Mock exam "Elliptic Partial Differential Equations", WS 2023/24

Duration of the exam 120 minutes. Each problem is worth of 25 points.
Full mark (Note 1,0) ≈ 90-95 points, pass mark (Note 4,0) ≈ 45-50 points.
No scripts, books, calculators, computers etc. are allowed.

Problem 1

Consider the differential operator L

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x)\, \partial_{ij} u + \sum_{k=1}^{n} b_k(x) \partial_k u,$$

where a_{ij} and b_k are continuous functions of x defined in an open subset D of $\mathbb{R}^n$. Assume that the operator L is elliptic at any point, that is, the matrix $\left(a_{ij}(x)\right)_{i,j=1}^{n}$ is positive definite at any point $x \in D$. Prove the maximum principle: if Ω is a bounded domain such that $\overline{\Omega} \subset D$ and a function $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies in Ω the inequality $Lu \geq 0$ then

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

Problem 2

State and prove the Lax-Milgram theorem (about bounded coercive bilinear forms in Hilbert spaces).

Problem 3

Consider in a bounded domain Ω a uniformly elliptic operator

$$Lu = \sum_{i,j=1}^{n} \partial_i \left(a_{ij} \partial_j u\right) \tag{1}$$

with measurable coefficients $a_{ij}(x)$. Let u be a solution of the weak Dirichlet problem

$$\begin{cases} Lu = f \text{ weakly in } \Omega, \\ u \in W_0^{1,2}(\Omega). \end{cases}$$

Prove that

$$\|u\|_{W^{1,2}} \leq \lambda \left(D + D^2\right) \|f\|_{L^2},$$

where λ is the ellipticity constant of L and $D = \operatorname{diam}(\Omega)$. *Hint*: State (without proof) and use the Friedrichs inequality.

Problem 4

State a theorem about existence of second order weak derivatives for weak solutions of the equations $Lu = f$, where L is a uniformly elliptic operator (1) with locally Lipschitz coefficients. Give the proof in a special case when $u \in W_c^{1,2}(\Omega)$ and $f \in L^2(\Omega)$.

Problem 5

State (without proof) the weak Harnack inequality for supersolutions. State and prove the oscillation inequality for weak solutions.