# Partial Differential Equations

Alexander Grigorian
Universität Bielefeld

WS 2015/16

# Contents

# Chapter 0

# Introduction

21.10.15

## 0.1 Examples of PDEs and their origin

Let $u = u(x_1, ..., x_n)$ be a real-valued function of $n$ independent real variables $x_1, ..., x_n$. Recall that, for any multiindex $\alpha = (\alpha_1, ..., \alpha_n)$ where $\alpha_i$ are non-negative integers, the expression $D^\alpha u$ denotes the following partial derivative of $u$:

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} ... \partial x_n^{\alpha_n}},$$

where $|\alpha| = \alpha_1 + ... + \alpha_n$ is the order of the derivative.

A *partial differential equation* (PDE) is an equation with an unknown function $u = u(x_1, .., x_n)$ of $n > 1$ independent variables, which contains partial derivatives of $u$. That is, a general PDE looks as follows:

$$F\left(D^\alpha u, D^\beta u, D^\gamma u, ...\right) = 0 \tag{0.1}$$

where $F$ is a given function, $u$ is unknown function, $\alpha, \beta, \gamma, ...$ are multiindices.

Of course, the purpose of studying of any equation is to develop methods of solving it or at least ensuring that it has solutions. For example, in the theory of *ordinary differential equations* (ODEs) one considers an unknown function $u(x)$ of a single real variable $x$ and a general ODE

$$F(u, u', u'', ...) = 0$$

and proves theorems about solvability of such an equation with initial conditions, under certain assumptions about $F$ (Theorem of Picard-Lindelöf). One also develops methods of solving explicitly certain types of ODEs, for example, linear ODEs.

In contrast to that, there is no theory of general PDEs of the form (0.1). The reason for that is that the properties of PDEs depend too much of the function $F$ and cannot be stated within a framework of one theory. Instead one develops theories for narrow classes of PDEs or even for single PDEs, as we will do in this course.

Let us give some examples of PDEs that arise in applications, mostly in Physics. These examples have been motivating development of Analysis for more than a century. In fact, a large portion of modern Analysis has emerged in attempts of solving those special PDEs.

### 0.1.1 Laplace equation

Let $\Omega$ be an open subset of $\mathbb{R}^n$ and let $u : \Omega \to \mathbb{R}$ be a function that twice continuously differentiable, that is, $u \in C^2(\Omega)$. By $\Delta u$ we denote the following function

$$\Delta u = \sum_{k=1}^{n} \partial_{x_k x_k} u,$$

that is, $\Delta u$ is the sum of all unmixed partial derivatives of $u$ of the second order. The differential operator

$$\Delta = \sum_{k=1}^{n} \partial_{x_k x_k}$$

is called the *Laplace operator*, so that $\Delta u$ is the result of application to $u$ of the Laplace operator.

The *Laplace equation* is a PDE of the form

$$\Delta u = 0.$$

Any function $u$ that satisfies the Laplace equation is called a *harmonic function.* Of course, any affine function

$$u(x) = a_1 x_1 + ... + a_n x_n + b$$

with real coefficients $a_1, ..., a_n, b$ is harmonic because all second order partial derivatives of $u$ vanish. However, there are more interesting examples of harmonic functions. For example, in $\mathbb{R}^n$ with $n \geq 3$ the function

$$u(x) = \frac{1}{|x|^{n-2}}$$

is harmonic away from the origin, where

$$|x| = \sqrt{x_1^2 + ... + x_n^2}$$

is the Euclidean norm of $x$. In $\mathbb{R}^2$ the function

$$u(x) = \ln|x|$$

is harmonic away from the origin.

It is easy to see that the Laplace operator $\Delta$ is linear, that is,

$$\Delta(u + v) = \Delta u + \Delta v$$

and

$$\Delta(cv) = c\Delta u$$

for all $u, v \in C^2$ and $c \in \mathbb{R}$. It follows that linear combinations of harmonic functions are harmonic.

A more general equation

$$\Delta u = f$$

where $f : \Omega \to \mathbb{R}$ is a given function, is called the *Poisson equation.* The Laplace and Poisson equations are most basic and most important examples of PDEs.

Let us discuss some origins of the Laplace and Poisson equations.

### Holomorphic function

Recall that a complex valued function $f(z)$ of a complex variable $z = x + iy$ is called holomorphic (or analytic) if it is $\mathbb{C}$-differentiable. Denoting $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$, we obtain functions $u(x, y)$ and $v(x, y)$ of two real variables $x, y$.

It is known from the theory of functions of complex variables that if $f$ is holomorphic then $u, v$ satisfy the Cauchy-Riemann equations

$$\begin{cases} \partial_x u = \partial_y v, \\ \partial_y u = -\partial_x v. \end{cases} \tag{0.2}$$

Assuming that $u, v \in C^2$ (and this is necessarily the case for holomorphic functions), we obtain from (0.2)

$$\partial_{xx} u = \partial_x \partial_y v = \partial_y \partial_x v = -\partial_{yy} u$$

whence

$$\Delta u = \partial_{xx} u + \partial_{yy} u = 0.$$

In the same way $\Delta v = 0$. Hence, both $u, v$ are harmonic functions.

This observation allows us to produce many examples of harmonic functions in $\mathbb{R}^2$ starting from holomorphic functions. For example, for $f(z) = e^z$ we have

$$e^z = e^{x+iy} = e^x (\cos y + i \sin y)$$

which yields the harmonic functions $u(x, y) = e^x \cos y$ and $v(x, y) = e^x \sin y$.

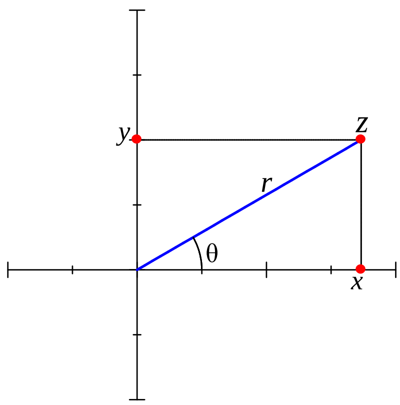
For $f(z) = z^2$ we have

$$z^2 = (x + iy)^2 = \left(x^2 - y^2\right) + 2xyi,$$

so that the functions $u = x^2 - y^2$ and $v = 2xy$ are harmonic.

For the function $f(z) = \ln z$ that is defined away from the negative part of the real axis, we have, using the polar form $z = re^{i\theta}$ of complex numbers that

$$\ln z = \ln r + i\theta.$$




Since $r = |z|$ and $\theta = \arg z = \arctan \frac{y}{x}$, it follows that the both functions $u = \ln |z| = \ln \sqrt{x^2 + y^2}$ and $u = \arctan \frac{y}{x}$ are harmonic.

**Gravitational field**

By Newton's law of gravitation of 1686, any two point masses $m, M$ are attracted each to other by the gravitational force $F = \gamma \frac{Mm}{r^2}$ where $r$ is the distance between the points and $\gamma$ is the gravitational constant. Assume that the point mass $M$ is located constantly at the origin of $\mathbb{R}^3$ and that the point mass $m$ is moving and let its current position be $x \in \mathbb{R}^3$. Taking for simplicity $\gamma = m = 1$, we obtain that the force acting at the moving mass is $F = \frac{M}{|x|^2}$ and it is directed from $x$ to the origin. The vector $\overrightarrow{F}$ of the force is then equal to

$$\overrightarrow{F} = \frac{M}{|x|^2}\left(-\frac{x}{|x|}\right) = -M\frac{x}{|x|^3}.$$

Any function $\overrightarrow{F}$ defined in a domain of $\mathbb{R}^n$ and taking values in $\mathbb{R}^n$ is called a *vector field*. The vector field $\overrightarrow{F}(x) = -M\frac{x}{|x|^3}$ in $\mathbb{R}^3$ is called the *gravitational field* of the point mass $M$.

A real-value function $U(x)$ in $\mathbb{R}^n$ is called a *potential* of a vector field $\overrightarrow{F}(x)$ in $\mathbb{R}^n$ if

$$\overrightarrow{F}(x) = -\nabla U(x),$$

where $\nabla U$ is the gradient of $U$ defined by

$$\nabla U = (\partial_{x_1} U, ..., \partial_{x_n} U).$$

Not every vector field has a potential; if it does then it is called *conservative*. Conservative fields are easier to handle as they can be described by one scalar function $U(x)$ instead of a vector function $\overrightarrow{F}(x)$.

It can be checked that the following function

$$U(x) = -\frac{M}{|x|}$$

is a potential of the gravitational field $\overrightarrow{F} = -M\frac{x}{|x|^3}$. It is called the *gravitational potential* of the point mass $M$ sitting at the origin.

If $M$ is located at another point $y \in \mathbb{R}^3$, then the potential of it is

$$U(x) = -\frac{M}{|x-y|}.$$

More generally, potential of a mass distributed in a closed region $D$ is given by

$$U(x) = -\int_D \frac{\rho(y)\,dy}{|x-y|}, \tag{0.3}$$

where $\rho(y)$ is the density of the matter at the point $y \in D$. In particular, the gravitational force of any mass is a conservative vector field.

As we have mentioned above, the function $\frac{1}{|x|^{n-2}}$ is harmonic in $\mathbb{R}^n$ away from the origin. As a particular case, we see that $\frac{1}{|x|}$ is harmonic in $\mathbb{R}^3$ away from the origin. It follows that the potential $U(x) = -\frac{M}{|x|}$ is harmonic away from the origin and the potential $U(x) = -\frac{M}{|x-y|}$ is harmonic away from $y$. One can deduce that also the function $U(x)$ given by (0.3) is harmonic away from $D$.

Historically, it was discovered by Pierre-Simon Laplace in 1784-85 that a gravitational field of any body is a conservative vector field and that its potential $U(x)$ satisfies in a free space the equation $\Delta U = 0$, which is called henceforth the Laplace equation. The latter can be used for actual computation of gravitational potentials even without knowledge of the density $\rho$.

#### Electric force

By Coulomb's law of 1784, magnitude of the electric force $F$ between two point electric charges $Q, q$ is equal to $k\frac{Qq}{r^2}$ where $r$ is the distance between the points and $k$ is the Coulomb constant. Assume that the point charge $Q$ is located at the origin and the point charge $q$ at a variable position $x \in \mathbb{R}^3$. Taking for simplicity that $k = q = 1$, we obtain $F = \frac{Q}{|x|^2}$ and that this force is directed from the origin to $x$ if $Q > 0$, and from $x$ to the origin if $Q < 0$ (indeed, if the both charges are positive then the electric force between them is repulsive, unlike the case of gravitation when the force is attractive). Hence, the vector $\overrightarrow{F}$ of the electric force is given by

$$\overrightarrow{F} = \frac{Q}{|x|^2}\frac{x}{|x|} = Q\frac{x}{|x|^3}.$$

This vector field is potential, and its potential is given by $U(x) = \frac{Q}{|x|}$.

If a distributed charge is located in a closed domain $D$ with the charge density $\rho$, then the electric potential of this charge is given by

$$U(x) = \int_D \frac{\rho(y)\,dy}{|x-y|},$$

which is a harmonic function outside $D$.

### 0.1.2 Wave equation

#### Electromagnetic fields

In the case of fast moving charges one should take into account not only their electric fields but also the induced magnetic fields. In general, an electromagnetic field is described by two vector fields $\overrightarrow{E}(x,t)$ and $\overrightarrow{B}(x,t)$ that depend not only on a point $x \in \mathbb{R}^3$ but also on time $t$. If a point charge $q$ moves with velocity $\overrightarrow{v}$, then the electromagnetic field exerts the following force on this charge:

$$\overrightarrow{F} = q\overrightarrow{E} + q\overrightarrow{v} \times \overrightarrow{B}$$

This force is also called the Lorentz force.

The evolution of the electromagnetic field $(\overrightarrow{E}, \overrightarrow{B})$ is described by *Maxwell's equations*:

$$\begin{cases} \operatorname{div} \overrightarrow{E} = 4\pi\rho \\ \operatorname{div} \overrightarrow{B} = 0 \\ \operatorname{rot} \overrightarrow{E} = -\frac{1}{c}\partial_t B \\ \operatorname{rot} \overrightarrow{B} = \frac{1}{c}\left(4\pi\overrightarrow{J} + \partial_t E\right) \end{cases} \tag{0.4}$$

where

- $c$ is the speed of light;
- $\rho$ is the charge density;
- $\overrightarrow{J}$ is the current density;
- $\operatorname{div}\overrightarrow{F}$ is the divergence of a vector field $\overrightarrow{F}=(F_1,...,F_n)$ in $\mathbb{R}^n$ given by

$$\operatorname{div}\overrightarrow{F}=\sum_{k=1}^{n}\partial_{x_k}F_k\ ;$$

- $\operatorname{rot}\overrightarrow{F}$ is the rotation (curl) of a vector field $\overrightarrow{F}=(F_1,F_2,F_3)$ in $\mathbb{R}^3$ given by

$$\operatorname{rot}\overrightarrow{F}=\det\begin{pmatrix} i & j & k \\ \partial_{x_1} & \partial_{x_2} & \partial_{x_3} \\ F_1 & F_2 & F_3\end{pmatrix}=\left(\partial_{x_2}F_3-\partial_{x_3}F_2,\ \partial_{x_3}F_1-\partial_{x_1}F_3,\ \partial_{x_1}F_2-\partial_{x_2}F_1\right).$$

The equations (0.4) were formulated by James Clerk Maxwell in 1873.

Assume for simplicity that $\rho=0$ and $\overrightarrow{J}=0$. Then we have from the third equation

$$\operatorname{rot}(\operatorname{rot}\overrightarrow{E})=-\frac{1}{c}\partial_t(\operatorname{rot}\overrightarrow{B})=-\frac{1}{c^2}\partial_{tt}\overrightarrow{E}.$$

On the other hand, there is a general identity for any $C^2$ vector field $\overrightarrow{F}$ in $\mathbb{R}^3$:

$$\operatorname{rot}(\operatorname{rot}\overrightarrow{F})=\nabla(\operatorname{div}\overrightarrow{F})-\Delta\overrightarrow{F},$$

where $\Delta\overrightarrow{F}=(\Delta F_1,\Delta F_2,\Delta F_3)$. Applying it to $\overrightarrow{E}$ and using that $\operatorname{div}\overrightarrow{E}=0$, we obtain that

$$\Delta\overrightarrow{E}=\frac{1}{c^2}\partial_{tt}\overrightarrow{E}.$$

Denoting by $u$ any component of $\overrightarrow{E}$ we obtain that $u$ satisfies the *wave equation*

$$\partial_{tt}u=c^2\Delta u,$$

that is,

$$\partial_{tt}u=c^2\left(\partial_{x_1x_1}u+\partial_{x_2x_2}u+\partial_{x_3x_3}u\right).$$

Similarly, any component of $\overrightarrow{B}$ satisfies the wave equation. In particular, if the electric force $\overrightarrow{E}$ is stationary, that is, does not depend on time, then we obtain the Laplace equation $\Delta u=0$.

### Vibrating string

Vibrating strings are used in many musical instruments, such as pianos, guitars, etc. The frequency of the sound produced by a vibrating string can be determined mathematically using the string equation that we are going to derive.

Assume that initially the string rests on the $x$-axis and denote by $u(x,t)$ the vertical displacement of the string at the point $x \in \mathbb{R}$ at time $t$. Assume also that the oscillations of the string from the horizontal position are small. Under this assumption the horizontal component of the tension force in the string will have the constant value that we denote by $T$.

Fix time $t$ and denote by $\alpha_x$ the angle between the tangential direction at the point $(x, u(x,t))$ and the $x$-axis. Denote by $T_x$ the magnitude of tension at the point $x$. Note that the direction of the tension is tangential to the string. Since the shape of the string is given by the graph of function $x \mapsto u(x,t)$, we have

$$\tan \alpha_x = \partial_x u.$$

Since the horizontal component of tension is $T_x \cos \alpha_x$, we obtain

$$T_x \cos \alpha_x = T.$$

The net force acting on the piece $(x, x+h)$ of the string in the vertical direction is equal to

$$\begin{aligned} T_{x+h} \sin \alpha_{x+h} - T_x \sin \alpha_x &= T \frac{\sin \alpha_{x+h}}{\cos \alpha_{x+h}} - T \frac{\sin \alpha_x}{\cos \alpha_x} \\ &= T \partial_x u(x+h,t) - T \partial_x u(x,t). \end{aligned}$$

By Newton's second law, the net force is equal to $ma$ where $m$ is the mass of the piece $(x, x+h)$ and $a$ is the acceleration in the vertical direction. Since $m = \rho h$ where $\rho$ is the linear density of the string and $a = \partial_{tt} u$, we obtain the equation

$$T \partial_x u(x+h,t) - T \partial_x u(x,t) = \rho h \partial_{tt} u.$$

Dividing by $h$ and letting $h \to 0$, we obtain

$$T \partial_{xx} u = \rho \partial_{tt} u,$$

that is,

$$\partial_{tt} u = c^2 \partial_{xx} u$$

where $c = \sqrt{T/\rho}$. This is the vibrating string equation that coincides with the 1-dimensional wave equation.

### Vibrating membrane

Similarly, consider a two-dimensional membrane, that initially rests on the $(x_1, x_2)$-plane and denote by $u(x,t)$ the vertical displacement of the membrane at the point $x \in \mathbb{R}^2$ at time $t$. Assuming that the oscillations of the membrane from the horizontal position are small, one obtains the following equation

$$\partial_{tt} u = c^2 \left( \partial_{x_1 x_1} u + \partial_{x_2 x_2} u \right),$$

which is a two-dimensional wave equation.

In general we will consider an $n$-dimensional wave equation

$$\partial_{tt} = c^2 \Delta u$$

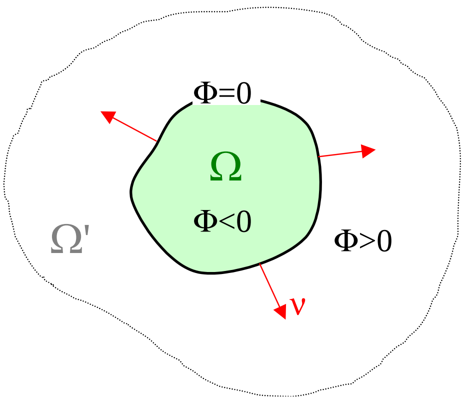
where $u = u(x,t)$ and $x \in \mathbb{R}^n$, $t \in \mathbb{R}$. Here $c$ is a positive constant, but we will see that $c$ is always the speed of wave propagation described by this equation. 23.10.15

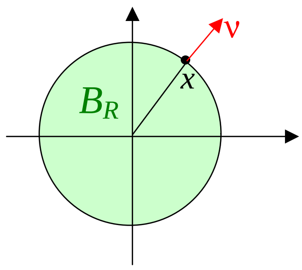
### 0.1.3 Divergence theorem

Recall the divergence theorem of Gauss. A bounded open set $\Omega \subset \mathbb{R}^n$ is called a *region* if there is a $C^1$ function $\Phi$ defined in an open neighborhood $\Omega'$ of $\overline{\Omega}$ such that

$$\begin{aligned} \Phi(x) &< 0 \text{ in } \Omega \\ \Phi(x) &= 0 \text{ on } \partial\Omega \\ \Phi(x) &> 0 \text{ in } \Omega' \setminus \overline{\Omega} \end{aligned}$$

and $\nabla\Phi \neq 0$ on $\partial\Omega$ (in words: $\Omega$ is a sublevel set of a $C^1$-function that is non-singular on $\partial\Omega$). The latter condition implies that $\partial\Omega$ is a $C^1$ hypersurface.




For any point $x \in \partial\Omega$ define the vector

$$\nu(x) = \frac{\nabla\Phi}{|\nabla\Phi|}.$$

The function $\nu : \partial\Omega \to \mathbb{R}^n$ is called the *outer unit normal vector field* on $\partial\Omega$.

For example, if $\Omega = B_R$ where

$$B_R = \{x \in \mathbb{R}^n : |x| < R\}$$

is the ball of radius $R$ centered at the origin, then $\Phi(x) = |x|^2 - R^2$ satisfies the above properties. Hence, the ball is a region. Since $\nabla\Phi = 2x$, we obtain that the outer unit normal vector field on $\partial B_R$ is

$$\nu(x) = \frac{x}{|x|}.$$

**Divergence theorem of Gauss.** *Let $\Omega$ be a region in $\mathbb{R}^n$ and $\nu$ the outer unit normal vector field on $\partial\Omega$. Then for any $C^1$ vector field $\overrightarrow{F} : \overline{\Omega} \to \mathbb{R}^n$ we have*

$$\int_\Omega \operatorname{div} \overrightarrow{F}(x)\, dx = \int_{\partial\Omega} \overrightarrow{F} \cdot \nu \, d\sigma, \tag{0.5}$$

*where $\sigma$ is the surface measure on $\partial\Omega$, $\operatorname{div}\overrightarrow{F}=\sum_{k=1}^{n}\partial_{x_k}F_k$ is the divergence of $\overrightarrow{F}$, and $\overrightarrow{F}\cdot\nu$ is the scalar product of the vectors $\overrightarrow{F},\nu$.*

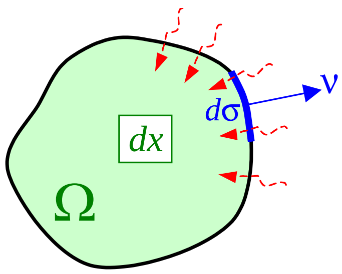
### 0.1.4 Heat equation

#### Heat conductivity

Let $u(x,t)$ denote the temperature in some medium at a point $x\in\mathbb{R}^3$ at time $t$. Fix a region $\Omega\subset\mathbb{R}^3$. The amount $Q$ of the heat energy that has flown into $\Omega$ through its boundary $\partial\Omega$ between the time moments $t$ and $t+h$ is equal to

$$Q=\int_t^{t+h}\left(\int_{\partial\Omega}k\partial_\nu u\,d\sigma\right)dt,$$

where $\nu$ is the outer unit normal vector field to $\partial\Omega$ and $k=k(x)$ is the thermal conductance of the material of the body.




Indeed, by the law of heat conductivity, discovered by Jean Baptiste Joseph Fourier in 1822, the influx of the heat energy through the surface element $d\sigma$ in unit time is proportional to the change of the temperature across $d\sigma$, that is to $\partial_\nu u$, and the coefficient of proportionality $k$ is determined by the physical properties of the material.

On the other hand, the amount of heat energy $Q'$ acquired by a region $\Omega\subset\mathbb{R}^3$ from time $t$ to time $t+h$ is equal to

$$Q'=\int_\Omega\left(u(x,t+h)-u(x,t)\right)c\rho dx,$$

where $\rho$ is the density of the material of the body and $c$ is its heat capacity (both $c$ and $\rho$ are functions of $x$). Indeed, the volume element $dx$ has the mass $\rho dx$, and increase of its temperature by one degree requires $c\rho dx$ of heat energy. Hence, increase of the temperature from $u(x,t)$ to $u(x,t+h)$ requires $\left(u(x,t+h)-u(x,t)\right)c\rho dx$ of heat energy.

By the law of conservation of energy, in the absence of heat sources we have $Q=Q'$, that is,

$$\int_t^{t+h}\left(\int_{\partial\Omega}k\partial_\nu u\,d\sigma\right)dt=\int_\Omega\left(u(x,t+h)-u(x,t)\right)c\rho dx.$$

Dividing by $h$ and passing to the limit as $h\to 0$, we obtain

$$\int_{\partial\Omega}k\partial_\nu u\,d\sigma=\int_\Omega(\partial_t u)\,c\rho dx.$$

Applying the divergence theorem to the vector field $\overrightarrow{F} = k\nabla u$, we obtain

$$\int_{\partial\Omega} k\partial_\nu u \, d\sigma = \int_{\partial\Omega} \overrightarrow{F} \cdot \nu = \int_\Omega \operatorname{div} \overrightarrow{F} dx = \int_\Omega \operatorname{div}(k\nabla u) \, dx,$$

which implies

$$\int_\Omega c\rho \, \partial_t u \, dx = \int_\Omega \operatorname{div}(k\nabla u) \, dx.$$

Since this identity holds for any region $\Omega$, it follows that the function $u$ satisfies the following *heat equation*

$$c\rho \, \partial_t u = \operatorname{div}(k\nabla u).$$

In particular, if $c, \rho$ and $k$ are constants, then, using that

$$\operatorname{div}(\nabla u) = \sum_{k=1}^n \partial_{x_k} (\nabla u)_k = \sum_{k=1}^n \partial_{x_k} \partial_{x_k} u = \Delta u,$$

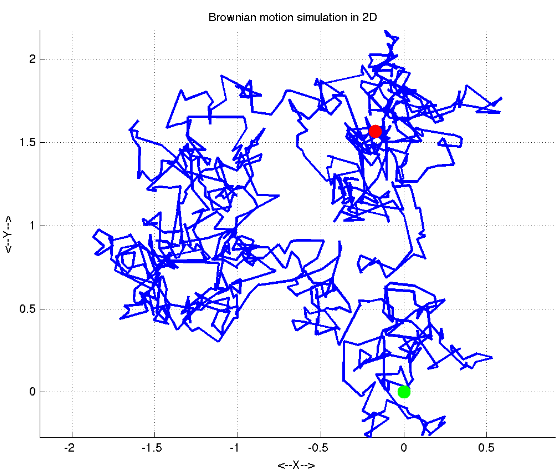
we obtain the simplest form of the heat equation

$$\partial_t u = a^2 \Delta u,$$

where $a = \sqrt{k/(c\rho)}$. In particular, if the temperature function $u$ is stationary, that is, time independent, then $u$ satisfies the Laplace equation $\Delta u = 0$.

### Stochastic diffusion

We consider here Brownian motion – an erratic movement of a microscopic particle suspended in a liquid, that was first observed by a botanist Robert Brown in 1828 (see a picture below). This irregular movement occurs as the result of a large number of random collisions that the particle experience from the molecules.




Brownian path in $\mathbb{R}^2$


The event $X_t \in \Omega$

Based on this explanation, Albert Einstein suggested in 1905 a mathematical model of Brownian motion. Assuming that the particle starts moving at time 0 at the origin of $\mathbb{R}^3$, denote its random position at time $t$ by $X_t$. One cannot predict the position of the particle deterministically as in classical mechanics, but can describe its movement stochastically, by means of *transition probability* $\mathbb{P}(X_t \in \Omega)$ for any open set $\Omega$ and any time $t$. The transition probability has a *density*: a function $u(x,t)$ such that, for any open set $\Omega \subset \mathbb{R}^3$,

$$\mathbb{P}(X_t \in \Omega) = \int_\Omega u(x,t)\, dx.$$

Einstein showed that the transition density $u(x,t)$ satisfies the following *diffusion equation*

$$\partial_t u = D\Delta u,$$

where $D > 0$ is the diffusion coefficient depending on the properties of the particle and surrounding medium. In fact, Einstein derived an explicit formula for $D$ and made a prediction that the mean displacement of the particle after time $t$ is $\sqrt{4Dt}$. The latter prediction was verified experimentally by Jean Perrin in 1908, for which he received a Nobel Prize for Physics in 1926. The experiment of Jean Perrin was considered as the final confirmation of the molecular structure of the matter.

Obviously, the diffusion equation is identical to the heat equation.

### 0.1.5 Schrödinger equation

In 1926, Erwin Schrödinger developed a new approach for describing motion of elementary particles in Quantum Mechanics. In this approach the movement of elementary particle is described stochastically, by means of the transition probability and its density. More precisely, the transition density of the particle is equal to $|\psi(x,t)|^2$ where $\psi(x,t)$ is a complex valued function that is called the *wave function* and that satisfies the following *Schrödinger equation*:

$$i\hbar\partial_t\psi = -\frac{\hbar^2}{2m}\Delta\psi + U\psi,$$

where $m$ is the mass of the particle, $U$ is the external potential field, $\hbar$ is the Planck constant, and $i = \sqrt{-1}$. For his discovery, Schrödinger received a Nobel Prize for Physics in 1933.

For $U = 0$ we rewrite this equation in the form

$$\partial_t\psi = i\frac{\hbar}{2m}\Delta\psi,$$

which looks similarly to the heat equation but with an imaginary coefficient in front of $\Delta\psi$.

*The main equations to be considered in this lecture course are the Laplace, heat and wave equations.*

## 0.2 Quasi-linear PDEs of second order and change of coordinates

In all the above examples the PDEs are of the *second order*, that is, the maximal order of partial derivatives involved in the equation is equal to 2. Although there are also important PDEs of higher order, we will restrict ourselves to those of the second order. Consider a second order PDE in $\mathbb{R}^n$ (or in a domain of $\mathbb{R}^n$) of the form

$$\sum_{i,j=1}^{n} a_{ij}(x)\,\partial_{x_i x_j} u + \Phi(x, u, \nabla u) = 0 \tag{0.6}$$

where $a_{ij}$ and $\Phi$ are given functions. If $\Phi = 0$ then this equation is called *linear*, because the expression in the left hand side is a linear function of the second derivatives $\partial_{x_i x_j} u$. With a general function $\Phi$, the equation is called quasi-linear.

A solution $u$ of (0.6) is always assumed to be $C^2$. Since $\partial_{x_i x_j} u = \partial_{x_j x_i} u$, it follows that we can assume that $a_{ij} = a_{ji}$, that is, the matrix $a = (a_{ij})$ is symmetric.

Let us make a linear change of the coordinates $x_1, ..., x_n$ and see how the PDE (0.6) changes. The goal of that is to try and find a change that simplifies (0.6). So, consider a linear transformation of coordinates

$$y = Mx$$

where $M = (M_{ij})_{i,j=1}^{n}$ is a non-singular matrix and $x$ and $y$ are regarded as columns. Explicitly we have, for any $k = 1, ..., n$,

$$y_k = \sum_{k=1}^{n} M_{ki} x_i \ .$$

The function $u(x)$ can be regarded also as a function of $y$ because $x$ is a function of $y$. By the chain rule we have

$$\partial_{x_i} u = \sum_k \frac{\partial y_k}{\partial x_i} \partial_{y_k} u = \sum_k M_{ki} \partial_{y_k} u$$

and

$$\begin{aligned}
\partial_{x_i x_j} u &= \partial_{x_j} \sum_k M_{ki} \partial_{y_k} u = \sum_k M_{ki} \partial_{x_j} (\partial_{y_k} u) = \sum_k M_{ki} \left( \sum_l M_{lj} \partial_{y_l} (\partial_{y_k} u) \right) \\
&= \sum_{k,l} M_{ki} M_{lj} \partial_{y_k y_l} u,
\end{aligned}$$

so that

$$\begin{aligned}
\sum_{i,j} a_{ij}(x)\,\partial_{x_i x_j} u &= \sum_{i,j} a_{ij}(x) \sum_{k,l} M_{ki} M_{lj} \partial_{y_k y_l} u = \sum_{k,l} \left( \sum_{i,j} a_{ij}(x)\, M_{ki} M_{lj} \right) \partial_{y_k y_l} u \\
&= \sum_{k,l} b_{kl}(y)\,\partial_{y_k y_l} u
\end{aligned}$$

where

$$b_{kl}(y) = \sum_{i,j} M_{ki} a_{ij}(x) M_{lj}.$$

For the matrices $a = (a_{ij})$ and $b = (b_{kl})$, we obtain the identity

$$b = MaM^T. \tag{0.7}$$

Hence, the change $y = Mx$ brings the PDE (0.6) to the form

$$\sum_{k,l=1}^{n} b_{kl}(y)\, \partial_{y_k y_l} u + \Psi(y, u, \nabla u) = 0, \tag{0.8}$$

where $b$ is given by (0.7). Moreover, the left hand sides of (0.6) and (0.8) are identical.

Now we fix a point $x_0$ and try to find $M$ so that the matrix $b$ at $y_0 = Mx_0$ is as simple as possible. Write for simplicity $a_{ij} = a_{ij}(x_0)$ and consider an auxiliary quadratic form

$$\sum_{i,j} a_{ij} \xi_i \xi_j = \xi^T a \xi, \tag{0.9}$$

where $\xi \in \mathbb{R}^n$ is a new variable (column) vector. The quadratic form (0.9) is called the *characteristic form* of (0.6) at $x_0$.

Consider in (0.9) the change $\xi = M^T \eta$:

$$\sum_{i,j} a_{ij} \xi_i \xi_j = \eta^T MaM^T \eta = \eta^T b \eta = \sum_{k,l} b_{kl} \eta_k \eta_l.$$

Hence, the change $\xi = M^T \eta$ in the characteristic form (0.9) of the PDE (0.6) results in the characteristic form of the PDE (0.8). Shortly, the change $y = Mx$ in the PDE is compatible with the change $\xi = M^T \eta$ in the characteristic form.

As it is known from Linear Algebra, by a linear change $\xi = M^T \eta$ any quadratic form can be reduced to a diagonal form; in other words, there a non-singular matrix $M$ such that the matrix $b = MaM^T$ is a diagonal matrix with diagonal elements $\pm 1$ and 0:

$$b = \operatorname{diag}(\underbrace{1, ...1}_{p}, \underbrace{-1, ..., -1}_{q}, 0, ..., 0).$$

One says that the matrix $a(x_0)$ has signature $(p, q)$. In this case we say that (0.8) is the *canonical form* of (0.6) at $x_0$.

**Definition.** We say that the PDE (0.6) has at the point $x_0$

- *elliptic type* if the matrix $a(x_0)$ has signature $(n, 0)$ (that is, the matrix $a(x_0)$ is positive definite);
- *hyperbolic type* if $a(x_0)$ has signature $(n-1, 1)$ or $(1, n-1)$
- *parabolic type* if $a(x_0)$ has signature $(n-1, 0)$ or $(0, n-1)$.

This classification is full in the case of dimension $n = 2$: indeed, in this case the only possibilities for signatures are $(2, 0)$, $(1, 1)$ and $(1, 0)$ and the symmetric ones, which gives us the above three cases. For a general dimension $n$ there are many other signatures that are not mentioned in the above Definition.

If the coefficients $a_{ij}(x)$ do not depend on $x$, then the canonical form (and, hence, the type) is the same at all points.

**Example.** The Laplace equation in $\mathbb{R}^n$ has the form

$$\partial_{x_1x_1}u + ... + \partial_{x_nx_n}u = 0,$$

whose characteristic form is

$$\xi_1^2 + ... + \xi_n^2.$$

It is already diagonal and has signature $(n, 0)$. Hence, the Laplace equation has elliptic type (at all points).

The $n$-dimensional wave equation

$$\partial_{tt}u = \Delta u$$

can be regarded as a PDE in $\mathbb{R}^{n+1}$ with the coordinates $(t, x_1, ..., x_n)$. It can be rewritten in the form

$$\partial_{tt}u - \partial_{x_1x_1}u - ... - \partial_{x_nx_n}u = 0,$$

and its characteristic form is

$$\xi_0^2 - \xi_1^2 - ... - \xi_n^2$$

has signature $(1, n)$. Hence, the wave equation is of hyperbolic type.

The $n$-dimensional heat equation

$$\partial_t u = \Delta u$$

can also be regarded as a PDE in $\mathbb{R}^{n+1}$ as follows

$$\partial_t u - \partial_{x_1x_1}u - ... - \partial_{x_nx_n}u = 0,$$

and its characteristic form is $-\xi_1^2 - ... - \xi_n^2$. It has signature $(0, n)$, and its type is parabolic.

**Example.** Let us bring to the canonical form the PDE in $\mathbb{R}^2$

$$\partial_{xx}u - 2\partial_{xy}u - 3\partial_{yy}u + \partial_y u = 0. \tag{0.10}$$

Here we use notation $(x, y)$ for the coordinates instead of $(x_1, x_2)$. Hence, the new coordinates will be denoted by $(x', y')$ instead of $(y_1, y_2)$.

The matrix $a$ of (0.10) is

$$a = \begin{pmatrix} 1 & -1 \\ -1 & -3 \end{pmatrix}$$

and the characteristic form of (0.10) is

$$\xi^2 - 2\xi\eta - 3\eta^2 = (\xi - \eta)^2 - 4\eta^2 = (\xi')^2 - (\eta')^2$$

where

$$\begin{aligned} \xi' &= \xi - \eta \\ \eta' &= 2\eta. \end{aligned}$$

In particular, we see that the signature of $a$ is $(1,1)$ so that the type of (0.10) is hyperbolic.

The inverse transformation is

$$\begin{aligned} \xi &= \xi' + \frac{1}{2}\eta' \\ \eta &= \frac{1}{2}\eta' \end{aligned}$$

whence we obtain

$$M^T = \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix} \quad \text{and} \quad M = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Therefore, the desired change of variables is

$$\begin{aligned} x' &= x \\ y' &= \frac{1}{2}x + \frac{1}{2}y \end{aligned}$$

Under this change we have

$$\partial_{xx} u - 2\partial_{xy} u - 3\partial_{yy} u = \partial_{x'x'} u - \partial_{y'y'} u$$

and

$$\partial_y u = \frac{\partial x'}{\partial y}\partial_{x'} u + \frac{\partial y'}{\partial y}\partial_{y'} u = \frac{1}{2}\partial_{y'} u.$$

Hence, the canonical form of (0.10) is

$$\partial_{x'x'} u - \partial_{y'y'} u + \frac{1}{2}\partial_{y'} u = 0.$$

**Example.** Let us show how to solve the PDE

$$\partial_{xy} u = 0$$

in $\mathbb{R}^2$ (and in any open convex subset of $\mathbb{R}^2$). We assume that $u \in C^2(\mathbb{R}^2)$. Since $\partial_y(\partial_x u) = 0$, we see that the function $\partial_x u$ is a constant as a function of $y$, that is,

$$\partial_x u(x,y) = f(x),$$

for some $C^1$ function $f$. Integrating this identity in $x$, we obtain

$$u(x,y) = \int f(x)\,dx + C,$$

where $C$ can depend on $y$. Renaming $\int f(x)\,dx$ back into $f(x)$ and denoting $C$ by $g(y)$, we obtain

$$u(x,y) = f(x) + g(y)$$

for arbitrary $C^2$ functions $f$ and $g$. Conversely, any function $u$ of this form satisfies $u_{xy} = 0$. Hence, the general solution of $u_{xy} = 0$ is given by

$$u(x, y) = f(x) + g(y).$$

This is a unique situation when a PDE can be explicitly solved. For other equations this is typically not the case.

The same argument works if $\Omega$ is a convex open subset of $\mathbb{R}^2$ and a function $u \in C^2(\Omega)$ satisfies $\partial_{xy} u = 0$ in $\Omega$. Denote by $I$ the projection of $\Omega$ onto the axis $x$ and by $J$ the projection of $\Omega$ onto the axis $y$, so that $I, J$ are open intervals. For any $x \in I$, the function $u(x, y)$ is defined for $y \in J_x$ where $J_x$ is the $x$-section of $\Omega$ (by convexity, $J_x$ is an open interval). Since $\partial_y(\partial_x u) = 0$ on $J_x$, we obtain that $\partial_x u$ as a function of $y$ is constant on $J_x$, that is,

$$\partial_x u(x, y) = \widetilde{f}(x)$$

for all $(x, y) \in \Omega$, where $\widetilde{f}$ is a function on $I$. For any $y \in J$, denote by $I_y$ the $y$-section of $\Omega$ and integrate the above identity in $x \in I_y$. We obtain

$$u(x, y) = f(x) + g(y)$$

for all $(x, y) \in \Omega$, for some function $g$ defined on $J$. It follows that $f \in C^2(I)$ and $g \in C^2(J)$.

**Example.** Let us find the general $C^2$ solution of the following PDE in $\mathbb{R}^2$:

$$c^2 \partial_{xx} u - \partial_{yy} u = 0 \tag{0.11}$$

where $c > 0$ is a constant. Let us show that it can be reduced to

$$\partial_{x'y'} u = 0.$$

Indeed, the characteristic form is

$$c^2\xi^2 - \eta^2 = (c\xi + \eta)(c\xi - \eta) = \xi'\eta'$$

where

$$\begin{aligned} \xi' &= c\xi + \eta \\ \eta' &= c\xi - \eta. \end{aligned}$$

It follows that

$$\begin{aligned} \xi &= \frac{1}{2c}(\xi' + \eta') \\ \eta &= \frac{1}{2}(\xi' - \eta') \end{aligned}$$

The matrix $M$ is therefore

$$M = \begin{pmatrix} \frac{1}{2c} & \frac{1}{2} \\ \frac{1}{2c} & -\frac{1}{2} \end{pmatrix}$$

and the change of coordinates is

$$\begin{aligned} x' &= \frac{1}{2c}x + \frac{1}{2}y = \frac{1}{2c}(x + cy) \\ y' &= \frac{1}{2c}x - \frac{1}{2}y = \frac{1}{2c}(x - cy). \end{aligned}$$

In the new coordinates the PDE becomes

$$\partial_{x'y'}u = 0$$

whose solution is

$$u = f(x') + g(y')$$

with arbitrary $C^2$ functions $f, g$. Hence, the solution of (0.11) is

$$\begin{aligned} u &= f\left(\frac{1}{2c}(x+cy)\right) + g\left(\frac{1}{2c}(x-cy)\right) \\ &= F(x+cy) + G(x-cy) \end{aligned}$$

where $F(s) = f\left(\frac{1}{2c}s\right)$ and $G(s) = g\left(\frac{1}{2c}s\right)$ are arbitrary $C^2$ functions on $\mathbb{R}$.

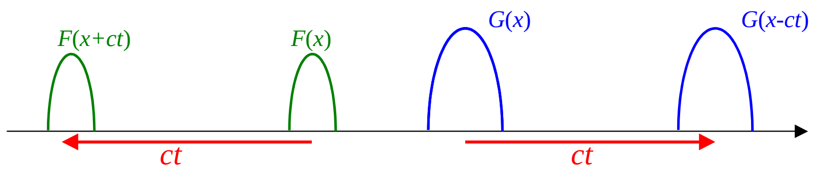
The equation (0.11) coincides with the one-dimensional wave equation

$$\partial_{tt}u = c^2\partial_{xx}u, \tag{0.12}$$

if we take $y = t$. Hence, the latter has the general solution

$$u(x,t) = F(x+ct) + G(x-ct). \tag{0.13}$$

Note that, for a fixed $t > 0$, the graph of $G(x-ct)$ as a function of $x$ is obtained from the graph of $G(x)$ by shifting to the right at distance $ct$, and the graph of $F(x+ct)$ is obtained from the graph of $F(x)$ by shifting to the left at distance $ct$.Hence, $u$ is the sum of two *waves* running at speed $c$: one to the right and the other to the left.




If $\Omega$ is a convex open subset in $\mathbb{R}^2$ and $u \in C^2(\Omega)$ satisfies (0.12) in $\Omega$ then we obtain similarly representation (0.13), where $F$ and $G$ are $C^2$ functions on the intervals $I$ and $J$ that are the projection of $\Omega$ onto the axis $x'$ and $y'$, respectively, where

$$x' = x + ct, \quad y' = x - ct.$$

In other words, $I$ consists of all possible values of $x + ct$ with $(x,t) \in \Omega$ and $J$ consists of all possible values of $x - ct$ with $(x,t) \in \Omega$.

# Chapter 1

# Laplace equation and harmonic functions

28.10.15

In this Chapter we are concerned with the Laplace equation $\Delta u = 0$ and Poisson equation $\Delta u = f$ in a bounded domain (=open set) $\Omega \subset \mathbb{R}^n$, where the function $u$ is always assumed to be $C^2$. We always assume that $n \geq 2$ unless otherwise specified.

As we already know, the family of harmonic functions is very large: for example, in $\mathbb{R}^2$ the real part of any analytic function is a harmonic function. In applications one needs to select one harmonic function by imposing additional conditions, most frequently – the boundary conditions.

**Definition.** Given a bounded domain $\Omega \subset \mathbb{R}^n$, a function $f : \Omega \to \mathbb{R}$ and a function $\varphi : \partial\Omega \to \mathbb{R}$, the *Dirichlet problem* is a problem of finding a function $u \in C^2(\Omega) \cap C(\overline{\Omega})$ that satisfies the following conditions:

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

In other words, one needs to solve the Poisson equation $\Delta u = f$ in $\Omega$ with the *boundary condition* $u = \varphi$ on $\partial\Omega$. In particular, if $f = 0$ then the problem (1.1) consists of finding a harmonic function in $\Omega$ with prescribed boundary condition.

We will be concerned with the questions of existence and uniqueness of solution to (1.1) as well as with various properties of solutions.

## 1.1 Maximum principle and uniqueness in Dirichlet problem

Here we will prove the uniqueness in the Dirichlet problem (1.1) using the *maximum principle*. Any $C^2$ function $u$ satisfying $\Delta u \geq 0$ in a domain $\Omega$ is called *subharmonic* in $\Omega$.

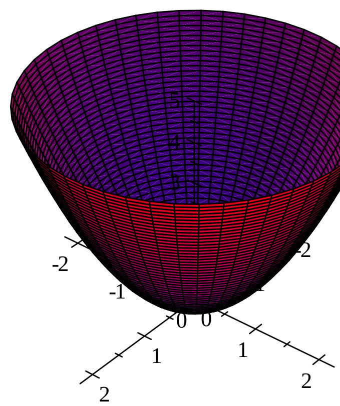
**Theorem 1.1** *(*Maximum principle*) Let $\Omega$ be a bounded domain. If $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is subharmonic in $\Omega$ then*

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u. \tag{1.2}$$

Since $\partial\Omega$ and $\overline{\Omega}$ are compact, the function $u$ attains its supremum on each of this sets, so that the both sides of (1.2) are well defined. Also, (1.2) can be rewritten in the form

$$\sup_{\Omega} u = \sup_{\partial\Omega} u. \tag{1.3}$$

Theorem 1.1 can be formulated as follows: any subharmonic function attains its maximum at the boundary.




Subharmonic function $f(x,y) = x^2 + y^2$

**Proof.** Assume first that $\Delta u > 0$ in $\Omega$. Let $z$ be a point of maximum of $u$ in $\overline{\Omega}$. If $z \in \partial\Omega$ then there is nothing to prove. Assume that $z \in \Omega$. Since $u$ takes a maximum at $z$, we all first derivatives $\partial_{x_i} u$ of $u$ vanish at $z$ and the second derivatives $\partial_{x_i x_i} u$ are at $z$ non-positive, that is,

$$\partial_{x_i x_i} u(z) \le 0.$$

Adding up for all $i$, we obtain that

$$\Delta u(z) \le 0,$$

which contradicts $\Delta u > 0$ in $\Omega$ and thus finishes the proof.

In the general case of $\Delta u \ge 0$, let us choose a function $v \in C^2(\mathbb{R}^n)$ such that $\Delta v > 0$. For example, we can take $v = |x|^2$ since

$$\Delta |x|^2 = \Delta\left(x_1^2 + ... + x_n^2\right) = 2n,$$

or $v = e^{Cx_1}$ since

$$\Delta e^{Cx_1} = \partial_{x_1 x_1} e^{Cx_1} = C^2 e^{Cx_1}$$

Consider for any $\varepsilon > 0$ the function $u + \varepsilon v$. Since

$$\Delta(u + \varepsilon v) = \Delta u + \varepsilon \Delta v > 0,$$

we obtain by the first part of the proof that

$$\max_{\overline{\Omega}}(u + \varepsilon v) = \max_{\partial\Omega}(u + \varepsilon v).$$

Passing to the limit as $\varepsilon \to 0$, we obtain (1.2), which finishes the proof. $\blacksquare$

A $C^2$ function $u$ is called *superharmonic* in $\Omega$ if $\Delta u \le 0$ in $\Omega$.

**Corollary 1.2** $(a)$ *(*Minimum principle*) Let $\Omega$ be a bounded domain. If $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is superharmonic in $\Omega$ then*

$$\min_{\overline{\Omega}} u = \min_{\partial\Omega} u. \tag{1.4}$$

$(b)$ (Maximum modulus principle) *If $u \in C^2(\Omega) \cap C(\overline{\Omega})$ is harmonic in $\Omega$ then*

$$\max_{\overline{\Omega}} |u| = \max_{\Omega} |u| \tag{1.5}$$

**Proof.** If $u$ is superharmonic then $-u$ is subharmonic. Applying Theorem 1.1 to $-u$, we obtain

$$\max_{\overline{\Omega}} (-u) = \max_{\partial\Omega} (-u),$$

whence (1.4) follows. If $u$ is harmonic, then it is subharmonic and superharmonic, so that both $u$ and $-u$ satisfy the maximum principle. Hence, (1.5) follows. ■

We use the maximum principle to prove uniqueness statement in the Dirichlet problem.

**Corollary 1.3** *The Dirichlet problem* (1.1) *has at most one solution $u$.*

**Proof.** Let $u_1$ and $u_2$ be two solutions of (1.1). The function $u = u_1 - u_2$ satisfies then

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

By the maximum principle (1.5) of Corollary 1.2 we obtain

$$\max_{\overline{\Omega}} |u| = \max_{\partial\Omega} |u| = 0$$

and, hence, $u \equiv 0$ in $\Omega$. It follows that $u_1 \equiv u_2$, which was to be proved. ■

In the next theorem we give an amazing application of the maximum principle.

**Theorem 1.4** (Fundamental theorem of Algebra) *Any polynomial*

$$P(z) = z^n + a_1 z^{n-1} + ... + a_n$$

*of degree $n \geq 1$ with complex coefficients $a_1, ..., a_n$ has at least one complex zero.*

**Proof.** We need to prove that there exists $z \in \mathbb{C}$ such that $P(z) = 0$. Assume from the contrary that $P(z) \neq 0$ for all $z \in \mathbb{C}$. Since $P(z)$ is a holomorphic function on $\mathbb{C}$, we obtain that $f(z) = \frac{1}{P(z)}$ is also a holomorphic function on $\mathbb{C}$. Note that

$$|P(z)| \to \infty \quad \text{as} \quad |z| \to \infty,$$

because

$$|P(z)| \sim |z|^n \quad \text{as} \quad |z| \to \infty.$$

It follows that

$$|f(z)| \to 0 \text{ as } |z| \to \infty. \tag{1.6}$$

We know that the function $u = \operatorname{Re} f$ is harmonic in $\mathbb{R}^2$. Applying the maximum principle to $u$ in the ball

$$B_R = \left\{z \in \mathbb{R}^2 : |z| < R\right\},$$

we obtain

$$\max_{B_R} |u| = \max_{\partial B_R} |u|,$$

in particular,

$$|u(0)| \leq \max_{\partial B_R} |u|. \tag{1.7}$$

On the other hand, by (1.6) we have

$$\max_{z \in \partial B_R} |u(z)| \leq \max_{z \in \partial B_R} |f(z)| \to 0 \text{ as } R \to \infty,$$

which together with (1.7) yields

$$|u(0)| \leq \lim_{R \to \infty} \max_{\partial B_R} |u| = 0$$

and, hence, $u(0) = 0$. In other words, we have $\operatorname{Re} f(0) = 0$. Similarly one obtains that $\operatorname{Im} f(0) = 0$ whence $f(0) = 0$, which contradicts to $f(z) = \frac{1}{P(z)} \neq 0$. ■

## 1.2 Representation of $C^2$ functions by means of potentials

Let us introduce the notation: in $\mathbb{R}^n$ with $n > 2$

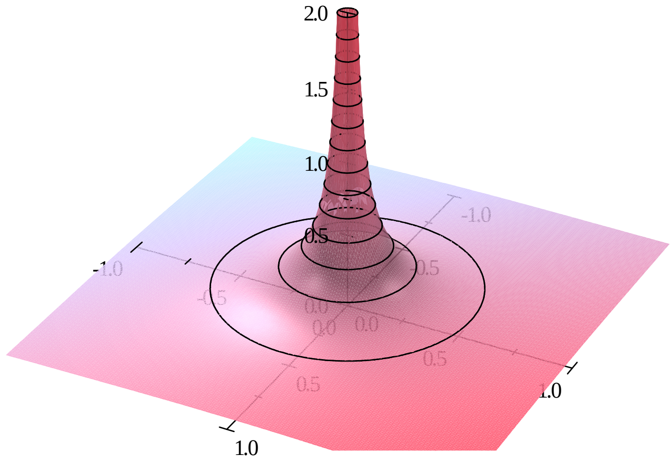
$$E(x) = \frac{1}{\omega_n (n-2) |x|^{n-2}},$$

where $\omega_n$ is the area of the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$ (for example, $\omega_3 = 4\pi$), and in $\mathbb{R}^2$

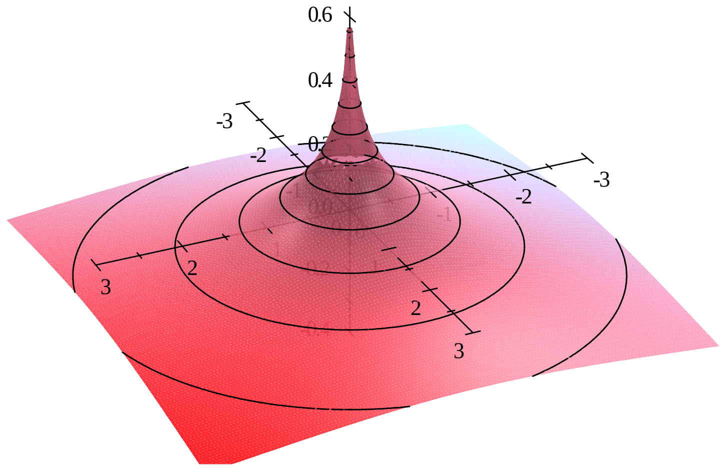
$$E(x) = \frac{1}{2\pi} \ln \frac{1}{|x|}.$$

**Definition.** The function $E(x)$ is called a fundamental solution of the Laplace operator in $\mathbb{R}^n$.

We already know that the function $E(x)$ is harmonic in $\mathbb{R}^n \setminus \{0\}$, but it has singularity at 0.




Functon $E(x)$ in the case $n>2$




Function $E(x)$ in the case $n=2$

Set also, for all $x,y\in\mathbb{R}^n$

$$E(x,y):=E(x-y).$$

If $\Omega$ is a region in $\mathbb{R}^n$ then as before we denote by $\nu$ the outer unit normal vector field on $\partial\Omega$ and by $\sigma$ the surface measure on $\partial\Omega$.

**Theorem 1.5** *Let $\Omega$ be a bounded region in $\mathbb{R}^n$. Then, for any function $u\in C^2\left(\overline{\Omega}\right)$ and any $y\in\Omega$*

$$u(y)=-\int_{\Omega}E(x,y)\,\Delta u(x)\,dx+\int_{\partial\Omega}E(x,y)\,\partial_\nu u(x)\,d\sigma(x)-\int_{\partial\Omega}\partial_\nu E(x,y)\,u(x)\,d\sigma(x), \tag{1.8}$$

*where in $\partial_\nu E(x,y)$ the derivative is taken with respect to the variable $x$.*

In the proof we will use the 2nd Green formula from Exercises: if $\Omega$ is a bounded region and $u, v \in C^2\left(\overline{\Omega}\right)$ then

$$\int_{\Omega} (v\Delta u - u\Delta v)\, dx = \int_{\partial\Omega} (v\partial_\nu u - u\partial_\nu v)\, d\sigma. \tag{1.9}$$

**Proof.** For simplicity of notation let $y = 0$, so that (1.8) becomes

$$u(0) = -\int_{\Omega} E(x)\, \Delta u(x)\, dx + \int_{\partial\Omega} E(x)\, \partial_\nu u(x)\, d\sigma(x) - \int_{\partial\Omega} \partial_\nu E(x)\, u(x)\, d\sigma(x)$$

or shorter:

$$u(0) = -\int_{\Omega} E\, \Delta u\, dx + \int_{\partial\Omega} (E\partial_\nu u - u\partial_\nu E)\, d\sigma. \tag{1.10}$$

Choose $\varepsilon > 0$ so that $\overline{B}_\varepsilon \subset \Omega$ and consider the set

$$\Omega_\varepsilon = \Omega \setminus \overline{B}_\varepsilon$$

that is a region (see Exercises). The functions $u, E$ belong to $C^2\left(\overline{\Omega}_\varepsilon\right)$ so that we can use the 2nd Green formula in $\Omega_\varepsilon$:

$$\int_{\Omega_\varepsilon} (E\Delta u - u\Delta E)\, dx = \int_{\partial\Omega_\varepsilon} (E\partial_\nu u - u\partial_\nu E)\, d\sigma. \tag{1.11}$$

Since $\Delta E = 0$ in $\Omega_\varepsilon$, the term $u\Delta E$ vanishes. 30.10.15

Next we let $\varepsilon \to 0$ in (1.9). In the left hand side we have

$$\int_{\Omega_\varepsilon} (E\Delta u - u\Delta E)\, dx = \int_{\Omega_\varepsilon} E\Delta u\, dx \to \int_{\Omega} E\Delta u\, dx. \tag{1.12}$$

Indeed, since $\Omega \setminus \Omega_\varepsilon = \overline{B}_\varepsilon$, we have

$$\begin{aligned} \left|\int_{\Omega} E\Delta u\, dx - \int_{\Omega_\varepsilon} E\Delta u\, dx\right| &= \left|\int_{\overline{B}_\varepsilon} E\Delta u\, dx\right| \\ &\leq \sup_{\overline{\Omega}} |\Delta u| \int_{\overline{B}_\varepsilon} E dx. \end{aligned}$$

Since $\Delta u$ is bounded, it suffices to verify that

$$\int_{\overline{B}_\varepsilon} E dx \to 0 \text{ as } \varepsilon \to 0.$$

The latter can be seen by means of integration in polar coordinates: since in the case $n > 2$

$$E(x) = \frac{1}{\omega_n (n-2)\, r^{n-2}},$$

and

$$\sigma(\partial B_r) = \omega_n r^{n-1},$$

we obtain

$$\begin{aligned}\int_{\overline{B}_\varepsilon} E dx &= \int_0^\varepsilon \left(\int_{\partial B_r} E d\sigma\right) dr \\ &= \int_0^\varepsilon \frac{1}{\omega_n (n-2) r^{n-2}} \omega_n r^{n-1} dr \\ &= \frac{1}{n-2} \int_0^\varepsilon r dr = \frac{\varepsilon^2}{2(n-2)} \to 0 \text{ as } \varepsilon \to 0.\end{aligned}$$

The case $n = 2$ is handled similarly (see Exercises).

In the right hand side of (1.11) the boundary $\partial\Omega_\varepsilon$ consists of two disjoint parts: $\partial\Omega$ and and $\partial B_\varepsilon$, so that

$$\int_{\partial\Omega_\varepsilon} = \int_{\partial\Omega} + \int_{\partial B_\varepsilon}.$$

Observe also that $\nu$ is outer normal to $\partial\Omega_\varepsilon$ with respect to $\Omega_\varepsilon$, which means that on $\partial B_\varepsilon$ the vector field $\nu$ ist inner with respect to $B_\varepsilon$.

Let us first show that

$$\int_{\partial B_\varepsilon} E \partial_\nu u d\sigma \to 0 \text{ as } \varepsilon \to 0. \tag{1.13}$$

Indeed, the function $\partial_\nu u = \nabla u \cdot \nu$ is bounded because $u \in C^1\left(\overline{\Omega}\right)$, and

$$\begin{aligned}\int_{\partial B_\varepsilon} E d\sigma &= \int_{\partial B_\varepsilon} \frac{1}{\omega_n (n-2) \varepsilon^{n-2}} d\sigma = \frac{1}{\omega_n (n-2) \varepsilon^{n-2}} \sigma(\partial B_\varepsilon) \\ &= \frac{1}{\omega_n (n-2) \varepsilon^{n-2}} \omega_n \varepsilon^{n-1} = \frac{\varepsilon}{(n-2)} \to 0.\end{aligned}$$

Let us compute the limit

$$\lim_{\varepsilon\to 0} \int_{\partial B_\varepsilon} u \partial_\nu E \, d\sigma.$$

Using again polar coordinates, observe that the direction of $\nu$ on $\partial B_\varepsilon$ is opposite to the radial direction, whence it follows that

$$\partial_\nu E = -\partial_r E = -\partial_r \left(\frac{1}{\omega_n (n-2) r^{n-2}}\right) = \frac{1}{\omega_n r^{n-1}}.$$

Consequently, we obtain

$$\int_{\partial B_\varepsilon} \partial_\nu E \, d\sigma = \frac{1}{\omega_n \varepsilon^{n-1}} \sigma(\partial B_\varepsilon) = 1.$$

Observe that

$$\begin{aligned}\int_{\partial B_\varepsilon} u(x) \partial_\nu E(x) \, d\sigma &= \int_{\partial B_\varepsilon} u(0) \partial_\nu E \, d\sigma + \int_{\partial B_\varepsilon} (u(x) - u(0)) \partial_\nu E \, d\sigma \\ &= u(0) + \int_{\partial B_\varepsilon} (u(x) - u(0)) \partial_\nu E \, d\sigma\end{aligned}$$

and

$$\begin{aligned}\left|\int_{\partial B_\varepsilon}(u(x)-u(0))\,\partial_\nu E\,d\sigma\right| &\leq \sup_{x\in\partial B_\varepsilon}|u(x)-u(0)|\int_{\partial B_\varepsilon}\partial_\nu E\,d\sigma\\ &= \sup_{x\in\partial B_\varepsilon}|u(x)-u(0)|\to 0 \ \text{ as } \varepsilon\to 0,\end{aligned}$$

which implies

$$\int_{\partial B_\varepsilon} u(x)\,\partial_\nu E(x)\,d\sigma \to u(0) \ \text{ as } \varepsilon\to 0. \tag{1.14}$$

Combining (1.11), (1.12), (1.13), (1.14), we obtain

$$\begin{aligned}\int_\Omega E\Delta u\,dx &= \int_{\partial\Omega}(E\partial_\nu u-u\partial_\nu E)\,d\sigma+\lim_{\varepsilon\to 0}\int_{\partial B_\varepsilon}(E\partial_\nu u-u\partial_\nu E)\,d\sigma\\ &= \int_{\partial\Omega}(E\partial_\nu u-u\partial_\nu E)\,d\sigma-u(0),\end{aligned}$$

whence (1.10) follows. ■

All the terms in the right hand side of (1.8) have physical meaning in the case of $n=3$. The term

$$\int_\Omega E(x,y)\,\Delta u(x)\,dx$$

is the electric potential of the change in $\Omega$ with the density $\Delta u$. Its is also called *Newtonian potential*, as in the case $\Delta u\geq 0$ it is also a gravitational potential.

The term

$$\int_{\partial\Omega} E(x,y)\,\partial_\nu u(x)\,d\sigma(x)$$

is the potential of a charge distributed on the surface $\partial\Omega$ with the density $\partial_\nu u$. It is also called the potential of a *single layer*.

The term

$$\int_{\partial\Omega}\partial_\nu E(x,y)\,u(x)\,d\sigma(x)$$

happens to be the potential of a *dipole field* distributed on the surface $\partial\Omega$ with the density $u$. It is also called the potential of a *double layer*.

## 1.3 Green function

Let $\Omega$ be a domain in $\mathbb{R}^n$. Assume that, for any $y\in\Omega$ there exists a function $h_y(x)\in C^2\left(\overline{\Omega}\right)$ such that

$$\begin{cases}\Delta h_y=0 \text{ in } \Omega\\ h_y(x)=E(x,y) \ \text{ for all } x\in\partial\Omega\end{cases} \tag{1.15}$$

**Definition.** Under the above assumption, the function

$$G(x,y):=E(x,y)-h_y(x)$$

is called the *Green function* (of the Laplace operator) in $\Omega$.

Note that $G(x,y)$ is defined for all $x \in \overline{\Omega}$ and $y \in \Omega$. By (1.15) we see that the function

$$x \mapsto G(x,y)$$

is harmonic in $\Omega \setminus \{y\}$, and that

$$G(x,y) = 0 \text{ for all } x \in \partial\Omega.$$

**Corollary 1.6** *Let $G$ be the Green function of a bounded region $\Omega \subset \mathbb{R}^n$. Then, for any function $u \in C^2\left(\overline{\Omega}\right)$ and any $y \in \Omega$,*

$$u(y) = -\int_{\Omega} G(x,y)\,\Delta u(x)\,dx - \int_{\partial\Omega} \partial_\nu G(x,y)\,u(x)\,d\sigma(x). \tag{1.16}$$

**Proof.** By Theorem (1.5) we have

$$u(y) = -\int_{\Omega} E(x,y)\,\Delta u(x)\,dx + \int_{\partial\Omega} \left(E(x,y)\,\partial_\nu u(x) - \partial_\nu E(x,y)\,u(x)\right) d\sigma(x). \tag{1.17}$$

By the 2nd Green formula (1.9) we have

$$\int_{\Omega} (h_y \Delta u - u \Delta h_y)\,dx = \int_{\partial\Omega} (h_y \partial_\nu u - u \partial_\nu h_y)\,d\sigma.$$

Using $\Delta h_y = 0$, rewrite this identity as follows:

$$0 = -\int_{\Omega} h_y \Delta u\,dx + \int_{\partial\Omega} (h_y \partial_\nu u - u \partial_\nu h_y)\,d\sigma.$$

Subtracting it from (1.17) we obtain

$$u(y) = -\int_{\Omega} G(x,y)\,\Delta u(x)\,dx + \int_{\partial\Omega} \left(G(x,y)\,\partial_\nu u(x) - \partial_\nu G(x,y)\,u(x)\right) d\sigma(x).$$

Finally, observing that $G(x,y) = 0$ at $\partial\Omega$, we obtain (1.16). ■

It is possible to show that if the Green function exists then necessarily $G(x,y) = G(y,x)$ for all $x, y \in \Omega$ and that $G(x,y) > 0$ provided $\Omega$ is connected (see Exercises).

Consider the Dirichlet problem

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega. \end{cases} \tag{1.18}$$

If $u \in C^2\left(\overline{\Omega}\right)$ solves this problem then by (1.16)

$$u(y) = -\int_{\Omega} G(x,y)\,f(x)\,dx - \int_{\partial\Omega} \partial_\nu G(x,y)\,\varphi(x)\,d\sigma(x). \tag{1.19}$$

The identity (1.19) suggests the following program for solving the Dirichlet problem:

1. construct the Green function of $\Omega$;
2. prove that (1.19) gives indeed a solution of (1.18) under certain assumptions about $f$ and $\varphi$.

We will realize this program in the case when $\Omega$ is a ball. For general domains $\Omega$ there are other methods of proving solvability of (1.18) without using the Green function.

## 1.4 The Green function in a ball

Consider in $\mathbb{R}^n$ the ball of radius $R > 0$:

$$B_R = \{x \in \mathbb{R}^n : |x| < R\}.$$

To construct the Green function of $B_R$, we will search the function $h_y$ in the form

$$h_y(x) = c_y E(x, y^*)$$

where $y^*$ is a point outside $\overline{B}_R$. Then $h_y$ is automatically harmonic in $\overline{B}_R$, but we need also to match the boundary condition

$$h_y(x) = E(x, y) \quad \text{for } x \in \partial B_R.$$

This is achieved by a careful choice of $y^*$ and $c_y$ using specific geometric properties of the ball.

For any $y \in \mathbb{R}^n \setminus \{0\}$ define $y^*$ as inversion of $y$ with respect to $B_R$, that is,

$$y^* = R^2 \frac{y}{|y|^2}.$$

In other words, the vector $y^*$ has the same direction as $y$ and $|y^*| = \frac{R^2}{|y|}$, that is,

$$|y|\,|y^*| = R^2. \tag{1.20}$$

Clearly, if $y \in B_R$ then $y^* \in \overline{B}_R^c$ and if $y \in \partial B_R$ then $y^* = y$.

**Theorem 1.7** *The Green function $G(x, y)$ of the ball $B_R$ exists and is given in the case $n > 2$ by the formulas*

$$G(x,y) = E(x,y) - \left(\frac{R}{|y|}\right)^{n-2} E(x,y^*) \quad \text{if } y \neq 0 \tag{1.21}$$

$$G(x,0) = \frac{1}{\omega_n(n-2)}\left(\frac{1}{|x|^{n-2}} - \frac{1}{R^{n-2}}\right), \quad \text{if } y = 0, \tag{1.22}$$

*and in the case $n = 2$ by the formulas*

$$G(x,y) = E(x,y) - E(x,y^*) - \frac{1}{2\pi}\log\frac{R}{|y|}, \quad \text{if } y \neq 0, \tag{1.23}$$

$$G(x,0) = \frac{1}{2\pi}\left(\ln\frac{1}{|x|} - \ln\frac{1}{R}\right), \quad \text{if } y = 0. \tag{1.24}$$

**Proof.** We give the proof in the case $n > 2$ leaving the case $n = 2$ to Exercises. In the both formulas (1.21)-(1.22) we have

$$G(x,y) = E(x,y) - h_y(x)$$

where

$$h_y(x) = \begin{cases} \left(\frac{R}{|y|}\right)^{n-2} E(x,y^*) & y \neq 0, \\ \frac{1}{\omega_n(n-2)R^{n-2}}, & y = 0. \end{cases}$$

We need to prove that $h_y(x)$ is harmonic in $B_R$ and that $G(x,y)=0$ if $x\in\partial\Omega$.

In the case $y=0$ the function $h_y(x)$ is constant and, hence, is harmonic; for $x\in\partial B_R$, that is, for $|x|=R$ we obviously have $G(x,0)=0$.

Consider the general case $y\in B_R\setminus\{0\}$. The function

$$h_y(x)=\left(\frac{R}{|y|}\right)^{n-2}E(x,y^*)$$

is harmonic away from $y^*$. Since $y^*$ lies outside $\overline{B}_R$, we see that $h_y$ is harmonic in $B_R$. It remains to show that $G(x,y)=0$ if $x\in\partial B_R$, which is equivalent to

$$\frac{1}{|x-y|^{n-2}}=\left(\frac{R}{|y|}\right)^{n-2}\frac{1}{|x-y^*|^{n-2}}$$

that is, to

$$\frac{|x-y^*|}{|x-y|}=\frac{R}{|y|}. \tag{1.25}$$

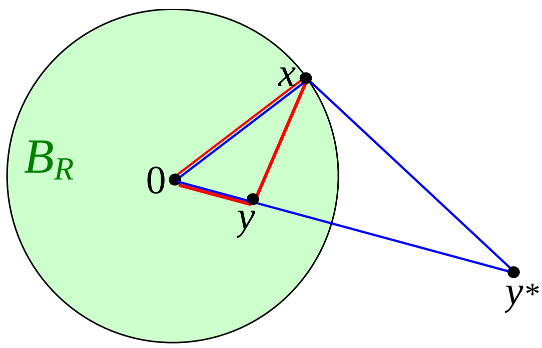
Indeed, we have

$$\begin{aligned}
|x-y^*|^2 &= |x|^2-2x\cdot y^*+|y^*|^2\\
&= |x|^2-2\frac{R^2}{|y|^2}x\cdot y+\frac{R^4}{|y|^2}\\
&= \frac{R^2}{|y|^2}\left(\frac{|x|^2|y|^2}{R^2}-2x\cdot y+R^2\right).
\end{aligned} \tag{1.26}$$

If $x\in\partial B_R$, that is, $|x|=R$, then we obtain from (1.26)

$$\begin{aligned}
|x-y^*|^2 &= \frac{R^2}{|y|^2}\left(|y|^2-2x\cdot y+|x|^2\right)\\
&= \frac{R^2}{|y|^2}|x-y|^2,
\end{aligned}$$

which finishes the proof.




Alternatively, one can prove (1.25) observing that the triangles $0xy$ and $0y^*x$ are similar. Indeed, they have a common angle at the vertex 0 and by (1.20)

$$\frac{|y^*|}{|x|}=\frac{|x|}{|y|},$$

where in the numerator we use the sides of the triangle $0y^*x$ and in the denominator – those of $0xy$. It follows from the similarity that also

$$\frac{|x-y^*|}{|x-y^*|}=\frac{|x|}{|y|},$$

which is equivalent to (1.25). ■

**Corollary 1.8** *We have, for all $y \in B_R$ and $x \in \overline{B}_R$, $x \neq y$, in the case $n > 2$*

$$G(x,y)=\frac{1}{\omega_n(n-2)}\left(\frac{1}{|x-y|^{n-2}}-\frac{1}{\left(\frac{|x|^2|y|^2}{R^2}-2x\cdot y+R^2\right)^{\frac{n-2}{2}}}\right) \tag{1.27}$$

*and in the case $n = 2$*

$$G(x,y)=\frac{1}{2\pi}\left(\ln\frac{1}{|x-y|}-\ln\frac{1}{\sqrt{\frac{|x|^2|y|^2}{R^2}-2x\cdot y+R^2}}\right) \tag{1.28}$$

**Proof.** Consider the case $n > 2$. If $y = 0$ then (1.27) obviously identical to (1.22). If $y \neq 0$ then we have by (1.21).

$$G(x,y)=\frac{1}{\omega_n(n-2)}\left(\frac{1}{|x-y|^{n-2}}-\left(\frac{R}{|y|}\right)^{n-2}\frac{1}{|x-y^*|^{n-2}}\right).$$

Substituting here $|x-y^*|$ from (1.26), we obtain (1.27). The case $n = 2$ is similar. ■ 04.11.15

**Corollary 1.9** *We have $G(x,y) = G(y,x)$ and $G(x,y) > 0$ for all distinct points $x, y \in B_R$.*

**Proof.** The symmetry $G(x,y) = G(y,x)$ is obvious from (1.27) and (1.28). Let us prove that $G(x,y) > 0$ for $x, y \in B_R$. By (1.27) it suffices to prove that

$$\frac{|x|^2|y|^2}{R^2}-2x\cdot y+R^2>|x-y|^2.$$

This inequality is equivalent to

$$\frac{|x|^2|y|^2}{R^2}-2x\cdot y+R^2>|x|^2-2x\cdot y+|y|^2,$$

which is equivalent to

$$|x|^2|y|^2+R^4-R^2|x|^2-R^2|y|^2>0$$

or to

$$\left(R^2-|x|^2\right)\left(R^2-|y|^2\right)>0,$$

and the latter is obviously the case. ■

## 1.5 Dirichlet problem in a ball and Poisson formula

**Theorem 1.10** *If $u \in C^2\left(\overline{B}_R\right)$ solves the Dirichlet problem*

$$\begin{cases} \Delta u = f & \text{in } B_R \\ u = \varphi & \text{on } \partial B_R \end{cases}$$

*then, for all $y \in B_R$,*

$$u(y) = -\int_{B_R} G(x,y) f(x)\, dx + \frac{1}{\omega_n R} \int_{\partial B_R} \frac{R^2 - |y|^2}{|x-y|^n} \varphi(x)\, d\sigma(x), \tag{1.29}$$

*where $G(x,y)$ is the Green function of $B_R$.*

**Proof.** By Corollary 1.6, we have, for any $y \in B_R$,

$$u(y) = -\int_{B_R} G(x,y) \Delta u(x)\, dx - \int_{\partial B_R} \partial_\nu G(x,y) u(x)\, d\sigma(x),$$

which implies

$$u(y) = -\int_{B_R} G(x,y) f(x)\, dx - \int_{\partial B_R} \partial_\nu G(x,y) \varphi(x)\, d\sigma(x).$$

Comparison with (1.29) shows that it remains to prove the identity:

$$-\partial_\nu G(x,y) = \frac{1}{\omega_n R} \frac{R^2 - |y|^2}{|x-y|^n},$$

where $x \in \partial B_R$ and $y \in B_R$.

Consider the case $n > 2$ (the case $n = 2$ is similar). By Theorem 1.7, we have in the case $y \neq 0$

$$G(x,y) = E(x,y) - cE(x,y^*),$$

where

$$c = \left(\frac{R}{|y|}\right)^{n-2},$$

and in the case $y = 0$

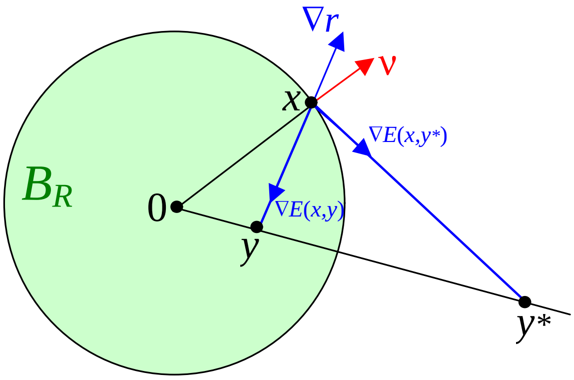
$$G(x,0) = \frac{1}{\omega_n (n-2)} \left(\frac{1}{|x|^{n-2}} - \frac{1}{R^{n-2}}\right).$$

In the case $y = 0$ we have, using the polar radius $r = |x|$, that

$$-\partial_\nu G(x,0) = -\partial_r G(x,0) = \left.\frac{1}{\omega_n r^{n-1}}\right|_{r=R} = \frac{1}{\omega_n R^{n-1}} = \frac{1}{\omega_n R} \frac{R^2 - |y|^2}{|x-y|^n}.$$

In the case $y \neq 0$ we use the polar coordinates with the pole in $y$, so that $r = |x-y|$ and

$$E(x,y) = \frac{1}{\omega_n (n-2) r^{n-2}}.$$




Since $\nabla r = \frac{x-y}{r}$ (see Exercises), we obtain by the chain rule

$$\nabla E(x,y) = \partial_r \left( \frac{1}{\omega_n (n-2) r^{n-2}} \right) \nabla r = -\frac{1}{\omega_n r^{n-1}} \frac{x-y}{r} = \frac{y-x}{\omega_n |x-y|^n}.$$

Since $\nu = \frac{x}{|x|}$, it follows that

$$\partial_\nu E(x,y) = \nabla E(x,y) \cdot \nu = \frac{y-x}{\omega_n |x-y|^n} \cdot \frac{x}{|x|} = \frac{x \cdot y - |x|^2}{\omega_n |x-y|^n |x|} = \frac{x \cdot y - R^2}{\omega_n R |x-y|^n}. \quad (1.30)$$

In the same way we have

$$\partial_\nu E(x,y^*) = \frac{x \cdot y^* - R^2}{\omega_n R |x-y^*|^n}. \quad (1.31)$$

Recall that

$$y^* = \frac{R^2}{|y|^2} y$$

and by (1.25)

$$|x-y^*| = \frac{R}{|y|} |x-y|.$$

Substituting these into (1.31), we obtain

$$\partial_\nu E(x,y^*) = \frac{x \cdot y \frac{R^2}{|y|^2} - R^2}{\omega_n R |x-y|^n (R/|y|)^n} = \frac{x \cdot y - |y|^2}{\omega_n R |x-y|^n} \frac{|y|^{n-2}}{R^{n-2}}$$

and

$$c \partial_\nu E(x,y^*) = \left( \frac{R}{|y|} \right)^{n-2} \partial_\nu E(x,y^*) = \frac{x \cdot y - |y|^2}{\omega_n R |x-y|^n}.$$

Combining with (1.30), we obtain

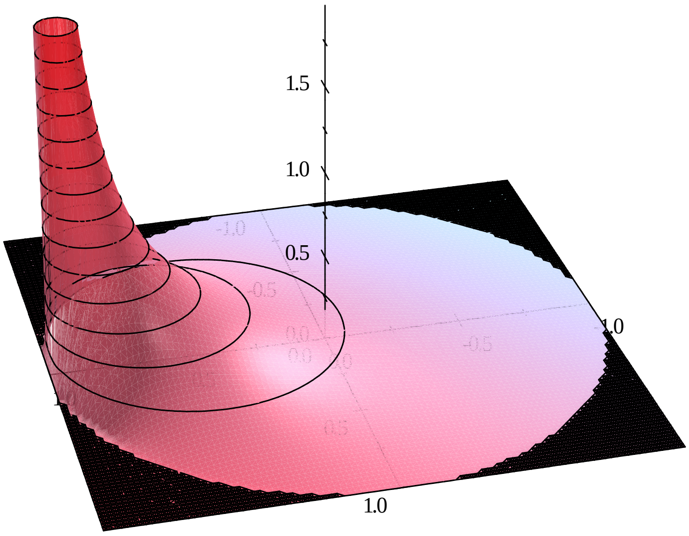
$$-\partial_\nu G(x,y) = -\partial_\nu E(x,y) + c \partial_\nu E(x,y^*) = \frac{R^2 - |y|^2}{\omega_n R |x-y|^n},$$

which was to be proved. ■

Let us interchange in (1.29) $x$ and $y$, and introduce the following function

$$K(x,y) = \frac{1}{\omega_n R} \frac{R^2 - |x|^2}{|x-y|^n}, \quad (1.32)$$

defined for $y \in \partial B_R$ and $x \in B_R$. This function is called the *Poisson kernel*.




The graph of the two-dimensional Poisson kernel $K(x,y)$ as a function of $x$.

**Theorem 1.11** (Poisson formula) *If $\varphi \in C(\partial B_R)$ then the Dirichlet problem*

$$\begin{cases} \Delta u = 0 & \text{in } B_R \\ u = \varphi & \text{on } \partial B_R \end{cases} \tag{1.33}$$

*has the following solution*

$$u(x) = \int_{\partial B_R} K(x,y)\,\varphi(y)\,d\sigma(y), \quad x \in B_R. \tag{1.34}$$

*More precisely, there exists a function $u \in C^2(B_R) \cap C(\overline{B}_R)$ that satisfies* (1.33), *and this function is given by* (1.34) *for all $x \in B_R$.*

**Proof.** It follows from (1.32) that the function $K(x,y)$ is $C^\infty$ as a function of $x \in B_R$, for any $y \in \partial B_R$. Therefore, the function $u(x)$ defined by (1.34) is also $C^\infty$ in $B_R$. Moreover, for any partial derivative $D^\alpha$ with respect to the variable $x$, we have

$$D^\alpha u(x) = \int_{\partial\Omega} D^\alpha K(x,y)\,\varphi(y)\,d\sigma(y).$$

Observe also that $K(x,y)$ as a function of $x$ is harmonic in $B_R$, that is, $\Delta K(x,y) = 0$. This can be checked directly, or one can see this as follows. By construction,

$$K(x,y) = -\partial_\nu G(x,y)$$

where $\partial_\nu$ is taken to the variable $y$. Therefore, $\Delta$ as an operator in $x$ and $\partial_\nu$ commute, and we obtain

$$\Delta K(x,y) = -\partial_\nu \Delta G(x,y) = 0.$$

It follows that

$$\Delta u(x) = \int_{\partial\Omega} \Delta K(x,y)\,\varphi(y)\,d\sigma(y) = 0,$$

which proves the harmonicity of $u$.

Now let us prove that $u \in C\left(\overline{B}_R\right)$ provided $u$ is defined on $\partial B_R$ by $u(x) = \varphi(x)$. It suffices to show that, for any $z \in \partial B_R$,

$$\lim_{\substack{x \to z \\ x \in B_R}} u(x) = \varphi(z).$$

We start with the observations that $K(x,y) > 0$ for $x \in B_R$, which is obvious from (1.32) and that

$$\int_{\partial B_R} K(x,y)\, d\sigma(y) = 1$$

for all $x \in B_R$. Indeed, the latter follows from the formula (1.29) of Theorem 1.10 with $u \equiv 1$.

It follows that

$$\varphi(z) = \int_{\partial B_R} K(x,y)\, \varphi(z)\, d\sigma(y)$$

and, hence,

$$u(x) - \varphi(z) = \int_{\partial B_R} K(x,y)\left(\varphi(y) - \varphi(z)\right) d\sigma(y),$$

$$|u(x) - \varphi(z)| \le \int_{\partial B_R} K(x,y)\, |\varphi(y) - \varphi(z)|\, d\sigma(y). \tag{1.35}$$

We will show that the right hand side of (1.35) goes to 0 as $x \to z$. The reason for that is the following: if the variable $y$ is close to $z$ then the integrand function is small because $\varphi(y)$ is close to $\varphi(z)$, while if $y$ is away from $z$ then $K(x,y)$ will be shown to be small.

To make this argument rigorous, let us choose some small $\delta > 0$ and split the integral in (1.35) into two parts:

$$\int_{\partial B_R} = \int_{\partial B_R \cap B_\delta(z)} + \int_{\partial B_R \setminus B_\delta(z)}. \tag{1.36}$$

The first integral is estimates as follows:

$$\begin{aligned} \int_{\partial B_R \cap B_\delta(z)} K(x,y)\, |\varphi(y) - \varphi(z)|\, d\sigma(y) &\le \sup_{y \in \partial B_R \cap B_\delta(z)} |\varphi(y) - \varphi(z)| \int_{\partial B_R} K(x,y)\, d\sigma(y) \\ &= \sup_{y \in \partial B_R \cap B_\delta(z)} |\varphi(y) - \varphi(z)|. \end{aligned}$$

By the continuity of $\varphi$, the last expression goes to 0 as $\delta \to 0$. In particular, for any $\varepsilon > 0$ there is $\delta > 0$ such that

$$\sup_{y \in \partial B_R \cap B_\delta(z)} |\varphi(y) - \varphi(z)| < \varepsilon/2,$$

and, hence, the first integral is bounded by $\varepsilon/2$.

The second integral in (1.36) is estimates as follows:

$$\begin{aligned} \int_{\partial B_R \setminus B_\delta(z)} K(x,y)\, |\varphi(y) - \varphi(z)|\, d\sigma(y) &\le 2 \sup |\varphi| \sup_{y \in \partial B_R \setminus B_\delta(z)} K(x,y)\, \sigma(\partial B_R) \\ &\le C \sup_{y \in \partial B_R \setminus B_\delta(z)} \frac{R^2 - |x|^2}{|x-y|^n} \end{aligned}$$

where $C = \frac{2}{\omega_n R} \sup|\varphi|\,\sigma(\partial B_R)$. As $x \to z$, we can assume that $|x-z| < \delta/2$. Since $|y-z| \geq \delta$, it follows then that $|x-y| \geq \delta/2$. Hence, the second integral is bounded by the expression

$$C\frac{R^2 - |x|^2}{(\delta/2)^n}$$

that goes to 0 as $x \to z$, because $|x| \to R$. In particular, the second integral is bounded by $\varepsilon/2$ if $x$ is close enough to $z$, which implies that

$$\int_{\partial B_R} K(x,y)\,|\varphi(y) - \varphi(z)|\,d\sigma(y) < \varepsilon$$

provided $x$ is close enough to $z$, which finishes the proof. ■

**Lemma 1.12** (Properties of Newtonian potential) *Let $f$ be a bounded function in $\mathbb{R}^n$ that has a compact support and is integrable. Then its Newtonian potential*

$$v(x) = \int_{\mathbb{R}^n} E(x,y)\,f(y)\,dy$$

*is a continuous function in $\mathbb{R}^n$. Moreover, if for some open set $\Omega \subset \mathbb{R}^n$ we have $f \in C^k(\Omega)$ then also $v \in C^k(\Omega)$. Furthermore, if $k \geq 2$ then $v$ satisfies in $\Omega$ the equation*

$$\Delta v = -f.$$

**Proof.** The proof is split into three steps. Let $S = \overline{\{x \in \mathbb{R}^n : f(x) \neq 0\}}$ be the support of $f$ so that we can write

$$v(x) = \int_S E(x,y)\,f(y)\,dy.$$

**Step1**: Let us prove that $v$ is well-defined and is continuos. Since the function $f$ is bounded and the function $y \mapsto E(x,y)$ is integrable in any bounded domain, in particular, in $S$, we see that the function $y \mapsto E(x,y)\,f(y)$ is also integrable in $S$ and, hence $v(x)$ is finite for any $x \in \mathbb{R}^n$. Let us show that, in fact $v$, is continuous in $\mathbb{R}^n$. Set $M = \sup|f|$.

Fix $z \in \mathbb{R}^n$ and show that $v(x) \to v(z)$ as $x \to z$. Fix some $\varepsilon > 0$ and write

$$\begin{aligned} v(x) - v(z) &= \int_{\mathbb{R}^n} (E(x,y) - E(z,y))\,f(y)\,dy \\ &= \int_{B_\varepsilon(z)} (E(x,y) - E(z,y))\,f(y)\,dy + \int_{S \setminus B_\varepsilon(z)} (E(x,y) - E(z,y))\,f(y)\,dy. \end{aligned}$$

For the first integral we have (assuming $n \geq 3$)

$$\left| \int_{B_\varepsilon(z)} E(z,y)\,f(y)\,dy \right| \leq M \int_{B_\varepsilon(z)} |E(z,y)|\,dy \leq C\varepsilon^2$$

and, assuming that $x \in B_\varepsilon(z)$,

$$\left| \int_{B_\varepsilon(z)} E(x,y)\,f(y)\,dy \right| \leq \left| \int_{B_{2\varepsilon}(x)} E(x,y)\,f(y)\,dy \right| \leq C\varepsilon^2.$$

To estimate the integral over $S \setminus B_\varepsilon(z)$, observe that the function $E(x,y)$ is continuous in $(x,y)$ in the domain $x \in \overline{B}_{\varepsilon/2}(z)$ and $y \in S \setminus B_\varepsilon(z)$. Since this domain is compact, this function is uniformly continuous. It follows that

$$E(x,y) \rightrightarrows E(z,y) \quad \text{as } x \to z,$$

where the convergence is uniform in $y$. It follows that

$$\int_{S \setminus B_\varepsilon(z)} (E(x,y) - E(z,y)) f(y)\, dy \to 0 \text{ as } x \to z.$$

We obtain that

$$\limsup_{x \to z} |v(x) - v(y)| \le 2C\varepsilon^2.$$

Since $\varepsilon > 0$ is arbitrary, it follows that

$$\lim_{x \to z} |v(x) - v(z)| = 0,$$

which proved the continuity of $v$ in $\mathbb{R}^n$.

**Step 2**: Assume that $f \in C_0^k(\Omega)$ where $C_0^k(\Omega)$ is a subset of $C^k(\Omega)$ that consists of functions $f$ with a compact support in $\Omega$. In this case we have also $f \in C_0^k(\mathbb{R}^n)$. Let us prove by induction in $k$ that $v \in C^k(\mathbb{R}^n)$. In the case $k=0$ we know already that $v \in C(\mathbb{R}^n)$. For induction step from $k-1$ to $k$, let us make change $z = x - y$ in the integral

$$v(x) = \int_{\mathbb{R}^n} E(x-y) f(y)\, dy = \int_{\mathbb{R}^n} E(z) f(x-z)\, dz$$

and compute the partial derivative $\partial_{x_i} v$ as follows: 06.11.15

$$\begin{aligned} \frac{v(x+te_i) - v(x)}{t} &= \int_{\mathbb{R}^n} E(z) \frac{f(x+te_i-z) - f(x-z)}{t} dz \\ &\to \int_{\mathbb{R}^n} E(z)\, \partial_{x_i} f(x-z)\, dz \quad \text{as } t \to 0 \end{aligned} \tag{1.37}$$

because

$$\frac{f(x+te_i-z) - f(x-z)}{t} \rightrightarrows \partial_{x_i} f(x-z) \quad \text{as } t \to 0,$$

where convergence is uniform with respect to $z$, and function $E(z)$ is integrable in bounded domains (note that integration in (1.37) can be restricted to a compact domain $\operatorname{supp} f(x-\cdot)$). Hence, $\partial_{x_i} v$ exists and

$$\partial_{x_i} v(x) = \int_{\mathbb{R}^n} E(z)\, \partial_{x_i} f(x-z)\, dz = \int_{\mathbb{R}^n} E(x-y)\, \partial_{y_i} f(y)\, dy. \tag{1.38}$$

In particular, $\partial_{x_i} v$ is the Newtonian potential of $\partial_{x_i} f$. Since $\partial_{x_i} f \in C_0^{k-1}(\mathbb{R}^n)$, we conclude by induction hypothesis that $\partial_{x_i} v \in C^{k-1}(\mathbb{R}^n)$. Since this is true for all $i = 1, ..., n$, it follows that $v \in C^k(\mathbb{R}^n)$.

It follows from (1.38) that, for any multiindex $\alpha$ with $|\alpha| \le k$,

$$D^\alpha v(x) = \int_{\mathbb{R}^n} E(x,y)\, D^\alpha f(y)\, dy.$$

Consequently, in the case $k \geq 2$, we have

$$\Delta v\left(x\right) = \int_{\mathbb{R}^n} E\left(x, y\right) \Delta f\left(y\right) dy.$$

Let us choose a large enough ball $B$ containing a point $x$ and $\operatorname{supp} f$. By Theorem 1.5, we have

$$f\left(x\right) = -\int_B E\left(x, y\right) \Delta f\left(y\right) dy + \int_{\partial B} E\left(x, y\right) \partial_\nu f\left(y\right) d\sigma\left(y\right) - \int_{\partial B} \partial_\nu E\left(x, y\right) f\left(y\right) d\sigma\left(y\right).$$

Since $f$ and $\partial_\nu f$ vanish on $\partial B$, we obtain

$$f\left(x\right) = -\int_B E\left(x, y\right) \Delta f\left(y\right) dy = -\Delta v\left(x\right),$$

that is $\Delta v = -f$.

**Step 3**: the general case. Assuming that $f \in C^k\left(\Omega\right)$, we prove that $v \in C^k\left(\Omega\right)$. It suffices to prove that, for any point $x_0 \in \Omega$, the function $v$ is of the class $C^k$ in a neighborhood of $x_0$. Besides, we will prove that if $k \geq 2$ then $\Delta v = -f$ in the neighborhood of $x$.

Without loss of generality, let us take $x_0 = 0$. Let $B_\varepsilon$ be a small ball centered at $x_0$ such that $B_{4\varepsilon} \subset \Omega$. Choose function $\varphi \in C^\infty\left(\mathbb{R}^n\right)$ such that $\varphi = 1$ on $B_{2\varepsilon}$ and $\varphi = 0$ outside $B_{3\varepsilon}$, and represent $v$ in the form

$$v = u + w,$$

where

$$u\left(x\right) = \int_{\mathbb{R}^n} E\left(x, y\right) \left(\varphi f\right)\left(y\right) dy, \qquad w\left(x\right) = \int_{\mathbb{R}^n} E\left(x, y\right) \left(1 - \varphi\right) f\left(y\right) dy.$$

Clearly, the function $\varphi f$ belong to $C_0^k\left(\mathbb{R}^n\right)$. By Step 2, we obtain that $u \in C^k\left(\mathbb{R}^n\right)$ and in the case $k \geq 2$ also $\Delta u = -\varphi f$, which implies $\Delta u = -f$ in $B_\varepsilon$, since $\varphi = 1$ in $B_\varepsilon$.

Note that $\left(1 - \varphi\right) f = 0$ in $B_{2\varepsilon}$ so that

$$w\left(x\right) = \int_{S \setminus B_{2\varepsilon}} E\left(x, y\right) \left(1 - \varphi\right) f\left(y\right) dy.$$

In the domain $x \in B_\varepsilon$ and $y \in B_{2\varepsilon}^c$ the function $E\left(x, y\right)$ is $C^\infty$ in $\left(x, y\right)$. Therefore, the function $w\left(x\right)$ belongs to $C^\infty\left(B_\varepsilon\right)$. Moreover, in $B_\varepsilon$ we have

$$\Delta w = \int_{S \setminus B_{2\varepsilon}} \Delta E\left(x, y\right) h\left(y\right) dy = 0.$$

Hence, we obtain that $v = u + w \in C^k\left(B_\varepsilon\right)$, and in $B_\varepsilon$

$$\Delta v = \Delta u + \Delta w = -f + 0 = -f,$$

which finishes the proof. $\blacksquare$

**Example.** Let us compute the integral

$$v(x) = \int_{B_R} E(x,y)\,dy, \tag{1.39}$$

that is, the Newtonian potential of the function $f = \mathbf{1}_{B_R}$. In the case $n = 3$ the function $-v(x)$ is the gravitational potential of the body $B_R$ in with the constant mass density 1. We assume further $n > 2$. By Lemma 1.12 with $f = 1_{B_R}$ we obtain that $v$ is a continuous function in $\mathbb{R}^n$. Besides, since $f \equiv 0$ in $\overline{B}_R^c$ then $v \in C^\infty\left(\overline{B}_R^c\right)$ and

$$\Delta v = 0 \text{ in } \overline{B}_R^c.$$

Since $f = 1 \in C^\infty(B_R)$ then $v \in C^\infty(B_R)$ and

$$\Delta v = -1 \text{ in } B_R.$$

Also it is easy to see that $v(x)$ depends only on $|x|$, because the integral in (1.39) does not change under rotations around 0. This allows to conclude that outside $\overline{B}_R$ we have

$$v(x) = C_1 |x|^{2-n} + C_2$$

(see Exercises), for some constants $C_1, C_2$. It is obvious from (1.39) that $v(x) \to 0$ as $|x| \to \infty$, which implies that $C_2 = 0$, that is,

$$v(x) = C_1 |x|^{2-n} \quad \text{outside } \overline{B}_R.$$

By continuity, we have also

$$v(x) = C_1 R^{2-n} \quad \text{for } x \in \partial B_R.$$

Hence, inside $B_R$ the function $v$ solves the following problem:

$$\begin{cases} \Delta v = -1 & \text{in } B_R \\ v = C_1 R^{2-n} & \text{on } \partial B_R \end{cases}$$

It is easy to see that the following function

$$v(x) = -\frac{|x|^2}{2n} + C_0 \tag{1.40}$$

satisfies $\Delta v = -1$, and the constant $C_0$ can be chosen to satisfies the boundary condition as follows:

$$-\frac{R^2}{2n} + C_0 = C_1 R^{2-n}. \tag{1.41}$$

By the uniqueness of solution of the Dirichlet problem, we conclude that $v(x)$ inside $B_R$ is indeed given by (1.40), although we do not know yet explicitly the values of $C_1, C_0$.

To determine them observe that

$$v(0) = \int_{B_R} E(y)\,dy = \int_0^R \frac{1}{\omega_n (n-2) r^{n-2}} \omega_n r^{n-1} dr = \frac{R^2}{2(n-2)},$$

which together with (1.40) at $x = 0$ implies

$$C_0 = \frac{R^2}{2(n-2)}.$$

Then we can determine $C_1$ from (1.41) as follows:

$$C_1 = \frac{R^n}{n(n-2)}.$$

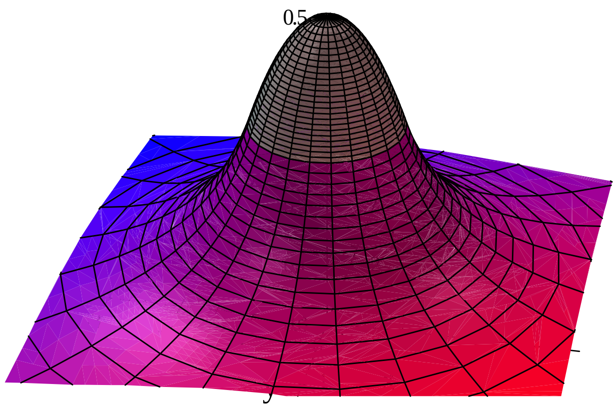
Hence, we obtain

$$v(x) = \begin{cases} \frac{R^n}{n(n-2)} |x|^{2-n}, & |x| \geq R \\ -\frac{|x|^2}{2n} + \frac{R^2}{2(n-2)}, & |x| \leq R. \end{cases}$$

Note that in domain $|x| \geq R$ we have

$$v(x) = \frac{\omega_n}{n} R^n \frac{1}{\omega_n (n-2) |x|^{n-2}} = \operatorname{vol}(B_R) E(x).$$

In other words, outside the ball $v(x)$ coincides with the Newtonian potential of a point mass $\operatorname{vol}(B_R)$ located at the center. This result was first obtained by Newton by an explicit computation of the integral (1.39) using clever geometric tricks.




Newtonian potential of a ball (inside the ball and outside the ball)

**Theorem 1.13** *Let $f$ be a bounded function in $B_R$ such that $f \in C^2(B_R)$, and let $\varphi \in C(\partial B_R)$. Then the Dirichlet problem*

$$\begin{cases} \Delta u = f & \text{in } B_R \\ u = \varphi & \text{on } \partial B_R \end{cases} \tag{1.42}$$

*has the following solution*

$$u(x) = -\int_{B_R} G(x,y) f(y)\, dy + \int_{\partial B_R} K(x,y) \varphi(y)\, d\sigma(y), \tag{1.43}$$

*where $G$ is the Green function of $B_R$ and $K$ is the Poisson kernel of $B_R$ (cf. (1.32)). More precisely, there exists a function $u \in C^2(B_R) \cap C\left(\overline{B}_R\right)$ that satisfies (1.42), and this function is given for any $x \in B_R$ by (1.43).*

**Remark.** The statement is also true if the condition $f \in C^2(B_R)$ is relaxed to $f \in C^\alpha(B_R)$ with arbitrary $\alpha > 0$, that is, if $f$ is Hölder continuous in $B_R$. However, the proof in that case is more complicated.

11.11.15

**Proof.** The case $f = 0$ was considered in Theorem 1.11. In the general case, extending $f$ to $\mathbb{R}^n$ by setting $f = 0$ in $\Omega^c$, consider the Newtonian potential of $-f$:

$$v(x) = -\int_{B_R} E(x,y) f(y)\,dy = -\int_{\mathbb{R}^n} E(x,y) f(y)\,dy. \tag{1.44}$$

By Lemma 1.12, we know that $v \in C^2(B_R) \cap C(\mathbb{R}^n)$ and $\Delta v = f$. Introduce a new unknown function

$$w = u - v$$

that has to be of the class $C^2(B_R) \cap C\left(\overline{B}_R\right)$ and to satisfy $\Delta w = 0$ in $B_R$ because

$$\Delta w = \Delta(u - v) = f - f = 0 \text{ in } B_R.$$

At the boundary we have

$$w = u - v = \varphi - v \text{ on } \partial B_R.$$

Hence, the Dirichlet problem (1.42) for $u$ is equivalent to the following Dirichlet problem for $w$:

$$\begin{cases} \Delta w = 0 & \text{in } B_R \\ w = \varphi - v & \text{on } \partial B_R. \end{cases}$$

Sine $v - \varphi$ is continuous, by Theorem 1.11 we conclude that the solution $w$ of this problem exists and is given by

$$\begin{aligned} w(x) &= \int_{\partial B_R} K(x,z)(\varphi - v)(z)\,d\sigma(z) \\ &= \int_{\partial B_R} K(x,z)\varphi(z)\,d\sigma(z) - \int_{\partial B_R} K(x,z) v(z)\,d\sigma(z). \end{aligned} \tag{1.45}$$

The second integral here is equal to

$$\begin{aligned} -\int_{\partial B_R} K(x,z) v(z)\,d\sigma(z) &= \int_{\partial B_R} K(x,z)\left(\int_{B_R} E(z,y) f(y)\,dy\right) dz \\ &= \int_{B_R}\left(\int_{\partial B_R} K(x,z) E(z,y)\,dz\right) f(y)\,dy \\ &= \int_{B_R} h(x,y) f(y)\,dy \end{aligned}$$

where

$$h(x,y) = \int_{\partial B_R} K(x,z) E(z,y)\,dz.$$

Fix $y \in B_R$. Then by Theorem 1.11, the function $h(x,y)$ as a function of $x$ solves the Dirichlet problem

$$\begin{cases} \Delta h(\cdot,y) = 0 & \text{in } B_R \\ h(\cdot,y) = E(\cdot,y) & \text{on } \partial B_R. \end{cases}$$

By uniqueness of solution of the Dirichlet problem, function $h(x,y)$ coincides with the function $h_y(x)$ from the Definition of the Green function, which implies that

$$G(x,y) = E(x,y) - h(x,y). \tag{1.46}$$

The function $u = v + w$ is clearly a solution of (1.42). Putting together (1.44), (1.45) and (1.46), we obtain

$$\begin{aligned} u(x) &= -\int_{\mathbb{R}^n} E(x,y) f(y)\, dy + \int_{B_R} h(x,y) f(y)\, dy + \int_{\partial B_R} K(x,z)\varphi(z)\, d\sigma(z) \\ &= -\int_{B_R} G(x,y) f(y)\, dy + \int_{\partial B_R} K(x,z)\varphi(z)\, d\sigma(z), \end{aligned}$$

which was to be proved. ■

## 1.6 Properties of harmonic functions

Here we obtain some consequences of Theorem 1.10. Let us restate it in the following form to be used below: if $u \in C^2\left(\overline{B}_R\right)$ and $\Delta u = 0$ in $B_R$ then, for any $y \in B_R$,

$$u(y) = \frac{1}{\omega_n R} \int_{\partial B_R} \frac{R^2 - |y|^2}{|x-y|^n} u(x)\, d\sigma(x). \tag{1.47}$$

We use the notation $B_R(z) = \{x \in \mathbb{R}^n : |x-y| < R\}$ for the ball of radius $R$ centered at $z \in \mathbb{R}^n$.

**Theorem 1.14** *If $u$ is a harmonic function in a domain $\Omega \subset \mathbb{R}^n$ then $u \in C^\infty(\Omega)$. Moreover, if $u \in C^2(\Omega)$ satisfies $\Delta u = f$ where $f \in C^\infty(\Omega)$ then also $u \in C^\infty(\Omega)$.*

Recall that by definition, a harmonic function is of the class $C^2$. This theorem tells that a posteriori it has to be $C^\infty$. Moreover, any function $u \in C^2$ is in fact of the class $C^\infty$ if $\Delta u \in C^\infty$. The latter property of added smoothness is called *hypoellipticity* of the Laplace operator. Typically, more general elliptic operator are also hypoelliptic.

**Proof.** Consider first the case when $u$ is harmonic in $\Omega$. In order to prove that $u \in C^\infty(\Omega)$, it suffices to prove that $u \in C^\infty(B_R(z))$ for any ball $B_R(z)$ such that $\overline{B}_R \subset \Omega$. Without loss of generality, take $z = 0$. By (1.47) we have an integral representation of $u(y)$ for any $y \in B_R$, which implies that $u \in C^\infty(B_R)$ because the kernel

$$\frac{R^2 - |y|^2}{|x-y|^n}$$

is $C^\infty$ in $y \in B_R$ provided $x \in \partial B_R$.

Assume now that $\Delta u = f$ in $\Omega$ with $f \in C^\infty(\Omega)$, and prove again that $u \in C^\infty(B_R)$ where $B_R$ is the ball as above. By Lemma 1.12, the Newtonian potential

$$v(x) = \int_{B_R} E(x,y) f(y)\, dy$$

is $C^\infty$ smooth in $B_R$ and $\Delta v = -f$ in $B_R$. Hence, the function $u + v$ is harmonic in $B_R$, which implies that $u + v \in C^\infty(B_R)$ and, hence, $u \in C^\infty(B_R)$. ■

**Theorem 1.15** (Mean-value theorem) *Let $u$ be a harmonic function in a domain $\Omega \subset \mathbb{R}^n$. Then, for any ball $B_R(z)$ such that $\overline{B}_R(z) \subset \Omega$, we have*

$$u(z) = \fint_{\partial B_R} u(x)\, d\sigma(x) \tag{1.48}$$

*and*

$$u(z) = \fint_{B_R(z)} u(x)\, dx. \tag{1.49}$$

Here we use the following notations for normalized integrals:

$$\fint_{\partial\Omega} u d\sigma := \frac{1}{\sigma(\partial\Omega)} \int_{\partial\Omega} u d\sigma$$

and

$$\fint_{\Omega} u dx = \frac{1}{\operatorname{vol}(\Omega)} \int_{\Omega} u dx.$$

Hence, the value of a harmonic function $u$ at the center of the ball is equal to the arithmetic mean of $u$ over the ball and over the sphere.

**Proof.** Without loss of generality we can assume that $z = 0$. Applying (1.47) with $y = 0$, we obtain

$$u(0) = \frac{1}{\omega_n R} \int_{\partial B_R} \frac{R^2 - 0^2}{|x-0|^n} u(x)\, d\sigma(x) = \frac{1}{\omega_n R^{n-1}} \int_{\partial B_R} u d\sigma. \tag{1.50}$$

Since $\omega_n R^{n-1} = \sigma(\partial B_R)$, we obtain (1.48). To prove (1.49) observe that in the polar coordinates

$$\int_{B_R} u(x)\, dx = \int_0^R \left( \int_{\partial B_r} u d\sigma \right) dr$$

Since by (1.50)

$$\int_{\partial B_r} u d\sigma = \omega_n r^{n-1} u(0),$$

we obtain

$$\int_{B_R} u(x)\, dx = \int_0^R \omega_n r^{n-1} u(0)\, dr = \frac{\omega_n}{n} R^n u(0). \tag{1.51}$$

Applying (1.51) with $u \equiv 1$, we obtain

$$\operatorname{vol}(B_R) = \frac{\omega_n}{n} R^n.$$

Hence, (1.51) implies

$$\int_{B_R} u(x)\, dx = \operatorname{vol}(B_R)\, u(0),$$

which is equivalent to (1.49). ■

**Theorem 1.16** (Harnack inequality) *Let $u$ be a non-negative harmonic function in a ball $B_R$. Then, for any $0 < r < R$,*

$$\sup_{B_r} u \le \left( \frac{R/r + 1}{R/r - 1} \right)^n \inf_{B_r} u. \tag{1.52}$$

It is important for applications, that the constant $C = \left(\frac{R/r+1}{R/r-1}\right)^n$ depends only on the ratio $R/r$. For example, if $R = 2r$ then $C = 3^n$.

**Proof.** By the maximum and minimum principles we have

$$\sup_{B_r} u = \max_{\partial B_r} u \quad \text{and} \quad \inf_{B_r} u = \min_{\partial B_r} u.$$

Let $y'$ be the point of maximum of $u$ at $\partial B_r$ and $y''$ – the point of minimum of $u$ at $\partial B_r$. Note that for any $y \in \partial B_r$ and for any $x \in \partial B_R$,

$$R - r \leq |x - y| \leq R + r.$$

It follows from (1.47) that

$$\begin{aligned} u(y') &= \frac{1}{\omega_n R} \int_{\partial B_R} \frac{R^2 - |y'|^2}{|x - y'|^n} u(x)\, d\sigma(x) \\ &\leq \frac{R^2 - r^2}{\omega_n R (R - r)^n} \int_{\partial B_R} u(x)\, d\sigma(x) \end{aligned}$$

and similarly

$$\begin{aligned} u(y'') &= \frac{1}{\omega_n R} \int_{\partial B_R} \frac{R^2 - |y''|^2}{|x - y''|^n} u(x)\, d\sigma(x) \\ &\geq \frac{R^2 - r^2}{\omega_n R (R + r)^n} \int_{\partial B_R} u(x)\, d\sigma(x). \end{aligned}$$

Therefore, we obtain

$$u(y') \leq \frac{(R + r)^n}{(R - r)^n} u(y''),$$

whence (1.52) follows. ■

## 1.7 Sequences of harmonic functions

**Theorem 1.17** (Harnack's first theorem) *Let $\{u_k\}_{k=1}^{\infty}$ be a sequence of harmonic functions in a domain $\Omega \subset \mathbb{R}^n$. If $u_k \rightrightarrows u$ in $\Omega$ as $k \to \infty$ then the function $u$ is also harmonic in $\Omega$.*

Let us recall for comparison, that uniform limits of continuous functions are again continuous, but uniform limits of $C^k$ functions where $k \geq 1$, do not have to be $C^k$. Hence, if $u$ is a uniform limit of harmonic functions $u_k$ then a priori we can only say that $u$ is continuous, whereas the harmonicity of $u$ and, in particular, the smoothness of $u$, are not at all obvious.

**Proof.** The function $u$ is continuous in $\Omega$ as a uniform limit of continuous functions. To prove that $u$ is harmonic in $\Omega$, it suffices to prove that $u$ is harmonic in any ball $B_R(z) \subset \Omega$. Assume without loss of generality that $z = 0$.

Denoting $\varphi_k = u_k|_{\partial B_R}$ and $\varphi = u|_{\partial B_R}$ we have

$$\varphi_k \rightrightarrows \varphi \text{ on } \partial B_R \text{ as } k \to \infty.$$

Let $v$ be the solution of the Dirichlet problem

$$\begin{cases} \Delta v = 0 & \text{in } B_R \\ v = \varphi & \text{on } \partial B_R \end{cases}$$

that exists by Theorem 1.11. Since $u_k - v$ is harmonic in $B_R$ and is continuous in $\overline{B}_R$, by the maximum principle (1.5) of Corollary 1.2, we obtain

$$\max_{\overline{B}_R} |u_k - v| = \max_{\partial B_R} |u_k - v| = \max_{\partial B_R} |\varphi_k - \varphi| .$$

Since the right hand side goes to 0 as $k \to \infty$, it follows that

$$u_k \rightrightarrows v \text{ in } B_R \ \text{ as } k \to \infty.$$

Since also $u_k \rightrightarrows u$, we conclude that $u = v$ and, hence, $u$ is harmonic in $B_R$. ■

**Theorem 1.18** (Harnack's second theorem) *Let $\{u_k\}_{k=1}^{\infty}$ be a sequence of harmonic functions in a connected domain $\Omega \subset \mathbb{R}^n$. Assume that this sequence is monotone increasing, that is, $u_{k+1}(x) \geq u_k(x)$ for all $k \geq 1, x \in \Omega$. The the function*

$$u(x) = \lim_{k \to \infty} u_k(x)$$

*is either identically equal to $\infty$ in $\Omega$, or it is a harmonic function in $\Omega$. Moreover, in the latter case the convergence $u_k \to u$ is locally uniform.*

**Proof.** By replacing $u_k$ with $u_k - u_1$, we can assume that all functions $u_k$ are non-negative. Consider the sets

$$F = \{x \in \Omega : u(x) < \infty\}$$

and

$$I = \{x \in \Omega : u(x) = \infty\}$$

so that $\Omega = F \sqcup I$. Let us prove that both $F$ and $I$ are open sets.

Indeed, take a point $x \in F$ and show that also $B_\varepsilon(x) \in F$ for some $\varepsilon > 0$. Choose $\varepsilon$ so that $B_{2\varepsilon}(x) \subset \Omega$. By the Harnack inequality, we have

$$\sup_{B_\varepsilon(x)} u_k \leq C \inf_{B_\varepsilon(x)} u_k \leq C u_k(x),$$

where $C = 3^n$. By passing to the limit as $k \to \infty$, we obtain

$$\sup_{B_\varepsilon(x)} u \leq C u(x).$$

Since $u(x) < \infty$, we obtain that also $\sup_{B_\varepsilon(x)} u < \infty$ and, hence, $B_\varepsilon(x) \subset F$. Hence, $F$ is open. 13.11.15

In the same way one proves that

$$\inf_{B_\varepsilon(x)} u \geq C^{-1} u(x),$$

which implies that $I$ is open.

Since $\Omega$ is connected and $\Omega = F \sqcup I$, it follows that either $I = \Omega$ or $F = \Omega$. In the former case we have $u \equiv \infty$ in $\Omega$, in the latter case: $u(x) < \infty$ for all $x \in \Omega$. Let us prove that in the latter case $u$ is harmonic. For that, we first show that the convergence $u_k \to u$ is locally uniform, that is, for any $x \in \Omega$ there is $\varepsilon > 0$ such that

$$u_k \rightrightarrows u \text{ in } B_\varepsilon(x) \quad \text{as } k \to \infty.$$

Then the harmonicity of $u$ will follow by Harnack's first theorem.

Choose again $\varepsilon > 0$ so that $B_{2\varepsilon}(x) \subset \Omega$. For any two indices $k > l$, apply the Harnack inequality to the non-negative harmonic function $u_k - u_l$:

$$\sup_{B_\varepsilon(x)} (u_k - u_l) \le C (u_k - u_l)(x).$$

Since $(u_k - u_l)(x) \to 0$ as $k, l \to \infty$, it follows that

$$u_k - u_l \rightrightarrows 0 \text{ in } B_\varepsilon(x) \text{ as } k, l \to \infty.$$

Hence, the sequence $\{u_k\}$ converges uniformly in $B_\varepsilon(x)$. Since $\{u_k\}$ convergence pointwise to $u$, it follows that

$$u_k \rightrightarrows u \text{ in } B_\varepsilon(x) \text{ as } k \to \infty,$$

which finishes the proof. ■

As an example of application of Harnack's second theorem, let us prove the following extension of Lemma 1.12.

**Corollary 1.19** *Let $f$ be a non-negative locally bounded measurable function on $\mathbb{R}^n$. Consider the Newtonian potential*

$$v(x) = \int_{\mathbb{R}^n} E(x, y) f(y) \, dy.$$

*Then either $v \equiv \infty$ in $\mathbb{R}^n$ or $v$ is a continuous function in $\mathbb{R}^n$. In the latter case, if $f \in C^2(\Omega)$ for some open set $\Omega \subset \mathbb{R}^n$, then also $v \in C^2(\Omega)$ and $\Delta v = -f$ in $\Omega$.*

**Proof.** Consider a sequence $\{B_k\}_{k=1}^\infty$ of balls $B_k = B_k(0)$ and set

$$v_k(x) = \int_{B_k} E(x, y) f(y) \, dy$$

so that

$$v(x) = \lim_{k \to \infty} v_k(x). \tag{1.53}$$

Since $v_k$ is the potential of the function $f_k = f \mathbf{1}_{B_k}$, by Lemma 1.12 we have $v_k \in C(\mathbb{R}^n)$. Let us show that if $v(x_0) < \infty$ at some point $x_0$ then $v(x)$ is a finite continuous function on $\mathbb{R}^n$. Choose $l$ so big that $B_l$ contains $x_0$. We have

$$v(x) = v_l(x) + \lim_{k \to \infty} (v_k - v_l)(x)$$

and

$$(v_k - v_l)(x) = \int_{B_k \setminus \overline{B}_l} E(x,y) f(y)\, dy.$$

Applying Lemma 1.12 to function $f\mathbf{1}_{B_k \setminus \overline{B}_l}$, we obtain that $v_k - v_l$ is a harmonic function in $B_l$, for all $k > l$. The sequence $\{v_k - v_l\}$ is monotone increasing in $k$ and is finite at $x_0 \in B_l$. Hence, by Harnack's second theorem, the limit $\lim_{k\to\infty}(v_k - v_l)$ is a harmonic function in $B_l$, which implies that $v$ is a continuous function in $B_l$. Since $l$ can be chosen arbitrarily big, we conclude that $v$ is continuous in $\mathbb{R}^n$.

Assuming that $v$ is finite and $f \in C^2(\Omega)$, repeat the above argument choosing $l$ so big that $\Omega \subset B_l$. As we have seen,

$$v = v_l + \text{a harmonic function in } B_l.$$

Since by Lemma 1.12 $v_l \in C^2(\Omega)$ and $\Delta v_l = -f$ in $\Omega$, it follows that also $v \in C^2(\Omega)$ and $\Delta v = -f$ in $\Omega$. ■

## 1.8 Discrete Laplace operator

A graph $G$ is a couple $(V, E)$ where $V$ is a set of *vertices*, that is, an arbitrary set, whose elements are called vertices, and $E$ is a set of *edges*, that is, $E$ consists of some unordered couples $(x, y)$ where $x, y \in V$. We write $x \sim y$ if $(x, y) \in E$ and say that $x$ is connected to $y$, or $x$ is adjacent to $y$, or $x$ is a neighbor of $y$. By definition, $x \sim y$ is equivalent to $y \sim x$.

A graph $G$ is called *locally finite* if each vertex has a finite number of edges. For each point $x$, define its degree

$$\deg(x) = \#\{y \in V : x \sim y\},$$

that is, $\deg(x)$ is the number of the edges with endpoint $x$. A graph $G$ is called *finite* if the number of vertices is finite. Of course, a finite graph is locally finite.

**Definition.** Let $(V, E)$ be a locally finite graph without isolated points (so that $0 < \deg(x) < \infty$ for all $x \in V$). For any function $f : V \to \mathbb{R}$, define the function $\Delta f$ by

$$\Delta f(x) := \frac{1}{\deg(x)} \sum_{y \in V: y \sim x} f(y) - f(x) = \frac{1}{\deg(x)} \sum_{y \in V: y \sim x} (f(y) - f(x)).$$

The operator $\Delta$ on functions on $V$ is called the *Laplace operator* of $(V, E)$.

The equation $\Delta u = 0$ is called the Laplace equation and its solutions are called harmonic functions on the graph. For example, a constant function is harmonic.

In what follows we always assume that $0 < \deg(x) < \infty$ for all $x \in V$, so that $\Delta$ is well-defined on functions on $V$.

One can regards a graph $(V, E)$ as an electrical network, where the edges are the wires that conduct electrical current, and the vertices are junctions. Assuming that the resistance of each edges is equal to one, we obtain by the Ohm's law, that the potential difference $u(y) - u(y)$ of two neighboring vertices $x$ and $y$ is equal to the current along the edge $xy$. By Kirchhoff's law, the sum of the currents incoming and outcoming at the same point $x$ is equal to 0, which implies

$$\sum_{y:y\sim x}(u(y)-u(x))=0,$$

which is equivalent to $\Delta u=0$. Hence, in the absence of the external sources of the current, the electrical potential of the network is a harmonic function.

**Definition.** A graph $G=(V,E)$ is called connected if any two vertices $x,y\in V$ can be connected by a finite chain $\{x_k\}_{k=0}^n$ such that

$$x=x_0\sim x_1\sim\ldots\sim x_{n-1}\sim x_n=y.$$

Choose a subset $\Omega$ of $V$ and consider the following Dirichlet problem:

$$\begin{cases}\Delta u(x)=f(x) & \text{for all } x\in\Omega,\\ u(x)=\varphi(x) & \text{for all } x\in\Omega^c,\end{cases}\tag{1.54}$$

where $u:V\to\mathbb{R}$ is an unknown function while the functions $f:\Omega\to\mathbb{R}$ and $\varphi:\Omega^c\to\mathbb{R}$ are given.

**Theorem 1.20** *Let $G=(V,E)$ be a connected graph, and let $\Omega$ be a finite subset of $V$ such that $\Omega^c$ is non-empty. Then, for all functions $f,\varphi$ as above, the Dirichlet problem* (1.54) *has a unique solution.*

Note that, by the second condition in (1.54), the function $u$ is already defined outside $\Omega$, so the problem is to construct an extension of $u$ to $\Omega$ that would satisfy the equation $\Delta u=f$ in $\Omega$.

Define the *vertex boundary* of $\Omega$ as follows:

$$\partial\Omega=\{y\in\Omega^c: y\sim x \text{ for some } x\in\Omega\}.$$

Observe that the Laplace equation $\Delta u(x)=f(x)$ for $x\in\Omega$ involves the values $u(y)$ at neighboring vertices $y$ of $x$, and any neighboring point $y$ belongs to either $\Omega$ or to $\partial\Omega$. Hence, the equation $\Delta u(x)=f(x)$ uses the prescribed values of $u$ only at the boundary $\partial\Omega$, which means that the second condition in (1.54) can be restricted to $\partial\Omega$ as follows:

$$u(x)=\varphi(x)\quad\text{for all } x\in\partial\Omega.$$

This condition (as well as the second condition in (1.54) is called the boundary condition.

If $\Omega^c$ is empty then the statement of Theorem 1.20 is not true. For example, in this case any constant function $u$ satisfies the same equation $\Delta u=0$ so that there is no uniqueness. One can show that the existence also fails in this case.

The proof of Theorem 1.20 is based on the following maximum principle. A function $u:V\to\mathbb{R}$ is called *subharmonic* in $\Omega$ if $\Delta u(x)\geq 0$ for all $x\in\Omega$, and *superharmonic* in $\Omega$ if $\Delta u(x)\leq 0$ for all $x\in\Omega$.

**Lemma 1.21** (A maximum/minimum principle) *Let $\Omega$ be a non-empty finite subset of $V$ such that $\Omega^c$ is non-empty. Then, for any function $u:V\to\mathbb{R}$, that is subharmonic in $\Omega$, we have*

$$\max_{\Omega}u\leq\sup_{\Omega^c}u.$$

*For any function $u : V \to \mathbb{R}$, that is superharmonic in $\Omega$, we have*

$$\min_{\Omega} u \geq \inf_{\Omega^c} u.$$

**Proof.** It suffices to prove the first claim. If $\sup_{\Omega^c} u = +\infty$ then there is nothing to prove. If $\sup_{\Omega^c} u < \infty$ then, by replacing $u$ by $u + \text{const}$, we can assume that $\sup_{\Omega^c} u = 0$. Set

$$M = \max_{\Omega} u$$

and show that $M \leq 0$, which will settle the claim. Assume from the contrary that $M > 0$ and consider the set

$$S := \{x \in V : u(x) = M\}. \tag{1.55}$$

Clearly, $S \subset \Omega$ and $S$ is non-empty.

**Claim 1.** *If $x \in S$ then all neighbors of $x$ also belong to $S$.*

Indeed, we have $\Delta u(x) \geq 0$ which can be rewritten in the form

$$u(x) \leq \frac{1}{\deg(x)} \sum_{y \sim x} u(y).$$

Since $u(y) \leq M$ for all $y \in V$ (note that $u(y) \leq 0$ for $y \in \Omega^c$), we have

$$\frac{1}{\deg(x)} \sum_{y \sim x} u(y) \leq M \frac{1}{\deg(x)} \sum_{y \sim x} = M.$$

Since $u(x) = M$, all inequalities in the above two lines must be equalities, whence it follows that $u(y) = M$ for all $y \sim x$. This implies that all such $y$ belong to $S$.

**Claim 2.** *Let $S$ be a non-empty set of vertices of a connected graph $(V, E)$ such that $x \in S$ implies that all neighbors of $x$ belong to $S$. Then $S = V$.*

Indeed, let $x \in S$ and $y$ be any other vertex. Then there is a path $\{x_k\}_{k=0}^n$ between $x$ and $y$, that is,

$$x = x_0 \sim x_1 \sim x_2 \sim \ldots \sim x_n = y.$$

Since $x_0 \in S$ and $x_1 \sim x_0$, we obtain $x_1 \in S$. Since $x_2 \sim x_1$, we obtain $x_2 \in S$. By induction, we conclude that all $x_k \in S$, whence $y \in S$.

It follows from the two claims that the set $S$ defined by (1.55) must coincide with $V$, which is not possible since $S \subset \Omega$ and $\Omega^c$ is non-empty. This contradiction shows that $M \leq 0$. ∎

**Proof of Theorem 1.20.** Let us first prove the uniqueness. If we have two solutions $u_1$ and $u_2$ of (1.54) then the difference $u = u_1 - u_2$ satisfies the conditions

$$\begin{cases} \Delta u(x) = 0 & \text{for all } x \in \Omega, \\ u(x) = 0 & \text{for all } x \in \Omega^c. \end{cases}$$

We need to prove that $u \equiv 0$. Since $u$ is both subharmonic and superharmonic in $\Omega$, Lemma 1.21 yields

$$0 = \inf_{\Omega^c} u \leq \min_{\Omega} u \leq \max_{\Omega} u \leq \sup_{\Omega^c} u = 0,$$

whence $u \equiv 0$.

Let us now prove the existence of a solution to (1.54) for all $f, \varphi$. For any $x \in \Omega$, rewrite the equation $\Delta u(x) = f(x)$ in the form

$$\frac{1}{\deg(x)} \sum_{y \in \Omega,\ y \sim x} u(y) - u(x) = f(x) - \frac{1}{\deg(x)} \sum_{y \in \Omega^c,\ y \sim x} \varphi(y), \tag{1.56}$$

where we have moved to the right hand side the terms with $y \in \Omega^c$ and used that $u(y) = \varphi(y)$. Denote by $\mathcal{F}$ the set of all real-valued functions $u$ on $\Omega$ and observe that the left hand side of (1.56) can be regarded as an operator in this space; denote it by $Lu$, that is,

$$Lu(x) = \frac{1}{\deg(x)} \sum_{y \in \Omega,\, y \sim x} u(y) - u(x),$$

for all $x \in \Omega$. Rewrite the equation (1.56) in the form $Lu = h$ where $h$ is the right hand side of (1.56), which is a given function on $\Omega$. Note that $\mathcal{F}$ is a linear space. Since the family $\left\{\mathbf{1}_{\{x\}}\right\}_{x \in \Omega}$ of indicator functions form obviously a basis in $\mathcal{F}$, we obtain that $\dim \mathcal{F} = \#\Omega < \infty$. Hence, the operator $L : \mathcal{F} \to \mathcal{F}$ is a linear operator in a finitely dimensional space, and the first part of the proof shows that $Lu = 0$ implies $u = 0$ (indeed, just set $f = 0$ and $\varphi = 0$ in (1.56)), that is, the operator $L$ is injective. By Linear Algebra, any injective operator acting in the spaces of equal dimensions, must be bijective (alternatively, one can say that the injectivity of $L$ implies that $\det L \neq 0$ whence it follows that $L$ is invertible and, hence, bijective). Hence, for any $h \in \mathcal{F}$, there is a solution $u = L^{-1}h \in \mathcal{F}$, which finishes the proof. ∎

18.11.15

## 1.9 Separation of variables in the Dirichlet problem

Here is an alternative method of solving the Dirichlet problem in the two-dimensional ball or annulus. Let $(r, \theta)$ be the polar coordinates. The Laplace equation $\Delta u = 0$ has in the polar coordinates the form

$$\partial_{rr} u + \frac{1}{r} \partial_r u + \frac{1}{r^2} \partial_{\theta\theta} u = 0 \tag{1.57}$$

(see Exercises). Let us first try to find a solution in the form $u = v(r)\, w(\theta)$. Substitution into (1.57) gives

$$v''w + \frac{1}{r} v'w + \frac{1}{r^2} vw'' = 0$$

that is

$$\frac{v'' + \frac{1}{r} v'}{\frac{1}{r^2} v} = -\frac{w''}{w}.$$

Since the left hand side here depends only on $r$ and the right hand side only on $\theta$, the two functions can be equal only if they both are constants. Denoting this constant by $\lambda$, we obtain two ODEs:

$$w'' + \lambda w = 0 \tag{1.58}$$

and

$$v'' + \frac{1}{r} v' - \frac{\lambda}{r^2} v = 0. \tag{1.59}$$

Since $w$ is a function of the polar angle $\theta$, the function $w(\theta)$ must be $2\pi$-periodic. Equation (1.58) has periodic solutions only if $\lambda \geq 0$. We have then

$$w(\theta) = C_1 \cos \sqrt{\lambda}\theta + C_2 \sin \sqrt{\lambda}\theta.$$

This function is $2\pi$-periodic if and only if $\sqrt{\lambda} = k$, where $k$ is a non-negative integer. Substituting $\lambda = k^2$ into (1.59), we obtain

$$v'' + \frac{1}{r}v' - \frac{k^2}{r^2}v = 0.$$

This is Euler equation that has the general solution:

$$v = C_1 r^k + C_2 r^{-k} \quad \text{if } k > 0$$

$$v = C_1 + C_2 \ln \frac{1}{r} \quad \text{if } k = 0.$$

Hence, for any $k \geq 0$ we obtain the following harmonic function

$$u_0 = \alpha_0 + \beta_0 \ln \frac{1}{r}, \quad \text{for } k = 0$$

and

$$u_k = \left(\alpha_k r^k + \beta_k r^{-k}\right)(a_k \cos k\theta + b_k \sin k\theta)$$

(we have seen already the harmonic functions $r^k \cos k\theta$ and $r^k \sin k\theta$). Each of these functions is harmonic in $\mathbb{R}^2 \setminus \{0\}$. If the series

$$\sum_{k=0}^{\infty} u_k$$

converges locally uniformly in some domain then the sum is also harmonic function in this domain by Harnack's first theorem. By choosing coefficients one can try to match the boundary conditions.

Let us illustrate this method for the Dirichlet problem in the disk $B_1 = \{x \in \mathbb{R}^2 : |x| < 1\}$ :

$$\begin{cases} \Delta u = 0 & \text{in } B_1 \\ u = f & \text{on } \partial B_1. \end{cases}$$

The function $f$ can be considered as a $2\pi$-periodic function of the polar angle, so we write $f(\theta)$. Since function $u$ has to be defined also at the origin, we drop from $u_0$ and $u_k$ the parts having singularities at 0, and search the solution in the form

$$u(r,\theta) = \frac{a_0}{2} + \sum_{k=1}^{\infty} r^k (a_k \cos k\theta + b_k \sin k\theta). \tag{1.60}$$

The boundary value of $u$ is attained for $r = 1$. Hence, function $f$ should have the following expansion in Fourier series

$$f(\theta) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos k\theta + b_k \sin k\theta). \tag{1.61}$$

It is known that any $2\pi$-periodic function $f$ that belongs to the Lebesgue class $L^2$, admits an expansion (1.61) into the Fourier series that converges to $f$ in the sense of $L^2$. The coefficients are computed as follows:

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos k\theta d\theta, \quad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin k\theta d\theta. \tag{1.62}$$

Moreover, if $f \in C^1(\mathbb{R})$ then the series (1.61) converges absolutely and uniformly.

**Proposition 1.22** *Assume that $f$ is a $2\pi$-periodic function on $\mathbb{R}$ that admits an absolutely convergent Fourier series* (1.61)*. Then the series* (1.60) *converges absolutely and uniformly for all $r \leq 1$ and $\theta \in \mathbb{R}$, its sum $u$ belongs to the class $C\left(\overline{B_1}\right)$, is harmonic in $B_1$, and is equal to $f$ at $\partial B_1$.*

**Proof.** Indeed, the absolut convergence of (1.61) is equivalent to

$$\sum_{k=1}^{\infty}(|a_k| + |b_k|) < \infty. \tag{1.63}$$

If so then the series (1.60) converges absolutely and uniformly for all $r \leq 1$ and $\theta \in \mathbb{R}$. Hence, the function $u$ is continuous in $\overline{B_1}$. In particular, on $\partial B_1$ we obtain $u = f$, just by taking $r = 1$ in (1.60). Since each term $r^k \cos k\theta$ and $r^k \sin k\theta$ is a harmonic function, the infinite sum $u$ is also harmonic in $B_1$, by Harnack's first theorem. ■

**Remark.** Differentiating the right hand side of (1.60) in $r$, we obtain that in $B_1$

$$\partial_r u(r,\theta) = \sum_{k=1}^{\infty} k r^{k-1}(a_k \cos k\theta + b_k \sin k\theta), \tag{1.64}$$

because the series in the right hand side converges absolutely and locally uniformly in $B_1$, that is, for all $r < 1$ and $\theta \in \mathbb{R}$. In the same way we have in $B_1$

$$\partial_\theta u = \sum_{k=1}^{\infty} k r^k (-a_k \sin k\theta + b_k \cos k\theta). \tag{1.65}$$

If we know in addition that

$$\sum_{k=1}^{\infty} k(|a_k| + |b_k|) < \infty \tag{1.66}$$

then the series in (1.64) and (1.65) converge absolutely and uniformly for $r \leq 1$ and $\theta \in \mathbb{R}$, which implies that $u \in C^1\left(\overline{B_1}\right)$.

## 1.10 Variational problem and the Dirichlet principle

Let $\Omega$ be a bounded domain and $\varphi$ be a continuous function on $\partial\Omega$. Consider the *variational problem*

$$\begin{cases} \int_\Omega |\nabla u|^2 dx \mapsto \min \\ u = \varphi \text{ on } \partial\Omega \end{cases} \tag{1.67}$$

where $u \in C^1(\overline{\Omega})$. In other words, we look for a function $u \in C^1(\overline{\Omega})$ with the given boundary value on $\partial\Omega$ that minimizes the *Dirichlet integral* $\int_\Omega |\nabla u|^2 \, dx$.

One of motivations for the problem (1.67) comes from the following geometric problem: construct a hypersurface $S$ in $\mathbb{R}^{n+1}$ over the base $\Omega$, whose boundary $\partial S$ is given and whose surface area $\sigma(S)$ is minimal. Indeed, let $S$ be the graph of a function $u$ in $\Omega$. The prescribed boundary of $\partial S$ amounts to the boundary condition $u = \varphi$ on $\partial\Omega$, while

$$\sigma(S) = \int_\Omega \sqrt{1 + |\nabla u|^2} dx.$$

Hence, we obtain the variational problem

$$\begin{cases} \int_\Omega \sqrt{1 + |\nabla u|^2} dx \mapsto \min. \\ u = \varphi \text{ on } \partial\Omega. \end{cases} \tag{1.68}$$

If we assume that $|\nabla u|$ is small, then

$$\sqrt{1 + |\nabla u|^2} \approx 1 + \frac{1}{2} |\nabla u|^2,$$

so that (1.68) becomes (1.67). Any function $u$ that solves (1.68) is called an *area minimizer*. As we will see, functions that solve (1.67) are harmonic. Hence, harmonic functions are approximately area minimizers.

Consider now the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \tag{1.69}$$

where the solution $u$ is sought in the class $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$.

**Theorem 1.23** (The Dirichlet principle) *Let $\Omega$ be a bounded region. Then a function $u$ is a solution of* (1.69) *if and only if $u$ is a solution of* (1.67).

Since solution to the Dirichlet problem is always unique, we see that also the variational problem has at most one solution. On the other hand, we know that if $\Omega$ is a ball then the Dirichlet problem does have a solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$. Under some additional assumption about $\varphi$ one obtains $u \in C^1(\overline{\Omega})$ (see, for example, the previous section), which then implies the existence of a solution of (1.67).

**Idea of proof.** Let us first prove a simplified version of this theorem, when solutions of both problems (1.67) and (1.69) are sought in the class $C^2(\overline{\Omega})$. Assume first that $u \in C^2(\overline{\Omega})$ is a solution of (1.67) and prove that $u$ is a solution of (1.69), that is, $\Delta u = 0$ in $\Omega$. Fix a function $w \in C_0^\infty(\Omega)$ and $t \in \mathbb{R}$ and consider the function $v = u + tw$. Since $v = u = \varphi$ on $\partial\Omega$, we conclude that

$$\int_\Omega |\nabla v|^2 \, dx \geq \int_\Omega |\nabla u|^2 \, dx.$$

Computing

$$|\nabla v|^2 = |\nabla(u + tw)|^2 = |\nabla u|^2 + 2t\nabla u \cdot \nabla w + |\nabla w|^2,$$

we obtain

$$\int_\Omega |\nabla u|^2\, dx + 2t\int_\Omega \nabla u\cdot\nabla w\, dx + t^2\int_\Omega |\nabla w|^2\, dx \geq \int_\Omega |\nabla u|^2\, dx$$

and, hence,

$$2t\int_\Omega \nabla u\cdot\nabla w\, dx + t^2\int_\Omega |\nabla w|^2\, dx \geq 0.$$

Assuming that $t>0$, divide by $t$ and obtain

$$2\int_\Omega \nabla u\cdot\nabla w\, dx + t\int_\Omega |\nabla w|^2\, dx \geq 0.$$

Letting $t\to 0$, we obtain

$$\int_\Omega \nabla u\cdot\nabla w\, dx \geq 0.$$

In the same way, considering $t<0$, we obtain

$$\int_\Omega \nabla u\cdot\nabla w\, dx \leq 0,$$

whence

$$\int_\Omega \nabla u\cdot\nabla w\, dx = 0. \tag{1.70}$$

By the Green formula we have

$$\int_\Omega w\Delta u\, dx = -\int_\Omega \nabla u\cdot\nabla w\, dx + \int_{\partial\Omega} w\partial_\nu u\, d\sigma. \tag{1.71}$$

By (1.70) and $w=0$ on $\partial\Omega$ we obtain

$$\int_\Omega w\Delta u\, dx = 0.$$

Since $w\in C_0^\infty(\Omega)$ is arbitrary, it follows that $\Delta u=0$ in $\Omega$.

Now assuming that $u\in C^2\left(\overline{\Omega}\right)$ is a solution of (1.69), let us show that $u$ is a solution of (1.67), that is, for any $v\in C^2\left(\overline{\Omega}\right)$ such that $v=\varphi$ on $\partial\Omega$,

$$\int_\Omega |\nabla v|^2\, dx \geq \int_\Omega |\nabla u|^2\, dx.$$

Set $w=v-u$ and write again

$$\int_\Omega |\nabla v|^2\, dx = \int_\Omega |\nabla u+\nabla w|^2\, dx = \int_\Omega |\nabla u|^2\, dx + 2\int_\Omega \nabla u\cdot\nabla w\, dx + \int_\Omega |\nabla w|^2\, dx.$$

Applying again the Green formula (1.71) and using that $\Delta u=0$ in $\Omega$ and $w=u-v=0$ on $\partial\Omega$, we obtain

$$\int_\Omega \nabla u\cdot\nabla w\, dx = 0.$$

It follows that

$$\int_\Omega |\nabla v|^2 \, dx = \int_\Omega |\nabla u|^2 \, dx + \int_\Omega |\nabla w|^2 \, dx \geq \int_\Omega |\nabla u|^2 \, dx,$$

which finishes the proof. ■

In the first part of this argument we used that a solution $u$ of the variational problem is of the class $C^2$ in order just to be able to write $\Delta u$. If we only know that $u \in C^1$ (and this is the minimal natural requirement for the problem (1.67)), then we cannot immediately apply $\Delta$ to $u$. In the both parts of the proof we used that $u, v \in C^2\left(\overline{\Omega}\right)$ in order to be able to use the Green formula. 20.11.15

In order to prove Theorem 1.23 under optimal requirements for $u$, as stated above, we need to do some preparations.

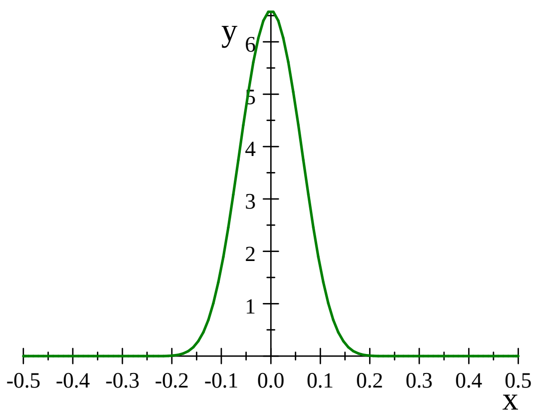
**Definition.** A function $\psi$ on $\mathbb{R}^n$ is called a *mollifier*, if $\psi$ is non-negative, $\psi \in C_0^\infty(B_1)$, and

$$\int_{\mathbb{R}^n} \psi(x) \, dx = 1.$$

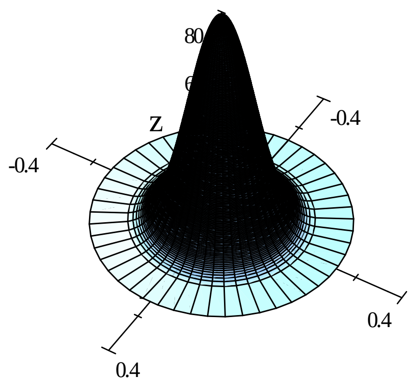
For example, the following function is a mollifier

$$\psi(x) = \begin{cases} c \exp\left(-\frac{1}{\left(\frac{1}{4} - |x|^2\right)^2}\right), & |x| < 1/2 \\ 0, & |x| \geq 1/2 \end{cases}$$

for an appropriate value of the constant $c$. Here are the graphs of this function in $\mathbb{R}^1$ and $\mathbb{R}^2$:




A mollifier in $\mathbb{R}^1$



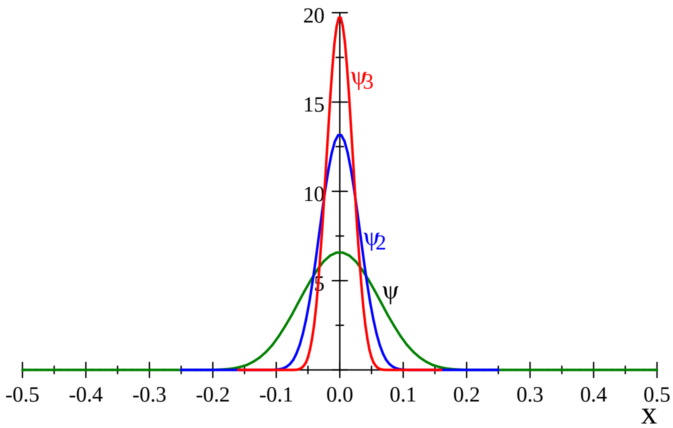

A mollifier in $\mathbb{R}^2$

Each mollifier gives rise to a sequence $\{\psi_k\}_{k=1}^\infty$ of mollifiers as follows:

$$\psi_k(x) = k^n \psi(kx). \tag{1.72}$$

Indeed, observe that $\psi_k \in C_0^\infty\left(B_{1/k}\right)$ and

$$\int_{\mathbb{R}^n} \psi_k(x)\,dx = \int_{\mathbb{R}^n} k^n \psi(kx)\,dx = \int_{\mathbb{R}^n} \psi(y)\,dy = 1. \tag{1.73}$$




Functions $\psi = \psi_1, \psi_2, \psi_3$ in $\mathbb{R}^1$

In the next lemma we develop a techniques of approximating continuous functions by smooth ones.

**Lemma 1.24** *Let $u$ be a locally integrable function in $\mathbb{R}^n$. For any $k \in \mathbb{N}$ set*

$$u_k = u * \psi_k = \int_{\mathbb{R}^n} u(x-y)\,\psi_k(y)\,dy. \tag{1.74}$$

*Then each $u_k$ is a $C^\infty$ function in $\mathbb{R}^n$. Moreover, if $u \in C(\Omega)$ then $u_k \to u$ locally uniformly in $\Omega$.*

**Proof.** Indeed, we have by change $z = x - y$

$$u_k(x) = \int_{\mathbb{R}^n} u(z)\,\psi_k(x-z)\,dz,$$

and the first claim follows from the fact that $\psi_k(x-z)$ is $C^\infty$-smooth in $x$ (cf. the proof of Lemma 1.12, Step 2).

Let us prove the second claim. For any $x \in \Omega$, we have by (1.73)

$$\begin{aligned} u(x) - u_k(x) &= \int_{\mathbb{R}^n} u(x)\,\psi_k(y)\,dy - \int_{\mathbb{R}^n} u(x-y)\,\psi_k(y)\,dy \\ &= \int_{B_{1/k}} \left(u(x) - u(x-y)\right)\psi_k(y)\,dy \end{aligned}$$

whence

$$|u(x) - u_k(x)| \le \sup_{y \in B_{1/k}} |u(x) - u(x-y)|.$$

Since $u$ is locally uniformly continuous in $\Omega$, we obtain that

$$\sup_{y \in B_{1/k}} |u(x) - u(x-y)| \to 0 \quad \text{as } k \to \infty$$

locally uniformly in $\Omega$, which implies that $u_k \to u$ locally uniformly in $\Omega$. ■

**Definition.** A function $u \in C(\Omega)$ is called *weakly harmonic* in $\Omega$ if, for any $w \in C_0^\infty(\Omega)$,

$$\int_\Omega u\Delta w\,dx = 0. \tag{1.75}$$

Observe that if $u$ is harmonic then $u$ is weakly harmonic because by the Green formula

$$\int_\Omega u\Delta w\,dx = \int_\Omega \Delta u\,w\,dx = 0.$$

Conversely, if $u$ is weakly harmonic and if $u \in C^2(\Omega)$ then $u$ is harmonic, because (1.75) implies then by the Green formula

$$\int_\Omega \Delta u\,w\,dx = \int_\Omega u\Delta w\,dx = 0,$$

and since $w \in C_0^\infty(\Omega)$ is arbitrary, we obtain $\Delta u = 0$ in $\Omega$. It turns out that the latter claim can be strengthened as follows.

**Lemma 1.25** (Weyl's lemma) *Let $\Omega$ be any open subset of $\mathbb{R}^n$. If $u \in C(\Omega)$ is weakly harmonic in $\Omega$ then $u$ is harmonic.*

**Proof.** We reducing $\Omega$, we can assume without loss of generality that $u$ is bounded. Extending $u$ to $\mathbb{R}^n$ by setting $u = 0$ in $\Omega^c$. Consider again the sequence $\{u_k\}$ given by (1.74) and show that if $u$ is weakly harmonic in $\Omega$ then also $u_k$ is weakly harmonic in $\Omega$. Indeed, for any $w \in C_0^\infty(\Omega)$ we have

$$\begin{aligned}
\int_\Omega u_k(x)\,\Delta w(x)\,dx &= \int_{\mathbb{R}^n}\left(\int_{B_{1/k}} u(x-y)\,\psi_k(y)\,dy\right)\Delta w(x)\,dx \\
&= \int_{B_{1/k}}\left(\int_{\mathbb{R}^n} u(x-y)\,\Delta w(x)\,dx\right)\psi_k(y)\,dy \\
&= \int_{B_{1/k}}\left(\int_{\mathbb{R}^n} u(z)\,\Delta w(z+y)\,dz\right)\psi_k(y)\,dy.
\end{aligned}$$

Since $y \in B_{1/k}$ and, hence, $|y| < 1/k$, the function $z \mapsto w(z+y)$ is supported in $\Omega$, provided $k$ is large enough, which implies by the weak harmonicity of $u$ in $\Omega$ that

$$\int_{\mathbb{R}^n} u(z)\,\Delta w(z+y)\,dz = 0.$$

It follows that

$$\int_\Omega u_k(x)\,\Delta w(x)\,dx = 0,$$

that is, $u_k$ is weakly harmonic in $\Omega$. Since $u_k \in C^\infty(\Omega)$, we obtain that $u_k$ is harmonic.

Finally, since $u_k \to u$ locally uniformly in $\Omega$, we obtain by Harnack's first theorem that $u$ is harmonic in $\Omega$. ■

The next lemma states two versions of the first Green formula.

**Lemma 1.26** *Let $\Omega$ be a bounded region.*
*(a) If $u \in C^2\left(\overline{\Omega}\right)$, $w \in C^1\left(\overline{\Omega}\right)$ then*

$$\int_\Omega w\Delta u\,dx = -\int_\Omega \nabla u \cdot \nabla w\,dx + \int_{\partial\Omega} w\partial_\nu u\,d\sigma. \tag{1.76}$$

*(b) If $u \in C^2(\Omega) \cap C^1\left(\overline{\Omega}\right)$, $w \in C^1(\Omega) \cap C\left(\overline{\Omega}\right)$ and $w = 0$ on $\partial\Omega$ then*

$$\int_\Omega w\Delta u\,dx = -\int_\Omega \nabla u \cdot \nabla w\,dx. \tag{1.77}$$

Recall for comparison that so far we required for the Green formula that $u, w \in C^2\left(\overline{\Omega}\right)$. Observe also that in the case $(b)$ the functions $w\Delta u$ and $\nabla u \cdot \nabla w$ are in $C(\Omega)$ but not necessarily in $C\left(\overline{\Omega}\right)$ so that the integrals in (1.77) are not necessarily well-defined or finite. The statement is that if one of the integrals is well-defined then so is the other, and their values are the same. In fact, one can prove that the formula (1.76) remains true also in the case $(b)$ without requirement $w = 0$ on $\partial\Omega$, but the argument is more technical than acceptable here.

**Proof.** $(a)$ If $u \in C^2\left(\overline{\Omega}\right)$ and $w \in C^1\left(\overline{\Omega}\right)$ then applying the divergence theorem with

$$\overrightarrow{F} = w\nabla u \in C^1\left(\overline{\Omega}\right),$$

we obtain

$$\int_\Omega \operatorname{div} \overrightarrow{F}\,dx = \int_{\partial\Omega} \overrightarrow{F} \cdot \nu\,d\sigma$$

that is

$$\int_\Omega (w\Delta u + \nabla u \cdot \nabla w)\,dx = \int_{\partial\Omega} w\partial_\nu u\,d\sigma,$$

which is equivalent to (1.76).

$(b)$ Assume now $u \in C^2(\Omega) \cap C^1\left(\overline{\Omega}\right)$ and $w \in C^1(\Omega) \cap C\left(\overline{\Omega}\right)$. Recall that by definition of a region, there exists a $C^1$ function $\Phi$ in a neighborhood of $\overline{\Omega}$ such that $\Phi < 0$ on $\Omega$, $\Phi = 0$ on $\partial\Omega$, $\Phi > 0$ outside $\overline{\Omega}$, and $\nabla\Phi \neq 0$ on $\partial\Omega$. For any $\varepsilon > 0$ consider the set

$$\Omega_\varepsilon = \{x : \Phi(x) < -\varepsilon\} = \{x : \Phi(x) + \varepsilon < 0\}.$$

Since $\nabla\Phi \neq 0$ also in a neighborhood of $\partial\Omega$, we see that for small enough $\varepsilon$ we have $\nabla\Phi \neq 0$ on $\partial\Omega_\varepsilon$, which implies that $\Omega_\varepsilon$ is also a region. Since $u \in C^2\left(\overline{\Omega_\varepsilon}\right)$ and $w \in C^1\left(\overline{\Omega_\varepsilon}\right)$, we obtain by (1.76)

$$\int_{\Omega_\varepsilon} w\Delta u\,dx = -\int_{\Omega_\varepsilon} \nabla u \cdot \nabla w\,dx + \int_{\partial\Omega_\varepsilon} w\partial_\nu u\,d\sigma. \tag{1.78}$$

Since $u \in C^1\left(\overline{\Omega}\right)$, we have

$$|\partial_\nu u| \le \sup_{\overline{\Omega}} |\nabla u| =: C < \infty.$$

If $\varepsilon \to 0$ then $\sup_{\partial\Omega_\varepsilon} |w| \to 0$ because $w \in C\left(\overline{\Omega}\right)$ and $w = 0$ on $\partial\Omega$. Hence,

$$\int_{\partial\Omega_\varepsilon} w\partial_\nu u\,d\sigma \to 0 \text{ as } \varepsilon \to 0.$$

Hence, letting $\varepsilon \to 0$ in (1.78), we obtain (1.77). More precisely, if one of the limits

$$\int_\Omega w\Delta u\,dx := \lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} w\Delta u\,dx$$

and

$$-\int_\Omega \nabla u\cdot\nabla w\,dx := -\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon}\nabla u\cdot\nabla w\,dx$$

exists then the other exists too, and their values are the same. ■

**Proof of Theorem 1.23.** Assume first that $u \in C^1\left(\overline{\Omega}\right)$ is a solution of (1.67) and prove that $u$ is a solution of (1.69). We need only to prove that $u$ is a harmonic function in $\Omega$. By Lemma 1.25 it suffices to prove that $u$ is weakly harmonic in $\Omega$. Fix a function $w \in C_0^\infty(\Omega)$ and $t \in \mathbb{R}$ and consider the function $v = u + tw$. Since $v = u = \varphi$ on $\partial\Omega$, we conclude that

$$\int_\Omega |\nabla v|^2\,dx \geq \int_\Omega |\nabla u|^2\,dx.$$

Using the the same argument as in the previous version of the proof, we conclude that

$$\int_\Omega \nabla u\cdot\nabla w\,dx = 0.$$

By the Green formula (1.76) (with swapped $u$ and $w$) we have

$$\int_\Omega u\,\Delta w\,dx = -\int_\Omega \nabla u\cdot\nabla w\,dx + \int_{\partial\Omega} u\partial_\nu w\,d\sigma = 0.$$

Hence, we obtain that that $u$ is weakly harmonic, which finishes this part of the proof. 25.11.15

Let $u$ be solution of (1.69) and let us show that $u$ solves also (1.67), that is, for any $v \in C^1\left(\overline{\Omega}\right)$ such that $v = \varphi$ on $\partial\Omega$,

$$\int_\Omega |\nabla v|^2\,dx \geq \int_\Omega |\nabla u|^2\,dx.$$

Set $w = v - u$ and write

$$\int_\Omega |\nabla v|^2\,dx = \int_\Omega |\nabla u + \nabla w|^2\,dx = \int_\Omega |\nabla u|^2\,dx + 2\int_\Omega \nabla u\cdot\nabla w\,dx + \int_\Omega |\nabla w|^2\,dx.$$

Since $u \in C^2(\Omega)\cap C^1\left(\overline{\Omega}\right)$, $w \in C^1\left(\overline{\Omega}\right)$, $w = u - v = 0$ on $\partial\Omega$, and $\Delta u = 0$ in $\Omega$, we obtain by (1.77) that

$$\int_\Omega \nabla u\cdot\nabla w\,dx = -\int_\Omega w\Delta u\,dx = 0.$$

It follows that

$$\int_\Omega |\nabla v|^2\,dx = \int_\Omega |\nabla u|^2\,dx + \int_\Omega |\nabla w|^2\,dx \geq \int_\Omega |\nabla u|^2\,dx,$$

which finishes the proof. ■

## 1.11 *Distributions

Denote by $\mathcal{D}$ the linear space $C_0^\infty(\mathbb{R}^n)$ with certain topology that we do not describe here. Elements of $\mathcal{D}$ are called test functions. A *distribution* is any linear continuous functional on $\mathcal{D}$. The set of all distributions is denoted by $\mathcal{D}'$. Clearly, this is a linear space (that is a dual space to $\mathcal{D}$). For any $f \in \mathcal{D}'$ and $\varphi \in \mathcal{D}$ the value $f(\varphi)$ is also denoted by $\langle f, \varphi \rangle$. One says that a sequence $\{f_k\}$ of distributions converges to a distribution $f$ if for any test function $\varphi$

$$\langle f_k, \varphi \rangle \to \langle f, \varphi \rangle \text{ as } k \to \infty.$$

Any locally integrable function $f$ in $\mathbb{R}^n$ determines a distribution, also denoted by $f$, using the rule

$$\langle f, \varphi \rangle = \int_{\mathbb{R}^n} f\varphi dx.$$

On the other hand, there are distributions that are not determined by functions. For example, denote by $\delta$ the distribution that is defined by

$$\langle \delta, \varphi \rangle = \varphi(0).$$

The distribution $\delta$ is called the Dirac-function (although it is not a function).

Let $\psi$ be a mollifier in $\mathbb{R}^n$, and $\psi_k$ be defined by (1.72), that is, $\psi_k(x) = k^n \psi(kx)$. By Lemma 1.24 we have the following: for any test function $\varphi$

$$\psi_k * \varphi(x) \to \varphi(x) \quad \text{as } k \to \infty.$$

Applying this to function $\varphi(-x)$ instead of $\varphi$, we obtain

$$\int_{\mathbb{R}^n} \psi_k(x+y)\varphi(y)\,dy \to \varphi(-x) \quad \text{as } k \to \infty.$$

In particular, for $x = 0$ we have

$$\langle \psi_k, \varphi \rangle \to \varphi(0) = \langle \delta, \varphi \rangle.$$

Hence, we can say that $\psi_k \to \delta$ the sense of distributions. A sequence that converges to $\delta$ is called *approximation of identity.*

One of huge advantages of the notion of distribution is that all partial derivatives $D^\alpha$ of all orders are well-defined on any distribution. Namely, for any $f \in \mathcal{D}'$ and for any multiindex $\alpha = (\alpha_1, ..., \alpha_n)$ define $D^\alpha f$ as distribution by the following identity:

$$\langle D^\alpha f, \varphi \rangle = (-1)^{|\alpha|} \langle f, D^\alpha \varphi \rangle \quad \forall \varphi \in \mathcal{D}. \tag{1.79}$$

This definition is compatible with the classical definition for functions in the following sense. If $f \in C^k(\mathbb{R}^n)$ then $D^\alpha f$ is defined as function for all $|\alpha| \le k$. By integration by parts formula, the following identity is true for any $\varphi \in \mathcal{D}$:

$$\int_{\mathbb{R}^n} (D^\alpha f)\,\varphi\,dx = (-1)^{|\alpha|} \int_{\mathbb{R}^n} f D^\alpha \varphi\,dx.$$

Hence, if we consider here $f$ and $D^\alpha f$ as distributions, then we obtain (1.79).

Using (1.79) we can compute the derivatives of the $\delta$-function as follows:

$$\langle D^\alpha \delta, \varphi \rangle = (-1)^{|\alpha|} D^\alpha \varphi(0).$$

It follows from (1.79) that, for the Laplace operator $\Delta$,

$$\langle \Delta f, \varphi \rangle = \langle f, \Delta \varphi \rangle. \tag{1.80}$$

A distribution $f$ is called harmonic if it satisfies the Laplace equation $\Delta f = 0$. By (1.80), $f \in \mathcal{D}'$ is harmonic if and only if

$$\langle f, \Delta \varphi \rangle = 0 \quad \forall \varphi \in \mathcal{D}. \tag{1.81}$$

Recall that a continuous function $f$ is called weakly harmonic if for all $\varphi \in \mathcal{D}$

$$\int_{\mathbb{R}^n} f \Delta \varphi \, dx = 0,$$

which can be equivalently written as (1.81). Hence, a continuous function $f$ is weakly harmonic if and only if $f$ is harmonic as a distribution. We have proved in Lemma 1.25 that any weakly harmonic function is harmonic. This lemma can be extended as follows: any harmonic distribution is in fact a harmonic function.

## 1.12 *Euler-Lagrange equation

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$. Consider a more general variational problem

$$\begin{cases} \int_\Omega \mathcal{L}(x, u, \nabla u) \, dx \mapsto \min \\ u = \varphi \quad \text{on } \partial\Omega \end{cases} \tag{1.82}$$

where $\mathcal{L}(x, p, q_1, ..., q_n)$ is a given function, called *Lagrangian*, and $u$ is an unknown function. If $u \in C^2(\Omega)$ is a solution of (1.82) then we can again compare $u$ with $v = u + tw$, where $w \in C_0^\infty(\Omega)$ and $t \in \mathbb{R}$. The function $tw$ is called a *variation* of $u$.

By the way, the branch of mathematics that studies variational problems is called variational calculus. The main idea here is the same as in the proof of Fermat's theorem in classical Analysis. In order to obtain points of minimum of a real valued function $F(z)$ of a variable $z \in \mathbb{R}^n$, let us compare $F(z)$ at the minimum point $z$ with $F(z + tw)$, where $w \in \mathbb{R}^n$ and $t \in \mathbb{R}$ (that is, $tw$ is an increment of the argument $z$). As we know from Analysis, if the function $F$ is differentiable, then the condition

$$F(z + tw) \geq F(z)$$

leads for $t \to 0$ to

$$F(z) + tw \cdot F'(z) + o(t) \geq F(z).$$

Since the latter has to be true both for $t > 0$ and $t < 0$, we obtain that $w \cdot F'(z) = 0$, and since this has to be true for all $w$, we obtain

$$F'(z) = 0.$$

This equation is a necessary condition for $z$ to be a point of minimum and it can be used to determine $z$ or at least candidates for $z$.

Returning to the variational problem and assuming that $\mathcal{L}$ is continuously differentiable in $p, q$ and that $t$ is small, we obtain as $t \to 0$

$$\mathcal{L}(x, u + tw, \nabla u + t\nabla w) = \mathcal{L}(x, u, \nabla u) + tw\partial_p \mathcal{L}(x, u, \nabla u) + t\nabla w \cdot \partial_q \mathcal{L}(x, u, \nabla u) + o(t).$$

The condition

$$\int_\Omega \mathcal{L}(x, u + tw, \nabla u + t\nabla w)\, dx \geq \int_\Omega \mathcal{L}(x, u, \nabla u)\, dx$$

implies

$$\int_\Omega t\left[w\partial_p \mathcal{L}(x, u, \nabla u) + \nabla w \cdot \partial_q \mathcal{L}(x, u, \nabla u)\right] dx \geq o(t),$$

and the fact, that this has to be true both for $t > 0$ and $t < 0$, implies that

$$\int_\Omega \left[w\partial_p \mathcal{L}(x, u, \nabla u) + \nabla w \cdot \partial_q \mathcal{L}(x, u, \nabla u)\right] dx = 0. \tag{1.83}$$

Consider a vector field

$$v = \partial_q \mathcal{L}(x, u, \nabla u).$$

Since

$$\operatorname{div}(wv) = \nabla w \cdot v + w \operatorname{div} v$$

(see Exercises) and by the divergence theorem

$$\int_\Omega \operatorname{div}(wv)\, dx = \int_{\partial\Omega} wv\, d\sigma = 0,$$

we obtain that

$$\int_\Omega \nabla w \cdot v\, dx = -\int_\Omega w \operatorname{div} v\, dx.$$

Substituting this into (1.83), we obtain

$$\int_\Omega w\left[\partial_p \mathcal{L}(x, u, \nabla u) - \operatorname{div} \partial_q \mathcal{L}(x, u, \nabla u)\right] dx = 0,$$

where div is taken with respect to $x$. Since $w$ is arbitrary, we obtain the $u$ satisfies the following PDE in $\Omega$:

$$\partial_p \mathcal{L}(x, u, \nabla u) = \operatorname{div} \partial_q \mathcal{L}(x, u, \nabla u),$$

or more explicitly

$$\partial_p \mathcal{L}(x, u, \nabla u) = \sum_{i=1}^n \partial_{x_i} \partial_{q_i} \mathcal{L}(x, u, \nabla u). \tag{1.84}$$

This PDE is called the *Euler-Lagrange equation* of the problem (1.82).

For example, the problem (1.67) corresponds to the Lagrangian

$$\mathcal{L}(x, p, q) = q_1^2 + \ldots + q_n^2.$$

Then $\partial_p \mathcal{L} = 0$, $\partial_{q_i} \mathcal{L} = 2q_i$, and (1.84) becomes

$$0 = \sum \partial_{x_i} (2\partial_{x_i} u),$$

which is equivalent to $\Delta u = 0$.

The variational problem (1.68) has the Lagrangian

$$\mathcal{L}(x, p, q) = \sqrt{1 + q_1^2 + ... + q_n^2}.$$

Since

$$\partial_{q_i} \mathcal{L} = \frac{q_i}{\sqrt{1 + q_1^2 + ... + q_n^2}},$$

we obtain the following Euler-Lagrange equation

$$\sum_{i=1}^{n} \partial_{x_i} \left( \frac{\partial_{x_i} u}{\sqrt{1 + |\nabla u|^2}} \right) = 0$$

that is called the *minimal surface equation.*

## 1.13 *Dirichlet problem in arbitrary domains (overview)

We discuss various methods of proof of the solvability of the Dirichlet problem in an arbitrary bounded open set $\Omega \subset \mathbb{R}^n$. In the case of a ball we have solved the Dirichlet problem by constructing the Green function. However, this method does not work for general domains because construction of the Green function in general domains requires a solution of a certain Dirichlet problem. We state below only the ideas of the methods, without rigorous statements.

**Perron's method.**

Let $u$ be a solution to the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases} \tag{1.85}$$

Observe that if $v$ is a superharmonic function in $\Omega$ such that $v \geq \varphi$ on $\partial\Omega$, then by the minimum principle we obtain $v \geq u$. It follows that

$$u(x) = \inf \{ v(x) : v \text{ is superharmonic in } \Omega \text{ and } v \geq \varphi \text{ on } \partial\Omega \}. \tag{1.86}$$

This formula can be used to define a function $u(x)$. Indeed, there are always superharmonic functions $v$ with $v \geq \varphi$ on $\partial\Omega$, for example, large enough constants, so that the right hand side of (1.86) always makes sense.

The main idea of Perron's method is a non-trivial fact that the function $u$ defined by (1.86) is always harmonic in $\Omega$. The next step is to show that $u$ satisfies the boundary condition, which can be done using certain assumptions about the boundary $\partial\Omega$, provided $\varphi \in C(\partial\Omega)$. For example, this method works if $\partial\Omega$ satisfies a so-called the *cone condition*, that is, if any point $x \in \partial\Omega$ can be touched from outside $\Omega$ by a solid cone. In particular, this is the case when $\Omega$ is a region.

**Brownian motion and Kakutani's formula.**

Let $\{X_t\}$ be Brownian motion in $\mathbb{R}^n$ (see Section 2.7 for more details). Then solution of (1.85) can be determined by *Kakutani's formula*:

$$u(x) = \mathbb{E}_x(\varphi(X_\tau))$$

where $x \in \Omega$ and $\tau$ is the first time when $X_t$ hits $\partial\Omega$ starting at $x$ at time 0. For example, if $\Omega$ is a ball centered at $x$, then $X_\tau$ is uniformly distributed on $\partial\Omega$ and we obtain the mean value property: $u(x)$ is the arithmetic mean of $\varphi$. In general, $u(x)$ is a *weighted mean* of $\varphi$ where the weight is given by the *exit measure* of Brownian motion, that is, by the distribution of $X_\tau$ on $\partial\Omega$. Similarly to the Perron method, one proves that $u$ is always a harmonic function in $\Omega$, and that $u = \varphi$ on $\partial\Omega$ provided $\partial\Omega$ satisfies the cone condition.

**Fredholm's method and integral equations.**

Assume that $\Omega$ is a region and let us look for the solution of (1.85) in the form

$$u(x) = -\int_{\partial\Omega} \partial_\nu E(x, y)\, v(y)\, d\sigma(y), \tag{1.87}$$

where $v$ is a new unknown function on $\partial\Omega$. This formula is motivated by the Poisson kernel of the ball that is equal to $\partial_\nu G(x, y)$ where $G$ is the Green function of the ball. Since we do not know the Green function of $\Omega$, we use in (1.87) the fundamental solution instead, but replace the boundary function $\varphi$ by a new unknown function.

It is easy to show that $u$ is a harmonic function in $\Omega$, assuming that $v$ is a reasonably good function. The main problem is to find $v$ so that $u$ satisfies the boundary condition $u = \varphi$ on $\partial\Omega$. The key observation is the following fact: for any $x \in \partial\Omega$

$$\lim_{z\in\Omega, z\to x} u(z) = \frac{1}{2}v(x) + u(x)$$

(consequently, $u$ is in general discontinuous at $\partial\Omega$). Then the boundary condition

$$\lim_{z\in\Omega, z\to x} u(z) = \varphi(x)$$

gives the *integral equation* for $v$

$$\frac{1}{2}v(x) - \int_{\partial\Omega} \partial_\nu E(x, y)\, v(y)\, d\sigma(y) = \varphi(x)$$

at $\partial\Omega$. The Fredholm theory develops methods for solving such integral equations. In particular, the celebrated *Fredholm alternative* asserts that the existence of solution of the integral equation for any right hand side $\varphi$ is equivalent to the uniqueness of solution of a certain dual integral equation. This is similar to the proof of existence of solution of the discrete Dirichlet problem when we first proved the uniqueness. However, the proof of the Fredholm alternative is much more complicated as it requires tools of *functional analysis*, that is, the theory of infinite dimensional linear spaces.

**The Dirichlet method and weak topology.**

We have learned in Theorem 1.23 that instead of solving (1.85) it suffices to solve the variational problem

$$\begin{cases} \int_\Omega |\nabla u|^2\, dx \mapsto \min \\ u = \varphi \text{ on } \partial\Omega. \end{cases} \tag{1.88}$$

If $u \in C^1(\Omega)$ and $w \in C_0^\infty(\Omega)$ then, applying the divergence theorem to the vector field $\nabla(wu)$, we obtain the identity

$$\int_\Omega w\nabla u\, dx = -\int_\Omega u\nabla w\, dx.$$

This identity is used to define the notion of a *weak gradient*. Namely, a vector field $F$ in $\Omega$ is called a *weak gradient* of $u$ in $\Omega$ if, for any $w \in C_0^\infty(\Omega)$,

$$\int_\Omega wF\, dx = -\int_\Omega u\nabla w\, dx.$$

The weak gradient (if it exists) will also be denoted by $\nabla u$. The advantage of the notion of weak gradient is that it can be defined for functions that are not necessarily pointwise differentiable.

Recall that the Lebesgue space $L^2(\Omega)$ consists of measurable functions $u$ in $\Omega$ that are square integrable, that is,

$$\int_\Omega u^2 dx < \infty.$$

It is known that $L^2(\Omega)$ is a Hilbert space with the inner product

$$(u, v)_{L^2} = \int_\Omega uv\, dx.$$

Define the *Sobolev space* $W^{1,2}(\Omega)$ as the subspace of $L^2(\Omega)$ that consists of functions $u$ possessing the weak gradient $\nabla u$ such that $|\nabla u| \in L^2(\Omega)$. The Sobolev space is a Hilbert space with respect to the inner product

$$(u, v)_{W^{1,2}} = \int_\Omega (uv + \nabla u \cdot \nabla v)\, dx. \tag{1.89}$$

Hence, the norm in $W^{1,2}(\Omega)$ is given by

$$\|u\|_{W^{1,2}}^2 = \int_\Omega \left(u^2 + |\nabla u|^2\right) dx.$$

We write shortly $W^{1,2} = W^{1,2}(\Omega)$. Consider also the subspace $W_0^{1,2}$ of $W^{1,2}$ that is the closure of $C_0^\infty(\Omega)$ in $W^{1,2}$. It is possible to prove that if $\Omega$ is bounded then $W_0^{1,2}$ admits also an equivalent norm

$$\|u\|_{W_0^{1,2}}^2 = \int_\Omega |\nabla u|^2\, dx,$$

which corresponds to the following inner product in $W_0^{1,2}$:

$$(u, v)_{W_0^{1,2}} = \int_\Omega \nabla u \cdot \nabla v\, dx.$$

Assume that the boundary function $\varphi$ extends to a function in $\Omega$ and that the extended function belongs to $W^{1,2}$. Then we understand the boundary condition of (1.88) in the generalized sense:

$$u - \varphi \in W_0^{1,2}. \tag{1.90}$$

Indeed, we consider the functions in $W_0^{1,2}$ as vanishing on $\partial\Omega$ in some generalized sense as they are obtained as limits of functions from $C_0^\infty(\Omega)$ vanishing on $\partial\Omega$ in the strong sense. Setting $v = u - \varphi$, we see that the variational problem (1.88) amounts to the following: find a function $v \in W_0^{1,2}$ where the functional

$$\Phi(v) := \int_\Omega |\nabla(v+\varphi)|^2 \, dx$$

attains its minimal value. It is easy to show that if $\|v\|_{W_0^{1,2}} \to \infty$ then $\Phi(v) \to \infty$ so that we can restrict the problem of finding the minimum of $\Phi$ to a ball

$$B_R = \left\{ v \in W_0^{1,2} : \|v\|_{W_0^{1,2}} \leq R \right\}$$

in $W_0^{1,2}$ of large enough radius $R$. It is also easy to see that $\Phi$ is a continuous functional in $W_0^{1,2}$. If this problem were in a finite dimensional Euclidean space then we could have concluded that $\Phi$ attains its minimum in the ball by the extreme value theorem, because the ball is compact. However, in the infinite dimensional space $W_0^{1,2}$ balls are not compact!

To overcome this difficulty, one introduces a so-called *weak* topology in $W_0^{1,2}$. In contrast to the norm topology, the ball $B_R$ happens to be compact in the weak topology, and function $\Phi$ is continuous in the weak topology (both statements are non-trivial). Hence, one obtains the existence of the minimum point of $\Phi$.

The function $u$ that one obtains in this way is an element of $W^{1,2}$. The one uses additional methods to show that this function is smooth enough in $\Omega$ and continuous up to $\partial\Omega$, in particular, that it solves (1.85). These methods belong to the *regularity theory.*

**The Riesz representation theorem and geometry of Hilbert spaces.**

Consider now the Dirichlet problem

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We will understand this problem also in a generalized sense as in the previous method. The boundary condition we understand in the sense

$$u \in W_0^{1,2}.$$

The equation $\Delta u = f$ is equivalent to the integral identity

$$\int_\Omega w \Delta u \, dx = \int_\Omega w f \, dx \quad \text{for any } w \in C_0^\infty(\Omega),$$

which is equivalent to

$$\int_\Omega \nabla u \cdot \nabla w \, dx = -\int_\Omega w f \, dx.$$

Since $u \in W_0^{1,2}$ and the class of test functions $w$ can also be extended from $C_0^\infty(\Omega)$ to its closure $W_0^{1,2}$, we restate the latter identity in the form

$$(u, w)_{W_0^{1,2}} = \Psi(w) \quad \text{for any } w \in W_0^{1,2}, \tag{1.91}$$

where

$$\Psi(w) := -\int_\Omega wf\, dx.$$

Clearly, $\Psi$ is a linear functional on $W_0^{1,2}$. One can show that it is continuous. Then one can apply the Riesz representation theorem: any continuous linear functional $\Psi$ on a Hilbert space has the form $\Psi(w) = (w, u)$ for some element $u$ of the Hilbert space. Hence, this element $u$ is our solution.

The proof of the Riesz representation theorem is based on the following geometric observation. The set null set of $\Psi$, that is, the set

$$N = \{w : \Psi(w) = 0\}$$

is a closed linear subspace of the given Hilbert space. The equation $\Psi(w) = (w, u)$ implies that $u$ must be orthogonal to $N$. In the theory of Hilbert spaces one proves the existence of a non-zero vector that is orthogonal to $N$. Then one finds $u$ as a multiple of this vector.

Finally one uses the regularity theory to show that $u$ is a smooth enough function.

# Chapter 2

# Heat equation

Our main subject here will be the heat equation

$$\partial_t u = \Delta u,$$

where $u = u(x,t)$, $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$. Here $n \geq 1$ is any natural number. In fact, the domain of the heat equation is $\mathbb{R}^{n+1}$ or a subset of $\mathbb{R}^{n+1}$.

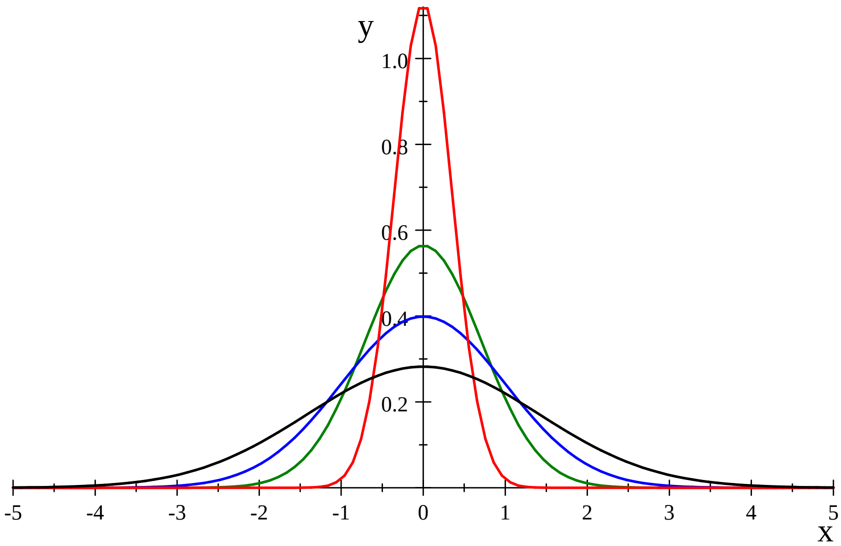
We have seen that in the study of the Laplace equation an important role was played by the fundamental solution. The heat equation possesses a similarly important solution.

## 2.1 Heat kernel

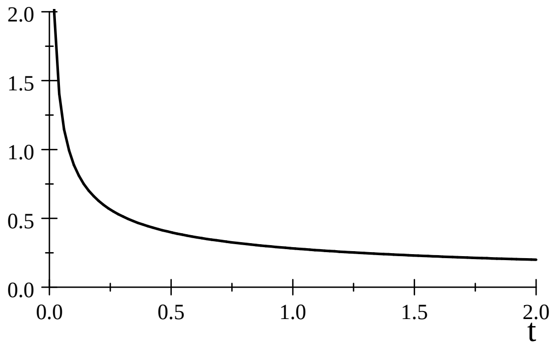
**Definition.** The following function

$$p_t(x) = p(t,x) := \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x|^2}{4t}\right), \tag{2.1}$$

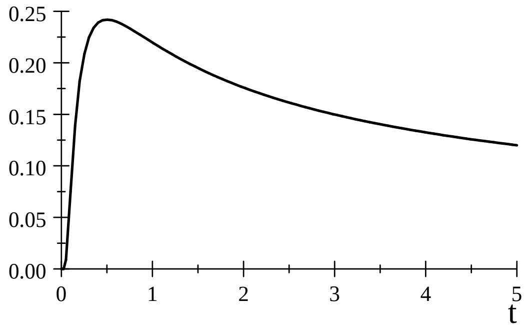
where $t > 0$ and $x \in \mathbb{R}^n$, is called the fundamental solution of the heat equation or the *heat kernel*. It is also called the *Gauss-Weierstrass* function.



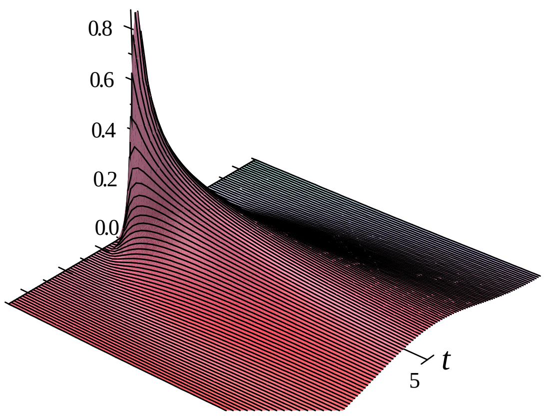
The graphs of the function $x \mapsto p_t(x)$ in $\mathbb{R}$ for $t = 1$, $t = \frac{1}{2}$, $t = \frac{1}{4}$ and $t = \frac{1}{16}$




The graph of the function
$t \mapsto p_t(0) = \frac{1}{(4\pi t)^{1/2}}$




The graph of the function
$t \mapsto p_t(1) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{1}{4t}\right)$




The graph of the function $(x, t) \mapsto p_t(x)$

The main properties of the heat kernel are stated in the following lemma.

**Lemma 2.1** *The function $p_t(x)$ is $C^\infty$ smooth in $\mathbb{R}^{n+1}_+ := \mathbb{R}^n \times (0, +\infty)$, positive, satisfies the heat equation*

$$\partial_t p_t = \Delta p_t, \tag{2.2}$$

*the identity*

$$\int_{\mathbb{R}^n} p_t(x) dx \equiv 1, \tag{2.3}$$

*and, for any $r > 0$,*

$$\int_{B_r^c} p_t(x) dx \to 0 \ \text{ as } t \to 0. \tag{2.4}$$

**Proof.** The smoothness and positivity of $p_t(x)$ are obvious. It is easier to verify the equation (2.2) using the function

$$u(x,t) := \ln p_t(x) = -\frac{n}{2} \ln t - \frac{|x|^2}{4t} + \ln \frac{1}{(4\pi)^{n/2}}.$$

Differentiating the identity $p_t = e^u$, we obtain

$$\partial_t p_t = e^u \partial_t u \quad \text{and} \quad \partial_{x_k x_k} p_t = \left(\partial_{x_k x_k} u + (\partial_{x_k} u)^2\right) e^u.$$

which implies

$$\partial_t p_t - \Delta p_t = e^u \left(\partial_t u - \Delta u - |\nabla u|^2\right).$$

Hence, the heat equation (2.2) is equivalent to

$$\partial_t u = \Delta u + |\nabla u|^2 . \tag{2.5}$$

Computing the derivatives of $u$,

$$\partial_t u = -\frac{n}{2t} + \frac{|x|^2}{4t^2}$$

and

$$\Delta u = -\frac{n}{2t}, \qquad \nabla u = -\frac{1}{2t}(x_1, ..., x_n), \qquad |\nabla u|^2 = \frac{|x|^2}{4t^2},$$

we obtain (2.5).

To prove (2.3), let us use the identity

$$\int_{-\infty}^{\infty} e^{-s^2} ds = \sqrt{\pi} \tag{2.6}$$

that implies by a change in the integral that

$$\int_{-\infty}^{\infty} e^{-s^2/4t} ds = \sqrt{4\pi t}.$$

Reducing the integration in $\mathbb{R}^n$ to repeated integrals, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^n} p_t(x)dx &= \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} \exp\left(-\frac{x_1^2 + \cdots + x_n^2}{4t}\right) dx_1 \cdots dx_n \\
&= \frac{1}{(4\pi t)^{n/2}} \prod_{k=1}^{n} \int_{\mathbb{R}} \exp\left(-\frac{x_k^2}{4t}\right) dx_k \\
&= \frac{1}{(4\pi t)^{n/2}} \left(\sqrt{4\pi t}\right)^n \\
&= 1.
\end{aligned}$$

Finally, to verify (2.4), let us make the change $y = t^{-1/2}x$ in the integral (2.4). Since $dy = t^{-n/2}dx$, the factor $t^{-n/2}$ cancels out and we obtain

$$\int_{\{x:|x|>r\}} p_t(x)dx = \frac{1}{(4\pi)^{n/2}} \int_{\{y:|y|>t^{-1/2}r\}} e^{-|y|^2/4} dy. \tag{2.7}$$

Since the integral in the right hand side is convergent and $t^{-1/2}r \to \infty$ as $t \to 0$, we obtain that the integral tends to 0 as $t \to 0$, which was to be proved. ■

## 2.2 Solution of the Cauchy problem

One of the most interesting and frequently used problems associated with the heat equation is the *Cauchy problem* (also known as the initial value problem): given a function $f(x)$ on $\mathbb{R}^n$, find $u(x,t)$ such that

$$\begin{cases} \partial_t u = \Delta u & \text{in } \mathbb{R}^{n+1}_+, \\ u|_{t=0} = f. \end{cases} \tag{2.8}$$

where $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times (0,\infty)$. The function $u$ is sought in the class $C^2(\mathbb{R}^{n+1}_+)$ so that the both derivatives $\partial_t u$ and $\Delta u$ make sense. The initial condition $u|_{t=0} = f$ can be understood in equivalent two ways:

$(i)$ $u \in C(\overline{\mathbb{R}}^{n+1}_+)$ where $\overline{\mathbb{R}}^{n+1}_+ = \mathbb{R}^n \times [0,+\infty)$ and $u(x,0) = f(x)$ for all $x \in \mathbb{R}^n$.

$(ii)$ We have
$$u(x,t) \to f(x) \text{ as } t \to 0+ \tag{2.9}$$
locally uniformly in $x \in \mathbb{R}^n$.

Indeed, if $(i)$ is satisfied then $u$ is locally uniformly continuous in $\overline{\mathbb{R}}^{n+1}_+$ whence $u(x,t) \to u(x,0) = f(x)$ as $t \to 0+$ locally uniformly in $x$. If $(ii)$ is satisfied then extending $u$ to $\overline{\mathbb{R}}^{n+1}_+$ by setting $u(x,0) = f(x)$, we obtain a continuous function in $\overline{\mathbb{R}}^{n+1}_+$.

**Theorem 2.2** *If $f$ is a bounded continuous function in $\mathbb{R}^n$ then the following function*

$$u(x,t) = (p_t * f)(x) = \int_{\mathbb{R}^n} p_t(x-y) f(y)\,dy \tag{2.10}$$

*is $C^\infty$ smooth in $\mathbb{R}^{n+1}_+$ and solves the Cauchy problem* (2.8)*. Moreover, the function $u$ is bounded and, for all $t > 0$ and $x \in \mathbb{R}^n$,*

$$\inf f \le u(x,t) \le \sup f. \tag{2.11}$$

**Remark.** Set
$$p(x) = p_1(x) = \frac{1}{(4\pi)^{n/2}} \exp\left(-\frac{|x|^2}{4}\right)$$
and observe that
$$p_t(x) = \frac{1}{\left(\sqrt{t}\right)^n} p\left(\frac{x}{\sqrt{t}}\right). \tag{2.12}$$
In particular, if we denote $k = \frac{1}{\sqrt{t}}$, then
$$p_t(x) = k^n p(kx),$$
which is the same rule that was used in Lemma 1.24 to create a sequence $\{\psi_k\}$ of mollifiers from a mollifier $\psi$. The function $p(x)$ is not a mollifier because its support

is unbounded, but it has many properties of mollifiers. In particular, the fact that the function $u(x,t)$ satisfies the initial condition (2.9) can be reformulated as follows:

$$p_t * f \to f \text{ as } t \to 0$$

that is similar to the statement of Lemma 1.24

$$\psi_k * f \to f \quad \text{as } k \to \infty.$$

**Proof.** Changing $z = x - y$ in (2.10) we can write

$$u(x,t) = \int_{\mathbb{R}^n} p_t(z) f(x-z)\, dz. \tag{2.13}$$

Since $f$ is bounded and $p_t$ is integrable, the integral here is convergent. The positivity of the heat kernel and (2.3) imply that

$$u(x) \le \sup f \int_{\mathbb{R}^n} p_t(z) dz = \sup f$$

and in the same way $u \ge \inf f$, which proves (2.11).

The function $u(x,t)$ from (2.13) is obviously continuous in $(x,t) \in \mathbb{R}^{n+1}_+$ because it is obtained by integrating of a continuous function $p_t(z) f(x-z)$.

Observe that for any partial derivative $D^\alpha$ in $(t,x)$ the following integral

$$\int_{\mathbb{R}^n} D^\alpha p_t(x-y) f(y)\, dy$$

converges, because $D^\alpha p_t(x-y)$ decays for large $|y|$ as $\exp\left(-\frac{|y|^2}{4t}\right)$ and $f(y)$ is bounded. Therefore, $D^\alpha u$ also exists and is given by

$$D^\alpha u(x,t) = \int_{\mathbb{R}^n} D^\alpha p_t(x-y) f(y)\, dy.$$

In particular, $u \in C^\infty\left(\mathbb{R}^{n+1}_+\right)$. It follows also that

$$(\partial_t - \Delta) u(x,t) = \int_{\mathbb{R}^n} (\partial_t - \Delta) p_t(x-y) f(y)\, dy = 0,$$

because $p_t$ solves the heat equation (cf. (2.2)).

Let us verify (2.9). The proof is very similar to that of Lemma 1.24. By (2.3), we have

$$f(x) = \int_{\mathbb{R}^n} p_t(z) f(x)\, dz,$$

which together with (2.13) yields

$$u(x,t) - f(x) = \int_{\mathbb{R}^n} p_t(z) \left(f(x-z) - f(x)\right) dz.$$

Since $f$ is continuous at $x$, for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|z| < \delta \Rightarrow |f(x-z) - f(x)| < \varepsilon.$$

Furthermore, since $f$ is locally uniformly continuous, $\delta$ can be chosen the same for all $x$ varying in a bounded set. Then we have

$$\begin{aligned}
|u(x,t) - f(x)| &\leq \left| \int_{B_\delta} p_t(z)(f(x-z) - f(x))dz \right| \\
&\quad + \left| \int_{B_\delta^c} p_t(z)(f(x-z) - f(x))dz \right| \\
&\leq \varepsilon \int_{\mathbb{R}^n} p_t(z)\,dz + 2\sup|f| \int_{B_\delta^c} p_t(z)dz.
\end{aligned}$$

By (2.3) we have $\int_{\mathbb{R}^n} p_t(z)\,dz = 1$ and by (2.4) $\int_{B_\delta^c} p_t(z)dz \to 0$ as $t \to 0$. In particular, if $t$ is sufficiently small then

$$2\sup|f| \int_{B_\delta^c} p_t(z)dz \leq \varepsilon,$$

which implies

$$|u(x,t) - f(x)| \leq 2\varepsilon.$$

Hence, (2.9) follows. The convergence is locally uniform in $x$ because $\delta$ can be chosen locally uniformly. ■

**Remark.** It is clear from the proof that if $f(x)$ is uniformly continuous in $\mathbb{R}^n$ then $u(t,x) \to f(x)$ uniformly in $x \in \mathbb{R}^n$.

27.11.15

## 2.3 Maximum principle and uniqueness in Cauchy problem

The Cauchy problem (2.8) is called bounded if the initial function $f$ is bounded and the solution $u$ must also be bounded. Theorem 2.2 claims the existence of solution of the bounded Cauchy problem for a continuous initial function $f$.

The uniqueness in the bounded Cauchy problem will follow from the *maximum principle*, which is of its own interest. Let $U \subset \mathbb{R}^n$ be a bounded open set. Fix some positive real $T$ and consider the *cylinder* $\Omega = U \times (0,T)$ as a subset in $\mathbb{R}^{n+1}$. The boundary $\partial\Omega$ is the union of three parts: the top $U \times \{T\}$, the bottom $U \times \{0\}$ and the lateral boundary $\partial U \times [0,T]$ (where $\partial U$ is the boundary of $U$ in $\mathbb{R}^n$). Define the *parabolic* boundary $\partial_p\Omega$ of the cylinder $\Omega$ as the union of its bottom and the lateral boundary, that is

$$\partial_p\Omega := (U \times \{0\}) \cup (\partial U \times [0,T])$$

(see Fig. 2.1). Note that $\partial_p\Omega$ is a closed subset of $\mathbb{R}^{n+1}$.

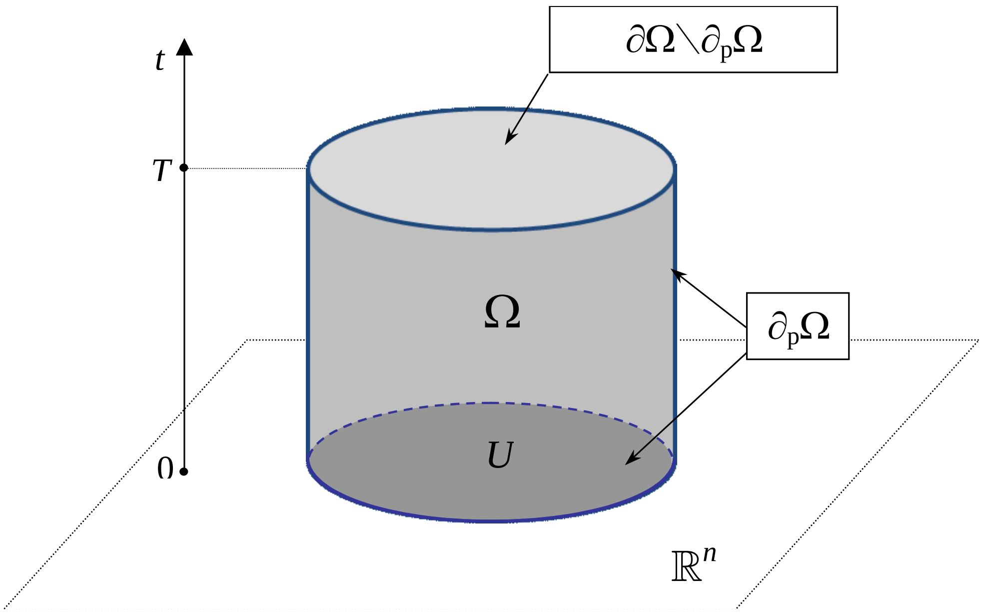



Figure 2.1: The parabolic boundary $\partial_p\Omega$

**Lemma 2.3** (Parabolic maximum principle)*Let $\Omega$ be a cylinder as above. If $u \in C^2(\Omega) \cap C(\overline{\Omega})$ and*

$$\partial_t u - \Delta u \leq 0 \text{ in } \Omega \tag{2.14}$$

*then*

$$\sup_{\Omega} u = \sup_{\partial_p \Omega} u. \tag{2.15}$$

*In particular, if $u \leq 0$ on $\partial_p\Omega$ then $u \leq 0$ in $\Omega$.*

By changing $u$ to $-u$, we obtain the *minimum principle*: if

$$\partial_t u - \Delta u \geq 0 \text{ in } \Omega \tag{2.16}$$

then

$$\inf_{\Omega} u = \inf_{\partial_p \Omega} u.$$

In particular, if $u$ solves the heat equation in $\Omega$ then the maximum and minimum of $u$ in $\overline{\Omega}$ are attained also in $\partial_p\Omega$.

**Remark.** Solutions to the heat equation are sometimes called *caloric* functions (analogously to harmonic functions). Any function that satisfies (2.14) is called a *subsolution* of the heat equation or *subcaloric* function, any function that satisfies (2.16) is called a *supersolution* of the heat equation or *supercaloric* function (analogously to sub- and superharmonic functions). Hence, subcaloric functions satisfy the maximum principle, and supercaloric functions satisfy the minimum principle.

**Proof.** By hypotheses, $u \in C^2(U \times (0,T))$. Let us assume first a bit more, that $u \in C^2(U \times (0,T])$, that is, $u$ is $C^2$ up to the top of the cylinder (in the end we will get rid of this assumption). The $u$ satisfies $\partial_t u - \Delta u \leq 0$ in $U \times (0,T]$. Note that we still assume $u \in C(\overline{\Omega})$.

Consider first a particular case when $u$ satisfies a *strict* inequality in $U \times (0, T]$:

$$\partial_t u - \Delta u < 0. \tag{2.17}$$

Let $(x_0, t_0)$ be a point of maximum of function $u$ in $\overline{\Omega}$. Let us show that $(x_0, t_0) \in \partial_p \Omega$, which will imply (2.15). If $(x_0, t_0) \notin \partial_p \Omega$ then $(x_0, t_0)$ lies either inside $\Omega$ or at the top of $\Omega$. In the both cases, $x_0 \in \Omega$ and $0 < t_0 \leq T$. Since the function $x \mapsto u(t_0, x)$ in $\overline{U}$ attains the maximum at $x = x_0$, we have

$$\partial_{x_j x_j} u(x_0, t_0) \leq 0 \text{ for all } j = 1, ..., n$$

whence $\Delta u(x_0, t_0) \leq 0$.

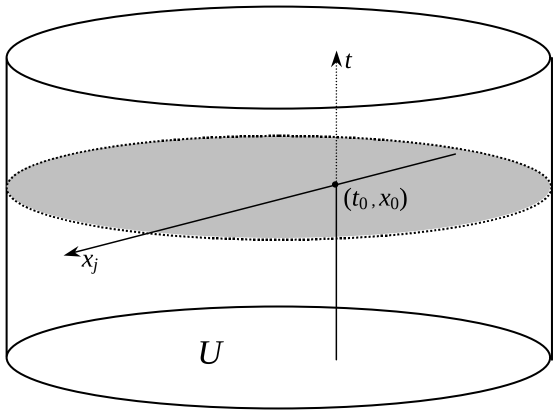



Figure 2.2: The restriction of $u(t, x)$ to the lines in the direction $x_j$ and in the direction of $t$ (downwards) attains the maximum at $(t_0, x_0)$.

On the other hand, the function $t \mapsto u(t, x_0)$ in $(0, t_0]$ attains its maximum at $t = t_0$ whence

$$\partial_t u(x_0, t_0) \geq 0$$

(if $t_0 < T$ then, in fact, $\partial_t u(x_0, t_0) = 0$). Hence, we conclude that

$$(\partial_t u - \Delta u)(x_0, t_0) \geq 0,$$

which contradicts (2.17).

Consider now the general case, when $u$ satisfies $\partial_t u - \Delta u \leq 0$ in $U \times (0, T]$. Set $u_\varepsilon = u - \varepsilon t$ where $\varepsilon$ is a positive parameter. Clearly, we have

$$\partial_t u_\varepsilon - \Delta u_\varepsilon = (\partial_t u - \Delta u) - \varepsilon < 0.$$

Hence, the previous case applies to the function $u_\varepsilon$, and we conclude that

$$\sup_{\Omega} (u - \varepsilon t) = \sup_{\partial_p \Omega} (u - \varepsilon t).$$

Letting $\varepsilon \to 0$ we obtain (2.15).

Finally, let us prove (2.15) under the assumption that $u \in C^2(\Omega)$ (and, of course, $u \in C(\overline{\Omega})$). Choose some $T' < T$ and consider the cylinder $\Omega' = U \times (0, T')$. Then $u \in C^2(U \times (0, T'])$ and we obtain by the above proof that

$$\sup_{\Omega'} u = \sup_{\partial_p \Omega'} u.$$

Letting $T' \to T$, we obtain (2.15). ∎

**Remark.** As we see from the proof, the requirement that $u \in C^2(\Omega)$ is superfluous: it suffices for $u$ to have in $\Omega$ the first time derivative $\partial_t u$ and all second unmixed derivatives $\partial_{x_i x_i} u$.

**Remark.** The maximum principle remains true for a more general parabolic equation

$$\partial_t u = \sum_{i,j=1}^n a_{ij}(x)\, \partial_{x_i x_j} u + \sum_{k=1}^n b_k(x) \partial_{x_k} u,$$

where the right hand side is an elliptic operator.

Now we can prove the uniqueness result.

**Theorem 2.4** *For any continuous function $f(x)$, the Cauchy problem* (2.8) *has at most one bounded solution $u(t, x)$.*

**Proof.** Fix some $T > 0$ and consider the restricted Cauchy problem

$$\begin{cases} \partial_t u = \Delta u & \text{in } \mathbb{R}^n \times (0, T), \\ u|_{t=0} = 0. \end{cases} \tag{2.18}$$

It suffices to prove that if $u$ is a bounded solution of (2.18) then $u \equiv 0$. Since $T > 0$ is arbitrary, the uniqueness in (2.8) will follows.

Consider the function

$$v(x, t) = |x|^2 + 2nt,$$

that is non-negative and obviously satisfies the heat equation

$$\partial_t v = \Delta v.$$

Fix $\varepsilon > 0$ and compare $u$ and $\varepsilon v$ in a cylinder $\Omega = B_R \times (0, T)$, where $R$ is to be chosen. At the bottom of the cylinder (that is, at $t = 0$) we have $u = 0 \le \varepsilon v$. At the lateral boundary of the cylinder (that is, when $|x| = R$) we have $u \le C$ where $C := \sup |u|$, and $v \ge R^2$, hence, $\varepsilon v \ge \varepsilon R^2$. Choosing $R$ so big that $\varepsilon R^2 \ge C$, we obtain that $u \le \varepsilon v$ on the lateral boundary of $\Omega$.

Hence, the function $u - \varepsilon v$ satisfies the heat equation in $\Omega$ and $u - \varepsilon v \le 0$ on the parabolic boundary $\partial_p \Omega$. By Lemma 2.3, we conclude that $u - \varepsilon v \le 0$ in $\Omega$. Letting $R \to \infty$ we obtain $u - \varepsilon v \le 0$ in $\mathbb{R}^n \times (0, T)$. Letting $\varepsilon \to 0$, we obtain $u \le 0$. In the same way $u \ge 0$, whence $u \equiv 0$. ∎

**Remark.** We have proved a bit stronger property that was claimed in Theorem 2.4: the uniqueness of a bounded solution of the heat equation in a strip $\mathbb{R}^n \times (0, T)$.

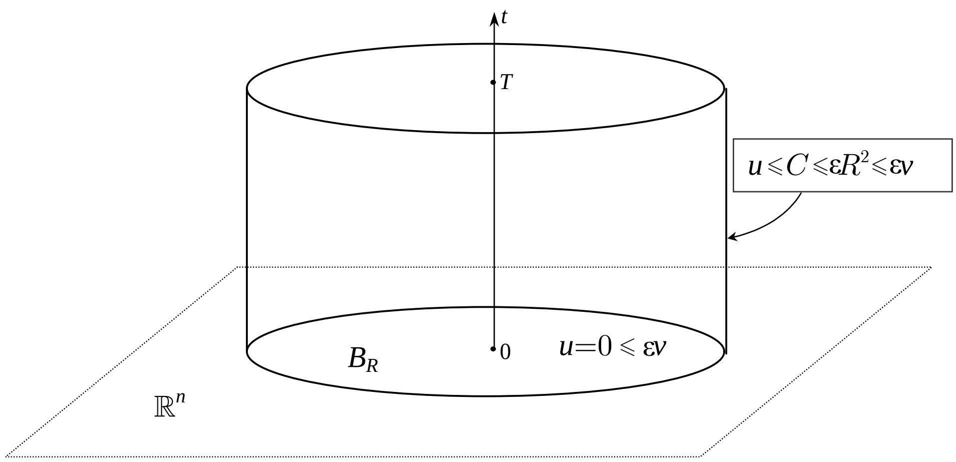



Figure 2.3: Comparison of functions $u$ and $\varepsilon v$ on $\partial_p \Gamma$

**Unbounded Cauchy problem.** In fact, the uniqueness class for solutions to the Cauchy problem is much wider than the set of bounded functions. For example, the Tikhonov theorem says that if $u(t,x)$ solves the Cauchy problem with the initial

$$|u(t,x)| \leq C \exp\left(C|x|^2\right) \tag{2.19}$$

for some constant $C$ and all $t > 0$, $x \in \mathbb{R}^n$, then $u \equiv 0$. On the other hand, one cannot replace here $|x|^2$ by $|x|^{2+\varepsilon}$ for $\varepsilon > 0$.

There is an example, also by Tikhonov, of a solution $u(t,x)$ to (2.18) that is not identical zero for $t > 0$. In fact, for any $t > 0$, the function $x \mapsto u(t,x)$ takes large positive and negative values and, of course, does not satisfy (2.19). This solution of the heat equation is non-physical as it cannot represent an actual physical temperature field.

Theorems 2.2 and 2.4 imply that, for any bounded continuous function $f$, the Cauchy problem has a unique bounded solution, given by (2.10). Let us show an amusing example of application of this result to the heat kernel. We use the notion of convolution $f * g$ of two functions in $\mathbb{R}^n$:

$$f * g(x) = \int_{\mathbb{R}^n} f(x-y)\, g(y)\, dy.$$

**Proposition 2.5** The following identity is true for all $t, s > 0$

$$p_t * p_s = p_{t+s}. \tag{2.20}$$

**Proof.** Let $f$ be a bounded non-negative continuous function in $\mathbb{R}^n$. By Theorem 2.2, the function $u_t = p_t * f$ solves the bounded Cauchy problem with the initial function $f$. Consider now the Cauchy problem with the initial function $u_s$. Obviously, the function $u_{t+s}$ gives the bounded solution to this problem at time $t$. On the other hand, the solution at time $t$ is given by $p_t * u_s$. Hence, we obtain the identity

$$u_{t+s} = p_t * u_s,$$

that is

$$p_{t+s} * f = p_t * (p_s * f).$$

By the associative law of convolution (which is a consequence of Fubini's theorem), we have

$$p_t * (p_s * f) = (p_t * p_s) * f,$$

whence

$$p_{t+s} * f = (p_t * p_s) * f.$$

Since this is true for all functions $f$ as above, we conclude that $p_{t+s} = p_t * p_s$. ■

The identity (2.20) can also be proved by a direct computation, but this is not very simple.

It follows from (2.20) that the one-parameter function family $\{p_t\}_{t>0}$ forms a *convolution semigroup,* that is a semigroup with respect to the operation of convolution; moreover, this semigroup is isomorphic to the additive semigroup of $\mathbb{R}_+$.

## 2.4 Mixed problem and separation of variables

Let $\Omega = U \times (0, T)$ be a cylinder in $\mathbb{R}^{n+1}$ based on a bounded domain $U \subset \mathbb{R}^n$. Consider the following *initial-boundary* problem (that is also called *mixed problem*) in $\Omega$:

$$\begin{cases} \partial_t u = \Delta u & \text{in } \Omega, \\ u = \varphi & \text{on } \partial_p \Omega, \end{cases} \tag{2.21}$$

where $\varphi$ is a given continuous function on the parabolic boundary $\partial_p\Omega$. Function $u$ should be in the class $C^2(\Omega) \cap C\left(\overline{\Omega}\right)$.

**Proposition 2.6** *If $u$ is a solution of* (2.21) *then in* $\Omega$

$$\inf \varphi \le u \le \sup \varphi. \tag{2.22}$$

*Consequently, the problem* (2.21) *has at most one solution.*

**Proof.** By the parabolic maximum principle, we have

$$\sup_\Omega u = \sup_{\partial_p\Omega} u = \sup \varphi$$

and similarly

$$\inf_\Omega u = \inf_{\partial_p\Omega} u = \inf \varphi,$$

whence (2.22) follows.

If $u_1, u_2$ are two solutions of (2.21) then $u = u_1 - u_2$ solves the problem

$$\begin{cases} \partial_t u = \Delta u & \text{in } \Omega \\ u = 0 & \text{on } \partial_p \Omega \end{cases}$$

It follows from (2.22) that $u \equiv 0$ in $\Omega$, whence also $u_1 \equiv u_2$. ■

For existence of solution of (2.21), we restrict ourself to the most important particular case, when $\varphi = 0$ on the lateral boundary $\partial U \times [0, T]$. We rewrite (2.21) in the form:

$$\begin{cases} \partial_t u = \Delta u & \text{in } U \times (0, T) \\ u(x, t) = 0 & \text{on } \partial U \times [0, T] \quad \text{(boundary condition)} \\ u(x, 0) = \varphi(x) & \text{in } U \quad \text{(initial condition)} \end{cases} \tag{2.23}$$

where $\varphi$ is now a given function on $\overline{U}$ such that $\varphi|_{\partial U} = 0$. The latter makes consistent
02.12.15 the boundary condition and initial condition.

We use the method of separation of variables as follows. Let us first look for a solution to the heat equation in the form $u(x,t) = v(x)w(t)$. Then the equation $\partial_t u = \Delta u$ becomes

$$vw' = (\Delta v)\,w$$

that is equivalent to

$$\frac{w'}{w} = \frac{\Delta v}{v}.$$

Since the left hand side is a function of $t$ and the right hand side is a function of $x$, the identity can hold only if they both are constant. Denote this constant by $-\lambda$, so that

$$\Delta v + \lambda v = 0 \quad \text{and } w' + \lambda w = 0.$$

In fact, we require that $v = 0$ on $\partial U$ because then also $u(x,t) = 0$ on $\partial U \times [0,T]$. Hence, we obtain for $v$ the following *eigenvalue problem*:

$$\begin{cases} \Delta v + \lambda v = 0 & \text{in } U \\ v = 0 & \text{on } \partial U. \end{cases} \tag{2.24}$$

Of course, we require that $v \in C^2(U) \cap C(\overline{U})$ and $v \not\equiv 0$ (clearly, the solution $v \equiv 0$ has no value for us). The question is to find non-trivial solutions $v$ to (2.24) as well as those values of $\lambda$ for which non-trivial solution exists.

**Definition.** If for some $\lambda$ (2.24) admits a non-trivial solution $v$, then this $\lambda$ is called an *eigenvalue* of (2.24) and the solution $v$ is called the *eigenfunction*.

This problem is similar to the eigenvalue problem in linear algebra: if $A$ is a linear operator in a linear space $V$ over $\mathbb{R}$ or $\mathbb{C}$ then $\lambda$ is an eigenvalue of $A$ if the equation $Av = \lambda v$ has a non-zero solution $v \in V$, that is called eigenvector. It is known that any operator in an $n$-dimensional space $V$ has at most $n$ eigenvalues (and at least 1 eigenvalue if $V$ is over $\mathbb{C}$). As we shall see later, the problem (2.24) has a *countable* set of eigenvalues that are positive real numbers. Moreover, they can be enumerated as an increasing sequence $\{\lambda_k\}_{k=1}^{\infty}$ such that $\lambda_k \to \infty$ as $k \to \infty$. Let $v_k$ be an eigenfunction that corresponds to $\lambda_k$.

Solving $w' + \lambda_k w = 0$ we obtain $w = Ce^{-\lambda_k t}$. Hence, for any $k \in \mathbb{N}$, we obtain the following solution to the heat equation:

$$u_k(x,t) = e^{-\lambda_k t} v_k(x)$$

that satisfies also the boundary condition $u_k = 0$ on $\partial U \times [0,\infty)$. Let us look for $u(x,t)$ in the form of a linear combination of all $u_k$:

$$u(x,t) = \sum_{k=1}^{\infty} c_k u_k(x,t),$$

for appropriate constants $c_k$. Note that $u_k(x,0) = v_k(x)$. Hence, for $t = 0$ we obtain the identity

$$\varphi(x) = \sum_{k=1}^{\infty} c_k v_k(x), \tag{2.25}$$

which can be used to determine $c_k$. However, the question arises why such an expansion is possible for a rather arbitrary function $\varphi$.

Recall again an analogy with linear algebra. Let $V$ be an $n$-dimensional linear space with an inner product $(\cdot,\cdot)$ (for example, $\mathbb{R}^n$ with the canonical inner product). A linear operator $A$ in $V$ is called symmetric if

$$(Ax, y) = (x, Ay) \text{ for all } x, y \in V.$$

For example, if $A$ is an operator in $\mathbb{R}^n$ that is represented by a matrix $(a_{ij})_{i,j=1}^n$ then the symmetry of $A$ means that the matrix $(a_{ij})$ is symmetric, that is, $a_{ij} = a_{ji}$. It is known that if $A$ is a symmetric operator in $V$ then there is an orthonormal basis $\{v_k\}_{k=1}^n$ in $V$ that consists of the eigenvectors of $A$ (diagonalization of $A$). In particular, any vector $x \in V$ has in this basis an expansion

$$x = \sum_{k=1}^n c_k v_k.$$

A similar theory can be developed for eigenfunctions of the problem (2.24). The role of the space $V$ is played by the *Lebesgue space* $L^2(U)$. By definition, $L^2(U)$ consists of Lebesgue measurable functions $f : U \to \mathbb{R}$ such that

$$\int_U f^2 dx < \infty.$$

Then $L^2(U)$ is a linear space over $\mathbb{R}$ with the inner product

$$(f, g)_{L^2} = \int_U f g dx.$$

Moreover, the space $L^2(U)$ is *complete* with respect to the norm $\|f\|_{L^2} = \sqrt{(f, f)}$, so that $L^2(U)$ is a *Hilbert space*.

Let us emphasize that $\infty$-dimensional spaces do not have to be complete (while finite dimensional spaces are always complete), and for the completeness of $L^2(U)$ it is important that the integrals in the definition of $L^2(U)$ are understood in the sense of Lebesgue.

The Laplace operator $\Delta$ cannot be regarded as an operator on the whole space $L^2(U)$ because $L^2(U)$ contains plenty of discontinuous functions. However, $\Delta$ acts on the dense subspace $C_0^\infty(U)$ of $L^2(U)$, and on this subspace $\Delta$ is symmetric! Indeed, for all $f, g \in C_0^\infty(U)$ we have by the Green formula

$$\int_U \Delta f \, g \, dx = \int_U f \Delta g \, dx,$$

which is equivalent to

$$(\Delta f, g)_{L^2} = (f, \Delta g)_{L^2}.$$

Using the symmetry of $\Delta$, one proves that $L^2(U)$ has an orthonormal basis $\{v_k\}$ that consists of the eigenfunctions of the problem (2.24).

The fact that $\{v_k\}$ is an orthonormal basis in $L^2(U)$ means that any function $\varphi \in L^2(U)$ admits an expansion (2.25) with $c_k = (\varphi, v_k)_{L^2}$, and the series converges in the norm of $L^2(U)$. Then we obtain the following candidate for solution of (2.23):

$$u(x,t) = \sum_{k=1}^{\infty} c_k e^{-\lambda_k t} v_k(x).$$

Of course, in order to prove that it is indeed a solution one needs to investigate the convergence of the series as well as that of its derivatives. So far we do not have tools to do so, and we postpone this task to one of the next chapters.

However, in the case when $n = 1$ and $U$ is an interval, this can be done now. Hence, let us assume that $n = 1$ and $U = (0, \pi)$. The mixed problem (2.23) becomes (with $T = \infty$)

$$\begin{cases} \partial_t u = \partial_{xx} u & \text{in } (0,\pi) \times (0,\infty) \\ u(0,t) = u(\pi,t) = 0 & \text{for } t \in [0,\infty) \\ u(x,0) = \varphi(x) & \text{for } x \in [0,\pi], \end{cases} \tag{2.26}$$

where $\varphi(x)$ is a given continuous function on $[0,\pi]$ that vanishes at $x = 0$ and $x = \pi$. The eigenvalue problem (2.24) becomes

$$\begin{cases} v'' + \lambda v = 0 & \text{in } (0,\pi) \\ v(0) = v(\pi) = 0. \end{cases}$$

If $\lambda < 0$ then setting $\lambda = -\alpha^2$ we obtain the general solution of $v'' - \alpha^2 v = 0$ in the form $v(x) = C_1 e^{\alpha x} + C_2 e^{-\alpha x}$, that cannot vanish at two points unless it is identical zero. In the case $\lambda = 0$ the general solution is $v(x) = C_1 + C_2 x$ that also cannot vanish at two points. Assume $\lambda > 0$. Then the general solution is

$$v(x) = C_1 \sin \sqrt{\lambda} x + C_2 \cos \sqrt{\lambda} x.$$

At $x = 0$ we obtain $v(0) = C_2$, whence $C_2 = 0$. Take without loss of generality that $C_1 = 1$ and, hence, $v(x) = \sin \sqrt{\lambda} x$. At $x = \pi$ we obtain $v(\pi) = \sin \sqrt{\lambda} \pi$ so that we obtain the equation for $\lambda$:

$$\sin \sqrt{\lambda} \pi = 0.$$

Solutions are $\sqrt{\lambda} = k \in \mathbb{N}$, that is, $\lambda_k = k^2$. Hence, we have determined the sequence of the eigenvalues $\lambda_k = k^2$, $k = 1, 2, \ldots$. The corresponding to $\lambda_k$ eigenfunction is $v_k = \sin kx$. Hence, the solution of (2.26) will be sought in the form

$$u(x,t) = \sum_{k=1}^{\infty} c_k e^{-k^2 t} \sin kx, \tag{2.27}$$

where $c_k$ are determined from

$$\varphi(x) = \sum_{k=1}^{\infty} c_k \sin kx. \tag{2.28}$$

Any function $\varphi \in L^2(0,\pi)$ allows such an expansion. Indeed, extend $\varphi$ to $(-\pi,\pi)$ oddly, by setting $\varphi(x) = -\varphi(-x)$ for $x < 0$, and then extend $\varphi$ to the whole $\mathbb{R}$ $2\pi$-periodically. Then $\varphi$ allows an expansion into a Fourier series

$$\varphi(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx),$$

where

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} \varphi(x) \cos kx \, dx = 0,$$

because $\varphi$ is odd. Therefore, only the terms $b_k \sin kx$ remain in the Fourier series. Renaming $b_k$ into $c_k$, we obtain (2.28). It follows that

$$c_k = \frac{1}{\pi} \int_{-\pi}^{\pi} \varphi(x) \sin kx \, dx = \frac{2}{\pi} \int_0^{\pi} \varphi(x) \sin kx \, dx. \tag{2.29}$$

If $\varphi \in C^1([0,\pi])$ and $\varphi(0) = \varphi(\pi) = 0$ then the extended function $\varphi$ belongs to $C^1(\mathbb{R})$ and, hence, the Fourier series (2.28) converges absolutely and uniformly.

**Proposition 2.7** *Assume that the series* (2.28) *converges absolutely, that is,* $\sum_{k=1}^{\infty} |c_k| < \infty$. *Then the series* (2.27) *determines a solution of* (2.26).

**Proof.** Since $\left|c_k e^{-kt} \sin kx\right| \leq |c_k|$, the series (2.27) converges absolutely and uniformly for all $x \in [0,\pi]$ and $t \geq 0$. Hence, $u \in C([0,\pi] \times [0,\infty))$. Let us show that $\partial_t u$ exists. The term-by-term differentiation in $t$ of the series (2.27) gives the series

$$\partial_t u(x,t) = -\sum_{k=1}^{\infty} c_k k^2 e^{-k^2 t} \sin kx, \tag{2.30}$$

where for justification we have to prove that the series in (2.30) converges in $(0,\pi) \times (0,+\infty)$ locally uniformly. Fix $\varepsilon > 0$ and observe that, for $t > \varepsilon$,

$$\left|c_k k^2 e^{-k^2 t} \sin kx\right| \leq |c_k| \, k^2 e^{-k^2 \varepsilon} \leq M_\varepsilon |c_k|$$

where

$$M_\varepsilon = \sup_{k \geq 1} k^2 e^{-k^2 \varepsilon} < \infty.$$

Hence, the series (2.30) converges absolutely and uniformly in $x \in [0,\pi]$ and $t > \varepsilon$. It follows that the sum of this series is a continuous function in this domain and it is equal to $\partial_t u$. Since $\varepsilon > 0$ is arbitrary, we obtain that (2.30) holds in $[0,\pi] \times (0,\infty)$. In the same way, we prove that for $x \in [0,\pi]$ and $t > 0$

$$\partial_x u(x,t) = \sum_{k=1}^{\infty} c_k k e^{-k^2 t} \cos kx$$

and

$$\partial_{xx} u(x,t) = -\sum_{k=1}^{\infty} c_k k^2 e^{-k^2 t} \sin kx. \tag{2.31}$$

Similar identities hold for all other partial derivatives of $u$ with respect to $x$ and $t$. It follows that $u \in C^\infty([0,\pi] \times (0,\infty))$. Comparison of (2.30) and (2.31) shows that $\partial_t u = \partial_{xx} u$. The boundary and initial conditions are obvious, so $u$ is a solution of (2.26). ■

**Example.** Consider the function $\varphi(x) = x(\pi - x)$ on $[0, \pi]$. Computing by (2.29) its Fourier coefficients yields

$$c_k = \frac{2}{\pi}\int_0^\pi x(\pi - x)\sin kx\,dx = \begin{cases} 0, & k \text{ even} \\ \frac{8}{\pi k^3}, & k \text{ odd} \end{cases} \tag{2.32}$$

Therefore, we obtain the solution $u$ of (2.26) as follows:

$$u(x,t) = \frac{8}{\pi}\sum_{k\text{ odd}} \frac{1}{k^3} e^{-k^2 t}\sin kx = \frac{8}{\pi}\left(e^{-t}\sin x + \frac{1}{27}e^{-9t}\sin 3x + \frac{1}{125}e^{-25t}\sin 5x + ...\right). \tag{2.33}$$

Note that by Proposition 2.6 we have $u \geq 0$ although this is not obvious from (2.33).

It follows from (2.33) that, for any $t \geq 0$,

$$\int_0^\pi u(x,t)\,dx = \frac{8}{\pi}\sum_{k\text{ odd}}\frac{1}{k^3}e^{-k^2 t}\underbrace{\int_0^\pi \sin kx\,dx}_{=2/k} = \frac{16}{\pi}\sum_{k\text{ odd}}\frac{1}{k^4}e^{-k^2 t},$$

which implies

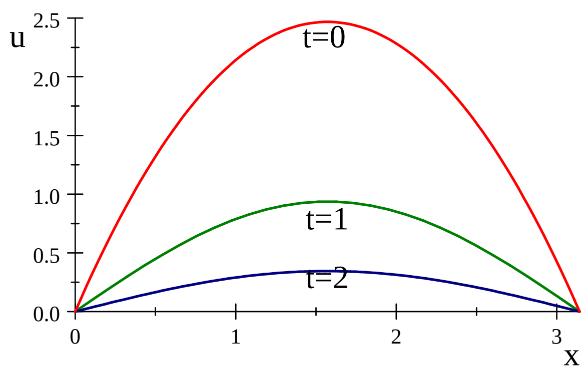
$$\int_0^\pi u(x,t)\,dx \sim \frac{16}{\pi}e^{-t} \quad \text{as } t \to \infty.$$

The physical meaning of this integral is the heat energy of the interval $[0, \pi]$ at time $t$. Due to the "cooling" condition at the boundary, the heat energy decays to 0 exponentially in $t \to \infty$.

It follows also from (2.33) that, for any $x \in (0, \pi)$,

$$u(x,t) \sim \frac{8}{\pi}e^{-t}\sin x \quad \text{as } t \to \infty.$$

Hence, for large $t$, the function $x \mapsto u(x,t)$ takes the shape of $\sin x$.




Solution $u(x,t)$ at $t = 0$, $t = 1$, $t = 2$

## 2.5 *Mixed problem with the source function

Consider now the Dirichlet problem in $(0,\pi)\times\mathbb{R}_+$ with the *source function* $f(x,t)$ at the right hand side:

$$\begin{cases} \partial_t u - \partial_{xx} u = f(x,t) & \text{in } (0,\pi)\times(0,\infty) \\ u(0,t) = u(\pi,t) = 0 & \text{for } t\in[0,+\infty) \\ u(x,0) = 0 & \text{for } x\in[0,\pi]. \end{cases} \tag{2.34}$$

Alongside with the method of separation of variables, we use also the method of *variation of constants*. Namely, we search for solution $u$ in the form (2.27) but now $c_k$ will be unknown functions of $t$:

$$u(x,t) = \sum_{k=1}^{\infty} c_k(t)\, e^{-k^2 t}\sin kx. \tag{2.35}$$

Assuming that we can differentiate the series term-by-term, we obtain

$$\partial_t u = \sum_{k=1}^{\infty}\left(c_k'(t) - c_k(t)\,k^2\right)e^{-k^2 t}\sin kx$$

and

$$\partial_{xx} u = -\sum_{k=1}^{\infty} c_k(t)\,k^2 e^{-k^2 t}\sin kx$$

whence

$$\partial_t u - \partial_{xx} u = \sum_{k=1}^{\infty} c_k'(t)\, e^{-k^2 t}\sin kx. \tag{2.36}$$

On the other hand, expanding the function $f(x,t)$ in a series in $\sin kx$ yields

$$f(x,t) = \sum_{k=1}^{\infty} f_k(t)\sin kx \tag{2.37}$$

where

$$f_k(t) = \frac{2}{\pi}\int_0^{\pi} f(x,t)\sin kx\,dx.$$

Comparing (2.36) and (2.37) we obtain the following equations for functions $c_k$:

$$c_k'(t)\, e^{-k^2 t} = f_k(t). \tag{2.38}$$

The initial condition $u|_{t=0}$ will be satisfied if we require that

$$c_k(0) = 0.$$

Hence, solving (2.38) with this initial condition, we obtain

$$c_k(t) = \int_0^t f_k(s)\, e^{k^2 s} ds. \tag{2.39}$$

Of course, in order to be rigorous, one needs to investigate the convergence of the series (2.35) as we did in Proposition 2.7, and verify that the series can be differentiated term-by-term. We skip this part but observe that if the series in (2.37) is finite then the series (2.35) is also finite, and no further justification is needed. Consider an example of this type.

**Example.** Let

$$f(x,t) = e^{-t}\sin x + t\sin 2x.$$

We obtain from (2.39)

$$c_1(t) = \int_0^t e^{-s}e^s ds = t$$

and

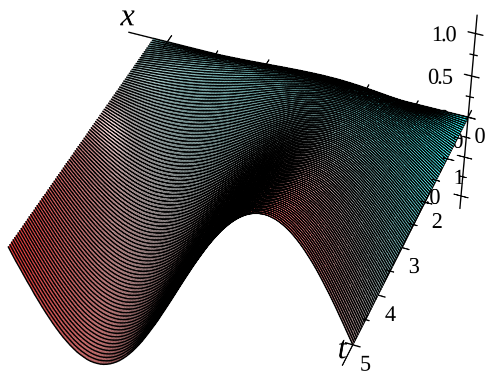
$$c_2(t) = \int_0^t se^{4s}ds = \left(\frac{1}{4}t - \frac{1}{16}\right)e^{4t} + \frac{1}{16},$$

while $c_k \equiv 0$ for all $k \geq 3$. Hence, the solution $u$ is

$$\begin{aligned} u(x,t) &= c_1(t)e^{-t}\sin x + c_2(t)e^{-4t}\sin 2x \\ &= te^{-t}\sin x + \left(\frac{1}{4}t - \frac{1}{16} + \frac{1}{16}e^{-4t}\right)\sin 2x. \end{aligned}$$

In particular, for $t \to \infty$ we obtain the following asymptotic as $t \to \infty$ for any $x \in (0,\pi)$:

$$u(x,t) \sim \begin{cases} te^{-t}, & x = \frac{\pi}{2} \\ \frac{1}{4}t\sin 2x, & x \neq \pi/2 \end{cases}$$




Solution $u(x,t)$

## 2.6 *Cauchy problem with source function and Duhamel's principle

Let $\varphi(x)$ be a function in some domain $D \subset \mathbb{R}^n$. Recall that the notation $\varphi \in C^k(D)$ means that $\varphi$ has in $D$ all partial derivatives of the order at most $k$ and all these derivatives are continuous in $D$. We write $\varphi \in C_b^k(D)$ if in addition all these derivatives are bounded in $D$. In particular, $C_b(D)$ is the set of all bounded continuous functions in $D$.

Let $f(x,t)$ be a function in some domain $D \subset \mathbb{R}^{n+1}$. We write $f \in C^{k,l}(D)$ if $f$ has all partial derivatives in $x$ of the order at most $k$ and in $t$ of the order at most $l$, and all these derivatives are continuous in $D$. We write $f \in C_b^{k,l}(D)$ if in addition all these derivatives are bounded in $D$. We use the convention that the derivative of the order zero is the function itself.

Given a function $f(x,t)$ in $\mathbb{R}_+^{n+1}$ and a function $\varphi(x)$ in $\mathbb{R}^n$, consider the following Cauchy problem

$$\begin{cases} \partial_t u - \Delta u = f & \text{in } \mathbb{R}_+^{n+1} \\ u|_{t=0} = \varphi \end{cases} \tag{2.40}$$

where the solution $u$ is sought in the class $C^{2,1}\left(\mathbb{R}_+^{n+1}\right) \cap C(\overline{\mathbb{R}}_+^{n+1})$.

**Lemma 2.8** *There is at most one solution $u$ of (2.40) that is bounded in any strip $\mathbb{R}^n \times (0,T)$ with $T < \infty$.*

**Proof.** Indeed, if $u_1, u_2$ are two solutions, then $u = u_1 - u_2$ is a bounded in $\mathbb{R}^n \times (0,T)$ solution of

$$\begin{cases} \partial_t u - \Delta u = 0 \\ u|_{t=0} = 0. \end{cases}$$

By Theorem 2.4 we obtain $u \equiv 0$ and, hence, $u_1 \equiv u_2$. ■

Let us use the following notations: $u_t(x) := u(x,t)$ and $f_t(x) = f(x,t)$.

**Theorem 2.9** (Duhamel's principle) *Assume that $\varphi \in C_b(\mathbb{R}^n)$ and $f \in C_b^{2,0}(\overline{\mathbb{R}}_+^{n+1})$. Then the problem (2.40) has the following solution*

$$u_t = p_t * \varphi + \int_0^t p_{t-s} * f_s \, ds. \tag{2.41}$$

*Moreover, the following estimate holds:*

$$\sup |u_t| \le \sup |\varphi| + t \sup |f|. \tag{2.42}$$

Since by (2.42) the solution $u$ is bounded in any strip $\mathbb{R}^n \times (0,T)$, we see by Lemma 2.8 that it is the unique solution of this class.

**Example.** Assume that $\varphi \equiv 0$. If $f \equiv 1$ then $p_{t-s} * 1 = 1$ and we obtain by (2.41) $u_t(x) = t$.

Consider one more example when $f_s(x) = p_s(x)$. Then

$$p_{t-s} * f_s = p_{t-s} * p_s = p_t$$

and we obtain from (2.41) that $u_t(x) = t p_t(x)$.

For the proof of Theorem 2.9 we need some lemmas. We use the following notation

$$P_t f = \begin{cases} p_t * f, & t > 0, \\ f, & t = 0. \end{cases}$$

If $f \in C_b(\mathbb{R}^n)$ then, for $t \ge 0$, the function $P_t f$ is also in $C_b(\mathbb{R}^n)$ so that $P_t$ can be considered as an operator in $C_b(\mathbb{R}^n)$. We consider $P_t f(x)$ as a function of $x$ and $t$. Note that, by Theorem 2.2, the function $P_t f(x)$ belongs to $C^\infty\left(\mathbb{R}_+^{n+1}\right) \cap C_b(\overline{\mathbb{R}}_+^{n+1})$. In the next statement we investigate the smoothness of $P_t f(x)$ in $\overline{\mathbb{R}}_+^{n+1}$.

**Lemma 2.10** *For any integer $k \geq 0$, if $f \in C_b^k(\mathbb{R}^n)$ then $P_t f(x) \in C_b^{k,0}(\overline{\mathbb{R}}_+^{n+1})$. Moreover, for any partial derivative $D^\alpha$ in $x$ of the order $|\alpha| \leq k$, and any $t \geq 0$,*

$$D^\alpha P_t f = P_t(D^\alpha f). \tag{2.43}$$

*Furthermore, if $f \in C_b^2(\mathbb{R}^n)$ then $P_t f(x) \in C_b^{2,1}(\overline{\mathbb{R}}_+^{n+1})$.*

**Proof.** The case $k = 0$ follows from Theorem 2.2 as it was already mentioned. Let $k = 1$. For any $t > 0$ we have

$$\begin{aligned} \partial_{x_i} P_t f &= \partial_{x_i} \int_{\mathbb{R}^n} p_t(y) f(x-y)\, dy \\ &= \int_{\mathbb{R}^n} p_t(y)\, \partial_{x_i} f(x-y)\, dy \end{aligned}$$

because the latter integral converges absolutely and uniformly in $x$, due to the boundedness of $\partial_{x_i} f$. Hence,

$$\partial_{x_i} P_t f = P_t(\partial_{x_i} f).$$

For $t = 0$ this identity is trivial. Since $\partial_{x_i} f \in C_b(\mathbb{R}^n)$, it follows that $P_t(\partial_{x_i} f) \in C_b(\overline{\mathbb{R}}_+^{n+1})$ and, hence, $P_t f \in C_b^{1,0}(\overline{\mathbb{R}}_+^{n+1})$.

For a general $k$ the result follows by induction.

If $f \in C_b^2(\mathbb{R}^n)$ then we obtain by Theorem 2.2 and (2.43) that, for $t > 0$,

$$\partial_t P_t f = \Delta P_t f = P_t(\Delta f).$$

Since $\Delta f \in C_b(\mathbb{R}^n)$, we have $P_t(\Delta f) \in C_b(\overline{\mathbb{R}}_+^{n+1})$, which implies that also $\partial_t P_t f \in C_b(\overline{\mathbb{R}}_+^{n+1})$. Hence, $P_t f \in C_b^{2,1}(\overline{\mathbb{R}}_+^{n+1})$. ∎

It follows from the estimate (2.11) of Theorem 2.2, that if $\{f_k\}$ is a sequence of functions from $C_b(\mathbb{R}^n)$ sucht that $f_k \rightrightarrows f$ in $\mathbb{R}^n$ then

$$P_t f_k(x) \rightrightarrows P_t f(x) \text{ in } \overline{\mathbb{R}}_+^{n+1}.$$

In the next lemma we prove a similar property with respect to the *local* uniform convergence.

**Lemma 2.11** *Let $\{f_k\}$ be a sequence of uniformly bounded continuous functions in $\mathbb{R}^n$. If $f_k(x) \to f(x)$ as $k \to \infty$ locally uniformly in $x \in \mathbb{R}^n$ then*

$$P_t f_k(x) \to P_t f(x)$$

*locally uniformly in $(x,t) \in \overline{\mathbb{R}}_+^{n+1}$.*

**Proof.** Fix some $x \in \mathbb{R}^n$ and choose $R$ large enough, in particular $R > 2|x|$. For any $\varepsilon > 0$ and for all large enough $k$ we have

$$\sup_{B_R} |f_k - f| < \varepsilon. \tag{2.44}$$

Set

$$\begin{aligned} g_k &= f_k \mathbf{1}_{B_R} \text{ and } h_k = f_k \mathbf{1}_{B_R^c} \\ g &= f \mathbf{1}_{B_R} \quad \text{and } h = f \mathbf{1}_{B_R^c} \end{aligned}$$

so that $g_k + h_k = f_k$ and $g + h = f$. Then we have

$$\begin{aligned} |P_t f_k - P_t f| &\leq |P_t g_k - P_t g| + |P_t h_k - P_t h| \\ &\leq |P_t g_k - P_t g| + |P_t h_k| + |P_t h| . \end{aligned}$$

By (2.44) we have

$$\sup_{\mathbb{R}^n} |g_k - g| < \varepsilon$$

whence it follows that

$$\sup_{t \geq 0} \sup_{x \in \mathbb{R}^n} |P_t g_k - P_t g| < \varepsilon.$$

Next, we have

$$P_t h_k (x) = \int_{B_R^c} p_t (x - y) f_k (y) \, dy \quad \text{if } t > 0$$

and

$$P_0 h_k (x) = h_k (x) = 0.$$

By $R > 2|x|$ we have

$$B_{R/2} (x) \subset B_R$$

and, hence,

$$B_R^c \subset B_{R/2} (x)^c .$$

Since $|f_k| \leq C$ where $C$ is the same constant for all $k$, we obtain

$$\begin{aligned} |P_t h_k (x)| &\leq C \int_{B_{R/2}^c (x)} p_t (x - y) \, dy \\ &= C \int_{B_{R/2}^c} p_t (z) \, dz \\ &= C' \int_{\left\{w : |w| > t^{-1/2} R/2\right\}} e^{-|w|^2/4} dw \\ &\to 0 \text{ as } R \to \infty, \end{aligned}$$

where the convergence is uniform in any bounded domain in $(x, t) \in \mathbb{R}^{n+1}_+$. In the same way $P_t h (x) \to 0$ as $R \to 0$, whence the claim follows. ■

Now we consider a function $f_s (x) = f (x, s)$ of $(x, s) \in \overline{\mathbb{R}}^{n+1}_+$. Then $P_t f_s (x)$ is a function of the triple

$$(x, t, s) \in \overline{\mathbb{R}}^{n+2}_+ := \{(x, t, s) : x \in \mathbb{R}^n, t, s \in [0, +\infty)\} .$$

**Lemma 2.12** *(a) If $f \in C_b(\overline{\mathbb{R}}^{n+1}_+)$ then $P_t f_s (x) \in C_b(\overline{\mathbb{R}}^{n+2}_+)$.*

*(b) If $f \in C_b^{2,0}(\overline{\mathbb{R}}^{n+1}_+)$ then $P_t f_s (x) \in C_b^{2,1,0}(\overline{\mathbb{R}}^{n+2}_+)$.*

Here the class $C_b^{2,1,0}$ means the existence of bounded continuous derivatives in $x$ of the order at most 2, in $t$ of the order at most 1 and in $s$ of the order 0.

**Proof.** $(a)$ For any $s \geq 0$, the function $P_t f_s$ is continuous in $(x,t) \in \overline{\mathbb{R}}_+^{n+1}$, and

$$\sup_{(x,t)\in\overline{\mathbb{R}}_+^{n+1}} |P_t f_s(x)| \leq \sup_{x\in\mathbb{R}^n} |f_s(x)| \leq \sup_{(x,s)\in\overline{\mathbb{R}}_+^{n+1}} |f(x,s)| < \infty.$$

It remains to prove that $P_t f_s(x)$ is jointly continuous in $(x,t,s)$. Since this function is continuous in $(x,t)$ for any $s \geq 0$, it suffices to show that it is also continuous in $s$, locally uniformly in $(x,t)$. Indeed, since the function $f(x,s)$ is bounded and locally uniformly continuous, the family $\{f_s\}_{s\geq 0}$ of functions on $\mathbb{R}^n$ is uniformly bounded and $f_s \to f_{s_0}$ as $s \to s_0$ locally uniformly in $x$. Hence, by Lemma 2.11, $P_t f_s \to P_t f_{s_0}$ locally uniformly in $(x,t)$, which finishes the proof.

$(b)$ By Lemma 2.10, for any partial derivative $D^\alpha$ in $x$ of the order $|\alpha| \leq 2$ we have

$$D^\alpha P_t f_s = P_t(D^\alpha f_s).$$

Since $D^\alpha f_s \in C_b(\overline{\mathbb{R}}_+^{n+1})$, we have by $(a)$ that also $D^\alpha P_t f_s \in C_b(\overline{\mathbb{R}}_+^{n+2})$.

For the time derivative $\partial_t$ we have

$$\partial_t P_t f_s = \Delta(P_t f_s) = P_t(\Delta f_s).$$

Since $\Delta f_s \in C_b(\overline{\mathbb{R}}_+^{n+1})$, we obtain $\partial_t P_t f_s \in C_b(\overline{\mathbb{R}}_+^{n+2})$. Hence, $P_t f_s \in C_b^{2,1,0}(\overline{\mathbb{R}}_+^{n+2})$. ■

**Proof of Theorem 2.9.** In the view of Theorem 2.2, it suffices to prove that the function

$$v_t(x) = v(t,x) := \int_0^t p_{t-s} * f_s(x)\,ds = \int_0^t P_{t-s} f_s\,(x)\,ds \tag{2.45}$$

is a solution of the Cauchy problem

$$\begin{cases} \partial_t v - \Delta v = f & \text{in } \mathbb{R}_+^{n+1} \\ v|_{t=0} = 0 & \text{in } \mathbb{R}^n. \end{cases}$$

By Lemma 2.12, the function $P_{t-s} f_s(x)$ belongs to $C_b^{2,1,0}$ in the domain $x \in \mathbb{R}^n$, $t \geq s \geq 0$. It follows from (2.45) that $v \in C(\overline{\mathbb{R}}_+^{n+1})$ and $v|_{t=0} = 0$. It follows also from (2.45) that

$$|v_t| \leq \int_0^t \sup_{\mathbb{R}^n} |P_{t-s} f_s|\,ds \leq \int_0^t \sup|f|\,ds = t\sup|f|,$$

which implies (2.42).

Let us show that $v \in C^{2,1}\left(\mathbb{R}_+^{n+1}\right)$ and that $v$ satisfies $\partial_t v - \Delta v = f$. Let us first compute $\partial_t v$. We have by (2.45)

$$\partial_t v = P_{t-s} f_s|_{s=t} + \int_0^t \partial_t(P_{t-s} f_s)\,ds = f_t + \int_0^t \Delta(P_{t-s} f_s)\,ds, \tag{2.46}$$

which is justified because $\partial_t(P_{t-s} f_s)$ belongs to $C_b$. It follows that $\partial_t v \in C(\overline{\mathbb{R}}_+^{n+1})$

Let $D^\alpha$ be any partial derivative in $x$ of the order $\leq 2$. By Lemma 2.10 we have $D^\alpha(P_{t-s} f_s) \in C_b$, whence by (2.45)

$$D^\alpha v = \int_0^t D^\alpha(P_{t-s} f_s)\,ds. \tag{2.47}$$

It follows that $D^{\alpha}v \in C(\overline{\mathbb{R}}^{n+1}_{+})$ and, hence, $v \in C^{2,1}(\overline{\mathbb{R}}^{n+1}_{+})$.

Finally, we have by (2.47)

$$\Delta v = \int_0^t \Delta \left(P_{t-s} f_s\right) ds,$$

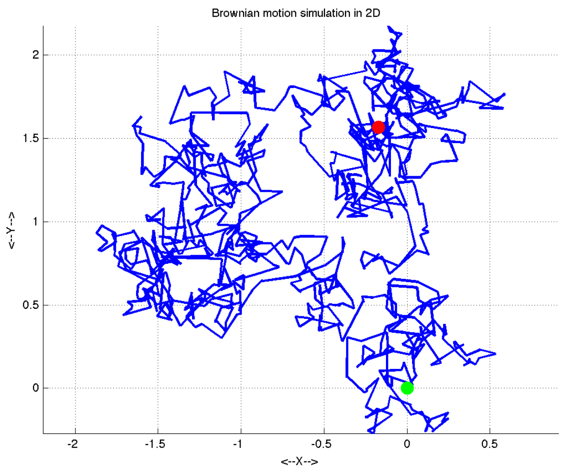
which together with (2.46) implies

$$\partial_t v - \Delta v = f_t,$$

which was to be proved. ■

## 2.7 *Brownian motion

Brownian motion in $\mathbb{R}^n$ is a diffusion process that is described by random continuous paths $\{X_t\}_{t\geq 0}$ in $\mathbb{R}^n$ and by the family $\{\mathbb{P}_x\}_{x\in\mathbb{R}^n}$ of probability measures, so that $\mathbb{P}_x$ is the probability measure on the set $\Omega_x$ of all continuous paths $\omega : [0,\infty) \to \mathbb{R}^n$ such that is $\omega(0) = x$.




Brownian path in $\mathbb{R}^2$

It suffices to define $\mathbb{P}_x$ first on subsets of $\Omega_x$ of the following type:

$$\{\omega \in \Omega_x : \omega(t_1) \in A_1, ..., \omega(t_k) \in A_k\}, \tag{2.48}$$

where $0 < t_1 < t_2 < ... < t_k$ is any finite sequence of reals and $A_1, ..., A_k$ is any sequence of Borel subsets of $\mathbb{R}^n$. Under certain consistency condition, $\mathbb{P}_x$ can be then extended to a $\sigma$-algebra $\mathcal{F}_x$ in $\Omega_x$ thus giving a probability space $(\Omega_x, \mathcal{F}_x, \mathbb{P}_x)$, for any $x \in \mathbb{R}^n$.

There are various ways of defining $\mathbb{P}_x$ on the sets (2.48), the most convenient of them being by means of the heat kernel $p_t(x)$. Let us write $p_t(x,y) = p_t(x-y)$ and set

$$\begin{aligned} &\mathbb{P}_x\left(\omega(t_1) \in A_1, ..., \omega(t_k) \in A_k\right) \\ &= \int_{A_k} ... \int_{A_1} p_{t_1}(x, x_1)\, p_{t_2-t_1}(x_1, x_2) ... p_{t_k - t_{k-1}}(x_{k-1}, x_k)\, dx_1 ... dx_k. \end{aligned} \tag{2.49}$$

The consistency condition that has to be verified is the following: if $A_i = \mathbb{R}^n$ for some $i$, then the condition $\omega(t_i) \in A_i$ can be dropped without affecting the probability, that is,

$$\mathbb{P}_x\left(\omega(t_1) \in A_1, ..., \omega(t_i) \in \mathbb{R}^n, ..., \omega(t_k) \in A_k\right) = \mathbb{P}_x(\omega(t_1) \in A_1, ..., \overset{i}{\checkmark}, ..., \omega(t_k) \in A_k), \tag{2.50}$$

where in the right hand side the $i$-th condition is omitted. Indeed, if $i = k$ then integrating in (2.49) first in $dx_k$ and using that

$$\int_{\mathbb{R}^n} p_{t_k - t_{k-1}}(x_{k-1}, x_k)\, dx_k = 1,$$

we obtain (2.50). If $i < k$ then integrating in (2.49) first in $dx_i$ and using

$$\int_{\mathbb{R}^n} p_{t_i - t_{i-1}}(x_{i-1}, x_i)\, p_{t_{i+1} - t_i}(x_i, x_{i+1})\, dx_i = p_{t_{i+1} - t_{i-1}}(x_{i-1}, x_{i+1}),$$

we again obtain (2.50) (in the case $i = 1$ use the convention $t_0 = 0$ and $x_0 = x$).

The random path $X_t$ is a random variable on $\Omega_x$ that is defined by $X_t(\omega) = \omega(t)$. It follows from (2.49) with $k = 1$ that

$$\mathbb{P}_x(X_t \in A) = \int_A p_t(x, y)\, dy = \int_A \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right) dy, \tag{2.51}$$

which gives the distribution function of $X_t$.


Event $X_t \in A$

The formula (2.51) can be extended as follows: for any bounded Borel function $f$ on $\mathbb{R}^n$,

$$\mathbb{E}_x(f(X_t)) = \int_{\mathbb{R}^n} p_t(x, y)\, f(y)\, dy. \tag{2.52}$$

Note that (2.51) is a particular case of (2.52) for $f = \mathbf{1}_A$. Comparison with Theorem 2.52 yields *Dynkin's formula*: the function

$$u(x, t) := \mathbb{E}_x(f(X_t))$$

is the solution of the Cauchy problem for the heat equation with the initial function $f$.

As it was already mentioned above, the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega \end{cases}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$ can be solved by means of *Kakutani's formula*

$$u(x) = \mathbb{E}_x(\varphi(X_\tau)), \tag{2.53}$$

where $\tau := \inf\{t > 0 : X_t \notin \Omega\}$ is the first exit time of $X_t$ from $\Omega$.

Consider a more general boundary value problem

$$\begin{cases} \Delta u + Vu = 0 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \tag{2.54}$$

where $V(x)$ is a given continuous function in $\Omega$. The operator $\Delta + V$ is called a stationary Schrödinger operator. Under certain natural assumptions about $V$ and $\varphi$, one can prove that the solution of (2.54) is given by the following *Feynman-Kac formula:*

$$u(x) = \mathbb{E}_x\left(\exp\left(\int_0^\tau V(X_t)dt\right)\varphi(X_\tau)\right). \tag{2.55}$$

Clearly, (2.53) is a particular case of (2.55) for $V = 0$.

# Chapter 3

# Wave equation

09.12.15

Here we will be concerned with the wave equation

$$\partial_{tt}u = \Delta u \tag{3.1}$$

where $u = u(x,t)$ is a function of $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$. Recall that the physical wave equation contains a parameter $c > 0$:

$$\partial_{tt}u = c^2 \Delta u. \tag{3.2}$$

The parameter $c$ plays an important physical role as the speed of wave. However, the change $s = ct$ reduces the latter PDE to $\partial_{ss}u = \Delta u$, which is equivalent to (3.1). Hence, all results for (3.1) can be reformulated for (3.1) using the change of time.

Note also that the change $s = -t$ brings (3.1) to the same form $\partial_{ss}u = \Delta u$, which means that the properties of the wave equation for $t > 0$ and for $t < 0$ are the same, unlike the heat equation.

One of the main problems associated with the wave equation is the Cauchy problem:

$$\begin{cases} \partial_{tt}u = \Delta u & \text{in } \mathbb{R}^{n+1}_+ \\ u|_{t=0} = g & \text{in } \mathbb{R}^n \\ \partial_t u|_{t=0} = h & \text{in } \mathbb{R}^n \end{cases} \tag{3.3}$$

where $g(x)$ and $h(x)$ are given function. Solution $u$ is sought in the class $u \in C^2(\overline{\mathbb{R}}^{n+1}_+)$.

The method of solving (3.3) depends on the dimension $n$, so we consider separately the cases $n = 1, 2, 3$.

## 3.1 Cauchy problem in dimension 1

Consider the Cauchy problem in the case $n = 1$:

$$\begin{cases} \partial_{tt}u = \partial_{xx}u & \text{in } \mathbb{R}^2_+ \\ u|_{t=0} = g & \text{in } \mathbb{R} \\ \partial_t u|_{t=0} = h & \text{in } \mathbb{R} \end{cases} \tag{3.4}$$

We have seen in Section 0.2 that the general $C^2$ solution of the wave equation

$$\partial_{tt}u = \partial_{xx}u$$

in $\mathbb{R}^2$ (or in $\mathbb{R}^2_+$) is given by (0.13), that is,

$$u(x,t) = F(x+t) + G(x-t), \tag{3.5}$$

where $F$ and $G$ are arbitrary $C^2$ functions on $\mathbb{R}$. Let us find $F$ and $G$ to satisfy the initial conditions

$$u(x,0) = g(x), \quad \partial_t u(x,0) = h(x).$$

Indeed, substituting into (3.5) $t = 0$ we obtain equation

$$g(x) = F(x) + G(x). \tag{3.6}$$

It follows from (3.5) that

$$\partial_t u = F'(x+t) - G'(x-t),$$

and setting $t = 0$ we obtain one more equation

$$h(x) = F'(x) - G'(x). \tag{3.7}$$

It follows from (3.6) that $g$ has to be $C^2$, and from (3.7) that $h$ has to be $C^1$.

Assuming $g \in C^2$ and $h \in C^1$, we solve the system (3.6)-(3.7) as follows. Differentiating (3.6) we obtain

$$g'(x) = F'(x) + G'(x),$$

which together with (3.7) gives

$$F'(x) = \frac{1}{2}\left(g'(x) + h(x)\right)$$

and

$$G'(x) = \frac{1}{2}\left(g'(x) - h(x)\right).$$

Therefore, we obtain

$$F(x) = \frac{1}{2}\left(g(x) + \int_0^x h(y)\,dy\right) + C \tag{3.8}$$

and

$$G(x) = \frac{1}{2}\left(g(x) - \int_0^x h(y)\,dy\right) - C, \tag{3.9}$$

so that $F$ and $G$ satisfy (3.6) and (3.7). Therefore, we obtain the following statement.

**Theorem 3.1** (D'Alembert's formula) *If $g \in C^2(\mathbb{R})$ and $h \in C^1(\mathbb{R})$ then the following function is a unique solution of (3.4):*

$$u(x,t) = \frac{1}{2}\left(g(x+t) + g(x-t)\right) + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,dy. \tag{3.10}$$

**Proof.** The uniqueness follows from the fact that functions $F$ and $G$ are determined uniquely, up to a constant $C$, that cancels out in (3.5). The function $u$ from (3.10) satisfies (3.5), that is,

$$u(x,t) = F(x+t) + G(x-t),$$

where the functions $F$ and $G$ are given by (3.8) and (3.9). It follows that $u \in C^2(\mathbb{R}^2)$, $u$ satisfies in $\mathbb{R}^2$ the wave equation, and $u$ satisfies the initial conditions by the choice of $F, G$. ∎

This argument shows in addition the following.

1. We have obtained a solution $u$ of the Cauchy problem (3.4) not only in $\mathbb{R}^2_+$ but in the whole $\mathbb{R}^2$.

2. As we see from (3.6) and (3.7), the conditions $g \in C^2$ and $h \in C^1$ are not only sufficient but also necessary for $F$ and $G$ to be in $C^2$; hence, they are also necessary for the existence of a $C^2$ solution.

**Example.** Consider the initial functions

$$g(x) = \sin x \quad \text{and } h(x) = x.$$

Then (3.10) gives

$$\begin{aligned} u(x,t) &= \frac{1}{2}(\sin(x+t) + \sin(x-t)) + \frac{1}{2}\left(\frac{(x+t)^2}{2} - \frac{(x-t)^2}{2}\right) \\ &= \sin x \cos t + xt. \end{aligned}$$

Before we construct solutions in higher dimension, let us discuss the uniqueness in arbitrary dimension.

## 3.2 Energy and uniqueness

We first prove the uniqueness in the setting of a mixed problem. Given a bounded region $U$ in $\mathbb{R}^n$ and $T > 0$, consider the mixed problem for the wave equation in the cylinder $\Omega = U \times (0,T)$:

$$\begin{cases} \partial_{tt} u = \Delta u & \text{in } \Omega \\ u = g & \text{on } \partial_p \Omega \\ \partial_t u|_{t=0} = h & \text{in } U \end{cases} \tag{3.11}$$

where $g$ and $h$ are given functions. Solution $u$ is sought in the class $C^2(\Omega) \cap C^1(\overline{\Omega})$.

**Theorem 3.2** *The problem* (3.11) *has at most one solution in* $C^2(\Omega) \cap C^1(\overline{\Omega})$.

**Proof.** It suffices to prove that if $g = 0$ and $h = 0$ then $u = 0$. Consider the energy of the solution $u$ at time $t$:

$$E(t) = \frac{1}{2}\int_U \left((\partial_t u)^2 + |\nabla u|^2\right) dx. \tag{3.12}$$

Obviously, $E(t)$ is a continuous function in $t \in [0,T]$. Differentiating $E$ in $t \in (0,T)$, we obtain

$$\begin{aligned} E'(t) &= \frac{1}{2}\int_U \left(\partial_t (\partial_t u)^2 + \partial_t (\nabla u \cdot \nabla u)\right) dx \\ &= \int_U (\partial_{tt} u\, \partial_t u + \nabla u \cdot \nabla \partial_t u)\, dx. \end{aligned}$$

Now we use the the second term the Green formula (1.77) of Lemma 1.26. We have $u \in C^2(U) \cap C^1(\overline{U})$ and $w := \partial_t u \in C^1(U) \cap C(\overline{U})$. Since $u = 0$ on the lateral boundary $\partial U \times [0,T]$, we obtain $w = \partial_t u = 0$ on $\partial U \times [0,T]$. Hence, by (1.77)

$$\int_U \nabla u \cdot \nabla \partial_t u\, dx = -\int_U \Delta u\, \partial_t u\, dx.$$

It follows that

$$E'(t) = \int_U (\partial_{tt} u\, \partial_t u - \Delta u\, \partial_t u)\, dx = \int_U (\partial_{tt} u - \Delta u)\, \partial_t u\, dx = 0.$$

Therefore, $E(t) = \text{const}$ on $[0,T]$. Since $E(0) = 0$ by the initial condition $u = 0$ and $\partial_t u = 0$ at $t = 0$, we conclude that $E(t) \equiv 0$. This implies that the functions $\partial_t u$ and $|\nabla u|$ are identically equal to zero in $\Omega$, whence $u \equiv \text{const}$ in $\Omega$. The initial condition $u = 0$ implies $u \equiv 0$ in $\Omega$, which was to be proved. ∎

The physical meaning of the energy (3.12) is as follows. If $u(x,t)$ is the displacement of a vibrating membrane over $U$, then $\frac{1}{2}(\partial_t u)^2$ is (the density of) the kinetic energy at the point $x$ at time $t$, while $\frac{1}{2}|\nabla u|^2$ is (the density of) the potential energy of tension, because the latter is proportional to the increase of the area

$$\sqrt{1 + |\nabla u|^2} - 1 \approx \frac{1}{2}|\nabla u|^2.$$

Now let us discuss uniqueness in the Cauchy problem:

$$\begin{cases} \partial_{tt} u = \Delta u & \text{in } \mathbb{R}^n \times (0,T), \\ u|_{t=0} = g & \text{in } \mathbb{R}^n, \\ \partial_t u|_{t=0} = h & \text{in } \mathbb{R}^n, \end{cases} \tag{3.13}$$

where $T \in (0,\infty]$ and $u \in C^2(\mathbb{R}^n \times [0,T))$.

**Theorem 3.3** (Uniqueness for the Cauchy problem the wave equation) *The problem* (3.13) *has at most one solution* $u \in C^2(\mathbb{R}^n \times [0,T))$.

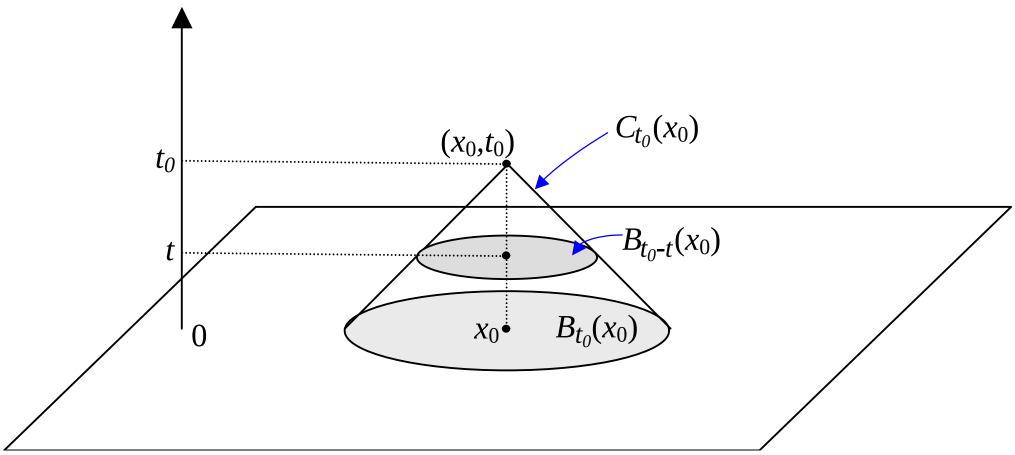
Note that, in contrast to the case of heat equation, there are no restrictions like boundedness of solution.

If $u(x,t)$ is a solution of (3.13), then for any open set $U \subset \mathbb{R}^n$ and any $t \in [0,T)$ define the energy of $u$ in $U$ at time $t$ by

$$E_U(t) = \frac{1}{2}\int_U \left((\partial_t u)^2 + |\nabla u|^2\right) dx.$$

For any $x_0 \in \mathbb{R}^n$ and $t_0 > 0$ define the *cone of dependence* by

$$C_{t_0}(x_0) = \left\{(x,t) \in \mathbb{R}^{n+1} : 0 \le t \le t_0,\ |x - x_0| \le t_0 - t\right\}.$$




Cone $C_{t_0}(x_0)$

Clearly, at each level $t \in [0, t_0]$, the cone $C_{t_0}(x_0)$ consists of the closed ball $\overline{B}_{t_0-t}(x_0)$. In particular, the base of the cone at $t = 0$ is the ball $\overline{B}_{t_0}(x_0)$, the top of the cone at $t = t_0$ is the point $x_0$.

The following theorem plays the main role in the proof of Theorem

**Theorem 3.4** (Domain of dependence) *If $u \in C^2(C_{t_0}(x_0))$ is a solution of the wave equation in $C_{t_0}(x_0)$ and if $u|_{t=0} = 0$ and $\partial_t u|_{t=0} = 0$ then $u \equiv 0$ in $C_{t_0}(x_0)$.*

**Proof of Theorem 3.3.** It suffices to prove that if $g = 0$ and $h = 0$ then $u = 0$. Choose any point $x_0 \in \mathbb{R}^n$ and $t_0 \in (0, T)$. Since $g = h = 0$ in $B_{t_0}(x_0)$, we obtain by Theorem 3.4 that $u = 0$ in the cone $C_{t_0}(x_0)$, in particular at $(x_0, t_0)$. Since $(x_0, t_0)$ is arbitrary, we obtain $u \equiv 0$, which was to be proved. ∎

**Proof of Theorem 3.4.** For simplicity of notation take $x_0 = 0$ and skip $x_0$ from all notation. Consider the energy of $u$ in the ball $B_{t_0-t}$ at time $t$:

$$F(t) := E_{B_{t_0-t}}(t) = \frac{1}{2}\int_{B_{t_0-t}} \left((\partial_t u)^2 + |\nabla u|^2\right) dx.$$

Obviously, $F(t)$ is a continuous function in $[0, t_0]$. By hypotheses, we have $F(0) = 0$. Let us show that $F'(t) \le 0$ for $t \in [0, t_0]$ which will then implies that $F(t) \equiv 0$ in $[0, t_0]$. In turn, this will yield that $\partial_t u = 0$ and $\nabla u = 0$ in $C_{t_0}$, that is, $u \equiv \text{const}$ in $C_{t_0}$, whence also $u \equiv 0$ in $C_{t_0}$ will follow.

In order to differentiate $F(t)$, consider first a simpler function

$$\Phi(r, t) = E_{B_r}(t) = \frac{1}{2}\int_{B_r} \left((\partial_t u)^2 + |\nabla u|^2\right) dx,$$

that is defined whenever $\overline{B}_r \times \{t\}$ lies in the domain of $u$. As in the proof of Theorem 3.2 we have

$$\begin{aligned}
\partial_t \Phi &= \frac{1}{2}\int_{B_r} \partial_t\left((\partial_t u)^2 + \nabla u\cdot\nabla u\right) dx \\
&= \int_{B_r} \left(\partial_{tt} u\,\partial_t u + \nabla u\cdot\nabla\partial_t u\right) dx \\
&= \int_{B_r} \left(\partial_{tt} u - \Delta u\right)\partial_t u\,dx + \int_{\partial B_r} \partial_\nu u\,\partial_t u\,d\sigma \\
&= \int_{\partial B_r} \partial_\nu u\,\partial_t u\,d\sigma.
\end{aligned}$$

Since

$$\partial_\nu u\,\partial_t u \le |\nabla u|\,|\partial_t u| \le \frac{1}{2}\left((\partial_t u)^2 + |\nabla u|^2\right),$$

we obtain the estimate

$$\partial_t \Phi \le \frac{1}{2}\int_{\partial B_r} \left((\partial_t u)^2 + |\nabla u|^2\right) d\sigma.$$

Next, representing integration over the ball $B_r$ as the repeated integral in radius and over the spheres, we obtain

$$\begin{aligned}
\partial_r \Phi &= \frac{1}{2}\partial_r \int_0^r \left(\int_{\partial B_s} \left((\partial_t u)^2 + |\nabla u|^2\right) d\sigma\right) ds \\
&= \frac{1}{2}\int_{\partial B_r} \left((\partial_t u)^2 + |\nabla u|^2\right) d\sigma \\
&\ge \partial_t \Phi. \qquad (3.14)
\end{aligned}$$

Now we can differentiate the function

$$F(t) = E_{B_{t_0-t}}(t) = \Phi(t_0 - t, t)$$

by the chain rule:

$$F' = -\partial_r \Phi(t_0 - t, t) + \partial_t \Phi(t_0 - t, t).$$

Using (3.14), we obtain $F' \le 0$, which was to be proved. ■

11.12.15

**Corollary 3.5** (Finite propagation speed) *Let $u \in C^2(\mathbb{R}^n \times [0,T))$ be a solution to the wave equation in $\mathbb{R}^n \times (0,T)$. If, for some $R > 0$,*

$$\operatorname{supp} u(x,0) \subset \overline{B}_R \quad \textit{and} \quad \operatorname{supp} \partial_t u(x,0) \subset \overline{B}_R \qquad (3.15)$$

*then, for any $0 < t < T$,*

$$\operatorname{supp} u(x,t) \subset \overline{B}_{R+t}. \qquad (3.16)$$

**Proof.** Fix some $t_0 \in (0, T_0)$ and a point $x_0 \notin \overline{B}_{R+t_0}$. It suffices to show that $u(x_0, t_0) = 0$. Indeed, the cone $C_{t_0}(x_0)$ is based on the ball $\overline{B}_{t_0}(x_0)$ and, due to condition $x_0 \notin \overline{B}_{R+t_0}$ we see that the balls $\overline{B}_{t_0}(x_0)$ and $\overline{B}_R$ are disjoint. Therefore,

$u$ and $\partial_t u$ vanish at $t = 0$ in $\overline{B}_{t_0}(x_0)$. By Theorem 3.4 we conclude that $u \equiv 0$ in $C_{t_0}(x_0)$, in particular, $u(x_0, t_0) = 0$, which was to be proved. ■

This statement shows clearly that the wave travels in time $t$ the distance at most $t$, that is, the speed of propagation of the wave is bounded by 1.

**Example.** Let us show in example, that the speed of wave can be exactly 1, that is, the value $R + t$ in (3.16) is sharp and cannot be reduced. Consider in the case $n = 1$ the solution $u(x, t) = F(x + t) + F(x - t)$ where $F$ is a non-negative $C^2$ function with $\operatorname{supp} F = [-R, R]$. Then $u(x, 0) = 2F(x)$ and $\partial_t u(x, 0) = 0$ so that the condition (3.15) is satisfied. At any time $t > 0$ we obtain

$$\operatorname{supp} u(x, t) = [-R - t, -R + t] \cup [-R + t, R + t],$$

that is, $\operatorname{supp} u(x, t)$ is the union of two intervals, and the external boundary points of them are $-R - t$, $R + t$, that is, the endpoints of the interval $[-R - t, R + t]$. Hence, the latter interval cannot be reduced.

**Remark.** Compare the result of Corollary 3.5 with the properties of the heat equation. If now $u(x, t)$ is a bounded solution of the Cauchy problem with the initial function $f$ with $\operatorname{supp} f \subset \overline{B}_R$ and $f \geq 0, f \not\equiv 0$, then by

$$u(x, t) = \int_{\mathbb{R}^n} \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x - y|^2}{4t}\right) f(y)\, dy$$

we see that $u(x, t) > 0$ for all $x \in \mathbb{R}^n$ and $t > 0$. Hence, for any $t > 0$ we have $\operatorname{supp} u(x, t) = \mathbb{R}^n$. This, of course, contradicts the physical meaning of $u$: the temperature cannot propagate instantaneously at infinite distance. This phenomenon reflects the fact that the heat equation describes the heat propagation only approximately. To overcome this difficulty, fix some $\varepsilon > 0$ to be considered as the error of measurement, and consider the notion of $\varepsilon$-support:

$$\operatorname{supp}_\varepsilon f := \{x \in \mathbb{R}^n : |f(x)| \geq \varepsilon\}.$$

Then one can prove the following: if $\operatorname{supp}_\varepsilon f \subset \overline{B}_R$ then $\operatorname{supp}_{2\varepsilon} u(\cdot, t) \subset \overline{B}_{\rho(t)}$ where

$$\rho(t) = \begin{cases} R + \sqrt{Ct \ln \frac{T}{t}}, & 0 < t < T, \\ 0 & t \geq T, \end{cases}$$

where $T > 0$ depends on the function $f$ and $C = C(n) > 0$ (see Exercises). We see that the heat travels in time $t$ the distance roughly $\sqrt{t}$, which matches experimental results.

## 3.3 Mixed problem for the wave equation

Let $U$ be a bounded domain in $\mathbb{R}^n$ and $\Omega = U \times (0, \infty)$. Consider the following mixed problem for the wave equation in $\Omega$:

$$\begin{cases} \partial_{tt} u = \Delta u & \text{in } \Omega \\ u = 0 & \text{on } \partial U \times [0, \infty) \\ u|_{t=0} = g & \text{in } U \\ \partial_t u|_{t=0} = h & \text{in } U \end{cases} \tag{3.17}$$

where $g$ and $h$ are given initial functions on $\overline{U}$. The solution is sought in the class $u \in C^2\left(\overline{\Omega}\right)$.

Note that $g$ and $h$ have to be compatible with the boundary condition $u = 0$ on $\partial U \times [0, \infty)$. The condition $u \in C^2\left(\overline{\Omega}\right)$ implies that

$$g \in C^2\left(\overline{U}\right) \quad \text{and } h \in C^1\left(\overline{U}\right). \tag{3.18}$$

Moreover, $u = 0$ on $\partial U \times [0, \infty)$ implies $g = 0$ on $\partial U$, but also $\partial_t u = 0$ and $\partial_{tt} u = 0$ on $\partial U \times [0, \infty)$. Hence, also $h = 0$ on $\partial U$. Since $\partial_{tt} u = \Delta u$ in $\overline{\Omega}$, we obtain that $\Delta u = 0$ on $\partial U \times [0, \infty)$, which at $t = 0$ amounts to $\Delta g = 0$ on $\partial U$. Hence, here are additional compatibility conditions for $g$ and $h$:

$$g = h = \Delta g = 0 \text{ on } \partial U. \tag{3.19}$$

Since (3.18) and (3.19) are necessary conditions for the existence of a solution $u \in C^2\left(\overline{\Omega}\right)$, we can further assume that $g$ and $h$ satisfy (3.18) and (3.19).

Using the method of separation of variables, search first for solutions of the wave equation in the form $u(x, t) = v(x) w(t)$. We obtain

$$vw'' = (\Delta v) w$$

and

$$\frac{\Delta v}{v} = \frac{w''}{w} = -\lambda$$

where $\lambda$ is a constant. Imposing also the boundary condition $v = 0$ on $\partial U$, we obtain the following eigenvalue problem

$$\begin{cases} \Delta v + \lambda v = 0 & \text{in } U \\ v|_{\partial U} = 0 \end{cases} \tag{3.20}$$

where we search for a non-zero solution $v$. This problem is the same as the one we obtained considering the wave equation. As before, denote by $\{v_k\}_{k=1}^{\infty}$ an orthonormal basis in $L^2(U)$ that consists of eigenfunctions, and by $\{\lambda_k\}_{k=1}^{\infty}$ the sequence of the corresponding eigenvalues in an increasing order. Recall also that all $\lambda_k > 0$.

For $w$ we obtain the equation

$$w'' + \lambda w = 0,$$

which gives us for any $\lambda = \lambda_k$ solution

$$w(t) = a_k \cos\sqrt{\lambda_k} t + b_k \sin\sqrt{\lambda_k} t.$$

Hence, we can search the solution $u$ of (3.17) in the form

$$u(x, t) = \sum_{k=1}^{\infty} \left(a_k \cos\sqrt{\lambda_k} t + b_k \sin\sqrt{\lambda_k} t\right) v_k(x). \tag{3.21}$$

If $v_k \in C^2\left(\overline{\Omega}\right)$ and the series (3.21) and all the series of its first and second derivatives converge uniformly in $\overline{\Omega}$, then we obtain $u \in C^2\left(\overline{\Omega}\right)$ and that $u$ satisfies the wave equation in $\Omega$ as well as the boundary condition $u = 0$ on $\partial U \times [0, \infty)$.

The coefficients $a_k$ and $b_k$ should be determined from the initial conditions. Assume that $g$ and $h$ have the following expansions

$$g(x) = \sum_{k=1}^{\infty} g_k v_k(x) \tag{3.22}$$

and

$$h(x) = \sum_{k=1}^{\infty} h_k v_k(x). \tag{3.23}$$

Setting in (3.21) $t = 0$ we obtain

$$g(x) = u(x, 0) = \sum_{k=1}^{\infty} a_k v_k(x)$$

whence we see that $a_k = h_k$. Differentiating (3.21) in $t$ and setting $t = 0$ we obtain

$$h(x) = \partial_t u(x, 0) = \sum_{k=1}^{\infty} \sqrt{\lambda_k} b_k v_k(x),$$

whence $b_k = h_k/\sqrt{\lambda_k}$. Hence, the solution $u$ becomes

$$u(x,t) = \sum_{k=1}^{\infty} \left( g_k \cos\sqrt{\lambda_k} t + \frac{h_k}{\sqrt{\lambda_k}} \sin\sqrt{\lambda_k} t \right) v_k(x).$$

In order to make the above argument rigorous, we have to justify all the steps, especially the convergence of the series locally uniformly. In general, this is quite a difficult task, as a priori we can only say that the series (3.22) and (3.23) converge in the norm of $L^2$, which is by far not enough.

However, we can justify this approach in the case $n = 1$. Let $U = (0, \pi)$, so that the mixed problem is

$$\begin{cases} \partial_{tt} u = \partial_{xx} u & \text{in } (0,\pi) \times (0,\infty) \\ u(0,t) = u(\pi, t) = 0 & \text{for } t \in [0,\infty) \\ u(x,0) = g(x) & \text{for } x \in [0,\pi] \\ \partial_t u(x,0) = h(x) & \text{for } x \in [0,\pi] \end{cases} \tag{3.24}$$

We know that the sequence of eigenvalues is $\lambda_k = k^2$ and the sequence of eigenfunctions is $v_k = \sin kx$. Assuming that

$$g(x) = \sum_{k=1}^{\infty} g_k \sin kx \tag{3.25}$$

and

$$h(x) = \sum_{k=1}^{\infty} h_k \sin kx, \tag{3.26}$$

we obtain the solution in the form

$$u(x,t) = \sum_{k=1}^{\infty} \left( g_k \cos kt + \frac{h_k}{k} \sin kt \right) \sin kx. \tag{3.27}$$

Before we justify the formula (3.27), let us discuss its physical meaning. Let $u(x,t)$ describe the vibration of the string initially located at the interval $[0,\pi]$. The value $u(x,t)$ is the vertical displacement of the string at point $x$ at time $t$. The boundary condition $u(0,t) = u(\pi,t)$ means that the endpoints of the string are fixed. The initial condition $u(x,0) = g(x)$ describes the initial vertical displacement of the string, and $\partial_t u(x,0) = h$ describes the initial speed of the string in the vertical direction.

While vibrating, the string produces a sound whose pitch is determined by the frequency of vibration. The term

$$(a_k \cos kt + b_k \sin kt) \sin kx = A_k \cos(kt + \varphi_k) \sin kx,$$

that corresponds to the sound of frequency $k$, is called the $k$-th *harmonic*. The amplitude of the $k$-th harmonic is $A_k = \sqrt{a_k^2 + b_k^2}$. If

$$u(x,t) = \sum_{k=1}^{\infty} (a_k \cos kt + b_k \sin kt) \sin kx,$$

then the sound produced by the string $u(x,t)$ ist superposition of the sounds of all integer frequencies $k$. The dominant frequency will be the one with the maximal amplitude. Typically this is the first harmonic, that is also called *fundamental tone*. The higher harmonics are called *overtones*. The timbre of the sound depends on the ratio of the amplitudes of the overtones to that of the fundamental tone.

**Proposition 3.6** *Assume that*

$$\sum_{k=1}^{\infty} \left(k^2 |g_k| + k |h_k|\right) < \infty. \tag{3.28}$$

*Then the function $u$ from* (3.27) *belongs to* $C^2([0,\pi] \times \mathbb{R})$ *and solves the mixed problem* (3.24).

**Proof.** The condition (3.28) implies that the series (3.27) converges absolutely and uniformly for all $x \in \mathbb{R}$ and $t \in \mathbb{R}$, as well as the series of its partial derivatives of the order $\leq 2$, which is enough to conclude that $u$ solves (3.24).

Indeed, each differentiation in $t$ or in $x$ results in an additional factor $k$ in the $k$-th term of (3.27), so that, for any derivative of at most second order, the additional factor is at most $k^2$. Hence, the convergence of the series of derivatives will follow from

$$\sum_{k=1}^{\infty} k^2 \left(|g_k| + \frac{|h_k|}{k}\right) < \infty$$

which is equivalent to (3.28). ∎

16.12.15

The condition (3.28) is too restrictive. Recall that $g \in C^1$ ensures only the convergence of $\sum |g_k|$, and to obtain the convergence of $\sum k^2 |g_k|$ we have to assume $g \in C^3$. Next theorem uses a different method to obtain (3.27) under optimal assumptions.

**Theorem 3.7** *Assume that*

$$g \in C^2([0,\pi]), \quad h \in C^1([0,\pi]) \tag{3.29}$$

*and*

$$g(0) = g(\pi) = g''(0) = g''(\pi) = h(0) = h(\pi) = 0. \tag{3.30}$$

*Then the series* (3.27) *converges absolutely and uniformly in* $[0,\pi] \times \mathbb{R}$*, its sums* $u$ *belongs to* $C^2([0,\pi] \times \mathbb{R})$ *and solves* (3.24).

**Remark.** The conditions (3.29) and (3.30) coincide with (3.18) and (3.19), respectively. Hence, these conditions are necessary for the existence of a $C^2$ solution.

**Proof.** Let us first prove the existence of solution of (3.24). First observe the following: if $f$ is a continuous function on the interval $[0,\pi]$ then the even extension by $f(-x) = f(x)$ defines a continuous function on $[-\pi,\pi]$. For the odd extension $f(-x) = -f(x)$ to be continuous on $[-\pi,\pi]$, it is necessary and sufficient that $f(0) = 0$.

Extend $g$ from $[0,\pi]$ oddly to $[-\pi,\pi]$. Due to the assumption $g(0) = 0$, the extended function $g$ is continuous on $[-\pi,\pi]$. With the odd extension of $g$, the derivative $g'$ extends evenly, so that $g'$ is also continuous on $[-\pi,\pi]$. Finally, the second derivative $g''$ extends oddly and, due to the hypothesis $g''(0) = 0$, the extended function $g''$ is continuous on $[-\pi,\pi]$. Hence, $g \in C^2[-\pi,\pi]$.

If $f$ is a continuous function on $[-\pi,\pi]$ then extend it $2\pi$-periodically by $f(x+2\pi k) = f(x)$ for any $x \in (-\pi,\pi]$ and $k \in \mathbb{Z}$. Observe that $f \in C(\mathbb{R})$ if and only if $f(-\pi) = f(\pi)$.

Now, extend $g$ $2\pi$-periodically from $[-\pi,\pi]$ to $\mathbb{R}$. Since $g(-\pi) = -g(\pi) = 0$ and, hence, $g(-\pi) = g(\pi)$, the extended function $g$ is continuous on $\mathbb{R}$. For the derivative $g'$ we have $g'(-\pi) = g'(\pi)$ since $g'$ is even, which implies that $g'$ is continuous on $\mathbb{R}$. For the second derivative $g''$ we have by (3.30) $g''(-\pi) = -g''(\pi) = 0$, so that $g''$ is also continuous on $\mathbb{R}$. Hence, $g \in C^2(\mathbb{R})$.

In the same way, extending $h$ oddly to $[-\pi,\pi]$ and then $2\pi$-periodically to $\mathbb{R}$, we obtain that $h \in C^1(\mathbb{R})$.

Now let us solve the Cauchy problem

$$\begin{cases} \partial_{tt}u = \partial_{xx}u & \text{in } \mathbb{R}^2_+ \\ u|_{t=0} = g \\ \partial_t u|_{t=0} = h \end{cases}$$

By Theorem 3.1 this problem has a solution $u \in C^2(\mathbb{R}^2)$. Let us show this the same function $u$ solves the mixed problem (3.24). Indeed, the wave equation and the initial conditions are true by definition of $u$. We need only to verify the boundary condition $u(0,t) = u(\pi,t) = 0$.

By Theorem 3.1, the solution is given by

$$u(x,t) = F(x+t) + G(x-t) \tag{3.31}$$

where

$$F(x) = \frac{1}{2}g(x) + \frac{1}{2}\int_0^x h(y)\,dy$$

and
$$G(x) = \frac{1}{2}g(x) - \frac{1}{2}\int_0^x h(y)\,dy.$$
Since $g$ and $h$ are odd functions, the function $\int_0^x h(y)\,dy$ is even, and we obtain
$$G(-x) = \frac{1}{2}g(-x) - \frac{1}{2}\int_0^{-x} h(y)\,dy = -\frac{1}{2}g(x) - \frac{1}{2}\int_0^x h(y)\,dy = -F(x),$$
that is,
$$G(-x) = -F(x).$$
Hence,
$$u(0,t) = F(t) + G(-t) = 0.$$
Since $g$ and $h$ are $2\pi$-periodic and $\int_{-\pi}^{\pi} h(y)\,dy = 0$, it follows that the function $F$ is $2\pi$-periodic. Hence, we obtain
$$u(\pi,t) = F(\pi+t) + G(\pi-t) = F(\pi+t-2\pi) - F(-\pi+t) = 0.$$
Hence, $u$ is a $C^2$ solution of (3.24).

Since $F$ is $2\pi$-periodic and $C^2$, it can be represented by an absolutely and uniformly convergent Fourier series:
$$F(x) = \frac{\alpha_0}{2} + \sum_{k=1}^{\infty} (\alpha_k \cos kx + \beta_k \sin kx).$$
It follows that
$$G(x) = -F(-x) = -\frac{\alpha_0}{2} - \sum_{k=1}^{\infty} (\alpha_k \cos kx - \beta_k \sin kx).$$
Hence, we obtain from (3.31)
$$\begin{aligned} u(x,t) &= \sum_{k=1}^{\infty} (\alpha_k \cos k(x+t) + \beta_k \sin k(x+t)) \\ &\quad - \sum_{k=1}^{\infty} (\alpha_k \cos k(x-t) - \beta_k \sin k(x-t)) \\ &= -\sum_{k=1}^{\infty} 2\alpha_k \sin kx \sin kt + \sum_{k=1}^{\infty} 2\beta_k \sin kx \cos kt \\ &= \sum_{k=1}^{\infty} (a_k \cos kt + b_k \sin kt) \sin kx, \end{aligned}$$
where $a_k = 2\beta_k$, $b_k = -2\alpha_k$ and the series converges absolutely and uniformly.

Since $F' \in C^1$, the Fourier series for $F'$ converges absolutely and uniformly; moreover, it is obtained by means of term by term differentiating of the Fourier series of $F$. It follows that the same is true for $u$: the Fourier series for $\partial_t u$ can be obtained by means of term by term differentiating of the series of $u$, that is,
$$\partial_t u = \sum_{k=1}^{\infty} \partial_t (a_k \cos kt + b_k \sin kt) \sin kx = \sum_{k=1}^{\infty} (-a_k k \sin kt + b_k k \cos kt) \sin kx.$$

Since the both functions $g, h$ are $2\pi$-periodic and odd, their Fourier series are sin-Fourier series as (3.25) and (3.26). Since $g, h \in C^1$, the series (3.25) and (3.26) converge absolutely and uniformly. Hence, the coefficients $a_k$ and $b_k$ of the above expansion of $u$ can be determined from the initial conditions as follows:

$$g(x) = u(x,0) = \sum_{k=1}^{\infty} a_k \sin kx$$

whence $a_k = g_k$, and

$$h(x) = \partial_t u(x,0) = \sum_{k=1}^{\infty} b_k k \sin kx,$$

whence $b_k k = h_k$. Hence, we obtain (3.27). ■

**Remark.** We have obtained in the proof that the series for $u$ can be differentiated in $t$ or in $x$ term by term. However, we cannot prove the same for the second derivatives unless we require $g \in C^3$ and $h \in C^2$. Note that we did not use the second derivatives of the series of $u$ because we employed a different method to prove that $u$ satisfies the wave equation.

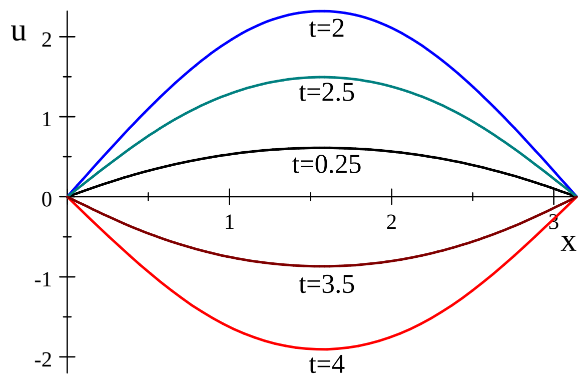
**Remark.** It is worth mentioning that the solution (3.27) is not only $2\pi$-periodic in $x$ but also $2\pi$-periodic in $t$.

**Example.** Consider the mixed problem (3.24) with $g \equiv 0$ and $h(x) = x(\pi - x)$ on $[0, \pi]$. These functions clearly satisfy (3.29) and (3.30). The coefficients $h_k$ of the sin-Fourier of $h$ were computed in (2.32):

$$h_k = \begin{cases} 0, & k \text{ even}, \\ \dfrac{8}{\pi k^3}, & k \text{ odd}. \end{cases}$$

Hence, we obtain the solution $u$ by (3.27):

$$\begin{aligned} u(x,t) &= \frac{8}{\pi} \sum_{k \text{ odd}} \frac{1}{k^4} \sin kt \sin kx \\ &= \frac{8}{\pi} \left( \sin t \sin x + \frac{1}{81} \sin 3t \sin 3x + \frac{1}{625} \sin 5t \sin 5x + \ldots \right). \end{aligned} \tag{3.32}$$




Function $x \mapsto u(x,t)$ at different moments of time.

In fact, already the first term in the series (3.32) provides a reasonable approximation to $u$, that is,

$$u(x,t) \approx \frac{8}{\pi} \sin t \sin x. \tag{3.33}$$

The error of approximation can be roughly estimated as follows. Using the inequality $|\sin kx| \le k |\sin x|$ that can be proved by induction in $k \in \mathbb{N}$, we obtain that, in the region $0 < x < \pi$ and $0 < t < \pi$,

$$|\sin kt \sin kx| \le k^2 \sin x \sin t$$

whence

$$\left| \sum_{k \text{ odd},\, k \ge 3} \frac{1}{k^4} \sin kt \sin kx \right| \le \left( \sum_{k \text{ odd},\, k \ge 3} \frac{1}{k^2} \right) \sin t \sin x = \left( \frac{1}{8}\pi^2 - 1 \right) \sin t \sin x < 0.24 \sin t \sin x$$

and

$$\left| u(x,t) - \frac{8}{\pi} \sin t \sin x \right| \le 0.24 \left( \frac{8}{\pi} \sin t \sin x \right).$$

Hence, the error of approximation in (3.33) is at most 24%, but in practice it is much less than that.

**Example.** Consider the initial conditions $g(x) = x(\pi - x)$ and $h \equiv 0$ on $[0, \pi]$. The function $g$ belongs to $C^\infty([0,\pi])$ and $g(0) = g(\pi) = 0$ but $g''(0)$ and $g''(\pi)$ do not vanish because $g''(x) \equiv -2$. Since the coefficients of the sin-Fourier series for this function are

$$g_k = \begin{cases} 0, & k \text{ even}, \\ \dfrac{8}{\pi k^3}, & k \text{ odd}, \end{cases}$$

the series (3.27) becomes

$$u(x,t) = \frac{8}{\pi} \sum_{k \text{ odd}} \frac{1}{k^3} \cos kt \sin kx. \tag{3.34}$$

This series converges absolutely and uniformly, and the same is true for its first derivatives. However, the series of the second derivative $\partial_{xx}$ is

$$\frac{8}{\pi} \sum_{k \text{ odd}} \partial_{xx} \left( \frac{1}{k^3} \cos kt \sin kx \right) = -\frac{8}{\pi} \sum_{k \text{ odd}} \frac{1}{k} \cos kt \sin kx,$$

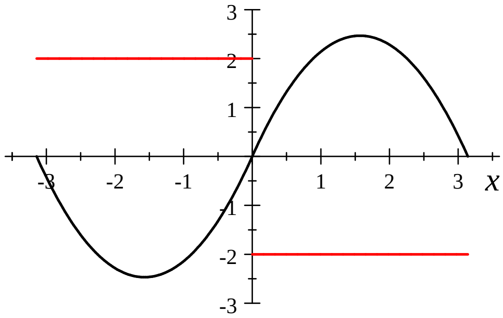
which does not converge uniformly and its sum is not a continuous function, although this is not quite obvious.

In fact, the function $u$ is not $C^2$ because $g$ does not satisfy the necessary condition (3.30). Let us see this directly from the representation:

$$u(x,t) = \frac{1}{2} \left( g(x+t) + g(x-t) \right),$$

where the function $g$ is extended oddly and $2\pi$-periodically. The extended function $g$ is no longer $C^2$ because its second derivative is

$$g''(x) = \begin{cases} -2, & x \in (0, \pi) \\ 2, & x \in (-\pi, 0) \end{cases}$$

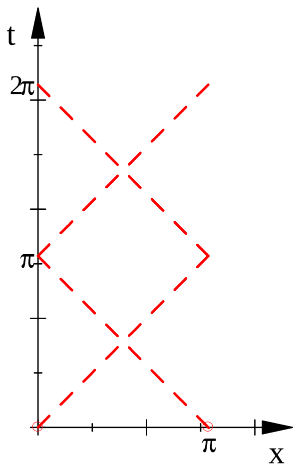



Oddly extended function $g(x) = x(\pi - x)$ (black) and its second derivative $g''$ (red)

Hence, $g''$ does not exists at any point $\pi k$, $k \in \mathbb{Z}$. It follows that function $u(x,t)$ does not have the second derivatives at the following sets:

$$x + t = \pi k \quad \text{and} \quad x - t = \pi k, \qquad k \in \mathbb{Z}.$$

Hence, we see that the singularities $x = 0$ and $x = \pi$ of $g''$ propagate and become the singularities of the second derivatives of $u$.




Singularities of the second derivatives of $u(x,t)$

Overall, the function $u$ from (3.34) is of the class $C^1([0,\pi] \times \mathbb{R})$ and of the class $C^2$ outside singularities. It satisfies the initial and boundary conditions, and satisfies the wave equation outside singularities. One can say that $u$ is a *weak* solution of the wave equation and of the mixed problem. In fact, there is a more general definition of a weak solution for the wave equation, which deals with functions that are only continuous (see Exercises).

In fact, the mixed problem (3.24) with the initial function $g(x) = x(\pi - x)$ has a perfect physical sense: this is the problem of vibration of a string having initially the shape of $g(x)$. In the absence of a $C^2$ solution, one accepts the function $u(x,t)$ from (3.34) as the solution of (3.24).

## 3.4 Spherical means

For solving the Cauchy problem in higher dimension, we use the method of *spherical means*. Given a continuous function $f$ in $\mathbb{R}^n$, fix $x \in \mathbb{R}^n$ and define for any $r > 0$ the function

$$F(x,r) = \fint_{\partial B_r(x)} f(y)\, d\sigma(y) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(x)} f(y)\, d\sigma(y). \tag{3.35}$$

The function $F(x,r)$ is called the *spherical mean* of $f$. We use also the simpler notation $F(r)$ instead of $F(x,r)$ in the case when the point $x$ is fixed.

**Lemma 3.8** *Fix $x \in \mathbb{R}^n$. If $f \in C^m(\mathbb{R}^n)$ where $m \geq 0$ then $F \in C^m([0,\infty))$. Furthermore, if $f \in C^2(\mathbb{R}^n)$ then, for all $r > 0$,*

$$F'(r) = \fint_{\partial B_r(x)} \partial_\nu f(y)\ d\sigma(y) = \frac{r}{n} \fint_{B_r(x)} \Delta f(y)\, dy, \tag{3.36}$$

*where $\nu$ is the outer normal unit vector field on $\partial B_r(x)$, and*

$$F''(r) = \fint_{\partial B_r(x)} \Delta f(y)\, d\sigma(y) - \frac{n-1}{n} \fint_{B_r(x)} \Delta f(y)\, dy. \tag{3.37}$$

*For $r = 0$ we have*

$$F(0) = f(x), \quad F'(0) = 0, \quad F''(0) = \frac{1}{n} \Delta f(x). \tag{3.38}$$

**Proof.** Making in (3.35) change $y = x + rz$, observing that $y \in \partial B_r(x) \Leftrightarrow z \in \partial B_1$ and $d\sigma(y) = r^{n-1} d\sigma(z)$, we obtain

$$F(r) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(x)} f(y)\, d\sigma = \frac{1}{\omega_n} \int_{\partial B_1} f(x + rz)\, d\sigma(z). \tag{3.39}$$

From this formula we see that $F$ is well-defined for all $r \geq 0$ (in fact, for all $r \in \mathbb{R}$). Moreover, if $f \in C^m(\mathbb{R}^n)$ then $F \in C^m([0,\infty))$. 18.12.15

Let $f \in C^2$. Differentiating (3.39) in $r > 0$, we obtain

$$\begin{aligned} F' &= \frac{1}{\omega_n} \int_{\partial B_1} \partial_r (f(x + rz))\, d\sigma(z) \\ &= \frac{1}{\omega_n} \int_{\partial B_1} (\nabla f)(x + rz) \cdot z\, d\sigma(z) \\ &= \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(x)} (\nabla f)(y) \cdot \frac{y - x}{r}\, d\sigma(y). \end{aligned}$$

Since $\frac{y-x}{r} = \nu$ is the outer normal unit vector field on $\partial B_r(x)$, we obtain that

$$(\nabla f)(y) \cdot \frac{y - x}{r} = \nabla f \cdot \nu = \partial_\nu f,$$

whence

$$F' = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(x)} \partial_\nu f\, d\sigma = \fint_{\partial B_r(x)} \partial_\nu f(y)\ d\sigma(y),$$

which proves the first identity in (3.36). Next, the Green formula yields

$$\begin{aligned} F' &= \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(x)} \partial_\nu f \, d\sigma \\ &= \frac{1}{\omega_n r^{n-1}} \int_{B_r(x)} \Delta f \ (y) \, dy \\ &= \frac{r}{n} \frac{1}{\omega_n r^n / n} \int_{B_r(x)} \Delta f \ (y) \, dy = \frac{r}{n} ⨍_{B_r(x)} \Delta f \ dy, \end{aligned}$$

which proves the second identity in (3.36). Rewrite the latter identity in the form

$$F' = \frac{1}{\omega_n r^{n-1}} G(r)$$

where

$$G(r) = \int_{B_r(x)} \Delta f \ (y) \, dy = \int_0^r \left( \int_{\partial_s B(x)} \Delta f (y) \, d\sigma (y) \right) ds.$$

We see that $G$ is differentiable in $r$ and

$$G' = \int_{\partial_r B(x)} \Delta f (y) \, d\sigma (y).$$

It follows that

$$\begin{aligned} F'' &= \frac{d}{dr} \left( \frac{1}{\omega_n r^{n-1}} G(r) \right) \\ &= \frac{1}{\omega_n r^{n-1}} G'(r) - \frac{n-1}{\omega_n r^n} G(r) \\ &= \frac{1}{\omega_n r^{n-1}} \int_{\partial_r B(x)} \Delta f (y) \, d\sigma (y) - \frac{n-1}{\omega_n r^n} \int_{B_r(x)} \Delta f \ (y) \, dy \\ &= ⨍_{\partial_r B(x)} \Delta f (y) \, d\sigma (y) - \frac{n-1}{n} ⨍_{B_r(x)} \Delta f \ (y) \, dy, \end{aligned}$$

that proves (3.37).

Taking limits in (3.35), (3.36), (3.37) as $r \to 0$ and using the continuity of $f$ and $\Delta f$, we obtain (3.38). ■

Now let us consider $F(x, r)$ as a function of $x$ and $r$.

**Lemma 3.9** *If $f \in C^m(\mathbb{R}^n)$ then $F$ as a function of $(x, r)$ belongs to $C^m(\mathbb{R}^n \times [0, \infty))$. If $f \in C^2(\mathbb{R}^n)$ then, for any $r \geq 0$,*

$$\Delta F(x, r) = ⨍_{\partial B_r(x)} \Delta f (y) \, dy. \tag{3.40}$$

**Proof.** This follows immediately from (3.39) and from

$$\Delta (f (x + rz)) = (\Delta f)(x + rz).$$

■

Let us consider the Cauchy problem in dimension $n$:

$$\begin{cases} \partial_{tt}u = \Delta u & \text{in } \mathbb{R}^{n+1}_+ \\ u|_{t=0} = g, \quad \partial_t u|_{t=0} = h & \text{in } \mathbb{R}^n \end{cases} \tag{3.41}$$

where $g, h$ are given functions in $\mathbb{R}^n$. We will assume that $u \in C^2(\overline{\mathbb{R}}^{n+1}_+)$ and, consequently, that

$$g \in C^2(\mathbb{R}^n), \quad h \in C^1(\mathbb{R}^n).$$

Assuming that the solution $u$ exists, we will deduce the formula for $u$. Define the spherical means

$$G(x,r) = \fint_{\partial B_r(x)} g(y)\, d\sigma(y),$$

$$H(x,r) = \fint_{\partial B_r(x)} h(y)\, d\sigma(y),$$

and

$$U(x,r,t) = \fint_{\partial B_r(x)} u(y,t)\, d\sigma(y), \tag{3.42}$$

where $x \in \mathbb{R}^n$ and $r > 0$. All these functions are also defined at $r = 0$ by continuity. We use the shorter notations $G(r), H(r), U(r,t)$ if $x$ is fixed.

Set

$$Q = \mathbb{R}_+ \times (0, \infty)$$

and denote the points of $Q$ by $(r,t)$ where $r, t > 0$.

**Proposition 3.10** (Euler-Poisson-Darboux equation) *If $u$ solves (3.41) then, for any fixed $x \in \mathbb{R}^n$, the function $U(r,t)$ belongs to $C^2(\overline{Q})$ and solves the following mixed problem*

$$\begin{cases} \partial_{tt}U = \partial_{rr}U + \frac{n-1}{r}\partial_r U & \text{in } Q, \\ U(0,t) = u(x,t) & \text{for all } t \geq 0, \\ U(r,0) = G(r) & \text{for all } r \geq 0, \\ \partial_t U(r,0) = H(r) & \text{for all } r \geq 0. \end{cases} \tag{3.43}$$

**Proof.** We have by (3.39)

$$U(r,t) = \frac{1}{\omega_n} \int_{\partial B_1} u(x + rz, t)\, d\sigma(z), \tag{3.44}$$

which implies that $U \in C^2(\overline{Q})$. By Lemma 3.8 we have

$$\partial_r U = \frac{r}{n} \fint_{B_r(x)} \Delta u(y,t)\, dy$$

and

$$\partial_{rr}U = \fint_{\partial B_r(x)} \Delta u(y,t)\, d\sigma(y) - \frac{n-1}{n} \fint_{B_r(x)} \Delta u(y,t)\, dy,$$

which implies

$$\begin{aligned}\partial_{rr}U+\frac{n-1}{r}\partial_r U &= \fint_{\partial B_r(x)}\Delta u\,(y,t)\,d\sigma\,(y)\\ &= \fint_{\partial B_r(x)}\partial_{tt}u\,(y,t)\,d\sigma\,(y)\\ &= \partial_{tt}U.\end{aligned}$$

The boundary condition $U(0,t)=u(x,t)$ from (3.38) or (3.44). The initial conditions follow from $u(x,0)=g(x)$ and $\partial_t u(x,0)=h(x)$. ■

## 3.5 Cauchy problem in dimension 3

Consider the Cauchy problem with $n=3$:

$$\begin{cases}\partial_{tt}u=\Delta u & \text{in } \mathbb{R}^4_+\\ u|_{t=0}=g,\quad \partial_t u|_{t=0}=h & \text{in } \mathbb{R}^3\end{cases}\tag{3.45}$$

As before, solution is sought in the class $u\in C^2(\overline{\mathbb{R}}^4_+)$, while $g\in C^2(\mathbb{R}^3)$, $h\in C^1(\mathbb{R}^3)$.

**Theorem 3.11** (Case $n=3$, Kirchhoff's formula) *If $u$ is a solution of* (3.45) *then, for all $x\in\mathbb{R}^3$ and $t>0$,*

$$u(x,t)=\fint_{\partial B_t(x)}\left(g(y)+t\partial_\nu g(y)+th(y)\right)d\sigma(y).\tag{3.46}$$

Recall that the ball $\overline{B}_t(x)$ is the bottom of the cone of dependence $C_t(x)$. As we know from Theorem 3.4, the value $u(x,t)$ is completely determined by the initial conditions in the ball $\overline{B}_t(x)$. The formula (3.46) shows that in the case of dimension 3 a stronger statement is true: $u(x,t)$ is completely determined by the initial conditions on the *sphere* $\partial B_t(x)$ (more precisely, in a little neighborhood of the sphere because one needs $\partial_\nu g$ as well). This is a specific property of wave propagation in the three dimensional space.

For comparison, recall D'Alembert's formula in dimension 1:

$$u(x,t)=\frac{1}{2}\left(g(x+t)+g(x-t)\right)+\frac{1}{2}\int_{x-t}^{x+t}h(y)\,dy.$$

In this case $B_t(x)=(x-t,x+t)$ and $\partial B_t(x)$ consists of two points $x-t,x+t$ so that we can rewrite this formula in the form

$$u(x,t)=\fint_{\partial B_t(x)}g\,d\sigma+t\fint_{B_t(x)}h(y)\,dy.$$

In particular, we see that the value $u(x,t)$ depends on the values of $h$ in the full "ball" $B_t(x)$.

**Proof.** We use the spherical means $U,G,$ $H$ as above. Consider also the functions

$$\widetilde{U}:=rU,\quad \widetilde{G}:=rG,\quad \widetilde{H}:=rH.$$

Using (3.43) and $n=3$, we obtain

$$\partial_{rr}\widetilde{U}=\partial_r\left(r\partial_r U+U\right)=r\partial_{rr}U+2\partial_r U=r\left(\partial_{rr}U+\frac{n-1}{r}\partial_r U\right)=r\partial_{tt}U=\partial_{tt}\widetilde{U}.$$

Therefore, $\widetilde{U}\in C^2(Q)$ where $Q=\mathbb{R}_+\times(0,\infty)$, and $\widetilde{U}$ solves the following mixed problem:

$$\begin{cases}\partial_{tt}\widetilde{U}=\partial_{rr}\widetilde{U} & \text{in } Q\\ \widetilde{U}(0,t)=0 & \text{for all } t\geq 0\\ \widetilde{U}(r,0)=\widetilde{G}(r) & \text{for all } r\geq 0\\ \partial_t\widetilde{U}(r,0)=\widetilde{H}(r) & \text{for all } r\geq 0\end{cases}$$

Since $\widetilde{U}$ is a solution of the wave equation in $Q$, it has to be of the form

$$\widetilde{U}(r,t)=\Phi(r+t)+\Psi(r-t),$$

for some $C^2$ functions $\Phi$ on $\mathbb{R}_+$ and $\Psi$ on $\mathbb{R}$. Let us use the boundary and initial values in order to determine $\Phi$ and $\Psi$. Setting $r=0$ and using $\widetilde{U}(0,t)=0$, we obtain

$$\Phi(t)=-\Psi(-t)\quad\text{for all } t\geq 0.$$

Setting $t=0$ we obtain

$$\Phi(r)+\Psi(r)=\widetilde{G}(r).$$

Differentiating $\widetilde{U}$ in $t$ and setting $t=0$ we obtain

$$\Phi'(r)-\Psi'(r)=\widetilde{H}(r).$$

Solving these two equations as in the proof of Theorem 3.1, we obtain

$$\Phi(r)=\frac{1}{2}\left(\widetilde{G}(r)+\int_0^r\widetilde{H}(s)\,ds\right),\qquad\Psi(r)=\frac{1}{2}\left(\widetilde{G}(r)-\int_0^r\widetilde{H}(s)\,ds\right).$$

In the range $0\leq r\leq t$ we have

$$\begin{aligned}\widetilde{U}(r,t)&=\Phi(r+t)+\Psi(-(t-r))\\&=\Phi(r+t)-\Phi(t-r)\\&=\frac{1}{2}\left(\widetilde{G}(r+t)+\int_0^{r+t}\widetilde{H}(s)\,ds\right)-\frac{1}{2}\left(\widetilde{G}(t-r)+\int_0^{t-r}\widetilde{H}(s)\,ds\right)\\&=\frac{1}{2}\left(\widetilde{G}(t+r)-\widetilde{G}(t-r)\right)+\frac{1}{2}\int_{t-r}^{t+r}\widetilde{H}(s)\,ds.\end{aligned}$$

Since

$$u(x,t)=\lim_{r\to 0}U(x,r,t)=\lim_{r\to 0}\frac{\widetilde{U}(x,r,t)}{r},$$

it follows that

$$\begin{aligned}u(x,t)&=\lim_{r\to 0}\left(\frac{\widetilde{G}(t+r)-\widetilde{G}(t-r)}{2r}+\frac{1}{2r}\int_{t-r}^{t+r}\widetilde{H}(s)\,ds\right)\\&=\widetilde{G}'(t)+\widetilde{H}(t)\\&=(tG)'+tH\\&=G+tG'+tH.\end{aligned}\tag{3.47}$$

By Lemma 3.8 we have

$$G'(t) = \fint_{\partial B_t(x)} \partial_\nu g(y)\ d\sigma(y),$$

whence (3.46) follows. ■

Finally, we can prove the existence of solution of (3.45).

**Theorem 3.12** (Kirchhoff's formula) *If $g \in C^3(\mathbb{R}^3)$ and $h \in C^2(\mathbb{R}^3)$ then the function*

$$u(x,t) = \fint_{\partial B_t(x)} (g(y) + t\partial_\nu g(y) + th(y))\, d\sigma(y) \tag{3.48}$$

*is a solution of* (3.45).

**Proof.** The formula (3.48) is equivalent to (3.47), that is,

$$u = \partial_t(tG) + tH = G + t\partial_t G + tH.$$

By Lemma 3.9, we have $G \in C^3$ and $H \in C^2$, whence $u \in C^2(\mathbb{R}^3 \times [0,\infty))$. At $t = 0$ we obtain by (3.38)

$$u(x,0) = G(x,0) = g(x).$$

Since

$$\partial_t u = 2\partial_t G + t\partial_{tt} G + t\partial_t H + H,$$

it follows by (3.38) that

$$\partial_t u(x,0) = H(x,0) = h(x).$$

Let us verify that $u$ satisfies the wave equation. It suffices to show that each of the functions $tH$ and $\partial_t(tG)$ satisfies the wave equation. Consider first the function

$$v(x,t) = tH(x,t).$$

It follows by Lemmas 3.8 and 3.9 that, for $t > 0$,

$$\begin{aligned}
\partial_{tt} v &= 2\partial_t H + t\partial_{tt} H \\
&= \frac{2t}{3}\fint_{B_t(x)} \Delta h\, dy + t\fint_{\partial B_t(x)} \Delta h\, d\sigma - \frac{2}{3}t\fint_{B_t(x)} \Delta h\, dy \\
&= t\fint_{\partial B_t(x)} \Delta h\, d\sigma \\
&= t\Delta H = \Delta v
\end{aligned}$$

that is, $v$ satisfies the wave equation.

Since the function $w = tG$ satisfies the wave equation $\partial_{tt} w = \Delta w$, differentiating this equation in $t$ and noticing that $\partial_t$ commutes with $\partial_{tt}$ and $\Delta$, we obtain that $\partial_t w$ also satisfies the wave equation, which finishes the proof. ■

## 3.6 Cauchy problem in dimension 2

06.01.16

Consider now the Cauchy problem with $n = 2$:

$$\begin{cases} \partial_{tt}u = \Delta u & \text{in } \mathbb{R}^3_+ \\ u|_{t=0} = g \\ \partial_t u|_{t=0} = h \end{cases} \tag{3.49}$$

Solution is sought in the class $u \in C^2(\overline{\mathbb{R}}^3_+)$.

**Theorem 3.13** (Poisson formula) *Let $g \in C^3(\mathbb{R}^2)$ and $h \in C^2(\mathbb{R}^2)$. Then (3.49) has the following solution:*

$$u(x,t) = \frac{1}{2}⨍_{B_t(x)} \frac{tg(y) + t\nabla g \cdot (y-x) + t^2 h(y)}{\sqrt{t^2 - |x-y|^2}} dy.$$

**Proof.** Let us extend (3.49) to a Cauchy problem in dimension 3. Indeed, any function $f(x_1, x_2)$ defined in $\mathbb{R}^2$ extends trivially to a function in $\mathbb{R}^3$ by setting

$$f(x_1, x_2, x_3) = f(x_1, x_2).$$

So, extend $u, g$ and $h$ to $\mathbb{R}^3$. In particular, we have $u(x_1, x_2, x_3, t) = u(x_1, x_2, t)$ and

$$\partial_{x_1x_1}u + \partial_{x_2x_2}u + \partial_{x_3x_3}u = \partial_{x_1x_1}u + \partial_{x_2x_2}u.$$

Hence, (3.49) is equivalent to the Cauchy problem in dimension 3

$$\begin{cases} \partial_{tt}u = \Delta u & \text{in } \mathbb{R}^4_+ \\ u|_{t=0} = g \\ \partial_t u|_{t=0} = h \end{cases} \tag{3.50}$$

Additional condition is that $u$ should not depend on $x_3$.

Denote points in $\mathbb{R}^3$ by $X = (x_1, x_2, x_3)$ and denote by $x$ the point $(x_1, x_2)$ in $\mathbb{R}^2$, that is, the projection of $X$ onto the plane $x_1, x_2$. The same convention we use for $Y$ and $y$. By Theorem 3.45 the problem (3.50) has solution

$$u(X,t) = ⨍_{\partial B_t(X)} (g + t\partial_\nu g + th)\, d\sigma(Y), \tag{3.51}$$

where $B_t(X)$ is a ball in $\mathbb{R}^3$ and $\nu$ is outer normal unit vector field on $\partial B_t(X)$. Using the fact that $g$ and $h$ do not depend on $x_3$, let us transform (3.51) to contain integration only in $\mathbb{R}^2$, and at the same time we check that $u$ does not depend on $x_3$. The sphere $\partial B_t(X)$ is given by the equation

$$(y_1 - x_1)^2 + (y_2 - x_2)^2 + (y_3 - x_3)^2 = t^2,$$

and it consists of two hemispheres that can be represented as the graphs of the following functions

$$y_3 = x_3 \pm \sqrt{t^2 - (y_1 - x_1)^2 - (y_2 - x_2)^2}$$

over the disk $D_t(x)$ in $\mathbb{R}^2$ of radius $t$ centered at $x$ (to distinguish the balls in $\mathbb{R}^3$ and $\mathbb{R}^2$, we refer to those on $\mathbb{R}^2$ as disks and denote them by $D$ rather than $B$).

If a surface $S$ in $\mathbb{R}^3$ is given by the graph of a function

$$y_3 = f(y), \quad y \in \Omega,$$

in domain $\Omega \subset \mathbb{R}^2$, then, for any continuous function $\Phi$ on $S$,

$$\int_S \Phi(Y)\, d\sigma(Y) = \int_\Omega \Phi(y, f(y)) \sqrt{1 + |\nabla f|^2} dy.$$

In our case $S$ is one of the two hemispheres of $\partial B_t(X)$, $\Omega = D_t(x)$,

$$f(y) = x_3 \pm \sqrt{t^2 - |y-x|^2}$$

and

$$\Phi = g + t\partial_\nu g + th.$$

Observe that $\partial_{x_3} g = 0$ and, at any point $Y \in \partial B_t(X)$, the normal vector $\nu$ is given by $\nu = \frac{Y-X}{t}$. Hence, we obtain

$$\begin{aligned} t\partial_\nu g &= t\nabla g \cdot \frac{Y-X}{t} = (\partial_{x_1} g, \partial_{x_2} g, \partial_{x_3} g) \cdot (Y - X) \\ &= (\partial_{x_1} g, \partial_{x_2} g) \cdot (y - x) \\ &= \nabla g \cdot (y - x), \end{aligned}$$

where from now on $\nabla$ denotes the gradient in $\mathbb{R}^2$. Since $g$ and $h$ depend only on $y$, we obtain

$$\Phi(Y) = g(y) + \nabla g \cdot (y - x) + th(y).$$

In particular, $\Phi$ does not depend on $y_3$, and in the expression $\Phi(Y) = \Phi(y, f(y))$ we do not have to substitute the value of $f(y)$.

We have for $i = 1, 2$

$$\partial_{y_i} f = \mp \frac{y_i - x_i}{\sqrt{t^2 - |y-x|^2}}$$

whence

$$\begin{aligned} 1 + |\nabla f|^2 &= 1 + \frac{(y_1 - x_1)^2}{t^2 - |y-x|^2} + \frac{(y_2 - x_2)^2}{t^2 - |y-x|^2} \\ &= \frac{t^2}{t^2 - |y-x|^2}. \end{aligned}$$

Hence, we obtain

$$\begin{aligned} \int_S \Phi(Y)\, d\sigma(Y) &= \int_{D_t(x)} \Phi(Y) \frac{t}{\sqrt{t^2 - |x-y|^2}} dy \\ &= \int_{D_t(x)} \frac{tg(y) + t\nabla g \cdot (y - x) + t^2 h(y)}{\sqrt{t^2 - |x-y|^2}} dy. \end{aligned}$$

Since the integral in (3.51) is crosses, we have to divide by the surface area of $\partial B_t(X)$ that is equal to $4\pi t^2$. Since we integrate over two hemispheres, we multiply by 2. Hence, we obtain

$$\begin{aligned} u(X,t) &= \frac{2}{4\pi t^2}\int_{D_t(x)} \frac{tg(y) + t\nabla g\cdot(y-x) + t^2 h(y)}{\sqrt{t^2 - |x-y|^2}}dy \\ &= \frac{1}{2}\fint_{D_t(x)} \frac{tg(y) + t\nabla g\cdot(y-x) + t^2 h(y)}{\sqrt{t^2 - |x-y|^2}}dy, \end{aligned}$$

where we have used that the area of $D_t(x)$ is equal to $\pi t^2$. Since the last integral does not depend on $x_3$, we can write $u(X,t) = u(x,t)$, which finishes the proof. ■

## 3.7 *Cauchy problem in higher dimensions

Similar formulas for solution of the Cauchy problem for the wave equation can be found in arbitrary dimension $n$, which we state without proof. Consider the Cauchy problem (3.41) in arbitrary dimension $n \geq 2$. As above, consider the spherical means

$$G(x,t) = \fint_{\partial B_t(x)} g d\sigma \quad \text{and} \quad H(x,t) = \fint_{\partial B_t(x)} g d\sigma.$$

As we know, in the case $n = 3$ the solution can be written in the form

$$u = \partial_t(tG) + tH = G + t\partial_t G + H. \tag{3.52}$$

**Theorem 3.14** *Let $n \geq 3$ be odd. If $g \in C^{\frac{n+3}{2}}(\mathbb{R}^n)$ and $h \in C^{\frac{n+1}{2}}(\mathbb{R}^n)$ then the following function is a solution of* (3.41)*:*

$$u = \frac{1}{(n-2)!!}\left[t\left(\frac{1}{t}\partial_t\right)^{\frac{n-1}{2}}\left(t^{n-2}G\right) + \left(\frac{1}{t}\partial_t\right)^{\frac{n-3}{2}}\left(t^{n-2}H\right)\right]. \tag{3.53}$$

Here $k!! = 1\cdot 3\cdot 5 ...\cdot k$ for the case of odd $k$ and $k!! = 2\cdot 4\cdot ...\cdot k$ in the case of even $k$.

Clearly, in the case $n = 3$ (3.53) coincides with (3.52). In the case $n = 5$ we have

$$u = \frac{1}{3}\left[t\left(\frac{1}{t}\partial_t\right)^2\left(t^3 G\right) + \left(\frac{1}{t}\partial_t\right)\left(t^3 H\right)\right].$$

Since

$$\left(\frac{1}{t}\partial_t\right)\left(t^3 G\right) = \frac{1}{t}\left(3t^2 G + t^3\partial_t G\right) = 3tG + t^2\partial_t G$$

$$t\left(\frac{1}{t}\partial_t\right)^2\left(t^3 G\right) = \partial_t\left(3tG + t^2\partial_t G\right) = 3G + 5t\partial_t G + t^2\partial_{tt} G$$

and

$$\left(\frac{1}{t}\partial_t\right)\left(t^3 H\right) = 3tH + t^2\partial_t H,$$

we obtain in the case $n = 5$ that

$$u = \frac{1}{3}\left[3G + 5t\partial_t G + t^2\partial_{tt}G + 3tH + t^2\partial_t H\right].$$

For the case of even $n$, we introduce the following notation:

$$\widetilde{G}(x,t) = \fint_{B_t(x)} \frac{g(y)}{\sqrt{t^2 - |x-y|^2}} dy$$

and

$$\widetilde{H}(x,t) = \fint_{B_t(x)} \frac{h(y)}{\sqrt{t^2 - |x-y|^2}} dy.$$

**Theorem 3.15** *Let $n \geq 2$ be even. If $g \in C^{\frac{n}{2}+2}(\mathbb{R}^n)$ and $h \in C^{\frac{n}{2}+1}(\mathbb{R}^n)$ then the following function is a solution of* (3.41)*:*

$$u = \frac{1}{n!!}\left[\partial_t\left(\frac{1}{t}\partial_t\right)^{\frac{n-2}{2}}\left(t^n\widetilde{G}\right) + \left(\frac{1}{t}\partial_t\right)^{\frac{n-2}{2}}\left(t^n\widetilde{H}\right)\right]. \tag{3.54}$$

# Chapter 4

# The eigenvalue problem

08.01.16
13.01.16

## 4.1 Distributions and distributional derivatives

Let $\Omega$ be an open subset of $\mathbb{R}^n$. Denote by $\mathcal{D}(\Omega)$ the linear topological space that as a set coincides with $C_0^\infty(\Omega)$, the linear structure in $\mathcal{D}(\Omega)$ is defined with respect to addition of functions and multiplication by scalars from $\mathbb{R}$, and the topology in $\mathcal{D}(\Omega)$ is defined by means of the following convergence: a sequence $\{\varphi_k\}$ of functions from $\mathcal{D}(\Omega)$ converges to $\varphi \in \mathcal{D}(\Omega)$ in the space $\mathcal{D}(\Omega)$ if the following two conditions are satisfied:

1. $\varphi_k \rightrightarrows \varphi$ in $\Omega$ and $D^\alpha \varphi_k \rightrightarrows D^\alpha \varphi$ for any multiindex $\alpha$ of any order;
2. there is a compact set $K \subset \Omega$ such that $\operatorname{supp} \varphi_k \subset K$ for all $k$.

It is possible to show that this convergence is indeed topological, that is, given by a certain topology.

Any linear topological space $\mathcal{V}$ has a dual space $\mathcal{V}'$ that consists of continuous linear functionals on $\mathcal{V}$.

**Definition.** Any linear continuos functional $f : \mathcal{D}(\Omega) \to \mathbb{R}$ is called a *distribution* in $\Omega$ (or generalized functions). The set of all distributions in $\Omega$ is denoted by $\mathcal{D}'(\Omega)$. If $f \in \mathcal{D}'(\Omega)$ then the value of $f$ on a *test function* $\varphi \in \mathcal{D}(\Omega)$ is denoted by $(f, \varphi)$.

Any locally integrable function $f : \Omega \to \mathbb{R}$ can be regarded as a distribution as follows: it acts on any test function $\varphi \in \mathcal{D}(\Omega)$ by the rule

$$(f, \varphi) := \int_\Omega f \varphi \, dx. \tag{4.1}$$

Note that two locally integrable functions $f, g$ correspond to the same distribution if and only if $f = g$ almost everywhere, that is, if the set

$$\{x \in \Omega : f(x) \neq g(x)\}$$

has measure zero. We write shortly in this case

$$f = g \text{ a.e.} \tag{4.2}$$

Clearly, the relation (4.2) is an equivalence relation, that gives rise to equivalence classes of locally integrable functions. The set of all equivalence classes of locally integrable functions is denoted[1] by $L^1_{loc}(\Omega)$. The identity (4.1) establishes the injective mapping $L^1_{loc}(\Omega) \to \mathcal{D}'(\Omega)$ so that $L^1_{loc}(\Omega)$ can be regarded as a subspace of $\mathcal{D}'(\Omega)$.

There are distributions that do not correspond to any function, that is, the difference $\mathcal{D}'(\Omega) \setminus L^1_{loc}(\Omega)$ is not empty. For example, define the *delta-function* $\delta_{x_0}$ for any $x_0 \in \Omega$ as follows:

$$(\delta_{x_0}, \varphi) = \varphi(x_0).$$

Although historically $\delta_{x_0}$ is called delta-*function*, it is a distribution that does not correspond to any function.

**Definition.** Let $f \in \mathcal{D}'(\Omega)$. A *distributional partial derivative* $\partial_{x_i} f$ is a distribution that acts on test functions $\varphi \in \mathcal{D}(\Omega)$ as follows:

$$(\partial_{x_i} f, \varphi) = -(f, \partial_{x_i}\varphi), \tag{4.3}$$

where $\partial_{x_i}\varphi$ is the classical (usual) derivative of $\varphi$.

Note that the right hand side of (4.3) makes sense because $\partial_{x_i}\varphi \in \mathcal{D}(\Omega)$. Moreover, the right hand side of (4.3) is obviously a linear continuous functions in $\varphi \in \mathcal{D}(\Omega)$, which means that $\partial_{x_i} f$ exists *always* as a distribution.

In particular, the above definition applies to $f \in L^1_{loc}(\Omega)$. Hence, any function $f \in L^1_{loc}(\Omega)$ has always all partial derivatives $\partial_{x_i} f$ as distributions.

Let us show that if $f \in C^1(\Omega)$ then its classical derivative $\partial_{x_i} f$ coincides with the distributional derivative. For that, it suffices to check that the classical derivative $\partial_{x_i} f$ satisfies the identity (4.3). Indeed, have, for any $\varphi \in \mathcal{D}(\Omega)$,

$$\begin{aligned}(\partial_{x_i} f, \varphi) &= \int_\Omega \partial_{x_i} f\, \varphi\, dx \\ &= -\int_\Omega f \partial_{x_i}\varphi dx = -(f, \partial_{x_i}\varphi)\end{aligned}$$

where we have used integration by parts and $\varphi \in C_0^\infty(\Omega)$.

Let again $f \in L^1_{loc}(\Omega)$. If there is a function $g \in L^1_{loc}(\Omega)$ such that

$$(g, \varphi) = -(f, \partial_{x_i}\varphi) \quad \forall \varphi \in \mathcal{D}(\Omega), \tag{4.4}$$

then we see that $g$ satisfies the definition of the distributional derivative $\partial_{x_i} f$. In this case, the distribution $\partial_{x_i} f$ is given by a function $g$. The distributional derivative that corresponds to a $L^1_{loc}$ function is called a *weak derivative*. In other words, a function $g \in L^1_{loc}(\Omega)$ is called a weak derivative of $f$ in $x_i$ if $g$ satisfies (4.4).

Let $f \in \mathcal{D}'(\Omega)$. Applying successively the definition of distributional partial derivatives, we obtain higher order distributional partial derivatives $D^\alpha f$ for any multiindex $\alpha = (\alpha_1, ..., \alpha_n)$. It follows from (4.3) by induction in $|\alpha|$ that

$$(D^\alpha f, \varphi) = (-1)^{|\alpha|}(f, D^\alpha \varphi) \quad \forall \varphi \in \mathcal{D}(\Omega). \tag{4.5}$$

**Example.** Consider the function $f(x) = |x|$ in $\mathbb{R}$. This functions has the following classical derivative:

$$f'(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases} \tag{4.6}$$

[1] Sometimes $L^1_{loc}(\Omega)$ is loosely used to denote the set of all locally integrable functions in $\Omega$. However, in a strict sense, the elements of $L^1_{loc}(\Omega)$ are not functions but equivalence classes of functions.

and is not differentiable at $x = 0$. Let us show that the function (4.6) is the distributional (and, hence, weak) derivative of $|x|$. Note that the value of $f'(x)$ at $x = 0$ does not matter because the set $\{0\}$ has measure 0. For any $\varphi \in \mathcal{D}(\Omega)$ we have

$$\begin{aligned}
(f, \varphi') &= \int_{-\infty}^{\infty} f\varphi' dx \\
&= \int_0^{\infty} x\varphi' dx - \int_{-\infty}^{0} x\varphi' dx \\
&= \int_0^{\infty} x d\varphi - \int_{-\infty}^{0} x d\varphi \\
&= [x\varphi(x)]_0^{\infty} - \int_0^{\infty} \varphi dx - [x\varphi(x)]_{-\infty}^{0} + \int_{-\infty}^{0} \varphi dx \\
&= -\int_{-\infty}^{\infty} f'\varphi dx \\
&= -(f', \varphi),
\end{aligned}$$

where we have used that $x\varphi(x)$ vanishes at $x = 0, \infty, -\infty$.

**Example.** Let $f(x)$ be a continuous function on $\mathbb{R}$. Assume that $f$ is continuously differentiable in $\mathbb{R} \setminus M$ where $M = \{x_1, ...x_N\}$ is a finite set, and that $f'(x)$ has right and left limits as $x \to x_i$ for any $i = 1, ..., N$. Then we claim that the classical derivative $f'(x)$, defined in $\mathbb{R} \setminus M$, is also a weak derivative of $f$ (again, the values of $f'$ at the points of $M$ do not matter since $M$ has measure 0). Indeed, assuming that $x_1 < x_2 < ... < x_N$ and setting $x_0 = -\infty$ and $x_{N+1} = +\infty$, we obtain, for any $\varphi \in \mathcal{D}(\mathbb{R})$,

$$\begin{aligned}
(f, \varphi') &= \int_{-\infty}^{\infty} f\varphi' dx = \sum_{k=0}^{N} \int_{x_k}^{x_{k+1}} f\varphi' dx \\
&= \sum_{k=0}^{N} [f\varphi]_{x_i}^{x_{i+1}} - \sum_{k=0}^{N} \int_{x_k}^{x_{k+1}} f'\varphi dx = -\int_{-\infty}^{\infty} f'\varphi dx = -(f', \varphi),
\end{aligned}$$

where we have used that

$$\sum_{k=0}^{N} [f\varphi]_{x_i}^{x_{i+1}} = f\varphi(x_1) + (f\varphi(x_2) - f\varphi(x_1)) + ... + (f\varphi(x_N) - f\varphi(x_{N-1})) - f\varphi(x_N) = 0.$$

**Example.** Consider the function

$$f(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}$$

as element of $L^1_{loc}(\mathbb{R})$. Let us compute its distributional derivative. For any $\varphi \in \mathcal{D}(\mathbb{R})$ we have

$$(f', \varphi) = -(f, \varphi') = -\int_{-\infty}^{\infty} f\varphi' dx = -\int_0^{\infty} \varphi' dx = \varphi(0).$$

It follows that $f' = \delta$, where $\delta$ is the delta-function at 0, that is, $(\delta, \varphi) = \varphi(0)$.

**Example.** Consider the delta-function $\delta_{x_0}$ at an arbitrary point $x_0 \in \Omega$. We have by (4.5)

$$(D^\alpha \delta_{x_0}, \varphi) = (-1)^{|\alpha|} (\delta_{x_0}, D^a \varphi) = (-1)^{|\alpha|} D^\alpha \varphi (x_0) .$$

Hence, the distribution $D^\alpha \delta_{x_0}$ acts on test functions using evaluation of $D^\alpha \varphi$ at $x_0$.

**Example.** For the Laplace operator $\Delta = \sum_{i=1}^n \partial_{x_i x_i}$ we obtain from (4.5) the identity

$$(\Delta f, \varphi) = (f, \Delta \varphi) .$$

Consequently, a distribution $f \in \mathcal{D}'(\Omega)$ is harmonic, if $\Delta f = 0$ that is, if for any $\varphi \in \mathcal{D}(\Omega)$

$$(f, \Delta \varphi) = 0.$$

If $f \in C(\Omega)$ then this was the definition of a weakly harmonic function.

**Example.** Consider a function $f(x) = |x|^\alpha$ in $\mathbb{R}^n$. Observe that

$$\int_{B_1} f(x)\, dx = \omega_n \int_0^1 r^\alpha r^{n-1} dr = \omega_n \int_0^1 r^{\alpha+n-1} dr = \omega_n \left[ \frac{r^{\alpha+n}}{\alpha+n} \right]_0^1 < \infty$$

provided $\alpha + n > 0$, and similarly

$$\int_{B_1} f(x)\, dx = \infty$$

if $\alpha + n \le 0$. So, assuming $\alpha > -n$, we obtain that $f \in L^1_{loc}(\mathbb{R}^n)$. In $\mathbb{R}^n \setminus \{0\}$ we have

$$\partial_{x_i} f = \alpha |x|^{\alpha-1} \partial_{x_i} |x| = \alpha |x|^{\alpha-1} \frac{x_i}{|x|}.$$

Since $|\partial_{x_i} f| \le |\alpha| |x|^{\alpha-1}$, we see that if $\alpha > -n+1$, then also $\partial_{x_i} f \in L^1_{loc}(\mathbb{R}^n)$. Let us show that in this case the classical derivative $\partial_{x_i} f$ is a weak derivative, that is, for any $\varphi \in \mathcal{D}(\mathbb{R}^n)$

$$(\partial_{x_i} f, \varphi) = -(f, \partial_{x_i} \varphi) .$$

Since in $\mathbb{R}^n \setminus \{0\}$

$$\partial_{x_i} f \, \varphi + f \partial_{x_i} \varphi = \partial_{x_i} (f\varphi) ,$$

it suffices to prove that

$$\int_{\mathbb{R}^n} \partial_{x_i} (f\varphi)\, dx = 0.$$

Let $\operatorname{supp} \varphi \in B_R$. For any $0 < r < R$ we have by the divergence theorem

$$\int_{B_R \setminus \overline{B}_r} \partial_{x_i} (f\varphi)\, dx = \int_{\partial\left(B_R \setminus \overline{B}_r\right)} f\varphi \nu_i d\sigma = \int_{\partial B_r} f\varphi \nu_i d\sigma,$$

where $\nu$ is the outer normal unit vector field on the boundary of $B_R \setminus \overline{B}_r$. Observe that $\varphi$ and $\nu_i$ are uniformly bounded, whereas

$$\int_{\partial B_r} f d\sigma = r^\alpha \omega_n r^{n-1} = \omega_n r^{\alpha+n-1} \to 0 \text{ as } r \to 0.$$

Hence, also

$$\int_{\partial B_r} f\varphi \nu_i d\sigma \to 0 \quad \text{as } r \to 0,$$

which implies that

$$\int_{\mathbb{R}^n} \partial_{x_i} (f\varphi)\, dx = \lim_{r\to 0} \int_{B_R \setminus \overline{B}_r} \partial_{x_i} (f\varphi)\, dx = 0.$$

## 4.2 Sobolev spaces

As before, let $\Omega$ be an open subset of $\mathbb{R}^n$. Fix $p \in [1, \infty)$. A Lebesgue measurable function $f : \Omega \to \mathbb{R}$ is called *p-integrable* if

$$\int_\Omega |f|^p \, dx < \infty.$$

Two measurable functions in $\Omega$ (in particular, $p$-integrable functions) are called equivalent if

$$f = g \text{ a.s.}$$

This is an equivalence relation, and the set of all equivalence classes of $p$-integrable functions in $\Omega$ is denoted by $L^p(\Omega)$. It follows from the Hölder inequality, that $L^p(\Omega) \subset L^1_{loc}(\Omega)$. In particular, all the elements of $L^p(\Omega)$ can be regarded as distributions.

The set $L^p(\Omega)$ is a linear space over $\mathbb{R}$. Moreover, it is a Banach space (=complete normed space) with respect to the norm

$$\|f\|_{L^p} := \left( \int_\Omega |f|^p \, dx \right)^{1/p}.$$

The Banach spaces $L^p(\Omega)$ are called *Lebesgue spaces.*

The case $p = 2$ is of special importance because the space $L^2(\Omega)$ has inner product

$$(f, g) = \int_\Omega f g \, dx,$$

whose norm coincides with $\|f\|_2$ as

$$\|f\|_{L^2} = \left( \int_\Omega f^2 dx \right)^{1/2} = \sqrt{(f, f)}.$$

Hence, $L^2(\Omega)$ is a Hilbert space.

**Definition.** The *Sobolev space* $W^{1,2}(\Omega)$ is a subspace of $L^2(\Omega)$ defined by

$$W^{1,2}(\Omega) = \left\{ f \in L^2(\Omega) : \partial_{x_i} f \in L^2(\Omega) \text{ for all } i = 1, ..., n \right\}$$

where $\partial_{x_i} f$ denotes distributional derivative. Similarly define the Sobolev space $W^{k,2}$ for arbitrary $k \in \mathbb{N}$:

$$W^{k,2}(\Omega) = \left\{ f \in L^2(\Omega) : D^\alpha f \in L^2(\Omega) \text{ for all } \alpha \text{ with } |\alpha| \le k \right\},$$

where $D^\alpha f$ is distributional derivative.

If $D^\alpha f \in L^2(\Omega)$ then $D^\alpha f$ is called a *weak derivative.* In words, $W^{k,2}(\Omega)$ is a subspace of $L^2(\Omega)$ that consists of functions having all weak partial derivatives of the order $\le k$ in $L^2(\Omega)$. In the notation $W^{k,2}$ the letter "W" stands for "weak", the number 2 refers to $L^2$ and the number $k$ means the order of derivatives.

If one uses the space $L^p$ instead of $L^2$ then one obtains more general Sobolev spaces $W^{k,p}$.

It is easy to see that $C_0^\infty(\Omega) \subset W^{k,p}(\Omega)$ for any $k$ and $p$. Since we need only the
15.01.16 spaces $W^{k,2}$, we are going to use a short notation $W^k := W^{k,2}$.

For convenience we will use a vector-valued space $\vec{L}^2(\Omega)$ that consists of sequences of $n$ functions $\vec{f} = (f_1, ..., f_n)$ such that each $f_i \in L^2(\Omega)$. The inner product in this space is defined by

$$\left(\vec{f}, \vec{g}\right) := \sum_{i=1}^{n} (f_i, g_i),$$

and the corresponding norm is

$$\left\|\vec{f}\right\|_{L^2} = \sum_{i=1}^{n} \|f_i\|_{L^2}^2.$$

If $f \in W^1$ then the weak gradient

$$\nabla f = (\partial_{x_1} f, \partial_{x_2} f, ..., \partial_{x_n} f)$$

belongs to $\vec{L}^2(\Omega)$. Define in $W^1$ the following inner product

$$(f,g)_{W^1} = \int_\Omega \left(fg + \sum_{i=1}^{n} \partial_{x_i} f\ \partial_{x_i} g\right) dx = (f,g) + (\nabla f, \nabla g),$$

where $(\cdot,\cdot)$ is the inner product in $L^2$. Clearly, $(f,g)_{W^1}$ satisfies all the axioms of an inner product. The associated norm is given by

$$\|f\|_{W^1}^2 = \int_\Omega \left(f^2 + \sum_{i=1}^{n} (\partial_{x_i} f)^2\right) dx = \|f\|_{L^2}^2 + \|\nabla f\|_{L^2}^2.$$

**Proposition 4.1** *The space $W^1(\Omega)$ with the above inner product is a Hilbert space.*

**Proof.** We need to prove that $W^1$ is complete, that is, any Cauchy sequence $\{f_k\}$ in $W^1$ converges to an element of $W^1$. The fact that the sequence $\{f_k\}$ is Cauchy means that

$$\|f_k - f_m\|_{W^1} \to 0 \ \text{ as } k, m \to \infty$$

Since for any $f \in W^1$

$$\|f\|_{L^2} \le \|f\|_{W^1} \text{ and } \|\partial_{x_i} f\|_{L^2} \le \|f\|_{W^1},$$

we obtain that all sequences $\{f_k\}$, $\{\partial_{x_i} f_k\}$ are Cauchy in $L^2$. Since $L^2$ is complete, it follows that $\{f_k\}$ converges in $L^2$ to a function $f \in L^2$ and $\{\partial_{x_i} f_k\}$ converges in $L^2$ to a function $g_i \in L^2$. Hence, we have

$$\|f_k - f\|_{L^2} \to 0 \ \text{ as } k \to 0$$

and, for any $i = 1, ..., n$,

$$\|\partial_{x_i} f_k - g_i\| \to 0 \ \text{ as } k \to 0.$$

Let us show that, in fact, $g_i = \partial_{x_i} f$. Indeed, by definition of the weak derivative, we have, for any $\varphi \in \mathcal{D}(\Omega)$

$$(\partial_{x_i} f_k, \varphi) = -(f_k, \partial_{x_i} \varphi).$$

Here the brackets are values of the distributions $\partial_{x_i} f_k$ and $f_k$ on the test functions, but they coincide with the inner product in $L^2$. Hence, passing to the limit as $k \to \infty$ we obtain

$$(g_i, \varphi) = -(f, \partial_{x_i}\varphi)$$

which means that $g_i = \partial_{x_i} f$. Consequently, $f \in W^1(\Omega)$.

Finally, we obtain

$$\begin{aligned}
\|f_k - f\|^2_{W^1} &= \|f_k - f\|^2_{L^2} + \sum_{k=1}^{n} \|\partial_{x_i} f_k - \partial_{x_i} f\|^2_{L^2} \\
&= \|f_k - f\|^2_{L^2} + \sum_{k=1}^{n} \|\partial_{x_i} f_k - g_i\|^2_{L^2} \to 0
\end{aligned}$$

as $k \to \infty$, which implies that $f_k \to f$ in $W^1$. $\blacksquare$

## 4.3 Weak Dirichlet problem

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$. We consider a weak version of the following Dirichlet problem:

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

We will understand the Laplace operator $\Delta u$ in distributional sense so that solution $u$ can be sought in the class $L^1_{loc}(\Omega)$. However, within such a general class it is impossible to understand the boundary condition $u = 0$ pointwise as typically the boundary $\partial\Omega$ has Lebesgue measure zero. We are going to reduce the class of functions $u$ that allows to make sense out of boundary condition.

**Definition.** Define the subspace $W^1_0(\Omega)$ of $W^1(\Omega)$ as the closure of $\mathcal{D}(\Omega)$ in $W^1(\Omega)$.

Note that $C_0^\infty(\Omega)$ is dense in $L^2(\Omega)$, but in general not in $W^1(\Omega)$, so that $W^1_0(\Omega)$ is a proper subspace of $W^1(\Omega)$. So, the weak Dirichlet problem is stated as follows:

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ u \in W^1_0(\Omega) \end{cases} \tag{4.7}$$

where the condition $u \in W^1_0(\Omega)$ replaces the boundary condition $u = 0$ on $\partial\Omega$, and the equation $\Delta u = f$ is understood in distributional sense. Since $u \in W^1$, we have, for any $\varphi \in \mathcal{D}(\Omega)$,

$$(\Delta u, \varphi) = \left(\sum_{i=1}^{n} \partial_{x_i}\partial_{x_i} u, \varphi\right) = -\sum_{i=1}^{n} (\partial_{x_i} u, \partial_{x_i}\varphi) = -(\nabla u, \nabla\varphi).$$

Hence, we can rewrite the problem (4.7) in the following form:

$$\begin{cases} (\nabla u, \nabla\varphi) = -(f, \varphi) \quad \forall \varphi \in \mathcal{D}(\Omega) \\ u \in W^1_0(\Omega) \end{cases} \tag{4.8}$$

We claim that (4.8) is equivalent to

$$\begin{cases} (\nabla u, \nabla \varphi) = -(f, \varphi) & \forall \varphi \in W_0^1(\Omega), \\ u \in W_0^1(\Omega) \end{cases} \tag{4.9}$$

that is, the class of test functions $\varphi \in \mathcal{D}(\Omega)$ can be extended to $W_0^1(\Omega)$. To prove this, observe that the functional $\varphi \mapsto (f, \varphi)$ is a linear bounded functional in $W^1(\Omega)$ because

$$|(f, \varphi)| \leq \|f\|_{L^2} \|\varphi\|_{L^2} \leq \|f\|_{L^2} \|\varphi\|_{W^1}$$

and also the functional $\varphi \mapsto (\nabla u, \nabla \varphi)$ is a linear bounded functional in $W^1(\Omega)$ because

$$|(\nabla u, \nabla \varphi)| \leq \|\nabla u\|_{L^2} \|\nabla \varphi\|_{L^2} \leq \|\nabla u\|_{L^2} \|\varphi\|_{W^1}.$$

Hence, in the both sides of the identity $(\nabla u, \nabla \varphi) = -(f, \varphi)$ we can pass to the limit along any sequence of functions $\varphi$ convergent in $W^1$. Since $W_0^1(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ in $W^1(\Omega)$, the validity of this identity for all $\varphi \in \mathcal{D}(\Omega)$ implies that for $\varphi \in W_0^1(\Omega)$.

**Theorem 4.2** *If $\Omega$ is a bounded domain and $f \in L^2(\Omega)$ then the weak Dirichlet problem* (4.9) *has a unique solution.*

Before the proof we need the following lemma.

**Lemma 4.3** (Friedrichs-Poincaré inequality) *Let $\Omega$ be a bounded open set in $\mathbb{R}^n$. Then, for any $\varphi \in \mathcal{D}(\Omega)$ and for any index $i = 1, ..., n$,*

$$\int_\Omega \varphi^2 dx \leq (\operatorname{diam} \Omega)^2 \int_\Omega (\partial_{x_i} \varphi)\, dx. \tag{4.10}$$

**Proof.** Consider first the case $n = 1$. Consider the interval $I = (\inf \Omega, \sup \Omega)$ that has the same diameter as $\Omega$, and observe that any $\varphi \in \mathcal{D}(\Omega)$ belongs also to $\mathcal{D}(I)$. Hence, for the sake of inequality

$$\int_\Omega \varphi^2 dx \leq (\operatorname{diam} \Omega)^2 \int_\Omega (\varphi')^2\, dx \tag{4.11}$$

we can replace $\Omega$ with $I$. Hence, assume in the sequel that $\Omega$ is an open bounded interval. By shifting we can assume that $\Omega$ is an interval $(0, l)$, where $l = \operatorname{diam} \Omega$. For any $x \in (0, l)$, we have using $\varphi(0) = 0$, the fundamental theorem of calculus, and Cauchy-Schwarz inequality inequality, that

$$\varphi^2(x) = \left( \int_0^x \varphi'(s)\, ds \right)^2 \leq \left( \int_0^l |\varphi'(s)|\, ds \right)^2 \leq l \int_0^l (\varphi')^2(s)\, ds.$$

Since the right hand side does not depend on $x$, integrating this inequality in $x$, we obtain

$$\int_0^l \varphi^2(x)\, dx \leq l^2 \int_0^l (\varphi')^2(s)\, ds,$$

which is exactly (4.11).

In the case $n > 1$, denote by $y$ the $(n-1)$-dimensional vector that is obtained from $x$ by removing the component $x_i$. Denote by $\Omega_y$ the 1-dimentional section of $\Omega$ at the level $y$. Since the function $\varphi$ as a function of $x_i$ alone belongs to $\mathcal{D}(\Omega_y)$, the 1-dimensional Friedrichs inequality in the direction $x_i$ yields

$$\int_{\Omega_y} \varphi^2 dx_i \leq (\operatorname{diam} \Omega_y)^2 \int_{\Omega_y} (\partial_{x_i} \varphi)^2 \, dx_i \leq (\operatorname{diam} \Omega)^2 \int_{\Omega_y} (\partial_{x_i} \varphi)^2 \, dx_i.$$

Integrating in $y$ and using Fubini's theorem, we obtain (4.10). ∎

**Proof of Theorem 4.2.** It follows from Lemma 4.3 that, for any $v \in W_0^1(\Omega)$

$$\int_\Omega v^2 dx \leq C \int_\Omega |\nabla v|^2 \, dx$$

where $C = (\operatorname{diam} \Omega)^2$. In particular,

$$\|v\|_{W^1}^2 = \|v\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 \leq (C+1) \|\nabla v\|_{L^2}^2$$

Since also

$$\|v\|_{W^1}^2 \geq \|\nabla v\|_{L^2}^2 ,$$

it follows that the expression $\|\nabla v\|_{L^2}$ is an equivalent norm in the space $W_0^1$. This norm comes from bilinear form $(\nabla u, \nabla v)$ that is hence an inner product, and $W_0^1$ with this inner product is a Hilbert space.

Let us use the Riesz representation theorem: in any Hilbert space $H$, for any linear bounded functional $l : H \to \mathbb{R}$, there exists exactly one element $u \in H$ such that, for all $\varphi \in H$,

$$(u, \varphi)_H = l(\varphi).$$

Using this theorem for $H = W_0^1$ with the inner product $(u, v)_H = (\nabla u, \nabla v)_{L^2}$ and for the functional $l(\varphi) = -(f, \varphi)$, we obtain the existence and uniqueness of solution $u$ of (4.9). ∎

20.01.16

## 4.4 The Green operator

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$. Define an operator $G : L^2(\Omega) \to L^2(\Omega)$ as follows: for any $f \in L^2(\Omega)$, the function $u = Gf$ is the solution of the weak Dirichlet problem

$$\begin{cases} \Delta u = -f & \text{in } \Omega \\ u \in W_0^1(\Omega) \end{cases}$$

that is,

$$\begin{cases} (\nabla u, \nabla \varphi) = (f, \varphi) & \forall \varphi \in W_0^1(\Omega) \\ u \in W_0^1(\Omega) \end{cases} \tag{4.12}$$

The operator $G$ is called the *Green operator*. Of course, we know that $u \in W_0^1(\Omega)$ and, hence, $Gf \in W_0^1(\Omega)$ so that $G$ could be considered as an operator from $L^2(\Omega)$ to $W_0^1(\Omega)$, but it will be more convenient for us to regard $G$ as an operator in $L^2$.

**Theorem 4.4** *The operator $G$ is bounded, self-adjoint and positive definite.*

**Proof.** The boundedness means that

$$\|Gf\|_{L^2} \leq C \|f\|_{L^2}$$

for some constant $C$ and all $f \in L^2(\Omega)$. Set $u = Gf$ so that $u$ satisfies (4.12). Substituting into (4.12) $\varphi = u$, we obtain

$$\int_\Omega |\nabla u|^2 \, dx = \int_\Omega f u \, dx \leq \left( \int_\Omega f^2 dx \right)^{1/2} \left( \int_\Omega u^2 dx \right)^{1/2}.$$

Since $u \in W_0^1(\Omega)$, we have by the Friedrichs inequality

$$\int_\Omega u^2 dx \leq C \int_\Omega |\nabla u|^2 \, dx,$$

where $C$ depends on $\Omega$ only. Combining the above two inequalities, we obtain

$$\frac{1}{C} \int_\Omega u^2 dx \leq \left( \int_\Omega f^2 dx \right)^{1/2} \left( \int_\Omega u^2 dx \right)^{1/2},$$

whence

$$\int_\Omega u^2 dx \leq C^2 \int_\Omega f^2 dx,$$

which is equivalent to the boundedness of $G$.

The fact that $G$ is self-adjoint means that

$$(Gf, g) = (f, Gg) \quad \forall f, g \in L^2(\Omega).$$

To prove this, set $u = Gf$ and $v = Gg$. Setting in (4.12) $\varphi = v$, we obtain

$$(\nabla u, \nabla v) = (f, v).$$

Similarly, using the weak Dirichlet problem for $v$, we obtain

$$(\nabla v, \nabla u) = (g, u).$$

Since the left hand sides of these identities coincide, we obtain that

$$(g, u) = (f, v),$$

which is equivalent to the self-adjointness of $G$.

The positive definiteness of $G$ means that $(Gf, f) > 0$ for all non-zero $f \in L^2(\Omega)$. Indeed, setting $u = Gf$ we obtain from (4.12) with $\varphi = u$

$$(\nabla u, \nabla u) = (f, u),$$

whence

$$(Gf, f) = (f, u) = (\nabla u, \nabla u) \geq 0.$$

Let us show that, in fact, $(\nabla u, \nabla u) > 0$. Indeed, if $(\nabla u, \nabla u) = 0$ then $\nabla u = 0$ a.e.. Hence, for any $\varphi \in W_0^1(\Omega)$, we obtain $(\nabla u, \nabla \varphi) = 0$ whence by (4.12) $(f, \varphi) = 0$. It follows that $f = 0$, which contradicts the assumption that $f$ is non-zero. ■

We are going to consider the eigenfunctions of the operator $G$, that is, non-zero functions $v \in L^2(\Omega)$ that satisfy $Gv = \mu v$ for some $\mu \in \mathbb{R}$. Since the operator $G$ is positive definite, we obtain that all its eigenvalues $\mu$ are positive.

Consider also the weak eigenvalue problem

$$\begin{cases} \Delta v + \lambda v = 0 & \text{in } \Omega \\ v \in W_0^1(\Omega) \end{cases}$$

that is equivalent to

$$\begin{cases} (\nabla v, \nabla \varphi) = \lambda (v, \varphi) & \forall \varphi \in \mathcal{D}(\Omega) \\ v \in W_0^1(\Omega) \end{cases} \tag{4.13}$$

As we already know, the class $\mathcal{D}(\Omega)$ of test functions $\varphi$ can be replaced by $W_0^1(\Omega)$.

**Lemma 4.5** *A function $v \in L^2(\Omega)$ is an eigenfunction of $G$ with the eigenvalue $\mu$ if and only if $v$ is an eigenfunction of* (4.13) *with $\lambda = \frac{1}{\mu}$.*

**Proof.** Let $v$ be an eigenfunction of $G$. Since $Gv \in W_0^1(\Omega)$ and $Gv = \mu v$, it follows that also $v \in W_0^1(\Omega)$. Setting $u = Gv$, we obtain from (4.12) that $u$ satisfies

$$(\nabla u, \nabla \varphi) = (v, \varphi) \quad \forall \varphi \in \mathcal{D}(\Omega).$$

Since $u = \mu v$, we obtain

$$\mu (\nabla v, \nabla \varphi) = (v, \varphi)$$

whence (4.13) follows with $\lambda = \frac{1}{\mu}$.

Let $v$ be an eigenvalue of (4.13). Setting $\varphi = v$ we obtain

$$\int_\Omega |\nabla v|^2 \, dx = \lambda \int_\Omega v^2 dx.$$

Since by Friedrichs inequality

$$\int_\Omega v^2 dx \le C \int_\Omega |\nabla v|^2 \, dx,$$

we obtain that $\lambda \ge \frac{1}{C}$, in particular, $\lambda > 0$. By (4.13), function $v$ solves the weak Dirichlet problem (4.12) with the right hand side $f = \lambda v$, which implies that $G(\lambda v) = v$, whence it follows that $Gv = \mu v$ with $\mu = \frac{1}{\lambda}$. ∎

## 4.5 Compact embedding theorem

Given two Banach spaces $X, Y$, an operator $A : X \to Y$ is called compact if, for any bounded sequence $\{x_k\} \subset X$, the sequence $\{Ax_k\}$ hat a convergence subsequence in $Y$.

Assume that the operator $A$ is bounded, that is, $\|A\| < \infty$. Then the sequence $\{Ax_k\}$ is bounded in $Y$. If $\dim Y < \infty$ then every bounded sequence in $Y$ has a convergent subsequence, which follows from theorem of Bolzano-Weierstrass. However, for infinite dimensional spaces this is not the case. For example, let $Y$ be an $\infty$-dimensional Hilbert space and let $\{v_k\}_{k=1}^{\infty}$ be an orthonormal sequence in $Y$. Then $\{v_k\}$ is bounded, but no subsequence is Cauchy because, for all distinct $k, m$, we have

$$\|v_k - v_m\|^2 = (v_k - v_m, v_k - v_m) = \|v_k\|^2 - 2(v_k, v_m) + \|v_m\|^2 = 2.$$

Hence, no subsequence of $\{v_k\}$ converges in $Y$. An explicit example is as follows: $Y = L^2(-\pi, \pi)$ and $v_k = \frac{1}{\sqrt{\pi}} \sin kx$.

Hence, the point of a compact operator is that it maps a bounded sequence into one that has a convergent subsequence.

The following are simple properties of compact operators that we mention without proof.

1. A compact operator is bounded.

2. Composition of a compact operator with a bounded operator is compact.

Out goal will be to prove that the Green operator in compact, which will allow then to invoke the Hilbert-Schmidt theorem about diagonalization of self-adjoint compact operators. A crucial step for that is the following theorem.

**Theorem 4.6** (Compact embedding theorem) *Let $\Omega$ be a bounded domain in $\mathbb{R}^n$. Then the natural embedding $W_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is a compact operator.*

Before the proof, let us revise some fact the theory of multidimensional Fourier series. For the further proof we need some knowledge of multidimensional Fourier series. Recall that any $f \in L^2(-\pi, \pi)$ allows expansion into the Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

that converges in $L^2(-\pi, \pi)$. Setting $c_k = \frac{1}{2}(a_k - ib_k)$ allows to rewrite the series as follows:

$$f(x) = c_0 + \sum_{k=1}^{\infty} 2\operatorname{Re}\left(c_k e^{ikx}\right) = c_0 + \sum_{k=1}^{\infty} \left(c_k e^{ikx} + \overline{c_k} e^{-ikx}\right) = \sum_{k\in\mathbb{Z}} c_k e^{ikx},$$

where $c_k$ is extended to all $k \in \mathbb{Z}$ in an obvious way.

Consider now $n$-dimensional cube $Q = (-\pi, \pi)^n$ and the space $L^2(Q)$ over $\mathbb{C}$. For any $\xi \in \mathbb{Z}^n$ consider the function $x \mapsto e^{i\xi\cdot x}$ (where $\xi \cdot x = \sum_{j=1}^n \xi_j x_j$) that is clearly in $L^2(Q)$. It is possible to prove that the sequence the sequence $\left\{e^{i\xi\cdot x}\right\}_{\xi\in\mathbb{Z}^n}$ is an orthogonal basis in $L^2(Q)$. The fact that this sequence is orthogonal is easy to verify: if $\xi$ and $\eta$ are distinct elements of $\mathbb{Z}^n$ then the inner product of $e^{i\xi\cdot x}$ and $e^{i\eta\cdot x}$ in $L^2(Q)$ is

$$\left(e^{i\xi\cdot x}, e^{i\eta\cdot x}\right) = \int_Q e^{i\xi\cdot x}\overline{e^{i\eta\cdot x}}dx = \int_Q e^{i\xi\cdot x}e^{-i\eta\cdot x}dx = \int_Q e^{i(\xi-\eta)\cdot x}dx = 0, \tag{4.14}$$

because the integral splits by Fubini's theorem into the product of the integrals

$$\int_{-\pi}^{\pi} e^{i\left(\xi_j-\eta_j\right)x_j}dx_j = \frac{1}{i\left(\xi_j-\eta_j\right)}\left[e^{i\left(\xi_j-\eta_j\right)x_j}\right]_{-\pi}^{\pi} = 0,$$

where the last computation is valid whenever $\xi_j \neq \eta_j$. Note also that

$$\left\|e^{i\xi\cdot x}\right\|_{L^2}^2 = \int_Q e^{i\xi\cdot x}\overline{e^{i\eta\cdot x}}dx = \int_Q dx = (2\pi)^n . \tag{4.15}$$

The fact that the sequence $\left\{e^{i\xi\cdot x}\right\}_{\xi\in\mathbb{Z}^n}$ is indeed a basis in $L^2(Q)$ is non-trivial and can be verified by induction in $n$.

Hence, any function $f \in L^2(Q)$ admits an expansion in this basis, and the coefficients of this expansion will be denoted by $\hat{f}(\xi)$, that is,

$$f(x) = \sum_{\xi\in\mathbb{Z}^n} \hat{f}(\xi)\, e^{i\xi\cdot x}. \tag{4.16}$$

The series (4.16) is called *$n$-dimensional Fourier series*, and it converges in the norm of $L^2(Q)$ for any $f \in L^2(Q)$. Taking an inner product of the series (4.16) with $e^{i\xi\cdot x}$ for some fixed $\xi \in \mathbb{Z}^n$ and using (4.14) and (4.15) we obtain that

$$\left(f, e^{i\xi\cdot x}\right) = \hat{f}(\xi)\left(e^{i\xi\cdot x}, e^{i\xi\cdot x}\right),$$

which implies the following explicit expression for $\hat{f}(\xi)$:

$$\hat{f}(\xi) = \frac{1}{(2\pi)^n}\int_Q f(x)\, e^{-i\xi\cdot x}dx. \tag{4.17}$$

Similarly, compute the norm $\|f\|_{L^2}^2$ by taking the inner product of the series (4.16) with itself term by term. Then (4.14) and (4.15) imply that

$$\begin{aligned}(f,f) &= \left(\sum_{\xi\in\mathbb{Z}^n}\hat{f}(\xi)\, e^{i\xi\cdot x}, \sum_{\eta\in\mathbb{Z}^n}\hat{f}(\eta)\, e^{i\eta\cdot x}\right) = \sum_{\xi,\eta\in\mathbb{Z}^n}\int_Q \hat{f}(\xi)\, e^{i\xi\cdot x}\overline{\hat{f}(\eta)}e^{-i\eta\cdot x}dx \\ &= (2\pi)^n\sum_{\xi\in\mathbb{Z}^n}\left|\hat{f}(\xi)\right|^2,\end{aligned}$$

and we obtain *Parseval's identity*:

$$\|f\|_{L^2(Q)}^2 = (2\pi)^n \sum_{\xi\in\mathbb{Z}^n}\left|\hat{f}(\xi)\right|^2.$$

Consider the following space of sequences on $\mathbb{Z}^n$:

$$l^2 = l^2(\mathbb{Z}^n) = \left\{ g : \mathbb{Z}^n \to \mathbb{C} \,\middle|\, \sum_{\xi\in\mathbb{Z}^n}|g(\xi)|^2 < \infty \right\}.$$

Then $l^2$ is a Hilbert space over $\mathbb{C}$ with the Hermitian inner product

$$(g,h)_{l^2} = \sum_{\xi\in\mathbb{Z}^n} g(\xi)\,\overline{h(\xi)}$$

and the corresponding norm

$$\|g\|_{l^2}^2 = \sum_{\xi\in\mathbb{Z}^n}|g(\xi)|^2 .$$

Hence, Parseval's identity can be restated as follows: for any $f \in L^2(Q)$ we have $\hat{f} \in l^2(\mathbb{Z}^n)$ and

$$\|f\|^2_{L^2(Q)} = (2\pi)^n \left\|\hat{f}\right\|^2_{l^2}. \tag{4.18}$$

The mapping $f \mapsto \hat{f}$ is called *discrete Fourier transform.* Let us denote it by $\mathcal{F}$, that is,

$$\begin{aligned} \mathcal{F} &: L^2(Q) \to l^2(\mathbb{Z}^n) \\ \mathcal{F}f &= \hat{f}. \end{aligned}$$

By (4.18) this mapping is isometry (up to the constant factor $(2\pi)^n$), in particular, injective. In fact, it is also surjective since for any $g \in l^2(\mathbb{Z}^n)$ the series

$$\sum_{\xi \in \mathbb{Z}^n} g(\xi) e^{i\xi \cdot x}$$

converges in $L^2(Q)$ and, hence, gives $\mathcal{F}^{-1}g$. Hence, $\mathcal{F}$ is an isomorphism of the Hilbert spaces $L^2(Q)$ and $l^2(\mathbb{Z}^n)$.

If $f \in \mathcal{D}(\Omega)$ then, for any multiindex $\alpha$, the partial derivative $D^\alpha f$ is also in $\mathcal{D}(Q)$, and the Fourier series of $D^\alpha f$ is given by

$$D^\alpha f(x) = \sum_{\xi \in \mathbb{Z}^n} (i\xi)^\alpha \hat{f}(\xi) e^{i\xi x}, \tag{4.19}$$

where

$$(i\xi)^\alpha := (i\xi_1)^{\alpha_1} \dots (i\xi_n)^{\alpha_n}.$$

Indeed, the Fourier coefficients of $D^\alpha f$ are given by

$$\begin{aligned} \int_Q D^\alpha f(x) e^{-i\xi \cdot x} dx &= (-1)^{|\alpha|} \int_Q f(x) D^\alpha e^{-i\xi \cdot x} dx \\ &= (-1)^{|\alpha|} \int_Q f(x) (-i\xi)^\alpha e^{-i\xi \cdot x} dx = (i\xi)^\alpha \hat{f}(\xi), \end{aligned}$$

where we have used integration by parts. As we see from (4.5), the differential operator $D^\alpha$ becomes in Fourier transform a multiplication operator by $(i\xi)^\alpha$, which can be written as follows:

$$\mathcal{F} \circ D^\alpha = (i\xi)^\alpha \circ \mathcal{F}.$$

The function $(i\xi)^\alpha$ is called the *symbol* of the differential operator $D^\alpha$.

It follows from Parseval's identity that

$$\|D^\alpha f\|^2_{L^2} = (2\pi)^n \sum_{\xi \in \mathbb{Z}^n} |\xi^\alpha|^2 \left|\hat{f}(\xi)\right|^2. \tag{4.20}$$

In particular, we have, for any $j = 1, ..., n$

$$\partial_{x_j} f = \sum_{\xi \in \mathbb{Z}^n} (i\xi_j) \hat{f}(\xi) e^{i\xi \cdot x}.$$

By Parseval's identity, we obtain

$$\|\nabla f\|_{L^2(\Omega)}^2 = \sum_{j=1}^{n} \left\|\partial_{x_j} f\right\|_{L^2}^2 = (2\pi)^n \sum_{\xi \in \mathbb{Z}^n} |\xi|^2 \left|\hat{f}(\xi)\right|^2 .$$

**Proof of Theorem 4.6.** Recall that $W_0^1(\Omega) \subset L^2(\Omega)$. By a natural embedding from $W_0^1(\Omega)$ to $L^2(\Omega)$ we mean the following trivial mapping: for each $f \in W_0^1(\Omega)$, its image is the same function $f$ but considered as an element of $L^2(\Omega)$. The fact that the embedding $W_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact means the following: for any sequence $\{f_k\}$ of functions from $W_0^1(\Omega)$ that is bounded in the norm of $W^1$, there is a subsequence that converges in $L^2(\Omega)$. Note that if a sequence $\{f_k\}$ is bounded in $L^2(\Omega)$ then it does not have to contain a subsequence convergent in $L^2(\Omega)$ as it was mentioned above. Hence, the point of this theorem is that the boundedness of $\{f_k\}$ in the norm of $W^1$ is a stronger hypothesis, that does imply the existence of a convergent subsequence in
22.01.16 $L^2$.

Now let $\{f_k\}$ be a bounded sequence in $W_0^1(\Omega)$. Since $\mathcal{D}(\Omega)$ is dense in $W_0^1(\Omega)$, we can choose for any $k$ a function $g_k \in \mathcal{D}(\Omega)$ such that $\|f_k - g_k\|_{W^1} < \frac{1}{k}$. Then $\{g_k\}$ is bounded in $W_0^1(\Omega)$, and if $\{g_k\}$ contains a subsequence $\{g_{k_j}\}$ that converges in $L^2(\Omega)$, then $\{f_{k_j}\}$ also converges in $L^2(\Omega)$ to the same limit because $\|f_k - g_k\|_{L^2} \to 0$ as $k \to \infty$. Renaming $g_k$ back to $f_k$, we can assume without loss of generality that all functions $f_k$ belong to $\mathcal{D}(\Omega)$.

Since $\Omega$ is bounded, $\Omega$ is contained in a cube $Q = (-a, a)^n$ for large enough $a$. Since $\mathcal{D}(\Omega) \subset \mathcal{D}(Q)$, we can forget about $\Omega$ and work with the domain $Q$ instead. Finally, without loss of generality, we can assume $Q = (-\pi, \pi)^n$. Hence, we assume in the sequel that all functions $f_k$ belong to $\mathcal{D}(Q)$ and the sequence $\{f_k\}$ is bounded in $W^1(Q)$, that is, there is a constant $C$ such that, for all $k \geq 1$,

$$\|f_k\|_{L^2(Q)}^2 < C \quad \text{and} \quad \|\nabla f_k\|_{L^2(Q)}^2 < C.$$

By Parseval's identity, it follows that, for all $k \geq 1$,

$$\sum_{\xi \in \mathbb{Z}^n} \left|\hat{f}_k(\xi)\right|^2 < C \quad \text{and} \quad \sum_{\xi \in \mathbb{Z}^n} |\xi|^2 \left|\hat{f}_k(\xi)\right|^2 < C. \tag{4.21}$$

We need to show that there exists a subsequence $\{f_{k_j}\}$ that converges in $L^2(Q)$, that is, this subsequence is a Cauchy sequence in $L^2(Q)$. In the view of Parseval's identity, the latter is equivalent to the fact that subsequence $\left\{\hat{f}_{k_j}\right\}$ is Cauchy in $l^2(\mathbb{Z}^n)$.

It follows from (4.21) that, for each $\xi \in \mathbb{Z}^n$, the sequence $\left\{\hat{f}_k(\xi)\right\}_{k=1}^{\infty}$ of complex numbers is bounded. By theorem of Bolzano-Weierstrass, this $\mathbb{C}$-valued sequence has a convergence subsequence $\left\{\hat{f}_{k_j}(\xi)\right\}$. Using the diagonal process, we will select a subsequence $\left\{\hat{f}_{k_j}\right\}$ that converges pointwise at all $\xi \in \mathbb{Z}^n$, not just at one $\xi$. Since the set $\mathbb{Z}^n$ is countable, we can enumerate all the elements of $\mathbb{Z}^n$ by $\xi_1, \xi_2, \ldots$. Choose first a subsequence of indices

$$k_{11}, \quad k_{12}, \quad k_{13}, \ldots \text{ such that } \left\{\hat{f}_{k_{1j}}(\xi_1)\right\}_{j=1}^{\infty} \text{ converges.} \tag{4.22}$$

Then from the sequence (4.22) choose a subsequence

$$k_{21},\quad k_{22},\quad k_{23},\ldots \text{ such that } \left\{\hat{f}_{k_{2j}}(\xi_2)\right\}_{j=1}^{\infty} \text{ converges.} \tag{4.23}$$

From the sequence (4.23) choose a subsequence

$$k_{31},\quad k_{32},\quad k_{33},\ldots \text{ such that } \left\{\hat{f}_{k_{3j}}(\xi_3)\right\}_{j=1}^{\infty} \text{ converges,} \tag{4.24}$$

and so on, for all $\xi_i$. We obtain a double sequence $\{k_{ij}\}$ of the indices with the above properties. We claim that the *diagonal* sequence

$$k_{11},\quad k_{22},\quad k_{33},\ldots$$

has the property that $\left\{\hat{f}_{k_{jj}}(\xi)\right\}_{j=1}^{\infty}$ converges at all $\xi \in \mathbb{Z}^n$. Indeed, the sequence $\{k_{jj}\}$ is a subsequence of any sequence (4.22), (4.23), (4.24), etc., provided we neglect the first $i-1$ terms. Since the convergence of a sequence does not depend on a finite number of terms, we obtain that $\left\{\hat{f}_{k_{jj}}(\xi_i)\right\}_{j=1}^{\infty}$ converges for any $\xi_i$. Since all $\xi_i$ exhaust $\mathbb{Z}^n$, we obtain that $\left\{\hat{f}_{k_{jj}}\right\}_{j=1}^{\infty}$ converges pointwise on $\mathbb{Z}^n$.

To simplify notation and without loss of generality, we can assume that the whole sequence $\left\{\hat{f}_k\right\}$ converges pointwise at all $\xi \in \mathbb{Z}^n$. Hence, for any $\xi$, the sequence $\left\{\hat{f}_k(\xi)\right\}$ of complex numbers is Cauchy. Let us prove that $\left\{\hat{f}_k\right\}$ is Cauchy in $l^2(\mathbb{Z}^n)$. Indeed, for all positive integers $k, m, r$ we have

$$\begin{aligned}\left\|\hat{f}_k - \hat{f}_m\right\|_{l^2}^2 &= \sum_{\xi\in\mathbb{Z}^m}\left|\hat{f}_k(\xi) - \hat{f}_m(\xi)\right|^2\\ &= \sum_{|\xi|<r}\left|\hat{f}_k(\xi) - \hat{f}_m(\xi)\right|^2 + \sum_{|\xi|\geq r}\left|\hat{f}_k(\xi) - \hat{f}_m(\xi)\right|^2.\end{aligned}$$

Since the first sum is finite and each summands goes to 0 as $k, m \to \infty$, the first sum goes to 0 as $k, m \to \infty$. The second sum we estimate as follows:

$$\sum_{|\xi|\geq r}\left|\hat{f}_k(\xi) - \hat{f}_m(\xi)\right|^2 \leq 2\sum_{|\xi|\geq r}\left|\hat{f}_k(\xi)\right|^2 + 2\sum_{|\xi|\geq r}\left|\hat{f}_m(\xi)\right|^2$$

and by (4.21)

$$\sum_{|\xi|\geq r}\left|\hat{f}_k(\xi)\right|^2 \leq \frac{1}{r^2}\sum_{|\xi|\geq r}|\xi|^2\left|\hat{f}_k(\xi)\right|^2 \leq \frac{C}{r^2}.$$

Hence,

$$\left\|\hat{f}_k - \hat{f}_m\right\|_{l^2}^2 \leq \sum_{|\xi|<r}\left|\hat{f}_k(\xi) - \hat{f}_m(\xi)\right|^2 + \frac{4C}{r^2},$$

which implies as $k, m \to \infty$ that

$$\limsup_{k,m\to\infty}\left\|\hat{f}_k - \hat{f}_m\right\|_{l^2}^2 \leq \frac{4C}{r^2}.$$

Since $r$ can be chosen arbitrarily large, it follows that

$$\lim_{k,m\to\infty} \left\| \hat{f}_k - \hat{f}_m \right\|_{l^2}^2 = 0,$$

which finishes the proof. ■

## 4.6 Eigenvalues and eigenfunctions of the weak Dirichlet problem

Now we can prove the main theorem in this chapter. Consider again the weak eigenvalue problem in a bounded domain $\Omega \subset \mathbb{R}^n$:

$$\begin{cases} \Delta v + \lambda v = 0 & \text{in } \Omega \\ v \in W_0^1(\Omega) \end{cases}$$

that is,

$$\begin{cases} (\nabla v, \nabla \varphi) = \lambda (v, \varphi) & \forall \varphi \in W_0^1(\Omega) \\ v \in W_0^1(\Omega) \end{cases} \tag{4.25}$$

**Theorem 4.7** *Let $\Omega$ be a bounded domain in $\mathbb{R}^n$. There is an orthonormal basis $\{v_k\}_{k=1}^{\infty}$ in $L^2(\Omega)$ that consists of eigenfunctions of (4.25). The corresponding eigenvalues $\lambda_k$ are positive reals, and the sequence $\{\lambda_k\}$ is monotone increasing and diverges to $+\infty$ as $k \to \infty$.*

**Proof.** We use the Green operator $G$ acting in $L^2(\Omega)$, that was constructed in Section 4.4. Recall that if $f \in L^2(\Omega)$ then the function $u = Gf$ solves the weak Dirichlet problem

$$\begin{cases} \Delta u = -f & \text{in } \Omega \\ u \in W_0^1(\Omega) \end{cases} \tag{4.26}$$

By Theorem 4.4, the operator $G$ is bounded, selfadjoint and positive definite, and by Lemma 4.5, function $v$ is an eigenfunction of (4.25) with eigenvalue $\lambda$ if and only if $v$ is an eigenfunction of the operator $G$ with the eigenvalue $\mu = \frac{1}{\lambda}$.

Hence, it suffices to prove that there is an orthonormal basis $\{v_k\}_{k=1}^{\infty}$ in $L^2(\Omega)$ that consists of eigenfunctions of $G$, and the corresponding sequence of eigenvalues $\{\mu_k\}$ is monotone decreasing and converges to 0.

The crucial observation is that the operator $G$ is compact. Indeed, let us represent $G$ as composition of two operators:

$$L^2(\Omega) \xrightarrow{\tilde{G}} W_0^1(\Omega) \xrightarrow{I} L^2(\Omega),$$

where $I$ is the natural embedding and $\tilde{G}$ is defined as follows: for any $f \in L^2(\Omega)$, the function $u = \tilde{G}f \in W_0^1(\Omega)$ is the solution of the weak Dirichlet problem (4.26). Of course, the function $Gf$ was also defined as solution of the same problem, so that $Gf = \tilde{G}f$, but $Gf$ is regarded as an element of $L^2(\Omega)$ whereas $\tilde{G}f$ is regarded as an element of $W_0^1(\Omega)$.

We claim that the operator $\tilde{G}$ is bounded, that is, the solution $u = \tilde{G}f$ satisfies the estimate

$$\|u\|_{W^1} \leq C \|f\|_{L^2} \tag{4.27}$$

for some constant $C$. Recall that by Theorem 4.4 the operator $G$ is bounded in $L^2(\Omega)$, which means that

$$\|u\|_{L^2} \leq C \|f\|_{L^2}. \tag{4.28}$$

Clearly, the required inequality (4.27) is stronger than (4.28). The inequality (4.27) is stated in Exercise 69 and can be proved using the same argument as (4.28).

The embedding operator $I$ is compact by Theorem 4.6. Hence, the composition $G = I \circ \tilde{G}$ is a compact operator.

Now we now that $G$ is a compact self-adjoint operator in $L^2(\Omega)$. We are left to apply the Hilbert-Schmidt theorem that claims the following: if $H$ is a separable $\infty$-dimensional Hilbert space and $A$ is a compact self-adjoint operator in $H$, then there exists an orthonormal basis $\{v_k\}_{k=1}^{\infty}$ in $H$ that consists of the eigenvectors of $A$, the corresponding eigenvalues $\mu_k$ are real, and the sequence $\{\mu_k\}$ goes to 0 as $k \to \infty$. Applying this for $A = G$, we obtain these statements for $G$. In addition, we know that the eigenvalues $\mu_k$ of $G$ are positive. Since the sequence $\{\mu_k\}$ converges to 0, it is possible to rearrange it to become monotone decreasing, which finishes the proof. ■

27.01.16

**Remark.** The fact that the sequence $\{v_k\}$ in Theorem 4.7 is orthogonal is a consequence of the following simple fact: if $v', v''$ are two eigenfunctions of (4.25) with distinct eigenvalues $\lambda', \lambda''$ then $v'$ and $v''$ are orthogonal, that is $(v', v'')_{L^2} = 0$ (cf. Exercise 64).

**Remark.** If we have a sequence $\{v_k\}$ of eigenfunctions of (4.25) that forms an orthogonal basis in $L^2(\Omega)$, then the corresponding sequence $\{\lambda_k\}$ of eigenvalues exhausts *all* the eigenvalues of (4.25). Indeed, if $\lambda$ is one more eigenvalue with the eigenfunction $v$ then the condition $\lambda \neq \lambda_k$ implies that $v$ is orthogonal to $v_k$. Hence, $v$ is orthogonal to all elements of the basis $\{v_k\}$, which implies that $v = 0$. This contradictions proves the claim.

**Remark.** Note that the sequence $\{\lambda_k\}$ can have repeated terms, as we will see in examples below. If a number $\lambda$ appears in $\{\lambda_k\}$ exactly $m$ times then $m$ is called the multiplicity of $\lambda$ (in particular, if $\lambda$ is not eigenvalue then $m = 0$). Since $\lambda_k \to \infty$ as $k \to \infty$, we see that the multiplicity is always finite.

The sequence $\{\lambda_k\}_{k=1}^{\infty}$ is called the *spectrum* of the Dirichlet problem in $\Omega$ or simply the spectrum of $\Omega$.

**Remark.** Consider the set $\Omega$ of the form $\Omega = U \times W$ where $U$ is an open subset of $\mathbb{R}^m$ and $W$ is an open subset of $\mathbb{R}^{n-m}$. The points of $\Omega$ are the couples $(x, y)$ where $x \in U$ and $y \in W$. Let us find eigenfunctions in $\Omega$ using the method of separation of variables. Namely, search for the eigenfunction $v$ of $\Omega$ in the form $v(x, y) = u(x) w(y)$, where $u$ and $w$ are functions in $U$ and $W$. Since

$$\Delta v = \Delta_x v + \Delta_y w = (\Delta u) w + u \Delta w,$$

the equation $\Delta v + \lambda v = 0$ becomes

$$(\Delta u) w + u \Delta w + \lambda u w = 0$$

that is,

$$\frac{\Delta u}{u}(x)+\frac{\Delta w}{w}(y)=-\lambda.$$

It follows that the both functions $\frac{\Delta u}{u}$ and $\frac{\Delta w}{w}$ must be constants, say

$$\frac{\Delta u}{u}=-\alpha \quad \text{and} \quad \frac{\Delta w}{w}=-\beta,$$

where $\alpha+\beta=\lambda$. The boundary $\partial\Omega$ consists of the union of $\partial U\times W$ and $U\times\partial W$. Therefore, to ensure the boundary condition $v=0$ on $\partial\Omega$, let us assume that

$$u|_{\partial U}=0 \;\text{ and }\; w|_{\partial W}=0.$$

Hence, $u$ is solution of the eigenvalue problem

$$\begin{cases} \Delta u+\alpha u=0 & \text{in } U \\ u|_{\partial U}=0 \end{cases}$$

and $w$ is solution of the eigenvalue problem

$$\begin{cases} \Delta w+\beta w=0 & \text{in } W \\ w|_{\partial W}=0 \end{cases}$$

Assuming that the first problem has the eigenvalues $\{\alpha_k\}_{k=1}^{\infty}$ and the eigenfunctions $\{u_k\}_{k=1}^{\infty}$ that form an orthonormal basis in $L^2(U)$, and the second problem has the eigenvalues $\{\beta_l\}_{l=1}^{\infty}$ and the eigenfunctions $\{w_l\}_{l=1}^{\infty}$ that form an orthonormal basis in $L^2(W)$, we obtain the following eigenfunctions of $\Omega$

$$v_{k,l}(x,y)=u_k(x)\,w_l(y)$$

and the eigenvalues

$$\lambda_{k,l}=\alpha_k+\beta_l.$$

It is possible to prove that the double sequence $\{u_k w_l\}$ is indeed a basis in $L^2(\Omega)$ so that we have found all eigenvalues of $\Omega$.

**Example.** Let us compute the eigenvalues of the Laplace operator in the interval $\Omega=(0,a)$. The eigenvalue problem is

$$\begin{cases} v''+\lambda v=0 \text{ in } (0,a) \\ v(0)=v(a)=0. \end{cases}$$

The ODE $v''+\lambda v=0$ has for positive $\lambda$ the general solution

$$v(x)=C_1\cos\sqrt{\lambda}x+C_2\sin\sqrt{\lambda}x.$$

At $x=0$ we obtain that $C_1=0$, and at $x=a$ we obtain that

$$\sin\sqrt{\lambda}a=0,$$

which gives all solutions

$$\lambda=\left(\frac{\pi k}{a}\right)^2,\quad k\in\mathbb{N}.$$

Hence, we obtain the sequence of eigenvalues $\lambda_k = \left(\frac{\pi k}{a}\right)^2$ and the corresponding eigenfunctions $v_k(x) = \sin\frac{\pi k x}{a}$. The sequence $\left\{\sin\frac{\pi k x}{a}\right\}$ is known to be an orthogonal basis in $L^2(0,a)$ (which follows from theory of Fourier series), which implies that we have found all the eigenvalues.

**Example.** Compute now the eigenvalues of the rectangle $\Omega = (0,a)\times(0,b)$. Using the notation of the previous Remark with $U = (0,a)$ and $W = (0,b)$, have the following eigenfunctions in $U$ and $W$

$$u_k(x) = \sin\frac{\pi k x}{a} \quad \text{and} \quad w_l(y) = \sin\frac{\pi l y}{b}$$

and eigenvalues

$$\alpha_k = \left(\frac{\pi k}{a}\right)^2, \quad \beta_l = \left(\frac{\pi l}{b}\right)^2,$$

where $k, l$ are arbitrary natural numbers. Hence, we obtain that $\Omega$ has the following eigenfunctions and eigenvalues:

$$\begin{aligned} v_{k,l}(x,y) &= \sin\frac{\pi k x}{a}\sin\frac{\pi l y}{b} \\ \lambda_{k,l} &= \pi^2\left(\left(\frac{k}{a}\right)^2 + \left(\frac{l}{b}\right)^2\right). \end{aligned}$$

For example, in the case $a = b = \pi$, the eigenvalues are

$$\lambda_{k,l} = k^2 + l^2,$$

that is, all sums of squares of two natural numbers. Setting $k, l = 1, 2, 3, \ldots$ we obtain

$$\lambda_{1,1} = 2, \quad \lambda_{1,2} = \lambda_{2,1} = 5, \quad \lambda_{2,2} = 8, \quad \lambda_{1,3} = \lambda_{3,1} = 10, \quad \lambda_{2,3} = \lambda_{3,2} = 13, \quad \lambda_{3,3} = 18, \ldots$$

The sequence of the eigenvalues in the increasing order is $\{2, 5, 5, 8, 10, 10, 13, 13, 18, \ldots\}$. In particular, the eigenvalues $5, 10, 13$ have multiplicity 2.

One can ask what is the multiplicity $m(\lambda)$ for an arbitrary number $\lambda$ in the sequence $\{\lambda_{k,l}\}$. Clearly, $m(\lambda)$ is equal to the number of ways $\lambda$ can be represented as a sum of squares of two positive integers. For example, $m(50) = 3$ because

$$50 = 5^2 + 5^2 = 1^2 + 7^2 = 7^2 + 1^2.$$

An explicit formula for $m(\lambda)$ is obtained in Number Theory, using decomposition of $\lambda$ into product of primes. In particular, $m(5^q) = q + 1$ if $q$ is an odd number. Consequently, $m(\lambda)$ can be arbitrarily large. For example, we have $m(125) = 4$, and the corresponding representations of 125 in the form $k^2 + l^2$ are

$$125 = 2^2 + 11^2 = 11^2 + 2^2 = 5^2 + 10^2 = 10^2 + 5^2.$$

**Example.** For a general $n$, consider the box

$$\Omega = (0,a_1)\times(0,a_2)\times\ldots\times(0,a_n),$$

where $a_1, ..., a_n$ are positive reals. Applying the method of separation of variables, we obtain the following eigenvalues and eigenfunctions of $\Omega$, parametrized by $n$ natural numbers $k_1, ..., k_n$:

$$\begin{aligned} v(x) &= \sin\frac{\pi k_1 x_1}{a_1} ... \sin\frac{\pi k_n x_n}{a_n} \\ \lambda_{k_1,...,k_n} &= \pi^2\left(\left(\frac{k_1}{a_1}\right)^2 + ... + \left(\frac{k_n}{a_n}\right)^2\right). \end{aligned}$$

## 4.7 Higher order weak derivatives

Our purpose is to investigate higher order differentiability of solutions of the weak Dirichlet problem. In particular, we will be able to prove that the eigenfunctions of the Dirichlet problem constructed in Theorem 4.7 as functions from $W_0^1(\Omega)$, are in fact $C^\infty$ functions.

Recall that the Sobolev space $W^k(\Omega)$ is defined by

$$W^k(\Omega) = \left\{f \in L^2(\Omega) : D^\alpha f \in L^2(\Omega) \quad \text{for all } \alpha \text{ with } |\alpha| \le k\right\}.$$

The space $W^k$ has an inner product

$$(f, g)_{W^k} = \sum_{|\alpha| \le k} (D^\alpha f, D^\alpha f)$$

and the associated norm

$$\|f\|^2_{W^k} := \sum \|D^\alpha f\|^2_{L^2}.$$

Similarly to Proposition 4.1 it is possible to prove that $W^k(\Omega)$ is a Hilbert space.

Similarly, define the space

$$W^k_{loc}(\Omega) = \left\{f \in L^2_{loc}(\Omega) : D^\alpha f \in L^2_{loc}(\Omega) \quad \text{for all } \alpha \text{ with } |\alpha| \le k\right\}.$$

### 4.7.1 Higher order derivatives in a cube

Let $Q = (-\pi, \pi)^n$ as above. The first main result is the following theorem.

**Theorem 4.8** *Let $u \in W^1(Q)$ and $U$ be an open subset of $Q$ such that $\overline{U} \subset Q$.*

*(a) If $\Delta u \in L^2(Q)$ then $u \in W^2(U)$ and*

$$\|u\|_{W^2(U)} \le C\left(\|u\|_{W^1(Q)} + \|\Delta u\|_{L^2(Q)}\right),$$

*where constant $C$ depends on $U$ and $n$. Consequently, $u \in W^2_{loc}(Q)$.*

*(b) If $\Delta u \in W^k(Q)$ then $u \in W^{k+2}(U)$ and*

$$\|u\|_{W^{k+2}(U)} \le C\left(\|u\|_{W^1(Q)} + \|\Delta u\|_{W^k(Q)}\right),$$

*where the constant $C$ depends on $U$,$n$,$k$. Consequently, $u \in W^{k+2}_{loc}(Q)$.*

In particular, if $u$ solves the weak Dirichlet problem

$$\begin{cases} \Delta u = f & \text{in } Q \\ u \in W_0^1(Q) \end{cases}$$

with $f \in L^2(Q)$ then, in fact, $u \in W_{loc}^2(Q)$. Moreover, if $f \in W^k(Q)$ then $u \in W_{loc}^{k+2}(Q)$.

The statement of Theorem 4.8 remains true if the cube $Q$ is replaced by any bounded domain $\Omega$, which will be stated and proved below as a Corollary. For the proof of Theorem 4.8 we will need two lemmas. We use the Fourier series in $L^2(Q)$ as above.

**Lemma 4.9** *Let $u \in L^2(Q)$ and assume that, for some multiindex $\alpha$,*

$$\sum_{\xi \in \mathbb{Z}^n} |\xi^\alpha|^2 |\hat{u}(\xi)|^2 < \infty. \tag{4.29}$$

*Then $D^\alpha u \in L^2(Q)$ and, moreover,*

$$D^\alpha u = \sum_{\xi \in \mathbb{Z}^n} (i\xi)^\alpha \hat{u}(\xi) e^{i\xi \cdot x} \tag{4.30}$$

*and*

$$\|D^\alpha u\|_{L^2}^2 = (2\pi)^n \sum_{\xi \in \mathbb{Z}^n} |\xi^\alpha|^2 |\hat{u}(\xi)|^2. \tag{4.31}$$

The function $(i\xi)^\alpha$ in (4.30) is called the *symbol* of the operator $D^\alpha$. Recall that we have already proved the identities (4.30) and (4.31) in the case $u \in C_0^\infty(Q)$ – see (4.19) and (4.20), respectively.

**Example.** Assume that

$$\sum_{\xi \in \mathbb{Z}^n} |\xi|^2 |\hat{u}(\xi)|^2 < \infty.$$

Then, for any $j = 1, ..., n$, we have

$$\sum_{\xi \in \mathbb{Z}^n} |\xi_j|^2 |\hat{u}(\xi)|^2 < \infty,$$

that is, the condition (4.29) holds for $\alpha = (0, ...1, ...0)$ where the 1 is at position $j$. By Lemma 4.9 we conclude that $\partial_{x_j} u \in L^2(Q)$,

$$\partial_{x_j} u = \sum i\xi_j \hat{u}(\xi) e^{i\xi \cdot x},$$

and

$$\left\|\partial_{x_j} u\right\|_{L^2}^2 = (2\pi)^n \sum_{\xi \in \mathbb{Z}^n} |\xi_j|^2 |\hat{u}(\xi)|^2.$$

It follows that $u \in W^1(Q)$ and

$$\|\nabla u\|_{L^2}^2 = \sum_{j=1}^n \left\|\partial_{x_j} u\right\|_{L^2}^2 = (2\pi)^n \sum_{\xi \in \mathbb{Z}^n} |\xi|^2 |\hat{u}(\xi)|^2.$$

**Example.** Assume now that

$$\sum_{\xi\in\mathbb{Z}^n} |\xi|^4 |\hat{u}(\xi)|^2 < \infty.$$

Then, for all $j = 1, ..., n$ we have

$$\sum_{\xi\in\mathbb{Z}^n} |\xi_j|^4 |\hat{u}(\xi)|^2 < \infty,$$

that is, the condition (4.29) holds for $\alpha = (0, ...2, ...0)$ where the 2 is at position $j$. By Lemma 4.9 we conclude that $\partial_{x_j x_j} u \in L^2(Q)$ and

$$\partial_{x_j xj} u = -\sum_{\xi\in\mathbb{Z}^n} \xi_j^2 \hat{u}(\xi) e^{i\xi\cdot x}.$$

In particular, it follows that $\Delta u \in L^2(Q)$ and

$$\Delta u = \sum_{j=1}^n \partial_{x_j x_j} u = -\sum_{\xi\in\mathbb{Z}^n} |\xi|^2 \hat{u}(\xi) e^{i\xi\cdot x},$$

whence by Parseval's identity

$$\|\Delta u\|_{L^2}^2 = (2\pi)^n \sum_{\xi\in\mathbb{Z}^n} |\xi|^4 |\hat{u}(\xi)|^2.$$

The function $-|\xi|^2$ on $\mathbb{Z}^n$ is called the symbol of $\Delta$.

**Proof of Lemma 4.9.** By the hypothesis (4.29), the following function

$$v(x) = \sum_{\xi\in\mathbb{Z}^n} (i\xi)^\alpha \hat{u}(\xi) e^{i\xi\cdot x} \tag{4.32}$$

belongs to $L^2(Q)$. Let us show that $D^\alpha u = v$. By definition, $D^\alpha u$ is a distribution that is defined by

$$(D^\alpha u, \varphi) = (-1)^{|\alpha|} (u, D^\alpha \varphi) \quad \forall \varphi \in \mathcal{D}(Q).$$

Hence, in order to prove that $D^\alpha u = v$, we need to verify that, for any $\varphi \in \mathcal{D}(Q)$,

$$\int_Q v\varphi dx = (-1)^{|\alpha|} \int_Q u D^\alpha \varphi dx. \tag{4.33}$$

Since the Fourier series (4.32) and

$$u(x) = \sum_{\xi\in\mathbb{Z}^n} \hat{u}(\xi) e^{i\xi\cdot x}$$

converge in $L^2(Q)$, we can compute both integrals in (4.33) by substituting the Fourier series of $u$ and $v$ and interchanging integration in $x$ with summation in $\xi$. We obtain

$$\begin{aligned}\int_Q uD^\alpha \varphi dx &= \sum_{\xi\in\mathbb{Z}^n} \hat{u}(\xi) \int_Q e^{i\xi\cdot x} D^\alpha \varphi(x)\, dx \\ &= \sum_{\xi\in\mathbb{Z}^n} \hat{u}(\xi)(-1)^{|\alpha|} \int_Q D^\alpha e^{i\xi\cdot x} \varphi(x)\, dx,\end{aligned}$$

where we have used integration by parts because $\varphi \in C_0^\infty(Q)$. Since

$$D^\alpha e^{i\xi\cdot x} = (i\xi)^\alpha e^{i\xi\cdot x},$$

we obtain

$$\begin{aligned}\int_Q uD^\alpha \varphi dx &= \sum_{\xi\in\mathbb{Z}^n} \hat{u}(\xi)(-1)^{|\alpha|} \int_Q (i\xi)^\alpha e^{i\xi\cdot x} \varphi(x)\, dx \\ &= (-1)^{|\alpha|} \int_Q \left( \sum_{\xi\in\mathbb{Z}^n} (i\xi)^\alpha \hat{u}(\xi) e^{i\xi\cdot x} \right) \varphi(x)\, dx \\ &= (-1)^{|\alpha|} \int_Q v\varphi dx,\end{aligned}$$

which proves (4.33). Then identities (4.30) and (4.31) follow from (4.32). ■

29.01.16

**Definition.** For any $u \in L^1_{loc}(\Omega)$, define the support $\operatorname{supp} u$ as the complement in $\Omega$ of the maximal open subset of $\Omega$ where $u = 0$ a.e..

Observe that the maximal open subset of $\Omega$ with this property exists since it is the union of all open subsets of $\Omega$ where $u = 0$ a.e..

By construction, $\operatorname{supp} u$ is a closed subset of $\Omega$ (by the way, the same construction can be used to define the support of any distribution). If $u$ is continuous then $\operatorname{supp} u$ coincides with the closure in $\Omega$ of the set where $u \neq 0$.

The following lemma is a partial converse of Lemma 4.9.

**Lemma 4.10** *Let $u \in L^2(Q)$ and assume that* $\operatorname{supp} u$ *is a compact subset*[2] *of $Q$.*

*(a) If $D^\alpha u \in L^2(Q)$ then* (4.29), (4.30) *and* (4.31) *hold.*

*(b) If $\Delta u \in L^2(Q)$ then*

$$\Delta u = -\sum_{\xi\in\mathbb{Z}^n} |\xi|^2 \hat{u}(\xi) e^{i\xi\cdot x}, \tag{4.34}$$

*where the series* (4.34) *converges in $L^2(Q)$, and*

$$\|\Delta u\|^2_{L^2} = (2\pi)^n \sum_{\xi\in\mathbb{Z}^n} |\xi|^4 |\hat{u}(\xi)|^2. \tag{4.35}$$

[2] Recall that the notion of compactness of a set does not depend on the choice of an ambient topological space. In the statement of Lemma 4.10 there are two natural choices of the ambient space: $\mathbb{R}^n$ or $Q$. Since a subset of $\mathbb{R}^n$ is compact if and only if it is bounded and closed, the phrase "$\operatorname{supp} u$ is a compact subset of $Q$" means that "$\operatorname{supp} u$ is a closed subset of $\mathbb{R}^n$ and $\operatorname{supp} u \subset Q$" (then $\operatorname{supp} u$ is automatically bounded and, hence, compact). However, this phrase **does not** mean that "$\operatorname{supp} u$ is a closed subset of the topological space $Q$" as there are closed (and obviously bounded) subsets of the topological space $Q$ that are not compact.

**Proof.** $(a)$ Let $U$ be an open neighborhood of $\operatorname{supp} u$ such that $\overline{U} \subset Q$. Let $\psi$ be a function from $\mathcal{D}(Q)$ such that $\psi = 1$ in $U$. Any function $\psi$ with this property is called a cutoff function of $U$ in $Q$. Denote by $h(\xi)$ the discrete Fourier transform of $D^\alpha u$. Observe that $\operatorname{supp} D^\alpha u \subset U$ because if $u = 0$ a.e. in an open set then also $D^\alpha u = 0$ a.e. in the same set. Since $\psi = 1$ on $U$, we have the identity

$$\psi D^\alpha u = D^\alpha u \text{ in } Q,$$

which implies

$$h(\xi) = \frac{1}{(2\pi)^n} \int_Q D^\alpha u \, e^{-i\xi\cdot x}\, dx = \frac{1}{(2\pi)^n} \int_Q D^\alpha u \, e^{-i\xi\cdot x} \psi(x)\, dx.$$

Since $\varphi(x) := e^{-i\xi\cdot x}\psi(x) \in \mathcal{D}(Q)$, we have by the definition of distributional Laplacian $D^\alpha u$ that

$$(D^\alpha u, \varphi) = (-1)^{|\alpha|} (u, D^\alpha \varphi),$$

whence

$$h(\xi) = \frac{(-1)^{|\alpha|}}{(2\pi)^n} \int_Q u D^\alpha \left(e^{-i\xi\cdot x}\psi(x)\right) dx. \tag{4.36}$$

Observe that $e^{-\iota\xi\cdot x}\psi = e^{-\iota\xi\cdot x}$ in $U$. Therefore, in $U$

$$D^\alpha \left(e^{-i\xi\cdot x}\psi\right) = D^\alpha e^{-i\xi\cdot x} = (-i\xi)^\alpha e^{-i\xi\cdot x} = (-1)^{|\alpha|} (i\xi)^\alpha e^{-i\xi\cdot x}.$$

Since the integration in (4.36) can be restricted to $U$, we obtain

$$h(\xi) = \frac{1}{(2\pi)^n} \int_Q u (i\xi)^\alpha e^{-i\xi\cdot x} dx = (i\xi)^\alpha \hat{u}(\xi),$$

which proves (4.30). Then (4.29) and (4.35) follow by Parseval's identity.

$(b)$ The proof is the same as that of $(a)$, we just replace everywhere $D^\alpha$ by $\Delta$. Let $\psi$ be the same cutoff function of $U$ in $Q$, and let $h(\xi)$ the discrete Fourier transform of $\Delta u$. Since $\operatorname{supp} \Delta u \subset U$ and $\psi = 1$ on $U$, we have the identity

$$\psi \Delta u = \Delta u \text{ in } Q,$$

which implies

$$h(\xi) = \frac{1}{(2\pi)^n} \int_Q \Delta u \, e^{-i\xi\cdot x}\, dx = \frac{1}{(2\pi)^n} \int_Q \Delta u \, e^{-i\xi\cdot x} \psi(x)\, dx.$$

Since $\varphi(x) := e^{-i\xi\cdot x}\psi(x) \in \mathcal{D}(Q)$, we have by the definition of distributional Laplacian $\Delta u$ that

$$(\Delta u, \varphi) = (u, \Delta \varphi),$$

whence

$$h(\xi) = \frac{1}{(2\pi)^n} \int_Q u \Delta \left(e^{-i\xi\cdot x}\psi(x)\right) dx. \tag{4.37}$$

Since $e^{-\iota\xi\cdot x}\psi = e^{-\iota\xi\cdot x}$ on $U$, it follows that in $U$

$$\Delta \left(e^{-i\xi\cdot x}\psi\right) = \Delta e^{-i\xi\cdot x} = -|\xi|^2 e^{-i\xi\cdot x}.$$

Since the integration in (4.37) can be restricted to $U$, we obtain

$$h(\xi) = -\frac{1}{(2\pi)^n}\int_Q u\,|\xi|^2 e^{-i\xi\cdot x}dx = -|\xi|^2\,\hat{u}(\xi),$$

which proves (4.34). Then (4.35) follows by Parseval's identity. ■

**Proof of Theorem 4.8.** $(a)$ Let $\psi$ be a cutoff function of $U$ in $Q$. Set $v = u\psi$. By the product rule for the Laplacian, we have

$$\Delta v = \Delta(\psi u) = \psi\Delta u + 2\nabla\psi\cdot\nabla u + \Delta\psi\, u.$$

Note that $\Delta u$, $\nabla u$ and $u$ are all in $L^2$, whereas $\psi, \nabla\psi$ and $\Delta\psi$ are in $\mathcal{D}(Q)$. It follows that $\Delta v \in L^2(Q)$ and, moreover,

$$\|\Delta v\|_{L^2(Q)} \le C\left(\|u\|_{W^1(Q)} + \|\Delta u\|_{L^2(Q)}\right),$$

where $C$ depends on $\sup|\nabla\psi|$ and $\sup|\Delta\psi|$ and, hence, on $U$.

Since $\operatorname{supp} v$ is a subset of $\operatorname{supp}\psi$ and, hence, is a compact subset of $Q$, we obtain by Lemma 4.10 that

$$\|\Delta v\|_{L^2}^2 = (2\pi)^n\sum_{\xi\in\mathbb{Z}^n}|\xi|^4\,|\hat{v}(\xi)|^2.$$

Since for all indices $j, l = 1, ..., n$ we have

$$|\xi_j\xi_l| \le \frac{1}{2}|\xi_j|^2 + \frac{1}{2}|\xi_l|^2 \le |\xi|^2,$$

we obtain

$$\sum_{\xi\in\mathbb{Z}^n}|\xi_j\xi_l|^2\,|\hat{v}(\xi)|^2 \le \sum_{\xi\in\mathbb{Z}^n}|\xi|^4\,|\hat{v}(\xi)|^2 < \infty.$$

Note that the function $-\xi_j\xi_l$ is the symbol of the operator $\partial_{x_jx_l}$. Hence, we conclude by Lemma 4.9 that the distributional derivative $\partial_{x_jx_l}v$ belongs to $L^2(Q)$ and

$$\left\|\partial_{x_jx_l}v\right\|_{L^2(Q)}^2 = (2\pi)^n\sum_{\xi\in\mathbb{Z}^n}|\xi_j\xi_l|^2\,|\hat{u}(\xi)|^2 \le \|\Delta v\|_{L^2(Q)}^2.$$

Similarly, since $|\xi_j| \le |\xi|^2$, we obtain

$$\sum_{\xi\in\mathbb{Z}^n}|\xi_j|^2\,|\hat{v}(\xi)|^2 \le \sum_{\xi\in\mathbb{Z}^n}|\xi|^4\,|\hat{v}(\xi)|^2 < \infty.$$

Hence, $\partial_{x_j}v \in L^2(Q)$ and

$$\left\|\partial_{x_j}v\right\|_{L^2(Q)}^2 = (2\pi)^n\sum_{\xi\in\mathbb{Z}^n}|\xi_j|^2\,|\hat{u}(\xi)|^2 \le \|\Delta v\|_{L^2(Q)}^2.$$

We conclude that $v \in W^2(Q)$ and

$$\|v\|_{W^2(Q)}^2 = \|v\|_{L^2}^2 + \sum_{j=1}^n\left\|\partial_{x_j}v\right\|_{L^2}^2 + \sum_{j,l=1}^n\left\|\partial_{x_jx_l}v\right\|_{L^2}^2 \le \|v\|_{L^2(Q)}^2 + C\,\|\Delta v\|_{L^2(Q)}^2.$$

Since $v = u$ in $U$, we obtain that $u \in W^2(U)$ and

$$\begin{aligned} \|u\|^2_{W^2(U)} &\leq \|v\|^2_{L^2(Q)} + C\|\Delta v\|^2_{L^2(Q)} \\ &\leq C'\left(\|u\|_{W^1(Q)} + \|\Delta u\|_{L^2(Q)}\right), \end{aligned}$$

which was to be proved.

($b$) Induction in $k$. The induction basis for $k = 0$ was proved in ($a$). For the inductive step from $k$ to $k+1$, choose a cube $Q' = (\pi - \varepsilon, \pi - \varepsilon)^n$ for some $\varepsilon > 0$, such that $\overline{U} \subset Q'$. Assume that $u \in W^1(Q)$ and $\Delta u \in W^{k+1}(Q)$. Since $\Delta u \in L^2(Q)$, by part ($a$) we have $u \in W^2(Q')$ and

$$\|u\|^2_{W^2(Q')} \leq C\left(\|u\|_{W^1(Q)} + \|\Delta u\|_{L^2(Q)}\right). \tag{4.38}$$

Set $v = \partial_{x_j} u$ and observe that $v \in W^1(Q')$ and $\Delta v = \partial_{x_j}\Delta u \in W^k(Q')$. By the inductive hypotheses applied to cube $Q'$ instead of $Q$, we obtain $v \in W^{k+2}(U)$ and

$$\begin{aligned} \|v\|_{W^{k+2}(U)} &\leq C\left(\|v\|_{W^1(Q')} + \|\Delta v\|_{W^k(Q')}\right) \\ &\leq C\left(\|u\|_{W^2(Q')} + \|\Delta u\|_{W^{k+1}(Q')}\right). \end{aligned}$$

Substituting here the estimate of $\|u\|_{W^2(Q')}$ from (4.38), we obtain

$$\|v\|_{W^{k+2}(U)} \leq C\left(\|u\|_{W^1(Q)} + \|\Delta u\|_{W^{k+1}(Q)}\right).$$

Finally, since this estimate holds for any partial derivative $v = \partial_{x_j} u$ of $u$, it follows that $u \in W^{k+3}(U)$ and

$$\|u\|_{W^{k+3}(U)} \leq C\left(\|u\|_{W^1(Q)} + \|\Delta u\|_{W^{k+1}(Q)}\right),$$

which proves the inductive step.

Finally, let us show that $u \in W^{k+2}_{loc}(Q)$ (both in the cases ($a$) and ($b$)). Indeed, since for any multiindex $\alpha$ of order $\leq k+2$ we have $D^\alpha u \in L^2(U)$ for any open set $U$ such that $\overline{U} \subset Q$, we see that $D^\alpha u \in L^2_{loc}(Q)$ and, hence, $u \in W^{k+2}_{loc}(Q)$. ■

03.02.16

### 4.7.2 Higher order derivatives in arbitrary domain

Our next task is to generalize Theorem 4.8 to general domains. For that we prove first two lemmas.

Let $f, g$ be distributions in $\Omega$. If $U$ is an open subset of $\Omega$ then we say that $f = g$ in $U$ if

$$(f, \varphi) = (g, \varphi) \quad \forall \varphi \in \mathcal{D}(U).$$

**Lemma 4.11** *Let $\Omega = U \cup V$ where $U, V$ are open domains in $\mathbb{R}^n$. If $f, g \in \mathcal{D}'(\Omega)$ and $f = g$ in $U$ and in $V$ then $f = g$ in $\Omega$.*

**Proof.** We have to prove that

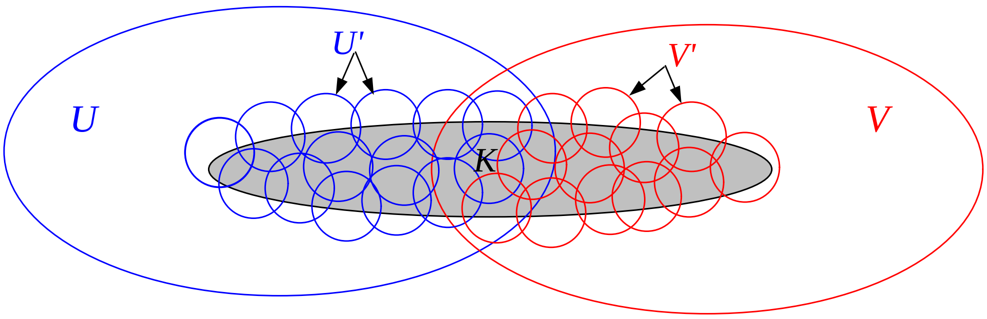
$$(f, \varphi) = (g, \varphi) \quad \forall \varphi \in \mathcal{D}(\Omega). \tag{4.39}$$

Fix $\varphi \in \mathcal{D}(\Omega)$ and denote $K = \operatorname{supp} \varphi$. If $K \subset U$ then $\varphi \in \mathcal{D}(U)$ and (4.39) holds by assumption that $f = g$ in $U$. In the same way (4.39) holds if $K \subset V$. However, if $K$ is not contained in $U$ or $V$, then additional argument is needed. In fact, it suffices to show that $\varphi$ can be represented in the form

$$\varphi = \varphi_1 + \varphi_2, \tag{4.40}$$

where $\varphi_1 \in \mathcal{D}(U)$ and $\varphi_2 \in \mathcal{D}(V)$. Then, adding up the identities (4.39) with $\varphi_1$ and $\varphi_2$, we obtain that for $\varphi$. The representation (4.40) is called *partition* of $\varphi$ subordinated to $U, V$.

Since $K \subset U \cup V$, for any point $x \in K$ there is a ball $B_x$ of small enough radius centered at $x$ such that $\overline{B}_x$ is contained in $U$ or in $V$. The family $\{B_x\}_{x \in K}$ is an open cover of $K$, so there exists a finite subcover, say $B_1, ... B_l$. Denote by $U'$ the union of all balls $B_j$ with $\overline{B}_j \subset U$, and by $V'$ – the union of all balls $B_j$ with $\overline{B}_j \subset V$ (some balls $B_j$ may be used in both $U'$ and $V'$).




Covering of the set $K$ (grey shaded) with $U'$ (the union of blue balls) and $V'$ (the union of red balls)

By construction we have

$$K \subset U' \cup V', \quad \overline{U'} \subset U, \quad \overline{V'} \subset V.$$

Therefore, there is a cutoff function $\psi_1$ of $U'$ in $U$, and a cutoff function $\psi_2$ of $V'$ in $V$. Set then

$$\varphi_1 = \psi_1 \varphi \quad \text{and} \quad \varphi_2 = (1 - \psi_1) \psi_2 \varphi.$$

Clearly, $\varphi_1 \in \mathcal{D}(U)$ and $\varphi_2 \in \mathcal{D}(V)$. Besides,

$$\begin{aligned} \varphi_1 + \varphi_2 &= (\psi_1 + \psi_2 - \psi_1 \psi_2) \varphi \\ &= (1 - (1 - \psi_1)(1 - \psi_2)) \varphi, \end{aligned}$$

which implies that

- $\varphi_1 + \varphi_2 = 0 = \varphi$ outside $K$;
- $\varphi_1 + \varphi_2 = \varphi$ on $V' \cup U'$ because on this set either $\psi_1 = 1$ or $\psi_2 = 1$.

Since $K$ is covered by $V' \cup U'$, we conclude that $\varphi_1 + \varphi_2 = \varphi$ everywhere, which finishes the proof. ■

**Lemma 4.12** *Let $\Omega = U \cup V$ where $U, V$ are open domains in $\mathbb{R}^n$. Let $u$ be a measurable function in $\Omega$. If $u \in W^k(U)$ and $u \in W^k(V)$ then $u \in W^k(\Omega)$. Besides, we have*

$$\|u\|^2_{W^k(\Omega)} \le \|u\|^2_{W^k(U)} + \|u\|^2_{W^k(V)}. \tag{4.41}$$

**Proof.** Obviously, if $u \in L^2(U)$ and $u \in L^2(V)$ then

$$\int_\Omega u^2 dx \le \int_U u^2 dx + \int_V u^2 dx < \infty$$

so that $u \in L^2(\Omega)$ and

$$\|u\|^2_{L^2(\Omega)} \le \|u\|^2_{L^2(U)} + \|u\|^2_{L^2(V)}.$$

Assume that, for some multiindex $\alpha$, we know that $D^\alpha u \in L^2(U)$ and $D^\alpha u \in L^2(V)$. Let us prove that $D^\alpha u \in L^2(\Omega)$. Denote by $v_1$ the function $D^\alpha u$ in $U$ and by $v_2$ the function $D^\alpha u$ in $V$. Observe then that $D^\alpha u$ in $U \cap V$ is equal simultaneously to $v_1$ and $v_2$ so that $v_1 = v_2$ in $U \cap V$. Let us define function $v$ in $U \cup V$ by

$$v(x) = \begin{cases} v_1(x), & x \in U, \\ v_2(x), & x \in V. \end{cases}$$

Clearly, $v$ is well-defined and $v \in L^2(\Omega)$. Then $D^\alpha u = v$ in $U$ and in $V$. Therefore, by Lemma 4.11 we conclude that $D^\alpha u = v$ in $\Omega$. It follows that

$$\|D^\alpha u\|^2_{L^2(\Omega)} \le \|D^\alpha u\|^2_{L^2(U)} + \|D^\alpha u\|^2_{L^2(V)}.$$

Summing up such identities over all multiindices $|\alpha| \le k$, we obtain (4.41). ■

**Theorem 4.13** *Let $\Omega$ be any bounded domain in $\mathbb{R}^n$. If $u \in W^1(\Omega)$ and $\Delta u \in W^k(\Omega)$ then, for any open subset $U$ of $\Omega$, such that $\overline{U} \subset \Omega$, we have $u \in W^{k+2}(U)$ and*

$$\|u\|_{W^{k+2}(U)} \le C \left( \|u\|_{W^1(\Omega)} + \|\Delta u\|_{W^k(\Omega)} \right),$$

*where the constant $C$ depends on $\Omega, U, n, k$. Consequently, $u \in W^{k+2}_{loc}(\Omega)$.*

**Proof.** For any point $x \in \Omega$ there exists $\varepsilon = \varepsilon(x) > 0$ such that the cube

$$Q_x := (x_1 - \varepsilon, x_1 + \varepsilon) \times ... \times (x_n - \varepsilon, x_n + \varepsilon)$$

is contained in $\Omega$. Denote by $U_x$ a similar cube where $\varepsilon$ is replaced by $\varepsilon/2$. Clearly, the family $\{U_x\}_{x \in \overline{U}}$ is an open covering of $\overline{U}$. By the compactness of $\overline{U}$, there is a finite subcover, denote its element by $U_1, ..., U_l$. Applying Theorem 4.8 in the corresponding cubes $Q_1, ..., Q_l$ (instead of $Q$), we obtain that $u \in W^{k+2}(U_j)$ and

$$\begin{aligned} \|u\|_{W^{k+2}(U_j)} &\le C_j \left( \|u\|_{W^1(Q_j)} + \|\Delta u\|_{W^k(Q_j)} \right) \\ &\le C \left( \|u\|_{W^1(\Omega)} + \|\Delta u\|_{W^k(\Omega)} \right), \end{aligned}$$

where $C = \max C_j$. Since $U \subset \bigcup_{j=1}^{l} U_j$, using Lemma 4.12, we obtain by induction in $l$ that $u \in W^{k+2}(U)$ and

$$\begin{aligned} \|u\|_{W^{k+2}(U)} &\leq \left(\sum_{j=1}^{l} \|u\|^2_{W^{k+2}(U_j)}\right)^{1/2} \\ &\leq \sum_{j=1}^{l} \|u\|_{W^{k+2}(U_j)} \\ &\leq C'\left(\|u\|_{W^1(\Omega)} + \|\Delta u\|_{W^k(\Omega)}\right), \end{aligned}$$

where $C' = lC$, which finishes the proof. ∎

**Corollary 4.14** *Let $\Omega$ be a bounded domain and $v \in W_0^1(\Omega)$ be an eigenfunction of the weak Dirichlet problem in $\Omega$ with the eigenvalue $\lambda$. Then $v \in W_{loc}^{\infty}(\Omega)$.*

**Proof.** It suffices to prove that $v \in W^k(U)$ for $k \in \mathbb{N}$ and for any open set $U$ such that $\overline{U} \subset Q$. Given $k$ and $U$, let us construct a sequence of open sets $U_0, ..., U_k$ such that $U_0 = \Omega$, $U_j \supset \overline{U}_{j+1}$, and $U_k = U$. Set

$$f = -\lambda v,$$

so that

$$\Delta v = f.$$

Since $v \in W^1(U_0)$ then also $f \in W^1(U_0)$. Therefore,

$$v \in W^1(U_0) \quad \text{and} \quad \Delta v \in W^1(U_0),$$

which implies by Theorem 4.13 that $v \in W^3(U_1)$. Hence, also $f \in W^3(U_1)$. Therefore,

$$v \in W^1(U_1) \quad \text{and} \quad \Delta v \in W^3(U_1),$$

which implies by Theorem 4.13 that $v \in W^5(U_2)$. Continuing further by induction, we obtain that $u \in W^{2k+1}(U_k)$, which finishes the proof. ∎

## 4.8 Sobolev embedding theorem

Recall that $C^m(\Omega)$ denotes the space of all $m$ times continuously differentiable functions in $\Omega$. Set

$$\|u\|_{C^m(\Omega)} = \sup_{\{\alpha:|\alpha|\leq m\}} \sup_{x\in\Omega} |D^\alpha u(x)|.$$

Note that $\|u\|_{C^m(\Omega)}$ can be equal to $\infty$. Define also the space $C_b^m(\Omega)$ as a subspace of $C^m(\Omega)$ with $\|u\|_{C^m(\Omega)} < \infty$. Then $C_b^m(\Omega)$ is a normed linear space with the norm $\|\cdot\|_{C^m(\Omega)}$. Moreover, it is a Banach space.

The following implications are trivial:

$$u \in C^m(\Omega) \Rightarrow u \in W_{loc}^m(\Omega)$$

and, if $\Omega$ is bounded, then

$$u \in C_b^m(\Omega) \Rightarrow u \in W^m(\Omega).$$

**Notational remark.** A better notion for $C^m(\Omega)$ would have been $C_{loc}^m(\Omega)$ and for $C_b^m(\Omega)$ – simply $C^m(\Omega)$. In this case the notation for $C^m$-spaces would have matched those for $W^k$-spaces. However, we use the notations that are commonly accepted in mathematics, even if they are not best possible.

The next theorem states a kind of converse to the above implications. It is one of the most amazing results of Analysis.

**Theorem 4.15** (Sobolev embedding theorem) *Let $\Omega$ be an open subset of $\mathbb{R}^n$ and let $m, k$ be non-negative integers such that*

$$k > m + \frac{n}{2}. \tag{4.42}$$

*If $u \in W_{loc}^k(\Omega)$ then $u \in C^m(\Omega)$.*

*Moreover, if $u \in W^k(\Omega)$ then, for any open set $U$ such that $\overline{U}$ is a compact subset of $\Omega$, we have $u \in C_b^m(U)$ and*

$$\|u\|_{C^m(U)} \leq C\|u\|_{W^k(\Omega)}, \tag{4.43}$$

*where the constant $C$ depends on $\Omega, U, k, m, n$.*

Note that $u$ is a priori an element of $L_{loc}^2(\Omega)$ and, hence, is the class of measurable functions defined almost everywhere. When we claim that $u \in C^m(\Omega)$ and, in particular, $u \in C(\Omega)$, we understand $u$ as a function defined pointwise. A precise meaning of that is as follows: if $u \in W_{loc}^k(\Omega)$ then $u$ as a class of functions has a representative, also denoted by $u$, such that this representative belongs to $C^m(\Omega)$.

The identification of $u \in W_{loc}^k(\Omega)$ with its $C^m$-representative allows to define an embedding (=injective linear mapping) of linear spaces

$$W_{loc}^k(\Omega) \hookrightarrow C^m(\Omega).$$

The estimate (4.43) implies that there is an embedding

$$W^k(\Omega) \hookrightarrow C_b^m(U)$$

05.02.16 of normed linear spaces, and this embedding is a bounded operator.

**Example.** Let $n = 1$. Then the condition (4.42) becomes $k > m + \frac{1}{2}$ that is equivalent to $k \geq m + 1$. Hence, if $u \in W_{loc}^k$ then $u \in C^{k-1}$, provided $k \geq 1$. In particular, any function from $W_{loc}^1$ has to be continuous. We have seen above that the continuous function $u(x) = |x|$ in $\mathbb{R}$ has the weak derivative $u' = \operatorname{sgn} x$ and, hence, belongs to $W_{loc}^1$. On the other hand, the function $u(x) = \mathbf{1}_{[0,\infty)}$ that has only one point of discontinuity at $x = 0$ has the distributional derivative $u' = \delta$ and, hence, is not in $W_{loc}^1$.

**Example.** For a general $n$ and for $m = 0$, the condition (4.42) becomes $k > \frac{n}{2}$. That is, if

$$k > \frac{n}{2} \tag{4.44}$$

then $u \in W^k_{loc}$ implies that $u$ is continuous. Let us show that the condition (4.44) is sharp. For that, consider in $\mathbb{R}^n$ the function $u(x) = |x|^\alpha$ where $\alpha$ is a real number. This function is clearly $C^\infty$ smooth outside the origin, but it is continuous in $\mathbb{R}$ if and only if $\alpha \geq 0$. We use without proof the fact that $u \in L^2_{loc}$ if and only if

$$\alpha > -\frac{n}{2}$$

(cf. Example at the end of Section 4.1). It is also possible to prove that any classical derivatives of $u$ of the order $k$ (which is defined outside 0) belongs to $L^2_{loc}$ if and only if

$$\alpha - k > -\frac{n}{2},$$

which is equivalent to

$$\alpha > k - \frac{n}{2}. \tag{4.45}$$

Under this condition the classical derivative coincides with the weak derivative, which therefore belongs to $L^2_{loc}$.

Hence, under the condition (4.45) we obtain $u \in W^k_{loc}$. If $k < \frac{n}{2}$ then there exists $\alpha < 0$ that satisfies (4.45). Then the function $u(x) = |x|^\alpha$ belongs to $W^k_{loc}$ but is not continuous at 0. This example shows that the condition (4.44), under which all functions from $W^k_{loc}$ are continuous, is sharp.

Before the proof of Theorem 4.15, let us state some consequences.

**Corollary 4.16** *Let $\Omega$ be a bounded domain in $\mathbb{R}^n$. Let $u$ be solution of the weak Dirichlet problem*

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ u \in W^1_0(\Omega) \end{cases}$$

*where $f \in L^2(\Omega)$. If in addition $f \in W^k_{loc}(\Omega)$ where*

$$k + 2 > m + \frac{n}{2}, \tag{4.46}$$

*then $u \in C^m(\Omega)$. Here $k, m$ are non-negative integers.*

In particular, the statement of Corollary 4.16 holds if $f \in C^k(\Omega)$.

**Proof.** Fix an open subset $U$ of $\Omega$ such that $\overline{U} \subset \Omega$. Then we have $f \in W^k(U)$. Since $u \in W^1(U)$ and $\Delta u \in W^k(U)$, we obtain by Theorem 4.13 that $u \in W^{k+2}_{loc}(U)$. By Theorem 4.15 and and (4.46), we conclude that $u \in C^m(U)$. Since $U$ is arbitrary, it follows that $u \in C^m(\Omega)$. ■

**Example.** Let $n = 2$. Then the condition $k + 2 > m + 1$ is equivalent to $k \geq m$. In the case $n = 3$ the condition

$$k + 2 > m + \frac{3}{2}$$

is also equivalent to $k \geq m$. Hence, in the both cases $n = 2, 3$ we obtain if $f \in W^k_{loc}(\Omega)$ then $u \in C^k(\Omega)$.

If $n = 4$ then the condition $k + 2 > m + 2$ is equivalent to $k \geq m + 1$.Hence, $f \in W^k_{loc}(\Omega)$ implies $u \in C^{k-1}(\Omega)$ provided $k \geq 1$.

**Corollary 4.17** *In any bounded domain $\Omega \subset \mathbb{R}^n$, all eigenfunctions of the weak Dirichlet problem belong to $C^\infty(\Omega)$.*

**Proof.** Let $v$ be an eigenfunction of the weak Dirichlet problem in $\Omega$. By Corollary 4.14, we have $v \in W^k_{loc}(\Omega)$ for any $k$. Hence, by Theorem 4.15 we conclude that $v \in C^m(\Omega)$ for any $m$, that is, $v \in C^\infty(\Omega)$. ∎

**Remark.** The question remains if the boundary condition $v \in W^1_0(\Omega)$ is the statement of the weak eigenvalue problem can be turned into the classical boundary condition $v = 0$ on $\partial\Omega$, which in particular requires the continuity of $v$ in $\overline{\Omega}$. This question is more difficult than the continuity of $v$ inside $\Omega$, because the answer depends on the properties of the boundary $\partial\Omega$.

In short, if the boundary is good enough, for example, if $\Omega$ is a region, then indeed $v \in C\left(\overline{\Omega}\right)$ and $v = 0$ on $\partial\Omega$ pointwise. A similar statement holds for weak solutions of the Dirichlet problem.

However, the study of the boundary behavior is outside the range of this course.

**Proof of Theorem 4.15.** The proof will be split in a few parts.

**Part 1**. Let $Q = (-\pi, \pi)^n$ be the cube as above. Assume first that $u \in L^2(Q)$ and that $\operatorname{supp} u$ is a compact subset of $Q$. We prove in this part that if $u \in W^k(Q)$ with $k > n/2$ then $u \in C(Q)$ and, moreover,

$$\|u\|_{C(Q)} \leq C \|u\|_{W^k(Q)} \tag{4.47}$$

for some constant $C = C(n,k)$ (which corresponds to the case $m = 0$).

By Lemma 4.10($a$), we have, for any multiindex $\alpha$ with $|\alpha| \leq k$ the identity (4.31), that is,

$$\sum_{\xi \in \mathbb{Z}^n} |\xi^\alpha|^2 |\hat{u}(\xi)|^2 = (2\pi)^{-n} \|D^\alpha u\|^2_{L^2} < \infty.$$

Applying this with $\alpha = (0, ..., 0, k, 0, ..., 0)$, where $k$ stands at position $i$, we obtain

$$\sum_{\xi \in \mathbb{Z}^n} |\xi_i|^{2k} |\hat{u}(\xi)|^2 = (2\pi)^{-n} \left\|\partial^k_{x_i} u\right\|^2_{L^2} < \infty.$$

Adding up in all $i = 1, ..., n$, we obtain

$$\sum_{\xi \in \mathbb{Z}^n} \left(|\xi_1|^{2k} + ... + |\xi_n|^{2k}\right) |\hat{u}(\xi)|^2 \leq \|u\|^2_{W^k}.$$

Observing that

$$|\xi|^{2k} = \left(\sum_{i=1}^n |\xi_i|^2\right)^k \leq C \sum_{i=1}^n |\xi_i|^{2k},$$

where $C = n^k$, we obtain

$$\sum_{\xi \in \mathbb{Z}^n} |\xi|^{2k} |\hat{u}(\xi)|^2 \leq C \|u\|^2_{W^k} < \infty. \tag{4.48}$$

On the other hand, we have by the Cauchy-Schwarz inequality,

$$\begin{aligned} \left(\sum_{\xi \in \mathbb{Z}^n \setminus \{0\}} |\hat{u}(\xi)|\right)^2 &= \left(\sum_{\xi \in \mathbb{Z}^n \setminus \{0\}} |\xi|^{-k} |\xi|^k |\hat{u}(\xi)|\right)^2 \\ &\leq \sum_{\xi \in \mathbb{Z}^n \setminus \{0\}} |\xi|^{-2k} \sum_{\xi \in \mathbb{Z}^n} |\xi|^{2k} |\hat{u}(\xi)|^2. \end{aligned} \tag{4.49}$$

If $k > \frac{n}{2}$ then $2k > n$. We claim that if $2k > n$ then

$$\sum_{\xi \in \mathbb{Z}^n \setminus \{0\}} |\xi|^{-2k} < \infty \tag{4.50}$$

(see Lemma 4.18 below). Combining this with (4.48) and (4.49), we obtain

$$\sum_{\xi \in \mathbb{Z}^n \setminus \{0\}} |\hat{u}(\xi)| \leq C' \|u\|_{W^k} < \infty.$$

In particular, this implies that the Fourier series

$$\sum_{\xi \in \mathbb{Z}^n} \hat{u}(\xi) e^{i\xi \cdot x}$$

converges absolutely and uniformly in $x \in Q$. Therefore, its sum is a continuous function in $Q$. On the other hand, we know that this series converges in $L^2$ to $u(x)$. Hence, $L^2$ function $u(x)$ has a continuous version that is the pointwise sum of the Fourier series. Besides, we have for the continuous function $u(x)$

$$\begin{aligned}
\sup_{x \in Q} |u(x)| &\leq \sum_{\xi \in \mathbb{Z}^n} \left| \hat{u}(\xi) e^{i\xi \cdot x} \right| \leq |\hat{u}(0)| + \sum_{\xi \in \mathbb{Z}^n \setminus \{0\}} |\hat{u}(\xi)| \\
&\leq \frac{1}{(2\pi)^n} \int_Q |u(x)| \, dx + C' \|u\|_{W^k} \\
&\leq \|u\|_{L^2} + C' \|u\|_{W^k} \\
&\leq C'' \|u\|_{W^k},
\end{aligned}$$

which proves (4.47).

**Part 2**. Let us extend the result of Part 1 to the case $m \geq 1$. Namely, in the setting of Part 1, assume that $u \in W^k(Q)$ with $k > m + \frac{n}{2}$ and prove that $u \in C^m(Q)$ and, moreover,

$$\|u\|_{C^m(Q)} \leq C \|u\|_{W^k(Q)}. \tag{4.51}$$

We still have (4.48), but instead of (4.49) we write

$$\begin{aligned}
\left( \sum_{\xi \in \mathbb{Z}^n \setminus \{0\}} |\xi|^m |\hat{u}(\xi)| \right)^2 &= \left( \sum_{\xi \in \mathbb{Z}^n \setminus \{0\}} |\xi|^{-k+m} |\xi|^k |\hat{u}(\xi)| \right)^2 \\
&\leq \sum_{\xi \in \mathbb{Z}^n \setminus \{0\}} |\xi|^{-2(k-m)} \sum_{\xi \in \mathbb{Z}^n} |\xi|^{2k} |\hat{u}(\xi)|^2.
\end{aligned}$$

Since $2(k-m) > n$, we obtain that

$$\sum_{\xi \in \mathbb{Z}^n \setminus \{0\}} |\xi|^{-2(k-m)} < \infty.$$

Combining this with (4.48) and noticing that $|\xi|^m = 0$ for $\xi = 0$, we obtain

$$\sum_{\xi \in \mathbb{Z}^n} |\xi|^m |\hat{u}(\xi)| \leq C \|u\|_{W^k} < \infty.$$

We claim that, for any $\alpha$ with $|\alpha| \le m$, the classical derivative $D^\alpha u$ exists and is given by the series

$$D^\alpha u(x) = \sum_{\xi \in \mathbb{Z}^n} (i\xi)^\alpha \hat{u}(\xi) e^{i\xi \cdot x}, \tag{4.52}$$

where the convergence is absolut and uniform. Indeed, since this series is obtained a term by term application of $D^\alpha$ to the series

$$u(x) = \sum_{\xi \in \mathbb{Z}^n} \hat{u}(\xi) e^{i\xi \cdot x},$$

it suffices to prove that the series (4.52) converges absolutely and uniformly in $x \in Q$ for all $|\alpha| \le m$. Observe that

$$\begin{aligned} |\xi^\alpha| &= |\xi_1|^{\alpha_1} \ldots |\xi_n|^{\alpha_n} \le (|\xi_1| + \ldots + |\xi_n|)^{|\alpha|} \\ &\le C \left(|\xi_1|^2 + \ldots + |\xi_n|^2\right)^{|\alpha|/2} = C |\xi|^{|\alpha|}. \end{aligned} \tag{4.53}$$

Therefore, for any $\alpha \ne 0$ with $|\alpha| \le m$,

$$\begin{aligned} \sum_{\xi \in \mathbb{Z}^n} \left|(i\xi)^\alpha \hat{u}(\xi) e^{i\xi \cdot x}\right| &\le C \sum_{\xi \in \mathbb{Z}^n} |\xi|^{|\alpha|} |\hat{u}(\xi)| \\ &\le C \sum_{\xi \in \mathbb{Z}^n} |\xi|^m |\hat{u}(\xi)| \\ &\le C' \|u\|_{W^k} < \infty, \end{aligned} \tag{4.54}$$

which proves (4.52). Besides, we obtain from (4.52) and (4.54) that

$$|D^\alpha u(x)| \le C' \|u\|_{W^k},$$

whence (4.51) follows.

**Part 3**. Assume that $u \in W^k(Q)$ and prove that $u \in C^m(Q)$ provided $k > m + \frac{n}{2}$. Besides, we prove that, for any open set $U$ such that $\overline{U} \subset Q$,

$$\|u\|_{C^m(U)} \le C \|u\|_{W^k(Q)}. \tag{4.55}$$

Let $\psi$ be a cutoff function of $U$ in $Q$. Then the function $v = \psi u$ has a compact support in $Q$ and $v \in W^k(Q)$. Indeed, to see the latter, let us use the Leibniz formula

$$D^\alpha (\psi u) = \sum_{\{\beta : \beta \le \alpha\}} \binom{\alpha}{\beta} D^{\alpha-\beta}\psi D^\beta u,$$

where $\beta \le \alpha$ means that $\beta_j \le \alpha_j$ for all $j = 1, \ldots, n$, and $\binom{\alpha}{\beta}$ is a polynomial coefficient defined by

$$\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!},$$

where $\alpha! = \alpha_1! \ldots \alpha_n!$. If $|\alpha| \le k$ then also $|\beta| \le k$ and $D^\beta u \in L^2_{loc}(Q)$. Since $D^{\alpha-\beta}\psi$ is supported in $\operatorname{supp}\psi$ and is bounded, we obtain that the product $D^{\alpha-\beta}\psi D^\beta u$ is supported in $\operatorname{supp}\psi$ and, hence, belongs to $L^2(Q)$. Hence, $D^\alpha(\psi v) \in L^2(Q)$, whence $v \in W^k(Q)$ follows.

By Part 2 we conclude that $v \in C^m(Q)$ and

$$\|v\|_{C^m(Q)} \le C \|u\|_{W^k(Q)}.$$

Since $u = v$ on $U$, we obtain (4.55).

**Part 4**. Let $\Omega$ be an arbitrary open set and $u \in W^k_{loc}(\Omega)$. Let $Q$ be any cube (of any size) such that $\overline{Q} \subset \Omega$. Then $u \in W^k(Q)$ and, hence, by Part 3, $u \in C^m(Q)$. Since such cubes $Q$ cover all the set $\Omega$, we conclude that $u \in C^m(\Omega)$.

Assume now that $u \in W^k(\Omega)$. Let $U$ be an open set such that $\overline{U}$ is a compact subset of $\Omega$. As in the proof of Theorem 4.13, choose for any point $x \in \Omega$ some $\varepsilon > 0$ such that the cube

$$Q_x := (x_1 - \varepsilon, x_1 + \varepsilon) \times \dots \times (x_n - \varepsilon, x_n + \varepsilon)$$

is contained in $\Omega$. Denote by $U_x$ a similar cube where $\varepsilon$ is replaced by $\varepsilon/2$. Clearly, the family $\{U_x\}_{x \in \overline{U}}$ is an open covering of $\overline{U}$. By the compactness of $\overline{U}$, there is a finite subcover, denote its element by $U_1, ..., U_l$. By (4.51), we have for any $j$

$$\|u\|_{C^m(U_j)} \le C_j \|u\|_{W^k(Q_j)}. \tag{4.56}$$

Since the union $\bigcup_{j=1}^l U_j$ covers $U$, taking (4.56) supremum in $j$, we obtain

$$\|u\|_{C^m(U)} \le C \|u\|_{W^k(\Omega)},$$

which finishes the proof. ■

To complete the proof of Theorem 4.15, it remains to prove the following lemma.

**Lemma 4.18** *For any $\gamma > n$ we have*

$$\sum_{\xi \in \mathbb{Z}^n \setminus \{0\}} |\xi|^{-\gamma} < \infty. \tag{4.57}$$

**Proof.** Let us first estimate the following number:

$$N(R) = \#\{\xi \in \mathbb{Z}^n : |\xi| < R\},$$

where $R > 0$. In other words, $N(R)$ is the number of integer points inside the ball $B_R$ of $\mathbb{R}^n$. With any $\xi \in \mathbb{Z}^n$, let us associate a unit cube

$$Q_\xi := \left\{x \in \mathbb{R}^n : \xi_j < x_j < \xi_j + 1, \quad \forall j = 1, ..., n\right\}.$$

In other words, $\xi$ is the bottom left corner of the cube $Q_\xi$. For any $x \in Q_\xi$, we have

$$|x - \xi| = \left(\sum_{j=1}^n |x_j - \xi_j|^2\right)^{1/2} \le \sqrt{n}.$$

Hence, if $\xi \in B_R$ then

$$|x| \le |\xi| + |x - \xi| < R + \sqrt{n},$$

which implies

$$Q_\xi \subset B_{R+\sqrt{n}}.$$

Since all the cubes $Q_\xi$ are disjoint and the volume of each cube $Q_\xi$ is equal to 1, we obtain

$$N(R) = \sum_{\xi \in B_R} \operatorname{vol}(Q_\xi) \le \operatorname{vol} B_{R+\sqrt{n}} = c_n \left(R+\sqrt{n}\right)^n,$$

where $c_n$ is the volume of the unit ball in $\mathbb{R}^n$. Assuming that $R$ is a positive integer and, in particular, $R \ge 1$, we obtain

$$N(R) \le CR^n,$$

for some constant $C = C(n)$. Therefore, we obtain

$$\begin{aligned}
\sum_{\xi \in \mathbb{Z}^n \setminus \{0\}} |\xi|^{-\gamma} &= \sum_{k=0}^{\infty} \sum_{\left\{\xi \in \mathbb{Z}^n : 2^k \le |\xi| < 2^{k+1}\right\}} |\xi|^{-\gamma} \\
&\le \sum_{k=0}^{\infty} \sum_{\left\{\xi \in \mathbb{Z}^n : 2^k \le |\xi| < 2^{k+1}\right\}} 2^{-k\gamma} \\
&= \sum_{k=0}^{\infty} 2^{-k\gamma} \left(N\left(2^{k+1}\right) - N\left(2^k\right)\right) \\
&\le \sum_{k=0}^{\infty} 2^{-k\gamma} N\left(2^{k+1}\right) \\
&\le C \sum_{k=0}^{\infty} 2^{-k\gamma} 2^{(k+1)n} \\
&= C 2^n \sum_{k=0}^{\infty} 2^{-(\gamma-n)k} < \infty,
\end{aligned}$$

where we have used that $\gamma > n$. ■

## 4.9 * Sobolev spaces of fractional orders

Let $u \in L^2(Q)$ and assume that $\operatorname{supp} u$ is a compact subset of $Q$. Combination of Lemmas 4.9 and 4.10 gives the following: $D^\alpha u \in L^2(Q)$ if and only if

$$\sum_{\xi \in \mathbb{Z}^n} |\xi^\alpha|^2 |\hat{u}(\xi)|^2 < \infty. \tag{4.58}$$

By (4.53) we have $|\xi^\alpha| \le C |\xi|^{|\alpha|}$. Hence, (4.58) holds for all multiindices $\alpha$ with $|\alpha| \le k$ provided

$$\sum_{\xi \in \mathbb{Z}^n} |\xi|^{2k} |\hat{u}(\xi)|^2 < \infty. \tag{4.59}$$

Hence, if (4.59) holds then $u \in W^k(Q)$ and

$$\|u\|_{W^k(Q)} \le C \sum_{\xi \in \mathbb{Z}^n} |\xi|^{2k} |\hat{u}(\xi)|^2.$$

On the other hand, by (4.48) we have the converse: is $u \in W^k(Q)$ then

$$\sum_{\xi \in \mathbb{Z}^n} |\xi|^{2k} |\hat{u}(\xi)|^2 \leq C \|u\|_{W^k(Q)}$$

and, in particular, (4.58) holds.

Hence, $u \in W^k(Q)$ is equivalent to (4.59), and

$$\|u\|^2_{W^k(Q)} \simeq \sum_{\xi \in \mathbb{Z}^n} |\xi|^{2k} |\hat{u}(\xi)|^2, \tag{4.60}$$

where the sign $\simeq$ means the equivalence of the two expressions in the sense that their ratio is bounded from above and below by positive constants.

Using (4.60) as motivation, we can introduce the norm $\|u\|_{W^s(Q)}$ for all positive *real* values of $s$ by setting

$$\|u\|^2_{W^s(Q)} = \sum_{\xi \in \mathbb{Z}^n} |\xi|^{2s} |\hat{u}(\xi)|^2,$$

and define the space $W^s(Q)$ as the set containing all $u \in L^2(Q)$ with compact[3] $\operatorname{supp} u$ and with $\|u\|_{W^s(Q)} < \infty$.

As in the proof of Theorem 4.15, one can show that if $u \in W^s(Q)$ and $s > m + \frac{n}{2}$ then $u \in C^m(Q)$.

Note that one can define also spaces $C^t(Q)$ for positive real values of parameter $t$. For simplicity, let us restrict ourselves to the case $0 < t < 1$. Then $C^t(Q)$ is the space of functions $u$ in $Q$ that are Hölder continuous with the Hölder exponent $t$, that is,

$$|u(x) - u(y)| \leq C |x - y|^t$$

for some constant $C$. The norm in $C^t(Q)$ is defined by

$$\|u\|_{C^t(Q)} = \|u\|_{C(Q)} + \sup \frac{|u(x) - u(y)|}{|x - y|^t}.$$

Then the following is true: if $u \in W^s(Q)$ and $s > t + \frac{n}{2}$, where $s, t$ are non-negative reals, then $u \in C^t(Q)$ and

$$\|u\|_{C^t}(u) \leq C \|u\|_{W^s(Q)}.$$

[3]One can extend this definition to allow in $W^s(Q)$ functions whose support is not necessarily compact. However, we skip this direction.