{ a.e. on } \{u = 0\}. \tag{1.42}$$

In particular, (1.42) implies the following: if u, v are two functions from $W_0^{1,2}(\Omega)$ and S is a subset of Ω then

$$u = v \text{ a.e. on } S \quad \Rightarrow \nabla u = \nabla v \text{ a.e. on } S.$$

Let us now prove that, for any $\alpha \in \mathbb{R}$,

$$\nabla u = 0 \text{ a.e. on } \{u = \alpha\}. \tag{1.43}$$

If the constant function $v \equiv \alpha$ were in $W_0^{1,2}$ then by

$$u = v \quad \text{on } \{u = \alpha\}$$

we could obtain

$$\nabla u = \nabla v = 0 \text{ a.e.on } \{u = \alpha\}$$

thus proving (1.43). However, the constant function is not in $W_0^{1,2}$ and we argue as follows. Choose a closed ball $K \subset \Omega$ and a function $v \in C_0^\infty(\Omega)$ such that $v = \alpha$ in K. Then

$$u = v \quad \text{on } \{u = \alpha\} \cap K$$

which implies that

$$\nabla u = \nabla v = 0 \text{ a.e. on } \{u = \alpha\} \cap K.$$

Covering Ω by a countable family of balls K, we obtain (1.43). ∎

23.10.23 Lecture 5

1.4.3 Sobolev inequality

Theorem 1.8 *Assume $1 \le p < n$. Then there is a constant $C = C(p,n)$ such that, for all $u \in W_0^{1,p}(\mathbb{R}^n)$,*

$$\left(\int_{\mathbb{R}^n} |u|^{\frac{pn}{n-p}} dx\right)^{\frac{n-p}{n}} \le C \int_{\mathbb{R}^n} |\nabla u|^p \, dx. \tag{1.44}$$

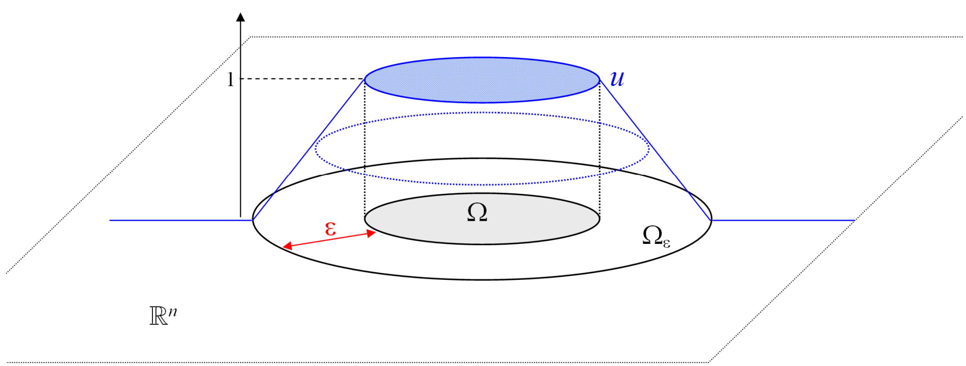
Remark. Let us explain the geometric meaning of (1.44) in the (main) case $p = 1$. In this case the Sobolev inequality becomes as follows: for $n > 1$ and for any $u \in W_0^{1,1}(\mathbb{R}^n)$,

$$\left(\int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}} dx\right)^{\frac{n-1}{n}} \le C \int_{\mathbb{R}^n} |\nabla u| \, dx, \tag{1.45}$$

with some constant $C = C(n)$. Fix an bounded open set $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$, and, for any $\varepsilon > 0$, denote by Ω_ε the open ε-neighborhood of Ω. Let u_ε be a continuous function in $\mathbb{R}^n$ such that

$$u_\varepsilon(x) = \begin{cases} 1, & x \in \Omega, \\ 0, & x \in (\Omega_\varepsilon)^c, \\ \text{linear in } \operatorname{dist}(x,\Omega), & x \in \Omega_\varepsilon \setminus \Omega. \end{cases}$$

It is possible to prove that u is a Lipschitz function and, hence, $u_\varepsilon \in W_0^{1,1}(\mathbb{R}^n)$.



Since $u_\varepsilon = 1$ in Ω and $|\Omega_\varepsilon \setminus \Omega| \to 0$ as $\varepsilon \to 0$, we obtain

$$\int_{\mathbb{R}^n} |u_\varepsilon|^{\frac{n}{n-1}} dx \to |\Omega| \quad \text{as } \varepsilon \to 0.$$

Since $|\nabla u| \sim \frac{1}{\varepsilon}$ in $\Omega_\varepsilon \setminus \Omega$ and $\nabla u = 0$ otherwise, we obtain

$$\int_{\mathbb{R}^n} |\nabla u| \, dx \sim \frac{1}{\varepsilon} |\Omega_\varepsilon \setminus \Omega| \to \sigma(\partial\Omega) \quad \text{as } \varepsilon \to 0,$$

which implies from (1.45) as $\varepsilon \to 0$ that

$$|\Omega|^{\frac{n-1}{n}} \le C\sigma\left(\partial\Omega\right). \tag{1.46}$$

This is a so called *isoperimetric inequality* that bounds the volume $|\Omega|$ from above by the boundary area of $\partial\Omega$. One can show that, conversely, the Sobolev inequality (1.45) can be derived from the isoperimetric inequality (1.46) so that these two statements are equivalent.

Let Ω be a ball of radius R. Then we have

$$\sigma\left(\partial\Omega\right) = \omega_n R^{n-1}$$

and

$$|\Omega| = \frac{\omega_n}{n} R^n,$$

whence it follows that

$$|\Omega|^{\frac{n-1}{n}} = c_n \sigma\left(\partial\Omega\right),$$

where $c_n = \left(\omega_n/n\right)^{\frac{n-1}{n}}/\omega_n$. It is possible to prove that the optimal constant C in (1.46) is exactly c_n, that is, among all domains Ω with a fixed boundary area $\sigma\left(\partial\Omega\right)$ (=perimeter), the ball has the maximal volume (the *isoperimetric property* of balls in $\mathbb{R}^n$).

In the proof of Theorem 1.8 we will use the following extended Hölder inequality.

Lemma 1.9 *For non-negative measurable functions* $\{f_i\}_{i=1}^m$ *on* $\mathbb{R}$*, we have*

$$\int_{\mathbb{R}} \prod_{i=1}^m f_i^{1/m} dt \le \prod_{i=1}^m \left(\int_{\mathbb{R}} f_i dt\right)^{1/m}. \tag{1.47}$$

Proof. For $m = 1$ the inequality (1.47) is trivial as the both sides are equal to $\int_{\mathbb{R}} f_1 dt$. In the case $m = 2$ (1.47) follows from the Cauchy-Schwarz inequality

$$\int_{\mathbb{R}} f_1^{1/2} f_2^{1/2} dt \le \left(\int_{\mathbb{R}} f_1 dt\right)^{1/2} \left(\int_{\mathbb{R}} f_2 dt\right)^{1/2}.$$

For a general m, we make the inductive step from $m-1$ to m by means of the Hölder inequality

$$\int fg \le \left(\int f^{\frac{m}{m-1}}\right)^{\frac{m-1}{m}} \left(\int g^m\right)^{\frac{1}{m}}. \tag{1.48}$$

Using (1.48) and the inductive hypothesis, we obtain

$$\begin{aligned}
\int_{\mathbb{R}} \underbrace{f_1^{\frac{1}{m}} \dots f_{m-1}^{\frac{1}{m}}}_{f} \underbrace{f_m^{\frac{1}{m}}}_{g} dt &\le \left(\int_{\mathbb{R}} \left(f_1^{\frac{1}{m}} \dots f_{m-1}^{\frac{1}{m}}\right)^{\frac{m}{m-1}} dt\right)^{\frac{m-1}{m}} \left(\int_{\mathbb{R}} \left(f_m^{\frac{1}{m}}\right)^m dt\right)^{\frac{1}{m}} \\
&= \left(\int_{\mathbb{R}} f_1^{\frac{1}{m-1}} \dots f_{m-1}^{\frac{1}{m-1}} dt\right)^{\frac{m-1}{m}} \left(\int_{\mathbb{R}} f_m dt\right)^{\frac{1}{m}} \\
&\le \left(\left(\int_{\mathbb{R}} f_1 dt\right)^{\frac{1}{m-1}} \dots \left(\int_{\mathbb{R}} f_{m-1} dt\right)^{\frac{1}{m-1}}\right)^{\frac{m-1}{m}} \left(\int_{\mathbb{R}} f_m dt\right)^{\frac{1}{m}} \\
&= \left(\int_{\mathbb{R}} f_1 dt\right)^{\frac{1}{m}} \dots \left(\int_{\mathbb{R}} f_{m-1} dt\right)^{\frac{1}{m}} \left(\int_{\mathbb{R}} f_m dt\right)^{\frac{1}{m}},
\end{aligned}$$

which is equivalent to (1.47). ∎

Proof of Theorem 1.8. *Step 0.* Let us first show that it suffices to prove (1.44) for $u \in \mathcal{D}(\mathbb{R}^n)$. Indeed, assuming that (1.44) is known to be true for all $u \in \mathcal{D}(\mathbb{R}^n)$. For any $u \in W_0^{1,p}(\mathbb{R}^n)$, choose a sequence $\{u_k\}$ from $\mathcal{D}(\mathbb{R}^n)$ such that $u_k \to u$ in the norm of $W^{1,p}$. Since $u_k \to u$ in L^p, choosing a subsequence we can assume that also $u_k \to u$ a.e.. By the assumption of validity of (1.44) for functions from $\mathcal{D}(\mathbb{R}^n)$, we have, for any k,

$$\left(\int_{\mathbb{R}^n} |u_k|^{\frac{pn}{n-p}}\, dx\right)^{\frac{n-p}{n}} \le C \int_{\mathbb{R}^n} |\nabla u_k|^p\, dx. \tag{1.49}$$

Since $u_k \overset{W^{1,p}}{\to} u$, we have

$$\int_{\mathbb{R}^n} |\nabla u_k|^p\, dx \to \int_{\mathbb{R}^n} |\nabla u|^p\, dx \ \text{ as } k \to \infty.$$

Since $u_k \overset{\text{a.e.}}{\to} u$, we obtain by Fatou's lemma that

$$\int_{\mathbb{R}^n} |u|^{\frac{pn}{n-p}}\, dx \le \liminf_{k\to\infty} \int_{\mathbb{R}^n} |u_k|^{\frac{pn}{n-p}}\, dx.$$

Hence, taking lim inf in (1.49), we obtain

$$\left(\int_{\mathbb{R}^n} |u|^{\frac{pn}{n-p}}\, dx\right)^{\frac{n-p}{n}} \le C \int_{\mathbb{R}^n} |\nabla u|^p\, dx,$$

which was to be proved.

Step 1. Let us prove (1.44) in the case $p = 1$ (and $n > 1$) for any $u \in C_0^1(\mathbb{R}^n)$ (and, hence, for any $u \in \mathcal{D}(\Omega)$). For $p = 1$ (1.44) becomes

$$\left(\int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}}\, dx\right)^{\frac{n-1}{n}} \le C \int_{\mathbb{R}^n} |\nabla u|\, dx. \tag{1.50}$$

Since u has a compact support, we have, for any index $i = 1, ..., n$,

$$u(x) = \int_{-\infty}^{x_i} \partial_i u\,(x_1, .., x_{i-1}, y_i, x_{i+1}, ..., x_n)\, dy_i,$$

which implies

$$|u(x)| \le \int_{-\infty}^{\infty} |\nabla u|\,(x_1, .., x_{i-1}, y_i, x_{i+1}, ..., x_n)\, dy_i. \tag{1.51}$$

Consider function $F = |\nabla u|$ and let us use the following notation: for any sequence $i_1, ..., i_k$ of distinct indices, set

$$F_{i_1 ... i_k} = \int_{\mathbb{R}} ... \int_{\mathbb{R}} F(x) dx_{i_1} dx_{i_2} ... dx_{i_k}.$$

By construction, $F_{i_1...i_k}$ is a function of all components x_j where $j \ne i_1, ..., i_k$. However, it will be convenient to consider $F_{i_1...i_k}$ as a function of all components of $x = (x_1, ..., x_n)$ that does not depend on $x_{i_1}, ..., x_{i_k}$.

Inequality (1.51) can be then rewritten in a short form

$$|u(x)| \le F_i(x).$$

Multiplying all these inequalities for $i = 1, ..., n$ and raising to the power $\frac{1}{n-1}$, we obtain

$$|u(x)|^{\frac{n}{n-1}} \le \prod_{i=1}^{n} F_i^{\frac{1}{n-1}}(x).$$

Let us integrate this inequality in x_1. Since F_1 does not depend on x_1, we obtain, using (1.47) with $m = n - 1$, that

$$\begin{aligned}
\int_{\mathbb{R}} |u(x)|^{\frac{n}{n-1}}\, dx_1 &\le F_1^{\frac{1}{n-1}} \int_{\mathbb{R}} \left(\prod_{i=2}^{n} F_i^{\frac{1}{n-1}} \right) dx_1 \\
&\le F_1^{\frac{1}{n-1}} \prod_{i=2}^{n} \left(\int_{\mathbb{R}} F_i dx_1 \right)^{\frac{1}{n-1}} \\
&= F_1^{\frac{1}{n-1}} \prod_{i=2}^{n} F_{1i}^{\frac{1}{n-1}}.
\end{aligned}$$

Now let us integrate the last inequality in x_2. Noticing that F_{12} does not depend on x_2 and using again (1.47), we obtain

$$\begin{aligned}
\int_{\mathbb{R}^2} |u(x)|^{\frac{n}{n-1}}\, dx_1 dx_2 &\le F_{12}^{\frac{1}{n-1}} \int_{\mathbb{R}} \left(F_1^{\frac{1}{n-1}} \prod_{i=3}^{n} F_{1i}^{\frac{1}{n-1}} \right) dx_2 \\
&\le F_{12}^{\frac{1}{n-1}} \left(\int_{\mathbb{R}} F_1 dx_2 \right)^{\frac{1}{n-1}} \prod_{i=3}^{n} \left(\int_{\mathbb{R}} F_{1i} dx_2 \right)^{\frac{1}{n-1}} \\
&= F_{12}^{\frac{1}{n-1}} F_{12}^{\frac{1}{n-1}} \prod_{i=3}^{n} F_{12i}^{\frac{1}{n-1}}.
\end{aligned}$$

Integrating the last inequality in x_3, noticing that F_{123} does not depend on x_3 and using (1.47), we obtain

$$\begin{aligned}
\int_{\mathbb{R}^3} |u(x)|^{\frac{n}{n-1}}\, dx_1 dx_2 dx_3 &\le F_{123}^{\frac{1}{n-1}} \int_{\mathbb{R}} \left(F_{12}^{\frac{1}{n-1}} F_{12}^{\frac{1}{n-1}} \prod_{i=4}^{n} F_{12i}^{\frac{1}{n-1}} \right) dx_3 \\
&\le F_{123}^{\frac{1}{n-1}} \left(\int_{\mathbb{R}} F_{12} dx_3 \right)^{\frac{1}{n-1}} \left(\int_{\mathbb{R}} F_{12} dx_3 \right)^{\frac{1}{n-1}} \prod_{i=4}^{n} \left(\int_{\mathbb{R}} F_{12i} dx_3 \right)^{\frac{1}{n-1}} \\
&= F_{123}^{\frac{1}{n-1}} F_{123}^{\frac{1}{n-1}} F_{123}^{\frac{1}{n-1}} \prod_{i=4}^{n} F_{123i}^{\frac{1}{n-1}}.
\end{aligned}$$

Continuing further by induction, we obtain that, for any $1 \le k \le n$,

$$\int_{\mathbb{R}^k} |u(x)|^{\frac{n}{n-1}}\, dx_1 ... dx_k \le F_{1...k}^{\frac{k}{n-1}} \prod_{i=k+1}^{n} F_{1...ki}^{\frac{1}{n-1}}.$$

In particular, for $k = n$ we obtain

$$\int_{\mathbb{R}^n} |u(x)|^{\frac{n}{n-1}} dx \leq F_{12...n}^{\frac{n}{n-1}} = \left(\int_{\mathbb{R}^n} |\nabla u| \, dx \right)^{\frac{n}{n-1}},$$

which proves (1.50) with $C = 1$.

Step 2. Let us prove now the Sobolev inequality (1.44) in the case $p > 1$, also for any $u \in C_0^1(\mathbb{R}^n)$. For that we will apply (1.50) to the function $|u|^\alpha$ with some $\alpha > 0$. Observe that, for any $\alpha > 1$, the function $|u|^\alpha$ belongs to $C_0^1(\mathbb{R}^n)$ because $|u|^\alpha = f(u)$ where the function $f(t) = |t|^\alpha$ is continuously differentiable in $\mathbb{R}$ and

$$f'(t) = \alpha |t|^{\alpha-1} \operatorname{sgn} t.$$

It follows that

$$\nabla |u|^\alpha = \alpha |u|^{\alpha-1} \operatorname{sgn} u \nabla u. \tag{1.52}$$

Applying (1.50) to the function $|u|^\alpha$ and using (1.52), we obtain

$$\left(\int_{\mathbb{R}^n} |u|^{\boxed{\frac{\alpha n}{n-1}}} dx \right)^{\frac{n-1}{n}} \leq \int_{\mathbb{R}^n} |\nabla |u|^\alpha| \, dx = \alpha \int_{\mathbb{R}^n} |u|^{\alpha-1} |\nabla u| \, dx. \tag{1.53}$$

By the Hölder inequality, we have

$$\int_{\mathbb{R}^n} |u|^{\alpha-1} |\nabla u| \, dx \leq \left(\int_{\mathbb{R}^n} |u|^{\boxed{\frac{(\alpha-1)p}{p-1}}} dx \right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^n} |\nabla u|^p \, dx \right)^{\frac{1}{p}}. \tag{1.54}$$

Choose α so that

$$\frac{(\alpha-1)p}{p-1} = \frac{\alpha n}{n-1}.$$

Solving this equation in α we obtain

$$\alpha \left(\frac{p}{p-1} - \frac{n}{n-1} \right) = \frac{p}{p-1},$$

$$\alpha \frac{n-p}{(p-1)(n-1)} = \frac{p}{p-1},$$

$$\alpha = \frac{(n-1)p}{n-p}.$$

Note that $\alpha > 1$ due to the assumption $1 < p < n$. For this α we have

$$\frac{\alpha n}{n-1} = \frac{pn}{n-p} =: q,$$

and we obtain from (1.53)-(1.54)

$$\left(\int_{\mathbb{R}^n} |u|^q \, dx \right)^{\frac{n-1}{n}} \leq \alpha \left(\int_{\mathbb{R}^n} |u|^q \, dx \right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^n} |\nabla u|^p \, dx \right)^{\frac{1}{p}}.$$

It follows that

$$\left(\int_{\mathbb{R}^n}|u|^q\,dx\right)^{\frac{n-1}{n}-\frac{p-1}{p}}\le\alpha\left(\int_{\mathbb{R}^n}|\nabla u|^p\,dx\right)^{\frac{1}{p}},$$

$$\left(\int_{\mathbb{R}^n}|u|^q\,dx\right)^{\frac{n-p}{np}}\le\alpha\left(\int_{\mathbb{R}^n}|\nabla u|^p\,dx\right)^{\frac{1}{p}}.$$

Rasing this inequality to the power p, we obtain (1.44) with

$$C=\alpha^p=\left(\frac{(n-1)\,p}{n-p}\right)^p.$$

■

26.10.23 Lecture 6

Now let us prove the Sobolev inequality in the form that was used in the proof of Theorem 1.3.

Corollary 1.10 *Let Ω be an open subset of $\mathbb{R}^n$. If $n>2$ then, for any $u\in W_0^{1,2}(\Omega)$,*

$$\int_\Omega|\nabla u|^2\,dx\ge c\left(\int_\Omega|u|^{\frac{2n}{n-2}}\,dx\right)^{\frac{n-2}{n}} \tag{1.55}$$

where $c=c(n)>0$. If $n=2$ and Ω is bounded then, for any $q\ge1$ and any $u\in W_0^{1,2}(\Omega)$,

$$\int_\Omega|\nabla u|^2\,dx\ge c\left(\int_\Omega|u|^{2q}\,dx\right)^{1/q}, \tag{1.56}$$

where $c=c_0\,|\Omega|^{-1/q}$ and $c_0=c_0(q)>0$.

Proof. Since $\mathcal{D}(\Omega)\subset\mathcal{D}(\mathbb{R}^n)$, it follows that $W_0^{1,2}(\Omega)\subset W_0^{1,2}(\mathbb{R}^n)$. More precisely, any function from $W_0^{1,2}(\Omega)$ that is extended by 0 outside Ω, belongs to $W_0^{1,2}(\mathbb{R}^n)$. Therefore, (1.55) is a particular case of (1.44) with $p=2$.

Assume $n=2$. By Exercise 12, we have, for any $p\in[1,2)$,

$$W_0^{1,2}(\Omega)\subset W_0^{1,p}(\Omega).$$

Hence, for any $u\in W_0^{1,2}(\Omega)$, we can apply the Sobolev inequality (1.44) with any $p\in[1,2)$ and obtain

$$\left(\int_\Omega|u|^{\frac{2p}{2-p}}\,dx\right)^{\frac{2-p}{2}}\le C\int_\Omega|\nabla u|^p\,dx.$$

By the Hölder inequality, we have

$$\begin{aligned}\int_\Omega|\nabla u|^p\,dx&=\int_\Omega 1\cdot|\nabla u|^p\,dx\le\left(\int_\Omega 1\cdot dx\right)^{1-\frac{p}{2}}\left(\int_\Omega|\nabla u|^{p\frac{2}{p}}\,dx\right)^{\frac{p}{2}}\\&=|\Omega|^{1-\frac{p}{2}}\left(\int_\Omega|\nabla u|^2\,dx\right)^{\frac{p}{2}}.\end{aligned}$$

It follows that

$$\left(\int_\Omega |u|^{\frac{2p}{2-p}}\,dx\right)^{\frac{2-p}{2}} \le C\,|\Omega|^{\frac{2-p}{2}}\left(\int_\Omega |\nabla u|^2\,dx\right)^{\frac{p}{2}}$$

and

$$\left(\int_\Omega |u|^{\frac{2p}{2-p}}\,dx\right)^{\frac{2-p}{p}} \le C\,|\Omega|^{\frac{2-p}{p}}\int_\Omega |\nabla u|^2\,dx.$$

Let us set $q=\frac{p}{2-p}$ and observe that q can be any number from $[1,\infty)$ as p is any number from $[1,2)$. Rewriting the above inequality in the form

$$\int_\Omega |\nabla u|^2\,dx \ge C^{-1}\,|\Omega|^{-1/q}\left(\int_\Omega |u|^{2q}\,dx\right)^{1/q},$$

we obtain (1.56). ■

1.4.4 Theorem of Lax-Milgram

Let H be a Hilbert space with an inner product $[\cdot,\cdot]$. The following theorem is a generalization of the Riesz representation theorem for non-symmetric bilinear forms.

Theorem 1.11 *Let $B(u,v)$ be a bilinear form in H. Assume that*

1. *B is bounded, that is, for some constant C,*

$$|B(u,v)| \le C\,\|u\|\,\|v\| \text{ for all } u,v\in H.$$

2. *B is coercive, that is, for some constant $c>0$,*

$$B(u,u)\ge c\,\|u\|^2 \text{ for all } u\in H.$$

Then, for any bounded linear functional l on H, the equation

$$B(u,v)=l(v)\quad \forall v\in H \tag{1.57}$$

has a unique solution $u\in H$. Moreover, for this solution we have

$$\|u\|\le c^{-1}\,\|l\|\,. \tag{1.58}$$

If the bilinear form $B(u,v)$ is symmetric then $B(u,v)$ is an inner product in H whose norm $B(u,u)^{1/2}$ is comparable with $\|u\|$. It follows that the linear space H with the inner product $B(u,v)$ is again a Hilbert space, and the solvability of the equation (1.57) is given by the Riesz representation theorem. The strength of Theorem 1.11 is that it works for *non-symmetric* forms B.

Proof. For any fixed $u\in H$, the function $v\mapsto B(u,v)$ is a bounded linear functional on H. Hence, by the Riesz representation theorem, the equation

$$[z,v]=B(u,v)\quad\forall v\in H$$

has a unique solution $z \in H$. Since z depends on u, we obtain a mapping $A : H \to H$, defined by $Au = z$. In other words, A is defined by the identity

$$[Au, v] = B(u, v) \quad \forall v \in H. \tag{1.59}$$

Operator A is called the *generator* of the bilinear form B. Clearly, the equation (1.57) is equivalent to

$$[Au, v] = l(v) \quad \forall v \in H. \tag{1.60}$$

Again by Riesz representation theorem, there is $w \in H$ such that

$$[w, v] = l(v) \quad \forall v \in H.$$

Therefore, in order to solve (1.60) it suffices to find u so that $Au = w$.

Hence, the question of solving of (1.57) amounts to verifying that A is bijective, so that the equation $Au = w$ has a solution $u = A^{-1}w$.

Let us prove that A is bijective in the following few steps.

Step 1. Operator A is linear. Indeed, for any $u_1, u_2 \in H$ and for all $v \in H$ we have by (1.59)

$$[A(u_1 + u_2), v] = B(u_1 + u_2, v) = B(u_1, v) + B(u_2, v) = [Au_1, v] + [Au_2, v],$$

which implies $Au_1 + Au_2 = A(u_1 + u_2)$. Similarly one shows that $A(\lambda u) = \lambda A(u)$ for any $\lambda \in \mathbb{R}$.

Step 2. Operator A is bounded. Indeed, it follows from (1.59) that, for all $u, v \in H$,

$$|[Au, v]| \le C \|u\| \|v\|.$$

Setting here $v = Au$, we obtain

$$\|Au\|^2 \le C \|u\| \|Au\|$$

whence $\|Au\| \le C\|u\|$, which proves the claim.

Step 3. Operator A is injective. Indeed, setting $v = u$ in (1.59), we obtain

$$[Au, u] = B(u, u) \ge c\|u\|^2. \tag{1.61}$$

In particular, $Au = 0$ implies $u = 0$, that is, A is injective. Moreover, applying Cauchy-Schwarz inequality to the left hand side of (1.61), we obtain

$$\|Au\| \|u\| \ge c\|u\|^2$$

and, hence,

$$\|Au\| \ge c\|u\| \quad \forall u \in H. \tag{1.62}$$

Step 4. The image $\operatorname{Im} A$ is dense in H. Indeed, if $\overline{\operatorname{Im} A} \neq H$ then there is a non-zero vector u in H that is orthogonal to $\operatorname{Im} A$. In particular, $[Au, u] = 0$, which by (1.61) is not possible.

Step 5. Operator A is surjective, that is, $\operatorname{Im} A = H$. In the view of Step4, it suffices to verify that $\operatorname{Im} A$ is a closed set. Indeed, let $\{w_k\}$ be a sequence of elements from

$\operatorname{Im} A$ that converges to $w \in H$. Let us show that $w \in \operatorname{Im} A$. We have $w_k = Au_k$ for some $u_k \in H$. It follows from (1.62) that, for all $k, l \in \mathbb{N}$,

$$\|w_k - w_l\| = \|A(u_k - u_l)\| \geq c\|u_k - u_l\|,$$

which implies that the sequence $\{u_k\}$ is Cauchy. Hence, there exists the limit

$$u := \lim_{k\to\infty} u_k.$$

By the boundedness of A we obtain

$$Au = \lim_{k\to\infty} Au_k = \lim_{k\to\infty} w_k = w$$

and, hence, $w \in \operatorname{Im} A$.

By Steps 3 and 5, we conclude that A is bijective and, hence, the equation (1.57) has a unique solution u.

Step 6. Finally, let us prove (1.58). Setting in (1.57) $v = u$ and using the coercive property of B, we obtain

$$c\|u\|^2 \leq B(u,u) = l(u) \leq \|l\|\|u\|,$$

whence $\|u\| \leq c^{-1}\|l\|$ follows. ■

1.4.5 Fredholm's alternative

Theorem 1.12 *Let K be a compact linear operator in a Hilbert space H. Set $A = I + K$. If the operator A is injective then A is surjective.*

Here I is the identity operator in H. Recall that K is a compact operator if, for any bounded sequence $\{x_i\} \subset H$, the sequence $\{Kx_i\}$ has a convergent subsequence.

Remark. Theorem 1.12 is a particular case of the following more general statement, a full Fredholm alternative: if $A = I + K$ where K is a compact operator in a Hilbert space then

1. $\dim\ker A = \dim\ker A^* < \infty$;
2. $\operatorname{Im} A = (\ker A^*)^\perp$.

In particular, if A is injective then $\ker A^* = \{0\}$ and, hence, $\operatorname{Im} A = H$, that is, A is surjective.

Remark. In a finite dimensional Euclidean space H, any linear operator $A : H \to H$ has this property: if A is injective then A is surjective, because each of this properties is equivalent to $\det A \neq 0$. In infinite dimensional spaces this is not the case for arbitrary bounded linear operators. For example, let $\{e_k\}_{k=1}^\infty$ be an orthonormal basis in H, and define the linear operator A by

$$Ae_k = e_{k+1} \text{ for all } k \geq 1.$$

(shift in the basis). Then, for any $x = \sum_{k=1}^{\infty} x_k e_k$, we have

$$Ax = \sum_{k=1}^{\infty} x_k e_{k+1}.$$

Consequently, if $Ax = 0$ then all $x_k = 0$ and, hence $x = 0$ so that A is injective. However, A is not surjective as $e_1 \notin \operatorname{Im} A$.

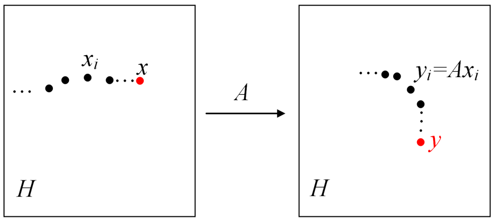
30.10.23 Lecture 7

Proof of Theorem 1.12. Assuming that $\ker A = 0$, we will prove that $\operatorname{Im} A = H$. The proof consists of a few steps.

Step 1. Let us show that if $\{x_i\}$ is a bounded sequence of elements of H and if the sequence $\{Ax_i\}$ converges then $\{x_i\}$ has a convergent subsequence. Indeed, by the compactness of K, the sequence $\{Kx_i\}$ has a convergent subsequence $\{Kx_{i_k}\}$. Since $\{Ax_{i_k}\}$ converges and $Ax_{i_k} = x_{i_k} + Kx_{i_k}$, it follows that also $\{x_{i_k}\}$ converges, which proves the claim.

Step 2. Let us prove that $\operatorname{Im} A$ is a closed subspace of H. The image of any linear operator is always a subspace, so we need to prove that $\operatorname{Im} A$ is closed. Let $\{y_i\}$ be a sequence of elements in $\operatorname{Im} A$ such that $y_i \to y \in H$ as $i \to \infty$. We need to prove that $y \in \operatorname{Im} A$.

Since $y_i \in \operatorname{Im} A$, we have $y_i = Ax_i$ for some $x_i \in H$.



It suffices to prove that the sequence $\{x_i\}$ has a convergent subsequence. Indeed, if this is known already, then passing to that subsequence, we can assume that $\{x_i\}$ converges. Setting $x = \lim x_i$ we obtain

$$y = \lim y_i = \lim Ax_i = Ax \in \operatorname{Im} A,$$

which will finish the proof.

By Step 1, in order to prove that $\{x_i\}$ has a convergent subsequence, it suffices to prove that $\{x_i\}$ is bounded, because we already know that $\{Ax_i\}$ converges (to y). Assume that $\{x_i\}$ is unbounded. Passing to a subsequence we can assume that $\|x_i\| \to \infty$. Setting $\widetilde{x}_i = \frac{x_i}{\|x_i\|}$, we have

$$A\widetilde{x}_i = \frac{Ax_i}{\|x_i\|} = \frac{y_i}{\|x_i\|} \to 0 \text{ as } i \to \infty.$$

Since the sequence $\{\widetilde{x}_i\}$ is bounded and $A\widetilde{x}_i$ converges, we conclude by Step 1 that $\{\widetilde{x}_i\}$ has a convergent subsequence. Passing to this subsequence, we can assume that

$\{\widetilde{x}_i\}$ converges, say, to $z \in H$. Clearly,

$$\|z\| = \lim_{i\to\infty} \|\widetilde{x}_i\| = 1$$

and

$$Az = \lim_{i\to\infty} A\widetilde{x}_i = 0,$$

that is, $z \in \ker A$. Since $\ker A = 0$, we obtain $z = 0$ which contradicts to $\|z\| = 1$. Hence, the sequence $\{x_i\}$ is bounded.

Step 3. Consider the sequence $\{V_k\}_{k=0}^{\infty}$ of subspaces

$$V_k := \operatorname{Im} A^k,$$

that is, $V_{k+1} = A(V_k)$. In particular, $V_0 = H$ and $V_1 = \operatorname{Im} A$. Clearly, we have $V_{k+1} \subset V_k$. By Step 2, V_1 is a closed subspace of V_0. In particular, V_1 is a Hilbert space. Since A can be considered as an operator in V_1, we conclude by Step 2 that $V_2 = A(V_1)$ is a closed subspace of V_1. Continuing by induction, we obtain that each V_{k+1} is a closed subspace of V_k.

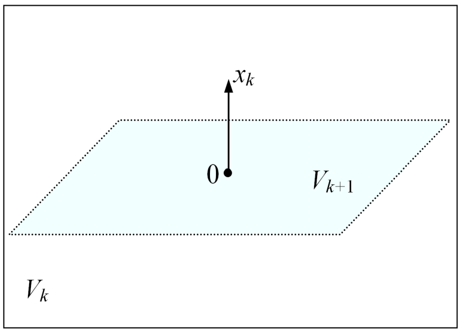
In this step let us prove that

$$V_{k+1} = V_k \text{ for some } k \geq 0.$$

Assume from the contrary that this is not the case, that is,

$$V_{k+1} \subsetneqq V_k \text{ for all } k \geq 0.$$

Hence, for any $k \geq 0$, there exists a non-zero vector $x_k \in V_k$ that is orthogonal to V_{k+1}; choose it so that $\|x_k\| = 1$. We will bring to contradiction the existence of such a sequence $\{x_k\}_{k=0}^{\infty}$.



Since the sequence $\{x_i\}$ is bounded, the sequence $\{Kx_i\}$ must have a convergent subsequence. However, we will show that the sequence $\{Kx_i\}$ cannot have a convergent subsequence, which will finish the proof. For that, we start with the identity

$$Kx_i - Kx_j = -(x_i - x_j) + A(x_i - x_j) = -x_i + (x_j + Ax_i - Ax_j).$$

Assuming that $j > i$ and, hence, $j \geq i + 1$, we obtain

$$x_j + Ax_i - Ax_j \in V_{i+1},$$

because $x_j \in V_j$, $Ax_i \in V_{i+1}$ and $Ax_j \in V_{j+1}$ and, hence, all these vectors are in V_{i+1}. Since $x_i \perp V_{i+1}$ it follows that

$$x_i \perp (x_j + Ax_i - Ax_j).$$

Hence, by Pythagoras' Theorem,

$$\|Kx_i - Kx_j\|^2 = \|x_i\|^2 + \|(x_j + Ax_i - Ax_j)\|^2 \geq 1.$$

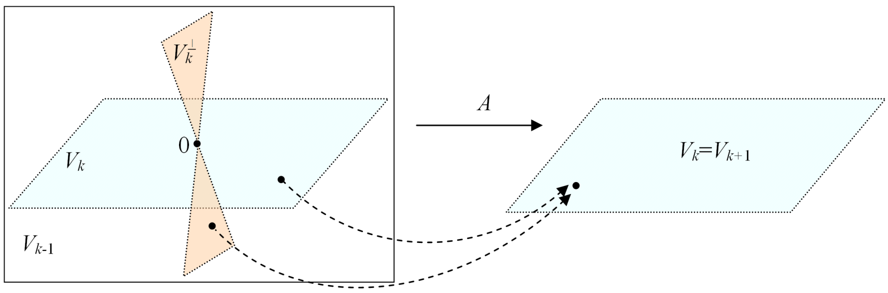
Consequently, no subsequence of $\{Kx_i\}$ is a Cauchy sequence, which was to be proved.

Step 4. Finally, let us prove that if A is injective then $\operatorname{Im} A = H$. Let k be the minimal non-negative integer such that $V_{k+1} = V_k$. We need to prove that $k = 0$, that is, $V_1 = V_0$, which is equivalent to $\operatorname{Im} A = H$.

Assume that $k \geq 1$ and bring this to contradiction. For that consider the following orthogonal decomposition of V_{k-1}:

$$V_{k-1} = V_k \oplus V_k^{\perp},$$

where $V_k^{\perp}$ is the orthogonal complement of V_k in V_{k-1}. Note that the subspace $V_k^{\perp}$ is non-trivial because by the minimality of k we have $V_k \subsetneqq V_{k-1}$.



Consider the mapping $A : V_{k-1} \to V_k$. Note that

$$A(V_k) = V_{k+1} = V_k.$$

However, $A\left(V_k^{\perp}\right)$ lies also in V_k, which implies that some of the points in V_k must have at least two preimages in V_{k-1}: one in V_k and another in $V_k^{\perp}$. Hence, the operator $A : V_{k-1} \to V_k$ is not injective. This contradiction shows that $k = 0$, which finishes the proof. ■

1.4.6 Existence

Consider again an operator

$$Lu = \sum_{i,j=1}^{n} \partial_i (a_{ij} \partial_j u) + \sum_{i=1}^{n} b_i(x) \partial_i u \tag{1.63}$$

in an open set $\Omega \subset \mathbb{R}^n$. As before, we assume that the coefficients a_{ij}, b_i are measurable functions, the second order part $\sum_{i,j=1}^{n} \partial_i (a_{ij} \partial_j u)$ is uniformly elliptic divergence form operator, and that all functions b_i are bounded, that is, there is a constant b such that

$$\sum_{i=1}^{n} |b_i| \leq b \text{ in } \Omega.$$

Theorem 1.13 *If Ω is bounded and L is the operator (1.63) in Ω then the Dirichlet problem*

$$\begin{cases} Lu = f & \text{weakly in } \Omega \\ u \in W_0^{1,2}(\Omega) \end{cases} \tag{1.64}$$

has a solution u for any $f \in L^2(\Omega)$.

Recall that by Theorem 1.3 the Dirichlet problem (1.64) has at most one solution, which together with Theorem 1.13 implies that (1.64) has exactly one solution.

Proof. Consider the following bilinear form on $W_0^{1,2}(\Omega)$:

$$B(u,v) := \int_\Omega \sum_{i,j=1}^n a_{ij}\partial_j u \partial_i v\,dx - \int_\Omega \sum_{i=1}^n b_i(\partial_i u)v\,dx.$$

As we know, the weak equation $Lu = f$ means that

$$B(u,\varphi) = -\int_\Omega f\varphi dx \quad \forall \varphi \in W_0^{1,2}(\Omega). \tag{1.65}$$

The bilinear form B is bounded because as we have seen in the proof of Lemma 1.1) (cf. (1.11))

$$\left|\int_\Omega \sum_{i,j=1}^n a_{ij}\partial_j u\partial_i v\,dx\right| \le \lambda \|u\|_{W^{1,2}} \|v\|_{W^{1,2}}$$

while

$$\left|\int_\Omega \sum_{i=1}^n b_i(\partial_i u)v\,dx\right| \le \int_\Omega \sum_{i=1}^n |b_i|\,|\nabla u|\,|v|\,dx \le b\|\nabla u\|_{L^2}\|v\|_{L^2} \le b\|u\|_{W^{1,2}}\|v\|_{W^{1,2}},$$

whence

$$|B(u,v)| \le (\lambda + b)\|u\|_{W^{1,2}}\|v\|_{W^{1,2}} \tag{1.66}$$

If the form B were coercive, that is, if for all $u \in W_0^{1,2}(\Omega)$

$$B(u,u) \ge c\|u\|_{W^{1,2}}^2 \tag{1.67}$$

with some positive constant c, then we could conclude by the Lax-Milgram theorem that the equation (1.65) has a solution $u \in W_0^{1,2}(\Omega)$, which yields also a solution of (1.64). However, the form B is not necessarily coercive.

We will use instead another bilinear form

$$B_C(u,v) = B(u,v) + C(u,v)_{L^2}$$

with some positive constant C, and show that B_C is coercive if C is large enough.

We start with the following inequality:

$$\begin{aligned} B(u,u) &= \int_\Omega \sum_{i,j=1}^n a_{ij}\partial_j u\partial_i u\,dx - \int_\Omega \sum_{i=1}^n b_i(\partial_i u)udx \\ &\ge \lambda^{-1}\int_\Omega |\nabla u|^2\,dx - b\int_\Omega |\nabla u|\,|u|\,dx. \end{aligned}$$

Note that, for any $\varepsilon > 0$,

$$|\nabla u|\,|u| \leq \varepsilon\,|\nabla u|^2 + \frac{1}{\varepsilon}u^2,$$

so that

$$\begin{aligned} B(u,u) &\geq \lambda^{-1}\int_\Omega |\nabla u|^2\,dx - \varepsilon b\int_\Omega |\nabla u|^2\,dx - \frac{b}{\varepsilon}\int_\Omega u^2 dx \\ &= c\int_\Omega |\nabla u|^2\,dx - \frac{b}{\varepsilon}\int_\Omega u^2 dx, \end{aligned}$$

where $c = \lambda^{-1} - b\varepsilon$.

Choosing ε small enough, say $\varepsilon = \frac{1}{2}b^{-1}\lambda^{-1}$, we can ensure that $c > 0$. It follows that

$$\begin{aligned} B(u,u) &\geq c\left(\int_\Omega |\nabla u|^2\,dx + \int_\Omega u^2 dx\right) - \left(\frac{b}{\varepsilon} + c\right)\int_\Omega u^2 dx \\ &\geq c\,\|u\|^2_{W^{1.2}} - C\,\|u\|^2_{L^2}\,, \end{aligned}$$

where $C = \frac{b}{\varepsilon} + c$.

Rewrite this inequality as follows:

$$B(u,u) + C\,\|u\|^2_{L^2} \geq c\,\|u\|^2_{W^{1,2}}\,,$$

that is,

$$B_C(u,u) \geq c\,\|u\|^2_{W^{1,2}}\,,$$

which means that the bilinear form B_C is coercive. Since B is bounded, the form B_C is also bounded.

Hence, let us consider instead of (1.65) an auxiliary problem:

$$B_C(u,\varphi) = -\int_\Omega f\varphi dx \quad \forall\varphi \in W_0^{1,2}(\Omega)\,. \tag{1.68}$$

By the Lax-Milgram theorem, the equation (1.68) has a unique solution $u \in W_0^{1,2}(\Omega)$. Moreover, for this solution we have

$$\|u\|_{W^{1,2}} \leq c^{-1}\,\|f\|_{L^2}\,, \tag{1.69}$$

because the norm of the functional $\ell(\varphi) = \int_\Omega f\varphi$ in $W_0^{1,2}(\Omega)$ is bounded by $\|f\|_{L^2}$.

Denote by R the resolvent operator of (1.68), that is, the operator

$$R : L^2(\Omega) \to W_0^{1,2}(\Omega)$$

$$Rf = u,$$

where u is the unique solution of (1.68). Obviously, R is a linear operator. Moreover, R is a bounded operator because by (1.69)

$$\|Rf\|_{W^{1.2}} \leq c^{-1}\,\|f\|_{L^2}\,.$$

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Now let us come back to the equation (1.65) and add $C(u,\varphi)_{L^2}$ to the both sides. We obtain an equivalent equation

$$B_C(u,\varphi) = -\int_\Omega f\varphi dx \ + C(u,\varphi)_{L^2},$$

that is,

$$B_C(u,\varphi) = -\int_\Omega (f - Cu)\varphi dx \quad \forall \varphi \in W_0^{1,2}(\Omega). \tag{1.70}$$

By the definition of the resolvent R, the equation (1.70) is equivalent to

$$u = R(f - Cu)$$

that is, to the equation

$$u + CRu = Rf. \tag{1.71}$$

Define the operator $K : L^2 \to L^2$ as composition of the following operators

$$L^2(\Omega) \overset{CR}{\to} W_0^{1,2}(\Omega) \overset{i}{\hookrightarrow} L^2(\Omega)$$

where i is the identical inclusion; that is,

$$K = i \circ (CR).$$

By the Compact Embedding Theorem, the operator i is compact. Since CR is bounded, we obtain that K is a compact operator. Let us rewrite (1.71) in the form

$$(I + K)u = Rf. \tag{1.72}$$

We consider this equation in the Hilbert space $L^2(\Omega)$, that is, the unknown function u is assumed to be in $L^2(\Omega)$.

Claim. *Solving (1.72) for $u \in L^2(\Omega)$ is equivalent to solving (1.71) for $u \in W_0^{1,2}(\Omega)$.*

Indeed, the direction (1.71)⇒(1.72) is trivial because if $u \in W_0^{1,2}(\Omega)$ then $u \in L^2(\Omega)$. For the opposite direction observe that if $u \in L^2(\Omega)$ solves (1.72) then

$$u = Rf - Ku = Rf - CRu \in W_0^{1,2}(\Omega)$$

by definition of the operator R.

Hence, it suffices to prove that the equation (1.72) has a solution $u \in L^2(\Omega)$ for any $f \in L^2(\Omega)$. For that, it suffices to prove that $I + K$ is surjective. By Fredholm's alternative, it suffices to prove that the operator $I+K$ is injective, that is, the equation

$$(I + K)u = 0$$

has the only solution $u = 0$. If $u \in L^2(\Omega)$ satisfies this equation then u satisfies also (1.71), (1.70), (1.65) and (1.64) with $f = 0$, that is,

$$\begin{cases} Lu = 0 \text{ weakly in } \Omega, \\ u \in W_0^{1,2}(\Omega). \end{cases}$$

By Theorem 1.3 this problem has the only solution $u = 0$, which finishes the proof. ■

1.5 Estimate of L^{∞}-norm of a solution

In this section we use the ∞-norm of a measurable function f in an open subset Ω of $\mathbb{R}^n$:

$$\|f\|_{L^\infty} := \operatorname*{esssup}_{\Omega} |f|\,.$$

The space $L^\infty(\Omega)$ consists of all measurable functions f on Ω with $\|f\|_{L^\infty} < \infty$. It is possible to prove that L^∞ is a linear space, $\|\cdot\|_{L^\infty}$ is a norm in $L^\infty(\Omega)$, and $L^\infty(\Omega)$ is a Banach space. The following extension of the Hölder inequality is obviously true:

$$\int_\Omega |fg|\,dx \le \|f\|_{L^\infty}\|g\|_{L^1}\,.$$

The Sobolev spaces $W^{k,p}(\Omega)$ are now defined by (1.5) also for $p = \infty$, as well as the spaces $W^{k,p}_{loc}(\Omega)$ (cf. (1.6)).

1.5.1 Operator without lower order terms

Theorem 1.14 *Let Ω be a bounded domain in $\mathbb{R}^n$ and let*

$$Lu = \sum_{i,j=1}^{n} \partial_i\left(a_{ij}\partial_j u\right)$$

be a divergence form uniformly elliptic operator in Ω with measurable coefficients. If u solves the Dirichlet problem

$$\begin{cases} Lu = -f \ \text{weakly in } \Omega \\ u \in W_0^{1,2}(\Omega) \end{cases} \tag{1.73}$$

where $f \in L^2(\Omega)$, then

$$\|u\|_{L^\infty} \le C\,|\Omega|^{2/n}\,\|f\|_{L^\infty}\,, \tag{1.74}$$

where $C = C(n,\lambda)$ and λ is the ellipticity constant of L.

Remark. In the proof we use the following *Faber-Krahn inequality*: if $u \in W_0^{1,2}(\Omega)$ and

$$U = \{x \in \Omega : u(x) \neq 0\}$$

then

$$\int_\Omega |\nabla u|^2\,dx \ge c\,|U|^{-2/n}\int_\Omega u^2\,dx, \tag{1.75}$$

where $c = c(n) > 0$. This inequality is proved in Exercise 11 in the case $n > 2$ and in Exercise 13 in the case $n = 2$. In fact, it is valid also in the case $n = 1$. Indeed, in this case any function from $W_0^{1,2}$ is continuous, the set U is open and, hence, consists of disjoint union of open intervals, say $U = \sqcup_j I_j$. In each interval I_j, the function u vanishes at the endpoints, which implies by Friedrichs' inequality that

$$\int_{I_j} |\nabla u|^2\,dx \ge |I_j|^{-2}\int_{I_j} u^2\,dx \ge |U|^{-2}\int_{I_j} u^2\,dx.$$

Summing up in all j, we obtain (1.75) with $n = 1$ and $c = 1$.

Remark. Denote by $\lambda_1(\Omega)$ the first (smallest) eigenvalue of the weak eigenvalue problem in Ω:

$$\begin{cases} \Delta v + \lambda v = 0 \text{ in } \Omega \\ v \in W_0^{1,2}(\Omega) \end{cases}$$

By the *Rayleigh principle*, we have

$$\lambda_1(\Omega) = \inf_{u \in W_0^{1,2}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega u^2 dx}.$$

Since $|U| \leq |\Omega|$, it follows from (1.75) that

$$\lambda_1(\Omega) \geq c|\Omega|^{-2/n}. \tag{1.76}$$

This inequality is related to the following *Faber-Krahn theorem*: if Ω^* denotes a ball of the same volume as Ω then

$$\lambda_1(\Omega) \geq \lambda_1(\Omega^*). \tag{1.77}$$

In other words, among all domains with the same volume, the minimal value of λ_1 is achieved on balls. This is related to isoperimetric property of balls: among all domains with the same volume, the minimal boundary area is achieved on balls.

Observe that if $\Omega^* = B_R$ then

$$\lambda_1(\Omega^*) = \lambda_1(B_R) = \frac{c'}{R^2}$$

where $c' = c'(n) > 0$. Since $|B_R| = c''R^n$, we obtain

$$\lambda_1(\Omega^*) = c|\Omega^*|^{-2/n},$$

which implies by (1.77) and $|\Omega^*| = |\Omega|$ that

$$\lambda_1(\Omega) \geq c|\Omega|^{-2/n}. \tag{1.78}$$

Of course, this looks the same as (1.76), except for the constant c in (1.78) is sharp and is achieved on balls, whereas the constant c in (1.76) was some positive constant. However, for our applications we do not need a sharp constant c.

Proof of Theorem 1.14. If $\|f\|_{L^\infty} = \infty$ then (1.74) is trivially satisfied. If $\|f\|_{L^\infty} = 0$ then by Theorem 1.2 we have $u = 0$ and (1.74) holds. Let now $0 < \|f\|_{L^\infty} < \infty$. Dividing u and f by $\|f\|_{L^\infty}$, we can assume without loss of generality that $\|f\|_{L^\infty} = 1$.

Fix $\alpha > 0$ and consider the function

$$v = (u - \alpha)_+ \in W_0^{1,2}(\Omega)$$

(cf. Lemma 1.4). Since the equation $Lu = -f$ holds weakly, we have the identity

$$\int_\Omega \sum_{i,j=1}^n a_{ij}\partial_j u \partial_i v\, dx = \int_\Omega f v\, dx. \tag{1.79}$$

Let us show that $\partial_j u$ here can be replaced by $\partial_j v$. Indeed, by Lemma 1.4, we have

$$\nabla v = \begin{cases} \nabla u & \text{a.e. on } \{u > \alpha\} = \{v > 0\}, \\ 0 & \text{a.e. on } \{u \leq \alpha\} = \{v = 0\}. \end{cases}$$

It follows that, for all $i, j = 1, ..., n$,

$$\partial_j u \partial_i v = \partial_j v \partial_i v \ \text{ a.e. in } \Omega, \tag{1.80}$$

because, on the set $\{v = 0\}$ we have $\partial_i v = 0$ a.e. so that the both sides of (1.80) vanish, while on $\{v > 0\}$ we have $\partial_j u = \partial_j v$ a.e..

Let us estimate the left hand side of (1.79) from below. Using (1.80) and the uniform ellipticity of L, we obtain

$$\int_\Omega \sum_{i,j=1}^n a_{ij} \partial_j u \partial_i v \, dx = \int_\Omega \sum_{i,j=1}^n a_{ij} \partial_j v \partial_i v \, dx \geq \lambda^{-1} \int_\Omega |\nabla v|^2 \, dx. \tag{1.81}$$

To estimate further the right hand side of (1.81) from below, consider the set

$$U_\alpha := \{u > \alpha\} = \{v \neq 0\},$$

and apply the Faber-Krahn inequality (1.75) to the function v:

$$\lambda^{-1} \int_\Omega |\nabla v|^2 \, dx \geq \lambda^{-1} c \, |U_\alpha|^{-2/n} \int_\Omega v^2 \, dx, \tag{1.82}$$

where $c = c(n) > 0$. Combining (1.79), (1.81) and (1.82), we obtain

$$c\lambda^{-1} |U_\alpha|^{-2/n} \int_\Omega v^2 \, dx \leq \int_\Omega f v \, dx.$$

Next, let us estimate the right hand side here from above using that $\|f\|_{L^\infty} = 1$ and the Cauchy-Schwarz inequality:

$$\int_\Omega f v \, dx \leq \int_\Omega v dx = \int_{U_\alpha} 1 \cdot v \, dx \leq |U_\alpha|^{1/2} \left(\int_\Omega v^2 dx \right)^{1/2}. \tag{1.83}$$

It follows that

$$c\lambda^{-1} |U_\alpha|^{-2/n} \int_\Omega v^2 \, dx \leq |U_\alpha|^{1/2} \left(\int_\Omega v^2 dx \right)^{1/2}$$

and, hence,

$$\left(\int_\Omega v^2 dx \right)^{1/2} \leq c^{-1} \lambda \, |U_\alpha|^{1/2 + 2/n}.$$

Let us rewrite this inequality in the form

$$\boxed{\int_\Omega (u - \alpha)_+^2 \, dx \leq K \, |U_\alpha|^p}, \tag{1.84}$$

where $K = (c^{-1}\lambda)^2$ and $p = 1 + 4/n$. It is important for what follows that $p > 1$.

Claim. *Assume that a measurable function u in Ω satisfies for any $\alpha > 0$ the inequality (1.84) with some K and $p > 1$. Then*

$$\operatorname*{esssup}_{\Omega} u \le C\,|\Omega|^{\frac{p-1}{2}}\,, \tag{1.85}$$

where $C = C\,(K, p)$.

In particular, if as above u is a solution of (1.73) with $\|f\|_{L^\infty} = 1$ then (1.84) holds with $p = 1 + 4/n$. Since $\frac{p-1}{2} = \frac{2}{n}$, we obtain by (1.85)

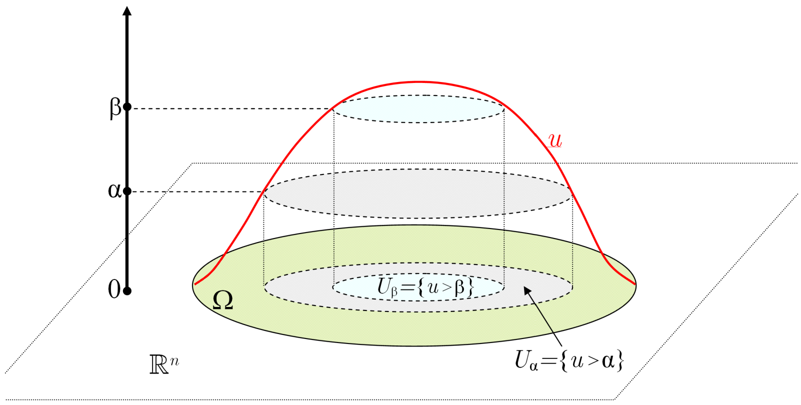
$$\operatorname*{esssup}_{\Omega} u \le C\,|\Omega|^{\frac{2}{n}}\,.$$

Since the same argument applies to $-u$, we obtain

$$\|u\|_{L^\infty} \le C\,|\Omega|^{2/n}\,,$$

which coincides with (1.74) when $\|f\|_{L^\infty} = 1$. The constant C here depends on K and p, that is, on λ and n.

Now let us prove the above Claim. For any $\beta > \alpha$ consider the set $U_\beta = \{u > \beta\}$.



Since $u - \alpha > \beta - \alpha$ on U_β, we obtain

$$\int_\Omega (u-\alpha)_+^2\,dx \ge \int_{U_\beta} (u-\alpha)_+^2\,dx \ge (\beta-\alpha)^2\,|U_\beta|\,,$$

which together with (1.84) implies

$$(\beta-\alpha)^2\,|U_\beta| \le K\,|U_\alpha|^p\,,$$

and, hence,

$$|U_\beta| \le \frac{K}{(\beta-\alpha)^2}\,|U_\alpha|^p\,. \tag{1.86}$$

Fix some $\alpha > 0$ (to be specified below) and consider a sequence $\{\alpha_k\}_{k=0}^\infty$ where

$$\alpha_k = \alpha\left(2 - 2^{-k}\right).$$

This sequence is increasing, $\alpha_0 = \alpha$ and $\alpha_k \to 2\alpha$ as $k \to \infty$.

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Set
$$m_k = |\{u > \alpha_k\}|$$
and observe that by (1.86)
$$m_k \leq \frac{K}{(\alpha_k - \alpha_{k-1})^2} m_{k-1}^p.$$
Since $\alpha_k - \alpha_{k-1} = \alpha 2^{-k}$, it follows that
$$m_k \leq K\alpha^{-2} 4^k m_{k-1}^p = 4^k A m_{k-1}^p \tag{1.87}$$
where $A = K\alpha^{-2}$.

We would like to make sure that $m_k \to 0$ as $k \to \infty$. Indeed, if this is already known then we argue as follows. Since $\alpha_k \nearrow 2\alpha$ as $k \to \infty$, the sequence of sets $\{u > \alpha_k\}$ is shrinking in k and
$$\bigcap_{k=0}^{\infty} \{u > \alpha_k\} = \{u \geq 2\alpha\}.$$
It follows that
$$|\{u \geq 2\alpha\}| = \lim_{k\to\infty} |\{u > \alpha_k\}| = \lim_{k\to\infty} m_k = 0$$
and, hence, $u \leq 2\alpha$ a.e., that is,
$$\operatorname{esssup} u \leq 2\alpha. \tag{1.88}$$

In order to prove that $m_k \to 0$, let us first iterate inequality (1.87):
$$\begin{aligned} m_k &\leq 4^k A m_{k-1}^p \\ &\leq 4^k A \left(4^{k-1} A m_{k-2}^p\right)^p \\ &= 4^{k+p(k-1)} A^{1+p} m_{k-2}^{p^2} \\ &\leq 4^{k+p(k-1)} A^{1+p} \left(4^{k-2} A m_{k-3}^p\right)^{p^2} \\ &= 4^{k+p(k-1)+p^2(k-2)} A^{1+p+p^2} m_{k-3}^{p^3} \\ &\quad \ldots \\ &\leq 4^{k+p(k-1)+\ldots+p^{k-1}} A^{1+p+p^2+\ldots+p^{k-1}} m_0^{p^k}. \end{aligned} \tag{1.89}$$
Next, let us use the identity
$$1 + p + p^2 + \ldots + p^{k-1} = \frac{p^k - 1}{p - 1} \tag{1.90}$$
and the following inequality[2]
$$k + p(k-1) + p^2(k-2) + \ldots + p^{k-1} \leq \frac{p^{k+1}}{(p-1)^2}. \tag{1.91}$$

---

[2] In fact, the following identity takes place:
$$k + p(k-1) + p^2(k-2) + \ldots + p^{k-1} = \frac{p^{k+1} - (k+1)p + k}{(p-1)^2}.$$

Indeed, dividing (1.90) by p^k we obtain

$$p^{-k} + p^{-(k-1)} + ... + p^{-1} = \frac{1}{p-1} - \frac{1}{p^k (p-1)},$$

which after differentiation in p and changing the sign yields

$$kp^{-(k+1)} + (k-1)\, p^{-k} + ... + p^{-2} = \frac{1}{(p-1)^2} + \left(\frac{1}{p^k (p-1)}\right)'.$$

Observing that the function $\frac{1}{p^k(p-1)}$ is decreasing on $\{p > 1\}$ so that its derivative is negative, and multiplying the resulting inequality by p^{k+1}, we obtain (1.91). Alternatively, (1.91) can be easily proved by induction.

Hence, we obtain from (1.89) that

$$m_k \leq 4^{\frac{p^{k+1}}{(p-1)^2}} A^{\frac{p^k-1}{p-1}} m_0^{p^k} = \left[4^{\frac{p}{(p-1)^2}} A^{\frac{1}{p-1}} m_0\right]^{p^k} A^{-\frac{1}{p-1}}. \tag{1.92}$$

Since $m_0 \leq |\Omega|$, in order to achieve that $m_k \to 0$, it suffices by (1.92) to have the inequality

$$4^{\frac{p}{(p-1)^2}} A^{\frac{1}{p-1}} |\Omega| < 1,$$

that is,

$$4^{\frac{p}{(p-1)^2}} K^{\frac{1}{p-1}} \alpha^{-\frac{2}{p-1}} |\Omega| < 1.$$

We can ensure this inequality by choosing α to satisfy, for example, the following equation:

$$4^{\frac{p}{(p-1)^2}} K^{\frac{1}{p-1}} \alpha^{-\frac{2}{p-1}} |\Omega| = \frac{1}{2}$$

that is,

$$\alpha = \left(2 \cdot 4^{\frac{p}{(p-1)^2}} K^{\frac{1}{p-1}} |\Omega|\right)^{\frac{p-1}{2}} = C_1 |\Omega|^{\frac{p-1}{2}},$$

where C_1 depends on K and p. As we have already seen, for this value of α we have (1.88), that is,

$$\operatorname{esssup} u \leq 2\alpha = 2C_1 |\Omega|^{\frac{p-1}{2}}, \tag{1.93}$$

which finishes the proof of (1.85) with $C = 2C_1$. ■

Remark. Theorem 1.14 provides a non-trivial estimate even in the case $L = \Delta$. Consider the following weak Dirichlet problem:

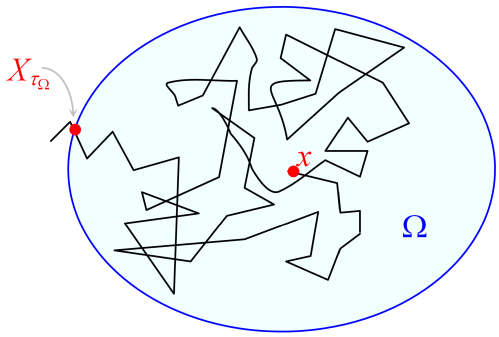
$$\begin{cases} \Delta u = -1 \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega. \end{cases} \tag{1.94}$$

We know that the solution $u(x)$ is a smooth function in Ω. In fact, it has the following probabilistic meaning: if $x \in \Omega$ is the starting point of Brownian motion $\{X_t\}$ in $\mathbb{R}^n$ then $u(x)$ is the *mean exit time* from Ω. In other words, if we define the first exist time τ_Ω from Ω by

$$\tau_\Omega = \inf\{t > 0 : X_t \notin \Omega\},$$

then

$$u(x) = \mathbb{E}_x \tau_\Omega. \tag{1.95}$$



More generally, the Dirichlet problem

$$\begin{cases} \Delta u = -f \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{cases}$$

has solution

$$u(x) = \mathbb{E}_x \int_0^{\tau_\Omega} f(X_t)\, dt,$$

which implies (1.95) for $f = 1$.

Let u be the solution of (1.94). Then by Theorem 1.14 we have

$$\sup_\Omega u \le C |\Omega|^{2/n},$$

that is, the mean exit time from Ω is bounded from above by $C |\Omega|^{2/n}$. In particular, if $\Omega = B_R$ then $|\Omega| = c_n R^n$ and we obtain the estimate

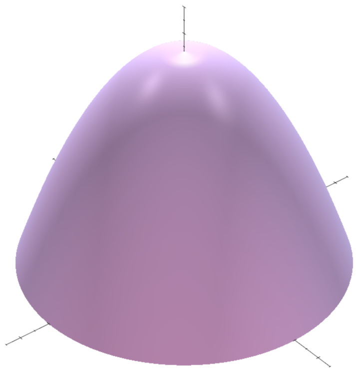
$$\sup_{B_R} u \le C' R^2. \tag{1.96}$$

Note that the classical Dirichlet problem

$$\begin{cases} \Delta u = -1 & \text{in } B_R \\ u = 0 & \text{on } \partial B_R \end{cases}$$

has an obvious solution

$$u(x) = \frac{R^2 - |x|^2}{2n}. \tag{1.97}$$



The graph of function (1.97), case $n = 2$

In particular, we see that

$$\sup_{B_R} u = u(0) = \frac{R^2}{2n},$$

which shows that the estimate (1.96) is optimal up to the value of the constant.

Let us emphasize the following probabilistic meaning of the latter identity: the mean exit time from the center of the ball is equal to $\frac{R^2}{2n}$, that is, $\mathbb{E}_0 \tau_{B_R} = \frac{R^2}{2n}$.

In particular, the mean exit time is not proportional to R as it would be in the case of a constant outward speed, but to R^2, which means that, in long term, the propagation of diffusion is very slow in comparison with a constant speed movement. This happens because Brownian particle does not go away in radial direction but spends a lot of time for moving also in angular directions. For example, an observer staying at the origin and watching in the direction of the particle, will have to turn around all the times in order to keep the particle in the view.

1.5.2 Operator with lower order terms

Now we state and prove a more general version of Theorem 1.14. Consider in Ω a more general operator

$$Lu = \sum_{i,j=1}^{n} \partial_i (a_{ij} \partial_j u) + \sum_{i=1}^{n} b_i \partial_i u \tag{1.98}$$

where the coefficients a_{ij} and b_i are measurable functions, the matrix (a_{ij}) is uniformly elliptic with the ellipticity constant λ, and all b_i are bounded, that is, there is a constant $b \geq 0$ such that

$$\sum_{i=1}^{n} |b_i| \leq b \quad \text{in } \Omega. \tag{1.99}$$

Definition. Given functions $u \in W^{1,2}_{loc}(\Omega)$ and $g \in L^2_{loc}(\Omega)$, we say that the inequality $Lu \geq g$ is satisfied weakly in Ω if, for any non-negative function $\varphi \in \mathcal{D}(\Omega)$,

$$-\int_\Omega \sum_{i,j=1}^{n} a_{ij} \partial_j u \partial_i \varphi \, dx + \int_\Omega \sum_{i=1}^{n} b_i \partial_i u \, \varphi \, dx \geq \int_\Omega g \varphi \, dx. \tag{1.100}$$

Similarly one defines the meaning of $Lu \leq g$.

Claim. *If $u \in W^{1,2}(\Omega)$ and $g \in L^2(\Omega)$ then the test function φ in (1.100) can be any non-negative function from $W_0^{1,2}(\Omega)$*

Recall that a similar result for the equality $Lu = g$ was proved in Lemma 1.1. For the inequality $Lu \geq g$ the proof is more difficult – see Exercise 27.

Theorem 1.15 *Let Ω be a bounded domain in $\mathbb{R}^n$ and let L be the operator (1.98). Assume*

$$|\Omega| < \delta, \tag{1.101}$$

where $\delta = c_n (\lambda b)^{-n}$ *with some* $c_n > 0$. *If* $u \in W^{1,2}(\Omega)$ *and* $f \in L^2(\Omega)$ *satisfy*

$$\begin{cases} Lu \geq -f \text{ weakly in } \Omega, \\ u_+ \in W_0^{1,2}(\Omega), \end{cases} \tag{1.102}$$

then, for any $q \in [2, \infty] \cap (n/2, \infty]$, *the following estimate holds:*

$$\operatorname*{esssup}_{\Omega} u \leq C |\Omega|^{\frac{2}{n} - \frac{1}{q}} \|f_+\|_{L^q} \tag{1.103}$$

with a constant $C = C(n, \lambda, q)$.

Remark. Theorem 1.15 extends Theorem 1.14 in four ways:

- we allow in operator L the lower order terms;
- we allow inequality $Lu \geq -f$ instead of equality;
- we allows a weaker boundary condition $u_+ \in W_0^{1,2}(\Omega)$ instead of $u \in W_0^{1,2}(\Omega)$;
- the main estimate in given in terms of $\|f_+\|_{L^q}$ instead of $\|f\|_{L^\infty}$, where q in particular can be ∞.

Let us explain how to deduce Theorem 1.14 from Theorem 1.15. Indeed, if all $b_i = 0$ and, hence, $b = 0$ then $\delta = \infty$ and the restriction (1.101) on $|\Omega|$ is void. Assuming that

$$\begin{cases} Lu = f \text{ weakly in } \Omega, \\ u \in W_0^{1,2}(\Omega), \end{cases} \tag{1.104}$$

and applying (1.103) with $q = \infty$, we obtain

$$\operatorname*{esssup}_{\Omega} u \leq C |\Omega|^{\frac{2}{n}} \|f_+\|_{L^\infty} \leq C |\Omega|^{\frac{2}{n}} \|f\|_{L^\infty}. \tag{1.105}$$

Applying this inequality to function $-u$, we obtain

$$\operatorname*{esssup}_{\Omega} (-u) \leq C |\Omega|^{2/n} \|f\|_{L^\infty},$$

whence it follows that

$$\operatorname*{esssup}_{\Omega} |u| \leq C |\Omega|^{2/n} \|f\|_{L^\infty}, \tag{1.106}$$

which is equivalent to the estimate (1.74) of Theorem 1.14.

Remark. Theorem 1.15 implies also the uniqueness result of Theorem 1.3 because if (1.104) holds with $f = 0$ then by (1.106) $u = 0$. Note that Theorem 1.3 does not follow from Theorem 1.14 because in the proof of the latter we used Theorem 1.3, whereas in the proof of Theorem 1.15 the uniqueness result of Theorem 1.3 will not be used.

Remark. Applying Theorem 1.15 with $f = 0$, we obtain the following *weak maximum principle*:

$$\text{if } Lu \geq 0 \text{ weakly in } \Omega \text{ and } u_+ \in W_0^{1,2}(\Omega) \text{ and then } u \leq 0 \text{ a.e. in } \Omega.$$

The condition $u_+ \in W_0^{1,2}(\Omega)$ can be regarded as a weak version of the boundary condition "$u_+ = 0$ on $\partial\Omega$", that is, a weak version of "$u \le 0$ on $\partial\Omega$".
Observe that if $a_{ij} \in C^1(\Omega)$ and $u \in C^2(\Omega) \cap C(\overline{\Omega})$ then Lu can be considered in the classical sense, and the maximum principle of Exercise 1 implies that

$$\text{if } Lu \ge 0 \text{ in } \Omega \text{ and } u \le 0 \text{ in } \partial\Omega \text{ then } u \le 0 \text{ in } \Omega.$$

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Proof of Theorem 1.15. Since f can be replaced in (1.102) by f_+, we can rename f_+ in f and assume without loss of generality that $f \ge 0$. If $\|f\|_{L^q} = \infty$ then there is nothing to prove. If $0 < \|f\|_{L^q} < \infty$ then dividing f and u by $\|f\|_{L^q}$, we can assume that $\|f\|_{L^q} = 1$. Finally, the case $\|f\|_{L^q} = 0$ amounts to the previous case as follows. Indeed, if $Lu \ge 0$ then also $Lu \ge -\varepsilon$ for any $\varepsilon > 0$. Applying (1.103) with $f = \varepsilon$, we obtain

$$\operatorname*{esssup}_{\Omega} u \le C\,|\Omega|^{\frac{2}{n}-\frac{1}{q}}\,\|\varepsilon\|_{L^q}.$$

Letting $\varepsilon \to 0$ we obtain $\operatorname{esssup}_\Omega u \le 0$, that is (1.103) with $f = 0$.

Hence, we assume in what follows that $f \ge 0$ and $\|f\|_{L^q} = 1$. As in the proof of Theorem 1.14, we will prove that, for any $\alpha > 0$

$$\int_\Omega (u-\alpha)_+^2\,dx \le K\,|U_\alpha|^p$$

where $U_\alpha = \{u > \alpha\}$, K, p are positive constants, and $p > 1$.

Fix some $\alpha > 0$ and consider a function

$$v := (u-\alpha)_+ = (u_+ - \alpha)_+.$$

Clearly, $v \ge 0$ and, by Lemma 1.4, $v \in W_0^{1,2}(\Omega)$ because $u_+ \in W_0^{1,2}(\Omega)$. Using v as a test function in the inequality $Lu \ge -f$, we obtain

$$-\int_\Omega \sum_{i,j=1}^n a_{ij}\partial_j u \partial_i v\,dx + \int_\Omega \sum_{i=1}^n b_i \partial_i u\, v\,dx \ge -\int_\Omega f v\,dx,$$

that we rewrite as follows:

$$\int_\Omega \sum_{i,j=1}^n a_{ij}\partial_j u \partial_i v\,dx \le \int_\Omega \sum_{i=1}^n b_i \partial_i u\, v\,dx + \int_\Omega f v\,dx. \tag{1.107}$$

We estimate the left hand side of (1.107) from below similarly to (1.81). In order to express $\partial_j u$ via $\partial_j v$ we use Exercise 15 that says the following: if $w \in W_{loc}^{1,2}(\Omega)$ then also $w_+ \in W_{loc}^{1,2}(\Omega)$ and

$$\nabla w_+ = \begin{cases} \nabla w & \text{a.e. on } \{w > 0\}, \\ 0 & \text{a.e. on } \{w \le 0\}. \end{cases}$$

Since $u \in W^{1,2}(\Omega)$ and, hence, also $u - \alpha \in W^{1,2}(\Omega)$, we can apply this identity to $w = u - \alpha$ and obtain

$$\nabla v = \nabla (u-\alpha)_+ = \begin{cases} \nabla (u-\alpha) = \nabla u & \text{a.e. on } \{u-\alpha>0\} = \{v>0\}, \\ 0 & \text{a.e. on } \{u-\alpha \le 0\} = \{v=0\}. \end{cases}$$

Note that if $u \in W_0^{1,2}(\Omega)$ then this identity is true also by Lemma 1.4. It follows that, for all $i, j = 1, ..., n$,

$$\partial_j u\, \partial_i v = \partial_j v\, \partial_i v \text{ a.e. in } \Omega.$$

Indeed, on the set $\{v=0\}$ we have $\partial_i v = 0$ a.e., while on the set $\{v>0\}$ we have

$$\partial_j v = \partial_j u \text{ a.e..}$$

Hence, we obtain

$$\int_\Omega \sum_{i,j=1}^n a_{ij}\partial_j u \partial_i v\, dx = \int_\Omega \sum_{i,j=1}^n a_{ij}\partial_j v \partial_i v\, dx \ge \lambda^{-1} \int_\Omega |\nabla v|^2 dx. \tag{1.108}$$

Now let us estimate from above the first term in the right hand side of (1.107). By the above argument, we have also

$$\partial_i u\, v = \partial_i v\, v \text{ a.e. in } \Omega.$$

Substituting into in the right hand side of (1.107), using $|\partial_i v| \le |\nabla v|$ and (1.99), we obtain

$$\int_\Omega \sum_{i=1}^n b_i \partial_i u\, v\, dx = \int_\Omega \sum_{i=1}^n b_i \partial_i v\, v\, dx \le b \int_\Omega |\nabla v|\, v dx.$$

Applying further the inequality

$$XY \le \frac{1}{2}\left(\varepsilon X^2 + \frac{1}{\varepsilon}Y^2\right),$$

that holds for all $X, Y \ge 0$ and $\varepsilon > 0$, we obtain

$$|\nabla v|\, v \le \frac{1}{2}\left(\varepsilon |\nabla v|^2 + \frac{1}{\varepsilon} v^2\right)$$

and, hence,

$$\int_\Omega \sum_{i=1}^n b_i \partial_i u\, v\, dx \le \frac{b}{2}\int_\Omega \left(\varepsilon |\nabla v|^2 + \frac{1}{\varepsilon}v^2\right) dx. \tag{1.109}$$

It follows from (1.107), (1.108) and (1.109) that

$$\lambda^{-1}\int_\Omega |\nabla v|^2 dx \le \frac{b\varepsilon}{2}\int_\Omega |\nabla v|^2 dx + \frac{b}{2\varepsilon}\int_\Omega v^2 dx + \int_\Omega f v dx.$$

Let us choose ε to satisfy the condition $b\varepsilon = \lambda^{-1}$, that is,

$$\varepsilon = \frac{1}{\lambda b}.$$

Then we obtain

$$\lambda^{-1}\int_\Omega |\nabla v|^2\,dx \le \frac{1}{2}\lambda^{-1}\int_\Omega |\nabla v|^2\,dx + \frac{\lambda b^2}{2}\int_\Omega v^2 dx + \int_\Omega fv dx,$$

whence

$$\int_\Omega |\nabla v|^2\,dx \le \lambda^2 b^2 \int_\Omega v^2 dx + 2\lambda \int_\Omega fv dx. \tag{1.110}$$

By the Faber-Krahn inequality (1.75) we have

$$\int_\Omega |\nabla v|^2\,dx \ge c\,|U_\alpha|^{-2/n}\int_\Omega v^2\,dx$$

because $\{v \neq 0\} = U_\alpha$ (note that $c = c\,(n) > 0$). Substituting into (1.110), we obtain

$$c\,|U_\alpha|^{-2/n}\int_\Omega v^2\,dx \le \lambda^2 b^2 \int_\Omega v^2 dx + 2\lambda\int_\Omega fv dx. \tag{1.111}$$

We would like to have here

$$c\,|U_\alpha|^{-2/n} > 2\lambda^2 b^2. \tag{1.112}$$

Since $|U_\alpha| \le |\Omega|$, it suffices to have

$$c\,|\Omega|^{-2/n} > 2\lambda^2 b^2,$$

which is equivalent to

$$|\Omega| < \left(\frac{c}{2\lambda^2 b^2}\right)^{n/2},$$

which in turn is equivalent to (1.101) with

$$\delta := \left(\frac{c}{2\lambda^2 b^2}\right)^{n/2} = c_n\,(\lambda b)^{-n}\,. \tag{1.113}$$

Hence, with this choice of δ, (1.112) follows from the hypothesis (1.101). Using (1.112), we obtain from (1.111) that

$$\frac{1}{2}c\,|U_\alpha|^{-2/n}\int_\Omega v^2\,dx \le 2\lambda\int_\Omega fv\,dx. \tag{1.114}$$

Applying in the right hand side the Hölder inequality with the Hölder exponents q and $q' = \frac{q}{q-1}$ and using$\|f\|_{L^q} = 1$, we obtain

$$\int_\Omega fv dx \le \|f\|_{L^q}\,\|v\|_{L^{q'}} = \left(\int_\Omega v^{q'} dx\right)^{1/q'}$$

(note that if $q = \infty$ then $q' = 1$). Since $q \ge 2$ and, hence, $q' \le 2$, applying the Hölder inequality with one of the Hölder exponents $\frac{2}{q'} \ge 1$, we obtain

$$\begin{aligned}\int_\Omega v^{q'} dx &= \int_{U_\alpha} v^{q'}\cdot 1 dx \le \left(\int_{U_\alpha}\left(v^{q'}\right)^{\frac{2}{q'}} dx\right)^{\frac{q'}{2}}\left(\int_{U_\alpha} 1 dx\right)^{1-\frac{q'}{2}}\\ &= \left(\int_\Omega v^2 dx\right)^{\frac{q'}{2}} |U_\alpha|^{1-\frac{q'}{2}}\,.\end{aligned}$$

It follows that

$$\int_\Omega fv\,dx \le \left(\int_\Omega v^2 dx\right)^{\frac{1}{2}} |U_\alpha|^{\frac{1}{q'}-\frac{1}{2}}.$$

Combining with (1.114), we obtain

$$\frac{1}{2}c\,|U_\alpha|^{-\frac{2}{n}}\int_\Omega v^2\,dx \le 2\lambda\left(\int_\Omega v^2 dx\right)^{\frac{1}{2}}|U_\alpha|^{\frac{1}{q'}-\frac{1}{2}},$$

whence

$$\left(\int_\Omega v^2 dx\right)^{\frac{1}{2}} \le 4c^{-1}\lambda\,|U_\alpha|^{\frac{2}{n}+\frac{1}{q'}-\frac{1}{2}},$$

and

$$\int_\Omega v^2 dx \le \left(4c^{-1}\lambda\right)^2 |U_\alpha|^{\frac{4}{n}+\frac{2}{q'}-1}. \tag{1.115}$$

Set

$$p = \frac{4}{n} + \frac{2}{q'} - 1$$

and observe that $p > 1$ because $q > \frac{n}{2}$ and, hence,

$$p - 1 = \frac{4}{n} + 2\left(1 - \frac{1}{q}\right) - 2 = \frac{4}{n} - \frac{2}{q} > 0.$$

Let us rewrite (1.115) in the form

$$\int_\Omega (u-\alpha)_+^2\,dx \le K\,|U_\alpha|^p,$$

where $p > 1$ and $K = \left(4c^{-1}\lambda\right)^2$. This inequality coincides with the inequality (1.84) from the proof of Theorem 1.14. Using the Claim from the proof of Theorem 1.14, we obtain (1.85), that is,

$$\operatorname*{esssup}_\Omega u \le C\,|\Omega|^{\frac{p-1}{2}} = C\,|\Omega|^{\frac{2}{n}-\frac{1}{q}},$$

where $C = C(K,p) = C(n,\lambda,q)$, which finishes the proof of (1.103). ■

Remark. If $n \le 3$ then the condition $q \in [2,\infty] \cap (\frac{n}{2},\infty]$ is satisfied for $q = 2$. Hence, the estimate (1.103) holds with $q = 2$. Consequently, the solution to the Dirichlet problem

$$\begin{cases} Lu = f \text{ weakly in } \Omega \\ u \in W_0^{1,2}(\Omega) \end{cases}$$

with any $f \in L^2(\Omega)$ satisfies the estimate

$$\|u\|_{L^\infty} \le C\,|\Omega|^{\frac{2}{n}-\frac{1}{2}}\,\|f\|_{L^2},$$

in particular, u is essentially bounded. In dimensions $n \ge 4$ there may exist unbounded solutions u with $f \in L^2(\Omega)$.

Remark. Let us discuss the restriction $|\Omega| < \delta$ that appears in the statement of Theorem 1.15. Consider the operator

$$L = \Delta + \sum_{i=1}^{n} b_i \partial_i u$$

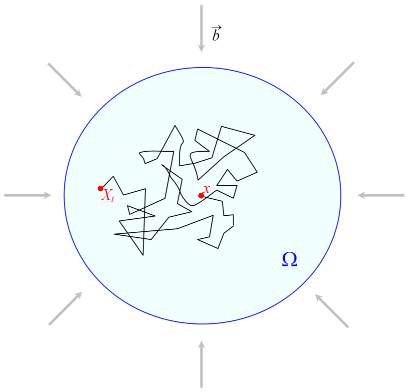
in a bounded domain $\Omega \subset \mathbb{R}^n$ and the Dirichlet problem

$$\begin{cases} Lu = -1 \text{ in } \Omega \\ u \in W_0^{1,2}(\Omega). \end{cases} \tag{1.116}$$

The estimate (1.103) of Theorem 1.15 yields, for $q = \infty$, that

$$u(x) \le C |\Omega|^{2/n} \text{ in } \Omega, \tag{1.117}$$

provided $|\Omega| < \delta$. The function $u(x)$ has the following probabilistic meaning. Operator L is the generator of a diffusion process $\{X_t\}$ with a drift $\overrightarrow{b} = (b_1, ..., b_n)$. In the case $\overrightarrow{b} \equiv 0$ this is Brownian motion, but in the case of non-zero $\overrightarrow{b}$ one can think of this diffusion process as Brownian motion under the wind with velocity $\overrightarrow{b}(x)$. The function $u(x)$ that solves (1.116) gives the mean exit time of this diffusion from Ω assuming that the starting point is x. The estimate (1.117) provides an upper bound for the mean exit time, saying that the exit on average occurs before time $C|\Omega|^{2/n}$.



The inward drift $\overrightarrow{b}$

However, if the drift $\overrightarrow{b}(x)$ is directed inwards the domain Ω, then one can imagine that the wind prevents the particle to escape from the domain, which may result in a longer exit time. As Theorem 1.15 says, this cannot happen if $|\Omega|$ is small enough, but, as we will see in example below, a longer exit time can actually occur if $|\Omega|$ is large enough (as for large domains/times the effect of drift becomes dominating over diffusion).

Example. Consider the case $n = 1$ with $\Omega = (-R, R)$ and

$$Lu = u'' + bu',$$

where

$$b(x) = -\operatorname{sgn} x = \begin{cases} 1, & x < 0, \\ 0, & x = 0, \\ -1, & x > 0. \end{cases}$$

Let us solve explicitly the Dirichlet problem

$$\begin{cases} Lu = -1 \ \text{ in } \ (-R, R) \\ u(-R) = u(R) = 0 \end{cases} \tag{1.118}$$

It suffices to solve the problem

$$\begin{cases} Lu = -1 \ \text{ in } \ (0, R) \\ u'(0) = u(R) = 0 \end{cases} \tag{1.119}$$

and then extend u evenly to $(-R, 0)$, that is, by setting $u(-x) = u(x)$. Since u satisfies in $(0, R)$ the equation

$$u'' - u' = -1, \tag{1.120}$$

in $(-R, 0)$ it will satisfy

$$u'' + u' = -1.$$

Due to the the boundary condition $u'(0) = 0$, the function u is a weak solution of $Lu = -1$ on $(-R, R)$.

The ODE (1.120) has the general solution[3]

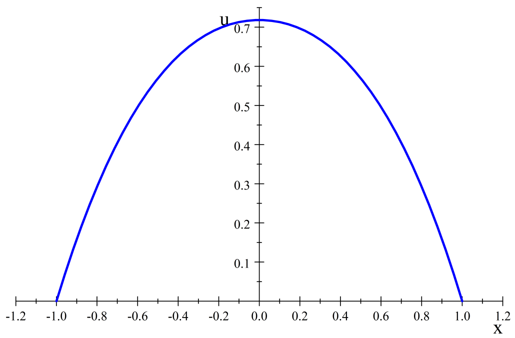
$$u(x) = c_1 + c_2 e^x + x.$$

The boundary conditions $u'(0) = u(R) = 0$ give the following equations for c_1 and c_2:

$$\begin{aligned} c_2 + 1 &= 0 \\ c_1 + c_2 e^R + R &= 0 \end{aligned}$$

whence $c_2 = -1$ and $c_1 = e^R - R$. Hence, (1.119) has solution

$$u(x) = \left(e^R - R\right) - (e^x - x).$$



Solution of (1.118) for $R = 1$

In particular, we have

$$\max u = u(0) = e^R - R - 1.$$

We see that for small R

$$\max u \approx \frac{R^2}{2}, \tag{1.121}$$

[3] Indeed, the homogeneous equation $u'' - u' = 0$ has two independent solutions $u_1 = 1$ and $u_2 = e^x$, so that its general solution is $c_1 + c_2 e^x$, while the equation $u'' - u' = -1$ has a special solution $u_0 = x$.

while for large R

$$\max u \approx e^R. \tag{1.122}$$

Note that the estimate (1.103) with $q = \infty$ gives in this case

$$\max u = \|u\|_{L^\infty} \leq CR^2, \tag{1.123}$$

provided $|\Omega| < \delta$, that is, when R is small enough. The estimate (1.123) agrees with (1.121), but (1.122) shows that (1.123) fails for large R. Hence, in general, the restriction $|\Omega| < \delta$ cannot be dropped.

Chapter 2

Higher order derivatives of weak solutions

13.11.23 **Lecture 11**

Recall the following property of the distributional Laplace operator in a domain of $\mathbb{R}^n$: if $u \in W^{1,2}_{loc}$ and $\Delta u \in L^2_{loc}$ then $u \in W^{2,2}_{loc}$. Moreover, if $\Delta u \in W^{k,2}_{loc}$ then $u \in W^{k+2,2}_{loc}$. In this Chapter we prove the same property for divergence form elliptic operators. The technique of Fourier series that worked for the Laplace operator, does not work for the operator with variable coefficients, so we use entirely different techniques based on *difference operators.*

2.1 Existence of 2nd order weak derivatives

Consider the operator

$$Lu = \sum_{i,j=1}^{n} \partial_i \left(a_{ij} \partial_j u\right) \tag{2.1}$$

in a domain $\Omega \subset \mathbb{R}^n$. As before, we assume that this operator is uniformly elliptic and the coefficients a_{ij} are measurable.

Recall that if $u \in W^{1,2}_{loc}(\Omega)$ and $f \in L^2_{loc}(\Omega)$ then we say that the equation $Lu = f$ holds weakly if, for any $\varphi \in \mathcal{D}(\Omega)$,

$$-\int_{\Omega} \sum_{i,j=1}^{n} a_{ij} \partial_j u \partial_i \varphi \, dx = \int_{\Omega} f \varphi \, dx. \tag{2.2}$$

Recall also that if in addition $u \in W^{1,2}(\Omega)$ and $f \in L^2(\Omega)$ then the identity (2.2) holds also for all $\varphi \in W^{1,2}_0(\Omega)$ (cf. Lemma 1.1).

Claim. *For any $u \in W^{1,2}_{loc}(\Omega)$ (and even for $u \in W^{1,1}_{loc}(\Omega)$) the expression Lu in (2.1) is well-defined in the distributional sense. The identity (2.2) for all $\varphi \in \mathcal{D}(\Omega)$ is equivalent to the fact that $Lu = f$ holds in the distributional sense.*

Note that, for a general distribution $u \in \mathcal{D}'(\Omega)$ the expression Lu is not well-defined because the product $a_{ij}\partial_j u$ of a measurable function a_{ij} and a distribution $\partial_j u$ does not makes sense in general[1].

Proof. The function $\partial_j u$ belongs to $L^2_{loc}(\Omega)$ and, since a_{ij} are bounded, the function $a_{ij}\partial_j u$ belongs also to $L^2_{loc}(\Omega)$, in particular, to $\mathcal{D}'(\Omega)$. Hence, $\partial_i(a_{ij}\partial_j u)$ is defined as an element of $\mathcal{D}'(\Omega)$, where ∂_i is understood in distributional sense. Consequently, Lu is defined as an element of $\mathcal{D}'(\Omega)$.

By definition of distributional derivative, we have, for any $\varphi \in \mathcal{D}(\Omega)$,

$$(Lu, \varphi) = \sum_{i,j=1}^n (\partial_i(a_{ij}\partial_j u), \varphi) = -\sum_{i,j=1}^n (a_{ij}\partial_j u, \partial_i\varphi) = -\sum_{i,j=1}^n \int_\Omega a_{ij}\partial_j u \partial_i \varphi \, dx.$$

Hence, the identity (2.2) becomes

$$(Lu, \varphi) = (f, \varphi) \quad \forall \varphi \in \mathcal{D}(\Omega),$$

which is equivalent to $Lu = f$.

For $u \in W^{1,1}_{loc}(\Omega)$ the proof is the same because L^2_{loc} can be replaced everywhere by L^1_{loc}. ■

Hence, from now on we understand the expression Lu as an element of $\mathcal{D}'(\Omega)$ for any $u \in W^{1,2}_{loc}$.

Denote by $W^{1,2}_c(\Omega)$ the set of functions from $W^{1,2}(\Omega)$ with compact support in Ω. By Exercise 8 we have

$$W^{1,2}_c(\Omega) \subset W^{1,2}_0(\Omega).$$

Claim. *If $u \in W^{1,2}_{loc}(\Omega)$, $f \in L^2_{loc}(\Omega)$ then $Lu = f$ holds if and only if the identity (2.2) holds for all $\varphi \in W^{1,2}_c(\Omega)$.*

Proof. Fix a function $\varphi \in W^{1,2}_c(\Omega)$ and let U be a precompact open set such that $\operatorname{supp}\varphi \subset U$ and $\overline{U} \subset \Omega$. Clearly, the integration in (2.2) can be restricted to U. Since $u \in W^{1,2}(U)$, $f \in L^2(U)$ and $\varphi \in W^{1,2}_0(U)$, we conclude that (2.2) holds by Lemma 1.1. ■

Now we can state the first main result of this Chapter.

Theorem 2.1 *Let L be the operator (2.1) and assume that all the coefficients a_{ij} of L are locally Lipschitz functions in Ω. If $u \in W^{1,2}_{loc}(\Omega)$ and $Lu \in L^2_{loc}(\Omega)$ then $u \in W^{2,2}_{loc}(\Omega)$.*

The notion of Lipschitz functions is explained below.

Remark. Assuming that $u \in W^{2,2}_{loc}$ and using formally a product rule for ∂_i, we have

$$Lu = \sum_{i,j} a_{ij}\partial_{ij}u + \sum_{i,j} \partial_i a_{ij}\partial_j u.$$

[1] A product av of a distribution $v \in \mathcal{D}'(\Omega)$ and a function a on Ω makes sense only if $a \in C^\infty(\Omega)$. In this case av is defined as an element of $\mathcal{D}'(\Omega)$ as follows:

$$(av, \varphi) = (v, a\varphi) \quad \forall \varphi \in \mathcal{D}(\Omega),$$

which makes sense because $a\varphi \in \mathcal{D}(\Omega)$.

Here $\partial_{ij}u \in L^2_{loc}$ and, hence, also $a_{ij}\partial_{ij}u \in L^2_{loc}$. If $Lu \in L^2_{loc}$ then one can expect that also $\partial_i a_{ij}\partial_j u \in L^2_{loc}$. Since $\partial_j u \in L^2_{loc}$, we expect that $\partial_i a_{ij} \in L^\infty_{loc}$. As we will see below, this conditions is satisfied when the coefficients a_{ij} are locally Lipschitz functions. Hence, the latter condition is to some extend necessary for $u \in W^{2,2}_{loc}$.

2.1.1 Lipschitz functions

Definition. Let S be a subset of $\mathbb{R}^n$. A function $f : S \to \mathbb{R}$ is called *Lipschitz* (or Lipschitz continuous) if there is a constant L such that

$$|f(x) - f(y)| \leq L\,|x - y| \quad \forall x, y \in S.$$

The constant L is called a Lipschitz constant of f on S.

The set of all Lipschitz functions on S is denoted by $Lip\,(S)$.

Definition. Let Ω be an open subset of $\mathbb{R}^n$. A function $f : \Omega \to \mathbb{R}$ is called *locally Lipschitz* if, for any point $x \in \Omega$, there is $\varepsilon > 0$ such that $B_\varepsilon(x) \subset \Omega$ and f is Lipschitz in $B_\varepsilon(x)$.

The set of all locally Lipschitz functions in Ω is denoted by $Lip_{loc}\,(\Omega)$. Let us list some simple properties of locally Lipschitz functions (see Exercises 35, 36 for the proof).

1. Any locally Lipschitz function in Ω is continuous in Ω.
2. If f, g are locally Lipschitz functions in Ω then $f + g$ and fg are also locally Lipschitz in Ω. Consequently, $Lip_{loc}\,(\Omega)$ is a vector space and even an subalgebra of $C\,(\Omega)$.
3. Any functions from $C^1\,(\Omega)$ is locally Lipschitz in Ω. Consequently, we have[2]

$$C^1\,(\Omega) \subset Lip_{loc}\,(\Omega) \subset C\,(\Omega)\,. \tag{2.3}$$

   It follows that Theorem 2.1 holds if all the coefficients a_{ij} belong to $C^1\,(\Omega)$.

4. If $f \in Lip_{loc}\,(\Omega)$ then $f \in Lip\,(K)$ on any compact subset K of Ω.

2.1.2 Difference operators

For the proof of Theorem 2.1 we need the notion and properties of *difference* operators. Fix a unit vector $e \in \mathbb{R}^n$, a non-zero real number h and denote by ∂_e^h an operator that acts on any function $f : \mathbb{R}^n \to \mathbb{R}$ by

$$\partial_e^h f(x) = \frac{f\,(x + he) - f(x)}{h},$$

that is,

$$\partial_e^h f = \frac{f\,(\cdot + he) - f}{h}.$$

[2]Both inclusions in (2.3) are strict. For example, function $|x|$ in $\mathbb{R}$ is Lipschitz but not C^1, whereas function $|x|^{1/2}$ is continuous but not locally Lipschitz.

Obliviously, if f is differentiable at some x then

$$\partial_e^h f(x) \to \partial_e f(x) \text{ as } h \to 0.$$

Claim. *If $f \in C_0^2(\mathbb{R}^n)$ then*

$$\partial_e^h f \rightrightarrows \partial_e f \quad \text{as } h \to 0$$

Proof. By the Taylor formula for the function $\psi(t) = f(x + te)$ we have

$$\psi(h) = \psi(0) + \psi'(0)h + \frac{1}{2}\psi''(\xi)h^2$$

for some ξ between 0 and t. Since

$$\psi'(t) = \sum_{i=1}^n \partial_i f(x + te) e_i = \partial_e f(x + te)$$

and similarly $\psi'' = \partial_{ee} f(x + te)$, we obtain

$$f(x + he) = f(x) + \partial_e f(x)h + \frac{1}{2}\partial_{ee} f(x + \xi h) h^2,$$

whence

$$\sup \left|\partial_e^h f - \partial_e f\right| \leq \frac{1}{2}\sup |\partial_{ee} f| h \to 0 \text{ as } h \to 0.$$

■

Our purpose is to use the differences $\partial_e^h f$ in order to make conclusions about the distributional derivative $\partial_e f$ provided $f \in L^2(\mathbb{R}^n)$.

Note that if f belongs to a function space $\mathcal{F}$ that is translation invariant[3], then also $\partial_e^h f \in \mathcal{F}$. All function spaces over $\mathbb{R}^n$ that we use: L^p, L^p_{loc}, $W^{k,p}$, $W^{k,p}_{loc}$, $W^{k,p}_0$ etc., are translation invariant.

Clearly, the operator ∂_e^h is linear. In the next lemma we state and prove some simple properties of difference operators.

Lemma 2.2 (Properties of the difference operators)

(a) Product rule*: for arbitrary functions f, g on $\mathbb{R}^n$ we have*

$$\partial_e^h(fg) = f(\cdot + he)\partial_e^h g + \left(\partial_e^h f\right) g. \tag{2.4}$$

(b) Integration by parts*: if $f, g \in L^2(\mathbb{R}^n)$ then*

$$\int_{\mathbb{R}^n} \left(\partial_e^h f\right) g\, dx = -\int_{\mathbb{R}^n} f\left(\partial_e^{-h} g\right) dx. \tag{2.5}$$

(c) Commutation with ∂_i: *If $f \in L^1_{loc}(\mathbb{R}^n)$ and the distributional derivative $\partial_i f$ belongs to $L^1_{loc}(\mathbb{R}^n)$ then*

$$\partial_e^h(\partial_i f) = \partial_i\left(\partial_e^h f\right).$$

[3] A space $\mathcal{F}$ of functions on $\mathbb{R}^n$ is translation invariant if $f \in \mathcal{F}$ implies that also $f(\cdot + v) \in \mathcal{F}$ for any vector $v \in \mathbb{R}^n$.

Proof. (a) We have

$$\begin{aligned}\partial_e^h (fg)(x) &= \frac{1}{h}\left(f(x+he)\,g(x+he) - f(x)g(x)\right) \\ &= \frac{1}{h} f(x+he)\left(g(x+he) - g(x)\right) \\ &\quad + \frac{1}{h}\left(f(x+he) - f(x)\right) g(x) \\ &= f(x+he)\,\partial_e^h g(x) + \partial_e^h f(x) g(x),\end{aligned}$$

which is equivalent to (2.4).

(b) Since all functions $f, \partial_e^h f, g, \partial_e^{-h} g$ are in L^2, the both integrals in (2.5) are convergent. We have

$$\begin{aligned}\int_{\mathbb{R}^n} \left(\partial_e^h f\right) g\,dx &= \frac{1}{h}\int_{\mathbb{R}^n} \left(f(x+he) - f(x)\right) g(x)\,dx \\ &= \frac{1}{h}\int_{\mathbb{R}^n} f(x+he)\,g(x)dx - \frac{1}{h}\int_{\mathbb{R}^n} f(x)g(x)\,dx \quad (\text{change } x+he \mapsto x) \\ &= \frac{1}{h}\int_{\mathbb{R}^n} f(x)g(x-he)\,dx - \frac{1}{h}\int_{\mathbb{R}^n} f(x)g(x)\,dx \\ &= -\int_{\mathbb{R}^n} f(x)\partial_e^{-h} g(x) dx.\end{aligned}$$

(c) We have

$$\begin{aligned}\partial_i\left(\partial_e^h f\right) &= \partial_i \frac{f(x+he) - f(x)}{h} \\ &= \frac{1}{h}\left(\partial_i f(x+he) - \partial_i f(x)\right) \\ &= \partial_e^h\left(\partial_i f\right).\end{aligned}$$

■

In the next lemma we prove an important test for $\partial_e f \in L^2$.

Lemma 2.3 *If $f \in L^2(\mathbb{R}^n)$ and there is a constant K such that*

$$\left\|\partial_e^h f\right\|_{L^2} \le K$$

for some unit vector e and all small enough $|h|$, then
(a) *the distributional derivative $\partial_e f$ belongs to $L^2(\mathbb{R}^n)$;*
(b) $\|\partial_e f\|_{L^2} \le K$;
(c) *$\partial_e^h f \rightharpoonup \partial_e f$ as $h \to 0$ where $\rightharpoonup$ means the weak convergence in $L^2(\mathbb{R}^n)$.*

Recall that a sequence $\{u_k\}$ of elements of a Hilbert space H converges weakly to $u \in H$ if

$$(u_k, v) \to (u, v) \quad \forall v \in H.$$

The weak convergence is denoted by $u_k \rightharpoonup u$, and it is generally weaker that the strong (norm) convergence $u_k \to u$.

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Proof. (a) Take any sequence $\{h_k\}$ of non-zero reals that converges to 0. The sequence $\left\{\partial_e^{h_k} f\right\}$ is bounded in L^2 by hypothesis. We use the fact that any norm-bounded sequence in a Hilbert space contains a *weakly* convergent subsequence. Hence, passing to a subsequence, we can assume that the sequence $\left\{\partial_e^{h_k} f\right\}$ converges weakly in L^2 to some function $g \in L^2$:

$$\partial_e^{h_k} f \rightharpoonup g \quad \text{as } k \to \infty. \tag{2.6}$$

By the definition of the weak convergence, we have, for any $\varphi \in L^2(\mathbb{R}^n)$,

$$\left(\partial_e^{h_k} f, \varphi\right) \to (g, \varphi) \quad \text{as } k \to \infty, \tag{2.7}$$

where $(\cdot, \cdot)$ is the inner product in $L^2(\mathbb{R}^n)$. Let us show that $\partial_e f = g$, which will settle the claim. Recall that the distributional derivative $\partial_e f$ is defined by

$$(\partial_e f, \varphi) = -(f, \partial_e \varphi) \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^n),$$

where $(\cdot, \cdot)$ now is the pairing of distributions and test functions. Hence, we need to verify that

$$(g, \varphi) = -(f, \partial_e \varphi).$$

Since f and g are L^2 functions, $(\cdot, \cdot)$ can be understood again as the inner product in L^2. Comparing to (2.6) we see that it remains to prove the following:

$$\left(\partial_e^{h_k} f, \varphi\right) \to -(f, \partial_e \varphi).$$

For any $\varphi \in \mathcal{D}(\mathbb{R}^n)$, we have by (2.5)

$$\left(\partial_e^{h_k} f, \varphi\right) = \int_{\mathbb{R}^n} \partial_e^{h_k} f \, \varphi dx = -\int_{\mathbb{R}^n} f \partial_e^{-h_k} \varphi dx = -(f, \partial_e^{-h_k} \varphi).$$

Since

$$\partial_e^{-h_k} \varphi \rightrightarrows \partial_e \varphi \quad \text{as } k \to \infty$$

it follows that also

$$\partial_e^{-h_k} \varphi \overset{L^2(\mathbb{R}^n)}{\longrightarrow} \partial_e \varphi \quad \text{as } k \to \infty$$

because all the supports of the functions $\partial_e^{-h_k} \varphi$ lie in a neighborhood of $\operatorname{supp} \varphi$. It follows that

$$\left(f, \partial_e^{-h_k} \varphi\right) \to (f, \partial_e \varphi) \quad \text{as } k \to \infty$$

and, hence,

$$\left(\partial_e^{h_k} f, \varphi\right) \to -(f, \partial_e \varphi) \quad \text{as } k \to \infty,$$

which was to be proved.

(b) Note that by (2.6)

$$\partial_e^{h_k} f \rightharpoonup \partial_e f \quad \text{as } k \to \infty. \tag{2.8}$$

Since $\left\|\partial_e^{h_k} f\right\| \leq K$ for all k large enough, the weak convergence (2.8) implies that also $\|\partial_e f\|_{L^2} \leq K$, which was claimed. Indeed, this is a general property of the weak convergence a Hilbert space H: if $u_k \rightharpoonup u$ in H and $\|u_k\| \le K$ then also $\|u\| \le K$. For the proof observe that, for any $v \in H$,

$$|(u_k, v)| \le \|u_k\| \, \|v\| \le K \, \|v\| \, ,$$

which implies as $k \to \infty$ that

$$|(u, v)| \le K \, \|v\| \, .$$

Setting here $v = u$ we obtain

$$\|u\|^2 \le K \, \|u\|$$

whence $\|u\| \le K$.

(c) We need to prove that, for any $\varphi \in L^2 (\mathbb{R}^n)$,

$$(\partial_e^h f, \varphi) \rightharpoonup (\partial_e f, \varphi) \text{ as } h \to 0. \tag{2.9}$$

Let us first prove this for all $\varphi \in \mathcal{D}(\Omega)$. Using the fact that

$$\partial_e^h \varphi \rightrightarrows \partial_e \varphi \text{ as } h \to 0$$

and arguing as in (a), we obtain

$$(\partial_e^h f, \varphi) = -\left(f, \partial_e^{-h} \varphi\right) \to -(f, \partial_e \varphi) = (\partial_e f, \varphi) \, ,$$

which was claimed.

Now let us prove (2.9) for any $\varphi \in L^2 (\mathbb{R}^n)$. For that take some $\psi \in \mathcal{D}(\mathbb{R}^n)$ (considered as an approximation to φ) and write

$$\begin{aligned} (\partial_e^h f, \varphi) - (\partial_e f, \varphi) &= (\partial_e^h f, \varphi) - (\partial_e^h f, \psi) + (\partial_e^h f, \psi) - (\partial_e f, \psi) + (\partial_e f, \psi) - (\partial_e f, \varphi) \\ &= \left(\partial_e^h f, \varphi - \psi\right) + (\partial_e^h f - \partial_e f, \psi) + (\partial_e f, \psi - \varphi). \end{aligned}$$

It follows that, for all small enough h,

$$\begin{aligned} \left|(\partial_e^h f, \varphi) - (\partial_e f, \varphi)\right| &\le \left\|\partial_e^h f\right\|_{L^2} \|\varphi - \psi\| + \left|(\partial_e^h f - \partial_e f, \psi)\right| + \|\partial_e f\|_{L^2} \|\varphi - \psi\|_{L^2} \\ &\le 2K \, \|\varphi - \psi\|_{L^2} + \left|(\partial_e^h f - \partial_e f, \psi)\right| . \end{aligned}$$

Letting $h \to 0$ we obtain that, by the first part of the proof,

$$(\partial_e^h f - \partial_e f, \psi) \to 0$$

whence

$$\limsup_{h \to 0} \left|(\partial_e^h f - \partial_e f, \psi)\right| \le 2K \, \|\varphi - \psi\|_{L^2} \, .$$

Since $\|\varphi - \psi\|_{L^2}$ can be made arbitrarily small by choice of $\psi \in \mathcal{D}(\mathbb{R}^n)$, we obtain (2.9). ■

For any open set $\Omega \subset \mathbb{R}^n$, denote by $Lip_c(\Omega)$ the set of Lipschitz functions in Ω with compact supports.

Corollary 2.4 (Lipschitz functions as elements of Sobolev spaces)

(a) *If* $f \in Lip_c(\mathbb{R}^n)$ *then* $f \in W^{1,2}(\mathbb{R}^n)$ *and* $f \in W^{1,\infty}(\mathbb{R}^n)$.

(b) *If* $f \in Lip_{loc}(\Omega)$ *then* $f \in W^{1,2}_{loc}(\Omega)$ *and* $f \in W^{1,\infty}_{loc}(\Omega)$.

Proof. (a) Let L be the Lipschitz constant of f, Then, for all x and all $h \neq 0$, we have

$$\left|\partial_e^h f(x)\right| = \left|\frac{f(x+he)-f(x)}{h}\right| \leq \frac{L\,|he|}{|h|} = L.$$

For all $|h|<1$ the support of $\partial_e^h f$ lie in 1-neighborhood of $\operatorname{supp} f$, which implies that also the L^2-norms $\left\|\partial_e^h f\right\|_{L^2}$ are uniformly bounded. By Lemma 2.3 we conclude that $\partial_e f \in L^2(\mathbb{R}^n)$. Since this is true for any unit vector e, it follows that $f \in W^{1,2}(\mathbb{R}^n)$.

Let us now show that $f \in W^{1,\infty}(\mathbb{R}^n)$. Since f is continuous and has compact support, we see that f is bounded, that is, $f \in L^\infty(\mathbb{R}^n)$. Let us show that

$$\|\partial_e f\|_{L^\infty} \leq L,$$

which will imply that $\partial_e f \in L^\infty(\mathbb{R}^n)$. Indeed, since $\left|\partial_e^h f(x)\right| \leq L$, we have, for any $\varphi \in \mathcal{D}(\mathbb{R}^n)$,

$$\left|\left(\partial_e^h f, \varphi\right)\right| \leq L\,\|\varphi\|_{L^1}.$$

Since $\partial_e^h f \rightharpoonup \partial_e f$ as $h \to 0$, it follows that

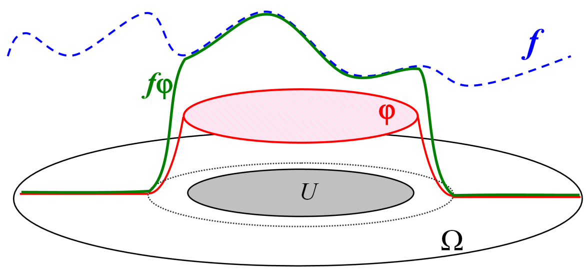
$$|(\partial_e f, \varphi)| \leq L\,\|\varphi\|_{L^1},$$

which implies that

$$\|\partial_e f\|_{L^\infty} = \sup_{\varphi \in L^1(\mathbb{R}^n)\setminus\{0\}} \frac{|(\partial_e f,\varphi)|}{\|\varphi\|_{L^1}} = \sup_{\varphi \in \mathcal{D}(\mathbb{R}^n)\setminus\{0\}} \frac{|(\partial_e f,\varphi)|}{\|\varphi\|_{L^1}} \leq L.$$

Hence, $\partial_e f \in L^\infty(\mathbb{R}^n)$ and $f \in W^{1,\infty}(\mathbb{R}^n)$.

(b) Let U be an arbitrary precompact open set such that $\overline{U} \subset \Omega$. Let φ be a cutoff function of U in Ω, that is, $\varphi \in \mathcal{D}(\Omega)$ and $\varphi \equiv 1$ on U. Since φ is Lipschitz, it follows that $f\varphi \in Lip_{loc}(\Omega)$.



Since $f\varphi$ has compact support in Ω, it follows that $f\varphi \in Lip_c(\Omega)$; extending $f\varphi$ by 0 outside Ω, we obtain that $f\varphi \in Lip_c(\mathbb{R}^n)$ (cf. Exercise 36). It follows by (a) that $f\varphi \in W^{1,2}(\mathbb{R}^n)$. Since $\varphi = 1$ in U, it follows that $f \in W^{1,2}(U)$ and $f \in W^{1,\infty}(U)$. Since U is arbitrary, we conclude that $f \in W^{1,2}_{loc}(\Omega)$ and $f \in W^{1,\infty}_{loc}(\Omega)$. ■

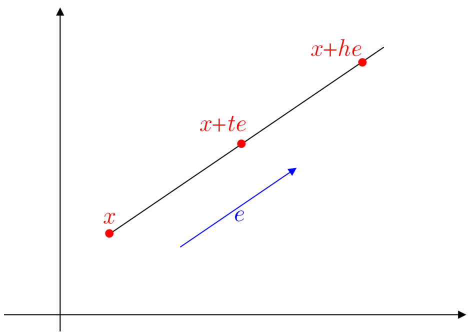
Lemma 2.5 *If $f \in W^{1,2}(\mathbb{R}^n)$ then*

$$\left\|\partial_e^h f\right\|_{L^2} \leq \|\partial_e f\|_{L^2}. \tag{2.10}$$

Proof. Note that the both sides of (2.10) are continuous functionals in $W^{1,2}(\mathbb{R}^n)$. Since C^1 functions are dense in $W^{1,2}(\mathbb{R}^n)$ (see Exercise[4] 4), it suffices to prove (2.10) assuming that $f \in C^1$.

Fix $x \in \mathbb{R}^n$, a unit vector $e \in \mathbb{R}^n$, and consider the following function of $t \in \mathbb{R}$:

$$g(t) = f(x+te).$$



By the chain rule we obtain

$$\frac{dg}{dt} = \sum_{i=1}^{n} \partial_{x_i} f(x+te)\, e_i = \partial_e f(x+te)$$

and

$$\begin{aligned}\partial_e^h f(x) &= \frac{1}{h}\left(f(x+he) - f(x)\right) = \frac{1}{h}\left(g(h) - g(0)\right)\\ &= \frac{1}{h}\int_0^h \frac{dg}{dt} dt = \frac{1}{h}\int_0^h \partial_e f(x+te)\, dt.\end{aligned}$$

It follows that

$$\left|\partial_e^h f(x)\right|^2 = \left(\frac{1}{h}\int_0^h \partial_e f(x+te)\, dt\right)^2 \le \frac{1}{h}\int_0^h \left|\partial_e f(x+te)\right|^2 dt.$$

Integrating over $\mathbb{R}^n$ and using Fubini's formula to interchange the integrals, we obtain

$$\begin{aligned}\left\|\partial_e^h f\right\|_{L^2}^2 &\le \frac{1}{h}\int_{\mathbb{R}^n}\left(\int_0^h \left|\partial_e f(x+te)\right|^2 dt\right) dx\\ &= \frac{1}{h}\int_0^h \left(\int_{\mathbb{R}^n} \left|\partial_e f(x+te)\right|^2 dx\right) dt \quad \text{(change } y = x+te\text{)}\\ &= \frac{1}{h}\int_0^h \left(\int_{\mathbb{R}^n} \left|\partial_e f(y)\right|^2 dy\right) dt \quad \text{(the internal integral does not depend on } t\text{)}\\ &= \frac{1}{h}\int_0^h \left\|\partial_e f\right\|_{L^2}^2 dt = \left\|\partial_e f\right\|_{L^2}^2,\end{aligned}$$

which finishes the proof. ■

[4] If $f \in W^{1,2}(\mathbb{R}^n)$ then, for any mollifier φ and any $\varepsilon > 0$, the convolution $f * \varphi_\varepsilon$ is a C^∞ function that also belongs to $W^{1,2}(\mathbb{R}^n)$, and $f * \varphi_\varepsilon \overset{W^{1,2}}{\longrightarrow} f$ as $\varepsilon \to 0$.

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2.1.3 Product rule for L

Consider in an open domain $\Omega \subset \mathbb{R}^n$ a uniformly elliptic operator

$$Lu = \sum_{i,j=1}^{n} \partial_i \left(a_{ij}\partial_j u\right), \tag{2.11}$$

with measurable coefficients a_{ij}.

Lemma 2.6 (Product rule for L) *If $u, v \in W^{1,2}_{loc}(\Omega)$ and $Lu, Lv \in L^2_{loc}(\Omega)$ then*

$$L(uv) = (Lu)v + u(Lv) + 2\sum_{i,j=1}^{n} a_{ij}\partial_j u \partial_i v. \tag{2.12}$$

Motivation. Before the proof in full generality, let us prove the formula (2.12) in a simpler setting assuming that $a_{ij} \in C^1(\Omega)$ and $u, v \in C^2(\Omega)$. Then all the derivatives involved are classical, and we obtain by the product rule

$$\begin{aligned}\partial_i \left(a_{ij}\partial_j (uv)\right) &= \partial_i \left(a_{ij}\partial_j u\, v\right) + \partial_i \left(a_{ij}\partial_j v\, u\right) \\ &= \partial_i \left(a_{ij}\partial_j u\right) v + a_{ij}\partial_j u \partial_i v \\ &\quad + \partial_i \left(a_{ij}\partial_j v\right) u + a_{ij}\partial_j v \partial_i u.\end{aligned}$$

Adding up in all i, j and using the symmetry of a_{ij}, we obtain that

$$L(uv) = (Lu)v + (Lv)u + 2\sum_{i,j=1}^{n} a_{ij}\partial_j u \partial_i v,$$

that is (2.13).

Note that under the assumptions $u, v \in W^{1,2}_{loc}(\Omega)$ the above argument does not work since $a_{ij}\partial_j u$ can be claimed only to belong to $L^2_{loc}(\Omega)$. Hence, the term $\partial_i \left(a_{ij}\partial_j u\right) v$ is meaningless as a product of a distribution $\partial_i \left(a_{ij}\partial_j u\right)$ with a measurable function v.

Proof of Lemma 2.6. We will use in the proof the following product rule from Exercise 21: if $u, v \in W^{1,2}_{loc}(\Omega)$ then $uv \in W^{1,1}_{loc}(\Omega)$ and

$$\partial_j (uv) = (\partial_j u) v + (\partial_j v) u. \tag{2.13}$$

Observe also that the expression $L(uv)$ is well-defined as a distribution sense because $uv \in W^{1,1}_{loc}(\Omega)$.

Using the distributional definition of L and the product rule (2.13), we obtain, for any $\varphi \in \mathcal{D}(\Omega)$,

$$\begin{aligned}(L(uv), \varphi) &= \sum_{i,j=1}^{n} \left(\partial_i \left(a_{ij}\partial_j (uv)\right), \varphi\right) \\ &= -\sum_{i,j=1}^{n} \left(a_{ij}\underline{\partial_j (uv)}, \partial_i \varphi\right) \\ &= -\sum_{i,j=1}^{n} \left(a_{ij}\underline{(\partial_j u)\, v}, \partial_i \varphi\right) - \sum_{i,j=1}^{n} \left(a_{ij}\underline{(\partial_j v) u}, \partial_i \varphi\right)\end{aligned} \tag{2.14}$$

Using again the product rule in the form

$$-v\,\partial_i\varphi = -\partial_i\,(v\varphi) + \partial_i v\,\varphi,$$

we obtain for the first term in (2.14):

$$\begin{aligned}-\sum_{i,j=1}^{n}(a_{ij}(\partial_j u)v,\partial_i\varphi) &= -\int_\Omega\sum_{i,j=1}^{n}a_{ij}\partial_j u\,\underline{v\,\partial_i\varphi}\,dx\\ &= -\int_\Omega\sum_{i,j=1}^{n}a_{ij}\partial_j u\,\underline{\partial_i(v\varphi)}dx + \int_\Omega\sum_{i,j=1}^{n}a_{ij}\partial_j u\,\underline{\partial_i v\,\varphi}\,dx. \end{aligned}\tag{2.15}$$

Next, recall that Lu satisfies the following identity:

$$-\int_\Omega\sum_{i,j=1}^{n}a_{ij}\partial_j u\,\partial_i\psi\,dx = \int_\Omega (Lu)\,\psi\,dx$$

for any test function $\psi \in W_c^{1,2}(\Omega)$. Since $v\varphi \in W_c^{1,2}(\Omega)$, setting here $\psi = v\varphi$, we obtain

$$-\int_\Omega\sum_{i,j=1}^{n}a_{ij}\partial_j u\,\partial_i\,(v\varphi)\,dx = \int_\Omega (Lu)\,v\varphi\,dx = (vLu,\varphi)\,.$$

Hence, (2.15) yields

$$-\sum_{i,j=1}^{n}(a_{ij}\,(\partial_j u)\,v,\partial_i\varphi) = (vLu,\varphi) + \left(\sum_{i,j=1}^{n}a_{ij}\partial_j u\,\partial_i v\,,\varphi\right). \tag{2.16}$$

By interchanging here u and v, we obtain that a similar identity holds for the second term in (2.14):

$$-\sum_{i,j=1}^{n}(a_{ij}u\,\partial_j v,\partial_i\varphi) = (uLv,\varphi) + \left(\sum_{i,j=1}^{n}a_{ij}\partial_j v\,\partial_i u\,,\varphi\right). \tag{2.17}$$

Noticing that $a_{ij} = a_{ji}$ and interchanging the indexes i and j in the last sum in (2.17), we obtain that it is equal to the last sum in (2.16). Hence, adding up (2.16), (2.17) and using (2.14), we obtain

$$(L\,(uv)\,,\varphi) = (vLu,\varphi) + (uLv,\varphi) + 2\left(\sum_{i,j=1}^{n}a_{ij}\partial_j u\,\partial_i v\,,\varphi\right),$$

which is equivalent to (2.12). ■

2.1.4 Proof of Theorem 2.1

Let L be the operator (2.11), where $a_{ij} \in Lip_{loc}(\Omega)$. We need to prove that

$$u \in W_{loc}^{1,2}(\Omega) \quad\text{and}\quad Lu \in L_{loc}^{2}(\Omega) \Rightarrow u \in W_{loc}^{2,2}(\Omega)\,.$$

Idea of proof. We need to prove that any second derivative $\partial_{ik}u$ belongs to $L^2_{loc}(\Omega)$. Denote $v = \partial_k u$. We know that $v \in L^2_{loc}(\Omega)$ and we need to prove that $\partial_i v \in L^2_{loc}(\Omega)$. By Lemma 2.3, if $v \in L^2(\mathbb{R}^n)$ then in order to prove that $\partial_e v \in L^2(\mathbb{R}^n)$ it suffices to verify that $\left\|\partial_e^h v\right\|_{L^2}$ is uniformly bounded for all small enough $|h|$. For that we need to obtain an upper bound for $\left\|\partial_e^h v\right\|_{L^2}$.

Let us first explain how to obtain such an estimate in a simpler situation. For that, we assume in addition that $a_{ij} \in C^1(\Omega)$ and $u \in C_0^3(\Omega)$, and obtain an upper bound for $\|\nabla v\|_{L^2}$, where $v = \partial_k u$ for a fixed index k (in the actual proof we use a similar argument to estimate $\left\|\partial_e^h v\right\|_{L^2}$).

Set $f = Lu$. Then we have the identity

$$-\int_\Omega \sum_{i,j=1}^n a_{ij}\partial_j u\, \partial_i \varphi\, dx = \int_\Omega f\varphi\, dx \quad \text{for all } \varphi \in W_0^{1,2}(\Omega).$$

Let us use here the test function $\varphi := \partial_k v = \partial_{kk} u \in C_0^1(\Omega)$. Since both functions $a_{ij}\partial_j u$ and $\partial_i v$ belong to $C_0^1(\Omega)$, we can use the integration by parts formula and obtain

$$\begin{aligned}
-\int_\Omega \sum_{i,j=1}^n a_{ij}\partial_j u\, \partial_i(\partial_k v)dx &= -\int_\Omega \sum_{i,j=1}^n \underline{a_{ij}\partial_j u}\, \partial_k(\underline{\partial_i v})\, dx \quad \text{(integrations by parts)}\\
&= \int_\Omega \sum_{i,j=1}^n \partial_k(\underline{a_{ij}\partial_j u})\underline{\partial_i v}\, dx \qquad \text{(product rule)}\\
&= \int_\Omega \sum_{i,j=1}^n a_{ij}\partial_k\partial_j u\, \partial_i v\, dx + \int_\Omega \sum_{i,j=1}^n (\partial_k a_{ij})\, \partial_j u\, \partial_i v\, dx.
\end{aligned}$$

Observing $\partial_k\partial_j u = \partial_j v$ and combining the above identities, we obtain

$$\int_\Omega \sum_{i,j=1}^n a_{ij}\partial_j v\, \partial_i v\, dx = \int_\Omega f\partial_k v\, dx - \int_\Omega \sum_{i,j=1}^n (\partial_k a_{ij})\, \partial_j u\, \partial_i v\, dx.$$

By the uniform ellipticity condition, we have

$$\int_\Omega \sum_{i,j=1}^n a_{ij}\partial_j v\, \partial_i v \geq \lambda^{-1}\|\nabla v\|_{L^2}^2,$$

and by the Cauchy-Schwarz inequality

$$\int_\Omega f\partial_k v\, dx \leq \|f\|_{L^2}\|\nabla v\|_{L^2}.$$

Since all $\partial_k a_{ij}$ are bounded on $\operatorname{supp} u$, we obtain

$$\begin{aligned}
\left|\int_\Omega \sum_{i,j=1}^n (\partial_k a_{ij})\, \partial_j u\, \partial_i v\, dx\right| &\leq \int_{\operatorname{supp} u} \sum_{i,j=1}^n |\partial_k a_{ij}|\, |\nabla u|\, |\nabla v|\, dx\\
&\leq C\|\nabla u\|_{L^2}\|\nabla v\|_{L^2},
\end{aligned}$$

where $C = \sup_{\operatorname{supp} u} \sum_{i,j=1}^{n} |\partial_k a_{ij}| < \infty$. Combining together all the above inequalities, we obtain

$$\lambda^{-1} \|\nabla v\|_{L^2}^2 \le \|f\|_{L^2} \|\nabla v\|_{L^2} + C \|\nabla u\|_{L^2} \|\nabla v\|_{L^2},$$

whence it follows that

$$\|\nabla v\|_{L^2} \le \lambda \left(\|f\|_{L^2} + C \|\nabla u\|_{L^2}\right).$$

Proof of Theorem 2.1. Set $f = Lu$. Consider first a special case when $u \in W_c^{1,2}(\Omega)$ and $f \in L^2(\Omega)$, and prove that in this case $u \in W^{2,2}(\Omega)$. It suffices to prove that all distributional derivatives $\partial_{ik} u$ belong to $L^2(\Omega)$.

Let us extend u to a function on $\mathbb{R}^n$ by setting $u = 0$ in Ω^c. Then we have $u \in W_c^{1,2}(\mathbb{R}^n)$. We will prove that all second order derivatives $\partial_k(\partial_i u)$ are in $L^2(\mathbb{R}^n)$. Since $\partial_i u \in L^2(\mathbb{R}^n)$, by Lemma 2.3 it suffices to verify that, for any unit vector e,

$$\left\|\partial_e^h(\partial_i u)\right\|_{L^2} \le K$$

for some constant K and for all small enough $|h|$. Since $\partial_e^h(\partial_i u) = \partial_i\left(\partial_e^h u\right)$, it suffices to prove that

$$\left\|\partial_i\left(\partial_e^h u\right)\right\|_{L^2} \le K.$$

Setting $v = \partial_e^h u$, we see that it suffices to prove that

$$\|\nabla v\|_{L^2} \le K. \tag{2.18}$$

We are going to show that (2.18) holds with

$$K = \lambda\left(\|f\|_{L^2} + C \|\nabla u\|_{L^2}\right),$$

where C depends on n and on the Lipschitz constant of the coefficients a_{ij} on $\operatorname{supp} u$.

For simplicity of notations, we write $\partial^h \equiv \partial_e^h$. We always assume that $|h|$ is small enough, in particular, $|h|$ is much smaller that the distance from $\operatorname{supp} u$ to the boundary of Ω. Clearly, we have then $v \in W_c^{1,2}(\Omega)$ and $\partial^{-h} v \in W_c^{1,2}(\Omega)$. Since $Lu = f$, we have, for any $\varphi \in W_0^{1,2}(\Omega)$,

$$-\int_\Omega \sum_{i,j=1}^{n} a_{ij} \partial_j u \, \partial_i \varphi \, dx = \int_\Omega f \varphi \, dx.$$

Setting here $\varphi = \partial^{-h} v$ (that is, $\varphi = \partial^{-h}(\partial^h u)$), we obtain

$$-\int_\Omega \sum_{i,j=1}^{n} a_{ij} \partial_j u \, \partial_i(\partial^{-h} v) dx = \int_\Omega f \, \partial^{-h} v \, dx.$$

On the left hand side, we apply the integration by parts formula[5] and the product rule for difference operators from Lemma 2.2:

[5] The integration by parts formula (2.5) of Lemma 2.2 was proved for functions from $L^2(\mathbb{R}^n)$. However, if both functions have compact supports in Ω then, for sufficiently small h, the integration in the both sides of (2.5) can be reduced to Ω.

$$
\begin{aligned}
-\int_\Omega \sum_{i,j=1}^n a_{ij}\partial_j u\,\partial_i(\partial^{-h}v)dx &= -\int_\Omega \sum_{i,j=1}^n a_{ij}\partial_j u\,\partial^{-h}(\partial_i v)dx \quad \text{(integration by parts)}\\
&= \int_\Omega \sum_{i,j=1}^n \partial^h(a_{ij}\partial_j u)\,\partial_i v\,dx \qquad \text{(product rule)}\\
&= \int_\Omega \sum_{i,j=1}^n a_{ij}(\cdot+eh)\partial^h\partial_j u\,\partial_i v\,dx + \int_\Omega \sum_{i,j=1}^n (\partial^h a_{ij})\partial_j u\,\partial_i v\,dx.
\end{aligned}
$$

Observing that $\partial^h\partial_j u = \partial_j\partial^h u = \partial_j v$ and combining the above identities, we obtain

$$
\int_\Omega \sum_{i,j=1}^n a_{ij}(\cdot+eh)\partial_j v\,\partial_i v\,dx = \int_\Omega f\,\partial^{-h}v\,dx - \int_\Omega \sum_{i,j=1}^n (\partial^h a_{ij})\partial_j u\,\partial_i v\,dx.
$$

By the uniform ellipticity we have

$$
\int_\Omega \sum_{i,j=1}^n a_{ij}(\cdot+eh)\partial_j v\,\partial_i v\,dx \geq \lambda^{-1}\|\nabla v\|_{L^2}^2.
$$

By the Cauchy-Schwarz inequality inequality and Lemma 2.5, we obtain

$$
\int_\Omega f\,\partial^{-h}v\,dx \leq \|f\|_{L^2}\left\|\partial^{-h}v\right\|_{L^2} \leq \|f\|_{L^2}\|\nabla v\|_{L^2}.
$$

Also we have

$$
\begin{aligned}
\left|\int_\Omega \sum_{i,j=1}^n (\partial^h a_{ij})\partial_j u\partial_i v\,dx\right| &\leq \int_{\operatorname{supp} u} \sum_{i,j=1}^n \left|\partial^h a_{ij}\right| |\nabla u|\,|\nabla v|\,dx\\
&\leq \sup_{\operatorname{supp} u} \sum_{i,j=1}^n \left|\partial^h a_{ij}\right| \|\nabla u\|_{L^2}\|\nabla v\|_{L^2}.
\end{aligned}
$$

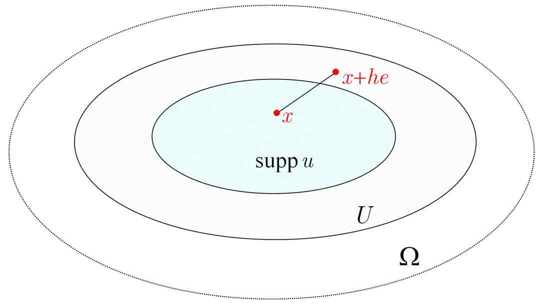
Let us fix a precompact open neighborhood U of $\operatorname{supp} u$ such that $\overline{U} \subset \Omega$. Since $a_{ij} \in Lip_{loc}(\Omega)$, it follows that $a_{ij} \in Lip(U)$ (cf. Exercise 36).

Let L be a common Lipschitz constant of all functions a_{ij} in U. Then, for any $x \in \operatorname{supp} u$ and small enough $|h|$, we have

$$
\left|\partial^h a_{ij}(x)\right| = \left|\frac{a_{ij}(x+he)-a_{ij}(x)}{h}\right| \leq L,
$$

whence

$$
\sup_{\operatorname{supp} u} \sum_{i,j=1}^n \left|\partial^h a_{ij}\right| \leq n^2 L =: C < \infty.
$$



Hence, combining all the above inequalities, we obtain

$$\lambda^{-1} \|\nabla v\|_{L^2}^2 \leq \|f\|_{L^2} \|\nabla v\|_{L^2} + C \|\nabla u\|_{L^2} \|\nabla v\|_{L^2}$$

whence it follows that

$$\|\nabla v\|_{L^2} \leq \lambda (\|f\|_{L^2} + C \|\nabla u\|_{L^2}) =: K,$$

which is equivalent to (2.18).

23.11.23 Lecture 14

Consider now a general case when $u \in W_{loc}^{1,2}(\Omega)$ and $Lu \in L_{loc}^2(\Omega)$. In order to prove that $u \in W_{loc}^{2,2}(\Omega)$ it suffices to prove that $u \in W^{2,2}(U)$ for any precompact domain U such that $\overline{U} \subset \Omega$. Fix U, choose a cutoff function η of U in Ω (as in the proof of Corollary 2.4(b)), and consider the function

$$w := u\eta \in W_c^{1,2}(\Omega).$$

By Lemma 2.6 we have

$$Lw = (Lu)\eta + uL\eta + 2\sum_{i,j=1}^n a_{ij}\partial_j u \, \partial_i \eta.$$

Observe that all the terms in the right hand side here belong to $L^2(\Omega)$. For example, the function

$$L\eta = \sum_{i,j=1}^n \partial_i (a_{ij}\partial_j \eta) = \sum_{i,j=1}^n \partial_i a_{ij} \, \partial_j \eta + \sum_{i,j=1}^n a_{ij} \, \partial_i \partial_j \eta$$

is compactly supported and belongs to $L^\infty(\Omega)$ because $\partial_i a_{ij} \in L_{loc}^\infty(\Omega)$ (by Corollary 2.4), which implies $uL\eta \in L^2(\Omega)$.

Hence, $Lw \in L^2(\Omega)$. By the above special case, we conclude that $w \in W^{2,2}(\Omega)$, which implies that $w \in W^{2,2}(U)$. Since $u = w$ on U, it follows $u \in W^{2,2}(U)$, which finishes the proof. ■

Proposition 2.7 *If $a_{ij} \in Lip_{loc}(\Omega)$ and $u \in W_{loc}^{2,2}(\Omega)$ then in the expression*

$$Lu = \sum_{i,j=1}^n \partial_i (a_{ij}\partial_j u)$$

the derivatives ∂_j and ∂_i act on functions from $W_{loc}^{1,2}(\Omega)$. In particular, this is the case under the hypothesis of Theorem 2.1.

Remark. If the both derivatives ∂_i and ∂_j in L act in $W^{1,2}_{loc}(\Omega)$ then one says that the operator L is understood in the *strong sense*. Recall that if $u \in W^{1,2}_{loc}(\Omega)$ then ∂_j acts in $W^{1,2}_{loc}(\Omega)$, whereas ∂_i acts in $\mathcal{D}'(\Omega)$; in that case we say (as before) that L is understood in the *weak sense*.

Recall for comparison that L is understood in the *classical sense* if the both operators ∂_i, ∂_j act in $C^1(\Omega)$, which is the case when $u \in C^2$ and $a_{ij} \in C^1$.

Proof. Let us first prove the following general fact:

$$v \in W^{1,2}_{loc}(\Omega) \quad \text{and} \quad a \in Lip_{loc}(\Omega) \;\Rightarrow\; av \in W^{1,2}_{loc}(\Omega).$$

Since $v \in L^2_{loc}(\Omega)$ and a is locally bounded, it follows that $av \in L^2_{loc}(\Omega)$. Let us prove that also $\partial_i(av) \in L^2_{loc}(\Omega)$ for any index i. Indeed, by Corollary 2.4 we have $a \in W^{1,2}_{loc}(\Omega)$, and by Exercise 21, we have $av \in W^{1,1}_{loc}(\Omega)$ and

$$\partial_i(av) = (\partial_i a)v + a\partial_i v.$$

By Corollary 2.4 we have also $a \in W^{1,\infty}_{loc}(\Omega)$ so that $\partial_i a$ is locally bounded and, hence, $(\partial_i a)v \in L^2_{loc}(\Omega)$. Since also $a\partial_i v \in L^2_{loc}(\Omega)$, it follows that $\partial_i(av) \in L^2_{loc}(\Omega)$, which was to be proved.

Now let us prove that the both operators ∂_i and ∂_j in Lu act on functions from $W^{1,2}_{loc}(\Omega)$. The operator ∂_j acts in $W^{1,2}_{loc}(\Omega)$ because $u \in W^{1,2}_{loc}(\Omega)$. Since $\partial_j u \in W^{1,2}_{loc}(\Omega)$ and $a_{ij} \in Lip_{loc}(\Omega)$, it follows that $a_{ij}\partial_j u \in W^{1,2}_{loc}(\Omega)$ and, hence, the operator ∂_i also acts in $W^{1,2}_{loc}(\Omega)$, which completes the proof. ■

2.2 Existence of higher order weak derivatives

As above, consider in a domain $\Omega \subset \mathbb{R}^n$ a uniformly elliptic operator

$$Lu = \sum_{i,j=1}^{n} \partial_i(a_{ij}\partial_j u). \tag{2.19}$$

Theorem 2.8 *Let $u \in W^{1,2}_{loc}(\Omega)$. If, for a non-negative integer k, we have*

$$a_{ij} \in C^{k+1}(\Omega) \quad \text{and} \quad Lu \in W^{k,2}_{loc}(\Omega)$$

then $u \in W^{k+2,2}_{loc}(\Omega)$.

Remark. In the case $k = 0$ this statement says the following: if $a_{ij} \in C^1(\Omega)$ and $Lu \in L^2_{loc}(\Omega)$ then $u \in W^{2,2}_{loc}(\Omega)$, which follows from Theorem 2.1 because $C^1(\Omega) \subset Lip_{loc}(\Omega)$.

For the proof we need the following lemma.

Lemma 2.9 *If $v \in W^{k,2}_{loc}(\Omega)$ and $a \in W^{k,\infty}_{loc}(\Omega)$ then $av \in W^{k,2}_{loc}(\Omega)$.*

Remark. In fact, we have proved this statement for $k = 1$ in the proof of Proposition 2.7,

Proof. Induction in k. For $k = 0$ the claim is obvious: if $v \in L^2_{loc}(\Omega)$ and $a \in L^\infty_{loc}(\Omega)$ then $av \in L^2_{loc}(\Omega)$. Let us make the inductive step from k to $k+1$, that is, assuming that

$$v \in W^{k+1,2}_{loc}(\Omega) \text{ and } \ a \in W^{k+1,\infty}_{loc},$$

let us prove that $av \in W^{k+1,2}_{loc}(\Omega)$. Since $av \in L^2_{loc}(\Omega)$, it suffices to prove that

$$\partial_i(av) \in W^{k,2}_{loc}(\Omega)$$

for any index i. Since both functions a, v belong to $W^{1,2}_{loc}(\Omega)$, we obtain by the product rule of Exercise 21, that $av \in W^{1,1}_{loc}(\Omega)$ and

$$\partial_i(av) = (\partial_i a)v + a\partial_i v.$$

Since $v \in W^{k,2}_{loc}(\Omega)$ and $\partial_i a \in W^{k,\infty}_{loc}(\Omega)$, we conclude by the inductive hypothesis that $(\partial_i a)v \in W^{k,2}_{loc}(\Omega)$. In the same way we obtain that $a\partial_i v \in W^{k,2}_{loc}(\Omega)$, whence it follows that $\partial_i(av) \in W^{k,2}_{loc}(\Omega)$, which completes the proof. ∎

Proof of Theorem 2.8. Induction in k. As it was mentioned above, the case $k = 0$ is covered by Theorem 2.1.

Assuming $k \geq 1$, let us make the inductive step from $k-1$ to k. Assuming that

$$a_{ij} \in C^{k+1}(\Omega) \quad \text{and} \quad Lu \in W^{k,2}_{loc}(\Omega),$$

we need to prove that $u \in W^{k+2,2}_{loc}(\Omega)$. Since also $a_{ij} \in C^k(\Omega)$ and $Lu \in W^{k-1,2}_{loc}(\Omega)$, and the inductive hypothesis yields that

$$u \in W^{k+1,2}_{loc}(\Omega).$$

In order to prove that $u \in W^{k+2,2}_{loc}(\Omega)$, it suffices to verify that any partial derivative $\partial_l u$ belongs to $W^{k+1,2}_{loc}(\Omega)$. For that, it will be sufficient to show that

$$L(\partial_l u) \in W^{k-1,2}_{loc}(\Omega). \tag{2.20}$$

Indeed, assuming that (2.20) is true, observing that $\partial_l u \in W^{k,2}_{loc}(\Omega) \subset W^{1,2}_{loc}(\Omega)$ and applying the inductive hypothesis to $\partial_l u$, we conclude that $\partial_l u \in W^{k+1,2}_{loc}(\Omega)$ thus finishing the proof.

It remains to prove (2.20). We have

$$L(\partial_l u) = \sum_{i,j=1}^n \partial_i (a_{ij}\partial_j\partial_l u) = \sum_{i,j=1}^n \partial_i (a_{ij}\partial_l\partial_j u).$$

Since both $\partial_j u$ and a_{ij} belong to $W^{1,2}_{loc}(\Omega)$, we have by the product rule in $W^{1,2}_{loc}(\Omega)$

$$\partial_l (a_{ij}\partial_j u) = a_{ij}\partial_l\partial_j u + (\partial_l a_{ij})\partial_j u,$$

whence

$$a_{ij}\partial_l\partial_j u = \partial_l (a_{ij}\partial_j u) - (\partial_l a_{ij})\partial_j u$$

and, hence,

$$L(\partial_l u) = \sum_{i,j=1}^{n} \partial_i \left[\partial_l (a_{ij}\partial_j u) - (\partial_l a_{ij})\partial_j u\right] = \partial_l (Lu) - \sum_{i,j=1}^{n} \partial_i (\partial_l a_{ij}\partial_j u).$$

Note that $\partial_l (Lu) \in W^{k-1,2}_{loc}(\Omega)$. Since

$$\partial_j u \in W^{k,2}_{loc}(\Omega) \quad \text{and} \quad \partial_l a_{ij} \in C^k(\Omega) \subset W^{k,\infty}_{loc}(\Omega),$$

it follows by Lemma 2.9 that the product $(\partial_l a_{ij})\partial_j u$ belongs to $W^{k,2}_{loc}(\Omega)$ whence $\partial_i (\partial_l a_{ij}\partial_j u) \in W^{k-1,2}_{loc}(\Omega)$. Hence, we obtain that $L(\partial_l u) \in W^{k-1,2}_{loc}(\Omega)$, which finishes the proof. ■

2.3 Operators with lower order terms

Here we extend the results of Theorems 2.1 and 2.8 to the operator with lower order terms. Consider in a domain $\Omega \subset \mathbb{R}^n$ the operator

$$Lu = \sum_{i,j=1}^{n} \partial_i (a_{ij}\partial_j u) + \sum_{i=1}^{n} b_i \partial_i u + cu, \tag{2.21}$$

where the coefficients a_{ij}, b_j, c are measurable functions in Ω. As before, for any $u \in W^{1,2}_{loc}(\Omega)$ the expression Lu is understood in the weak (distributional) sense. That is, the terms $a_{ij}\partial_j u$, $b_j \partial_j u$ and cu are elements of $L^2_{loc}(\Omega)$, while the terms $\partial_i (a_{ij}\partial_j u)$ are elements of $\mathcal{D}'(\Omega)$.

Theorem 2.10 *Let L be the operator (2.21). Assume that (a_{ij}) is uniformly elliptic and that the coefficients b_j and c are bounded in Ω. Let $u \in W^{1,2}_{loc}(\Omega)$.*

(a) Assume that a_{ij} are locally Lipschitz. If $Lu \in L^2_{loc}(\Omega)$ then $u \in W^{2,2}_{loc}(\Omega)$.

(b) Let k be a non-negative integer. If $a_{ij} \in C^{k+1}(\Omega)$, $b_j, c \in C^k(\Omega)$ and $Lu \in W^{k,2}_{loc}(\Omega)$ then $u \in W^{k+2,2}_{loc}(\Omega)$.

Proof. We use the operator L_0 defined by

$$L_0 u := \sum_{i,j=1}^{n} \partial_i (a_{ij}\partial_j u) = Lu - \left(\sum_{i=1}^{n} b_i \partial_i u + cu\right).$$

(a) If $u \in W^{1,2}_{loc}(\Omega)$ then

$$\sum_{i=1}^{n} b_i \partial_i u + cu \in L^2_{loc}(\Omega).$$

If $Lu \in L^2_{loc}(\Omega)$ then also $L_0 u \in L^2_{loc}(\Omega)$, and we conclude by Theorem 2.1 that $u \in W^{2,2}_{loc}(\Omega)$.

(b) Induction in k. The inductive basis $k = 0$ is covered by part (a). Inductive step from $k-1$ to k. By the inductive hypothesis we already know that $u \in W^{k+1,2}_{loc}(\Omega)$

and, hence, $\partial_j u \in W^{k,2}_{loc}(\Omega)$. Since b_j and c are in $C^k(\Omega) \subset W^{k,\infty}_{loc}(\Omega)$, it follows from Lemma 2.9 that

$$\sum_{i=1}^{n} b_i \partial_i u + cu \in W^{k,2}_{loc}(\Omega).$$

Since also $Lu \in W^{k,2}_{loc}(\Omega)$, we obtain $L_0 u \in W^{k,2}_{loc}(\Omega)$ and conclude by Theorem 2.8 that $u \in W^{k+2,2}_{loc}(\Omega)$. ■

27.11.23 Lecture 15

2.4 Sobolev embedding theorem and classical derivatives

Our next purpose is to conclude (under appropriate assumptions) that a solution u of $Lu = f$ is in some class C^m and, hence, is a classical solution. For that we use so called *embedding* theorems.

Recall Theorem 1.8 that says the following: if $1 \le p < n$ then, for any $u \in W^{1,p}(\mathbb{R}^n)$,

$$\left(\int_{\mathbb{R}^n} |u|^{\frac{pn}{n-p}} dx \right)^{\frac{n-p}{n}} \le C \int_{\mathbb{R}^n} |\nabla u|^p dx, \tag{2.22}$$

with some constant $C = C(n,p)$. Theorem 1.8 was stated for $u \in W^{1,p}_0(\mathbb{R}^n)$, but one can prove that $W^{1,p}_0(\mathbb{R}^n) = W^{1,p}(\mathbb{R}^n)$ (see Exercise 44) so that the subscript '0' can be omitted here.

Set

$$q := \frac{pn}{n-p} \tag{2.23}$$

and rewrite (2.22) as follows:

$$\|u\|_{L^q} \le C \|\nabla u\|_{L^p}, \tag{2.24}$$

which implies that $u \in L^q(\mathbb{R}^n)$. That is, if u and $\partial_i u$ are in L^p then, in fact, $u \in L^q$ with $q > p$.

We can write this as an inclusion

$$W^{1,p}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n),$$

or as an *embedding* (=injective linear mapping)

$$\boxed{W^{1,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)}, \tag{2.25}$$

given by $u \mapsto u$. Moreover, this embedding is a bounded operator because by (2.24)

$$\|u\|_{L^q} \le C \|u\|_{W^{1,p}}.$$

Corollary 2.11 (Sobolev embedding theorem I) *Let $k \in \mathbb{N}$ and $p, q \in [1, \infty)$.*
(a) If
$$\frac{1}{q} = \frac{1}{p} - \frac{k}{n} \tag{2.26}$$
then there is an embedding
$$W^{k,p}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n). \tag{2.27}$$
(b) If for some non-negative integer $m < k$
$$\frac{1}{q} = \frac{1}{p} - \frac{k-m}{n}, \tag{2.28}$$
then there is an embedding
$$W^{k,p}(\mathbb{R}^n) \hookrightarrow W^{m,q}(\mathbb{R}^n). \tag{2.29}$$

Note that when k gets larger, then q also gets larger, so that a higher degree of differentiability of u yields a higher degree of integrability of u.

Proof. (a) Induction basis for $k = 1$. In this case (2.26) becomes
$$\frac{1}{q} = \frac{1}{p} - \frac{1}{n} = \frac{n-p}{pn},$$
which implies that $p < n$ and $q = \frac{pn}{n-p}$, that is, (2.23). Hence, (2.27) is equivalent in this case to (2.25).

Induction step from k to $k+1$. Assume that $u \in W^{k+1,p}(\mathbb{R}^n)$, and prove that $u \in L^{q'}(\mathbb{R}^n)$ where
$$\frac{1}{q'} = \frac{1}{p} - \frac{k+1}{n}.$$
Indeed, both functions u and $\partial_i u$ belong to $W^{k,p}(\mathbb{R}^n)$, which implies by the induction hypothesis that
$$u \text{ and } \partial_i u \in L^q(\mathbb{R}^n), \tag{2.30}$$
where q is as in (2.26), that is,
$$\frac{1}{q} = \frac{1}{p} - \frac{k}{n}.$$
It follows from (2.30) that $u \in W^{1,q}(\mathbb{R}^n)$. Comparing q and q' we see that
$$\frac{1}{q'} = \frac{1}{q} - \frac{1}{n},$$
which implies that $q < n$ and $q' = \frac{qn}{n-q}$. By (2.25) we conclude $u \in L^{q'}(\mathbb{R}^n)$, which was to be proved.

(b) Let $u \in W^{k,p}(\mathbb{R}^n)$. In order to prove that $u \in W^{m,q}(\mathbb{R}^n)$, we need to verify that $D^\alpha u \in L^q(\mathbb{R}^n)$ for any $|\alpha| \le m$. Indeed, we have
$$D^\alpha u \in W^{k-m,p}(\mathbb{R}^n),$$
and by (a) we obtain that $D^\alpha u \in L^q(\mathbb{R}^n)$, where q satisfies (2.28), which completes the proof. ■

Corollary 2.11(b) can be applied provided

$$m < k < m + \frac{n}{p},$$

where the second inequality follows from (2.28) and $\frac{1}{q} > 0$. In the limiting case $q = \infty$ the identity (2.28) becomes

$$k = m + \frac{n}{p}.$$

Although the above statements do not cover the case $q = \infty$, one still can expect that, for this or a larger value of k, the following is true: if $u \in W^{k,p}(\mathbb{R}^n)$ then u has essentially *bounded* partial derivatives up to the order m. This idea is rigorously implemented in the next theorem that provides even the existence of *continuous* derivatives up to the order m.

Theorem 2.12 (Sobolev embedding theorem II) *Let Ω be an open subset of $\mathbb{R}^n$. Fix $p \in (1, \infty)$ and let m, k be non-negative integers such that*

$$k > m + \frac{n}{p}. \tag{2.31}$$

(a) *If $u \in W^{k,p}_{loc}(\Omega)$ then $u \in C^m(\Omega)$.*

(b) *If $u \in W^{k,p}(\Omega)$ then, for any precompact open set U such that $\overline{U} \subset \Omega$,*

$$\|u\|_{C^m(\overline{U})} \leq C \|u\|_{W^{k,p}(\Omega)}, \tag{2.32}$$

where the constant C depends on n, k, m, p, U, Ω.

Remark. Note that u is an element of $L^p_{loc}(\Omega)$ and, hence, is a equivalence *class* of measurable functions. However, when we say that $u \in C^m(\Omega)$ and, in particular, $u \in C(\Omega)$, then we understand u as a function that is defined pointwise. A precise meaning of the claim of Theorem 2.12(a) is as follows: if $u \in W^{k,p}_{loc}(\Omega)$ and k satisfies (2.31) then u as an equivalence class has a representative, also denoted by u, that belongs to $C^m(\Omega)$.

The identification of $u \in W^{k,p}_{loc}(\Omega)$ with its C^m-representative allows to define an embedding

$$W^{k,p}_{loc}(\Omega) \hookrightarrow C^m(\Omega).$$

Since $C^m(\Omega) \subset W^{m,\infty}_{loc}(\Omega)$, it also follows that

$$W^{k,p}_{loc}(\Omega) \hookrightarrow W^{m,\infty}_{loc}(\Omega).$$

The estimate (2.32) means that the embedding of the normed (Banach) spaces

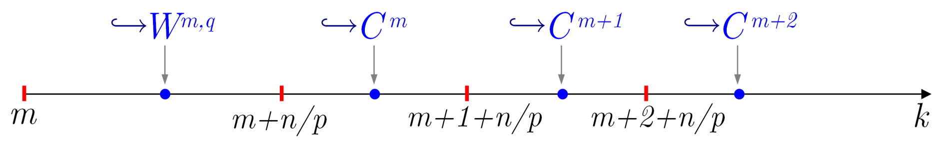
$$W^{k,p}(\Omega) \hookrightarrow C^m(\overline{U})$$

is a bounded operator. Recall that the norm in $C^m(\overline{U})$ is defined by

$$\|u\|_{C^m(U)} = \max_{|\alpha| \leq m} \max_{x \in \overline{U}} |D^\alpha u(x)|.$$

The next diagram combines Corollary 2.11 and Theorem 2.12.

Embedding of $W^{k,p}$ *depending on* k



Example. Let $n = 1$. Then the condition (2.31) becomes $k > m + \frac{1}{p}$ that is equivalent to $k \geq m + 1$. Hence,

$$u \in W^{k,p}_{loc} \Rightarrow u \in C^{k-1},$$

provided $k \geq 1$. In particular, any function from $W^{1,p}_{loc}$ is continuous.

For example, the continuous function $u(x) = |x|$ in $\mathbb{R}$ has the weak derivative $u' = \operatorname{sgn} x$ and, hence, belongs to $W^{1,p}_{loc}$. On the other hand, the function $u(x) = \mathbf{1}_{[0,\infty)}$ that has only one point of discontinuity at $x = 0$ has the distributional derivative $u' = \delta$ and, hence, is not in $W^{1,p}_{loc}$.

Example. For a general n and for $m = 0$, the condition (2.31) becomes

$$k > \frac{n}{p}. \tag{2.33}$$

That is, if (2.33) holds then

$$u \in W^{k,p}_{loc}(\mathbb{R}^n) \Rightarrow u \in C(\mathbb{R}^n).$$

Let us show that the condition (2.33) is sharp. For that, consider in $\mathbb{R}^n$ the function

$$u(x) = |x|^a$$

where a is a real number. This function is clearly C^∞ smooth outside the origin, but it is not continuous at 0 if $a < 0$. Let us verify that $u \in L^p_{loc}(\mathbb{R}^n)$ if and only if

$$a > -\frac{n}{p}. \tag{2.34}$$

Indeed, integrating in the polar coordinates (r, θ) we obtain, for any $R > 0$,

$$\int_{B_R} |u|^p \, dx = \int_0^R \int_{\mathbb{S}^{n-1}} r^{\alpha p} r^{n-1} dr d\theta = \omega_n \int_0^R r^{ap+n-1} dr = \omega_n R^{ap+n} < \infty$$

provided $ap + n > 0$, which is equivalent to (2.34). Similarly, any classical derivatives $D^\alpha u$ of the order $|\alpha| = k$ (which is defined outside 0) belongs to $L^p_{loc}(\mathbb{R}^n)$ provided

$$a > k - \frac{n}{p}, \tag{2.35}$$

because
$$|D^\alpha u| \leq \text{const}\, |x|^{a-k}.$$
In this case the classical derivative coincides with the weak derivative that, hence, belongs to $L^p_{loc}(\mathbb{R}^n)$. Hence, under the condition (2.35) we obtain $u \in W^{k,p}_{loc}(\mathbb{R}^n)$.

If $k < \frac{n}{p}$ then there exists a negative number a that satisfies (2.35). Then the function $u(x) = |x|^a$ belongs to $W^{k,p}_{loc}(\mathbb{R}^n)$ but is not continuous at 0. In the borderline case $k = \frac{n}{p}$ there is also an example of a function $u \in W^{k,p}_{loc}(\mathbb{R}^n)$ that is not continuous.

These examples show that the condition $k > \frac{n}{p}$, under which all functions from $W^{k,p}_{loc}$ are continuous, is sharp.

Combining Theorems 2.10 and 2.12 (case $p = 2$), we obtain the following.

Corollary 2.13 (Existence of classical derivatives of a weak solution) *Let L be the operator*
$$Lu = \sum_{i,j=1}^n \partial_i (a_{ij}\partial_j u) + \sum_{i=1}^n b_i \partial_i u + cu,$$
where (a_{ij}) is uniformly elliptic and b_j, c are bounded in Ω. Let $u \in W^{1,2}_{loc}(\Omega)$. Assume that
$$a_{ij} \in C^{k+1}(\Omega) \ \text{ and } \ b_i, c \in C^k(\Omega),$$
where k is a non-negative integer such that
$$k > m + \frac{n}{2} - 2,$$
where m is a non-negative integer. Then
$$Lu \in W^{k,2}_{loc}(\Omega) \Rightarrow u \in C^m(\Omega).$$
Consequently, if $a_{ij}, b_i, c \in C^\infty(\Omega)$ then
$$Lu \in C^\infty(\Omega) \Rightarrow u \in C^\infty(\Omega).$$

Proof. Indeed, by Theorem 2.10 we have $u \in W^{k+2,2}_{loc}(\Omega)$. Since
$$k + 2 > m + \frac{n}{2},$$
Theorem 2.12 with $p = 2$ yields $u \in C^m(\Omega)$.

The second statement follows from the first one as $C^\infty(\Omega) \subset W^{k,2}_{loc}(\Omega)$ for any k and, hence, $u \in C^m(\Omega)$ for any m. ∎

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Proof of Theorem 2.12. We split the proof into a series of claims.

Claim 1. *For any $u \in \mathcal{D}(B_R)$ and*
$$k > \frac{n}{p}, \tag{2.36}$$

we have

$$|u(0)| \leq C\|u\|_{W^{k,p}(B_R)}, \tag{2.37}$$

where the constant C depends on n, k, p, R (which is a particular case of (2.32) for $m = 0$).

We use for the proof the polar coordinates (r, θ) centered at the origin $0 \in \mathbb{R}^n$, where $r \geq 0$ and $\theta \in \mathbb{S}^{n-1}$. The relations between the Cartesian and polar coordinates are given by the identities

$$x_j = r f_j(\theta),$$

where f_j are the smooth functions of $\theta \in \mathbb{S}^{n-1}$ such that $|f_j| \leq 1$ (for example, in the case $n = 2$ we have $f_1(\theta) = \cos\theta$ and $f_2(\theta) = \sin\theta$).

Considering u as a function of r and θ (away from the origin), we obtain by the chain rule

$$\partial_r u = \sum_j \partial_{x_j} u \frac{\partial x_j}{\partial r} = \sum_j f_j(\theta)\partial_j u. \tag{2.38}$$

It follows by induction in $k \in \mathbb{N}$ that

$$\partial_r^k u = \sum_{j_1,\ldots,j_k} f_{j_1}(\theta) \ldots . f_{j_k}(\theta)\, \partial_{j_1 \ldots j_k} u,$$

whence

$$\left|\partial_r^k u\right| \leq c' \sum_{|\alpha|=k} |D^\alpha u|,$$

where $c' = c'(n, k)$ (note that any derivative $D^\alpha = \frac{\partial^k}{\partial x_1^{\alpha_1} \ldots . \partial x_n^{\alpha_n}}$ can be represented in many ways as $\partial_{j_1 \ldots j_k}$ but the number of such representations is bounded by a constant depending on n and k only).

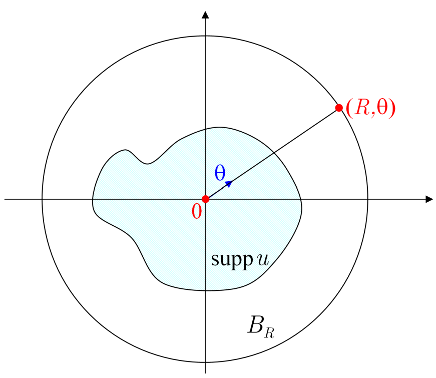
By the Hölder inequality, we obtain

$$\left|\partial_r^k u\right|^p \leq c \sum_{|\alpha|=k} |D^\alpha u|^p, \tag{2.39}$$

where $c = c(n, k, p)$. In particular, we see that the function $\partial_r^k u$ is bounded in $\mathbb{R}^n \setminus \{0\}$ (note that this function is not defined at 0), which allows to integrate $\partial_r^k u$ in r over the interval $[0, R]$.

For any $\theta \in \mathbb{S}^{n-1}$ we have $u(R, \theta) = 0$ whence by the fundamental theorem of calculus

$$u(0) = u(0) - u(R,\theta) = -\int_0^R \partial_r u(r, \theta)\, dr.$$



Integration by parts yields

$$\begin{aligned} u(0) &= -\left[\partial_r u(r,\theta)\, r\right]_0^R + \int_0^R r\partial_r^2 u(r,\theta)\, dr = \int_0^R r\partial_r^2 u(r,\theta)\, dr \\ &= \frac{1}{2}\int_0^R \partial_r^2 u(r,\theta)\, dr^2 = -\frac{1}{2}\int r^2 \partial_r^3 u(r,\theta)\, dr \\ &= -\frac{1}{6}\int \partial_r^3 u(r,\theta)\, dr^3 = \frac{1}{6}\int r^3 \partial_r^4 u(r,\theta)\, dr \\ &= \ldots \end{aligned}$$

Continuing by induction, we arrive at

$$u(0) = \frac{(-1)^k}{(k-1)!}\int_0^R r^{k-1}\partial_r^k u(r,\theta)\, dr.$$

Integrating this identity in θ over $\mathbb{S}^{n-1}$ and using

$$r^{n-1} dr d\theta = dx,$$

where dx denotes the Lebesgue measure, we obtain

$$\omega_n u(0) = \frac{(-1)^k}{(k-1)!}\int_{\mathbb{S}^{n-1}}\int_0^R r^{k-1}\partial_r^k u(r,\theta)\, dr d\theta = \frac{(-1)^k}{(k-1)!}\int_{B_R} r^{k-n}\partial_r^k u\, dx,$$

where ω_n is the surface measure of $\mathbb{S}^{n-1}$. The Hölder inequality yields then

$$|u(0)| \le C\left(\int_{B_R} r^{(k-n)q} dx\right)^{1/q}\left(\int_{B_R} \left|\partial_r^k u\right|^p dx\right)^{1/p}, \tag{2.40}$$

where $q = \frac{p}{p-1}$ is the Hölder conjugate of p.

We want the first integral in (2.40) to be finite, that is, the function r^{k-n} belong to $L^q_{loc}(\mathbb{R}^n)$. As we have seen above, the latter is the case provided

$$k - n > -\frac{n}{q}$$

which is equivalent to

$$k > n\left(1 - \frac{1}{q}\right) = \frac{n}{p},$$

that is, to (2.36). Hence, the first integral in (2.40) is just a constant depending on n, k, p, R. By (2.39), we have

$$\left(\int_{B_R} \left|\partial_r^k u\right|^p dx\right)^{1/p} \le \left(c\int_{B_R} \sum_{|\alpha|=k} |D^\alpha u|^p dx\right)^{1/p} \le c^{1/p}\|u\|_{W^{k,p}(B_R)}.$$

Substituting this inequality into (2.40) we obtain (2.37).

For the next Claims 2-4, Ω is a bounded open set in $\mathbb{R}^n$.

Claim 2. *For any $u \in \mathcal{D}(\Omega)$ and $k > n/p$, we have*

$$\sup_{\Omega} |u| \leq C\|u\|_{W^{k,p}(\Omega)} \tag{2.41}$$

where the constant C depends on n, k, p and $\operatorname{diam}\Omega$.

Indeed, let x be a point of maximum of $|u|$ and $R = \operatorname{diam}\Omega$. Since $u \in \mathcal{D}(B_R(x))$, applying Claim 1 in the ball $B_R(x)$ and using x as the origin, we obtain

$$|u(x)| \leq C\|u\|_{W^{k,p}(B_R(x))},$$

whence (2.41) follows.

Claim 3. *Assume that $u \in W_c^{k,p}(\Omega)$, where $k > n/p$. Then $u \in C(\Omega)$ and the estimate (2.41) holds.*

Let us extend u to all $\mathbb{R}^n$ by setting $u = 0$ outside Ω. Since

$$W_c^{k,p}(\Omega) \subset W_0^{k,p}(\Omega) \subset W_0^{k,p}(\mathbb{R}^n),$$

(see Exercise 42), we have $u \in W^{k,p}(\mathbb{R}^n)$.

Let φ be a mollifier, that is, $\varphi \in \mathcal{D}(B_1(0))$, $\varphi \geq 0$ and $\int_{\mathbb{R}^n} \varphi dx = 1$. For any $j \in \mathbb{N}$ consider the function

$$\varphi_j(x) = j^n \varphi(jx)$$

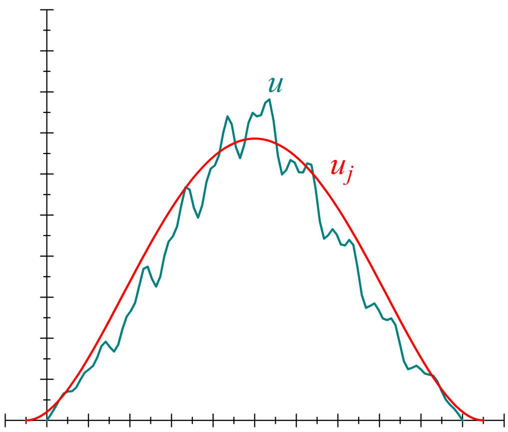
that is also a mollifier with $\operatorname{supp}\varphi_j \subset B_{1/j}(0)$. Set

$$u_j := u * \varphi_j = \int_{\mathbb{R}^n} u(\cdot - y)\varphi_j(y)dy.$$

Then $u_j \in C^\infty(\mathbb{R}^n)$ (see Exercise 39) and

$$u_j \overset{W^{k,p}}{\longrightarrow} u \quad \text{as} \quad j \to \infty \tag{2.42}$$

(see Exercises 4, 40, 41).



Since $\operatorname{supp}\varphi_j \subset B_{1/j}$, the support of u_j lies in the closed $(1/j)$-neighborhood of $\operatorname{supp} u$ (Exercise 39). Hence, if j is large enough, then $\operatorname{supp} u_j \subset \Omega$, that is, $u_j \in \mathcal{D}(\Omega)$. Applying (2.41) to the difference $u_i - u_j$, we obtain

$$\sup_{\Omega} |u_i - u_j| \leq C\|u_i - u_j\|_{W^{k,p}(\Omega)}. \tag{2.43}$$

By (2.42) we have

$$\|u_i - u_j\|_{W^{k,p}(\Omega)} \to 0,$$

which together with (2.43) implies that the sequence $\{u_j\}$ is Cauchy with respect to sup-norm in Ω. Hence, $\{u_j\}$ converges uniformly to a function $\widetilde{u} \in C(\Omega)$:

$$u_j \rightrightarrows \widetilde{u} \text{ as } j \to \infty. \tag{2.44}$$

On the other hand, it follows from (2.42) that there is a subsequence of $\{u_j\}$ that converges to u a.e.. Comparing to (2.44) we conclude that $u = \widetilde{u}$ a.e., that is, the function u has a continuos version $\widetilde{u}$, that will be now denoted also by u.

Since each u_j satisfies (2.41), that is,

$$\sup_{\Omega} |u_j| \leq C\|u_j\|_{W^{k,p}(\Omega)}$$

passing to the limit as $j \to \infty$ and using that $\{u_j\}$ converges to u both in sup-norm and in $W^{k,p}$-norm, we obtain that u also satisfies (2.41).

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Claim 4. *Assume that $u \in W_c^{k,p}(\Omega)$, where $k > m + n/p$ and m is a positive integer. Then $u \in C^m(\Omega)$ and*

$$\|u\|_{C^m(\Omega)} \leq C\|u\|_{W^{k,p}(\Omega)}, \tag{2.45}$$

where the constant C depends on n, k, m, p, and $\operatorname{diam}\Omega$.

If α is a multiindex with $|\alpha| \leq m$ then $D^\alpha u \in W_c^{k-m,p}(\Omega)$. Since $k - m > n/p$, we conclude by Claim 3 that $D^\alpha u \in C(\Omega)$ and

$$\sup_{\Omega} |D^\alpha u| \leq C\|D^\alpha u\|_{W^{k-m,p}(\Omega)} \leq C\|u\|_{W^{k,p}(\Omega)}. \tag{2.46}$$

The fact that the weak derivatives $D^\alpha u$ are continuous for all $|\alpha| \leq m$ implies that they are actually classical derivatives. Let us prove this for the first derivative $\partial_i u$. We have as above in Claim 3 that $u_j = u * \varphi_j \in C^\infty(\Omega)$ and

$$u_j \rightrightarrows u \tag{2.47}$$

as $j \to \infty$. Applying Claim 3 to $\partial_i u$, we obtain (using Exercise 41) that

$$\partial_i u_j = (\partial_i u) * \varphi_j \rightrightarrows \partial_i u. \tag{2.48}$$

Comparing (2.47) and (2.48) we conclude that $\partial_i u$ is the classical derivative of u.

By induction, we obtain that also $D^\alpha u$ with $|\alpha| \leq m$ is the classical derivative of u. Hence, $u \in C^m(\Omega)$, and (2.45) follows from (2.46).

Finally, let us prove the statements of Theorem 2.12.

(a) Assume $u \in W^{k,p}_{loc}(\Omega)$ where $k > m + n/p$. Fix a precompact open set U, such that $\overline{U} \subset \Omega$, and choose another precompact open set Ω' such that $\overline{U} \subset \Omega'$ and $\overline{\Omega'} \subset \Omega$ (we need Ω' because Ω may be unbounded). Choose also a function $\eta \in \mathcal{D}(\Omega')$ such that $\eta \equiv 1$ on U. Then

$$\eta u \in W^{k,p}_c(\Omega')$$

and, by Claim 4, we conclude that

$$\eta u \in C^m(\Omega').$$

Consequently, $u \in C^m(U)$ because $u = \eta u$ in U. Since U may be chosen arbitrarily, we conclude that $u \in C^m(\Omega)$.

(b) If $u \in W^{k,p}(\Omega)$ then applying (2.45) to the function $\eta u \in W^{k,p}_c(\Omega')$, we obtain

$$\|u\|_{C^m(U)} \le \|\eta u\|_{C^m(\Omega')} \le C\|\eta u\|_{W^{k,p}(\Omega')} \le C'\|u\|_{W^{k,p}(\Omega)},$$

which finishes the proof. ■

Remark. The statement of Theorem 2.12 is true also for $p = 1$ and $p = \infty$.

In the case $p = 1$ the condition (2.31) becomes $k > m + n$, but in this case it can be relaxed to

$$k \ge m + n.$$

Indeed, in the above proof the assumption $p > 1$ was used only in the Hölder inequality (2.40). If $p = 1$ then we replace (2.40) by a trivial inequality

$$|u(0)| \le C\left|\int_{B_R} r^{k-n}\partial_r^k u\,dx\right| \le C\sup_{B_R} r^{k-n}\int_{B_R}\left|\partial_r^k u\right|dx,$$

where the sup-term is finite provided $k \ge n$. Hence, Claim 1 works if $k \ge n$, and the rest of the proof works if $k \ge m + n$.

Hence, if $k \ge n$ then setting $m = k - n$, we obtain the embedding

$$W^{k,1}_{loc}(\Omega) \hookrightarrow C^{k-n}(\Omega).$$

In the case $p = \infty$ the condition (2.31) becomes $k > m$ that is,

$$k \ge m + 1.$$

For the proof in this case, observe that if $u \in W^{k,\infty}_{loc}(\Omega)$ then $u \in W^{k,p}_{loc}(\Omega)$ for any $p < \infty$. Choose p so large that $k > m + n/p$. Applying Theorem 2.12 with this p we obtain that $u \in C^m(\Omega)$.

Hence, if $k \ge 1$ then setting $m = k - 1$, we obtain the following embedding:

$$W^{k,\infty}_{loc}(\Omega) \hookrightarrow C^{k-1}(\Omega).$$

Alternatively, this can be proved using that $W^{1,\infty}_{loc}(\Omega) = Lip_{loc}(\Omega)$ (see Exercise 43).

2.5 Non-divergence form operator

Recall that for a divergence form uniformly elliptic operator

$$Lu = \sum_{i,j=1}^{n} \partial_i (a_{ij} \partial_j u) + \sum_{j=1}^{n} b_j \partial_j u$$

in a domain $\Omega \subset \mathbb{R}^n$, the equation $Lu = f$ is understood in the weak sense if ∂_j acts in $W^{1,2}_{loc}$ (while ∂_i acts in L^2_{loc}), and $Lu = f$ is understood in the strong sense if both ∂_j and ∂_i act in $W^{1,2}_{loc}$. In particular, in the case of a weak solution u must be in $W^{1,2}_{loc}$ while in the case of a strong solution u must be in $W^{2,2}_{loc}$.

Consider now a *non-divergence* form elliptic operator

$$Lu = \sum_{i,j=1}^{n} a_{ij} \partial_{ij} u + \sum_{j=1}^{n} b_j \partial_j u \tag{2.49}$$

in a domain $\Omega \subset \mathbb{R}^n$. In this case the notion of a weak solution is not defined, while the notion of a strong solution makes sense as follows.

Definition. We say that the equation $Lu = f$ is satisfied in Ω in the *strong* sense if $u \in W^{2,2}_{loc}(\Omega)$ (so that ∂_{ij} and ∂_j act in $W^{2,2}_{loc}$) and $Lu(x) = f(x)$ holds for almost all $x \in \Omega$.

We say that the equation $Lu = f$ is satisfied in Ω in the *classical* sense if $u \in C^2(\Omega)$ and $Lu(x) = f(x)$ holds for all $x \in \Omega$.

Example. Consider in $\mathbb{R}$ the function $u(x) = |x|$. Obviously, we have $u''(x) = 0$ for all $x \neq 0$, in particular, for almost all $x \in \mathbb{R}$. However, this function is not a strong solution of $u'' = 0$ because $u \notin W^{2,2}_{loc}(\Omega)$. Indeed, for distributional derivatives we have $u' = \operatorname{sgn} x \in L^2_{loc}$ and $u'' = 2\delta \notin L^2_{loc}$.

In fact, every strong solution of $\Delta u = 0$ in $\mathbb{R}^n$ is also a weak solution, and we obtain by Corollary 2.13 that $u \in C^\infty(\mathbb{R}^n)$.

Consider the Dirichlet problem

$$\begin{cases} Lu = f & \text{in } \Omega, \\ u \in W^{1,2}_0(\Omega), \end{cases} \tag{2.50}$$

where L is the operator (2.49) and the equation $Lu = f$ is understood in the strong (or classical) sense.

Theorem 2.14 *Let L be the operator (2.49) in a bounded domain $\Omega \subset \mathbb{R}^n$. Assume that (a_{ij}) is uniformly elliptic in Ω, a_{ij} are Lipschitz in Ω, b_j are bounded and measurable. Then, for any $f \in L^2(\Omega)$, the Dirichlet problem (2.50) has a unique strong solution.*

If in addition all the functions a_{ij}, b_j, f belong to $C^\infty(\Omega)$, then the solution u of (2.50) belongs to $C^\infty(\Omega)$, and the equation $Lu = f$ is satisfied in the classical sense.

Proof. By Corollary 2.4 we have $a_{ij} \in W^{1,2}_{loc}$. If $u \in W^{2,2}_{loc}(\Omega)$ then $\partial_j u \in W^{1,2}_{loc}$ and, by the product rule,

$$\partial_i (a_{ij} \partial_j u) = (\partial_i a_{ij}) \partial_j u + a_{ij} \partial_{ij} u.$$

Therefore, for $u \in W^{2,2}_{loc}(\Omega)$, we have

$$\begin{aligned} Lu &= \sum_{i,j=1}^{n} a_{ij}\partial_{ij}u + \sum_{j=1}^{n} b_j\partial_j u \\ &= \sum_{i,j=1}^{n} \partial_i\left(a_{ij}\partial_j u\right) - \sum_{i,j=1}^{n}\left(\partial_i a_{ij}\right)\partial_j u + \sum_{j=1}^{n} b_j\partial_j u \\ &= \widetilde{L}u, \end{aligned}$$

where $\widetilde{L}$ is a divergence form operator defined by

$$\widetilde{L}u = \sum_{i,j=1}^{n} \partial_i\left(a_{ij}\partial_j u\right) + \sum_{j=1}^{n} \widetilde{b}_j\partial_j u$$

with

$$\widetilde{b}_j = b_j - \sum_{i=1}^{n} \partial_i a_{ij}.$$

Since functions a_{ij} are Lipschitz in Ω, the weak derivatives $\partial_i a_{ij}$ are bounded in Ω (see Corollary 2.4(a) and Exercise 38). Since also b_j are bounded in Ω, we obtain that the coefficients $\widetilde{b}_j$ are bounded in Ω.

The above computation shows that $Lu = \widetilde{L}u$ for $u \in W^{2,2}_{loc}(\Omega)$. In particular, the strong Dirichlet problem (2.50) is equivalent to the strong Dirichlet problem

$$\begin{cases} \widetilde{L}u = f & \text{in } \Omega, \\ u \in W^{1,2}_0(\Omega), \end{cases} \tag{2.51}$$

where u is sought in the class $W^{2,2}_{loc}(\Omega)$. However, unlike the operator L, the divergence form operator $\widetilde{L}$ can be regarded also in a weak sense, that is, on functions from $W^{1,2}_{loc}(\Omega)$.

Hence, consider (2.51) first in the weak sense. By Theorem 1.13, the weak Dirichlet problem (2.51) has a solution u. Since a_{ij} are locally Lipschitz, we obtain by Theorem 2.10 that $u \in W^{2,2}_{loc}(\Omega)$. Hence, the same function u is a strong solution of the Dirichlet problem (2.50), which proves the existence of solution of (2.50).

Since any strong solution u of (2.50) is a strong and, hence, a weak solution of (2.51), we obtain by Theorem 1.3 the uniqueness of u.

If $a_{ij}, b_j, f \in C^\infty(\Omega)$ then by Corollary 2.13 the solution u of (2.51) belongs to C^∞ and, hence, $Lu = f$ is satisfied in the classical sense. ■

Remark. Theorem 1.15 yields the following estimate of the solution u of (2.51):

$$\|u\|_{L^\infty} \leq C\,|\Omega|^{\frac{2}{n}-\frac{1}{q}}\,\|f\|_{L^q}$$

with any $q \in [2,\infty] \cap (n/2,\infty]$, provided

$$|\Omega| < \delta,$$

where $\delta = c_n\lambda^{-n}\widetilde{b}^{-n}$ depends on the ellipticity constant λ of (a_{ij}) and on the constant

$$\widetilde{b} := \sup_{\Omega}\sum_{j=1}^{n}|\widetilde{b}_j| \leq \sup_{\Omega}\left(\sum_{j=1}^{n}|b_j| + \sum_{i,j=1}^{n}|\partial_i a_{ij}|\right) \leq b + n^2\ell,$$

where $b = \sup_\Omega \sum_{j=1}^n |b_j|$ and ℓ is a common Lipschitz constant of all a_{ij}. Hence, the same estimate holds for the solution u of (2.50). Note that $\widetilde{b}$ may be non-zero even if $b = 0$ because of $\ell \neq 0$.

Example. Let us give an example to show that the uniqueness statement of Theorem 2.14 fails if the coefficients a_{ij} are not Lipschitz. Consider the operator $L = \sum_{i,j=1}^n a_{ij}\partial_{ij}$ in $\mathbb{R}^n$ $(n > 2)$ with the coefficients

$$a_{ij}(x) = \begin{cases} \delta_{ij} + c\frac{x_i x_j}{|x|^2}, & x \neq 0, \\ \delta_{ij}, & x = 0, \end{cases}$$

where c is a positive constant. It is easy to verify that L is uniformly elliptic. Consider the following Dirichlet problem in a ball B_R:

$$\begin{cases} Lu = 0 \ \text{ in } B_R \\ u \in W_0^{1,2}(B_R) \end{cases} \tag{2.52}$$

where L is understood in the strong sense, that is, $u \in W_{loc}^{2,2}(B_R)$. If the coefficients a_{ij} were Lipschitz as in the statement of Theorem 2.14 then this problem would have a unique strong solution $u = 0$.

However, the coefficients a_{ij} are not Lipschitz near 0 (not even continuous), and the problem (2.52) can have a non-zero solution. Indeed, it is possible to prove that if

$$1 > s > 2 - \frac{n}{2}$$

and $c = \frac{n-2+s}{1-s}$ then the function

$$u(x) = |x|^s - R^s$$

belongs to $W^{2,2}(B_R) \cap W_0^{1,2}(B_R)$ and solves in B_R the equation $Lu = 0$ (see Exercise 51 for details). Hence, u is a non-zero strong solution of the Dirichlet problem (2.52) so that the uniqueness fails in this case.

Chapter 3

Hölder continuity for divergence form equations

07.12.23 **Lecture 18**

In this Chapter we will consider again a divergence form uniformly elliptic operator

$$Lu = \sum_{i,j=1}^{n} \partial_i \left(a_{ij} \partial_j u\right) \tag{3.1}$$

with measurable coefficients and will prove that any weak solution u of $Lu = 0$ is, in fact, a continuous function! Moreover, we will prove that weak solutions are *Hölder continuous.*

Definition. A function f on a set $S \subset \mathbb{R}^n$ is called *Hölder continuous* with the Hölder exponent $\alpha > 0$ if there is a constant C such that

$$|f(x) - f(y)| \le C\,|x - y|^{\alpha}$$

for all $x, y \in S$.

For example, f is Lipschitz if and only if f is Hölder continuous with $\alpha = 1$.

Definition. Let S be a subset of $\mathbb{R}^n$. We say that a function f on S is *locally Hölder continuous* in S with the Hölder exponent $\alpha > 0$ if, for any point $x \in S$, there exists $\varepsilon > 0$ such that f is Hölder continuous in $B_\varepsilon(x) \cap S$ with the exponent α.

It is easy to prove that if f is locally Hölder continuous in S then f is Hölder continuous on any compact subset of S with the same Hölder exponent (the proof is the same as that in the case of Lipschitz functions). In particular, if S is compact then any locally Hölder continuous function on S is Hölder continuous.

The set of all locally Hölder continuous functions on S with the Hölder exponent $\alpha \in (0,1)$ is denoted by $C^{\alpha}(S)$ (or sometimes by $C^{0,\alpha}(S)$).

Theorem 3.1 (Theorem of de Giorgi) *If $u \in W^{1,2}_{loc}(\Omega)$ is a weak solution of $Lu = 0$ in Ω then $u \in C^{\alpha}(\Omega)$ where $\alpha = \alpha(n, \lambda) > 0$ (where λ is the constant of ellipticity of L).*

In particular, weak solutions are always continuous functions. For comparison, let us observe that in order to obtain the continuity of a weak solution u by Corollary 2.13, we have to assume that $a_{ij} \in C^{k+1}$ with non-negative $k > \frac{n}{2} - 2$. Theorem 3.1 ensures the continuity of u *without* any assumption about a_{ij} except for uniform ellipticity and measurability.

Theorem 3.1 was proved by Ennio de Giorgi in 1957, which opened a new era in the theory of elliptic PDEs. A year later John Nash proved the Hölder continuity for solutions of parabolic equation $\partial_t u = Lu$, which contained the theorem of de Giorgi as a particular case for time-independent solutions.

We will prove Theorem 3.1 after a long preparatory work.

3.1 Mean value inequality for subsolutions

Let L be the operator (3.1) defined in a domain Ω of $\mathbb{R}^n$. We always assume that L is uniformly elliptic with the ellipticity constant λ and that the coefficients a_{ij} are measurable.

Definition. A function $u \in W^{1,2}_{loc}(\Omega)$ is called a *subsolution* of L in Ω (or that of the equation $Lu = 0$ in Ω) if it satisfies the inequality $Lu \geq 0$ weakly, that is, if

$$\int_\Omega \sum_{i,j=1}^n a_{ij} \partial_j u \, \partial_i \varphi \, dx \leq 0 \tag{3.2}$$

for any non-negative function $\varphi \in \mathcal{D}(\Omega)$. Similarly, u is called a *supersolution* if it satisfies $Lu \leq 0$ weakly.

If $u \in W^{1,2}(\Omega)$ is a subsolution in Ω then (3.2) is satisfied also for any non-negative function $\varphi \in W^{1,2}_0(\Omega)$ (Exercise 27).

For example, any subharmonic function is a subsolution of the Laplace operator Δ. Sometimes subsolutions of L are also referred to as L-subharmonic functions.

Theorem 3.2 (The mean-value inequality for subsolutions) *Let $B_R \subset \Omega$ and let $u \in W^{1,2}(B_R)$ be a subsolution of L in B_R. Then*

$$\operatorname*{esssup}_{B_{R/2}} u \leq \frac{C}{R^{n/2}} \left(\int_{B_R} u_+^2 dx \right)^{1/2}, \tag{3.3}$$

where $C = C(n, \lambda)$.

Since $|B_R| = \text{const}\, R^n$, the inequality (3.3) is equivalent to

$$\operatorname*{esssup}_{B_{R/2}} u \leq C \left(⨍_{B_R} u_+^2 dx \right)^{1/2}, \tag{3.4}$$

where the constants C in (3.3) and (3.4) may be different (but both depend only on n and λ). The value

$$\left(⨍_{B_R} u_+^2 dx \right)^{1/2}$$

is called the *quadratic mean* of u_+ in B_R. Hence, $\operatorname{esssup}_{B_{R/2}} u$ is bounded by the quadratic mean of u_+ in B_R, up to a constant factor C.

Corollary 3.3 *If $u \in W^{1,2}_{loc}(\Omega)$ solves $Lu = 0$ in Ω then $u \in L^{\infty}_{loc}(\Omega)$.*

Proof. Indeed, in any ball B_R such that $\overline{B}_R \subset \Omega$ we have $u \in W^{1,2}(B_R)$ and, by Theorem 3.2,

$$\operatorname*{esssup}_{B_{R/2}} u \le \frac{C}{R^{n/2}} \|u\|_{L^2(B_R)}.$$

Applying the same inequality to $-u$, we conclude that

$$\|u\|_{L^\infty\left(B_{B/2}\right)} \le \frac{C}{R^{n/2}} \|u\|_{L^2(B_R)} < \infty.$$

Hence, $u \in L^\infty\left(B_{R/2}\right)$, which implies $u \in L^\infty_{loc}(\Omega)$. ■

Recall that, for a harmonic function u in B_R, we have the mean value property

$$u(0) = \fint_{B_R} u dx.$$

Using the Cauchy-Schwarz inequality, we obtain

$$u(0) \le \fint_{B_R} u_+ dx \le \left(\fint_{B_R} u_+^2 dx\right)^{1/2}. \tag{3.5}$$

Fix a point $z \in B_{R/2}$. Applying (3.5) to the ball $B_{R/2}(z)$ instead of $B_R(0)$ and noticing that $B_{R/2}(z) \subset B_R(0)$, we obtain

$$u(z) \le \left(\fint_{B_{R/2}(z)} u_+^2 dx\right)^{1/2} \le \left(2^n \fint_{B_R} u_+^2 dx\right)^{1/2},$$

which proves (3.4) for harmonic functions. Using the maximum principle, one can extend this inequality also to subharmonic functions.

The proof of (3.3) for a general operator L is much more complicated because we do not have the mean value property in general. The proof uses some ideas from the proof of Theorem 1.14. Recall that Theorem 1.14 says the following: if u solves the weak Dirichlet problem

$$\begin{cases} Lu = -f & \text{in } \Omega, \\ u \in W_0^{1,2}(\Omega), \end{cases}$$

then

$$\|u\|_{L^\infty} \le C |\Omega|^{2/n} \|f\|_{L^\infty},$$

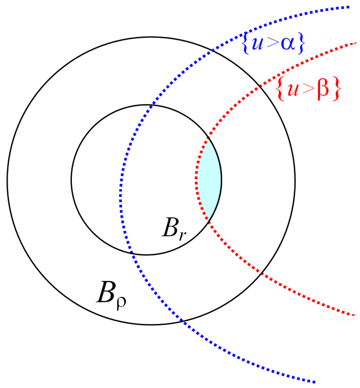
where $C = C(n, \lambda)$. In particular, $u \in L^\infty(\Omega)$ provided $f \in L^\infty(\Omega)$. The essential difference between Theorem 1.14 and the present setting is that in Theorem 1.14 $u \in W_0^{1,2}$ whereas now $u \in W^{1,2}$ or even $u \in W^{1,2}_{loc}$.

Proof. The proof consists of two parts.

Part 1. Fix two values $0 < \alpha < \beta$ as well as $0 < r < \rho < R$ and set

$$a = \int_{B_\rho} (u - \alpha)_+^2 \, dx \quad \text{and} \quad b = \int_{B_r} (u - \beta)_+^2 \, dx. \tag{3.6}$$

Clearly, $b \le a$.



The purpose of the first part of the proof to obtain a stronger inequality showing that b is essentially smaller than a. In the second part of the proof we will use an iteration procedure similarly to the proof of Theorem 1.14.

Consider the function

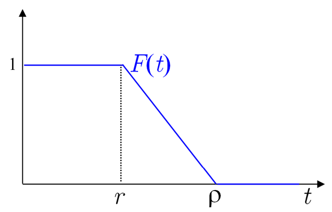
$$v = (u - \beta)_+$$

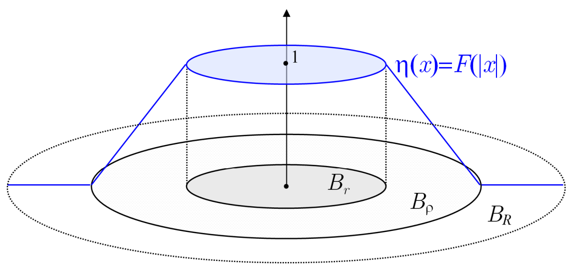
that belongs to $W^{1,2}(B_R)$ (see Lemma 1.4 and Exercise 16). Unlike the proof of Theorem 1.14, we cannot claim that $v \in W_0^{1,2}$ and, hence, cannot use v as a test function φ in (3.2). To overcome this difficulty, we use instead the function $\varphi = v\eta^2$, where

$$\eta(x) = \begin{cases} 1, & |x| \le r, \\ \frac{\rho - |x|}{\rho - r}, & r < |x| < \rho, \\ 0, & |x| \ge \rho. \end{cases}$$

In other words, $\eta(x) = F(|x|)$ where

$$F(t) = \begin{cases} 1, & t \le r, \\ \frac{\rho - t}{\rho - r}, & r < t < \rho, \\ 0, & t \ge \rho. \end{cases}$$





Clearly, $F(t)$ is a Lipschitz function with the Lipschitz constant $\frac{1}{\rho - r}$, and $|x|$ is a Lipschitz function with the Lipschitz constant 1. Hence, the composition $\eta = F \circ \eta$ is a Lipschitz function with the Lipschitz constant $\frac{1}{\rho - r}$. Since η is bounded, it follows that η^2 is also a Lipschitz function.

Set $\varphi = v\eta^2$ and let us verify that $\varphi \in W^{1,2}(B_R)$. Indeed, we have clearly $v\eta^2 \in L^2(B_R)$ and, by the product rule,

$$\partial_i \left(v\eta^2\right) = (\partial_i v)\eta^2 + v(\partial_i \eta^2) = (\partial_i v)\eta^2 + 2v\eta\partial_i\eta, \tag{3.7}$$

where η^2 and $\eta\partial_i\eta$ are bounded while $\partial_i v$ and v belong to $L^2(B_R)$, whence also $\partial_i \left(v\eta^2\right) \in L^2(B_R)$.

Since $\operatorname{supp}\varphi \subset \operatorname{supp}\eta = \overline{B}_\rho$, we have $\varphi \in W_c^{1,2}(B_R)$ and, hence, $\varphi \in W_0^{1,2}(B_R)$. Since $\varphi \geq 0$, the function φ can be used as a test function in (3.2).

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Substituting $\varphi = v\eta^2$ unto (3.2), that is,

$$\int_\Omega \sum_{i,j=1}^n a_{ij}\partial_j u\, \partial_i \varphi\, dx \leq 0, \tag{3.8}$$

yields

$$\int_{B_R} \sum_{i,j=1}^n a_{ij}\partial_j u\, \partial_i (v\eta^2) dx \leq 0. \tag{3.9}$$

Since by (3.7)

$$\partial_i(v\eta^2) = (\partial_i v)\eta^2 + 2v\eta\partial_i\eta,$$

we obtain from (3.9) that

$$\int_{B_R} \sum_{i,j=1}^n a_{ij}\partial_j u\, \partial_i v\, \eta^2 dx \leq -2\int_{B_R} \sum_{i,j=1}^n a_{ij}\partial_j u\, v\eta\, \partial_i\eta\, dx. \tag{3.10}$$

Recall that $\partial_j u \partial_i v = \partial_j v \partial_i v$ because on the set $\{u \leq \beta\}$ we have $v = 0$ and, hence, $\partial_i v = 0$, while on the set $\{u > \beta\}$ we have $\partial_j u = \partial_j v$. Hence, the left hand side of (3.10) is equal to

$$\int_{B_R} \sum_{i,j=1}^n a_{ij}\partial_j v\, \partial_i v\, \eta^2 dx \geq \lambda^{-1} \int_{B_R} |\nabla v|^2 \eta^2 dx.$$

Since in the same way $\partial_j u\, v = \partial_j v\, v$, the right hand side of (3.10) is equal to

$$\begin{aligned} -2\int_{B_R} \sum_{i,j=1}^n a_{ij}\partial_j v\, \partial_i\eta\, v\eta\, dx &\leq 2\lambda \int_{B_R} |\nabla v|\, |\nabla\eta|\, v\eta\, dx \\ &\leq 2\lambda \left(\int_{B_R} |\nabla v|^2 \eta^2 dx\right)^{1/2} \left(\int_{B_R} |\nabla \eta|^2 v^2 dx\right)^{1/2}. \end{aligned}$$

Therefore, (3.10) implies

$$\lambda^{-1}\int_{B_R} |\nabla v|^2 \eta^2 dx \leq 2\lambda \left(\int_{B_R} |\nabla v|^2 \eta^2 dx\right)^{1/2} \left(\int_{B_R} |\nabla \eta|^2 v^2 dx\right)^{1/2},$$

whence

$$\left(\int_{B_R} |\nabla v|^2 \eta^2 dx\right)^{1/2} \le 2\lambda^2 \left(\int_{B_R} |\nabla \eta|^2 v^2 dx\right)^{1/2}$$

and

$$\int_{B_R} |\nabla v|^2 \eta^2 dx \le 4\lambda^4 \int_{B_R} |\nabla \eta|^2 v^2 dx. \tag{3.11}$$

This inequality is called a *Caccioppoli inequality*. It is obtain from (3.8) by using a test function $\varphi = v\eta^2$.

Next, we will use (3.11) in order to estimate the integral of $|\nabla(v\eta)|^2$. We have

$$\nabla(v\eta) = \eta\nabla v + v\nabla\eta$$

and

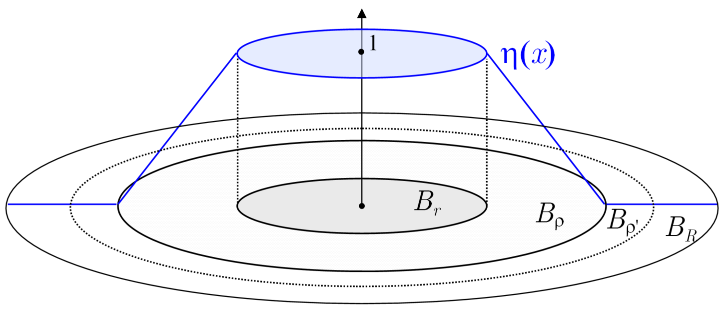
$$|\nabla(v\eta)|^2 \le (|\eta\nabla v| + |v\nabla\eta|)^2 \le 2|\nabla v|^2 \eta^2 + 2|\nabla\eta|^2 v^2.$$

Combining with (3.11) yields

$$\begin{aligned}\int_{B_R} |\nabla(v\eta)|^2 dx &\le 2\int_{B_R} |\nabla v|^2 \eta^2 dx + 2\int_{B_R} |\nabla\eta|^2 v^2 dx \\ &\le \left(8\lambda^4 + 2\right)\int_{B_R} |\nabla\eta|^2 v^2 dx.\end{aligned}$$

Setting $C = 8\lambda^4 + 2$ and observing that $|\nabla\eta| = 0$ outside B_ρ and $|\nabla\eta| \le \frac{1}{\rho - r}$ in B_ρ, we obtain that

$$\int_{B_R} |\nabla(v\eta)|^2 dx \le \frac{C}{(\rho - r)^2}\int_{B_\rho} v^2 dx. \tag{3.12}$$



Choose some ρ' that is a bit larger than ρ. Since $\operatorname{supp}\eta = \overline{B}_\rho \subset B_{\rho'}$, we have by the above argument that $v\eta \in W_0^{1,2}(B_{\rho'})$. Applying the Faber-Krahn inequality (1.75) to the function $v\eta$ in $B_{\rho'}$, we obtain

$$\int_{B_{\rho'}} |\nabla(v\eta)|^2 dx \ge c\,|U|^{-2/n}\int_{B_{\rho'}} (v\eta)^2 dx, \tag{3.13}$$

where $c = c(n) > 0$ and

$$U := \{x \in B_{\rho'} : v\eta(x) > 0\}.$$

Since $\eta = 0$ outside B_ρ and $\eta > 0$ in B_ρ, we see that

$$U = \{x \in B_\rho : v(x) > 0\} = \{x \in B_\rho : u(x) > \beta\}.$$

For the same reason the integration over $B_{\rho'}$ in (3.13) can be replaced by that over B_ρ, so that

$$\int_{B_\rho} |\nabla(v\eta)|^2\, dx \geq c\, |U|^{-2/n} \int_{B_\rho} (v\eta)^2 dx. \tag{3.14}$$

Combining with (3.12) and using that $\eta = 1$ on B_r, we obtain

$$|U|^{-2/n} \int_{B_\rho} (v\eta)^2 dx \leq \frac{C}{(\rho - r)^2} \int_{B_\rho} v^2 dx,$$

where we have absorbed c and C into a single constant C.

Since $\eta = 1$ on B_r, it follows that

$$\int_{B_r} v^2 dx \leq \frac{C}{(\rho - r)^2} |U|^{2/n} \int_{B_\rho} v^2 dx.$$

Finally, since $v = (u - \beta)_+ \leq (u - \alpha)_+$, we obtain, using the notations a and b from (3.6),

$$\begin{aligned} b = \int_{B_r} (u - \beta)_+^2\, dx &\leq \frac{C}{(\rho - r)^2} |U|^{2/n} \int_{B_\rho} (u - \alpha)_+^2\, dx \\ &= \frac{C}{(\rho - r)^2} |U|^{2/n}\, a. \end{aligned} \tag{3.15}$$

Let us estimate $|U|$ from above as follows. Since $u > \beta$ on U and, hence, $u - \alpha > \beta - \alpha$ on U, we obtain

$$a = \int_{B_\rho} (u - \alpha)_+^2\, dx \geq \int_U (u - \alpha)_+^2\, dx \geq \int_U (\beta - \alpha)^2\, dx = (\beta - \alpha)^2 |U|,$$

whence

$$|U| \leq \frac{a}{(\beta - \alpha)^2} \quad \text{and} \quad |U|^{2/n} \leq \frac{a^{2/n}}{(\beta - \alpha)^{4/n}}.$$

Substituting this into (3.15) yields

$$b \leq \frac{C}{(\rho - r)^2 (\beta - \alpha)^{4/n}} a^{1+2/n}. \tag{3.16}$$

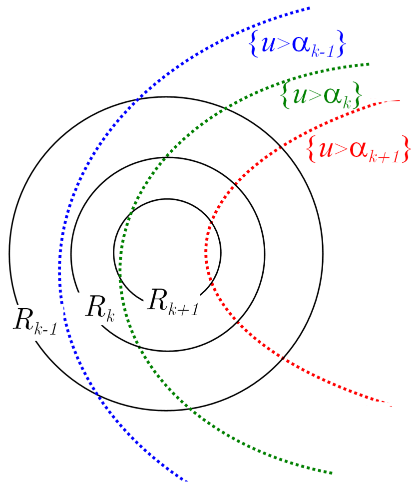
Part 2. Consider now a decreasing sequence $\{R_k\}_{k=0}^\infty$ of radii:

$$R_k = \frac{1}{2}\left(1 + \frac{1}{2^k}\right) R.$$

Clearly, $R_0 = R$ and $R_k \searrow \frac{R}{2}$ as $k \to \infty$. Also, fix some $\alpha > 0$ and consider an increasing sequence $\{\alpha_k\}_{k=0}^\infty$:

$$\alpha_k = \left(2 - \frac{1}{2^k}\right) \alpha.$$

Clearly, $\alpha_0 = \alpha$ and $\alpha_k \nearrow 2\alpha$ as $k \to \infty$.



Set

$$m_k = \int_{B_{R_k}} (u - \alpha_k)_+^2 \, dx.$$

Since the sequence $\{B_{R_k}\}$ of balls is shrinking and the sequence $\left\{(u - \alpha_k)_+\right\}$ of function is monotone decreasing, we see that the sequence $\{m_k\}$ is monotone decreasing.

Our aim is to choose α so that $m_k \to 0$ as $k \to \infty$. Since

$$\lim_{k\to\infty} m_k = \int_{B_{R/2}} (u - 2\alpha)_+^2 \, dx,$$

in this case we will obtain that

$$\int_{B_{R/2}} (u - 2\alpha)_+^2 \, dx = 0,$$

whence

$$\operatorname*{esssup}_{B_{R/2}} u \le 2\alpha, \tag{3.17}$$

which will lead us to the desired estimate (3.3).

Applying (3.16) to the pair m_{k-1}, m_k instead of a, b, we obtain

$$m_k \le \frac{C}{(R_{k-1} - R_k)^2 (\alpha_k - \alpha_{k-1})^{4/n}} m_{k-1}^{1+2/n}.$$

Since $R_{k-1} - R_k = \frac{1}{2}\left(2^{-k}R\right)$ and $\alpha_k - \alpha_{k-1} = 2^{-k}\alpha$, it follows that

$$m_k \le \frac{C4^{(1+2/n)k}}{R^2\alpha^{4/n}} m_{k-1}^{1+2/n}.$$

Denoting

$$p = 1 + \frac{2}{n} \quad \text{and} \quad A = \frac{C}{R^2\alpha^{4/n}}, \tag{3.18}$$

rewrite this inequality in the form

$$m_k \le 4^{pk} A \, m_{k-1}^p. \tag{3.19}$$

This inequality is similar to the inequality (1.87) obtained in the proof of Theorem 1.14:

$$m_k \le 4^k A\, m_{k-1}^p. \tag{3.20}$$

The difference between (3.20) and (3.19) is only that (3.19) uses 4^p instead of 4, which does not make any difference for the next argument. Indeed, iterating (3.20), we obtained in the proof of Theorem 1.14 the estimate (1.92), that is,

$$m_k \le \left[4^{\frac{p}{(p-1)^2}} A^{\frac{1}{p-1}} m_0\right]^{p^k} 4^{\frac{-(k+1)p+k}{(p-1)^2}} A^{-\frac{1}{p-1}}.$$

Hence, iterating in the same way (3.19) and replacing everywhere 4 by 4^p, we obtain that

$$m_k \le \left[4^{p\frac{p}{(p-1)^2}} A^{\frac{1}{p-1}} m_0\right]^{p^k} 4^{p\frac{-(k+1)p+k}{(p-1)^2}} A^{-\frac{1}{p-1}}. \tag{3.21}$$

We would like to derive from (3.21) that $m_k \to 0$ as $k \to \infty$. This will be the case whence the term in the square brackets is smaller than 1. Since

$$m_0 = \int_{B_R} (u-\alpha)_+^2\, dx \le \int_{B_R} u_+^2 dx,$$

it suffices to have the following inequality

$$4^{\frac{p^2}{(p-1)^2}} A^{\frac{1}{p-1}} \int_{B_R} u_+^2 dx < 1.$$

Substituting from (3.18) $A = \frac{C}{R^2\alpha^{4/n}}$ and $p = 1 + 2/n$, we replace this inequality by the equality

$$4^{\frac{p^2}{(p-1)^2}} \left(\frac{C}{R^2\alpha^{4/n}}\right)^{n/2} \int_{B_R} u_+^2 dx = \frac{1}{2},$$

which allows us to determine the desired value of α as follows:

$$\alpha^2 = \frac{C'}{R^n} \int_{B_R} u_+^2 dx.$$

Substituting into (3.17), we obtain

$$\operatorname*{esssup}_{B_{R/2}} u \le \frac{C''}{R^{n/2}} \left(\int_{B_R} u_+^2 dx\right)^{1/2},$$

which finishes the proof. ■

3.2 Weak Harnack inequality for positive supersolutions

Theorem 3.4 *Let $B_{3R} \subset \Omega$ and assume that $u \in W^{1,2}(B_{3R})$ is a non-negative supersolution of L in B_{3R}, that is, $Lu \le 0$ in B_{3R}. Fix some $a > 0$ and set*

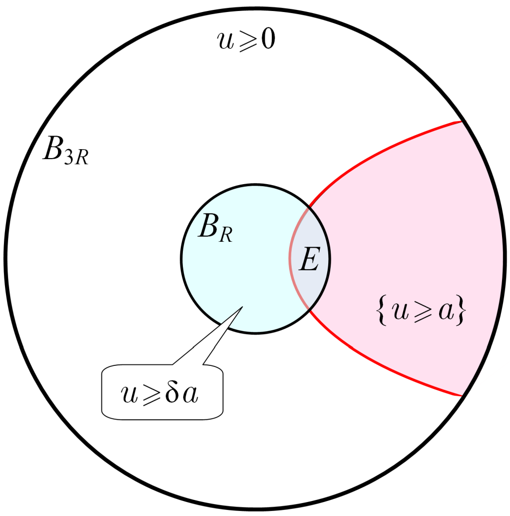
$$E = \{x \in B_R : u(x) \ge a\}.$$

For any $\varepsilon > 0$ there exists $\delta = \delta(n, \lambda, \varepsilon) > 0$ such that if

$$|E| \ge \varepsilon |B_R| \tag{3.22}$$

then

$$\operatorname*{essinf}_{B_R} u \ge \delta a. \tag{3.23}$$



Recall that any positive harmonic function u in a ball B_{3R} satisfies the Harnack inequality

$$\sup_{B_R} u \le C \inf_{B_R} u,$$

where $C = C(n)$. In particular, for any $0 < a \le \sup_{B_R} u$, we have

$$\inf_{B_R} u \ge \delta a,$$

where $\delta = C^{-1}$, which looks similarly to (3.23). However, for the Harnack inequality we do not need to know that the measure of the set $E = \{u \ge a\} \cap B_R$ is positive – in fact, it suffices to know that this set is non-empty as the latter will imply $a \le \sup_{B_R} u$. This is the reason why Theorem 3.4 is referred to as a *weak* Harnack inequality.

Before the proof, let us derive from Theorem 3.4 the following mean value inequality for supersolutions.

Corollary 3.5 (Mean-value inequality for supersolutions) *Let $B_{3R} \subset \Omega$ and assume that $u \in W^{1,2}(B_{3R})$ is a non-negative supersolution of L in B_{3R}. Then*

$$\operatorname*{essinf}_{B_R} u \geq c \left(\fint_{B_R} u^{-1} dx \right)^{-1}, \tag{3.24}$$

where $c = c(n, \lambda) > 0$.

The value

$$\left(\fint_{\Omega} u^p dx \right)^{1/p}$$

is called the p-mean of function u in Ω. If $p = 1$ then this is the arithmetic mean, if $p = 2$ – the quadratic mean. For example, the quadratic mean was used in the mean-value inequality for subharmonic functions. If $p = -1$ as in (3.24) then the p-mean is called the *harmonic mean*. Hence, for a non-negative supersolution, $\operatorname{essinf}_{B_R} u$ is bounded from below by the harmonic mean of u in B_R, up to a constant factor.

Proof. If $\fint_{B_R} u^{-1} dx = \infty$ then (3.24) holds trivially. Assume that this integral is finite. For any $a > 0$, we have

$$|\{u < a\} \cap B_R| = \left| \left\{ \frac{1}{u} > \frac{1}{a} \right\} \cap B_R \right| \leq \frac{1}{1/a} \int_{B_R} \frac{1}{u} dx = a\mu(B) \fint_{B_R} \frac{1}{u} dx.$$

Choosing

$$a = \frac{1}{2} \left(\fint_{B_R} \frac{1}{u} dx \right)^{-1},$$

we obtain

$$|\{u < a\} \cap B_R| \leq \frac{1}{2} \mu(B)$$

and, hence,

$$|\{u \geq a\} \cap B_R| \geq \frac{1}{2} \mu(B).$$

Applying Theorem 3.4 with $\varepsilon = 1/2$, we obtain

$$\operatorname*{essinf}_{B_R} u \geq \delta a = \frac{\delta}{2} \left(\fint_{B_R} \frac{1}{u} dx \right)^{-1},$$

which proves (3.24) with $c = \frac{1}{2}\delta(n, \lambda, \frac{1}{2})$. ∎

18.12.23 Lecture 20

Proof of Theorem 3.4. Assuming that u is a non-negative supersolution of L in B_{3R} and that, for the the set

$$E = \{x \in B_R : u(x) \geq a\},$$

we have

$$|E| \geq \varepsilon |B_R| \tag{3.25}$$

for some $\varepsilon > 0$, we need to prove that

$$\operatorname*{essinf}_{B_R} u \geq \delta a, \tag{3.26}$$

where $\delta = \delta(n, \lambda, \varepsilon) > 0$. Without loss of generality, we can assume that $\operatorname{essinf}_{B_{3R}} u > 0$. Indeed, if $\operatorname{essinf}_{B_{3R}} u = 0$ then consider the function $u + m$ for a positive m. Clearly, $L(u + m) \leq 0$ so that $u + m$ is also a supersolution, and $\operatorname{essinf}_{B_{3R}}(u + m) > 0$. Applying (3.26) to the function $u + m$ and observing that

$$u \geq a \quad \Leftrightarrow \quad u + m \geq a + m,$$

we obtain

$$\operatorname*{essinf}_{B_R} (u + m) \geq \delta (a + m).$$

Letting $m \longrightarrow 0$, we obtain (3.23). Hence, we can assume that $\operatorname{essinf}_{B_{3R}} u > 0$.

Besides, by replacing u by u/a, we can also assume that $a = 1$. In this case we have

$$E = \{u \geq 1\} \cap B_R$$

and, assuming (3.25), we need to prove that

$$\operatorname*{essinf}_{B_R} u \geq \delta,$$

where $\delta = \delta(n, \lambda, \varepsilon) > 0$.

The main idea of the proof is to consider the function

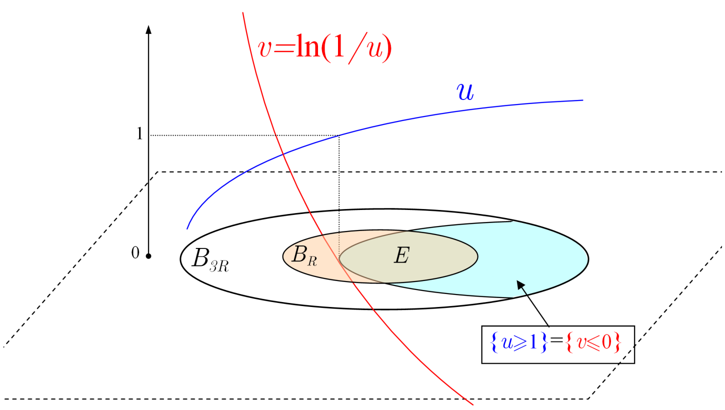
$$v = \ln \frac{1}{u}.$$

In terms of this function, we have

$$E = \{v \leq 0\} \cap B_R, \qquad |E| \geq \varepsilon |B_R|,$$

and we need to prove that

$$\operatorname*{esssup}_{B_R} v \leq C = C(n, \lambda, \varepsilon). \tag{3.27}$$



The plan of the proof is as follows. Firstly, we will verify that v is a subsolution of L, which will imply by Theorem 3.2 that

$$\operatorname*{esssup}_{B_R} v \leq \frac{C}{R^{n/2}} \left(\int_{B_{2R}} v_+^2 \, dx \right)^{1/2}.$$

Secondly, using a certain *Poincaré inequality* (similar to the Friedrichs inequality), we will deduce that

$$\int_{B_{2R}} v_+^2 \, dx \leq \ldots \int_{B_{2R}} |\nabla v|^2 \, dx.$$

Thirdly, using again specific properties of Lv, we will obtain an upper bound for

$$\int_{B_{2R}} |\nabla v|^2 \, dx,$$

which together with the previous estimates will yield (3.27).

Step 1. Now let us prove that v is a subsolution of L in B_{3R}. Let us first verify that $v \in W^{1,2}(B_{3R})$. On the set $\{u \leq 1\}$ function v is non-negative. Since u is separated from 0, we see that in this case

$$0 \leq v < \text{const}.$$

On the set $\{u > 1\}$ function v is negative and

$$|v| = -\ln \frac{1}{u} = \ln u \leq u.$$

Hence, in the both cases

$$|v| \leq \text{const} + u,$$

which implies $v \in L^2(B_{3R})$. Since $\left(\ln \frac{1}{t}\right)' = -\frac{1}{t}$ is a bounded function outside a neighborhood of 0, that is, in the range of u, we obtain by the chain rule of Exercise 17, that

$$\partial_j v = \partial_j \ln \frac{1}{u} = -\frac{\partial_j u}{u} \in L^2(B_{3R}).$$

Hence, $v \in W^{1,2}(B_{3R})$.

In the same way also the function $\frac{1}{u}$ belongs to $W^{1,2}(B_R)$, which will be used below. Indeed, $\frac{1}{u}$ is essentially bounded and, hence, is in $L^2(B_{3R})$, and by the same chain rule

$$\partial_j \left(\frac{1}{u} \right) = -\frac{\partial_j u}{u^2} \in L^2(B_{3R}).$$

Now let us verify that v is a subsolution of L, that is, $Lv \geq 0$ in B_{3R}. In fact, this is shown in Exercise 47 using the chain rule for L, but we give here a direct independent proof.

Motivation. The motivation for $Lv \geq 0$ comes from the following observation. In the simplest case $n = 1$ and $L = \frac{d^2}{dx^2}$, if $u \in C^2(\mathbb{R})$, $u > 0$ and $u'' \leq 0$ then we have

$$v'' = \left(\ln \frac{1}{u} \right)'' = \left(-\frac{u'}{u} \right)' = \frac{(u')^2 - u''u}{u^2} \geq \frac{(u')^2}{u^2} \geq 0.$$

If $n > 1$, $L = \Delta$, $u \in C^2(\mathbb{R}^n)$, $u > 0$ and $\Delta u \le 0$ then similarly

$$\Delta v = \sum_{i=1}^{n} \partial_{ii} \ln \frac{1}{u} = \sum_{i=1}^{n} \frac{(\partial_i u)^2 - (\partial_{ii} u)\, u}{u^2} = \frac{|\nabla u|^2 - (\Delta u)\, u}{u^2} \ge \frac{|\nabla u|^2}{u^2} \ge 0.$$

Noticing that $|\nabla v| = \left|\frac{\nabla u}{u}\right|$, we obtain from the above computation that

$$\Delta v \ge |\nabla v|^2 . \tag{3.28}$$

In fact, the above computation shows that (3.28) is equivalent to $\Delta u \le 0$.

In the present general case, we have to verify that, for any non-negative test function $\varphi \in \mathcal{D}\left(B_{3R}\right)$

$$-\int_{B_{3R}} \sum_{i,j=1}^{n} a_{ij} \partial_j v\, \partial_i \varphi\, dx \ge 0.$$

Since a part of the following computation will also be used below for a different purpose, we will do it for non-negative functions $\varphi \in Lip_c(B_{3R})$. Since $\partial_j v = -\frac{\partial_j u}{u}$, we have

$$-\int_{B_{3R}} \sum_{i,j=1}^{n} a_{ij} \partial_j v\, \partial_i \varphi\, dx = \int_{B_{3R}} \sum_{i,j=1}^{n} a_{ij} \frac{\partial_j u}{u} \partial_i \varphi\, dx = \int_{B_{3R}} \sum_{i,j=1}^{n} a_{ij} \partial_j u \frac{\partial_i \varphi}{u}\, dx. \tag{3.29}$$

The next idea is to use the function φ/u as a test function in inequality $Lu \le 0$. Since u is a supersolution, we have

$$\int_{B_{3R}} \sum_{i,j=1}^{n} a_{ij} \partial_j u \partial_i \psi\, dx \ge 0 \tag{3.30}$$

for any non-negative $\psi \in W_0^{1,2}(B_{3R})$. Let us verify that the function $\psi = \varphi/u$ belongs to $W_0^{1,2}(B_{3R})$. Since $\varphi \in W^{1,\infty}(B_{3R})$ and $1/u \in W^{1,2}(B_{3R})$, the function $\psi = \varphi/u$ belongs to $W^{1,2}(B_{3R})$ (as it was done in the proof of Theorem 3.2). Since $\operatorname{supp} \psi \subset \operatorname{supp} \varphi \subset B_{3R}$, we have $\psi \in W_c^{1,2}\left(B_{3R}\right)$. Since also $\psi \ge 0$, we can use this function in (3.30) and obtain that

$$\int_{B_{3R}} \sum_{i,j=1}^{n} a_{ij} \partial_j u\, \partial_i \left(\frac{\varphi}{u}\right)\, dx \ge 0. \tag{3.31}$$

By the product rule we have

$$\partial_i \left(\frac{\varphi}{u}\right) = \partial_i \left(\varphi \frac{1}{u}\right) = \frac{\partial_i \varphi}{u} - \varphi \frac{\partial_i u}{u^2}.$$

Substituting

$$\frac{\partial_i \varphi}{u} = \partial_i \left(\frac{\varphi}{u}\right) + \varphi \frac{\partial_i u}{u^2}$$

into (3.29) and using (3.31), we obtain

$$\begin{aligned} -\int_{B_{3R}} \sum_{i,j=1}^{n} a_{ij} \partial_j v\, \partial_i \varphi\, dx &= \int_{B_{3R}} \sum_{i,j=1}^{n} a_{ij} \partial_j u \left(\partial_i \left(\frac{\varphi}{u}\right) + \frac{\partial_i u}{u^2} \varphi\right) dx \\ &\ge \int_{B_{3R}} \sum_{i,j=1}^{n} a_{ij} \partial_j u \frac{\partial_i u}{u^2} \varphi\, dx = \int_{B_{3R}} \sum_{i,j=1}^{n} a_{ij} \partial_j u\, \partial_i u \frac{\varphi}{u^2}\, dx \ge 0, \end{aligned} \tag{3.32}$$

where we have also used the ellipticity of L. Hence, v is a subsolution of L.

Noticing that in the right hand side of (3.32) $\partial_j u/u = -\partial_j v$ and $\partial_i u/u = -\partial_i v$, we obtain that

$$-\int_{B_{3R}} \sum_{i,j=1}^{n} a_{ij}\partial_j v\, \partial_i \varphi\, dx \geq \int_{B_{3R}} \sum_{i,j=1}^{n} a_{ij}\partial_j v\, \partial_i v\, \varphi\, dx, \tag{3.33}$$

which is an analog of (3.28)[1]. The inequality (3.33) will be used below.

Applying the mean value inequality of Theorem 3.2 to a subsolution v, we obtain

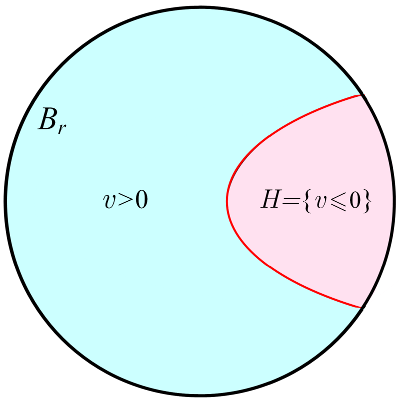
$$\operatorname*{esssup}_{B_R} v \leq \frac{C}{R^{n/2}} \left(\int_{B_{2R}} v_+^2 dx \right)^{1/2}, \tag{3.34}$$

where $C = C(n, \lambda)$, which completes the first step towards the proof of the bound (3.27).

Step 2. In order to estimate the integral in (3.34), we need the following fact.

Poincaré inequality. *Let $v \in W^{1,2}(B_r)$ and consider the set*

$$H = \{x \in B_r : v(x) \leq 0\}.$$



Then

$$\int_{B_r} v_+^2 dx \leq C \frac{r^2 |B_r|}{|H|} \int_{B_r} |\nabla v_+|^2 dx \tag{3.35}$$

where $C = C(n)$.

Comment. Recall that the Friedrichs inequality says that if $v \in W_0^{1,2}(B_r)$ then

$$\int_{B_r} v^2 dx \leq C r^2 \int_{B_r} |\nabla v|^2 dx. \tag{3.36}$$

For an arbitrary function $v \in W^{1,2}(B_r)$ this type of inequality cannot be true because if $v \equiv 1$ then the right hand side vanish while the left hand side is positive. Assume for simplicity that $v \geq 0$. Then (3.35) amounts to

$$\int_{B_r} v^2 dx \leq C \frac{r^2 |B_r|}{|H|} \int_{B_r} |\nabla v|^2 dx,$$

[1] Indeed, observing that the left hand side of (3.33) is equal to (Lv, φ) where Lv is regarded as distribution, we can rewrite (3.33) as follows: $Lv \geq \sum_{i,j=1}^{n} a_{ij}\partial_j v\, \partial_i v$.

where $H = \{v = 0\}$. Hence, if v vanishes on a large enough set (in the sense that $|H| \geq c\,|B_r|$), then we obtain again (3.36). As we see, the validity of (3.36) or similar inequalities depends on the property of v to vanish on certain sets.

The proof of (3.35) is non-trivial and will be given below (see Theorem 3.10 and Corollary 3.11).

Now let us apply (3.35) for the function $v = \ln \frac{1}{u}$ in the ball B_{2R}, that is, for $r = 2R$. Since

$$E = \{v \leq 0\} \cap B_R \subset \{v \leq 0\} \cap B_{2R} = H,$$

we have

$$|H| \geq |E| \geq \varepsilon\,|B_R| = \varepsilon 2^{-n}\,|B_{2R}|\,.$$

Then (3.35) yields

$$\int_{B_{2R}} v_+^2 dx \leq C\frac{R^2}{\varepsilon}\int_{B_{2R}} |\nabla v_+|^2\,dx \leq C_\varepsilon R^2 \int_{B_{2R}} |\nabla v|^2\,dx,$$

where C_ε depends on ε, n, λ. Combining with (3.34), we obtain

$$\operatorname*{esssup}_{B_R} v \leq \frac{C}{R^{n/2}}\left(C_\varepsilon R^2 \int_{B_{2R}} |\nabla v|^2\,dx\right)^{1/2}. \tag{3.37}$$

Step 3. In this step we estimate the integral

$$\int_{B_{2R}} |\nabla v|^2\,dx.$$

Let $\eta \in Lip_c(B_{3R})$ be such that $\eta \equiv 1$ on B_{2R} (we will specify η below). Using in (3.33) the function $\varphi := \eta^2 \in Lip_c(B_{3R})$, we obtain

$$\int_{B_{3R}} \sum_{i,j=1}^n a_{ij}\partial_j v\,\partial_i v\,\eta^2\,dx \leq -\int_{B_{3R}} \sum_{i,j=1}^n a_{ij}\partial_j v\,\partial_i\left(\eta^2\right)\,dx. \tag{3.38}$$

Using the uniform ellipticity of (a_{ij}), we estimate the left hand side of (3.38) as follows:

$$\int_{B_{3R}} \sum_{i,j=1}^n a_{ij}\partial_j v\,\partial_i v\,\eta^2\,dx \geq \lambda^{-1}\int_{B_{3R}} |\nabla v|^2\,\eta^2 dx,$$

while the right hand side of (3.38) is estimated as follows:

$$-\int_{B_{3R}} \sum_{i,j=1}^n a_{ij}\partial_j v\,\partial_i\left(\eta^2\right)\,dx \leq \lambda\int_{B_{3R}} |\nabla v|\left|\nabla\eta^2\right|dx = 2\lambda\int_{B_{3R}} |\nabla v|\,\eta\,|\nabla\eta|\,dx.$$

Hence, we obtain

$$\begin{aligned}\int_{B_{3R}} |\nabla v|^2\,\eta^2 dx &\leq 2\lambda^2\int_{B_{3R}} |\nabla v|\,\eta\,|\nabla\eta|\,dx \\ &\leq 2\lambda^2\left(\int_{B_{3R}} (|\nabla v|\,\eta)^2\,dx\right)^{1/2}\left(\int_{B_{3R}} |\nabla\eta|^2\,dx\right)^{1/2},\end{aligned}$$

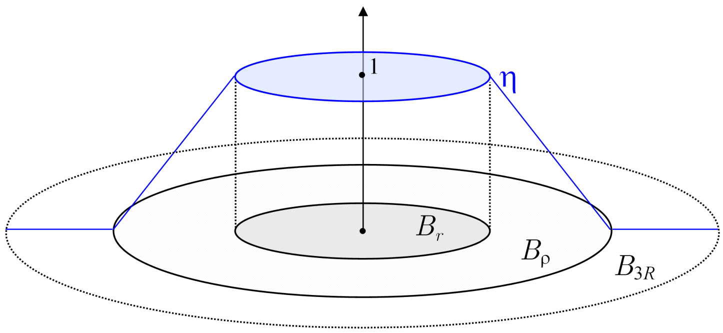
whence

$$\int_{B_{3R}} |\nabla v|^2 \eta^2 dx \le 4\lambda^4 \int_{B_{3R}} |\nabla \eta|^2 dx. \tag{3.39}$$

Let us now specify η as follows:

$$\eta(x) = \begin{cases} 1, & |x| \le r, \\ \frac{\rho-|x|}{\rho-r}, & r < |x| < \rho, \\ 0, & |x| \ge \rho, \end{cases}$$

where $r = 2R$ and $\rho = \frac{5}{2}R < 3R$.



A "bump" function η

Since $\eta = 1$ on B_{2R} and $|\nabla\eta| \le \frac{1}{\rho-r}$, where $\rho - r = R/2$, we obtain from (3.39) that

$$\int_{B_{2R}} |\nabla v|^2 dx \le 4\lambda^2 \frac{|B_{3R}|}{(\rho-r)^2} = CR^{n-2},$$

where $C = C(n,\lambda)$. Finally, substituting this estimate into (3.37), we obtain

$$\operatorname*{esssup}_{B_R} v \le \frac{C}{R^{n/2}} \left(R^2 C_\varepsilon R^{n-2}\right)^{1/2} = C(n,\lambda,\varepsilon),$$

which finishes the proof of (3.27). ■

21.12.23 Lecture 21

3.3 Oscillation inequality and Theorem of de Giorgi

Define the oscillation of a function u in a domain D by

$$\operatorname*{osc}_D u = \operatorname*{esssup}_D u - \operatorname*{essinf}_D u.$$

Observe that, for all real a, b,

$$\operatorname*{osc}_D (au+b) = |a| \operatorname*{osc}_D u.$$

As above, let

$$Lu = \sum_{i,j=1}^{n} \partial_i (a_{ij} \partial_j u)$$

be a uniformly elliptic operator with measurable coefficients in a domain $\Omega \subset \mathbb{R}^n$.

The next theorem is a simple consequence of the weak Harnack inequality, but it provides a key argument for the proof of De Giorgi's theorem.

Theorem 3.6 (Oscillation inequality) *Let $B_{3R} \subset \Omega$ and assume that $u \in W^{1,2}(B_{3R})$ is a weak solution of $Lu = 0$ in B_{3R}. Then*

$$\operatorname*{osc}_{B_R} u \leq \gamma \operatorname*{osc}_{B_{3R}} u, \tag{3.40}$$

where $\gamma = \gamma(n, \lambda) < 1$.

Proof. If $\operatorname{osc}_{B_{3R}} u = 0$ or ∞ then there is nothing to prove. If $0 < \operatorname{osc}_{B_{3R}} u < \infty$, then, by adding a constant to u and rescaling u, we can assume that

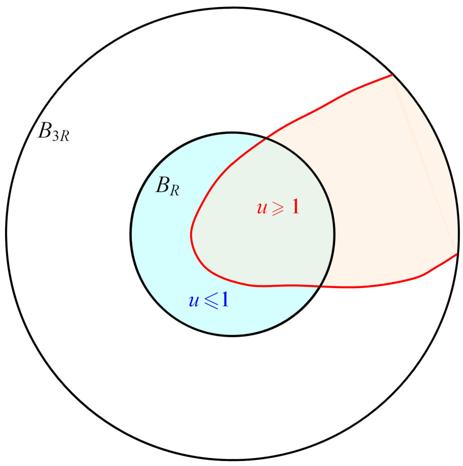
$$\operatorname*{essinf}_{B_{3R}} u = 0 \quad \text{and} \quad \operatorname*{esssup}_{B_{3R}} u = 2.$$

Consider the two sets

$$\{u \geq 1\} \cap B_R \quad \text{and} \quad \{u \leq 1\} \cap B_R. \tag{3.41}$$

One of these sets has measure $\geq \frac{1}{2} |B_R|$. Assume that this is the first set, that is,

$$|\{u \geq 1\} \cap B_R| \geq \frac{1}{2} |B_R|.$$



Applying Theorem 3.4 with $a = 1$ and $\varepsilon = \frac{1}{2}$, we obtain that

$$\operatorname*{essinf}_{B_R} u \geq \delta = \delta(n, \lambda, \frac{1}{2}).$$

It follows that

$$\operatorname*{osc}_{B_R} u \leq 2 - \delta = \frac{2 - \delta}{2} \operatorname*{osc}_{B_{3R}} u,$$

which proves (3.40) with $\gamma = \frac{2-\delta}{2} < 1$.

Assume now that the second set in (3.41) has measure at most $\frac{1}{2}|B_R|$, that is,

$$|\{u \le 1\} \cap B_R| \ge \frac{1}{2}|B_R|.$$

Consider the function $v = 2 - u$. For this function, the oscillation in any domain is equal to that of u. Also we have $Lv = 0$ in B_{3R} and

$$u \le 1 \Leftrightarrow v \ge 1,$$

which implies

$$|\{v \ge 1\} \cap B_R| \ge \frac{1}{2}|B_R|.$$

Applying the same argument as above, we obtain that

$$\operatorname*{osc}_{B_R} v \le \gamma \operatorname*{osc}_{B_{3R}} v,$$

which implies the same inequality for u, thus finishing the proof. ■

Recall that $C^\alpha(S)$ denotes the set of all locally Hölder continuous functions on a set S with the Hölder exponent α. Assume that S is a compact. Then $C^\alpha(S)$ coincides with the set of all Hölder continuous functions on S with the Hölder exponent α (cf. Exercise 36, where this was proved for $\alpha = 1$, but the case of any α is similar). The set $C^\alpha(S)$ is obviously a linear space. The following expression

$$\sup_{\substack{x,y \in S \\ x \ne y}} \frac{|u(x) - u(y)|}{|x - y|^\alpha}$$

is the minimal Hölder constant of u and, hence, is finite for any function $u \in C^\alpha(S)$. Moreover, this expression as a functional on the space $C^\alpha(S)$ is a seminorm that is called the *Hölder seminorm*. It gives rise to the following C^α*-norm*:

$$\|u\|_{C^\alpha(S)} := \sup_S |u| + \sup_{\substack{x,y \in S \\ x \ne y}} \frac{|u(x) - u(y)|}{|x - y|^\alpha}.$$

One can show that $C^\alpha(S)$ with this norm is a Banach space.

Theorem 3.7 (Theorem of De Giorgi) *If $u \in W^{1,2}(\Omega)$ and $Lu = 0$ weakly in Ω then $u \in C^\alpha(\Omega)$ where $\alpha = \alpha(n, \lambda) > 0$. Moreover, for any precompact open set U, such that $\overline{U} \subset \Omega$,*

$$\|u\|_{C^\alpha(\overline{U})} \le C \|u\|_{L^2(\Omega)}, \tag{3.42}$$

where $C = C(n, \lambda, \rho)$ and $\rho = \operatorname{dist}(U, \partial\Omega)$.

For the proof of Theorem 3.7 we need the following lemma that will be proved below.

Lemma 3.8 *Let U be a domain in $\mathbb{R}^n$ and u be a function from $L^2(U)$ such that, for some positive α, ε, A,*

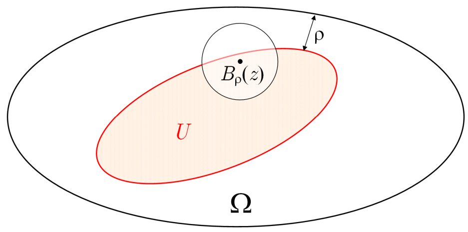
$$|u(x) - u(y)| \le A\,|x - y|^{\alpha} \text{ for almost all } x, y \in U \text{ s.t. } |x - y| < \varepsilon.$$

Then there exists a continuous version $\widetilde{u}$ of u.

Remark. The expression "for almost all $x, y \in U$" has the following rigorous meaning: for almost all points $(x, y) \in U \times U$. Hence, here we use the Lebesgue measure in $\mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$.

Proof of Theorem 3.7. The proof consists of four steps.

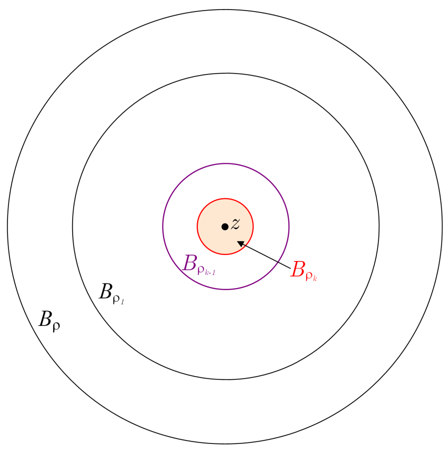
Step 1. Set $\rho = \operatorname{dist}(\overline{U}, \partial\Omega)$ and observe that, for any $z \in \overline{U}$, the ball $B_\rho(z)$ is contained in Ω.



For any non-negative integer k, set

$$\rho_k = 3^{-k}\rho.$$

Fix a point $z \in \overline{U}$ and consider the sequence of balls $B_{\rho_k}(z)$.



By Theorem 3.6, we have

$$\operatorname*{osc}_{B_{\rho_k}(z)} u \le \gamma \operatorname*{osc}_{B_{\rho_{k-1}}(z)} u, \tag{3.43}$$

which implies by induction that, for all $k \ge 1$,

$$\operatorname*{osc}_{B_{\rho_k}(z)} u \le \gamma^{k-1} \operatorname*{osc}_{B_{\rho_1}(z)} u \le 2\gamma^{k-1} \operatorname*{esssup}_{B_{\rho_1}(z)} |u|\,.$$

Applying the mean value inequality of Theorem 3.2 in the ball $B_\rho(z)$ with functions u and $-u$, we obtain that

$$\operatorname*{esssup}_{B_{\rho_1}(z)} |u| \le \operatorname*{esssup}_{B_{\rho/2}(z)} |u| \le \frac{C}{\rho^{n/2}} \|u\|_{L^2(B_\rho(z))} \le C\,\|u\|_{L^2(\Omega)}\,,$$

where $c = c(n, \lambda)$ and $C = C(n, \lambda, \rho)$. Combining the above inequalities, we obtain

$$\boxed{\operatorname*{osc}_{B_{\rho_k}(z)} u \leq C\gamma^k \|u\|_{L^2(\Omega)}}. \tag{3.44}$$

Note that, without application of Theorem 3.2, we obtain from (3.43)

$$\operatorname*{osc}_{B_{\rho_k}(z)} u \leq \gamma^k \operatorname*{osc}_{B_\rho(z)} u \leq 2\gamma^k \|u\|_{L^\infty(\Omega)}. \tag{3.45}$$

Step 2. Let us prove that, for almost all $x, y \in U$ with

$$0 < |x - y| \leq \rho/2, \tag{3.46}$$

the following inequality holds

$$\boxed{|u(x) - u(y)| \leq C|x - y|^\alpha \|u\|_{L^2(\Omega)}}, \tag{3.47}$$

for some $\alpha > 0$ and $C = C(n, \lambda, \rho)$.

Idea of proof. One of the difficulties in the proof of (3.47) is that this inequality has to be proved for *almost all* x, y. In order to show the idea of the proof let us first assume that u is continuous and prove (3.47) for *all* $x, y \in U$ satisfying

$$0 < |x - y| \leq \rho.$$

Fix such a pair x, y and find a non-negative integer k such that

$$\rho_{k+1} < |x - y| \leq \rho_k.$$

Since $y \in \overline{B}_{\rho_k}(x)$, using the continuity of u and (3.44), we obtain

$$|u(x) - u(y)| \leq \operatorname*{osc}_{B_{\rho_k}(x)} u \leq C\gamma^k \|u\|_{L^2(\Omega)}.$$

Setting $\alpha = \log_3 \frac{1}{\gamma} > 0$, we obtain $\gamma = 3^{-\alpha}$ and

$$\gamma^k = 3^{-ka} = \left(\frac{\rho_k}{\rho}\right)^\alpha \leq \left(\frac{3|x - y|}{\rho}\right)^\alpha.$$

It follows that

$$|u(x) - u(y)| \leq C\left(\frac{3|x - y|}{\rho}\right)^\alpha \|u\|_{L^2(\Omega)},$$

which is equivalent to (3.47).

For any couple $x, y \in U$ satisfying (3.46) there is a non-negative integer k such that

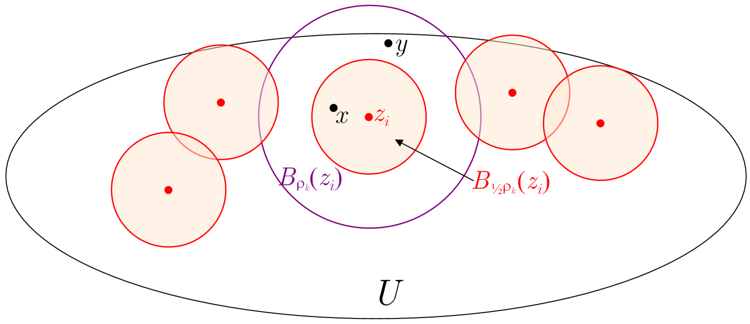
$$\frac{1}{2}\rho_{k+1} < |x - y| \leq \frac{1}{2}\rho_k. \tag{3.48}$$

Let us fix k and prove (3.47) for almost all $x, y \in U$ satisfying (3.48).[2]

The compact set $\overline{U}$ can be covered by a finite number of balls of radius $\frac{1}{2}\rho_k$, say $B_{\frac{1}{2}\rho_k}(z_i)$ where $z_i \in \overline{U}$. Then, for any $x \in U$ there is z_i such that $x \in B_{\frac{1}{2}\rho_k}(z_i)$; moreover, it follows from (3.48) that $y \in B_{\rho_k}(z_i)$. Hence, for any couple $x, y \in U$ satisfying

[2]Indeed, if we know already that the set S_k of points $(x, y) \in U \times U$ satisfying (3.48) and not satisfying (3.47) has measure 0 in $\mathbb{R}^{2n}$, then the set of points $(x, y) \in U \times U$ satisfying (3.46) and not satisfying (3.47) is $\bigcup_{k=0}^\infty S_k$, which also has measure zero.

(3.48) there is z_i such that $x, y \in B_{\rho_k}(z_i)$.



Therefore, it suffices to prove (3.47) for almost all $x, y \in B_{\rho_k}(z_i)$ satisfying (3.48). Applying (3.44) with $z = z_i$, we obtain that, for almost all $x, y \in B_{\rho_k}(z_i)$,

$$|u(x) - u(y)| \leq \operatorname*{osc}_{B_{\rho_k}(z_i)} u \leq C\gamma^k \|u\|_{L^2(\Omega)} . \tag{3.49}$$

Let us estimate γ^k via $|x - y|$ using (3.48). Setting

$$\alpha := \log_3 \frac{1}{\gamma} > 0,$$

we obtain $\gamma = 3^{-\alpha}$ and

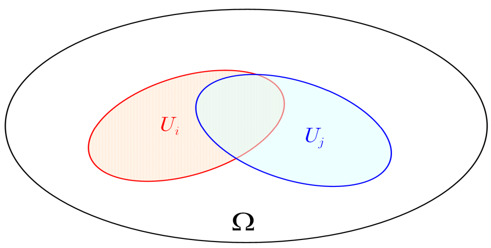
$$\gamma^k = 3^{-\alpha k} = \left(\frac{\rho_k}{\rho}\right)^\alpha = \left(\frac{3\rho_{k+1}}{\rho}\right)^\alpha \leq \left(\frac{6|x-y|}{\rho}\right)^\alpha, \tag{3.50}$$

where we have used (3.48). Substituting this into (3.49), we obtain (3.47).

Alternatively, if we use (3.45) instead of (3.44) and (3.49), then we obtain, for almost all $x, y \in U$ satisfying (3.46) that

$$\boxed{|u(x) - u(y)| \leq 2\left(6\frac{|x-y|}{\rho}\right)^\alpha \|u\|_{L^\infty(\Omega)}}. \tag{3.51}$$

Step 3. By (3.47), the function u in U satisfies the hypotheses of Lemma 3.8 with $\varepsilon = \rho/2$. Hence, u has a continuous version $\widetilde{u}$ in U. Since Ω can be covered by a countable family $\{U_i\}$ of precompact open sets U_i such that $\overline{U}_i \subset \Omega$, we obtain in any set U_i a continuous version of u denoted by $\widetilde{u}_i$.



In any intersection $U_i \cap U_j$ we have $\widetilde{u}_i = u = \widetilde{u}_j$ a.e., which implies that $\widetilde{u}_i \equiv \widetilde{u}_j$ pointwise in $U_i \cap U_j$. Hence, we can now define a continuous function $\widetilde{u}$ in the entire set Ω by setting

$$\widetilde{u}(x) = \widetilde{u}_i(x) \text{ if } x \in U_i.$$

It follows that $\widetilde{u} = u$ a.e.in each U_i, whence $\widetilde{u} = u$ a.e.in Ω.

Step 4. Now we prove the estimate (3.42). Let us rename $\widetilde{u}$ back to u so that now u is continuous in Ω. By Theorem 3.2 we have, for any $x \in U$,

$$u(x) \leq \sup_{B_{\rho/2}(x)} u \leq C \|u\|_{L^2(B_\rho(x))} \leq C \|u\|_{L^2(\Omega)},$$

where $C = C(n, \lambda, \rho)$. Applying the same estimate to $-u$, we obtain the same inequality for $-u(x)$, which implies that

$$\sup_U |u| \leq C \|u\|_{L^2(\Omega)}.$$

By inequality (3.47) of Step 2, we have, for all $x, y \in U$ such that $0 < |x - y| \leq \rho/2$,

$$|u(x) - u(y)| \leq C |x - y|^\alpha \|u\|_{L^2(\Omega)}$$

(it was proved above for almost all x, y but now, due to the continuity of u, we obtain that it holds for all x, y). Hence, we obtain

$$\sup_{\substack{x,y \in U, \\ 0<|x-y|\leq\rho/2}} \frac{|u(x) - u(y)|}{|x - y|^\alpha} \leq C \|u\|_{L^2(\Omega)}.$$

Observe that

$$\sup_{\substack{x,y \in U, \\ |x-y|>\rho/2}} \frac{|u(x) - u(y)|}{|x - y|^\alpha} \leq 2 \left(\frac{2}{\rho}\right)^\alpha \sup_U |u| \leq C \|u\|_{L^2(\Omega)}.$$

Finally, combining all these estimates, we obtain

$$\|u\|_{C^\alpha(\overline{U})} = \sup_U |u| + \sup_{\substack{x,y \in U, \\ x \neq y}} \frac{|u(x) - u(y)|}{|x - y|^\alpha} \leq C \|u\|_{L^2(\Omega)},$$

which finishes the proof of (3.42). ■

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Corollary 3.9 *Under the hypotheses of Theorem 3.7, it is also true that, for any precompact open set U such that $\overline{U} \subset \Omega$ and for all $x, y \in U$,*

$$|u(x) - u(y)| \leq C \left(\frac{|x - y|}{\rho}\right)^\alpha \|u\|_{L^\infty(\Omega)}, \tag{3.52}$$

where $\rho = \operatorname{dist}\left(\overline{U}, \partial\Omega\right)$ and the constant C depends only on n, λ (and does not depend on ρ).

Proof. Indeed, if $|x-y|<\rho/2$ then (3.52) was proved at the end of Step 2 for *almost all* x,y satisfying the above restrictions (see (3.51)). Since u is now continuous, the inequality (3.52) holds for *all* such x,y. If $|x-y|\geq\rho/2$ then (3.52) follows from

$$|u(x)-u(y)|\leq 2\,\|u\|_{L^\infty(\Omega)}\,.$$

∎

Proof of Lemma 3.8. Assuming that a function $u\in L^2(U)$ is "almost" Hölder in the sense that

$$|u(x)-u(y)|\leq A\,|x-y|^\alpha \text{ for almost all } x,y\in U \text{ s.t. } |x-y|<\varepsilon, \tag{3.53}$$

for some positive constants α,ε,A, we need to prove that there exists a continuous version $\widetilde{u}$ of u.

Choose a mollifier φ, that is, a function $\varphi\in C_0^\infty(\mathbb{R}^n)$ such that

$$\varphi\geq 0,\quad \operatorname{supp}\varphi\subset B_1 \text{ and } \int_{\mathbb{R}^n}\varphi dx=1.$$

Set for any positive integer k

$$\varphi_k(x)=k^n\varphi(kx)\,, \tag{3.54}$$

so that

$$\operatorname{supp}\varphi_k\subset B_{1/k} \text{ and } \int_{\mathbb{R}^n}\varphi_k dx=1. \tag{3.55}$$

Let us extend the function u to all $\mathbb{R}^n$ by setting $u=0$ outside U. Then $u\in L^2(\mathbb{R}^n)$, and we can consider the mollification of u, that is, the sequence of functions $\{u_k\}_{k=1}^\infty$ defined by

$$u_k(x)=u*\varphi_k(x)=\int_{\mathbb{R}^n}u(x-y)\,\varphi_k(y)dy.$$

It is known that $u_k\in C^\infty(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$ and

$$u_k\xrightarrow{L^2}u \text{ as } k\to\infty \tag{3.56}$$

(cf. Exercises 4, 39, 40). The idea of what follows is to show that the limit

$$\widetilde{u}(x):=\lim_{k\to\infty}u_k(x) \tag{3.57}$$

exists for all $x\in U$, and

$$|\widetilde{u}(x)-\widetilde{u}(y)|\leq A\,|x-y|^\alpha \text{ for all } x,y\in U \text{ s.t. } |x-y|<\varepsilon/2 \tag{3.58}$$

(note that (3.58) holds for *all* x,y in contrast to (3.53) that holds for *almost all* x,y). Consequently, $\widetilde{u}$ is continuous. By (3.56), there is a subsequence $\{u_{k_i}\}$ such that

$$u_{k_i}\to u \text{ a.e..}$$

Comparing to (3.57) we conclude that $\widetilde{u}=u$ a.e.; that is, $\widetilde{u}$ is a continuous version of u.

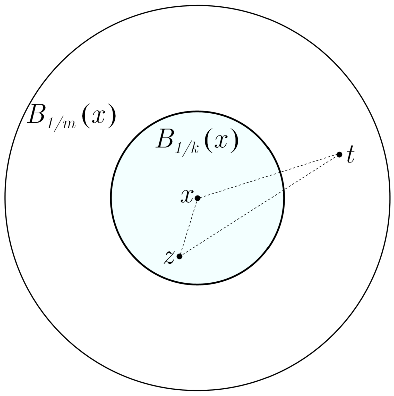
Now let us prove the existence of $\lim_{k\to\infty} u_k(x)$. It suffices to prove that, for any $x \in U$, the sequence $\{u_k(x)\}$ is Cauchy. Fix some $x \in U$. Since $\operatorname{supp}\varphi_k \subset B_{1/k}$, let us rewrite the definition of u_k in the form

$$u_k(x) = \int_{B_{1/k}(0)} u(x-y)\,\varphi_k(y)dy = \int_{B_{1/k}(x)} u(z)\,\varphi_k(x-z)\,dz. \tag{3.59}$$

For all k, m we have, using (3.59) and (3.55),

$$u_k(x) = \int_{B_{1/k}(x)} u(z)\varphi_k(x-z)\cdot 1\,dz = \int_{B_{1/m}(x)} \left(\int_{B_{1/k}(x)} u(z)\varphi_k(x-z)dz\right)\varphi_m(x-t)dt,$$

where $z \in B_{1/k}(x)$ and $t \in B_{1/m}(x)$.



Similarly, we have

$$u_m(x) = \int_{B_{1/m}(x)} u(t)\varphi_m(x-t)dt = \int_{B_{1/k}(x)} \left(\int_{B_{1/m}(x)} u(t)\varphi_m(x-t)dt\right)\varphi_k(x-z)dz.$$

Using Fubini's theorem we obtain

$$u_k(x) - u_m(x) = \iint_{B_{1/k}(x)\times B_{1/m}(x)} (u(z)-u(t))\,\varphi_k(x-z)\varphi_m(x-t)dzdt. \tag{3.60}$$

If k and m are large enough then the balls $B_{1/k}(x)$ and $B_{1/m}(x)$ lie in U. Since $z \in B_{1/k}(x)$ and $t \in B_{1/m}(x)$, we have

$$|z-t| \le \frac{1}{k} + \frac{1}{m} < \varepsilon$$

(where ε is from (3.53)), provided k, m are large enough. Hence, for almost all z, t in the domain of integration in (3.60), we have

$$|u(z) - u(t)| \le A\,|z-t|^\alpha \le A\left(\frac{1}{k} + \frac{1}{m}\right)^\alpha.$$

Substituting into (3.60) and using (3.55), we obtain

$$|u_k(x) - u_m(x)| \le A\left(\frac{1}{k} + \frac{1}{m}\right)^\alpha \to 0 \text{ as } k, m \to \infty.$$

Therefore, the sequence $\{u_k(x)\}$ is Cauchy for any $x \in U$ and, hence, there exists the limit

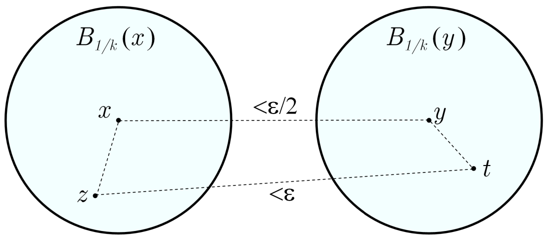
$$\widetilde{u}(x) := \lim_{k\to\infty} u_k(x).$$

Let us now show that $\widetilde{u}$ satisfies (3.58). For all $x, y \in U$ we have

$$u_k(x) = \int_{B_{1/k}(x)} u(z)\varphi_k(x-z)dz = \int_{B_{1/k}(y)} \left(\int_{B_{1/k}(x)} u(z)\varphi_k(x-z)dz\right)\varphi_k(y-t)dt$$

and

$$u_k(y) = \int_{B_{1/k}(y)} u(t)\varphi_k(y-t)dt = \int_{B_{1/k}(x)} \left(\int_{B_{1/k}(y)} u(t)\varphi_k(y-t)dt\right)\varphi_k(x-z)dz.$$



Hence, using Fubini's theorem, we obtain

$$u_k(x) - u_k(y) = \iint_{B_{1/k}(x)\times B_{1/k}(y)} (u(z) - u(t))\,\varphi_k(x-z)\varphi_k(y-t)dz\,dt. \tag{3.61}$$

Fix some $x, y \in U$ such that $|x-y| < \varepsilon/2$. If k is large enough then both balls $B_{1/k}(x)$ and $B_{1/k}(y)$ lie in U. For all $z \in B_{1/k}(x)$ and $t \in B_{1/k}(y)$ we have by the triangle inequality

$$|z-t| < |x-y| + \frac{2}{k} < \varepsilon$$

provided k is large enough. Hence, for almost all z, t in the domain of integration in (3.61), we have

$$|u(z) - u(t)| \le A\,|z-t|^\alpha \le A\left(|x-y| + \frac{2}{k}\right)^\alpha.$$

Substituting into (3.61), we obtain

$$|u_k(x) - u_k(y)| \le A\left(|x-y| + \frac{2}{k}\right)^\alpha.$$

Letting $k \to \infty$ we obtain

$$|\widetilde{u}(x) - \widetilde{u}(y)| \le A\,|x-y|^\alpha,$$

which finishes the proof. ■

3.4 Poincaré inequality

We start with the following more general version of the Poincaré inequality.

Theorem 3.10 *Let $p \in [1,\infty)$. For any ball B_R in $\mathbb{R}^n$ and any $f \in W^{1,p}(B_R)$, the following inequality is true:*

$$\int_{B_R}\int_{B_R} |f(x) - f(y)|^p \, dxdy \le CR^{n+p} \int_{B_R} |\nabla f|^p \, dx, \tag{3.62}$$

where $C = C(n,p)$.

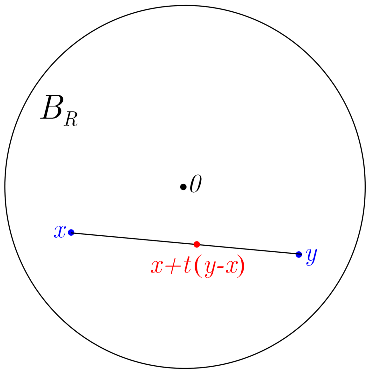
Dividing the both sides of (3.62) by $|B_R|^2$ and recalling that $|B_R| = c_n R^n$, we can rewrite it in the following form:

$$⨍_{B_R}⨍_{B_R} |f(x) - f(y)|^p \, dxdy \le CR^p ⨍_{B_R} |\nabla f|^p \, dx.$$

Proof. Let us first prove (3.62) for $f \in C^1(B_R)$. Fix some $x, y \in B_R$ and consider the function

$$\varphi(t) = f(x + t(y-x)), \quad t \in [0,1],$$

that is well defined and differentiable because $x + t(y-x) = (1-t)x + ty \in B_R$.



Applying the fundamental theorem of calculus and the chain rule, we obtain

$$\begin{aligned}
|f(y) - f(x)| = |\varphi(1) - \varphi(0)| &= \left|\int_0^1 \varphi'(t)dt\right| \\
&= \left|\int_0^1 \partial_t [f(x + t(y-x))] \, dt\right| \\
&= \left|\int_0^1 \nabla f(x + t(y-x)) \cdot (y-x) \, dt\right| \\
&\le \int_0^1 |\nabla f|(x + t(y-x)) \, |y-x| \, dt \\
&\le 2R \int_0^1 |\nabla f|(x + t(y-x)) \, dt.
\end{aligned}$$

Next, applying the Hölder inequality, we obtain

$$|f(y) - f(x)| \le 2R \left(\int_0^1 |\nabla f|^p \left(x + t \left(y - x \right) \right) dt \right)^{1/p}.$$

Raising this inequality to the power p and integrating over $(x, y) \in B_R \times B_R$, we obtain

$$\int_{B_R} \int_{B_R} |f(x) - f(y)|^p \, dxdy \le (2R)^p \int_{B_R} \int_{B_R} \int_0^1 |\nabla f|^p \left(x + t \left(y - x \right) \right) dtdxdy. \quad (3.63)$$

In the view of (3.63), in order to prove (3.62) it suffices to show that

$$\int_{B_R} \int_{B_R} \int_0^1 |\nabla f|^p \left(x + t \left(y - x \right) \right) dtdxdy \le CR^n \int_{B_R} |\nabla f|^p \, dx.$$

Set $F = |\nabla f|^p$ and rewrite this inequality as follows:

$$\int_{B_R} \int_{B_R} \int_0^1 F \left(x + t \left(y - x \right) \right) dtdxdy \le CR^n \int_{B_R} F dx. \quad (3.64)$$

Let us prove (3.64) for any non-negative $F \in C(B_R)$, with a constant $C = C(n)$. Let us extend F to the entire $\mathbb{R}^n$ by setting $F = 0$ outside B_R. By Fubini's theorem, the integrations in the left hand side of (3.64) are all interchangeable. In the integral

$$\int_{B_R} F \left(x + t \left(y - x \right) \right) dy$$

let us make change $z = y - x$, so that

$$\int_{B_R} F \left(x + t \left(y - x \right) \right) dy = \int_{B_R(-x)} F \left(x + tz \right) dz \le \int_{B_{2R}} F \left(x + tz \right) dz$$

and, hence,

$$\int_{B_R} \int_{B_R} \int_0^1 F \left(x + t \left(y - x \right) \right) dtdxdy \le \int_{B_{2R}} \int_{B_R} \int_0^1 F \left(x + tz \right) dtdxdz.$$

Next, in the integral

$$\int_{B_R} F \left(x + tz \right) dx,$$

let us make change $x' = x + tz$ so that

$$\int_{B_R} F \left(x + tz \right) dx = \int_{B_R(tz)} F \left(x' \right) dx' \le \int_{\mathbb{R}^n} F \left(x' \right) dx' = \int_{B_R} F \left(x' \right) dx'.$$

It follows that

$$\begin{aligned} \int_{B_R} \int_{B_R} \int_0^1 F \left(x + t \left(y - x \right) \right) dtdxdy &\le \int_{B_{2R}} \int_{B_R} \int_0^1 F \left(x' \right) dt \, dx' dz \\ &= 1 \cdot |B_{2R}| \int_{B_R} F \left(x' \right) dx' \\ &= CR^n \int_{B_R} F(x) dx, \end{aligned}$$

which finishes the proof of (3.64) for $f \in C^1(B_R)$.

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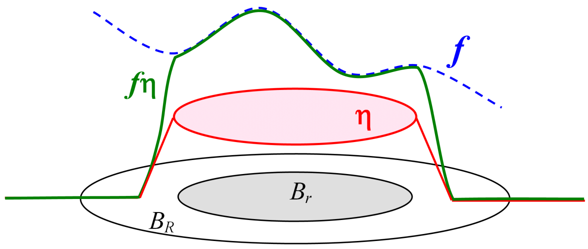
Let now $f \in W^{1,p}(B_R)$. It suffices to prove that, for any $r < R$,

$$\int_{B_r}\int_{B_r} |f(x) - f(y)|^p \, dxdy \le C r^{n+p} \int_{B_r} |\nabla f|^p \, dx, \tag{3.65}$$

and then let $r \to R$. Let $\eta \in Lip_c(B_R)$ be a cutoff function of B_r in B_R. Then $f\eta \in W_c^{1,p}(B_R) \subset W_0^{1,p}(B_R)$ and, setting $f\eta = 0$ in B_R^c, we obtain $f\eta \in W_0^{1,p}(\mathbb{R}^n)$.



Since $f = f\eta$ in B_r, the function f in (3.65) can be replaced by $f\eta$. Hence, renaming $f\eta$ back into f, we can assume that $f \in W_0^{1,p}(\mathbb{R}^n)$.

Consider mollifications $f_k = f * \varphi_k$ where $\{\varphi_k\}$ is a sequence of mollifiers defined by (3.54). Then $f_k \in C^\infty(\mathbb{R}^n)$ and, hence, by the first part of the proof, we have

$$\int_{B_r}\int_{B_r} |f_k(x) - f_k(y)|^p \, dxdy \le C r^{n+p} \int_{B_r} |\nabla f_k|^p \, dx. \tag{3.66}$$

By Exercise 41, we have

$$f_k \overset{W^{1,p}(\mathbb{R}^n)}{\longrightarrow} f \quad \text{as } k \to \infty,$$

in particular,

$$\int_{B_r} |\nabla f_k|^p \, dx \to \int_{B_r} |\nabla f|^p \, dx.$$

Since $f_k \overset{L^p}{\to} f$, there is a subsequence of k such that $f_k(x) \to f(x)$ for almost all $x \in \mathbb{R}^n$, whence

$$f_k(x) - f_k(y) \to f(x) - f(y) \text{ for almost all } x, y \in \mathbb{R}^n.$$

By Fatou's lemma, we obtain

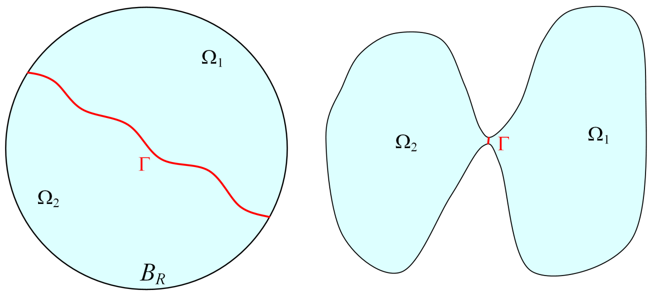
$$\begin{aligned}
\int_{B_r}\int_{B_r} |f(x) - f(y)|^p \, dxdy &\le \liminf_{k\to\infty} \int_{B_r}\int_{B_r} |f_k(x) - f_k(y)|^p \, dxdy \\
&\le \liminf_{k\to\infty} C r^{n+p} \int_{B_r} |\nabla f_k|^p \, dx \\
&= C r^{n+p} \int_{B_r} |\nabla f|^p \, dx,
\end{aligned}$$

which proves (3.65). ■

Remark. In the case $p = 1$, the Poincaré inequality (3.62) has the following geometric meaning. Let Γ be a smooth hypersurface that divides a ball B_R into two open subsets Ω_1 and Ω_2. We claim that

$$\sigma(\Gamma) \geq c \frac{\min(|\Omega_1|, |\Omega_2|)}{R}, \tag{3.67}$$

where $c = c(n) > 0$. That is, the surface measure $\sigma(\Gamma)$ of Γ cannot be too small in comparison to the volumes of Ω_1 and Ω_2. In other words, a ball has no *bottleneck*.

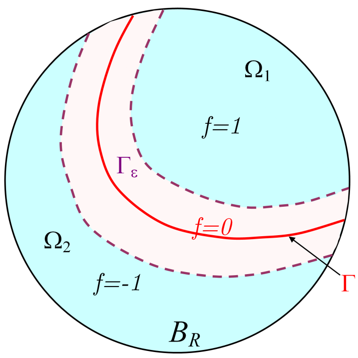


A ball and another domain with a bottleneck

Let us sketch the proof of (3.67). Fix some $\varepsilon > 0$ and define a Lipschitz function f_ε in B_R as follows:

$$f_\varepsilon(x) = \begin{cases} \min(1, \frac{1}{\varepsilon} dist(x, \Gamma)), & x \in \Omega_1, \\ -\min(1, \frac{1}{\varepsilon} dist(x, \Gamma)), & x \in \Omega_2. \end{cases}$$

Denote by Γ_ε the ε-neighborhood of Γ. Then $f_\varepsilon = 1$ in $\Omega_1 \setminus \Gamma_\varepsilon$, $f_\varepsilon = -1$ in $\Omega_2 \setminus \Gamma_\varepsilon$, while $f_\varepsilon = \pm\frac{1}{\varepsilon} dist(x, \Gamma)$ if $x \in \Gamma_\varepsilon$. In particular, $f = 0$ on Γ.



By (3.62) with $p = 1$ we have

$$\int_{B_R} |\nabla f_\varepsilon| \, dx \geq \frac{c}{R^{n+1}} \int_{B_R} \int_{B_R} |f_\varepsilon(x) - f_\varepsilon(y)| \, dx dy, \tag{3.68}$$

for some $c = c(n) > 0$. One can show that $|\nabla f_\varepsilon| = \frac{1}{\varepsilon}$ in Γ_ε so that

$$\int_{B_R} |\nabla f_\varepsilon| \, dx = \frac{1}{\varepsilon} |\Gamma_\varepsilon| \to 2\sigma(\Gamma) \text{ as } \varepsilon \to 0,$$

where we have used that $|\Gamma_\varepsilon| \sim 2\varepsilon\sigma(\Gamma)$. As $\varepsilon \to 0$, we have, for any $x \in B_R \setminus \Gamma$,

$$f_\varepsilon(x) \to f(x) = \begin{cases} 1, & x \in \Omega_1, \\ -1, & x \in \Omega_2, \end{cases}$$

which implies

$$\begin{aligned}\iint\limits_{B_R\times B_R} |f_\varepsilon(x) - f_\varepsilon(y)|\,dxdy \to &\iint\limits_{B_R\times B_R} |f(x) - f(y)|\,dxdy \\ &= \left[\iint\limits_{\Omega_1\times\Omega_2} + \iint\limits_{\Omega_2\times\Omega_1} + \iint\limits_{\Omega_1\times\Omega_1} + \iint\limits_{\Omega_2\times\Omega_2}\right] |f(x) - f(y)|\,dxdy \\ &= \int_{\Omega_1}\int_{\Omega_2} \underbrace{|f(x) - f(y)|}_{=2} dxdy + \int_{\Omega_2}\int_{\Omega_1} \underbrace{|f(x) - f(y)|}_{=2} dxdy \\ &= 4\,|\Omega_1|\,|\Omega_2|\,.\end{aligned}$$

Hence, letting $\varepsilon \to 0$ in (3.68), we obtain

$$\sigma(\Gamma) \geq c\frac{|\Omega_1|\,|\Omega_2|}{R^{n+1}}.$$

Since $|\Omega_1| + |\Omega_2| = |B_R| = c_n R^n$, it follows that

$$\sigma(\Gamma) \geq \frac{c}{R}\frac{|\Omega_1|\,|\Omega_2|}{|\Omega_1| + |\Omega_2|} \geq \frac{c}{R}\min(|\Omega_1|\,,|\Omega_2|),$$

which was claimed.

Now let us prove the Poincaré inequality in the form that was used in the proof of Theorem 3.4.

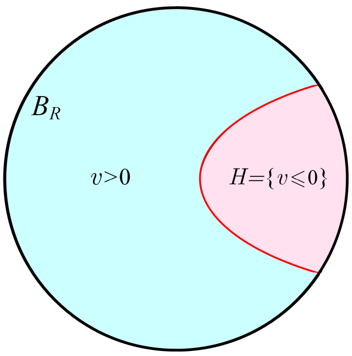
Corollary 3.11 *Let $v \in W^{1,2}(B_R)$ and consider the set*

$$H = \{x \in B_R : v(x) \leq 0\}\,.$$

Then

$$\int_{B_R} v_+^2 dx \leq C\frac{R^2\,|B_R|}{|H|}\int_{B_R} |\nabla v_+|^2\,dx$$

where $C = C(n)$.



Proof. By Exercise 16, we have $v_+ \in W^{1,2}(B_R)$. Renaming v_+ into v, we can assume that $v \geq 0$. Hence, we prove that

$$\int_{B_R} v^2 dx \leq C \frac{R^2 |B_R|}{|H|} \int_{B_R} |\nabla v|^2 dx,$$

where $H = \{v = 0\}$. By (3.62) with $p = 2$ we have

$$\int_{B_R} \int_{B_R} (v(x) - v(y))^2 \, dx dy \leq C R^{n+2} \int_{B_R} |\nabla v|^2 dx.$$

Restricting the integration in $y \in B_R$ in the left hand side to $y \in H$ and noticing that $v(y) = 0$, we obtain

$$\int_H \int_{B_R} v(x)^2 dx dy \leq C R^{n+2} \int_{B_R} |\nabla v|^2 dx$$

whence

$$|H| \int_{B_R} v(x)^2 dx \leq C R^{n+2} \int_{B_R} |\nabla v|^2 dx.$$

Finally, it remains to observe that $R^{n+2} = c R^2 |B_R|$. ∎

Remark. Here is yet another form of the Poincaré inequality in the case $p = 2$: for any ball B_R in $\mathbb{R}^n$ and for any $f \in W^{1,2}(B_R)$,

$$\int_{B_R} \left(f - \overline{f}\right)^2 dx \leq C R^2 \int_{B_R} |\nabla f|^2 dx, \tag{3.69}$$

where $C = C(n)$ and

$$\overline{f} := \fint_{B_R} f(x) dx$$

(see Exercise 66). In particular, if

$$\int_{B_R} f dx = 0 \tag{3.70}$$

then $\overline{f} = 0$ and (3.69) becomes

$$\int_{B_R} f^2 dx \leq C R^2 \int_{B_R} |\nabla f|^2 dx, \tag{3.71}$$

which has the same shape as the Friedrichs inequality in B_R. However, the Friedrichs inequality holds for $f \in W_0^{1,2}(B_R)$ while the Poincaré inequality in the form (3.71) holds for $f \in W^{1,2}(B_R)$ satisfying (3.70).

3.5 Hölder continuity for inhomogeneous equations

As above, consider in a domain $\Omega \subset \mathbb{R}^n$ a divergence form uniformly elliptic operator

$$Lu = \sum_{i,j=1}^{n} \partial_i \left(a_{ij} \partial_j u \right)$$

with measurable coefficients.

Theorem 3.12 *Let $u \in W^{1,2}(\Omega)$ be a weak solution of $Lu = f$ in Ω, where $f \in L^q(\Omega)$ with*

$$q \in [2, \infty] \cap (n/2, \infty]. \tag{3.72}$$

Then $u \in C^\beta(\Omega)$ where $\beta = \beta(n, \lambda, q) > 0$.

Remark. Assume that Ω is bounded (then $f \in L^q(\Omega)$ implies $f \in L^2(\Omega)$). By Theorem 1.15, if u is a solution of the Dirichlet problem

$$\begin{cases} Lu = f \quad \text{weakly in } \Omega, \\ u \in W_0^{1,2}(\Omega), \end{cases}$$

where $f \in L^q(\Omega)$ with q as in (3.72) then $u \in L^\infty(\Omega)$ and, moreover,

$$\|u\|_{L^\infty} \leq C \, |\Omega|^{\frac{2}{n} - \frac{1}{q}} \, \|f\|_{L^q} \, .$$

Theorem 3.12 says that also $u \in C^\beta(\Omega)$.

Remark. Note that if $f \in L^q$ with $q < n/2$ then there may exist a solution $u \in W^{1,2}(\Omega)$ of $Lu = f$ that does not admit a continuous version (see Exercise 52).

Proof. Fix a precompact open set U such that $\overline{U} \subset \Omega$. Recall that, in Step 1 of the proof of Theorem 3.7, we have proved the inequality (3.44): if $Lu = 0$ in Ω then, for any $z \in U$ and any $k \in \mathbb{N}$,

$$\operatorname*{osc}_{B_{\rho_k}(z)} u \leq C\gamma^k, \tag{3.73}$$

where $\rho = \operatorname{dist}(U, \partial\Omega)$, $\rho_k = 3^{-k}\rho$, $\gamma = \gamma(n, \lambda) \in (0, 1)$, and C depends on u and U, but does not depend on z, k. In the next steps of the proof, we have used only (3.73) and showed that it implies that $u \in C^\alpha(\Omega)$ with $\alpha = \log_3 \frac{1}{\gamma}$.

Hence, here it is also sufficient to verify that a solution u of $Lu = f$ satisfies (3.73). In fact, we will prove that, for all small enough $r > 0$ and all $z \in U$,

$$\operatorname*{osc}_{B_r(z)} u \leq Cr^\beta, \tag{3.74}$$

where $\beta = \beta(n, \lambda, q) \in (0, 1)$ and C does not depend on z, r (in fact, C will depend on n, λ, q as well as on $\|u\|_{L^2}$ and $\|f\|_{L^q}$). Indeed, (3.74) implies (3.73) because setting in (3.74) $r = \rho_k$ we obtain

$$\operatorname*{osc}_{B_{\rho_k}(z)} u \leq C\rho_k^\beta = C\left(\rho^\beta 3^{-k\beta}\right) = (C\rho^\beta)\gamma^k,$$

where $\gamma = 3^{-\beta}$. By the argument of Theorem 3.7, it follows that $u \in C^\alpha(\Omega)$ with $\alpha = \log_3 \frac{1}{\gamma} = \beta$.

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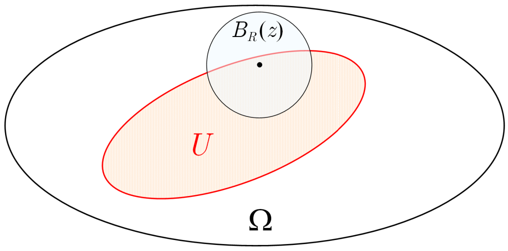
Hence, let us prove (3.74). First we choose some positive $R < \operatorname{dist}(U, \partial\Omega)$ so that $B_R := B_R(z) \subset \Omega$.



Let v be the solution of the Dirichlet problem in B_R:

$$\begin{cases} Lv = f \text{ weakly in } B_R \\ v \in W_0^{1,2}(B_R) \end{cases}$$

that exists by Theorem 1.2. Consider the difference

$$w := u - v \in W^{1,2}(B_R)$$

that satisfies

$$Lw = 0 \quad \text{weakly in } B_R.$$

Then $u = v + w$ and, hence, for any positive $r < R$,

$$\operatorname*{osc}_{B_r} u \le \operatorname*{osc}_{B_r} v + \operatorname*{osc}_{B_r} w.$$

Let us estimate the term $\operatorname{osc}_{B_r} v$ simply by $\|v\|_{L^\infty}$:

$$\operatorname*{osc}_{B_r} v \le 2\,\|v\|_{L^\infty(B_R)}\,.$$

Then we apply Theorem 1.15 to estimate $\|v\|_{L^\infty(B_R)}$ as follows:

$$\|v\|_{L^\infty(B_R)} \le C\,|B_R|^{\frac{2}{n}-\frac{1}{q}}\,\|f\|_{L^q(B_R)}$$

that is,

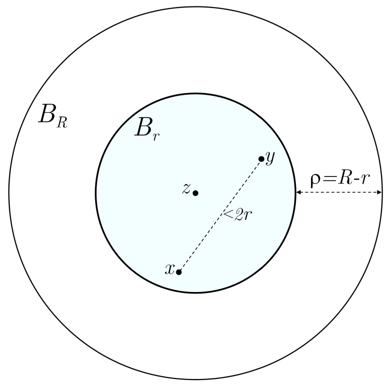
$$\boxed{\|v\|_{L^\infty(B_R)} \le CR^{2-\frac{n}{q}}\,\|f\|_{L^q(\Omega)}}, \tag{3.75}$$

where $C = C(n, \lambda, q)$.

Next, we estimate $\operatorname{osc}_{B_r} w$ by means of Corollary 3.9. By Theorem 3.7, we know that $w \in C^\alpha(B_R)$ where $\alpha = \alpha(n, \lambda) > 0$. Applying Corollary 3.9 to the subset B_r of B_R, we obtain that, for all $x, y \in B_r$,

$$|w(x) - w(y)| \le C\left(\frac{|x-y|}{\rho}\right)^\alpha \|w\|_{L^\infty(B_R)}\,, \tag{3.76}$$

where $\rho = \operatorname{dist}(B_r, \partial B_R) = R - r$ and $C = C(n, \lambda)$.



Assume further that $r \le R/2$ so that $\rho \ge \frac{1}{2}R$. Since $|x - y| < 2r$, we obtain from (3.76) that

$$\operatorname*{osc}_{B_r} w \le C\left(\frac{r}{R}\right)^{\alpha} \|w\|_{L^\infty(B_R)}.$$

Applying the same argument to $R/2$ instead of R, we obtain the following: if $r \le R/4$ then

$$\boxed{\operatorname*{osc}_{B_r} w \le C\left(\frac{r}{R}\right)^{\alpha} \|w\|_{L^\infty\left(B_{R/2}\right)}}.$$

Let us estimate $\|w\|_{L^\infty\left(B_{R/2}\right)}$ as follows. By the mean value inequality of Theorem 3.2, we have

$$\|w\|_{L^\infty\left(B_{R/2}\right)} \le \frac{C}{R^{n/2}} \|w\|_{L^2(B_R)}.$$

Since $w = u - v$, we have

$$\begin{aligned} \|w\|_{L^2(B_R)} &\le \|u\|_{L^2(B_R)} + \|v\|_{L^2(B_R)} \\ &\le \|u\|_{L^2(\Omega)} + CR^{n/2} \|v\|_{L^\infty(B_R)}, \end{aligned}$$

and, hence,

$$\boxed{\|w\|_{L^\infty\left(B_{R/2}\right)} \le \frac{C}{R^{n/2}} \|u\|_{L^2(\Omega)} + C \|v\|_{L^\infty(B_R)}}.$$

Combining the above inequalities, we obtain

$$\begin{aligned} \operatorname*{osc}_{B_r} u &\le \operatorname*{osc}_{B_r} v + \operatorname*{osc}_{B_r} w \\ &\le 2\|v\|_{L^\infty(B_R)} + C\left(\frac{r}{R}\right)^{\alpha} \left(\frac{C}{R^{n/2}} \|u\|_{L^2(\Omega)} + \|v\|_{L^\infty(B_R)}\right) \\ &\le C\|v\|_{L^\infty(B_R)} + C\left(\frac{r}{R}\right)^{\alpha} \frac{1}{R^{n/2}} \|u\|_{L^2(\Omega)}. \end{aligned} \tag{3.77}$$

Finally, substituting (3.75) into (3.77), we obtain

$$\boxed{\operatorname*{osc}_{B_r} u \le CR^{2-\frac{n}{q}} \|f\|_{L^q(\Omega)} + C\left(\frac{r}{R}\right)^{\alpha} \frac{1}{R^{n/2}} \|u\|_{L^2(\Omega)}}. \tag{3.78}$$

Let us emphasize that that $C = C(n, \lambda, q)$ and the norms of f and u here do not depend on R, r, z (in contrast to the norms of v and w from the previous estimates).

So far R and r are arbitrary positive numbers such that

$$R < \operatorname{dist}(U, \partial\Omega) \quad \text{and} \quad r \leq R/4. \tag{3.79}$$

Now, for any $r > 0$, we choose $R = R(r)$ so that

$$R^{2-n/q} = \left(\frac{r}{R}\right)^{\alpha} \frac{1}{R^{n/2}},$$

that is,

$$R = r^{\frac{\alpha}{2-n/q+\alpha+n/2}}.$$

Note that $2 - n/q > 0$ as $q > n/2$. Clearly, we have

$$0 < \frac{\alpha}{2 - n/q + \alpha + n/2} < 1.$$

Therefore, if $r \to 0$ then $R \to 0$ and $R/r \to \infty$. Hence, if r is small enough then the both conditions (3.79) are satisfied. For these values of r and R, we obtain from (3.78) that, for any $z \in U$,

$$\begin{aligned} \operatorname*{osc}_{B_r(z)} u &\leq CR^{2-\frac{n}{q}} \left(\|f\|_{L^q(\Omega)} + \|u\|_{L^2(\Omega)} \right) \\ &= Cr^{\beta} \left(\|f\|_{L^q(\Omega)} + \|u\|_{L^2(\Omega)} \right), \end{aligned}$$

where

$$\beta = \frac{(2 - n/q)\,\alpha}{2 - n/q + \alpha + n/2} > 0,$$

thus proving (3.74). ∎

3.6 Applications to semi-linear equations

Consider a divergence form uniformly elliptic operator

$$Lu = \sum_{i,j=1}^{n} \partial_i \left(a_{ij} \partial_j u\right)$$

in a bounded domain Ω with measurable coefficients. Given a function $f(x, v)$ on $\Omega \times \mathbb{R}$, consider the following *semi-linear* Dirichlet problem

$$\begin{cases} Lu = f(x, u) \text{ in } \Omega \\ u \in W_0^{1,2}(\Omega), \end{cases} \tag{3.80}$$

where the operator Lu is understood weakly as before. We assume that function f is such that the composition $f(x, u(x))$ belongs to $L^2(\Omega)$ whenever $u \in L^2(\Omega)$. Our goal is to investigate the solvability of the problem (3.80).

For that, fix first a function $v \in L^2(\Omega)$ and consider the following linear Dirichlet problem

$$\begin{cases} Lu = f(x, v) \text{ in } \Omega \\ u \in W_0^{1,2}(\Omega), \end{cases} \tag{3.81}$$

By Theorem 1.2, it has a unique solution u. Hence, we obtain the mapping

$$T : L^2(\Omega) \to L^2(\Omega)$$
$$T(v) = u.$$

The problem (3.80) amounts then to solving of the equation $T(u) = u$. Hence, we face the problem of finding a *fixed point* of the mapping T.

3.6.1 Fixed point theorems

Let us discuss some fixed point theorems, that is, the statements that ensure the existence of a fixed point of a mapping under certain hypotheses. In this section X is a Banach space. We use the following theorem without proof.

Theorem 3.13 *(*Fixed point theorem of Schauder*) Let K be a compact convex subset of a Banach space X. If $T : K \to K$ is a continuous mapping then T has a fixed point, that is, there exists a point $x \in K$ such that $T(x) = x$.*

If $X = \mathbb{R}^n$ then then K can be any bounded closed convex subset of $\mathbb{R}^n$. In this case Theorem 3.13 is referred to as the fixed point theorem of Brouwer. In fact, theorem of Schauder is normally proved by using theorem of Brouwer and finite dimensional approximations of K.

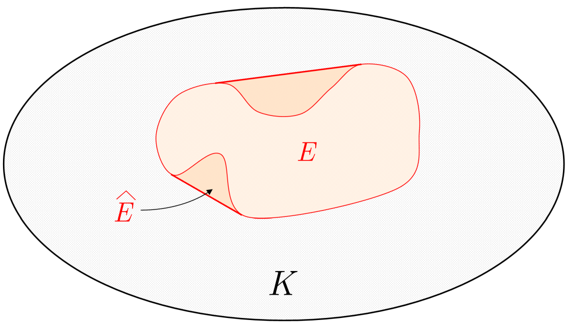
The following is an *alternative version* of the fixed point theorem of Schauder that we prove using Theorem 3.13.

Theorem 3.14 *Let K be a closed convex subset of a Banach space X and $T : K \to K$ is a continuous mapping such that the image $T(K)$ is precompact. Then T has a fixed point.*

Proof. Denote $E = T(K)$ so that E is a precompact subset of K. Let $\widehat{E}$ denote the convex hull of E, that is, $\widehat{E}$ consists of all finite convex combinations of the points of E, that is

$$\widehat{E} = \left\{ \sum_{k=1}^{n} c_k x_k : n \in \mathbb{N},\ x_k \in E,\ c_k \geq 0,\ \sum_{k=1}^{n} c_k = 1 \right\}.$$

In fact, $\widehat{E}$ is the minimal convex set that contains E.



We will show below that $\widehat{E}$ is also precompact. Since $E \subset K$ and K is convex, we have $\widehat{E} \subset K$. Since K is closed, the closure $\overline{\widehat{E}}$ is contained in K.

The restricted mapping $T|_{\overline{\widehat{E}}}$ has the image in $E \subset \overline{\widehat{E}}$ so that $T|_{\overline{\widehat{E}}}$ can be regarded as a mapping from $\overline{\widehat{E}}$ to itself. Since $\overline{\widehat{E}}$ is a compact convex set, we obtain by Theorem 3.13 that $T|_{\overline{\widehat{E}}}$ has a fixed point, which finishes the proof.

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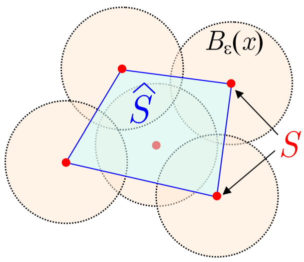
It remains to prove that $\widehat{E}$ is precompact. Since E is precompact, there exists for any $\varepsilon > 0$ a finite ε-net S, that is, a finite sequence S of points in E such that

$$E \subset \bigcup_{x \in S} B_\varepsilon(x). \tag{3.82}$$

It follows that

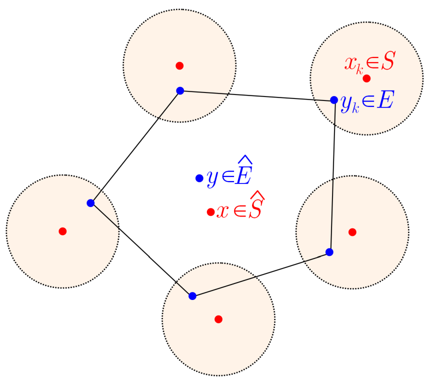
$$\widehat{E} \subset \bigcup_{x \in \widehat{S}} B_\varepsilon(x). \tag{3.83}$$



Indeed, every point $y \in \widehat{E}$ is a convex combination of some points $\{y_k\}$ of E, that is,

$$y = \sum_k c_k y_k,$$

where $y_k \in E$, $c_k \geq 0$, $\sum_k c_k = 1$. By (3.82), each y_k lies in some ball $B_\varepsilon(x_k)$ with $x_k \in S$.



Then the point

$$x := \sum_k c_k x_k$$

belongs to $\widehat{S}$ and

$$\|y-x\| = \left\|\sum_k c_k y_k - \sum_k c_k x_k\right\| = \left\|\sum_k c_k (y_k - x_k)\right\| < \sum_k c_k \varepsilon = \varepsilon,$$

that is, $y \in B_\varepsilon(x)$, which proves (3.83).

The set $\widehat{S}$ is in general infinite, and we need to replace it in (3.83) by a finite set in order to obtain a finite ε-net. Since the sequence S is finite, its convex hull $\widehat{S}$ is a bounded subset of a finite dimensional subspace of X. Therefore, $\widehat{S}$ is precompact and, hence, there exists a finite ε-net Z of $\widehat{S}$. In particular, each $x \in \widehat{S}$ lies in some ball $B_\varepsilon(z)$ with $z \in Z$, which implies

$$B_\varepsilon(x) \subset B_{2\varepsilon}(z).$$

It follows that

$$\widehat{E} \subset \bigcup_{z\in Z} B_{2\varepsilon}(z),$$

that is Z is a finite 2ε-net of $\widehat{E}$, which proves that $\widehat{E}$ is precompact. ■

Definition. A mapping $T : X \to X$ is called *compact* if, for any bounded set $E \subset X$, the image $T(E)$ is precompact.

Note that if T is linear and compact then T is also bounded and, hence, continuous. However, in general a compact mapping T does not have to be continuous.

Theorem 3.15 *(*Fixed point theorem of Leray-Schauder*) Let $T : X \to X$ be a compact, continuous mapping. Assume that*

$$\textit{the set } \{x \in X : x = \alpha T(x) \textit{ for some } \alpha \in (0,1)\} \textit{ is bounded.} \tag{LS}$$

Then T has a fixed point.

The condition (*LS*) is called the Leray-Schauder condition.

Remark. Let us say that $x \in X$ is an eigenvector of T if $T(x) = \lambda x$ for some $\lambda \in \mathbb{R}$, where λ is called an eigenvalue. The condition (*LS*) means that all the eigenvectors of T with eigenvalues > 1 are uniformly bounded.

Example. Consider an affine mapping $T(x) = x + b$ with some non-zero $b \in X$. The equation $x = \alpha T(x)$ is equivalent to $x = \alpha(x + b)$, that is, to

$$x = \frac{\alpha b}{1-\alpha}.$$

This can be satisfied with any $\alpha \in (0,1)$, and the norm of x is clearly unbounded as $\alpha \to 1$. Hence, condition (*LS*) fails. Obviously, T has no fixed point in this case.

Example. Let $T(x)$ be a continuous function on $X = \mathbb{R}$ that satisfies the condition (*LS*). Let us prove directly that T has a fixed point. By (*LS*), there exists $R > 0$ such that

$$\text{if } x = \alpha T(x) \text{ for some } \alpha \in (0,1) \text{ then } |x| < R. \tag{3.84}$$

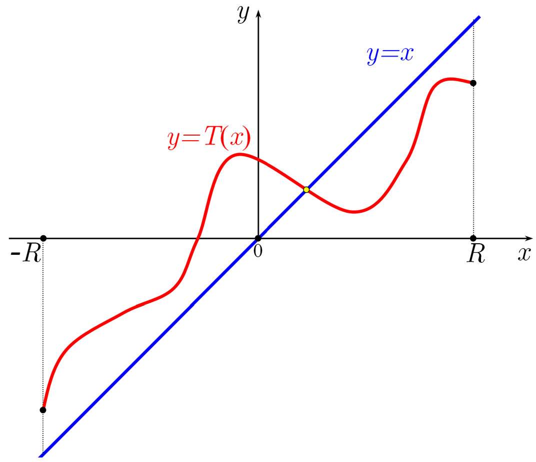
We claim that in this case

$$T(R) \le R \quad \text{and} \quad T(-R) \ge -R. \tag{3.85}$$

Indeed, if $T(R) > R$ then $R = \alpha T(R)$ with some $\alpha \in (0,1)$. Using (3.84) we obtain $|R| < R$, which is wrong.

If $T(-R) < -R$ then $(-R) = \alpha T(-R)$ with some $\alpha \in (0,1)$, whence by (3.84) $|R| < -R$.

This contradiction proves (3.85).



Next, consider the function $f(x) = x - T(x)$. It follows from (3.85) that

$$f(R) \ge 0 \quad \text{and} \quad f(-R) \le 0,$$

which implies by the intermediate value theorem that $f(x) = 0$ for some $x \in [-R, R]$, that is, $T(x) = x$.

Proof of Theorem 3.15. The condition (*LS*) means that there $R > 0$ such that

$$\text{if } x = \alpha T(x) \text{ for some } \alpha \in (0,1) \text{ then } \|x\| < R.$$

By dividing the norm in X by R, we can assume without loss of generality that $R = 1$. Hence, we assume that

$$\text{if } x = \alpha T(x) \text{ for some } \alpha \in (0,1) \text{ then } \|x\| < 1. \tag{3.86}$$

Consider a mapping $S : X \to X$ defined by

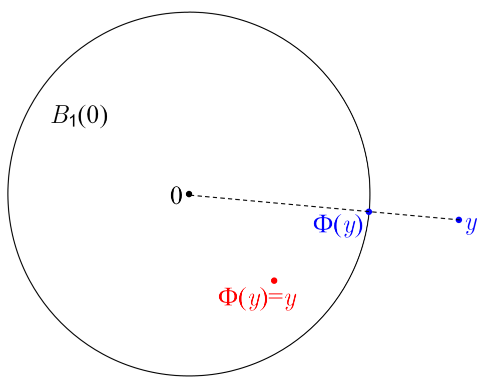
$$S(x) = \begin{cases} T(x), & \text{if } \|T(x)\| \le 1 \\ \frac{T(x)}{\|T(x)\|}, & \text{if } \|T(x)\| > 1. \end{cases} \tag{3.87}$$

Consequently, $\|S(x)\| \le 1$ for all $x \in X$. We claim that S is continuous and compact. To see that, let use represent S in the form of composition

$$S = \Phi \circ T,$$

where $\Phi : X \to X$ is defined by

$$\Phi(y) = \begin{cases} y, & \text{if } \|y\| \le 1, \\ \frac{y}{\|y\|}, & \text{if } \|y\| > 1. \end{cases}$$



Mapping Φ is continuous because

$$\Phi(y) = \varphi(\|y\|)\, y$$

where φ is the following function on $[0,\infty)$:

$$\varphi(t)=\begin{cases}1 & t\leq 1,\\ \frac{1}{t}, & t>1.\end{cases}$$

Since φ is continuous, Φ is also continuous, whence also $S=\Phi\circ T$ is continuous.

Let us show that S is compact. Since T is compact, for any bounded set $E\subset X$, the image $T(E)$ is precompact. Since Φ is continuous, the set $S(E)=\Phi(T(E))$ is precompact as a continuous image of a precompact set. Hence, S is compact.

Denote by B the closed unit ball of radius 1 in X centered at the origin. Since $\|S(x)\|\leq 1$ for all $x\in X$, we have $S(X)\subset B$ and, in particular, $S(B)\subset B$. Hence, S can be regarded as a mapping from B to B. Since B is convex and closed, and $S(B)$ is precompact, we obtain by Theorem 3.14 that S has a fixed point $x\in B$.

Let us verify that x is also a fixed point of T. Indeed, if $T(x)\in B$ then by (3.87) $T(x)=S(x)$ and, hence, $T(x)=x$. Assume now that $T(x)\notin B$, that is, $\|T(x)\|>1$. In this case we obtain from (3.87)

$$x=S(x)=\frac{T(x)}{\|T(x)\|}, \tag{3.88}$$

that is, $x=\alpha T(x)$ where $\alpha=\frac{1}{\|T(x)\|}<1$. By (3.86) we obtain $\|x\|<1$, whereas (3.88) implies $\|x\|=1$. This contradiction shows that the case $T(x)\notin B$ is impossible, which finishes the proof. ∎

3.6.2 A semi-linear Dirichlet problem

Consider a divergence form uniformly elliptic operator

$$Lu=\sum_{i,j=1}^{n}\partial_i\left(a_{ij}\partial_j u\right)$$

in a bounded domain Ω with measurable coefficients, and the following *semi-linear* Dirichlet problem

$$\begin{cases}Lu=f(x,u)\text{ in }\Omega\\ u\in W_0^{1,2}(\Omega),\end{cases} \tag{SD}$$

where the operator Lu is understood weakly as before. Function $f=f(x,u)$ is defined in $\Omega\times\mathbb{R}$, and we assume that it is Borel measurable. Then, for any measurable function u on Ω, the composite function $f(x,u(x))$ is also measurable.

We assume in addition that f satisfies the following two conditions:

$$|f(x,v)|\leq C\left(1+|v|^{\gamma}\right), \tag{3.89}$$

for all $v\in\mathbb{R}$ and $x\in\Omega$, and

$$|f(x,v_1)-f(x,v_2)|\leq C\,|v_1-v_2| \tag{3.90}$$

for all $v_1,v_2\in\mathbb{R}$ and $x\in\Omega$, where γ,C are positive constants.

Theorem 3.16 *Assume that the above hypotheses* (3.89) *and* (3.90) *hold with* $\gamma < 1$. *Then the following is true.*

(*a*) *The problem* (SD) *has a solution* u.

(*b*) *If in addition* $|\Omega|$ *is small enough then the solution* u *is unique.*

(*c*) *If in addition* $\gamma < \frac{4}{n}$ *then* $u \in C^{\beta}(\Omega)$ *for some* $\beta = \beta(n, \lambda, \gamma) > 0$.

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Remark. In part (*b*), without restriction on $|\Omega|$ there is no uniqueness for the problem (SD). Indeed, even in the one dimensional case, the Dirichlet problem

$$\begin{cases} u'' = -u \\ u(0) = u(\pi) = 0 \end{cases}$$

has two solutions $u \equiv 0$ and $u(x) = \sin x$. Although the function $f(x, u) = -u$ does not satisfy (3.89), it is easy to modify it to satisfy (3.89) with any $\gamma > 0$:

$$f(x, u) := -\min(u_+, 1).$$

Then the problem

$$\begin{cases} u'' = f(x, u) \\ u(0) = u(\pi) = 0 \end{cases}$$

still has two solutions $u \equiv 0$ and $u(x) = \sin x$ because both solutions take values in $[0, 1]$, and for $u \in [0, 1]$ we have $f(x, u) = -u$.

Similarly, if Ω is a bounded domain in $\mathbb{R}^n$ and u is an eigenfunction of the Laplace operator in Ω, that is,

$$\begin{cases} \Delta u = -\lambda u \text{ in } \Omega, \\ u \in W_0^{1,2}(\Omega), \end{cases} \tag{3.91}$$

then we obtain again an example of non-uniqueness because $u \not\equiv 0$ and the problem (3.91) has also a solution $u \equiv 0$.

Remark. In part (*c*), the restriction $\gamma < 4/n$ is not optimal. In fact, if (3.89) holds with $\gamma \leq 1$ then any solution u of (SD) is Hölder continuous (see Exercise 70). In particular, all the eigenfunctions of L are Hölder continuous (see Exercise 59). On the other hand, if $\gamma > \frac{n}{n-4}$ then solution u does not have to be continuous (see Exercise 61).

Proof of Theorem 3.16. For any $v \in L^2(\Omega)$, the function

$$F_v(x) := f(x, v(x)) \tag{3.92}$$

belongs to $L^2(\Omega)$, because by (3.89) and $\gamma < 1$

$$|F_v(x)| \leq C(1 + |v|^{\gamma}) \leq C(2 + |v|) \in L^2(\Omega). \tag{3.93}$$

(*a*) For any $v \in L^2(\Omega)$, consider the following *linear* Dirichlet problem

$$\begin{cases} Lu = F_v \text{ in } \Omega \\ u \in W_0^{1,2}(\Omega) \end{cases} \tag{3.94}$$

that has a unique solution u by Theorem 1.2. Define the mapping

$$T \ : \ L^2(\Omega) \to L^2(\Omega)$$
$$T(v) = u$$

that is, for any $v \in L^2(\Omega)$, the function $T(v)$ is defined as the solution u of (3.94) considered as an element of $L^2(\Omega)$. Clearly, if u solves (SD) then

$$T(u) = u.$$

Conversely, if $u \in L^2(\Omega)$ is a fixed point of T, then necessarily $u \in W_0^{1,2}(\Omega)$ because the range of T lies in $W_0^{1,2}(\Omega)$, and u solves the equation $Lu = F_u$, which is equivalent to (SD).

Hence, the existence of solution of (SD) is equivalent to the existence of a fixed point of the mapping $T : L^2(\Omega) \to L^2(\Omega)$. Let us first prove that T is continuous and compact. Clearly, T is the composition of the following mappings:

$$\begin{array}{ccc} L^2(\Omega) & \to & L^2(\Omega) \to W_0^{1,2}(\Omega) \to L^2(\Omega) \\ v & \mapsto & F_v \quad\quad F_v \mapsto u \quad\quad\quad u \mapsto u \end{array}$$

where u is the solution of the Dirichlet problem (3.94).

The mapping $v \mapsto F_v$, given by (3.92), is continuous because by (3.90)

$$\|F_{v_1} - F_{v_2}\|_{L^2} \leq C \|v_1 - v_2\|_{L^2}. \tag{3.95}$$

Besides, the mapping $v \mapsto F_v$ is bounded in the sense that image of any bounded set is bounded, because by (3.93)

$$\|F_v\|_{L^2} \leq C + C \|v\|_{L^2}.$$

By the properties of the linear Dirichlet problem (3.94), the mapping $F_v \mapsto u$ is linear and bounded because

$$\|u\|_{W^{1,2}} \leq C \|F_v\|_{L^2}, \tag{3.96}$$

where $C = C(\lambda, \operatorname{diam}(\Omega))$ (cf. Exercise 22), which implies that it is continuous.

Finally, the identical mapping $u \mapsto u$ from $W_0^{1,2}(\Omega)$ to L^2 is continuous and compact, the latter by the compact embedding theorem. Hence, we conclude that T is continuous as a composition of continuous mappings, and compact as a composition of bounded and compact mappings.

In order to apply Leray-Schauder theorem for existence of a fixed point of T, we need to prove that, for some $R > 0$,

$$\text{if } v = \alpha T(v) \text{ for some } \alpha \in (0,1) \ \text{ then } \ \|v\| \leq R.$$

Since the function $u = T(v)$ solves the Dirichlet problem

$$\begin{cases} Lu = F_v \ \text{ in } \Omega, \\ u \in W_0^{1,2}(\Omega), \end{cases}$$

and $v = \alpha T(v) = \alpha u$, we obtain by (3.96)

$$\|v\|_{W^{1,2}} = \alpha \|u\|_{W^{1,2}} \leq \alpha C \|F_v\|_{L^2} \leq C \|F_v\|_{L^2},$$

whence

$$\int_\Omega v^2 dx \leq C \int_\Omega F_v^2 dx.$$

On the other hand, it follows from (3.93) that

$$\int_\Omega F_v^2 dx \leq C \int_\Omega (1 + |v|^\gamma)^2 dx \leq C + C \int_\Omega |v|^{2\gamma} dx,$$

where the value of the constant C is changed at each occurrence. Hence, we obtain

$$\int_\Omega v^2 dx \leq C + C \int_\Omega |v|^{2\gamma} dx. \tag{3.97}$$

By Young's inequality, we have, for any $\varepsilon > 0$,

$$|v|^{2\gamma} = \frac{1}{\varepsilon} \varepsilon v^{2\gamma} \leq \frac{1}{\varepsilon^p} + \left(\varepsilon v^{2\gamma}\right)^q$$

where p, q is a pair of Hölder conjugate exponents. Choose $q = \frac{1}{\gamma}$ and, hence, $p = \frac{1}{1-\gamma}$, so that

$$|v|^{2\gamma} \leq \frac{1}{\varepsilon^p} + \varepsilon^q v^2$$

and, hence,

$$\int_\Omega |v|^{2\gamma} dx \leq C_\varepsilon + \varepsilon^q \int_\Omega v^2 dx.$$

Substitution into (3.97) yields

$$\int_\Omega v^2 dx \leq C_\varepsilon + C\varepsilon^q \int_\Omega v^2 dx.$$

Choosing ε so small that $\varepsilon^q \leq \frac{1}{2C}$, we obtain

$$\int_\Omega v^2 dx \leq 2C_\varepsilon,$$

that is, $\|v\|_{L^2} \leq R := \sqrt{2C_\varepsilon}$.

By a fixed point Theorem of Leray-Schauder we conclude that T has a fixed point and, hence, the Dirichlet problem (SD) has a solution.

(b) Let us show that if $|\Omega|$ is small enough then the mapping T is a contraction in $L^2(\Omega)$. This will imply by the Banach fixed point theorem that T has a unique fixed point, that is, both uniqueness and existence. Let v_1 and v_2 be two functions from $L^2(\Omega)$, set

$$u_1 = T(v_1) \text{ and } u_2 = T(v_2).$$

We need to prove that, for some $\theta < 1$,

$$\|u_1 - u_2\| \leq \theta \|v_1 - v_2\|.$$

The function $u = u_1 - u_2$ satisfies the equation

$$Lu = Lu_1 - Lu_2 = f(x, v_1) - f(x, v_2),$$

that is, for any $\varphi \in W_0^{1,2}(\Omega)$, we have

$$\int_\Omega \sum_{i,j=1}^n a_{ij}\partial_j u\, \partial_i \varphi\, dx = -\int_\Omega (f(x, v_1) - f(x, v_2))\, \varphi dx. \tag{3.98}$$

By (3.90) we have

$$|f(x, v_1) - f(x, v_2)| \le C |v_1 - v_2|.$$

Hence, setting in (3.98) $\varphi = u$ and using the uniform ellipticity of (a_{ij}), we obtain

$$\lambda^{-1} \int_\Omega |\nabla u|^2\, dx \le C \int_\Omega |v_1 - v_2|\, |u|\, dx. \tag{3.99}$$

By the Faber-Krahn inequality, we have

$$\int_\Omega |\nabla u|^2\, dx \ge c_n |\Omega|^{-2/n} \int_\Omega u^2 dx = c_n |\Omega|^{-2/n} \|u\|_{L^2}^2.$$

On the other hand, by the Cauchy-Schwarz inequality,

$$\int_\Omega |v_1 - v_2|\, |u|\, dx \le \|v_1 - v_2\|_{L^2} \|u\|_{L^2}.$$

Substituting into (3.99), we obtain

$$|\Omega|^{-2/n} \|u\|_{L^2}^2 \le C \|v_1 - v_2\|_{L^2} \|u\|_{L^2},$$

where C depends on λ, n and on the constant in (3.90). It follows that

$$\|u_1 - u_2\|_{L^2} = \|u\|_{L^2} \le C |\Omega|^{2/n} \|v_1 - v_2\|_{L^2},$$

If $|\Omega|$ is small enough then $\theta := C |\Omega|^{2/n} < 1$, that is, T is a contraction, which was to be proved.

(c) By Theorem 3.12, a solution of (SD) is Hölder continuous, provided $F_u \in L^q(\Omega)$ with

$$q \in [2, \infty] \cap (n/2, \infty]. \tag{3.100}$$

We have by (3.89)

$$\int_\Omega |F_u|^q\, dx \le C \int_\Omega (1 + |u|^\gamma)^q dx \le C + C \int_\Omega |u|^{\gamma q}\, dx.$$

Since $u \in L^2(\Omega)$, we see that $\int_\Omega |u|^{\gamma q}\, dx < \infty$ provided $\gamma q = 2$. Set $q = 2/\gamma$ and verify that this q satisfies (3.100). Indeed, we have $q > 2$ because $\gamma < 1$, and $q > n/2$ because $\gamma < 4/n$. Hence, q satisfies (3.100), and we conclude that $u \in C^\beta(\Omega)$ with some $\beta = \beta(n, \lambda, \gamma) > 0$. ■

Chapter 4

Boundary behavior of solutions

Consider again in a bounded domain $\Omega \subset \mathbb{R}^n$ the weak Dirichlet problem

$$\begin{cases} Lu = f, \\ u \in W_0^{1,2}(\Omega), \end{cases}$$

where

$$Lu = \sum_{i,j=1}^{n} \partial_i \left(a_{ij} \partial_j u \right) \tag{4.1}$$

is a uniformly elliptic operator in Ω with measurable coefficients. We know by Theorem 3.12 that if $f \in L^q(\Omega)$ where

$$q \in [2, +\infty] \cap (n/2, \infty],$$

then $u \in C^\beta(\Omega)$ with some $\beta > 0$, in particular, u is continuous in Ω. We can ask if u takes the boundary value in the classical sense, that is, if, for a given point $x_0 \in \partial\Omega$,

$$\lim_{\substack{x \to x_0 \\ x \in \Omega}} u(x) = 0.$$

The answer to this question depends in the properties of the boundary $\partial\Omega$ near x_0.

The aim of this Chapter is to prove the following: if $\partial\Omega$ is "good" enough in some sense then, in fact, $u \in C(\overline{\Omega})$ and $u = 0$ on $\partial\Omega$ in the classical sense.

25.01.24 Lecture 27

Let Ω be a bounded domain of $\mathbb{R}^n$ and let Γ be a subset of the boundary $\partial\Omega$.

Definition. Let $u \in W^{1,2}(\Omega)$. We say that "$u = 0$ weakly on Γ" if there exists a sequence $\{\varphi_k\} \subset C^1(\overline{\Omega})$ such that

$$\varphi_k \overset{W^{1,2}(\Omega)}{\longrightarrow} u \;\text{ and }\; \varphi_k|_\Gamma = 0. \tag{4.2}$$

As a motivation for this definition, let us prove the following statement.

Claim. *If $u \in W^{1,2}(\Omega)$ then $u = 0$ weakly on $\partial\Omega \Leftrightarrow u \in W_0^{1,2}(\Omega)$.*

Proof. Indeed, if there exists a sequence $\{\varphi_k\}$ as in (4.2) then $\varphi_k|_{\partial\Omega} = 0$, which implies that $\varphi_k \in W_0^{1,2}(\Omega)$ (Exercise 28). Since $\varphi_k \overset{W^{1,2}(\Omega)}{\to} u$ it follows that also $u \in W_0^{1,2}(\Omega)$.

Conversely, if $u \in W_0^{1,2}(\Omega)$ then there exists a sequence $\{\varphi_k\} \subset C_0^\infty(\Omega) \subset C^1(\overline{\Omega})$ such that $\varphi_k \overset{W^{1,2}(\Omega)}{\to} u$. Since all φ_k vanish on $\partial\Omega$, we obtain that $u = 0$ weakly on $\partial\Omega$. ■

In this Chapter we will prove results of the following type: if $u \in W^{1,2}(\Omega)$, $Lu = f$ weakly in Ω and $u = 0$ weakly on Γ then, under certain assumptions about Γ and f, $u \in C(\Omega \cap \Gamma)$ and $u|_\Gamma = 0$.

4.1 Flat boundary

We use here the following subsets of $\mathbb{R}^n$: the upper and lower halfspaces

$$\mathbb{R}_+^n = \{x \in \mathbb{R}^n : x_n > 0\} \quad \text{and} \quad \mathbb{R}_-^n = \{x \in \mathbb{R}^n : x_n < 0\},$$

both being open subsets of $\mathbb{R}^n$. Their common boundary is the hyperplane

$$H = \{x \in \mathbb{R}^n : x_n = 0\},$$

that is a subspace of $\mathbb{R}^n$ isomorphic to $\mathbb{R}^{n-1}$.

Consider an open set $\Omega_+ \subset \mathbb{R}_+^n$ such that a part of the boundary $\partial\Omega_+$ lies on the hyperplane H. More precisely, assume that there is a non-empty open subset Γ of H (considering H as $\mathbb{R}^{n-1}$) such that $\partial\Omega_+ \cap H = \overline{\Gamma}$.

Let L be a uniformly elliptic operator in Ω_+ with measurable coefficients. Assume that u satisfies the following conditions:

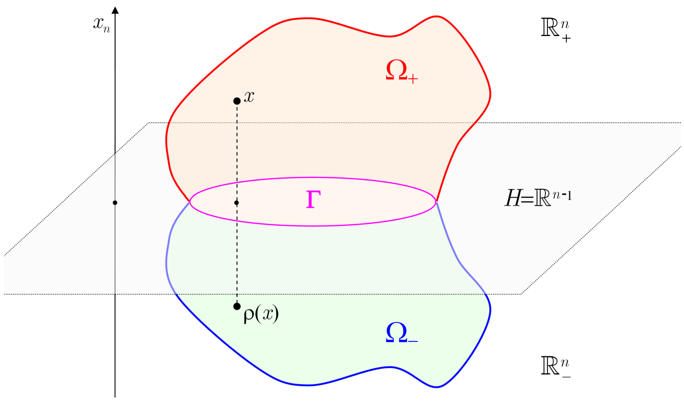
$$\begin{cases} Lu = f \text{ weakly in } \Omega_+, \\ u \in W^{1,2}(\Omega_+), \\ u = 0 \text{ weakly on } \Gamma, \end{cases} \tag{4.3}$$

where so far $f \in L^2(\Omega_+)$. We will investigate the (Hölder) continuity of u at Γ.

Define a *mirror reflection* in H as a mapping $\rho : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\rho(x_1, ..., x_{n-1}, x_n) = (x_1, ..., x_{n-1}, -x_n).$$

Clearly, ρ is involution, that is, $\rho^{-1} = \rho$, and any point of H is a fixed point of ρ.



Set $\Omega_- = \rho(\Omega_+)$ so that Ω_- is an open subset of $\mathbb{R}^n_-$. Observe that the set Γ is contained in the both boundaries $\partial\Omega_+$ and $\partial\Omega_-$. Consider the set

$$\Omega = \Omega_+ \cup \Omega_- \cup \Gamma$$

that is an open subset of $\mathbb{R}^n$ invariant under the mapping ρ. Note that all points of Γ are interior points of Ω.

Our plan is to extends u, f, L from Ω_+ to Ω, so that the problem amounts to investigation of the continuity of u at interior points of Ω, which can be handled by means of Theorem 3.12.

A function $v : \Omega \to \mathbb{R}$ is called *even* if

$$v(\rho(x)) = v(x) \text{ for all } x \in \Omega,$$

and *odd* if

$$v(\rho(x)) = -v(x) \text{ for all } x \in \Omega.$$

Note that an odd function vanishes at Γ.

Any function $v : \Omega_+ \to \mathbb{R}$ admits even and odd extensions to the entire set Ω. Indeed, to extend v to Ω oddly, we set

$$v(\rho(x)) = -v(x) \text{ for all } x \in \Omega_+$$

and $v(x) = 0$ for $x \in \Gamma$; to extend v evenly, we set

$$v(\rho(x)) = v(x) \text{ for all } x \in \Omega_+$$

whereas the values of $v(x)$ for $x \in \Gamma$ can be chosen arbitrarily.

Let us extend both functions u and f to Ω oddly. To extend the coefficients of L, we use the following rules:

(i) a_{ij} extends to Ω evenly if $i, j < n$ or $i = j = n$;

(ii) a_{ij} extends to Ω oddly if $i < n$, $j = n$ or $i = n$, $j < n$;

(iii) for any $x \in \Gamma$ set $(a_{ij}(x)) = \mathrm{id}$.

In other words, a_{ij} extends evenly if the number of values n in the pair (i, j) is even, and oddly otherwise.

Note that $a_{ij} = 0$ on Γ for the values i, j as in (ii); since in this case $i \neq j$, the vanishing on a_{ij} on Γ is compatible with $(a_{ij}) = \mathrm{id}$ as is required by (iii).

In order to formalize the above rules, let us use the following notation:

$$\sigma_i = \begin{cases} 1, & i < n, \\ -1, & i = n. \end{cases}$$

It follows from (i)-(ii) that

$$a_{ij}(\rho(x)) = \sigma_i \sigma_j a_{ij}(x) \text{ for all } x \in \Omega_+. \tag{4.4}$$

Hence, the operator L as well as the functions u and f are now defined on Ω.

Theorem 4.1 (Extension of (4.3) under reflection) *Let u satisfy (4.3) in Ω_+. Then the extended L, u, f satisfy the following conditions:*

(a) *the operator L is uniformly elliptic in Ω;*

(b) *$u \in W^{1,2}(\Omega)$;*

(c) *$Lu = f$ weakly in Ω.*

Proof. (a) In view of (4.4), in order to prove that L is uniformly elliptic, it suffices to prove the following: if (a_{ij}) is a symmetric matrix such that, for any $\xi \in \mathbb{R}^n$,

$$\lambda^{-1} |\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \leq \lambda |\xi|^2 \tag{4.5}$$

then the same holds for the matrix $(\sigma_i\sigma_j a_{ij})$. We have

$$\sum_{i,j=1}^{n} (\sigma_i\sigma_j a_{ij}) \xi_i\xi_j = \sum_{i,j=1}^{n} a_{ij}\eta_i\eta_j,$$

where $\eta_i = \sigma_i\xi_i$, that is, $\eta = \left(\xi_1, ..., \xi_{n-1}, -\xi_n\right)$. By (4.5) we have

$$\lambda^{-1} |\eta|^2 \leq \sum_{i,j=1}^{n} a_{ij}\eta_i\eta_j \leq \lambda |\eta|^2 . \tag{4.6}$$

Since $|\eta| = |\xi|$, we obtain

$$\lambda^{-1} |\xi| \leq \sum_{i,j=1}^{n} (\sigma_i\sigma_j a_{ij}) \xi_i\xi_j \leq \lambda |\xi|^2 ,$$

which proves the uniform ellipticity of $(\sigma_i\sigma_j a_{ij})$, with the same ellipticity constant λ.

(b) Since $u = 0$ weakly on Γ, there exists a sequence $\{\varphi_k\} \subset C^1(\overline{\Omega}_+)$ such that

$$\varphi_k \overset{W^{1,2}(\Omega_+)}{\to} u \text{ and } \varphi_k|_\Gamma = 0.$$

Let us extend φ_k to Ω oddly. Then the condition $\varphi_k|_\Gamma = 0$ implies that $\varphi_k \in C^1(\overline{\Omega})$. Observe that the derivative $\partial_i\varphi_k$ extends oddly to Ω if $i < n$, and evenly if $i = n$.

Since u also extends oddly, we have

$$\varphi_k \overset{L^2(\Omega)}{\to} u. \tag{4.7}$$

Denote $v_i = \partial_{x_i} u$ in Ω_+ and extend v_i oddly to Ω if $i < n$, and evenly if $i = n$. Since

$$\partial_i\varphi_k \overset{L^2(\Omega_+)}{\to} v_i$$

we obtain that also

$$\partial_i\varphi_k \overset{L^2(\Omega)}{\to} v_i. \tag{4.8}$$

It follows from (4.7) and (4.8) that $v_i = \partial_i u$ in Ω, so that $u \in W^{1,2}(\Omega)$.

(c) Let us show that $Lu = f$ weakly in Ω, that is, for any $\varphi \in \mathcal{D}(\Omega)$,

$$\int_\Omega \left[\sum_{i,j=1}^n a_{ij}\partial_j u\, \partial_i \varphi + f\varphi\right] dx = 0. \tag{4.9}$$

We will reduce here the domain of integration Ω to Ω_+. For that we split the integral $\int_\Omega$ into the sum $\int_{\Omega_+} + \int_{\Omega_-}$, and in the integral $\int_{\Omega_-} ...dy$ we make change $y = \rho(x)$ where $x \in \Omega_+$, thus reducing it to an integral over Ω_+.

Using the substitution rule, we obtain

$$\int_{\Omega_-} f(y)\varphi(y)dy = \int_{\Omega_+} f(\rho(x))\,\varphi(\rho(x))\,|\det J_\rho|\, dx,$$

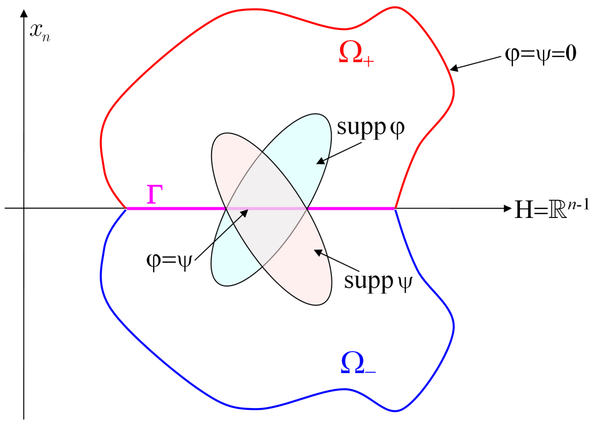
where $J_\rho = \frac{dy}{dx}$ is the Jacobi matrix of ρ. Since $J_\rho = \operatorname{diag}(1, ..., 1, -1)$, we have $\det J_\rho = -1$ and, hence,

$$\int_{\Omega_-} f(y)\varphi(y)dy = \int_{\Omega_+} f(\rho(x))\,\varphi(\rho(x))\, dx. \tag{4.10}$$

Consider the function

$$\psi(x) := \varphi(\rho(x)) = \varphi(x_1, ..., x_{n-1}, -x_n) \quad \text{for all } x \in \Omega, \tag{4.11}$$

that is a reflection of φ in H. Clearly, $\psi \in \mathcal{D}(\Omega)$.



Using that

$$f(\rho(x)) = -f(x),$$

we obtain

$$\int_{\Omega_-} f(y)\varphi(y)dy = -\int_{\Omega_+} f(x)\psi(x)dx.$$

It follows that

$$\int_\Omega f\varphi dx = \int_{\Omega_+} f\varphi dx + \int_{\Omega_-} f\varphi dy = \int_{\Omega_+} f\varphi dx - \int_{\Omega_+} f\psi dx = \int_{\Omega_+} f\,(\varphi - \psi)\, dx. \tag{4.12}$$

Similarly, let us reduce $\int_\Omega \sum_{i,j} a_{ij}\partial_j u\, \partial_i\varphi dx$ to an integral over Ω_+. As in (4.10) we have

$$\int_{\Omega_-} \sum_{i,j} \left(a_{ij}\partial_j u \partial_i \varphi\right)(y)dy = \int_{\Omega_+} \sum_{i,j} \left(a_{ij}\partial_j u\, \partial_i \varphi\right)(\rho(x))dx.$$

Let us compute all the derivatives in the right hand side using the chain rule. We obtain

$$\left(\partial_i \varphi\right)(\rho(x)) = \left(\partial_i \varphi\right)(x_1, ..., x_{n-1}, -x_n) = \sigma_i \partial_i \left[\varphi\left(x_1, ..., x_{n-1}, -x_n\right)\right] = \sigma_i \partial_i \psi(x)$$

and similarly

$$\left(\partial_j u\right)(\rho(x)) = \left(\partial_j u\right)(x_1, ..., x_{n-1}, -x_n) = \sigma_j \partial_j \left[u\left(x_1, ..., x_{n-1}, -x_n\right)\right] = -\sigma_j \partial_j u(x),$$

where we have used the fact that u is odd. Using also (4.4), we obtain

$$\left(a_{ij}\partial_j u\, \partial_i \varphi\right)(\rho(x)) = -\sigma_i\sigma_j a_{ij}(x)\sigma_j\partial_j u(x)\sigma_i\partial_i\psi(x) = -\left(a_{ij}\partial_j u \partial_i \psi\right)(x),$$

as $\sigma_i^2 = \sigma_j^2 = 1$. Hence, we obtain

$$\int_{\Omega_-} \left(a_{ij}\partial_j u\, \partial_i \varphi\right)(y)dy = -\int_{\Omega_+} \left(a_{ij}\partial_j u \partial_i \psi\right)(x)dx,$$

which implies

$$\begin{aligned}\int_\Omega \sum_{i,j} a_{ij}\partial_j u\, \partial_i\varphi dx &= \int_{\Omega_+} \sum_{i,j} a_{ij}\partial_j u\, \partial_i\varphi dx + \int_{\Omega_-} \sum_{i,j} a_{ij}\partial_j u\, \partial_i\varphi dy \\ &= \int_{\Omega_+} \sum_{i,j} a_{ij}\partial_j u\, \partial_i \left(\varphi - \psi\right) dx.\end{aligned}$$

Combining with (4.12), we obtain

$$\int_\Omega \left[\sum_{i,j} a_{ij}\partial_j u\, \partial_i\varphi + f\varphi\right] dx = \int_{\Omega_+} \left[\sum_{i,j} a_{ij}\partial_j u\, \partial_i \left(\varphi - \psi\right) + f\left(\varphi - \psi\right)\right] dx. \tag{4.13}$$

Let us verify that

$$\varphi - \psi \in W_0^{1,2}\left(\Omega_+\right).$$

Since $\varphi - \psi \in C^\infty\left(\overline{\Omega}_+\right)$, we have

$$\varphi - \psi \in W^{1,2}\left(\Omega_+\right).$$

We claim that $\varphi - \psi = 0$ on $\partial\Omega_+$. Indeed, for $x \in \Gamma$ we have by (4.11)

$$\psi(x) = \varphi(\rho(x)) = \varphi(x),$$

that is, $\varphi = \psi$ in Γ, while

$$\varphi = \psi = 0 \text{ on } \partial\Omega_+ \setminus \Gamma$$

because $\partial\Omega_+ \setminus \Gamma \subset \partial\Omega$ and $\varphi, \psi \in C_0^\infty(\Omega)$. Hence, $\varphi - \psi \in W_0^{1,2}\left(\Omega_+\right)$ by Exercise[1] 28.

Since $Lu = f$ weakly in Ω_+, using $\varphi - \psi$ as a test function, we obtain that the right hand side of (4.13) vanishes, whence (4.9) follows. ■

[1] It follows from Exercise 28, that if $g \in W^{1,2}(\Omega) \cap C\left(\overline{\Omega}\right)$ and $g = 0$ on $\partial\Omega$ then $g \in W_0^{1,2}(\Omega)$.

29.01.24 **Lecture 28**

Corollary 4.2 (Boundary regularity: flat boundary) *Let L be the operator (4.1) in Ω_+ that is uniformly elliptic and with measurable coefficients. Let u satisfy (4.3), where $f \in L^q(\Omega_+)$ and $q \in [2,\infty] \cap (n/2,\infty]$. Then $u \in C^\beta(\Omega_+ \cup \Gamma)$ for some $\beta = \beta(n,\lambda,q) > 0$, and $u|_\Gamma = 0$.*

The exact meaning of the statement is as follows: there exists a version $\widetilde{u}$ of u such that $\widetilde{u} \in C^\beta(\Omega_+ \cup \Gamma)$ and $\widetilde{u}|_\Gamma = 0$.

Proof. By Theorem 3.12, u has in Ω_+ a continuous version so that from now on let u denote this continuous version. Let us extend u, f, L to $\Omega = \Omega_+ \cup \Omega_- \cup \Gamma$ as in Theorem 4.1. By construction, the function u is continuous in Ω_+ and Ω_-, but not yet in Ω.

By Theorem 4.1 we have $u \in W^{1,2}(\Omega)$ and $Lu = f$ weakly in Ω. Since $f \in L^q(\Omega)$, we conclude by Theorem 3.12 that the function u in Ω has a continuous version $\widetilde{u} \in C^\beta(\Omega)$. In particular, $\widetilde{u} \in C^\beta(\Omega_+ \cup \Gamma)$.

Let us verify that $\widetilde{u}|_\Gamma = 0$. Since both u and $\widetilde{u}$ are continuous in Ω_+ and Ω_-, they coincide pointwise in Ω_+ and Ω_-. Since u is odd, it follows that, for any $x \in \Omega_+$,

$$\widetilde{u}(\rho(x)) = u(\rho(x)) = -u(x) = -\widetilde{u}(x).$$

By the continuity of $\widetilde{u}$ in Ω, this identity extends to all $x \in \Gamma$, whence $\widetilde{u}|_\Gamma = 0$. ■

4.2 Boundary as a graph

Let V be an open subset of $\mathbb{R}^{n-1}$ and I be a non-empty open interval in $\mathbb{R}$. Consider the cylinder $Q = V \times I$ that is an open subset of $\mathbb{R}^n$. Given a function $h : V \to I$, consider its *graph*

$$\Gamma_h = \{(z,t) \in Q : \ t = h(z)\}$$

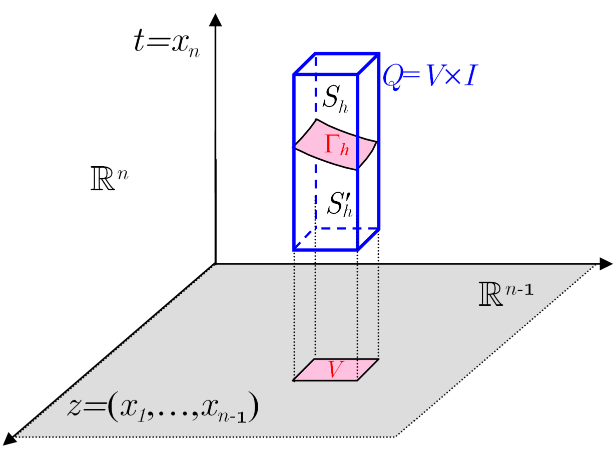
its *supergraph*

$$S_h = \{(z,t) \in Q : \ t > h(z)\},$$

and its *subgraph*

$$S_h' = \{(z,t) \in Q : \ t < h(z)\},$$

Here $z \in \mathbb{R}^{n-1}$, $t \in \mathbb{R}$, and we consider the pair (z,t) as the point $(z_1, ..., z_{n-1}, t) \in \mathbb{R}^n$.



In this section we consider the domain $\Omega = S_h$ and a part of its boundary $\Gamma = \Gamma_h$. In particular, if $h \equiv 0$ then $\Omega \subset \mathbb{R}_+$ and $\Gamma = V \subset \mathbb{R}^{n-1}$, which fits into the setting of the previous section.

Theorem 4.3 (Boundary regularity: a graph boundary) *Let $h \in C^1(V; I)$ and assume that*

$$\sup_V |\nabla h| < \infty.$$

Let L be a uniformly elliptic operator (4.1) in $\Omega = S_h$ with measurable coefficients. Let u satisfy

$$\begin{cases} Lu = f \text{ weakly in } \Omega, \\ u \in W^{1,2}(\Omega), \\ u = 0 \text{ weakly on } \Gamma, \end{cases} \tag{4.14}$$

where $\Gamma = \Gamma_h$, $f \in L^q(\Omega)$ and $q \in [2, \infty] \cap (n/2, \infty]$. Then $u \in C^\beta(\Omega \cup \Gamma)$ where $\beta = \beta(n, \lambda, \gamma, \sup |\nabla h|) > 0$, and $u|_\Gamma = 0$.

Proof. Let us consider the following mapping $\Psi : V \times \mathbb{R} \to V \times \mathbb{R}$:

$$\Psi(x) = (x_1, ..., x_{n-1}, x_n - h(x_1, ..., x_{n-1})). \tag{4.15}$$

Clearly, Ψ has the inverse mapping

$$\Psi^{-1}(y) = (y_1 ..., y_{n-1}, y_n + h(y_1, ..., y_{n-1})) \tag{4.16}$$

and, hence, Ψ is a C^1-diffeomorphism of $V \times \mathbb{R}$ into itself. Since

$$\Gamma = \{x \in Q : x_n = h(x_1, ..., x_{n-1})\} \quad \text{and} \quad \Omega = \{x \in Q : x_n > h(x_1, ..., x_{n-1})\}$$

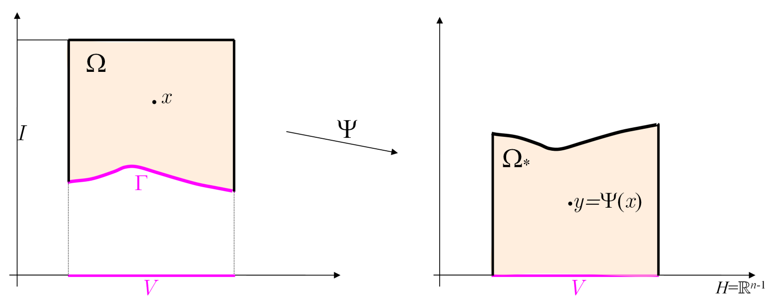
we see that

$$\Psi(\Gamma) = V \subset \mathbb{R}^{n-1} =: H \quad \text{and} \quad \Psi(\Omega) \subset \mathbb{R}^n_+.$$

Set

$$\Omega_* = \Psi(\Omega)$$

so that Ω_* satisfies the conditions for Ω_+ from the previous section, with $\partial\Omega_* \cap H = \overline{V}$.



The mapping Ψ is called *straightening* of Γ as it straightens the piece Γ of the boundary $\partial\Omega$ into a flat piece V. We regard Ψ as a C^1-diffeomorphism between Ω and

Ω_*. Denote points in Ω by x, points in Ω_* by y, and write Ψ in the form $y = \Psi(x)$. We will need the Jacobi matrices of Ψ and Ψ^{-1}. Using (4.15) and (4.16), we see that

$$J_\Psi = \left(\frac{\partial y_k}{\partial x_i}\right) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ -\partial_1 h & \cdots & -\partial_{n-1} h & 1 \end{pmatrix}$$

and

$$J_{\Psi^{-1}} = \left(\frac{\partial x_i}{\partial y_k}\right) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ \partial_1 h & \cdots & \partial_{n-1} h & 1 \end{pmatrix},$$

which implies

$$\det J_\Psi = \det J_{\Psi^{-1}} = 1.$$

Set

$$K = \max(1, \sup |\nabla h|)$$

so that all the entries of the both matrices J_Ψ and $J_{\Psi^{-1}}$ are bounded by K.

Given a function $u : \Omega \to \mathbb{R}$, define its *push-forward* function $u_* : \Omega_* \to \mathbb{R}$ as follows:

$$u_*(\Psi(x)) = u(x) \quad \text{for all } x \in \Omega,$$

which is equivalent to

$$u_*(y) = u\left(\Psi^{-1}(y)\right) \quad \text{for all } y \in \Omega_*.$$

Let us prove some properties of push-forward.

(a) If $u \in L^p(\Omega)$ then $u_* \in L^p(\Omega_*)$. Indeed, changing $y = \Psi(x)$ in the integral, we obtain

$$\int_{\Omega_*} |u_*(y)|^p \, dy = \int_\Omega |u_*(\Psi(x))|^p \, |\det J_\Psi| \, dx = \int_\Omega |u(x)|^p \, dx.$$

It follows also that

$$\|u\|_{L^p(\Omega)} = \|u_*\|_{L^p(\Omega_*)},$$

that is, push-forward is an isometry of $L^p(\Omega)$ and $L^p(\Omega_*)$.

(b) If $u \in W^{1,2}(\Omega)$ then $u_* \in W^{1,2}(\Omega_*)$. Indeed, observe that, by the chain rule,

$$\partial_{y_k} u_*(y) = \partial_{y_k}\left[u\left(\Psi^{-1}(y)\right)\right] = \sum_{i=1}^n (\partial_{x_i} u)\left(\Psi^{-1}(y)\right) \frac{\partial x_i}{\partial y_k} = \sum_{i=1}^n (\partial_{x_i} u)_*(y) \frac{\partial x_i}{\partial y_k}. \tag{4.17}$$

Since $\partial_{x_i} u \in L^2(\Omega)$, we obtain by ($a$) that $(\partial_{x_i} u)_* \in L^2(\Omega_*)$. Since all partial derivatives $\frac{\partial x_i}{\partial y_k}$ are bounded by K, we obtain that $(\partial_{x_i} u)_* \frac{\partial x_i}{\partial y_k}$ belongs to $L^2(\Omega_*)$, whence $\partial_{y_k} u_* \in L^2(\Omega_*)$. Hence, $u_* \in W^{1,2}(\Omega_*)$.

It follows from (4.17) that

$$\|\partial_{y_k} u_*\|_{L^2} \le K \sum_{i=1}^n \|\partial_{x_i} u\|_{L^2} \le K\sqrt{n} \left(\sum_{i=1}^n \|\partial_{x_i} u\|_{L^2}^2\right)^{1/2} = K\sqrt{n}\, \|\nabla u\|_{L^2}$$

and

$$\|\nabla u_*\|_{L^2}^2 = \sum_{k=1}^{n} \|\partial_{y_k} u_*\|_{L^2}^2 \leq n \left(K\sqrt{n}\, \|\nabla u\|_{L^2}\right)^2,$$

whence

$$\|\nabla u_*\|_{L^2} \leq Kn \, \|\nabla u\|_{L^2}.$$

Consequently, we obtain

$$(Kn)^{-1} \|u\|_{W^{1,2}(\Omega)} \leq \|u_*\|_{W^{1,2}(\Omega_*)} \leq Kn \, \|u\|_{W^{1,2}(\Omega)}. \tag{4.18}$$

(c) If $u = 0$ weakly on Γ then $u_* = 0$ weakly on V. Since $u = 0$ weakly on Γ, there is a sequence $\{\varphi_k\} \subset C^1\left(\overline{\Omega}\right)$ such that $\varphi_k|_\Gamma = 0$ and

$$\|u - \varphi_k\|_{W^{1.2}(\Omega)} \to 0 \text{ as } k \to \infty.$$

Since $(\varphi_k)_* \in C^1\left(\overline{\Omega}_*\right)$ and $(\varphi_k)_*|_V = 0$. Besides by (4.18) we obtain

$$\|u_* - (\varphi_k)_*\|_{W^{1.2}(\Omega)} \to 0 \text{ as } k \to \infty,$$

whence we conclude that $u_* = 0$ weakly on V.

(d) By Exercise 3 we have the following property of push-forward. Let

$$L = \sum_{i,j=1}^{n} \partial_i \left(a_{ij} \partial_j\right)$$

be an operator in Ω and assume that $Lu = f$ weakly in Ω. Then

$$L_* u_* = f_* \text{ weakly in } \Omega_*, \tag{4.19}$$

where the operator L_* is given by

$$L_* = \frac{1}{\sqrt{D}} \sum_{l,k=1}^{n} \partial_{y_k} \left(b_{kl} \sqrt{D} \partial_{y_l}\right)$$

with the coefficients

$$b_{kl}(y) = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial y_k}{\partial x_i} \frac{\partial y_l}{\partial x_j}.$$

and $D = (\det J_\Psi)^{-2}$. Since $D = 1$, we have

$$L_* = \sum_{l,k=1}^{n} \partial_{y_k} \left(b_{kl} \partial_{y_l}\right).$$

Let us show that the operator L_* is uniformly elliptic in Ω_*. For any $\xi \in \mathbb{R}^n$, we have

$$\begin{aligned}
\sum_{k,l=1}^{n} b_{kl} \xi_k \xi_l &= \sum_{k,l=1}^{n} \sum_{i,j=1}^{n} a_{ij} \frac{\partial y_k}{\partial x_i} \frac{\partial y_l}{\partial x_j} \xi_k \xi_l \\
&= \sum_{i,j=1}^{n} a_{ij} \left(\sum_{k=1}^{n} \frac{\partial y_k}{\partial x_i} \xi_k\right) \left(\sum_{l=1}^{n} \frac{\partial y_l}{\partial x_j} \xi_l\right).
\end{aligned}$$

Set

$$\eta_i = \sum_{k=1}^{n} \frac{\partial y_k}{\partial x_i} \xi_k \tag{4.20}$$

so that

$$\sum_{k,l=1}^{n} b_{kl} \xi_k \xi_l = \sum_{i,j=1}^{n} a_{ij} \eta_i \eta_j. \tag{4.21}$$

By the uniform ellipticity of (a_{ij}), we have

$$\lambda^{-1} |\eta|^2 \leq \sum_{i,j=1}^{n} a_{ij} \eta_i \eta_j \leq \lambda |\eta|^2 . \tag{4.22}$$

Since the coefficients $\frac{\partial y_k}{\partial x_i}$ are bounded by K, we obtain from (4.20)

$$|\eta_i|^2 \leq K^2 \left(\sum_{k=1}^{n} |\xi_k| \right)^2 \leq K^2 n \sum_{k=1}^{n} |\xi_k|^2 = K^2 n |\xi|^2 ,$$

whence

$$|\eta|^2 = \sum_{i=1}^{n} |\eta_i|^2 \leq K^2 n^2 |\xi|^2 .$$

By inverting (4.20) we obtain

$$\xi_k = \sum_{i=1}^{n} \frac{\partial x_i}{\partial y_k} \eta_i$$

whence in the same way

$$|\xi|^2 \leq K^2 n^2 |\eta|^2 .$$

Hence, we have

$$\left(K^2 n^2\right)^{-1} |\xi|^2 \leq |\eta|^2 \leq K^2 n^2 |\xi|^2 .$$

Combining with (4.21) and (4.22), we obtain

$$\lambda_*^{-1} |\xi|^2 \leq \sum_{k,l=1}^{n} b_{kl} \xi_k \xi_l \leq \lambda_* |\xi|^2 ,$$

where $\lambda_* = \lambda K^2 n^2$. Hence, L_* is uniformly elliptic with the ellipticity constant λ_*.

Now let u satisfy (4.14) with $f \in L^q(\Omega)$. By the above properties of push-forward, we obtain that u_* satisfies the following conditions:

$$\begin{cases} L_* u_* = f_* \text{ weakly in } \Omega_*, \\ u_* \in W^{1,2}(\Omega_*), \\ u_* = 0 \text{ weakly on } V, \end{cases}$$

where $f_* \in L^q(\Omega_*)$. Since $\Omega_* \subset \mathbb{R}^n_+$ and $\partial\Omega_* \cap H = \overline{V}$, we conclude by Corollary 4.2 that $u_* \in C^\beta(\Omega_* \cup V)$ for some

$$\beta = \beta(n, \lambda_*, q) = \beta(n, \lambda, q, K) > 0,$$

and that $u_*|_V = 0$. Since $u = u^* \circ \Psi$, it follows that also $u \in C^\beta(\Omega \cup \Gamma)$ and, for any $x \in \Gamma$,

$$u(x) = u^*(\Psi(x)) = 0,$$

because $\Psi(x) \in V$, which finishes the proof. ■

Remark. The statement and proof of Theorem 4.3 (with necessary modifications) remain valid if h is a Lipschitz function rather than C^1.

01.02.24 **Lecture 29**

4.3 Domains with C^1 boundary

Given two sets $A \subset \mathbb{R}^{n-1}$ and $B \subset \mathbb{R}$, define the product $A \times_i B$ with respect to the coordinate x_i in $\mathbb{R}^n$ as follows:

$$A \times_i B = \{(x_1, ..., x_n) \in \mathbb{R}^n : (x_1, ...\hat{x}_i..., x_n) \in A,\ x_i \in B\},$$

where the notation $\hat{x}_i$ means that x_i is omitted, that is,

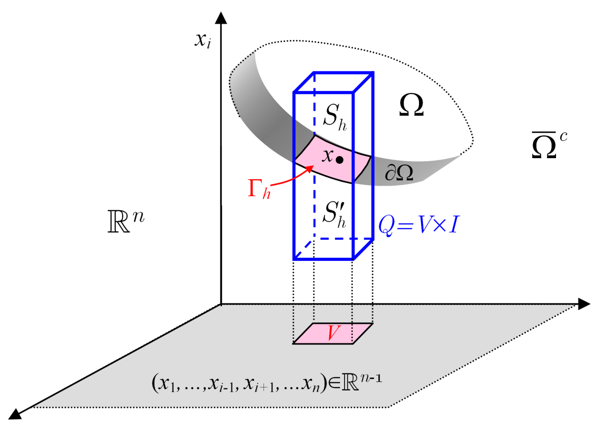
$$(x_1, ...\hat{x}_i..., x_n) = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_n).$$

A (open) *cylinder* in $\mathbb{R}^n$ with respect to the coordinate x_i is any set Q of the form $Q = V \times_i I$ where V is an open subset of $\mathbb{R}^{n-1}$ and I is an open interval in $\mathbb{R}$.

Definition. Let Ω be an open subset of $\mathbb{R}^n$. We say that the boundary of Ω belongs to the class C^1 (or simply Ω belongs to C^1) if the following two conditions are satisfied:

(i) any ball $B_\varepsilon(x)$ with $x \in \partial\Omega$ and $\varepsilon > 0$ has a non-empty intersection with $\overline{\Omega}^c$;

(ii) for any point $x \in \partial\Omega$ there exist a connected cylinder $Q = V \times_i I$ containing x and a C^1-function $h : V \to I$ such that $\partial\Omega \cap Q = \Gamma_h$ (that is, $\partial\Omega$ is locally a C^1 graph).



Without loss of generality, we can assume that V (and, hence, Q) is connected.

Claim. *It follows from* (i) *and* (ii) *that* $\Omega \cap Q$ *coincides either with the supergraph* S_n *or with the subgraph* S_h'.

Proof. Let us skip for simplicity the index h. We clearly have

$$Q = S \sqcup S' \sqcup \Gamma.$$

Since S is an image of Q under a continuous mapping, it follows that S is connected. Since $S \subset (\partial\Omega)^c$ and $(\partial\Omega)^c$ is a disjoint union $\Omega \sqcup \overline{\Omega}^c$ of open sets, it follows from the connectedness that $S \subset \Omega$ or $S \subset \overline{\Omega}^c$. The same argument applies also to S': either $S' \subset \Omega$ or $S' \subset \overline{\Omega}^c$.

However, S and S' cannot both be contained in the same of the two sets Ω or $\overline{\Omega}^c$. Indeed, if S and S' are both contained in Ω then any point x on Γ has in a small enough neighborhood no points from $\overline{\Omega}^c$, which contradicts (i). If S and S' are contained in $\overline{\Omega}^c$, and any point $x \in \Gamma$ has in a small enough neighborhood no points from Ω, which contradicts the definition of the boundary.

Hence, there remain only two possibilities:

- either $S \subset \Omega$ and $S' \subset \overline{\Omega}^c$
- or $S' \subset \Omega$ and $S \subset \overline{\Omega}^c$.

In the first case we have $\Omega \cap Q = S$, and in the second case $\Omega \cap Q = S'$. ∎

The next statement provides a large class of examples of domains with C^1 boundary. Recall that a bounded open set Ω is called a *region* if there exists a C^1 function F defined in an open neighborhood Ω' of $\overline{\Omega}$ such that

$$\Omega = \{x \in \Omega' : F(x) < 0\},$$

$$\partial\Omega = \{x \in \Omega : F(x) = 0\},$$

and

$$\nabla F \neq 0 \ \text{ on } \ \partial\Omega.$$

For example, a ball $B_R = \{x \in \mathbb{R}^n : |x| < R\}$ is a region with function

$$F(x) = |x|^2 - R^2.$$

Lemma 4.4 *If* Ω *is a region then* Ω *has* C^1 *boundary.*

Proof. Fix some point $z \in \partial\Omega$. By the hypothesis $\nabla F(z) \neq 0$, the point z cannot be a local maximum of F. Since $F(z) = 0$, it follows that any neighborhood of z contains points x with $F(x) > 0$, that is, the points from $\overline{\Omega}^c$.

Since $\nabla F(z) \neq 0$, there is an index $i = 1, 2..., n$ such that $\partial_i F(z) \neq 0$. By the theorem of implicit function, the equation

$$F(x_1, x_2, ..., x_n) = 0$$

can be resolved in a neighborhood of z with respect to x_i as follows: there is a cylinder $Q = V \times_i I$ containing z and a C^1 function $h : V \to I$ such that, for all $x \in Q$,

$$F(x_1, ..., x_n) = 0 \Leftrightarrow x_i = h(x_1, ...\hat{x}_i...x_n).$$

Consequently, we have $\partial\Omega \cap Q = \Gamma_h$ and, hence, Ω is a domain with C^1 boundary. ∎

Theorem 4.5 (Boundary regularity: C^1 boundary) *Let Ω be a bounded domain with C^1 boundary and L be a uniformly elliptic operator (4.1) with measurable coefficients in Ω. Let Γ be an open subset of the boundary $\partial\Omega$, and assume that a function u satisfies the following conditions:*

$$\begin{cases} Lu = f \text{ weakly in } \Omega, \\ u \in W^{1,2}(\Omega), \\ u = 0 \text{ weakly on } \Gamma, \end{cases} \tag{4.23}$$

where $f \in L^q(\Omega)$ with $q \in [2, \infty] \cap (n/2, \infty]$. Then $u \in C(\Omega \cup \Gamma)$ and $u|_\Gamma = 0$.

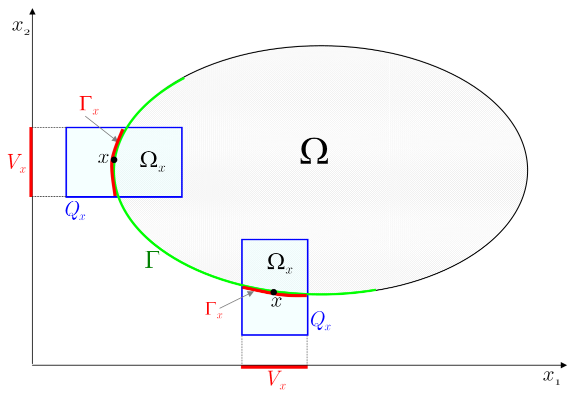
In the special case when $\Gamma = \partial\Omega$ we obtain that if u solves the weak Dirichlet problem

$$\begin{cases} Lu = f \text{ weakly in } \Omega, \\ u \in W_0^{1,2}(\Omega), \end{cases}$$

then $u \in C(\overline{\Omega})$ and $u|_{\partial\Omega} = 0$.

Proof. By Theorem 3.12, we can assume that $u \in C(\Omega)$. It suffices to prove that, in a neighborhood of any point $x \in \Gamma$, u extends continuously to Γ and $u(x) = 0$.

By definition of C^1 boundary, for any $x \in \partial\Omega$ there is a cylinder $Q_x = V_x \times_i I_x$ such that $\Gamma_x := \partial\Omega \cap Q_x$ is the graph of a C^1 function $h_x : V_x \to I_x$. Besides, by the above claim, the set $\Omega_x := \Omega \cap Q_x$ is either supergraph or subgraph of h_x in Q_x.



By reducing V_x we can assume that V_x is connected (for example, a ball) and $|\nabla h_x|$ is bounded.

Let $x \in \Gamma$. Since Γ is an open subset of $\partial\Omega$, again by reducing V_x, we can assume that $\Gamma_x \subset \Gamma$. Since $u = 0$ weakly on Γ, it follows that also $u = 0$ weakly on Γ_x. Hence, we can apply Theorem 4.3 in Ω_x, which yields $u \in C(\Omega_x \cup \Gamma_x)$ and $u|_{\Gamma_x} = 0$, which was to be proved. ■

Remark. The statement of Theorem 4.5 remains valid if the boundary $\partial\Omega$ is Lipschitz rather than C^1. Besides, by slightly modifying the argument, one can prove that u is locally Hölder continuous on $\Omega \cup \Gamma$.

4.4 Classical solutions

Now we can prove a result about existence of a classical solution.

Theorem 4.6 *Assume that Ω is a bounded domain with C^1 boundary and let k be an integer such that $k > n/2$. Consider in Ω a uniformly elliptic operator*

$$Lu = \sum_{i,j=1}^{n} \partial_i \left(a_{ij}\partial_j u\right)$$

with coefficients $a_{ij} \in C^{k+1}(\overline{\Omega})$. Then, for all $f \in C^k(\overline{\Omega})$ and $g \in C^2(\overline{\Omega})$, the classical Dirichlet problem

$$\begin{cases} Lu = f \text{ in } \Omega \\ u = g \text{ on } \partial\Omega \end{cases} \tag{4.24}$$

has exactly one solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$.

Remark. The assumptions of this theorem about functions a_{ij}, f, g are not quite optimal. They are to illustrate the method of obtaining classical solutions by means of weak solutions. Note that this theorem provides a new result even for $L = \Delta$.

Proof. Consider first the weak Dirichlet problem

$$\begin{cases} Lu = f \text{ weakly in } \Omega, \\ u - g \in W_0^{1,2}(\Omega). \end{cases} \tag{4.25}$$

By Exercise 7, if $f \in L^2(\Omega)$ and $g \in W^{1,2}(\Omega)$ (which is the case under the present assumptions) then the problem (4.25) has a unique weak solution $u \in W^{1,2}(\Omega)$.

Since $f \in C^k(\overline{\Omega})$, we have also $f \in W^{k,2}(\Omega)$, that is, $Lu \in W^{k,2}(\Omega)$. Since $a_{ij} \in C^{k+1}(\Omega)$ and

$$k > 2 + (\frac{n}{2} - 2),$$

we obtain by Corollary 2.13 that $u \in C^2(\Omega)$. Hence, u is a classical solution of $Lu = f$ in Ω.

In order to investigate the behavior of u on $\partial\Omega$, let us rewrite (4.25) in terms of the function $v = u - g$ as follows:

$$\begin{cases} Lv = f - Lg \text{ in } \Omega, \\ v \in W_0^{1,2}(\Omega). \end{cases} \tag{4.26}$$

Since $g \in C^2(\overline{\Omega})$ and $a_{ij} \in C^1(\overline{\Omega})$, it follows that $Lg \in C(\overline{\Omega})$, whence

$$f - Lg \in C(\overline{\Omega}) \subset L^\infty(\Omega)$$

(consequently, the problem (4.26) has a unique weak solution v, which provides an alternative proof of the existence and uniqueness of solution u of (4.25)). By Theorem 4.5 we obtain $v \in C(\overline{\Omega})$ and $v = 0$ on $\partial\Omega$. It follows that also $u \in C(\overline{\Omega})$ and $u = g$ on $\partial\Omega$, that is, u satisfies the boundary condition in the classical sense.

Hence, u is a classical solution of (4.24). Finally, the uniqueness of the classical solution of (4.24) in the class $C^2(\Omega) \cap C(\overline{\Omega})$ follows from the maximum principle of Exercise 1. ■

Recall from PDE the following result for the Laplace operator: let $f \in C^2(B_R)$ be bounded and let $g \in C(\partial B_R)$. Then the Dirichlet problem

$$\begin{cases} \Delta u = f & \text{in } B_R \\ u = g & \text{on } \partial B_R \end{cases} \tag{4.27}$$

has exactly one classical solution $u \in C^2(B_R) \cap C\left(\overline{B}_R\right)$. Of course, the requirements here are much milder than those in Theorem 4.6 because this is very special situation of $L = \Delta$ and $\Omega = B_R$ where one can expect better results than in general.

There is one more serious distinction between these two results. If u is the classical solution of (4.27), it may not be a weak solution in any sense, because, as we have seen on examples, the classical solution of (4.27) with arbitrary continuous function g on $\partial\Omega$ may have infinite energy:

$$\int_{B_R} |\nabla u|^2 \, dx = \infty,$$

and, hence, may be not in $W^{1,2}(B_R)$. Hence, for the methods based on weak solutions, one need to impose additional restriction on g.

Chapter 5

* Harnack inequality

5.1 Statement of the Harnack inequality (Theorem of Moser)

Consider again in a domain $\Omega \subset \mathbb{R}^n$ a uniformly elliptic operator in divergence form

$$L = \sum_{i,j=1}^{n} \partial_i \left(a_{ij} \partial_j \right)$$

with measurable coefficients. Recall that if $u \in W^{1,2}_{loc}(\Omega)$ is a weak solution of $Lu = 0$ in Ω then by Theorem 3.7 u is Hölder continuous in Ω.

Definition. We say that a function u is *L-harmonic* in Ω if u is the continuous version of a weak solution $u \in W^{1,2}_{loc}(\Omega)$ of $Lu = 0$ in Ω.

The main result of this Chapter is the following theorem.

Theorem 5.1 *If u is a non-negative L-harmonic function in a ball $B_{2R} \subset \Omega$ then*

$$\sup_{B_R} u \leq C \inf_{B_R} u \tag{5.1}$$

where $C = C(n, \lambda)$.

The inequality (5.1) is called the *Harnack inequality*, analogously to the classical Harnack inequality for harmonic functions that holds with the constant $C = 3^n$. This inequality for uniformly elliptic operators in divergence form with measurable coefficients was first proved by Jürgen Moser in 1961.

Recall the weak Harnack inequality of Theorem 3.4 that we now reformulate in the following form[1]:

Weak Harnack inequality *Let $B_{4R} \subset \Omega$ and assume that $u \in W^{1,2}(B_{4R})$ is L-harmonic in B_{4R}. Choose some $a > 0$ and set*

$$E = \{ x \in B_R : u(x) \geq a \} .$$

If for some $\varepsilon > 0$

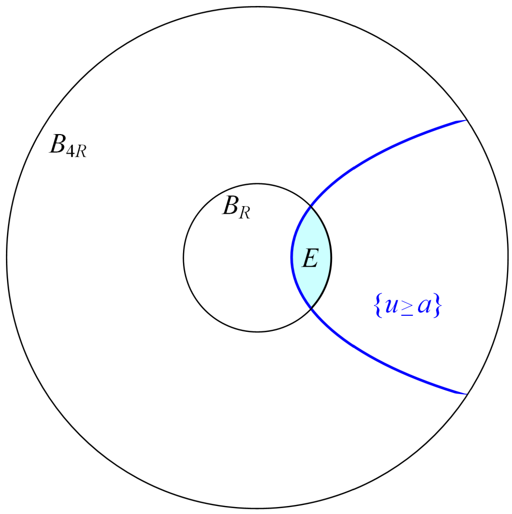
$$|E| \geq \varepsilon |B_R| ,$$

[1] In comparison with Theorem 3.4, we replace B_{3R} by B_{4R} and supersolution by solution.

then

$$\inf_{B_R} u \geq \delta a, \tag{5.2}$$

where $\delta = \delta(n, \lambda, \varepsilon) > 0$.



The Harnack inequality (should it be already proved) implies the weak Harnack inequality as follows: if the set E has positive measure then we conclude that

$$a \leq \sup_{B_R} u,$$

and then (5.2) follows from (5.1).

However, in the proof of Theorem 5.1 we will use the weak Harnack inequality. Moreover, we will use only the following properties of L-harmonic functions (apart from continuity):

(i) the weak Harnack inequality;

(ii) if u is L-harmonic then also the function $au + b$ is L-harmonic for arbitrary $a, b \in \mathbb{R}$.

If these two properties hold for any other operator L then also the Harnack inequality holds for L.

The method of derivation of the Harnack inequality from the weak Harnack inequality was invented by Eugene Landis in 1970s as an alternative to a more complicated method of Moser that involved a difficult lemma of John-Nirenberg.

5.2 Lemmas of growth

For the proof of Theorem 5.1 we need some lemmas. The first lemma is an extension of the weak Harnack inequality.

Lemma 5.2 (Reiteration of the weak Harnack inequality) *Let u be a non-negative L-harmonic function in some ball $B_R(x)$. Consider a ball $B_r(y)$ where*

$$y \in B_{\frac{1}{9}R}(x) \quad \text{and} \quad r \leq \frac{2}{9}R.$$

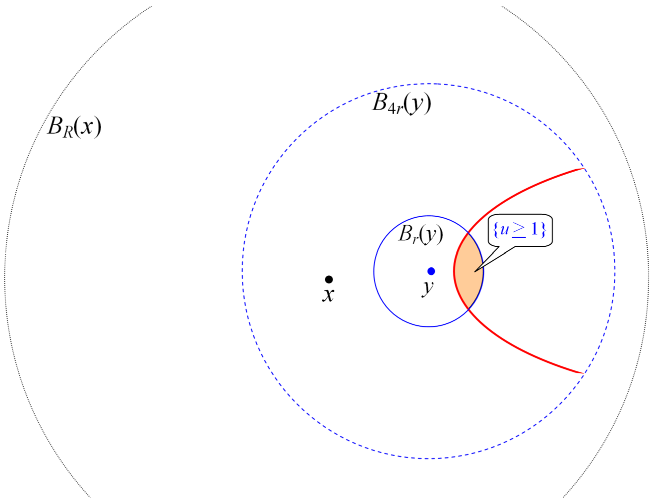
If

$$\frac{|\{u \geq 1\} \cap B_r(y)|}{|B_r(y)|} \geq \theta > 0 \tag{5.3}$$

then

$$u(x) \geq \left(\frac{r}{R}\right)^s \delta,$$

where $\delta = \delta(n, \lambda, \theta) > 0$ and $s = s(n, \lambda) > 0$.



Proof. Note that

$$B_{4r}(y) \subset B_R(x)$$

because

$$|x - y| + 4r < \frac{1}{9}R + \frac{8}{9}R = R.$$

Applying the weak Harnack inequality in $B_r(y)$ and using (5.3), we obtain that

$$\inf_{B_r(y)} u \geq \delta_1 := \delta(n, \lambda, \theta).$$

It follows that

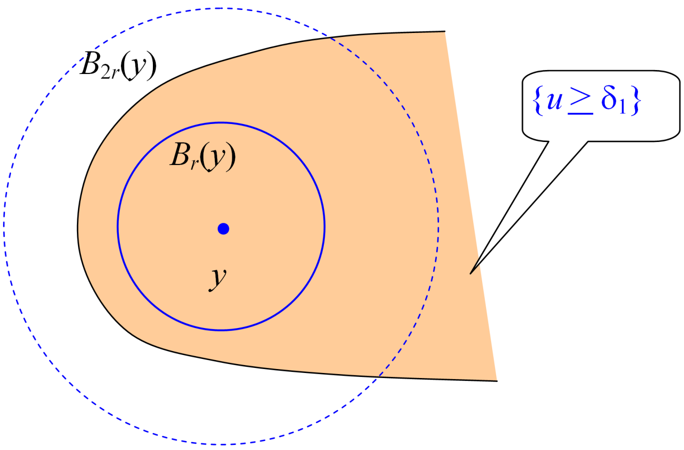
$$\frac{|\{u \geq \delta_1\} \cap B_{2r}(y)|}{|B_{2r}(y)|} \geq \frac{|B_r|}{|B_{2r}|} = 2^{-n}.$$

If $B_{8r}(y) \subset B_R(x)$ then applying the weak Harnack inequality in $B_{2r}(y)$, we obtain that

$$\inf_{B_{2r}(y)} u \geq \delta_1 \delta\left(n, \lambda, 2^{-n}\right) = \varepsilon \delta_1,$$

where

$$\varepsilon := \delta\left(n, \lambda, 2^{-n}\right).$$



It follows that

$$\frac{|\{u \geq \varepsilon\delta_1\} \cap B_{4r}(y)|}{|B_{4r}(y)|} \geq \frac{|B_{2r}|}{|B_{4r}|} = 2^{-n}.$$

Therefore, if $B_{16r}(y) \subset B_R(x)$ then

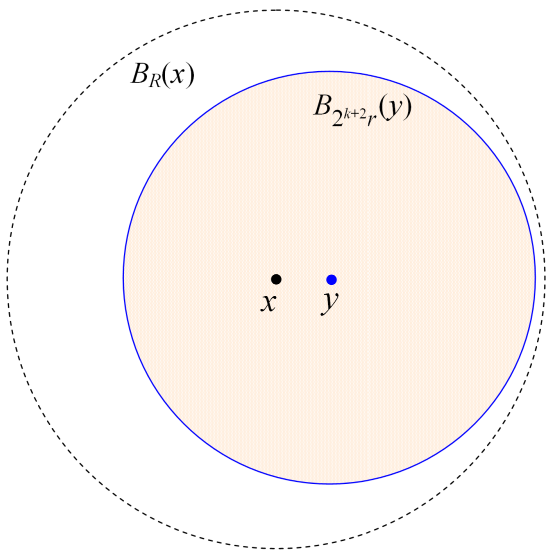
$$\inf_{B_{4r}} u \geq (\delta_1 \varepsilon)\, \varepsilon = \varepsilon^2 \delta_1.$$

We continue by induction and obtain the following statement for any positive integer k:

$$\text{if } B_{2^{k+2}r}(y) \subset B_R(x) \text{ then } \inf_{B_{2^k r}} u \geq \varepsilon^k \delta_1. \tag{5.4}$$

Let k be the maximal integer such that

$$B_{2^{k+2}r}(y) \subset B_R(x).$$



Then

$$2^{k+2}r + |x-y| \le R$$

while

$$2^{k+3}r + |x-y| > R.$$

It follows that

$$2^k r > \frac{R-|x-y|}{8} \ge |x-y|$$

where we have used that $R > 9\,|x-y|$. Hence, for this value of k, we have

$$x \in B_{2^k r}(y).$$

Then by (5.4)

$$u(x) \ge \varepsilon^k \delta_1.$$

On the other hand, we have

$$2^k r < 2^{k+2}r + |x-y| \le R$$

whence

$$k \le \log_2 \frac{R}{r}.$$

It follows that

$$u(x) \ge \varepsilon^{\log_2 \frac{R}{r}} \delta_1 = \delta_1 2^{\log_2 \varepsilon \log_2 \frac{R}{r}} = \delta_1 \left(\frac{R}{r}\right)^{\log_2 \varepsilon} = \delta_1 \left(\frac{r}{R}\right)^s$$

with $s = \log_2 \frac{1}{\varepsilon} > 0$, which finishes the proof. ■

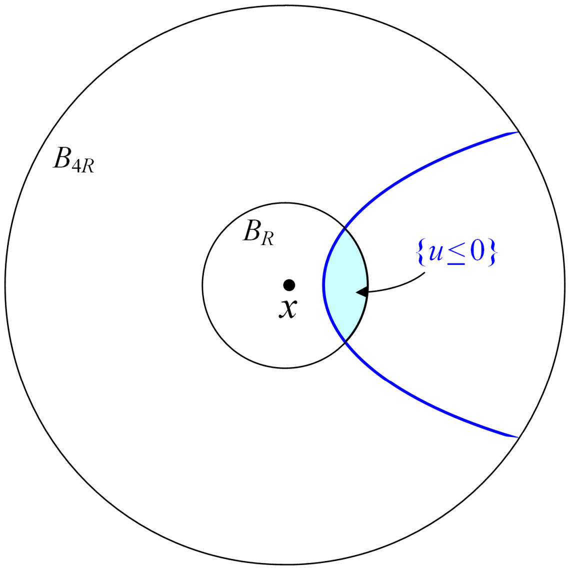
Lemma 5.3 (Alternative form of the weak Harnack inequality) *Let u be an L-harmonic function in some ball $B_{4R}(x)$. If*

$$\frac{|\{u \le 0\} \cap B_R(x)|}{|B_R|} \ge \theta > 0,$$

then

$$\sup_{B_{4R}(x)} u \ge (1+\delta)\, u(x), \tag{5.5}$$

where $\delta = \delta(n, \lambda, \theta) > 0$ is the same as in the weak Harnack inequality.



Proof. If $u(x) \leq 0$ then (5.5) is trivially satisfied. Assume that $u(x) > 0$. By rescaling, we can assume also that

$$\sup_{B_{4R}(x)} u = 1.$$

Consider the function $v = 1 - u$ that is a non-negative L-harmonic function in $B_{4R}(x)$. Observe also, that

$$u \leq 0 \Leftrightarrow v \geq 1.$$

Hence, we obtain that

$$\frac{|\{v \geq 1\} \cap B_R(x)|}{|B_R|} \geq \theta.$$

By the weak Harnack inequality, we conclude that

$$\inf_{B_R(x)} v \geq \delta,$$

where $\delta = \delta(n, \lambda, \theta) > 0$. It follows that $v(x) \geq \delta$ and, hence

$$u(x) \leq 1 - \delta < \frac{1}{1+\delta} = \frac{1}{1+\delta} \sup_{B_{4R}} u,$$

which is equivalent to (5.5). ∎

Lemma 5.4 (Lemma of growth in a thin domain) *There exists $\varepsilon = \varepsilon(n, \lambda) > 0$ such that the following is true: if u is an L-harmonic function in a ball $B_R(x)$ and if*

$$\frac{|\{u > 0\} \cap B_R|}{|B_R|} \leq \varepsilon$$

then

$$\sup_{B_R} u \geq 4u(x).$$

Corollary 5.5 *Under the same assumptions, choose some $a \in \mathbb{R}$ and assume that*

$$\frac{|\{u > a\} \cap B_R|}{|B_R|} \leq \varepsilon.$$

Then

$$\sup_{B_R} u \geq a + 4\,(u(x) - a)\,.$$

Proof. Indeed, just apply Lemma 5.4 to the L-harmonic function $v = u - a$. ■

Proof of Lemma 5.4. The value of ε will be determined later. So far consider ε as given. Consider any ball $B_r(y) \subset B_R(x)$ such that

$$\frac{|B_r|}{|B_R|} = 2\varepsilon,$$

which is equivalent to $\left(\frac{r}{R}\right)^n = 2\varepsilon$ and, hence, to

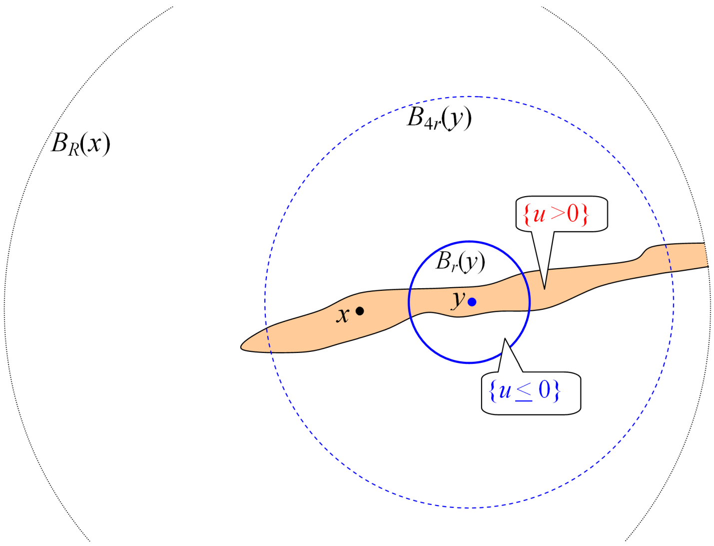
$$r = (2\varepsilon)^{1/n}\,R.$$

Then

$$\frac{|\{u > 0\} \cap B_r(y)|}{|B_r|} \leq \frac{|\{u > 0\} \cap B_R(x)|}{|B_R|}\frac{|B_R|}{|B_r|} \leq \varepsilon\frac{1}{2\varepsilon} = \frac{1}{2}.$$

It follows that

$$\frac{|\{u \leq 0\} \cap B_r(y)|}{|B_r|} \geq \frac{1}{2}.$$



If $B_{4r}(y) \subset B_R(x)$ then we can apply Lemma 5.3 and obtain that

$$\sup_{B_{4r}(y)} u \geq (1 + \delta)\,u(y),$$

where $\delta = \delta\left(n, \lambda, \frac{1}{2}\right) > 0$. By slightly reducing δ, we obtain the following claim.

Claim. *If $B_{4r}(y) \subset B_R(x)$ and $r = (2\varepsilon)^{1/n} R$ then there exists $y' \in B_{4r}(y)$ such that*

$$u(y') \geq (1+\delta)\, u(y),$$

where $\delta > 0$ depends on n, λ.

Let us apply the Claim first for $y = x$. Assuming that ε is small enough, we obtain $4r < R$ and, hence, $B_{4r}(x) \subset B_R(x)$. Hence, we obtain by Claim a point $x_1 \in B_{4r}(x)$ such that

$$u(x_1) \geq (1+\delta)\, u(x).$$

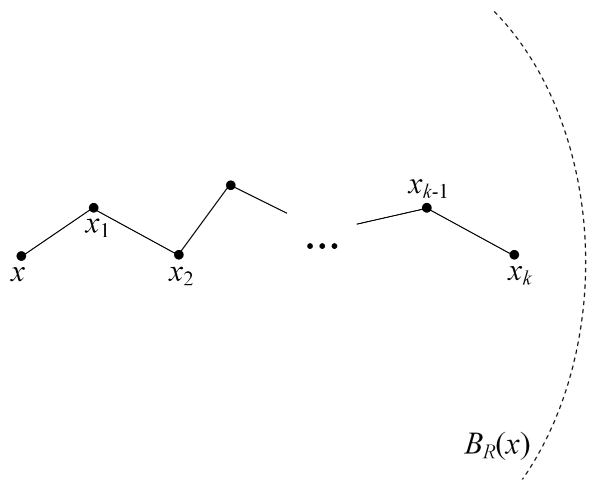
If $B_{4r}(x_1) \subset B_R(x)$ then we apply Claim again and obtain that there is $x_2 \in B_{4r}(x_1)$ such that

$$u(x_2) \geq (1+\delta)\, u(x_1).$$

We continue construction of the sequence $\{x_k\}$ by induction: as long as $B_{4r}(x_k) \subset B_R(x)$, we obtain $x_{k+1} \in B_{4r}(x_k)$ such that

$$u(x_{k+1}) \geq (1+\delta)\, u(x_k).$$

If, for some k, $B_{4r}(x_k)$ is not contained in $B_R(x)$ then we stop the construction.



By construction, if x_k exists then $x_k \in B_R(x)$ and

$$u(x_k) \geq (1+\delta)^k\, u(x_k). \tag{5.6}$$

Besides, we have

$$|x_{l+1} - x_l| < 4r \text{ for all } l \leq k-1,$$

which implies that

$$|x_k - x| < 4kr. \tag{5.7}$$

Let us prove by induction in k the following claim:

$$\text{if } 4kr < R \text{ then } x_k \text{ exists.}$$

We know already that x_1 exists. Let us prove the induction step, that is,

$$\text{if } 4(k+1)\, r < R \text{ then } x_{k+1} \text{ exists.}$$

Indeed, if $4(k+1)r < R$ then also $4kr < R$ and we obtain the inductive hypothesis that x_k exists. It follows from (5.7) that

$$B_{4r}(x_k) \subset B_{4(k+1)r}(x).$$

Since $4(k+1)r < R$, we see that $B_{4r}(x_k) \subset B_R(x)$, and this construction can be continued so that x_{k+1} exists, which finishes the inductive proof.

Let us choose the maximal integer k with $4kr < R$. Then we have

$$4(k+1)r \geq R$$

and, hence,

$$k \geq \frac{R}{4r} - 1 = \frac{1}{4(2\varepsilon)^{1/n}} - 1.$$

It follows from (5.6) that

$$u(x_k) \geq (1+\delta)^{\frac{1}{4(2\varepsilon)^{1/n}}-1} u(x).$$

Finally, choosing ε small enough (depending only on δ and n, that is, on λ and n), we obtain

$$\sup_{B_R(x)} u \geq u(x_k) \geq 4u(x),$$

which was to be proved. ■

5.3 Proof of the Harnack inequality

Here we prove Theorem 5.1. Observe first that it suffices to prove the following version of the Harnack inequality: there exists a constant C, depending on n, λ and such that if u is a non-negative L-harmonic function on a ball $B_{KR}(x)$ (where $K = 18$) then

$$\sup_{B_R(x)} u \leq Cu(x).$$

Without loss of generality, we can assume that

$$\sup_{B_R(x)} u = 2, \tag{5.8}$$

and we need to prove that

$$u(x) \geq c \tag{5.9}$$

for some positive constant $c = c(n, \lambda)$. Let us construct a sequence $\{x_k\}_{k\geq 1}$ of points such that

$$x_k \in B_{2R}(x) \quad \text{and} \quad u(x_k) = 2^k. \tag{5.10}$$

A point x_1 with $u(x_1) = 2$ exists in $\overline{B}_R(x)$ by assumption (5.8). Assume that x_k satisfying (5.10) is already constructed. Then, for small enough $r > 0$, we have

$$\sup_{B_r(x_k)} u \leq 2^{k+1}.$$

Set

$$r_k = \sup\left\{r \in (0, R] : \sup_{B_r(x_k)} u \le 2^{k+1}\right\}.$$

If $r_k = R$ then we stop the process without constructing x_{k+1}. If $r < R$ then we necessarily have

$$\sup_{B_r(x_k)} u = 2^{k+1}$$

(note that $B_r(x_k) \subset B_R(x_k) \subset B_{4R}(x)$ so that u is defined in $B_r(x_k)$). Therefore, there exists $x_{k+1} \in \overline{B}_{r_{k+1}}(x_k)$ such that $u(x_{k+1}) = 2^{k+1}$.

If $x_{k+1} \in B_{2R}(x)$ then we keep x_{k+1} and go to the next step. If $x_{k+1} \notin B_{2R}(x)$ then we disregard x_{k+1} and stop the process.

Hence, we obtain a sequence of balls $\{B_{r_k}(x_k)\}$ such that

$$r_k \le R, \quad x_k \in B_{2R}(x), \quad u(x_k) = 2^k$$

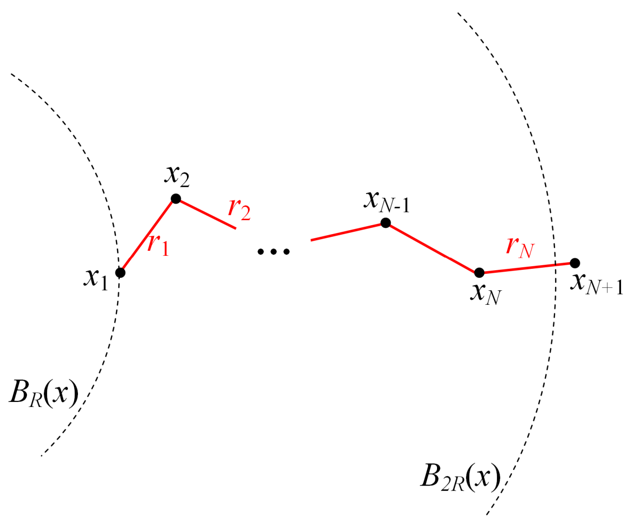
and

$$\sup_{B_{r_k}(x_k)} u \le 2^{k+1}. \tag{5.11}$$

Moreover, we have also

$$|x_{k+1} - x_k| \le r_k.$$

The sequence $\{x_k\}$ cannot be infinite because $u(x_k) \to \infty$ whereas u is bounded in $\overline{B}_{2R}(x)$ as a continuous function. Let N be the largest value of k in this sequence. Then we have either $r_N = R$ or $r_N < R$ and $x_{N+1} \notin B_{2R}(x)$ (where x_{N+1} is the disregarded point).



In the both cases we clearly have $r_1 + ... + r_N \ge R$.

In any ball $B_{r_k}(x_k)$ we have by (5.11)

$$\begin{aligned}\sup_{B_{r_k}(x_k)} u &\le 2^{k+1} < 2^{k-1} + 4\left(2^k - 2^{k-1}\right) \\ &= a + 4\left(u(x_k) - a\right),\end{aligned}$$

where $a = 2^{k-1}$. By Corollary 5.5, we conclude that

$$\frac{|\{u > a\} \cap B_{r_k}(x_k)|}{|B_{r_k}|} > \varepsilon,$$

that is,

$$\frac{\left|\left\{u \geq 2^{k-1}\right\} \cap B_{r_k}(x_k)\right|}{|B_{r_k}|} \geq \varepsilon.$$

Now let us apply Lemma 5.2 with $B_r(y) = B_{r_k}(x_k)$. Since u is non-negative and L-harmonic in $B_{KR}(x)$, the following conditions need to be satisfied:

$$r_k \leq \frac{2}{9}KR \ \text{ and } \ |x_k - x| \leq \frac{1}{9}KR.$$

Since $r_k \leq R$ and $|x_k - x| \leq 2R$, the both conditions are satisfied if $K = 18$. By Lemma 5.2, we obtain that

$$u(x) \geq \left(\frac{r_k}{R}\right)^s \delta 2^{k-1}, \tag{5.12}$$

where $\delta = \delta(n, \lambda, \varepsilon) > 0$ and $s = s(n, \lambda) > 0$.

The question remains how to estimate $\left(\frac{r_k}{R}\right)^s 2^{k-1}$ from below, given the fact that we do not know much about the sequence $\{r_k\}$: the only available information is (**??**). The following trick was invented by Landis. The condition (**??**) implies that there exists $k \leq N$ such that

$$r_k \geq \frac{R}{k(k+1)}. \tag{5.13}$$

Indeed, if for all $k \leq N$ we have

$$r_k < \frac{R}{k(k+1)},$$

then it follows that

$$\sum_{k=1}^{N} r_k < \sum_{k=1}^{\infty} \frac{R}{k(k+1)} = R,$$

which contradicts (**??**). Hence, choose k that satisfies (5.13). For this k we obtain from (5.12) that

$$u(x) \geq \delta \left(\frac{r_k}{R}\right)^s 2^{k-1} \geq \delta \frac{2^{k-1}}{(k(k+1))^s}.$$

The next observation is that although we do not know the value of k, nevertheless we can obtain a lower bound of $u(x)$ independent of k because

$$m := \inf_{k \geq 1} \frac{2^{k-1}}{(k(k+1))^s} > 0.$$

Hence, we conclude that

$$u(x) \geq \delta m =: c,$$

which finishes the proof of (5.9).

Finally, let us prove that if u is non-negative and L-harmonic function in a ball B_{2R} then

$$\sup_{B_R} u \leq C \inf_{B_R} u.$$

Assume without loss of generality that the center of the ball B_R is 0. Let a be a point in $\overline{B}_R$ where u takes the maximal value and b be the point in $\overline{B}_R$ where u takes the minimal value. We need to prove that

$$u(a) \leq Cu(b)$$

for some $C = C(n, \lambda)$. It suffices to prove that

$$u(a) \leq Cu(0) \quad \text{and } u(0) \leq Cu(b).$$

Set $r = R/K$ (where $K = 18$ as above) and connect 0 and a by a sequence $\{x_j\}_{j=0}^K$ of points such that

$$x_0 = 0, \quad x_K = a, \quad |x_j - x_{j+1}| \leq r.$$

For that, it suffices to choose all x_k on the interval $[0, a]$ dividing this interval into K equal parts.

Since $x_j \in \overline{B}_R$, the ball $B_{Kr}(x_j) = B_R(x_j)$ is contained in $B_{2R}(0)$. By the form of the Harnack inequality that we proved above, we conclude that

$$\sup_{B_r(x_j)} u \leq Cu(x_j).$$

Since $x_{j+1} \in \overline{B}_r(x_j)$, it follows that

$$u(x_{j+1}) \leq Cu(x_j)$$

and, hence,

$$u(a) \leq C^K u(0).$$

The inequality for $u(b)$ is proved in the same way.

5.4 Convergence theorems

Theorem 5.6 *Let $\{u_k\}_{k=1}^\infty$ be a sequence of L-harmonic functions in a domain $\Omega \subset \mathbb{R}^n$. If*

$$u_k \stackrel{L^2_{loc}(\Omega)}{\longrightarrow} u \quad \text{as } k \to \infty$$

then the function u is also L-harmonic in Ω. Moreover, the sequence $\{u_k\}$ converges to u locally uniformly.

Proof. Let us show that the sequence $\{u_k\}$ converges also in $W^{1,2}_{loc}(\Omega)$. For that it suffices to show that the sequence of $\{\nabla u_k\}$ is Cauchy in $L^2\left(B_{R/2}\right)$ in any ball $B_{R/2}$ such that $\overline{B}_R \subset \Omega$. For that we use the inequality (3.11) from the proof of Theorem 3.2:

$$\int_{B_R} |\nabla v|^2 \eta^2 dx \leq 4\lambda^4 \int_{B_R} |\nabla \eta|^2 v^2 dx, \tag{5.14}$$

where v is any L-harmonic function[2] in Ω and η is any Lipschitz function with compact support in B_R; in particular, choose η to be the following bump function:

$$\eta(x)=\begin{cases}1, & |x|\le r,\\ \frac{\rho-|x|}{\rho-r}, & r<|x|<\rho,\\ 0, & |x|\ge\rho.\end{cases} \tag{5.15}$$

where $0<r<\rho<R$. Take $r=\frac{1}{2}R$ and $\rho=\frac{3}{4}R$. Then it follows from (5.14) that

$$\int_{B_{R/2}}|\nabla v|^2\,dx\le\frac{C}{R^2}\int_{B_R}v^2dx. \tag{5.16}$$

Let us apply this inequality to $v=u_k-u_l$. Since

$$\|u_k-u_l\|_{L^2(B_R)}\to 0\ \text{ as } k,l\to\infty,$$

it follows from (5.16) that

$$\|\nabla u_k-\nabla u_l\|_{L^2\left(B_{R/2}\right)}\to 0\ \text{ as } k,l\to\infty.$$

Hence, ∇u_k converges in $L^2_{loc}(\Omega)$, which implies that $u\in W^{1,2}_{loc}$ and $u_k\to u$ in $W^{1,2}_{loc}(\Omega)$.

Since each u_k satisfies the identity

$$\int_\Omega\sum_{i,j=1}^n a_{ij}\partial_j u_k\partial_i\varphi=0$$

for all $\varphi\in\mathcal{D}(\Omega)$, passing to the limit as $k\to\infty$, we obtain the same identity for u, whence $Lu=0$ follows.

The last claim follows from Theorem 3.2 that implies that, for any ball $\overline{B}_R\subset\Omega$,

$$\sup_{B_{R/2}}|u-u_k|\le\frac{C}{R^{n/2}}\|u-u_k\|_{L^2(B_R)}.$$

Since $\|u-u_k\|_{L^2(B_R)}\to 0$ as $k\to\infty$, it follows that also

$$\sup_{B_{R/2}}|u-u_k|\to 0,$$

which means that $u_k\to u$ locally uniformly. ■

Theorem 5.7 *Let $\{u_k\}_{k=1}^\infty$ be a sequence of L-harmonic functions in a connected domain $\Omega\subset\mathbb{R}^n$. Assume that this sequence is monotone increasing, that is, $u_{k+1}(x)\ge u_k(x)$ for all $k\ge 1, x\in\Omega$. Then the function*

$$u(x):=\lim_{k\to\infty}u_k(x)$$

is either identically equal to ∞ in Ω, or it is an L-harmonic function in Ω. Moreover, in the latter case the sequence $\{u_k\}$ converges to u locally uniformly.

[2]In fact, (5.14) was proved for $v=u_+$ where u is L-harmonic function. Applying (5.14) also to $v=u_-$, we obtain the same inequality with $v=u$.

Proof. By replacing u_k with $u_k - u_1$, we can assume that all functions u_k are non-negative. Consider the sets

$$F = \{x \in \Omega : u(x) < \infty\}$$

and

$$I = \{x \in \Omega : u(x) = \infty\}$$

so that $\Omega = F \sqcup I$. Let us prove that both F and I are open sets.

Indeed, take a point $x \in F$ and show that also $B_\varepsilon(x) \in F$ for some $\varepsilon > 0$. Choose ε so that $B_{2\varepsilon}(x) \subset \Omega$. By the Harnack inequality, we have

$$\sup_{B_\varepsilon(x)} u_k \leq C \inf_{B_\varepsilon(x)} u_k \leq C u_k(x).$$

By passing to the limit as $k \to \infty$, we obtain

$$\sup_{B_\varepsilon(x)} u \leq C u(x).$$

Since $u(x) < \infty$, we obtain that also $\sup_{B_\varepsilon(x)} u < \infty$ and, hence, $B_\varepsilon(x) \subset F$. Hence, F is open.

In the same way one proves that

$$\inf_{B_\varepsilon(x)} u \geq C^{-1} u(x),$$

which implies that I is open.

Since Ω is connected and $\Omega = F \sqcup I$, it follows that either $I = \Omega$ or $F = \Omega$. In the former case we have $u \equiv \infty$ in Ω, in the latter case: $u(x) < \infty$ for all $x \in \Omega$. Let us prove that in the latter case u is L-harmonic. For that, we first show that the convergence $u_k \to u$ is locally uniform, that is, for any $x \in \Omega$ there is $\varepsilon > 0$ such that

$$u_k \rightrightarrows u \text{ in } B_\varepsilon(x) \ \text{ as } k \to \infty.$$

Then the L-harmonicity of u will follow by Theorem 5.6.

Choose again $\varepsilon > 0$ so that $B_{2\varepsilon}(x) \subset \Omega$. For any two indices $k > l$, apply the Harnack inequality to the non-negative L-harmonic function $u_k - u_l$:

$$\sup_{B_\varepsilon(x)} (u_k - u_l) \leq C (u_k - u_l)(x).$$

Since $(u_k - u_l)(x) \to 0$ as $k, l \to \infty$, it follows that

$$u_k - u_l \rightrightarrows 0 \text{ in } B_\varepsilon(x) \text{ as } k, l \to \infty.$$

Hence, the sequence $\{u_k\}$ converges uniformly in $B_\varepsilon(x)$. Since $\{u_k\}$ convergence pointwise to u, it follows that

$$u_k \rightrightarrows u \text{ in } B_\varepsilon(x) \text{ as } k \to \infty,$$

which finishes the proof. ■

Theorem 5.8 *If $\{u_k\}$ is a sequence of L-harmonic functions in Ω that is bounded in $L^2(\Omega)$, then there is a subsequence $\{u_{k_i}\}$ that converges to an L-harmonic function locally uniformly.*

Proof. Consider any ball $\overline{B}_R \subset \Omega$. Let us apply the inequality (3.12) from the proof of Theorem 3.2 that says the following: v is L-harmonic in Ω then

$$\int_{B_R} |\nabla(v\eta)|^2 \, dx \leq \frac{C}{(\rho - r)^2} \int_{B_\rho} v^2 dx$$

where we take $0 < r < \rho < R$ and function η is defined by (5.15). Taking $r = \frac{1}{2}R$ and $\rho = \frac{3}{4}R$, and applying this to $v = u_k$,

$$\int_{B_R} |\nabla(u_k\eta)|^2 \, dx \leq \frac{C}{R^2} \int_{B_R} u_k^2 dx.$$

Since the right hand side is uniformly bounded for all k, so is the left hand side. Therefore, the sequence $\{u_k\eta\}_{k=1}^{\infty}$ is bounded in $W^{1,2}(B_R)$. Since $u_k\eta \in W_0^{1,2}(B_R)$, we obtain by the compact embedding theorem that this sequence has a convergent subsequence in $L^2(B_R)$. Since $\eta = 1$ on $B_{R/2}$, it follows that $\{u_k\}$ has a convergence subsequence in $L^2\left(B_{R/2}\right)$.

Covering Ω by a countable family of the balls and using the diagonal process, we conclude that $\{u_k\}$ has a subsequence that converges in $L^2_{loc}(\Omega)$ to some function u. By Theorem 5.6 we conclude that u is L-harmonic and the convergence is locally uniform. ■

5.5 Liouville theorem

Theorem 5.9 *If u is a non-negative L-harmonic function in $\mathbb{R}^n$ then $u \equiv \text{const}$.*

Proof. By subtracting from u the constant $\inf_{\mathbb{R}^n} u$, we can assume without loss of generality that $\inf_{\mathbb{R}^n} u = 0$. We can apply the Harnack inequality to u in any ball B_R because u is L-harmonic and non-negative in B_{2R} for any $R > 0$. Hence, we obtain

$$\sup_{B_R} u \leq C \inf_{B_R} u,$$

where C does not depend on R. Letting $R \to \infty$, we see that the right hand side goes to 0. Hence, the left hand side also goes to 0, and we conclude that $u \equiv 0$. ■

5.6 Green function

We state the next theorem without proof.

Theorem 5.10 *Let Ω be a bounded domain in $\mathbb{R}^n$. Then there exists a function $G(x, y)$ on $\Omega \times \Omega$ with the following properties:*

1. *$G(x,y)$ is jointly continuous in $(x,y) \in \Omega \times \Omega \setminus \operatorname{diag}$.*
2. *$G(x,y) \geq 0$.*
3. *$G(x,y) = G(y,x)$.*
4. *For any function $f \in L^2(\mathbb{R}^n)$, the following function*
$$u(x) = \int_\Omega G(x,y)\, f(y)dy$$
is a weak solution of the Dirichlet problem
$$\begin{cases} Lu = -f \ \text{in } \Omega, \\ u \in W_0^{1,2}(\Omega). \end{cases}$$
5. *Assume $n > 2$. Then, for any compact set $K \subset \Omega$, there are positive constants $c_1, c_2 > 0$ such that*
$$c_1 |x-y|^{2-n} \leq G(x,y) \leq c_2 |x-y|^{2-n} \tag{5.17}$$
for all $x, y \in K$.

This theorem was proved by Walter Littman, Guido Stampacchia, and Hans Weinberger in 1963. The Harnack inequality of Theorem 5.1 was used to prove the estimate (5.17).

5.7 Boundary regularity

Let Ω be a bounded domain in $\mathbb{R}^n$ and consider the following Dirichlet problem in Ω:
$$\begin{cases} Lu = 0 \text{ in } \Omega \\ u - g \in W_0^{1,2}(\Omega) \end{cases} \tag{5.18}$$
where $g \in C^1(\overline{\Omega})$ is a given function.

Definition. We say that a point $z \in \partial\Omega$ is *regular* for (5.18) if, for any $g \in C^1(\overline{\Omega})$, the (continuous version of the) solution u of (5.18) satisfies
$$\lim_{\substack{x \to z \\ x \in \Omega}} u(x) = g(z). \tag{5.19}$$

Fix a point z on the boundary $\partial\Omega$ and, for any integer $k \geq 1$, consider the following sets:
$$E_k(z) = B_{2^{-k}}(z) \cap \Omega^c.$$

Theorem 5.11 *Assume $n > 2$. Then a point $z \in \partial\Omega$ is regular for (5.18) if and only if*
$$\sum_{k=1}^{\infty} 2^{k(n-2)} \operatorname{cap}(E_k(z)) = \infty. \tag{5.20}$$

This theorem was proved by W.Littman, G.Stampacchia, and H.F.Weinberger in 1963 using their estimate (5.17) of the Green function. For the case $L = \Delta$, Theorem 5.11 was first proved by Norbert Wiener in 1924. The condition (5.20) for regularity is called *Wiener's criterion.*

One of the consequences of Theorem 5.11 is that the notion of regularity of $z \in \partial\Omega$ does not depend on the choice of the operator L as long as it in the divergence form and uniformly elliptic.

Chapter 6

* Equations in non-divergence form

6.1 Strong and classical solutions

Consider in a domain $\Omega \subset \mathbb{R}^n$ a non-divergence form operator

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x) \partial_{ij} u$$

with measurable coefficients $a_{ij} \in C^{\infty}(\Omega)$. Assume that L is uniformly elliptic with the ellipticity constant λ. Given a function $f \in L^p_{loc}(\Omega)$, where $p \geq 1$, we say that u is a *strong* solution of $Lu = f$ in Ω if $u \in W^{2,p}_{loc}(\Omega)$ and the equation

$$\sum_{i,j=1}^{n} a_{ij}(x) \partial_{ij} u(x) = f(x) \tag{6.1}$$

is satisfied for almost all $x \in \Omega$. Here $\partial_{ij} u$ is the weak derivative of u that obviously belongs to $L^p_{loc}(\Omega)$. Here we consider only strong solutions of the class $W^{2,n}_{loc}$, that is, $p = n$. By the Sobolev embedding theorem, we have

$$W^{2,n}_{loc}(\Omega) \hookrightarrow C(\Omega),$$

so that all strong solutions are continuous functions.

Assume now that the coefficients a_{ij} are continuous in Ω. Given a function $f \in C(\Omega)$, we say that u is a *classical* solution of $Lu = f$ in Ω if $u \in C^2(\Omega)$ and the equation (6.1) is satisfied for all $x \in \Omega$. Of course, any classical solution is also strong.

If u is a solution of $Lu = 0$ (either strong or classical) then we refer to u as an *L-harmonic function.*

6.2 Theorem of Krylov-Safonov

The main results of this Chapter are stated in the next two theorems that were proved by Nikolai Krylov and Michail Safonov in 1980 based on the previous work of Eugene Landis.

Theorem 6.1 (Estimate of the Hölder norm) *If u is an L-harmonic function in Ω then $u \in C^{\alpha}(\Omega)$ with some $\alpha = \alpha(n, \lambda) > 0$. Moreover, for any compact set $K \subset \Omega$,*

$$\|u\|_{C^{\alpha}(K)} \leq C \|u\|_{C(\Omega)}, \tag{6.2}$$

where $C = C(n, \lambda, \operatorname{dist}(K, \partial\Omega))$.

Of course, if u is a classical solution then $u \in C^2(\Omega)$ and, hence, $u \in C^{\alpha}(\Omega)$ with any $\alpha < 1$. However, even in this case the estimate (6.2) of the Hölder norm is highly non-trivial, because α and C do not depend on a particular solution u.

Theorem 6.2 (The Harnack inequality) *If u is a non-negative L-harmonic function in a ball $B_{2R} \subset \Omega$ then*

$$\sup_{B_R} u \leq C \inf_{B_R} u$$

where $C = C(n, \lambda)$.

In this Chapter we will prove restricted versions of Theorems 6.1 and 6.2 assuming that $a_{ij} \in C^{\infty}(\Omega)$ and that the L-harmonic functions are classical solutions of $Lu = 0$. Passage from C^{∞} coefficients to the general case can be done by using approximation techniques that we do not consider here.

6.3 Weak Harnack inequality

From now on we assume that $a_{ij} \in C^{\infty}(\overline{\Omega})$ and that any L-harmonic function u is classical, that is, belongs to $C^2(\Omega)$. In fact, by Corollary 2.13, we have $u \in C^{\infty}(\Omega)$.

As in the case of the divergence form operator, we will concentrate on the proof of the weak Harnack inequality for L-harmonic functions. Then both Theorems 6.1 and 6.2 follow in the same way as for the divergence form case. Hence, our main goal is the following theorem.

Theorem 6.3 (Weak Harnack inequality for non-divergence form operator) *Let u be a non-negative L-harmonic function in a ball $B_{4R} \subset \Omega$. Choose any $a > 0$ and consider the set*

$$E = \{u \geq a\} \cap B_R.$$

If, for some $\theta > 0$,

$$|E| \geq \theta |B_R|,$$

then

$$\inf_{B_R} u \geq \delta a,$$

where $\delta = \delta(n, \lambda, \theta) > 0$.

We present here the proof devised by E.Landis shortly after Krylov and Safonov announced the proofs of Theorems 6.1 and 6.2. This proof has advantage that it is in many ways similar to the proof in the divergence form case.

6.4 Classical solution of the Dirichlet problem

In the present setting of a non-divergence form operator, the proof of the Harnack inequality uses a highly non-trivial theorem of Alexandrov-Pucci that we state below and that provides an estimate of solution of the corresponding Dirichlet problem. We precede it by the statement of the existence result that we also need.

Theorem 6.4 *Let $B_R \subset \Omega$ and $f \in C^\infty\left(\overline{B}_R\right)$. Then the classical Dirichlet problem*

$$\begin{cases} Lu = f \text{ in } B_R \\ u = 0 \text{ on } \partial B_R \end{cases} \tag{6.3}$$

has a solution $u \in C^2\left(B_R\right) \cap C\left(\overline{B}_R\right)$.

Approach to the proof. Rewrite the operator L in the form

$$\begin{aligned} Lu &= \sum_{i,j=1}^n \partial_i\left(a_{ij}\partial_j u\right) - \sum_{j=1}^n \left(\sum_{i=1}^n \partial_i a_{ij}\right)\partial_j u \\ &= \sum_{i,j=1}^n \partial_i\left(a_{ij}\partial_j u\right) + \sum_{j=1}^n b_j \partial_j u, \end{aligned}$$

where

$$b_j = \sum_{i=1}^n \partial_i a_{ij}.$$

Then we need the classical solvability of the Dirichlet problem for the divergence form operator with lower order terms and with smooth coefficients.

Since L has now a divergence form, we can consider first the weak Dirichlet problem

$$\begin{cases} Lu = f \text{ weakly in } B_R, \\ u \in W_0^{1,2}\left(B_R\right). \end{cases}$$

By Theorem 2.14, this problem has a solution $u \in C^\infty\left(B_R\right)$, that is hence a classical solution of $Lu = f$.

We need still to ensure the boundary condition $u = 0$ in the classical sense. For the operators without lower order terms b_j the corresponding result is contained in Theorem 4.6. With the terms b_j one basically has to repeat all the theory of Hölder regularity (both interior and up to the boundary) and then to arrive to a version of Theorem 4.6 for the operator with lower order terms. We skip this part. ∎

Theorem 6.5 (Theorem of Alexandrov-Pucci) *If u is a classical solution of the Dirichlet problem (6.3) with $f \in C(\overline{\Omega})$ then the following estimate is true:*

$$\sup_{B_R} |u| \le CR \|f\|_{L^n(B_R)},$$

where $C = C(n, \lambda)$.

We present this theorem without proof.

6.5 Three lemmas

In this section we prove three lemmas needed for the proof of the weak Harnack inequality.

Lemma 6.6 *Let u be an L-harmonic function in Ω and assume that $u \geq 0$ in a ball $B_{4R}(z) \subset \Omega$. Choose any $a > 0$ and consider the set*

$$E = \{u \geq a\} \cap B_R(z).$$

If the set E contains a ball $B_r(y)$ then

$$\inf_{B_R(z)} u \geq c \left(\frac{r}{R}\right)^s a,$$

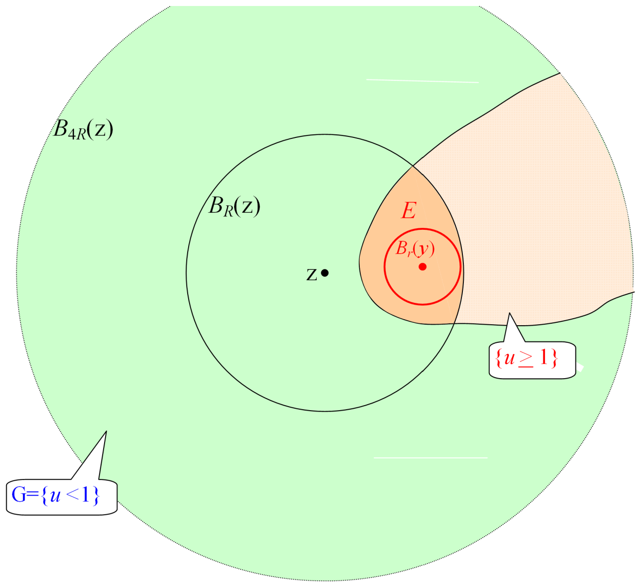
where $s = s(n, \lambda) > 0$ and $c = c(n, \lambda) > 0$.

Proof. Without loss of generality, we can take $a = 1$, so that

$$E = \{u \geq 1\} \cap B_R(z).$$

Assume also for simplicity that y is the origin of $\mathbb{R}^n$. Consider the set

$$G = \{u < 1\} \cap B_{4R}(z).$$



Fix some $s > 0$ to be chosen later, and consider the following function

$$w(x) = \left(\frac{1}{|x|^s} - \frac{1}{(3R)^s}\right) r^s$$

Since the origin is at y, outside the ball $B_r(y)$ we have $|x| \geq r$, whence

$$w(x) \leq 1 \quad \text{outside } B_r(y).$$

Since by hypotheses $B_r(y) \subset E$ and hence $B_r(y) \cap G = \emptyset$, it follows that

$$w(x) \leq 1 \text{ on } \overline{G}.$$

Since on $\partial B_{4R}(z)$ we have $|x| \geq 3R$, it follows that

$$w(x) \leq 0 \ \text{ on } \partial B_{4R}(z).$$

Recall that by Exercise 5 (b) we have in $\mathbb{R}^n \setminus \{0\}$

$$L|x|^{-s} > 0$$

provided $s > n\lambda^2 - 2$. Choose one of such values of s, for example, $s = n\lambda^2$. Since $G \subset \mathbb{R}^n \setminus \{0\}$, we obtain

$$Lw > 0 \text{ in } G.$$

As we have seen above, the values of w on ∂G are as follows:

$$\begin{aligned} w &\leq 1 \ \text{ on } \partial G \cap B_{4R}(z) \\ w &\leq 0 \ \text{ on } \partial G \cap \partial B_{4R}(z). \end{aligned}$$

Let us compare w with u in G. The function u satisfies

$$Lu = 0 \ \text{ in } G$$

and the boundary conditions:

$$\begin{aligned} u &\geq 1 \text{ on } \partial G \cap B_{4R}(z), \\ u &\geq 0 \text{ on } \partial G \cap \partial B_{4r}(z). \end{aligned}$$

Using the comparison principle of Exercise 2, we conclude that

$$u \geq w \ \text{ in } G.$$

It follows that

$$\inf_{B_R(z)} u = \inf_{B_R(z) \cap G} u \geq \inf_{B_R(z) \cap G} w \geq \inf_{B_R(z)} w.$$

Since in $B_R(z)$ we have $|x| \leq 2R$, it follows that in $B_R(z)$

$$w(x) \geq \left(\frac{1}{(2R)^s} - \frac{1}{(3R)^s} \right) r^s = c \left(\frac{r}{R} \right)^s,$$

where $c = 2^{-s} - 3^{-s} > 0$. We conclude that

$$\inf_{B_R(z)} u \geq c \left(\frac{r}{R} \right)^s,$$

which was to be proved. ■

Lemma 6.7 (Lemma of growth in a thin domain) *Let u be a non-negative L-harmonic function in a ball $B_R \subset \Omega$. There exists $\varepsilon = \varepsilon(n, \lambda) > 0$ with the following property: if for some $a > 0$*

$$\frac{|\{u < a\} \cap B_R|}{|B_R|} \leq \varepsilon,$$

then

$$\inf_{B_{R/4}} u \geq \frac{1}{2} a.$$

Restating this lemma in terms of the function $v = a - u$ with $a = \sup_{B_R} u$ yields the following: if v is L-harmonic in B_R and

$$\frac{|\{v > 0\} \cap B_R|}{|B_R|} \leq \varepsilon$$

then

$$\sup_{B_R} u \geq 2 \sup_{B_R/4} u.$$

This formulation matches that of Lemma 5.4 for the divergence form operators (except for the value 2 instead of 4, which is unimportant).

Proof. Assume that the ball B_R is centered at the origin. Without loss of generality set $a = 1$, and consider the set

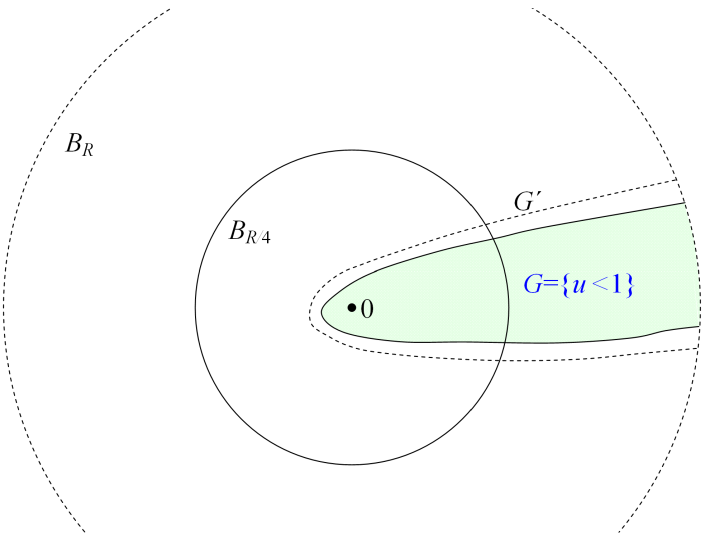
$$G = \{u < 1\} \cap B_R.$$

Since $|G| < \varepsilon |B_R|$, there exists an open set G' in B_R such that

$$\overline{G} \cap B_R \subset G'$$

and

$$|G'| < 2\varepsilon |B_R| \tag{6.4}$$



Choose a function $f \in C^{\infty}\left(\overline{B}_R\right)$ such that

$$0 \leq f \leq 1, \quad f = 1 \text{ on } G, \quad f = 0 \text{ outside } G'.$$

By Theorem 6.4, the following Dirichlet problem

$$\begin{cases} Lv = -f \text{ in } B_R \\ v = 0 \text{ on } \partial B_R \end{cases}$$

has a classical solution $v \in C^2(B_R) \cap C\left(\overline{B}_R\right)$. Since $Lv \leq 0$, it follows by the minimum principle that $v \geq 0$ in B_R. By Theorem 6.5 of Alexandrov and Pucci,

$$\sup_{B_R} v \leq CR \|f\|_{L^n(B_R)} \leq CR \left|G'\right|^{1/n} \leq C' R^2 \varepsilon^{1/n}, \tag{6.5}$$

where we have also used (6.4). Consider now the function

$$w(x) = c_1 - c_2 |x|^2 - c_3 v(x)$$

where c_1, c_2, c_3 are positive constant to be chosen. We would like w to satisfy the same conditions as in the previous proof:

(i) $Lw \geq 0$ in G

(ii) $w \leq 1$ in $\overline{G}$

(iii) $w \leq 0$ on ∂B_R

We have in G

$$\begin{aligned} Lw &= -c_2 L|x|^2 - c_3 Lv \\ &= -2c_2 \sum_{i=1}^{n} a_{ii}(x) + c_3 f \\ &\geq -2c_2 \lambda n + c_3 f \\ &\geq -2c_2 \lambda n + c_3, \end{aligned}$$

where we have used that $f = 1$ on G. Hence, in order to satisfy (i), the constants c_2 and c_3 should satisfy

$$c_3 \geq 2c_2 \lambda n.$$

In G we have $w(x) \leq c_1$; hence, (ii) is satisfied if

$$c_1 \leq 1.$$

Finally, on ∂B_R we have $|x| = R$ and, hence,

$$w(x) \leq c_1 - c_2 R^2.$$

Hence, to satisfy (iii) we should have

$$c_1 \leq c_2 R^2.$$

Therefore, we choose c_1, c_2, c_3 as follows:

$$\begin{aligned} c_1 &= 1 \\ c_2 &= R^{-2} \\ c_3 &= 2c_2\lambda n = \frac{2\lambda n}{R^2}. \end{aligned}$$

Comparing w with u as in the previous proof, we obtain again that $u \geq w$ in G. Hence, we have

$$\inf_{B_{R/4}} u = \inf_{B_{R/4} \cap G} u \geq \inf_{B_{R/4} \cap G} w \geq \inf_{B_{R/4}} w.$$

In $B_{R/4}$ we have, using (6.5),

$$\begin{aligned} w(x) &\geq c_1 - c_2 (R/4)^2 - c_3 \sup v \\ &\geq c_1 - c_2 (R/4)^2 - c_3 C' R^2 \varepsilon^{1/n} \\ &= 1 - \frac{1}{16} - 2\lambda n C' \varepsilon^{1/n}. \end{aligned}$$

Choosing ε small enough depending on λ and n, we obtain

$$\inf_{B_{R/4}} w \geq \frac{1}{2},$$

which finishes the proof. ∎

Lemma 6.8 *Under conditions of Lemma 6.7, if*

$$\frac{\left|\{u < a\} \cap B_{R/4}\right|}{\left|B_{R/4}\right|} \leq \varepsilon$$

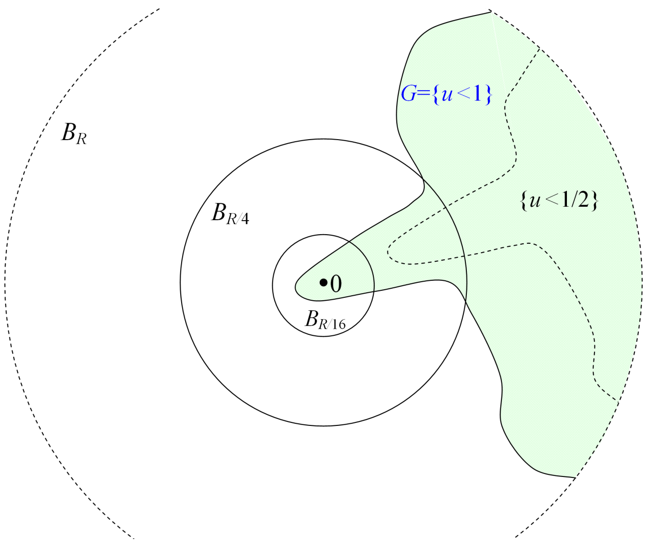
then

$$\inf_{B_{R/4}} u \geq \gamma a,$$

where $\gamma = \gamma(n, \lambda) > 0$.

Proof. Let $a = 1$ and let ε be from Lemma 6.7. Applying Lemma 6.7 to the ball $B_{R/4}$ instead of B_R, we obtain that

$$\inf_{B_{R/16}} u \geq \frac{1}{2}.$$



Hence, the set $\left\{u \geq \frac{1}{2}\right\} \cap B_{R/4}$ contains the ball $B_{R/16}$. Applying Lemma 6.6, we obtain

$$\inf_{B_{R/4}} u \geq c\left(\frac{R/16}{R/4}\right)^s \frac{3}{4} = c4^{-s}\frac{1}{2} =: \gamma,$$

which finishes the proof. ∎

6.6 Proof of the weak Harnack inequality

Set without loss of generality $a = 1$. Let u be a non-negative L-harmonic function in a ball $B_{4R} \subset \Omega$. Assuming that the set

$$E = \{u \geq 1\} \cap B_R$$

satisfies the condition

$$|E| \geq \theta\,|B_R|\,,$$

where $\theta > 0$, we need to prove that

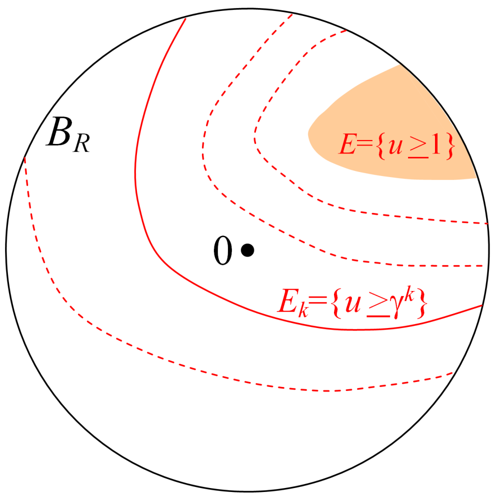
$$\inf_{B_R} u \geq \delta,$$

where $\delta = \delta\,(n, \lambda, \theta) > 0$.

Consider for any non-negative integer k the set

$$E_k = \left\{u \geq \gamma^k\right\} \cap B_R,$$

where $\gamma \in (0, 1)$ is the constant from Lemma 6.8.



The main part of the proof is contained in the following claim.

Claim. *There exist $\beta = \beta\,(n, \lambda) > 0$ and a positive integer $l = l\,(n, \lambda, \theta)$ such that, for any $k \geq 0$ the following dichotomy holds:*

(i) *either*

$$|E_{k+1}| \geq (1+\beta) |E_k|$$

(ii) *or*

$$E_{k+l} = B_R.$$

Let us first show how this Claim allows to finish the proof. Since the function u in B_R is bounded, the case (1) cannot holds for all k. Let N be the minimal value of k such that (i) does not holds for $k = N$. In other words, (i) holds for $k = 0, ..., N-1$ but does not holds for $k = N$. Hence, (ii) holds for $k = N$.

It follows that

$$|E_N| \geq (1+\beta) |E_{N-1}| \geq ... \geq (1+\beta)^N |E_0| .$$

Since $|E_N| \leq |B_R|$ and $|E_0| = |E| \geq \theta |B_R|$, it follows that

$$(1+\beta)^N \leq \frac{1}{\theta}$$

whence

$$N \leq \frac{\ln \frac{1}{\theta}}{\ln (1+\beta)}.$$

On the other hand, applying (ii) for $k = N$, we obtain

$$E_{N+l} = B_R$$

that is,

$$\inf_{B_R} u = \inf_{E_{N+l}} u \geq \gamma^{N+l} \geq \gamma^{\frac{\ln \frac{1}{\theta}}{\ln(1+\beta)}+l} =: \delta,$$

which finished the proof of the weak Harnack inequality.

Now let us prove the above Claim. It suffices to prove it for the special case $k = 0$, that is,

(i) either $|E_1| \geq (1+\beta) |E_0|$

(ii) or $E_l = B_R$.

Indeed, if it is proved for $k = 0$, then for a general k consider the function $v = u/\gamma^k$. Consider the sets

$$\widetilde{E}_j = \left\{v \geq \gamma^j\right\} \cap B_R$$

where j is a non-negative integer. Clearly, we have

$$E_{k+j} = \left\{u \geq \gamma^{k+j}\right\} \cap B_R = \left\{v \geq \gamma^j\right\} \cap B_R = \widetilde{E}_j.$$

In particular, $E_k = \widetilde{E}_0$ and $E_{k+1} = \widetilde{E}_1$. Hence, applying the special case of the Claim to function v, we obtain the general case of the Claim for function u.

Hence, let us prove the above special case $k = 0$. Let us reformulate it in the following equivalent way:

(i) either $|E_1| \geq (1+\beta)|E_0|$

(ii) or $\inf_{B_R} u \geq \delta$, where $\delta = \delta(n, \lambda, \theta) > 0$.

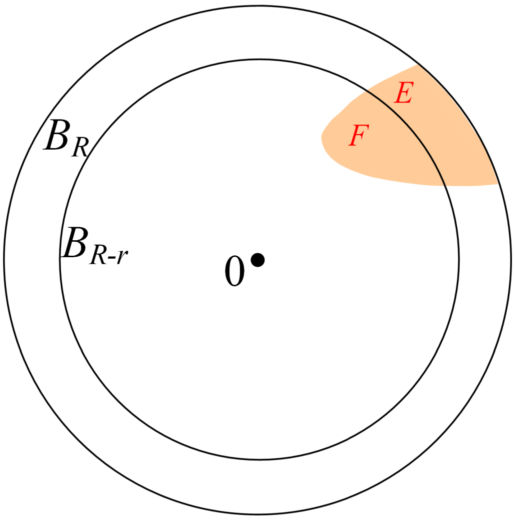
Indeed, if the latter condition holds then we find l such that $\gamma^l \leq \delta$, and obtain $E_l = B_R$.

Choose $r < R$ such that

$$|E \cap B_{R-r}| = \frac{1}{2}|E| \tag{6.6}$$

and set

$$F := E \cap B_{R-r} = \{u \geq 1\} \cap B_{R-r}.$$

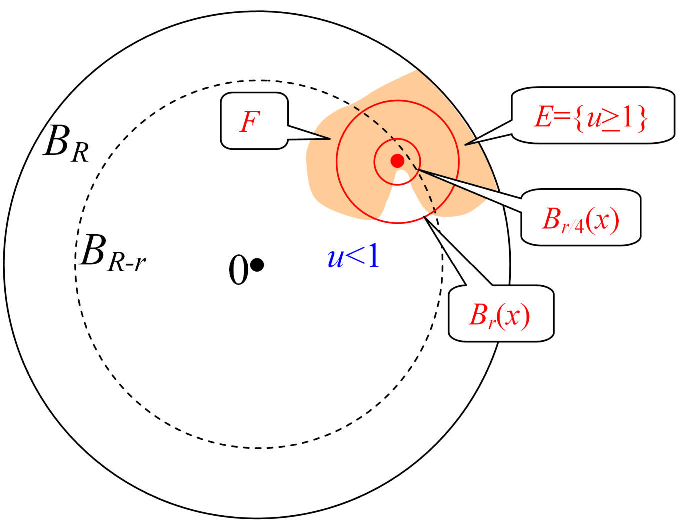


Consider two cases.

Case 1. Assume that there exists $x \in F$ such that

$$\frac{|\{u<1\} \cap B_r(x)|}{|B_r|} \leq \varepsilon,$$

where $\varepsilon = \varepsilon(n, \lambda) > 0$ is the constant from Lemma 6.7.



Then by Lemma 6.7 we have

$$\inf_{B_{r/4}(x)} u \geq \frac{1}{2}.$$

Note that $B_{r/4}(x) \subset B_R$. Hence, in B_R there is a ball $B_{r/4}(x)$ where $u \geq \frac{1}{2}$. Applying Lemma 6.6, we conclude that

$$\inf_{B_R} u \geq c \left(\frac{r/4}{R}\right)^s \frac{1}{2}.$$

From (6.6) we have

$$|B_R| - |B_{R-r}| = |B_R \setminus B_{R-r}| \geq |E \setminus B_{R-r}| = \frac{1}{2}|E| \geq \frac{1}{2}\theta |B_R|$$

which implies after division by B_R that

$$1 - \left(\frac{R-r}{R}\right)^n \geq \frac{1}{2}\theta.$$

It follows that

$$\frac{r}{R} \geq 1 - \left(1 - \frac{1}{2}\theta\right)^{1/n}.$$

Hence, we obtain

$$\inf_{B_R} u \geq \frac{c}{2} 4^{-s} \left(1 - \left(1 - \frac{1}{2}\theta\right)^{1/n}\right)^s =: \delta > 0,$$

which means that the alternative (ii) takes places.

Case 2 (main). Assume that, for any $x \in F$, we have

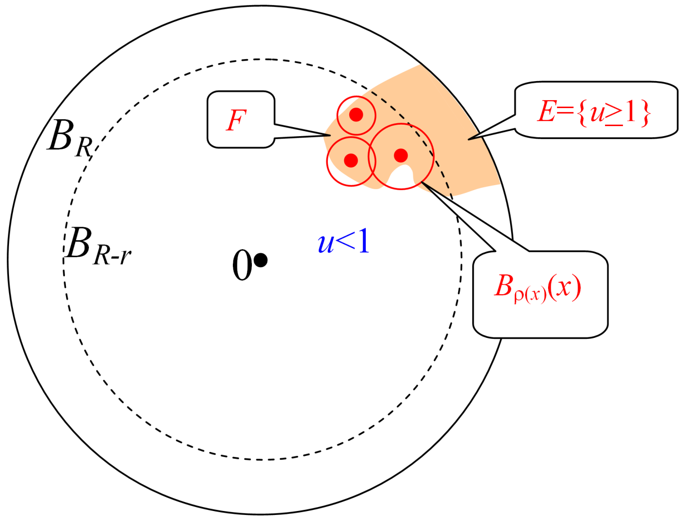
$$\frac{|\{u < 1\} \cap B_r(x)|}{|B_r|} \geq \varepsilon.$$

For any $x \in F$ and $\rho > 0$ consider the quotient:

$$\frac{|\{u < 1\} \cap B_\rho(x)|}{|B_\rho|}.$$

As $\rho \to 0$, this quotient goes to 0 for almost all $x \in F$ because in F we have $u \geq 1$. On the other hand, for $\rho = r$, this quotient is $\geq \varepsilon$. Hence, for almost all $x \in F$, there exists $\rho(x) \in (0, r)$ such that

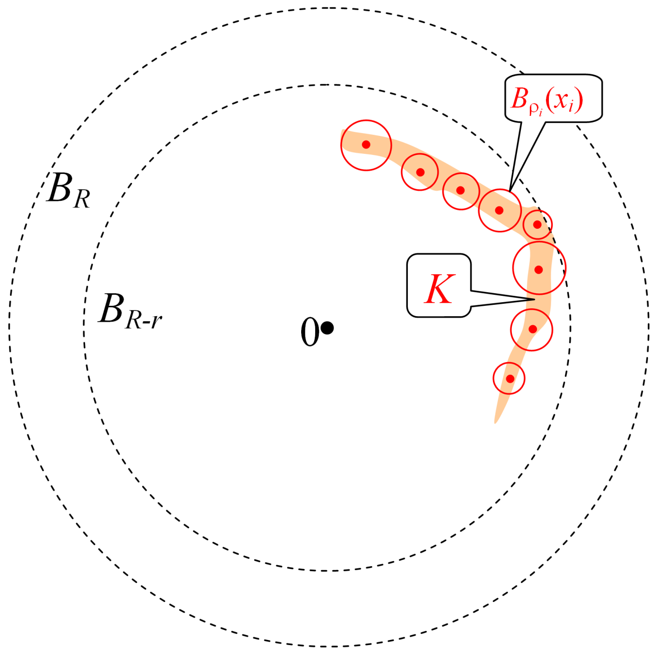
$$\frac{\left|\{u < 1\} \cap B_{\rho(x)}(x)\right|}{\left|B_{\rho(x)}\right|} = \varepsilon. \tag{6.7}$$



Denote this set of points x by F', so that $F' \subset F$ and $|F'| = |F|$. By the property of the Lebesgue measure, there is a compact set $K \subset F'$ such that

$$|K| \geq \frac{1}{2}|F'| = \frac{1}{2}|F| = \frac{1}{4}|E|.$$

The family of ball $\{B_{\rho(x)}(x)\}_{x \in K}$ forms an open covering of K. Choose a finite subcover $\{B_{\rho_i}(x_i)\}$ where $\rho_i = \rho(x_i)$. By the standard ball covering argument, we can pass to a subsequence and, hence, assume that the balls $\{B_{\rho_i}(x_i)\}$ are disjoint while $\{B_{3\rho_i}(x_i)\}$ cover K.



Observe that $x_i \in B_{R-r}$, whence

$$|x_i| + 4\rho_i \leq R - r + 4\rho_i \leq R + 3r \leq R + 3R = 4R.$$

Therefore, $B_{4\rho_i}(x_i) \subset B_{4R}$. We can apply in $B_{4\rho_i}(x_i)$ Lemma 6.8 because by (6.7)

$$\frac{\left|\{u<1\} \cap B_{\rho_i}(x_i)\right|}{\left|B_{\rho_i}\right|} = \varepsilon, \tag{6.8}$$

which yields

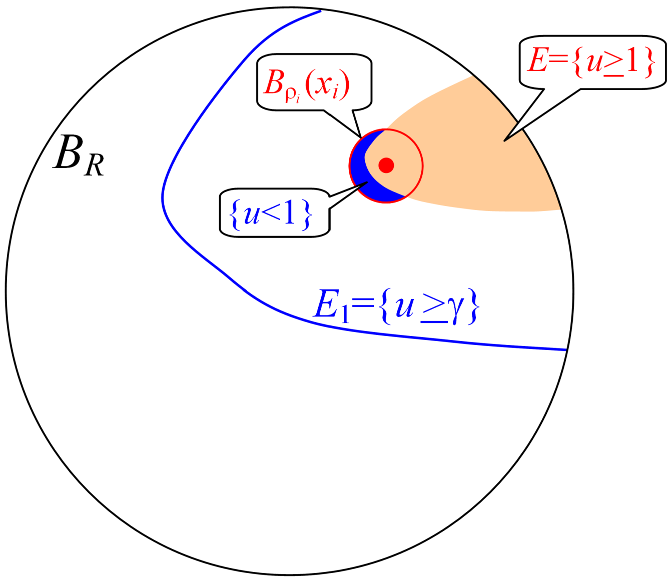
$$\inf_{B_{\rho_i}(x_i)} u \geq \gamma. \tag{6.9}$$

By construction, all balls $B_{\rho_i}(x_i)$ are contained B_R, which implies by (6.9) that

$$(E_1 \setminus E) \cap B_{\rho_i(x_i)} = \{\gamma \leq u < 1\} \cap B_{\rho_i(x_i)} = \{u<1\} \cap B_{\rho_i}(x_i).$$

Combining with (6.8), we obtain

$$\left|(E_1 \setminus E) \cap B_{\rho_i(x_i)}\right| = \varepsilon \left|B_{\rho_i}(x_i)\right|.$$



Adding up in i and using that all balls $B_{\rho_i}(x_i)$ are disjoint, we obtain

$$\begin{aligned} |E_1 \setminus E| &\geq \sum_i \varepsilon \left|B_{\rho_i}(x_i)\right| \\ &= 3^{-n} \sum_i \varepsilon \left|B_{3\rho_i}(x_i)\right| \\ &\geq 3^{-n} \varepsilon |K| \geq 3^{-n} \frac{\varepsilon}{4} |E|, \end{aligned}$$

whence

$$|E_1| \geq \left(1 + 3^{-n}\frac{\varepsilon}{4}\right) |E|,$$

thus proving the alternative (i) with $\beta = 3^{-n}\frac{\varepsilon}{4}$.