Holonomy groups

in Riemannian geometry

Lecture 1

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Parallel translation in $\mathbb{R}^n$

- $\gamma\colon [0,1] \to \mathbb{R}^n$ arbitrary (smooth) curve
- $v\colon [0,1] \to \mathbb{R}^n$ vector field along γ

Then v is *parallel*, if $\dot{v} = \frac{dv}{dt} \equiv 0$.

Parallel translation on curved spaces

- $S \subset \mathbb{R}^n$ oriented hypersurface (e.g. $S^2 \subset \mathbb{R}^3$)
- n unit normal vector along S
- $\gamma\colon [0,1] \to S$ curve
- $v\colon [0,1] \to \mathbb{R}^n$ vector field along γ s.t.

$$v(t) \in T_{\gamma(t)}S \quad \Leftrightarrow \quad \big\langle v(t), n(\gamma(t))\big\rangle = 0 \qquad \forall t \qquad (1)$$

(1) $\Rightarrow$ v can not be constant in t. The eqn $\dot{v} = 0$ is replaced by

$$\mathrm{proj}_{TS}\dot{v} = 0 \quad \Leftrightarrow \quad \dot{v} - \langle \dot{v}, n(\gamma)\rangle n(\gamma) = 0.$$

Differentiating (1) we obtain a first order ODE for *parallel* v:

$$\boxed{\dot{v} + \langle v, \tfrac{d}{dt}n(\gamma)\rangle n(\gamma) = 0}$$

Parallel transport

Parallel transport is a linear isomorphism

$$P_\gamma\colon T_{\gamma(0)}S \to T_{\gamma(1)}S, \qquad v_0 \mapsto v(1)$$

where v is the solution of the problem

$$\dot{v} + \langle v, \tfrac{d}{dt}n(\gamma)\rangle n(\gamma) = 0, \qquad v(0) = v_0.$$

P_γ is an isometry, since

$$v, w \text{ are parallel} \Rightarrow \langle v(t), w(t)\rangle \text{ is constant in } t$$

Holonomy group

- $s \in S$ basepoint
- $Hol_s := \{P_\gamma \mid \gamma(0) = s = \gamma(1)\} \subset SO(T_s S)$ *based holonomy group*
- $Hol_{s'}$ is conjugated to Hol_s ("Holonomy group does not depend on the choice of the basepoint")
- Holonomy group is intrinsic to S, i.e. depends on the Riemannian metric on S but not on the embedding $S \subset \mathbb{R}^n$
- Ex: $Hol(S^2) = SO(2)$

Properties:

- ◇ definition generalises to any Riemannian manifold (M, g)
- ◇ encodes both local and global features of the metric
- ◇ "knows" about additional structures compatible with metric

Classification of holonomy groups

Berger's list, 1955

Assume M is a simply–connected irreducible nonsymmetric Riemannian mfld of dimension n. Then $Hol(M)$ is one of the following:

	Holonomy	Geometry	Extra structure
•	$SO(n)$		
•	$U(n/2)$	Kähler	complex
•	$SU(n/2)$	Calabi–Yau	complex + hol. vol.
•	$Sp(n/4)$	hyperKähler	quaternionic
•	$Sp(1)Sp(n/4)$	quaternionic Kähler	"twisted" quaternionic
•	G_2 (n=7)	exceptional	"octonionic"
•	$Spin(7)$ (n=8)	exceptional	"octonionic"

Plan

- General theory (torsion, Levi–Civita connection, Riemannian curvature, holonomy)
- Proof of Berger's theorem (Olmos 2005)
- Properties of manifolds with non–generic holonomies (some constructions, examples, curvature tensors. . .)

Holonomy groups

in Riemannian geometry

Lecture 2

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Smooth manifold comes equipped with a collection of charts $(U_\alpha, \varphi_\alpha)$, where $\{U_\alpha\}$ is an open covering and the maps $\varphi_\beta \circ \varphi_\alpha^{-1} \colon \mathbb{R}^n \to \mathbb{R}^n$ are smooth.

A Lie group G is a group which has a structure of a smooth mfld such that the structure maps, i.e. $m \colon G \times G \to G$, $\cdot^{-1} \colon G \to G$, are smooth.

$\mathfrak{g} := T_e G$ is a *Lie algebra*, i.e. a vector space endowed with a map $[\cdot,\cdot] \colon \Lambda^2 \mathfrak{g} \to \mathfrak{g}$ satisfying the Jacobi identity:

$$\big[\xi, [\eta, \zeta]\big] + \big[\eta, [\zeta, \xi]\big] + \big[\zeta, [\xi, \eta]\big] = 0.$$

Ex.

G	$GL_n(\mathbb{R})$	$GL_n(\mathbb{C})$	$SO(n)$	$U(n)$
$\mathfrak{g}$	$\operatorname{End} \mathbb{R}^n$	$\operatorname{End} \mathbb{C}^n$	$\{A^t = -A\}$	$\{\bar{A}^t = -A\}$

Identification: $\mathfrak{g} \cong \{$left-invariant vector fields on $G\}$

- $\xi_1, \ldots, \xi_n$ a basis of $\mathfrak{g}$
- $\omega_1, \ldots, \omega_n$ dual basis

$\omega := \sum \omega_i \otimes \xi_i \in \Omega^1(G; \mathfrak{g})$ canonical 1-form with values in $\mathfrak{g}$, which satisfies the Maurer–Cartan equation

$$d\omega + \frac{1}{2}[\omega \wedge \omega] = \sum_i d\omega_i \otimes \xi_i + \frac{1}{2}\sum_{i,j} \omega_i \wedge \omega_j \otimes [\xi_i, \xi_j] = 0.$$

Vector bundles

A vector bundle E over M satisfies:

- E is a manifold endowed with a submersion $\pi\colon E \to M$
- $\forall m \in M\ E_m := \pi^{-1}(m)$ has the structure of a vector space
- $\forall m \in M \quad \exists U \ni m$ s.t. $\quad \pi^{-1}(U) \cong U \times E_m$

$\Gamma(E) = \{s\colon M \to E \mid \pi \circ s = id_M\}$ space of sections of E

Ex.

E	$\Gamma(E)$	
TM	$\mathfrak{X}(M)$	vector fields
$\Lambda^k T^*M$	$\Omega^k(M)$	differential k-forms
$T^p_q(M) := \bigotimes^p TM \otimes \bigotimes^q T^*M$	?	tensors of type (p, q)

de Rham complex

Exterior derivative $d\colon \Omega^k \to \Omega^{k+1}$ is the unique map with the properties:

- df is the differential of f for $f \in \Omega^0(M) = C^\infty(M)$
- $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$, if $\alpha \in \Omega^p$
- $d^2 = 0$

Thus, we have the de Rham complex:

$$0 \to \Omega^0 \to \Omega^1 \to \cdots \to \Omega^n \to 0, \qquad n = \dim M.$$

Betti numbers:

$$b_k = \dim H^k(M;\mathbb{R}) = \dim \frac{\operatorname{Ker} d\colon \Omega^k \to \Omega^{k+1}}{\operatorname{im} d\colon \Omega^{k-1} \to \Omega^k}.$$

Lie bracket of vector fields

A vector field can be viewed as an $\mathbb{R}$–linear derivation of the algebra $C^\infty(M)$. Then $\mathfrak{X}(M)$ is a Lie algebra:

$$[v,w]\cdot f = v\cdot(w\cdot f) - w\cdot(v\cdot f).$$

The exterior derivative and the Lie bracket are related by

$$2d\omega(v,w) = v\cdot\omega(w) - w\cdot\omega(v) - \omega([v,w])$$

Rem. “2” is optional in the above formula.

Lie derivative

For $v \in \mathfrak{X}(M)$ let φ_t be the corresponding 1-parameter (semi)group of diffeomorphisms of M, i.e.

$$\frac{d}{dt}\varphi_t(m) = v(\varphi_t(m)), \qquad \varphi_0 = id_M.$$

The *Lie derivative* of a tensor S is defined by

$$\mathcal{L}_v S = \frac{d}{dt}\Big|_{t=0} \varphi_t^* S$$

In particular, this means:

$$\mathcal{L}_v f(m) = \frac{d}{dt}\Big|_{t=0} f(\varphi_t(m)) = df_m\big(v(m)\big), \qquad \text{if } f \in C^\infty(M),$$
$$\mathcal{L}_v w(m) = \frac{d}{dt}\Big|_{t=0} \big(d\varphi_t\big)_m^{-1} w(\varphi_t(m)), \qquad \text{if } w \in \mathfrak{X}(M)$$

Properties of the Lie derivative

- $\mathcal{L}_v(S \otimes T) = (\mathcal{L}_v S) \otimes T + S \otimes (\mathcal{L}_v T)$
- $\mathcal{L}_v w = [v, w]$ for $w \in \mathfrak{X}(M)$
- $[\mathcal{L}_v, \mathcal{L}_w] = \mathcal{L}_{[v,w]}$
- Cartan formula

$$\boxed{\mathcal{L}_v \omega = \imath_v d\omega + d(\imath_v \omega)} \qquad \text{where } \omega \in \Omega(M).$$

- $[\mathcal{L}_v, d] = 0$ on $\Omega(M)$

Connections on vector bundles

Def. *A connection* on E is a linear map $\nabla\colon \Gamma(E) \to \Gamma(T^*M \otimes E)$ satisfying the Leibnitz rule:

$$\nabla(fs) = df \otimes s + f\nabla s, \qquad \forall f \in C^\infty(M) \quad \text{and } \forall s \in \Gamma(E)$$

For $v \in \mathfrak{X}(M)$ we write

$$\nabla_v s = v \cdot \nabla s, \qquad \text{where "} \cdot \text{"is a contraction.}$$

Then

$$\nabla_{\alpha v}(\beta s) = \alpha \nabla_v(\beta s) = \alpha(v \cdot \beta)\nabla_v s + \alpha\beta\nabla_v s.$$

Curvature

Prop. *For $v, w \in \mathfrak{X}(M)$ and $s \in \Gamma(E)$ the expression*

$$\nabla_v(\nabla_w s) - \nabla_w(\nabla_v s) - \nabla_{[v,w]} s$$

is $C^\infty(M)$–linear in v, w, and s.

Def. The unique section $R = R(\nabla)$ of $\Lambda^2 T^*M \otimes \operatorname{End}(E)$ satisfying

$$R(\nabla)(v \wedge w \otimes s) = \nabla_v(\nabla_w s) - \nabla_w(\nabla_v s) - \nabla_{[v,w]} s$$

is called the *curvature* of the connection ∇.

Choose local coordinates $(x_1, \dots, x_n)$ on M

$v_i := \frac{\partial}{\partial x_i} \qquad \Rightarrow \qquad [v_i, v_j] = 0$

Then $R(v_i, v_j)s = \nabla_v(\nabla_w s) - \nabla_w(\nabla_v s)$

Think of $\nabla_{v_i} s$ as "partial derivative" of s

Curvature measures how much "partial derivatives" of sections of E fail to commute.

Twisted differential forms

Denote $\Omega^k(E) := \Gamma(\Lambda^k T^* M \otimes E)$

Then $\nabla \colon \Omega^0(E) \to \Omega^1(E)$ extends uniquely to $d^\nabla \colon \Omega^k(E) \to \Omega^{k+1}(E)$ via the rule

$$d^\nabla(\omega \otimes s) = d\omega \otimes s + (-1)^k \omega \wedge \nabla s$$

We obtain the sequence

$$\Omega^0(E) \xrightarrow{\nabla = d^\nabla} \Omega^1(E) \xrightarrow{d^\nabla} \Omega^2(E) \xrightarrow{d^\nabla} \dots \xrightarrow{d^\nabla} \Omega^n(E) \qquad (1)$$

Then $$\boxed{(d^\nabla \circ d^\nabla)\sigma = R(\nabla) \cdot \sigma}$$

Curvature measures the extend to which sequence (1) fails to be a complex.

Principal bundles

Let G be a Lie group

A principal bundle P over M satisfies:

- P is a manifold endowed with a submersion $\pi \colon P \to M$
- G acts on P on the right and $\pi(p \cdot g) = \pi(p)$
- $\forall m \in M$ the group G acts freely and transitively on $P_m := \pi^{-1}(m)$. Hence $P_m \cong G$
- Local triviality: $\forall m \in M \quad \exists U \ni m$ s.t. $\quad \pi^{-1}(U) \cong U \times G$

Example: Frame bundle

Let $E \to M$ be a vector bundle. A *frame* at a point m is a linear isomorphism $p \colon \mathbb{R}^k \to E_m$.

$$Fr(E) := \bigcup_{m,p} \{(m,p) \mid p \text{ is a frame at } m\}$$

(i) $GL(k;\mathbb{R}) = Aut(\mathbb{R}^k)$ acts freely and transitively on $Fr_m(E)$:

$$p \cdot g = p \circ g.$$

(ii) A *moving frame* on $U \subset M$ is a set $\{s_1, \ldots, s_k\}$ of pointwise linearly independent sections of E over U. This gives rise to a section s of $Fr(E)$ over U:

$$s(m)x = \sum x_i s_i(m), \qquad x \in \mathbb{R}^k.$$

By *(i)* this defines a trivialization of $Fr(E)$ over U.

Frame bundle: variations

If in addition E is

- *oriented*, i.e. $\Lambda^{\text{top}} E$ is trivial, $Fr^+(E)$ is a principal $GL^+(k;\mathbb{R})$–bundle
- *Euclidean* Fr_O is a principal $O(k)$–bundle
- *Hermitian* Fr_U is a principal $U(k)$–bundle
- *quaternion–Hermitian* is a principal $Sp(k)$–bundle
- ......

Def. Let G be a subgroup of $GL(n;\mathbb{R})$, $n = \dim M$. A *G–structure* on M is a principal G–subbundle of $Fr_M = Fr(TM)$.

- orientation $\Leftrightarrow$ $GL^+(n;\mathbb{R})$–structure
- Riemannian metric $\Leftrightarrow$ $O(n)$–structure
- ......

Associated bundle

$P \to M$ principal G–bundle
V G–representation, i.e. a homomorphism $\rho\colon G \to GL(V)$ is given

$$P \times_G V := \big(P \times V\big)/G, \qquad \text{action: } (p, v)\cdot g = (pg, \rho(g^{-1})v)$$

is called the *bundle associated to P with fibre V*.

Ex. For $P = Fr_M$, $G = GL(n;\mathbb{R})$, and $E = P \times_G V$ we have

- $E = TM$ for $V = \mathbb{R}^n$ (tautological representation)
- $E = T^*M$ for $V = \big(\mathbb{R}^n\big)^*$
- $E = \Lambda^k T^*M$ for $V = \Lambda^k\big(\mathbb{R}^n\big)^*$

Sections of associated bundles correspond to equivariant maps:

$$\big\{f\colon P \to V \mid f(pg) = \rho(g^{-1})f(p)\big\} \equiv \Gamma(E)$$
$$f \mapsto s_f, \qquad s_f(m) = [p, f(p)], \quad p \in P_m$$

Connection as horizontal distribution

For $\xi \in \mathfrak{g}$ the Killing vector at $p \in P$ is given by

$$K_\xi(p) := \frac{d}{dt}\Big|_{t=0} (p \cdot \exp t\xi)$$

$\mathcal{V}_p = \{K_\xi(p) \mid \xi \in \mathfrak{g}\} \cong \mathfrak{g}$ is called *vertical space* at p

Def. A connection on P is a subbundle $\mathcal{H}$ of TP satisfying

(i) $\mathcal{H}$ is G–invariant, i.e. $\mathcal{H}_{pg} = (R_g)_* \mathcal{H}_p$

(ii) $TP = \mathcal{V} \oplus \mathcal{H}$

$\mathcal{H}$ is called a *horizontal* bundle.

Connection as a 1–form

Given a connection on P, define $\omega \in \Omega^1(P; \mathfrak{g})$ as follows

$$T_pP \to \mathcal{V}_p \cong \mathfrak{g}$$

ω is called the *connection form* and satisfies:

(a) $\omega(K_\xi) = \xi$

(b) $R_g^* \omega = ad_{g^{-1}}\, \omega$, where ad denotes the adjoint representation

Prop. *Every $\omega \in \Omega^1(P; \mathfrak{g})$ satisfying (a) and (b) defines a connection via*

$$\mathcal{H} = \operatorname{Ker} \omega.$$

Horizontal lift

$\mathrm{Ker}(\pi_*)_p = \mathcal{V}_p$. Hence $(\pi_*)_p \colon \mathcal{H}_p \to T_{\pi(p)}M$ is an isomorphism. In particular, $\mathcal{H} \cong \pi^*TM$. Hence, we have

Prop. *For any $w \in \mathfrak{X}(M)$ there exists $\tilde{w} \in \mathfrak{X}(P)$ s.t.*

(i) *$\tilde{w}$ is G–invariant and horizontal*

(ii) $(\pi_*)_p \tilde{w} = w(\pi(p))$

Vice versa, if $\tilde{w} \in \mathfrak{X}(P)$ is G–invariant and horizontal, then $\exists!\ w \in \mathfrak{X}(M)$ s.t. $\pi_* \tilde{w} = w$.

Invariant and equivariant forms

$\tilde{\alpha} \in \Omega^k(P)$ is called *basic* if $\imath_v \tilde{\alpha} = 0$ for any vertical vector field v.

Then $\forall \alpha \in \Omega^k(M)$ the form $\tilde{\alpha} = \pi^*\alpha$ is G–invariant and basic. On the other hand, any G–invariant and basic k–form $\tilde{\alpha}$ on P induces a k–form on M. **Notice:** no connection required here.

V is a representation of G
$\tilde{\alpha} \in \Omega^k(P; V)$ is G–equivariant if $R_g^* \tilde{\alpha} = \rho(g^{-1})\tilde{\alpha}$.

Ex. Connection 1-form is an equivariant form for $V = \mathfrak{g}$.

For basic and equivariant forms we have the identification

$$\Omega^k_{G,bas}(P, V) \cong \Omega^k(M; E), \qquad \pi^*\alpha \leftarrow\!\shortmid \alpha$$

Curvature tensor

Prop. *Let ω be a connection form. The 2–form $\tilde{F}_\omega = d\omega + \frac{1}{2}[\omega \wedge \omega]$ is basic and G–equivariant, i.e. $R_g^* \tilde{F} = ad_{g^{-1}} \tilde{F}$.*

Cor. *Denote $ad\,P := P \times_{G,ad} \mathfrak{g}$. Then there exists $F \in \Omega^2(M; ad\,P)$ s.t. $\pi^* F = \tilde{F}$.*

The 2-form F is called the *curvature form* of the connection ω. The defining equation for F is often written as

$$d\omega = -\frac{1}{2}[\omega \wedge \omega] + F$$

and is called the *structural equation*.

Covariant differentiation

$P \to M$ G–bundle, $\rho\colon G \to GL(V)$, $E := P \times_G V$,
$f\colon P \to V$ equivariant map, i.e. section of E.

Def. $\nabla f = d^h f = df\big|_{\mathcal{H}}$ is called the covariant derivative of f.

Rem. Denote $\tau = d\rho_e\colon \mathfrak{g} \to \mathfrak{gl}(V) = \mathrm{End} V$. Then for a vertical vector $K_\xi(p)$ we have: $df(K_\xi(p)) = -\tau(\xi) f(p)$, that is all information about df is contained in $d^h f$.

Prop.

$$\nabla f = df + \omega \cdot f$$

Here "$\cdot$" means the action of $\mathfrak{g}$ on V via the map τ.

Prop. *$\nabla f \in \Omega^1(P;V)$ is G–equivariant and basic form.*

Thus ∇f can be interpreted as an element of $\Omega^1(M;E)$ and we have a diagram

$$\begin{array}{ccc} Map^G(P;V) & \xrightarrow{\nabla} & \Omega^1_{G,bas}(P;V) \\ \Vert & & \Vert \\ \Gamma(E) & \xrightarrow{\nabla^E} & \Omega^1(M;E) \end{array} \qquad \begin{array}{ccc} f & \longmapsto & \nabla f \\ \downarrow & & \downarrow \\ s_f & \longmapsto & \nabla^E s_f \end{array}$$

Prop. *∇^E is a connection on E.*

Bianchi identity

ω connection on P, F curvature
$ad\, P$ has an induced connection ∇

Theorem (Bianchi identity)

$$\boxed{d^{\nabla} F = 0}$$

Proof. For $\tilde{\varphi} \in \Omega^k(P;\mathfrak{g})$ denote $D\tilde{\varphi} = d\tilde{\varphi} + [\omega \wedge \tilde{\varphi}]$

Step 1. For any $\varphi \in \Omega^k(M; ad\, P)$ we have $\widetilde{d^{\nabla}\varphi} = D\tilde{\varphi}$.

Can assume $\varphi = s \cdot \varphi_0$, where $\varphi_0 \in \Omega^k(M)$ and

$$\Gamma(ad\, P) \ni s \leftrightsquigarrow f \in Map^G(P;\mathfrak{g}).$$

Then

$$\begin{aligned} \widetilde{d^{\nabla}\varphi} &= \widetilde{\nabla s} \wedge \tilde{\varphi}_0 + \tilde{s} \cdot d\tilde{\varphi}_0 \\ &= (df + [\omega, f]) \wedge \tilde{\varphi}_0 + f\, d\tilde{\varphi}_0 \\ &= d(f\tilde{\varphi}_0) + [\omega \wedge f\tilde{\varphi}_0] \\ &= D\varphi \end{aligned}$$

Proof of the Bianchi identity (continued)

Step 2. $D\tilde{F} = 0$, where $\tilde{F} = d\omega + \frac{1}{2}[\omega \wedge \omega]$.

$$\begin{aligned} d\tilde{F} &= \tfrac{1}{2}\big([d\omega \wedge \omega] - [\omega \wedge d\omega]\big) \\ &= [d\omega \wedge \omega] \\ &= [\tilde{F} \wedge \omega] - \tfrac{1}{2}[[\omega \wedge \omega] \wedge \omega] \end{aligned}$$

$$\text{Jacobi identity} \implies [[\omega \wedge \omega] \wedge \omega] = 0$$

Thus, $D\tilde{F} = 0 \iff d^{\nabla}F = 0.$ □

Horizontal lift of a curve

$\gamma\colon [0,1] \to M$ (piecewise) smooth curve, $p_0 \in P_{\gamma(0)}$.

Prop. *[KN, Prop. II.3.1] For any γ there exists a unique horizontal lift of γ through p_0, i.e. a curve $\Gamma\colon [0,1] \to P$ with the following properties:*

(i) $\dot{\Gamma}(t) \in \mathcal{H}_{\Gamma(t)}$ *for any $t \in [0,1]$ ("Γ is horizontal")*

(ii) $\Gamma(0) = p_0$

(iii) $\pi \circ \Gamma = \gamma$

Sketch of the proof. Let Γ_0 be an arbitrary lift of γ, $\Gamma_0(0) = p_0$. Then $\Gamma = \Gamma_0 \cdot g$ for some curve $g\colon [0,1] \to G$. Hence,

$$\dot{\Gamma} = \dot{\Gamma}_0 \cdot g + \Gamma_0 \cdot \dot{g} \quad \implies \quad \omega(\dot{\Gamma}) = ad_{g^{-1}}\omega(\dot{\Gamma}_0) + g^{-1}\dot{g}.$$

Then there exists a unique curve g, $g(0) = e$, such that
$g^{-1}\dot{g} + ad_{g^{-1}}\omega(\dot{\Gamma}_0) = 0 \quad \iff \quad \omega(\dot{\Gamma}) = 0.$ □

Parallel transport

$\gamma\colon [0,1] \to M,\ \gamma(0) = m,\ \gamma(1) = n$
Parallel transport $\Pi_\gamma\colon P_m \to P_n$ is defined by

$$\Pi_\gamma(p) = \Gamma(1),$$

where Γ is the horizontal lift of γ satisfying $\Gamma(0) = p$.

Prop.

(i) Π_γ *commutes with the action of* G *for any curve* γ

(ii) Π_γ *is bijective*

(iii) $\Pi_{\gamma_1 * \gamma_2} = \Pi_{\gamma_1} \circ \Pi_{\gamma_2}, \quad \Pi_{\gamma^{-1}} = \Pi_\gamma^{-1}$

Holonomy group

Denote $\Omega_m := \{\text{piecewise smooth loops in } M \text{ based at } m\}$

$$\boxed{Hol_p(\omega) := \{g \in G \mid \exists \gamma \in \Omega_m \text{ s.t. } \Pi_\gamma(p) = pg\}}$$

Prop.

(i) Hol_p *is a Lie group*

(ii) $Hol_{pg} = Ad_{g^{-1}}(Hol_p)$

Proof. Group structure follows from *(iii)* of the previous Prop.
For the structure of Lie group see [Kobayashi–Nomizu, Thm 4.2].
Statement *(ii)* follows from the observation

$$\Gamma \text{ is horizontal} \implies R_g \circ \Gamma \text{ is also horizontal.}$$

□

Reduction of connections

Let $H \subset G$ be a Lie subgroup and $Q \subset P$ be a principal H–bundle ("structure group reduces to H").

Def. A connection $\mathcal{H}$ on P reduces to Q if $\mathcal{H}_q \subset T_qQ \quad \forall q \in Q$.

Prop. *A connection reduces to Q $\iff$ $\imath^*\omega$ takes values in $\mathfrak{h}$, where $\imath\colon Q \hookrightarrow P$.*

Proof. $(\Rightarrow)$:

$$\begin{array}{ccc} T_qQ \cong \mathcal{H}_q \oplus \mathfrak{h} & \xrightarrow{(0,\, id)} & \mathfrak{h} \\ \big\downarrow & & \big\downarrow \\ T_qP & \xrightarrow{\ \omega\ } & \mathfrak{g} \end{array}$$

$(\Leftarrow)$: $\imath^*\omega$ is a connection on Q, hence $TQ = \mathcal{H}^Q \oplus \mathfrak{h}$. Since $\mathcal{H}^Q \subset \mathcal{H}^P$ and $\operatorname{rk}\mathcal{H}^P = \dim M = \operatorname{rk}\mathcal{H}^Q$, we obtain $\mathcal{H}^Q = \mathcal{H}^P$. □

Lie groups | Vector bundles | Principal bundles | Connections on G–bundles | Holonomy | Torsion

Reduction theorem

For $p_0 \in P$ define the *holonomy bundle* through p_0 as follows:

$Q(p_0) := \{p \in P \mid \exists$ a horizontal curve Γ s.t. $\Gamma(0) = p_0,\ \Gamma(1) = p\}$.

Theorem ("Reduction theorem")

Put $H = Hol_{p_0}(P, \omega)$. Then the following holds:

(i) Q is a principal H–bundle

(ii) connection ω reduces to Q

Proof. *(i):* $p \in Q,\ g \in H \ \Rightarrow\ pg \in Q$ (by the def of H).

Exercise: Show that $Hol_p(\omega) = H \quad \forall p \in Q$.

From the def of Q follows, that H acts transitively on fibres.

Local triviality: Use parallel transport over coordinate chart U wrt segments to obtain a local section of Q (see [KN, Thm II.7.1] for details).

(ii): Follows immediately from the def of Q. □

Parallel transport and covariant derivative

Let $\Gamma\colon [0,1] \to P$ be a horizontal lift of γ
$\Gamma_E(t) := [\Gamma(t), v], \quad v \in V, \;\; E = P \times_G V$
$\Gamma_E\colon [0,1] \to E$ is called the horizontal lift of γ to E
$\Pi_t\colon E_{\gamma(t)} \to E_m$ parallel transport in E, $m = \gamma(0)$

Lem. $\nabla_w s = \lim_{t\to 0} \frac{1}{t}\Big(\Pi_t s\big(\gamma(t)\big) - s(m)\Big)$, *where* $w = \dot\gamma(0)$.

Proof. Let $s \leftrightsquigarrow f$, i.e. $[p, f(p)] = s(\pi(p))$. First observe that

$$\Pi^E_\gamma [p, v] = [\Pi_\gamma p, v].$$

Since $\big[\Gamma(t), f(\Gamma(t))\big] = s(\gamma(t))$, we obtain

$$\Pi_t s = \big[p, f(\Gamma(t))\big].$$

⇓ to be continued ⇓

Lem. $\nabla_w s = \lim_{t\to 0} \frac{1}{t}\Big(\Pi_t s\big(\gamma(t)\big) - s(m)\Big)$, *where* $w = \dot\gamma(0)$.

Proof. Let $s \leftrightsquigarrow f$, i.e. $[p, f(p)] = s(\pi(p))$. First observe that

$$\Pi^E_\gamma [p, v] = [\Pi_\gamma p, v].$$

Since $\big[\Gamma(t), f(\Gamma(t))\big] = s(\gamma(t))$, we obtain

$$\Pi_t s = \big[p, f(\Gamma(t))\big].$$

Then

$$\begin{aligned}
\nabla_w s &= [p, df(\tilde w)] \\
&= \big[p, \tfrac{d}{dt}\big|_{t=0} f \circ \Gamma(t)\big] \\
&= \lim_{t\to 0} \tfrac{1}{t}\Big(\big[p, f(\Gamma(t))\big] - [p, f(p)]\Big) \\
&= \lim_{t\to 0} \tfrac{1}{t}\Big(\Pi_t s\big(\gamma(t)\big) - s(m)\Big).
\end{aligned}$$

□

Rem. Let $w \in \mathfrak{X}(M)$. If $s \leftrightsquigarrow f$, then $\nabla_w s \leftrightsquigarrow df(\tilde{w})$.

Lem. *Let $s \in \Gamma(E)$, $s_0 = s(m)$. Assume $\nabla s = 0$. Then for any loop γ based at m we have $\Pi_\gamma^E s_0 = s_0$.*

Proof. Let Γ be a horizontal lift of γ. Then $f \circ \Gamma = const$. Hence $\Pi_t s(\gamma(t)) = [p, f \circ \Gamma]$ does not depend on t. □

V is a G–representation, $H = Stab_\eta$, where $\eta \in V$.
$Q \subset P$ is a principal H–subbundle
The constant function $q \mapsto \eta$ can be extended to an equivariant function η on P

Theorem

ω reduces to Q $\iff$ $\nabla^E \eta = 0$.

Proof. ($\Rightarrow$): $\forall q \in Q \; d\eta\big|_{\mathcal{H}_q} = 0$, since η is constant on Q and $\mathcal{H} \subset TQ$.

($\Leftarrow$): For any $q \in Q$ we have

$$[q, \eta] = \Pi_\gamma^E [q, \eta] = \big[\Pi_\gamma q, \eta\big] = [qg, \eta] = [q, \rho(g^{-1})\eta].$$

Hence $Hol_q(\omega) \subset H$. Then the holonomy bundle through q is contained in Q. Therefore, ω reduces to Q. □

Ambrose–Singer theorem

Theorem (Ambrose–Singer)

Let Q be the holonomy bundle through p_0, $\tilde F \in \Omega^2(P;\mathfrak{g})$ curvature of ω. Then

$$\mathfrak{hol}_{p_0} = \operatorname{span}\{\tilde F_q(w_1, w_2) \mid q \in Q,\ w_1, w_2 \in \mathcal{H}_q\}.$$

Sketch of the proof. Can assume $Q = P$. Denote

$$\mathfrak{g}' = \operatorname{span}\{\tilde F_q(w_1, w_2) \mid q \in Q,\ w_1, w_2 \in \mathcal{H}_q\} \subset \mathfrak{g}.$$

Further, $S_p := \mathcal{H}_p \oplus \{K_\xi(p) \mid \xi \in \mathfrak{g}'\}$. Then the distribution S is integrable. If $P_0 \ni p_0$ is a maximal integral submanifold, then $P_0 = P$, since each horizontal curve must lie in P_0. Then $\dim\mathfrak{g} = \dim P - \dim M = \dim P_0 - \dim M = \dim\mathfrak{g}'$. Hence $\mathfrak{g} = \mathfrak{g}'$. □

From now on $P = Fr(M)$ is the principal $G = GL_n(\mathbb{R})$–bundle of linear frames

Def. A *canonical 1–form* $\theta \in \Omega^1(P;\mathbb{R}^n)$ is given by

$$\theta(v) = p^{-1}(d\pi(v)), \qquad v \in T_pP.$$

Rem. θ is defined for bundles of linear frames only.

θ is G–equivariant in the following sense: $R_g^*\theta = g^{-1}\theta$. Indeed, for any $v \in T_pP$ we have

$$R_g^*\theta(v) = (pg)^{-1}\big(d\pi(R_g v)\big) = g^{-1}p^{-1}\big(d\pi(v)\big) = g^{-1}\theta(v).$$

Torsion

ω is a connection on $Fr(M)$. In particular, ω is $\mathfrak{gl}_n(\mathbb{R})$–valued. Thus, we have induced connections on TM, T^*M, $\Lambda^k T^*M \ldots$

Def. $\Theta = d\theta + \frac{1}{2}[\omega, \theta] \in \Omega^2(Fr(M); \mathbb{R}^n)$ is called the *torsion form* of ω.

Rem. $[\omega, \theta](v, w) = \omega(v)\theta(w) - \omega(w)\theta(v)$.

Prop. *Θ is horizontal and equivariant. Hence there exists $T \in \Omega^2(M; TM)$ s.t. $2\Theta = \pi^* T$.*

T can be viewed as a skew–symmetric linear map $TM \otimes TM \to TM$ and is called the *torsion tensor*.

Theorem

For $v, w \in \mathfrak{X}(M)$ we have

$$\boxed{T(v, w) = \nabla_v w - \nabla_w v - [v, w]}$$

Proof. Represent v, w by equivariant functions $f_v, \ f_w \colon Fr \to \mathbb{R}^n$. Then $\nabla_v w$ is represented by $df_w(\tilde{v})$.

For the bundle of frames, $f_w = \theta(\tilde{w})$. Hence $\nabla_v w = p\big(\tilde{v} \cdot \theta(\tilde{w})\big)$. Therefore we obtain

$$\begin{aligned} T(v, w) &= p\big(2\Theta(\tilde{v}, \tilde{w})\big) \\ &= p\big(\tilde{v} \cdot \theta(\tilde{w}) - \tilde{w} \cdot \theta(\tilde{v}) - \theta([\tilde{v}, \tilde{w}])\big) \\ &= \nabla_v w - \nabla_w v - [v, w]. \end{aligned}$$

The last equality follows from $[\tilde{v}, \tilde{w}]^h = \widetilde{[v, w]}$ (*exercise*). □

Denote
$\Gamma(T^*M) \xrightarrow{\nabla} \Gamma(T^*M \otimes T^*M) \xrightarrow{\mathrm{Alt}} \Omega^2(M), \quad \alpha \mapsto \mathrm{Alt}(\nabla\alpha).$

Theorem

$$\mathrm{Alt}(\nabla\alpha) = d\alpha - \alpha \circ T$$

In particular, for torsion–free connections $\mathrm{Alt}(\nabla\alpha) = d\alpha$.

Proof. This follows from the previous Thm with the help of the formulae $v \cdot \alpha(w) = \nabla_v(\alpha(w)) = (\nabla_v\alpha)(w) + \alpha(\nabla_v w)$. □

Holonomy groups

in Riemannian geometry

Lecture 3

November 3, 2011

Recap of the previous lecture

$$Fr(M) := \bigcup_{m,p} \{(m,p) \mid p\colon \mathbb{R}^n \xrightarrow{\cong} T_mM\}$$ frame bundle;

$$\theta(v) = p^{-1}(d\pi(v)),\ v \in T_pFr(M)$$ canonical 1–form

$$\Theta = d\theta + \tfrac{1}{2}[\omega, \theta] \in \Omega^2(Fr(M); \mathbb{R}^n),$$ torsion form

$$\exists T \in \Omega^2(M; TM), \text{s.t.} \qquad 2\Theta = \pi^*T,$$ torsion tensor

$$T(v,w) = \nabla_v w - \nabla_w v - [v,w], \quad v,w \in \mathfrak{X}(M)$$

$$\mathrm{Alt}(\nabla\alpha) = d\alpha - \alpha \circ T, \quad \alpha \in \Omega^1(M)$$

Curvature tensor

For $P = Fr(M)$ we have $ad\,P = \mathrm{End}(TM)$. Then the curvature can be viewed as a skew–symmetric map

$$TM \otimes TM \to \mathrm{End}(TM), \qquad (v, w) \mapsto R(v, w).$$

R is called the *curvature tensor*.

Theorem (KN, Thm. II.5.1)

For $v, w, x \in \mathfrak{X}(M)$ we have

$$R(v, w)x = [\nabla_v, \nabla_w]x - \nabla_{[v,w]}x.$$

Theorem

For any G–bundle P the space $\mathcal{A}(P)$ of all connections is an affine space modelled on $\Omega^1(M; ad\,P)$.

Proof. Pick an arbitrary connection ω on P. Then for any $\omega' \in \mathcal{A}(P)$, the 1-form $\xi = \omega - \omega'$ is basic and ad–equivariant. Vice versa, for any basic and equivariant 1–form ξ, the form $\omega' = \omega - \xi$ is a connection. Hence, the statement of the thm. □

Assume $G \subset GL_n(\mathbb{R})$ and therefore $\mathfrak{g} \subset \mathfrak{gl}_n(\mathbb{R}) \cong (\mathbb{R}^n)^* \otimes \mathbb{R}^n$.
$Fr(M) \supset P$ is a G–bundle, $\omega, \omega' \in \mathcal{A}(P)$, $\xi = \omega - \omega'$.
For any $p \in P$, the map $\theta_p \colon \mathcal{H}_p \to \mathbb{R}^n$ is an isomorphism.
Therefore we can write

$$\xi_p \in (\mathbb{R}^n)^* \otimes \mathfrak{g}, \qquad T_p \colon \Lambda^2\mathbb{R}^n \cong \Lambda^2\mathcal{H}_p \xrightarrow{\Theta_p} \mathbb{R}^n.$$

Then

$$\Theta' - \Theta = \tfrac{1}{2}[\xi, \theta] \iff \big(T'_p - T_p\big)x \wedge y = \tfrac{1}{2}\big(\xi_p(x)y - \xi_p(y)x\big).$$

Consider the G–equivariant homomorphism

$$\delta \colon (\mathbb{R}^n)^* \otimes \mathfrak{g} \hookrightarrow (\mathbb{R}^n)^* \otimes (\mathbb{R}^n)^* \otimes \mathbb{R}^n \longrightarrow \Lambda^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n.$$

Then, $T' - T = \delta\xi$.

Prop. *P has a torsion–free connection if and only if $T_p \in \operatorname{Im}\delta$ for all $p \in P$.*

Torsion Levi–Civita con–n Decomposable metrics Symmetric spaces Berger Thm

(M, g) Riemannian manifold (by default, M **is oriented**)
$Fr(M) \supset P$ is the $G = SO(n)$–bundle of orthonormal oriented frames

We have the commutative diagram of $SO(n)$–representations:

$$\begin{array}{ccc} \mathfrak{so}(n) & \hookrightarrow & \mathfrak{gl}_n(\mathbb{R}) = \operatorname{End}\mathbb{R}^n \\ \cong\downarrow & & \downarrow\cong \\ \Lambda^2\mathbb{R}^n & \hookrightarrow & \mathbb{R}^n \otimes \mathbb{R}^n \end{array} \qquad (\mathbb{R}^n)^* \cong \mathbb{R}^n.$$

Prop. *The map $\delta_{\mathfrak{so}(n)} \colon \mathbb{R}^n \otimes \Lambda^2\mathbb{R}^n \to \Lambda^2\mathbb{R}^n \otimes \mathbb{R}^n$ is an isomorphism.*

Proof. For $a = \sum a_{ijk} e_i \otimes e_j \wedge e_k$ we have (*exercise*):

$$\delta a = \frac{1}{2}\sum (a_{ijk} - a_{jik}) e_i \wedge e_j \otimes e_k.$$

Hence, if $a \in \operatorname{Ker}\delta$, then $a_{ijk} = a_{jik} = -a_{jki} = -a_{kji} = a_{kij} = a_{ikj} = -a_{ijk} \implies a = 0$. □

The Levi–Civita connection

Theorem ("Fundamental theorem of Riemannian geometry")

Any $SO(n)$–subbundle of $Fr(M)$ admits a unique torsion–free connection.

Theorem ("Fundamental theorem", reformulation)

For any Riemannian metric g there exists a unique torsion–free connection on $Fr(M)$ such that $\nabla g = 0$.

The unique connection in the "Fundamental thm" is called the *Levi–Civita* (or *Riemannian*) connection. The corresponding curvature tensor is called *Riemannian curvature tensor*.

For any $p \in P$ we have

$$R_p \colon \Lambda^2\mathbb{R}^n \cong \Lambda^2\mathcal{H}_p \longrightarrow \mathfrak{so}(n) \cong \Lambda^2\mathbb{R}^n.$$

Theorem ("algebraic Bianchi identity")

$R_p(x,y)z + R_p(y,z)x + R_p(z,x)y = 0$ *for all* $x, y, z \in \mathbb{R}^n$.

Proof. $d\theta + \frac{1}{2}[\omega, \theta] = \Theta = 0 \Rightarrow [d\omega, \theta] - [\omega, d\theta] = 0$. This implies the *first Bianchi identity*:

$$\begin{aligned}[R,\theta] &= [d\omega,\theta] + \tfrac{1}{2}\big[[\omega\wedge\omega],\theta\big] \\ &= [\omega, d\theta] + \tfrac{1}{2}\big[[\omega\wedge\omega],\theta\big] \\ &= -\tfrac{1}{2}\big[\omega,[\omega,\theta]\big] + \tfrac{1}{2}\big[[\omega\wedge\omega],\theta\big] \\ &= 0.\end{aligned}$$

$[R,\theta](px, py, pz) = 0 \iff$ algebraic Bianchi identity. □

Cor. $\langle R_p(x,y)z, t\rangle = \langle R_p(z,t)x, y\rangle$, *i.e.* $R_p \in S^2(\Lambda^2\mathbb{R}^n)$.

Proof. *Exercise.* □

Observation: If $V = V_1 \oplus V_2$ as G–representation, then $E = E_1 \oplus E_2$, where $E_i := P \times_G V_i$.

Determine irreducible components of the $SO(n)$–representation

$$\mathfrak{R} = \big\{ R \in \Lambda^2(\mathbb{R}^n)^* \otimes \mathfrak{so}(n) \mid R \text{ satisfies alg. Bianchi id.} \big\}.$$

We can decompose

$$\operatorname{End} \mathbb{R}^n = \mathfrak{so}(n) \oplus \operatorname{Sym} \mathbb{R}^n = \mathfrak{so}(n) \oplus \operatorname{Sym}_0 \mathbb{R}^n \oplus \mathbb{R},$$

where $\operatorname{Sym}_0 \mathbb{R}^n = \operatorname{Ker}(\operatorname{tr}\colon \operatorname{Sym} \mathbb{R}^n \to \mathbb{R})$. In other words,

$$\mathbb{R}^n \otimes \mathbb{R}^n \cong \Lambda^2 \mathbb{R}^n \oplus S^2_0 \mathbb{R}^n \oplus \mathbb{R}. \tag{1}$$

Prop. (1) *is decomposition into irreducible components if* $n \neq 4$. *For* $n = 4$ *we have in addition* $\Lambda^2 \mathbb{R}^4 = \Lambda^2_+ \mathbb{R}^4 \oplus \Lambda^2_- \mathbb{R}^4$.

Here: $*\colon \Lambda^m \mathbb{R}^{2m} \to \Lambda^m \mathbb{R}^{2m}$ is the Hodge operator, $*^2 = id$
$\Lambda^m_\pm \mathbb{R}^{2m}$ are eigenspaces corresponding to $\lambda = \pm 1$.

Think of $\bigotimes^4 \mathbb{R}^n$ as the space of quadrilinear forms on $(\mathbb{R}^n)^*$.
Consider the map

$$b(R)(\alpha, \beta, \gamma, \delta) = \tfrac{1}{3}\Big(R(\alpha, \beta, \gamma, \delta) + R(\beta, \gamma, \alpha, \delta) + R(\gamma, \alpha, \beta, \delta) \Big)$$

(cyclic permutation in the first 3 variables; *Bianchi map*). Then

- b is $SO(n)$–invariant
- $b^2 = b$
- $b\colon S^2(\Lambda^2 \mathbb{R}^n) \to S^2(\Lambda^2 \mathbb{R}^n)$

Hence, we have

$$S^2(\Lambda^2 \mathbb{R}^n) = \operatorname{Ker} b \oplus \operatorname{Im} b = \mathfrak{R} \oplus \Lambda^4 \mathbb{R}^n.$$

The *Ricci contraction* is the $SO(n)$–equivariant map

$$c\colon S^2(\Lambda^2\mathbb{R}^n) \to S^2\mathbb{R}^n, \qquad c(R)(x,y) = \operatorname{tr} R(x,\cdot,y,\cdot)$$

The *Kulkarni–Nomizu product* of $h, k \in S^2\mathbb{R}^n$ is the 4–tensor $h \owedge k$ given by

$$\begin{aligned} h \owedge k(\alpha,\beta,\gamma,\delta) = {} & h(\alpha,\gamma)k(\beta,\delta) + h(\beta,\delta)k(\alpha,\gamma) \\ & - h(\alpha,\delta)k(\beta,\gamma) - h(\beta,\gamma)k(\alpha,\delta). \end{aligned}$$

Prop.

- $h \owedge k = k \owedge h$;
- $h \owedge k \in \operatorname{Ker} b = \mathfrak{R}$;
- $q \owedge q = 2\, id_{\Lambda^2\mathbb{R}^n}$, *where* q *=standard scalar product on* $\mathbb{R}^n$.

Lem. *If* $n \geq 3$, *the map* $q \owedge \cdot\colon S^2\mathbb{R}^n \to \mathfrak{R}$ *is injective and its adjoint is the restriction of the Ricci contraction* $c\colon \mathfrak{R} \to S^2\mathbb{R}^n$.

Components of the Riemannian curvature tensor

Theorem

We have the following decomposition:

$$\mathfrak{R} \cong \mathbb{R} \oplus S^2_0\mathbb{R}^n \oplus \mathcal{W},$$

where $\mathcal{W} = \operatorname{Ker} c \cap \operatorname{Ker} b$. *If* $n \geq 5$, *each component is irreducible.*

Explicitly:

- $\frac{1}{n}\operatorname{tr} c(R) + c(R)_0$ are the components of R in $\mathbb{R} \oplus S^2_0\mathbb{R}^n$;
- the inclusions of the first two spaces are given by

$$\mathbb{R} \ni 1 \mapsto q \owedge q, \qquad S^2_0\mathbb{R}^n \ni h \mapsto q \owedge h. \tag{2}$$

Def. For the Riemannian curvature tensor R we define:

- $Ric(R) = c(R)$ *Ricci curvature*;
- $s = \operatorname{tr} c(R)$ *scalar curvature*, Ric_0 *traceless Ricci curvature*;
- $W(R) \in \operatorname{Ker} c \cap \operatorname{Ker} b$ *Weyl tensor*.

From (2) follows that $R = \lambda\, q \owedge q + \mu\, Ric_0 \owedge q + W$. The coefficients $\lambda,\ \mu$ can be determined from the equality $c(q \owedge h) = (n-2)h + (\operatorname{tr} h)q$. Hence, we obtain

$$\boxed{R = \frac{s}{2n(n-1)}\, q \owedge q + \frac{1}{n-2}\, Ric_0 \owedge q + W.}$$

Observe: Ric is a symmetric quadratic form on the tangent bundle.

Def. A Riemannian mfld (M, g) is called *Einstein*, if there exists $\lambda \in \mathbb{R}$ such that

$$Ric(g) = \lambda g.$$

Local expressions

Choose local coordinates $(x_1, \dots, x_n)$ on M and write:

$$\nabla_{\partial/\partial x_i} \tfrac{\partial}{\partial x_j} = \sum_k \Gamma_{ij}^k \tfrac{\partial}{\partial x_k}, \qquad g_{ij} = g\big(\tfrac{\partial}{\partial x_i}, \tfrac{\partial}{\partial x_j}\big), \quad (g^{ij}) = (g_{ij})^{-1}$$

Local functions Γ_{ij}^k are called *Chistoffel symbols*.

Theorem ([KN, Prop. III.7.6 + Cor. IV.2.4])

$$\Gamma_{ij}^k = \frac{1}{2} \sum_l g^{kl} \big(\partial_i g_{lj} + \partial_j g_{il} - \partial_l g_{ij}\big),$$

$$T\big(\tfrac{\partial}{\partial x_i}, \tfrac{\partial}{\partial x_j}\big) = \sum_k (\Gamma_{ij}^k - \Gamma_{ji}^k) \tfrac{\partial}{\partial x_k},$$

$$R\big(\tfrac{\partial}{\partial x_i}, \tfrac{\partial}{\partial x_j}\big) \tfrac{\partial}{\partial x_k} = \sum_l R_{ijk}^l \tfrac{\partial}{\partial x_l},$$

$$R_{ijk}^l = \big(\partial_j \Gamma_{ki}^l - \partial_k \Gamma_{ji}^l\big) + \sum_m \big(\Gamma_{ki}^m \Gamma_{jm}^l - \Gamma_{ji}^m \Gamma_{km}^l\big)$$

Low dimensions

$n = 2$. The curvature tensor is determined by the scalar curvature:

$$S^2(\Lambda^2\mathbb{R}^2) = \mathbb{R}q \owedge q, \qquad R = \frac{s}{4} q \owedge q.$$

Notice: Einstein $\Leftrightarrow$ constant sc. curvature

$n = 3$. The curvature tensor is determined by the Ricci curvature:

$$S^2(\Lambda^2\mathbb{R}^3) = \mathbb{R}q \owedge q \oplus S^2_0(\mathbb{R}^3) \owedge q, \qquad R = \frac{s}{12} q \owedge q + Ric_0 \owedge q.$$

$n = 4$. Recall: $\Lambda^2\mathbb{R}^4 = \Lambda^2_+ \oplus \Lambda^2_-$. Then

$$S^2_0(\mathbb{R}^4) \cong \Lambda^2_+ \otimes \Lambda^2_-, \qquad \mathcal{W} \cong S^2_0(\Lambda^2_+) \oplus S^2_0(\Lambda^2_-).$$

Hence, the Weyl tensor splits: $W = W^+ + W^-$, $W^\pm \in S^2_0(\Lambda^2_\pm)$.
If we consider R as a linear symmetric map of $\Lambda^2\mathbb{R}^4 = \Lambda^2_+ \oplus \Lambda^2_-$, we have

$$R = \left(\begin{array}{c|c} W^+ + \frac{s}{12} id & Ric_0 \\ \hline Ric_0^* & W^- + \frac{s}{12} id \end{array}\right)$$

Two Riemannian metrics g and g' are *conformally equivalent* if $g' = e^\varphi g$ for some $\varphi \in C^\infty(M)$. The class $[g]$ is called the conformal class of g.

conformal class $\iff$ $CO(n) = O(n) \times \mathbb{R}_+$–structure on M

Prop. *The Weyl tensor is conformally invariant.*

Proof. $g' \sim g$; ω', ω corresponding LC connections, $\omega' = \omega + \xi$.
Recall: $0 = T' - T = \delta\xi$, where
$\delta\colon (\mathbb{R}^n)^* \otimes \mathfrak{co}(n) \to \Lambda^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n$, $\mathfrak{co}(n) = \mathfrak{so}(n) \oplus \mathbb{R}$. Since $\delta\colon (\mathbb{R}^n)^* \otimes \mathfrak{so}(n) \to \Lambda^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n$ is an isomorphism, we have $\xi \in \operatorname{Ker}\delta \cong (\mathbb{R}^n)^*$. Then

$$\begin{aligned} \tilde{F}' - \tilde{F} &= d\omega' - d\omega + \tfrac{1}{2}[\omega' \wedge \omega'] - \tfrac{1}{2}[\omega \wedge \omega] \\ &= d\xi + [\omega \wedge \xi] + \tfrac{1}{2}[\xi \wedge \xi] \\ &= \nabla\xi + \tfrac{1}{2}[\xi \wedge \xi]. \end{aligned}$$

Hence, $R' - R$ takes values in $(\mathbb{R}^n)^* \otimes (\mathbb{R}^n)^*$ and thus belongs to $\mathbb{R} \oplus S^2_0(\mathbb{R}^n)$. □

Geodesics

Def. A curve $\gamma\colon \mathbb{R} \to M$ is called *geodesic* if $\nabla_{\dot\gamma(t)}\dot\gamma(t) = 0$ for all t, i.e. if the vector field $\dot\gamma$ is parallel along γ.

Choose local coordinates $(x_1, \cdots, x_n)$ and write $\gamma: \; x_i = x_i(t)$.

$$\nabla_{\dot\gamma(t)}\dot\gamma(t) = 0 \quad \Longleftrightarrow \quad \frac{d^2x_i}{dt^2} + \sum_{j,k} \Gamma^i_{jk}\dot x_i \dot x_j = 0, \quad i = 1, \ldots, n.$$

Cor. *For any $m \in M$ and any $\mathrm{v} \in T_mM$ there exists a unique geodesic γ such that $\gamma(0) = m$ and $\dot\gamma(0) = \mathrm{v}$.*

Rem. γ is not necessarily defined on the whole real line.

Def. (M, g) is called *complete*, if each geodesic is defined on the whole $\mathbb{R}$.

Def (Exponential map). For $m \in M$ we define

$$\exp\colon T_mM \to M \qquad \exp(t\mathrm{v}) = \gamma_{\mathrm{v}}(t).$$

Rem. In general, $\exp$ is defined on $B_\varepsilon(0)$ only.

Since $\exp_* = \mathrm{id}$ at m, $\exp$ is a diffeomorphism between some neighbourhoods of $0 \in T_mM$ and $m \in M$.

Def (Normal coordinates). The map

$$M \xrightarrow{\exp^{-1}} T_mM \xrightarrow{p} \mathbb{R}^n, \qquad p \text{ is an isometry},$$

defined in a neighbourhood of m is called *normal coordinate system*.

Theorem (Gauss Lemma)

$$g_{\exp_m(\mathrm{v})}\big((\exp_m)_*\mathrm{v}, (\exp_m)_*\mathrm{v}\big) = g_m(\mathrm{v},\mathrm{v}), \qquad \text{for all } \mathrm{v} \in T_mM.$$

Recall: A solution to the equation

$$\ddot{J} + R(J, \dot\gamma_{\mathrm{v}})\dot\gamma_{\mathrm{v}} = 0, \qquad J \in \Gamma(\gamma_{\mathrm{v}}^* TM)$$

is called a *Jacobi vector field* along γ. If J_{v} is the unique Jacobi vector field satisfying $J_{\mathrm{v}}(0) = m,\ \dot{J}_{\mathrm{v}}(0) = \mathrm{v}$, then

$$(exp_m)_*\mathrm{v} = J_{\mathrm{v}}(1).$$

Def. $\mathrm{Hol}_p^0 = \big\{ g \mid \Pi_\gamma(p) = pg,\ \gamma \text{ is contractible} \big\} \subset \mathrm{Hol}_p$ is called the *restricted holonomy group* at $p \in P$.

Hol_p^0 is the identity component of Hol_p.

Consider $\mathbb{R}^n$ as an $H = \mathrm{Hol}_p$–representation and write

$$\mathbb{R}^n = V_0 \oplus V_1 \oplus \cdots \oplus V_k. \qquad (3)$$

Here V_0 is a trivial representation (may be 0), all $V_i,\ i \geq 1$, are irreducible. All V_i are pairwise orthogonal.

Prop. *Under* (3), $H^0 = \mathrm{Hol}_p^0$ *is isomorphic to a product*

$$\{e\} \times H_1 \times \cdots \times H_k.$$

Prop. *Under* (3), $H^0 = \mathrm{Hol}_p^0$ *is isomorphic to a product*

$$\{e\} \times H_1 \times \cdots \times H_k.$$

Proof. Let P be the holonomy bundle through $p \in Fr(M)$. Then, $\forall q \in P$ and $\forall x, y \in \mathbb{R}^n$ we have $R_q(x,y) \in \mathfrak{h}$. Hence

$$R_q(x,y)(V_i) \subset V_i.$$

Write $x = \sum x_i, \ y = \sum y_i$ with $x_i, y_i \in V_i$. Then

$$\big\langle R(x,y)u, v\big\rangle = \big\langle R(u,v)x, y\big\rangle = \sum_i \big\langle R(u,v)x_i, y_i\big\rangle$$
$$= \sum_i \big\langle R(x_i, y_i)u, v\big\rangle,$$

i.e. $R(x,y) = \sum_i R(x_i, y_i)$. By the Ambrose–Singer thm,

$$\mathfrak{h} = 0 \oplus \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_k, \qquad \text{with } \mathfrak{h}_i \subset \mathrm{End}\, V_i.$$

This implies the statement of the Proposition. □

Prop. *Under* (3), M *is locally isomorphic to a Riemannian product*

$$M_0 \times M_1 \times \cdots \times M_k, \qquad \textit{where } M_0 \textit{ is flat.}$$

Proof. Denote $E_i := P \times_H V_i$, where P is the holonomy bundle. Then $TM = \bigoplus_i E_i$. Each distribution E_i is integrable:

$$v, w \in \Gamma(E_i) \Rightarrow \ \nabla_v w \in \Gamma(E_i) \ \Rightarrow \ [v,w] = \nabla_v w - \nabla_w v - 0 \in \Gamma(E_i).$$

From the Frobenius thm, in a neigbhd of m we may choose coordinates

$$x_1^1, \dots x_1^{r_1}; \dots ; x_k^1, \dots x_k^{r_k}$$

s.t. $\frac{\partial}{\partial x_i^j}$ is belongs to E_i. If $v = \frac{\partial}{\partial x_i^j}, \ w = \frac{\partial}{\partial x_s^t}, \ i \neq s$, then $\nabla_v w = \nabla_w v$ belongs to $E_s \cap E_i = 0$. Hence,

$$\frac{\partial}{\partial x_s^t} g\Big(\frac{\partial}{\partial x_i^{j_1}}, \frac{\partial}{\partial x_i^{j_2}}\Big) = g\big(\nabla_w v_i^{j_1}, v_i^{j_2}\big) + g\big(v_i^{j_1}, \nabla_w v_i^{j_2}\big) = 0$$

provided $s \neq i$. Hence, the restriction of g to E_i depends on x_i^j only. □

Def. Under the circumstances of the previous Proposition, M is called *locally reducible*. M is called *locally irreducible* if the holonomy representation is irreducible.

Cor. *M is locally irreducible iff M is locally a Riemannian product.*

Theorem (de Rham decomposition theorem)

Let M be connected, simply connected, and complete. If the holonomy representation is reducible, then M is isometric to a Riemannian product.

Proof. [KN, Thm. IV.6.1] □

Symmetric spaces

Def. (M, g) is called *symmetric* if $\forall m \in M$ $\exists$ an isometry $s = s_m$ with the following properties:

$$s(m) = m, \qquad (s_*)_m = -\mathrm{id} \quad \text{on } T_mM.$$

Prop. *Let M be symmetric. Then*

(i) s_m *is a local geodesic symmetry, i.e.* $s_m(\exp_m(\mathrm{v})) = \exp_m(-\mathrm{v})$ *whenever* $\exp_m$ *is defined on* $\pm\mathrm{v}$*;*

(ii) (M, g) *is complete;*

(iii) $s_m^2 = \mathrm{id}_M$.

Proof. *(i):* s_m is isometry $\Rightarrow$ $s_m(\exp_m(\mathrm{v})) = \exp_m(s_*\mathrm{v}) = \exp_m(-\mathrm{v})$. *(ii):* If $\gamma\colon(-\varepsilon, \varepsilon) \to M$, $\gamma(0) = m$ is a geodesic, then $s_m(\gamma(t)) = \gamma(-t)$ $\Rightarrow s_{\gamma(\tau/2)}(\gamma(t)) = \gamma(\tau - t) \Rightarrow s_{\gamma(\tau/2)} \circ s_m(\gamma(t)) = \gamma(\tau + t)$ whenever $\tau/2, t, \tau + t \in (-\varepsilon, \varepsilon)$. Since $s_{\gamma(\tau/2)} \circ s_m$ is globally defined, γ extends to $(0, +\infty)$. □

Prop. *A Riemannian symmetric space M is homogeneous, i.e. the group of isometries acts transitively on M.*

Proof. If γ is a geodesic, then $\gamma(t_1)$ is mapped to $\gamma(t_2)$ by s_m with $m = \gamma(\frac{t_1+t_2}{2})$.
For any $(p, q) \in M \times M$ there exists a sequence of geodesic segments put end to end which joins p and q (in fact, there is a single geodesic). Then the composition of reflections in the corresponding middle points maps p to q. □

Rem. In fact, we have shown, that the identity component G of the isometry group acts transitively.

Pick $m \in M$ and denote $K = Stab_m \subset G$. Then $M \cong G/K$.
Observe, that G is endowed with the involution

$$\sigma\colon G \to G, \qquad f \mapsto s_m \circ f \circ s_m$$

Theorem ([Helgason. Diff geom and symm spaces, IV.4])

(i) *Let G be a connected Lie group with an involution σ and a left invariant metric which is also right–invariant under $\hat{K} = \{\sigma(g) = g\}$. Let K be a closed subgroup of G s.t. $\hat{K}^0 \subset K \subset \hat{K}$. Then $M = G/K$ is a symmetric space with its induced metric.*

(ii) *Every symmetric space arises as in (i).*

(iii) *We have the Cartan decomposition: $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ with*

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}.$$

Moreover, $T_m M \cong \mathfrak{m}$.

(iv) $\mathrm{Hol}_m \subset K$.

Rem. Holonomy groups of Riemannian symmetric spaces were classified by Cartan (see [Besse. Einstein mflds, 7.H, 10.K])

Theorem

For a Riemannian mfld M the following conditions are equivalent:

(i) $\nabla R = 0$;

(ii) *the local geodesic symmetry s_m is an isometry for any $m \in M$.*

Def. (M, g) is called *locally symmetric*, if *(i)* $\Leftrightarrow$ *(ii)* holds.

Proof. *(ii)$\Rightarrow$(i):*
s_m isometry $\Rightarrow$ s_m preserves ∇R. On the other hand, since ∇R is of order 5, we must have $s_m^*(\nabla R)_m = -(\nabla R)_m$. Hence, $(\nabla R)_m = 0 \ \forall m$.

$\nabla R = 0 \ \Rightarrow \ s_m$ is isometry:

$\gamma = \gamma_{\mathrm{w}}$ geodesic through m, $(e_1, \dots, e_n)$ orthonormal frame of $T_m M$. Define $E_i \in \Gamma(\gamma^* TM) : \nabla_{\dot\gamma} E_i = 0, \ E_i(0) = e_i$.

$\nabla R = 0 \ \Rightarrow \ R(E_i, \dot\gamma)\dot\gamma$ is parallel along $\gamma \Rightarrow$
$R(E_i, \dot\gamma)\dot\gamma = \sum_j r_{ij} E_j$ with $r_{ij} = \langle R(E_i, \dot\gamma)\dot\gamma, E_j\rangle$, which is constant in t.

Write $J_{\mathrm{v}}(t) = \sum a^i_{\mathrm{v}}(t) E_i(t)$. Then a_{v} satisfies ODE with constant coefficients $\ddot a_{\mathrm{v}} + r a_{\mathrm{v}} = 0$.

Similarly, for $\bar\gamma = \gamma_{-\mathrm{w}}$ put $\bar E_i : \ \nabla_{\dot{\bar\gamma}} \bar E_i = 0, \ \bar E_i(0) = -e_i$; $\bar J_{\mathrm{v}} = \sum \bar a^i_{\mathrm{v}} \bar E_i$. Then $\ddot{\bar a}_{\mathrm{v}} + r \bar a_{\mathrm{v}} = 0$ (with the same matrix r!). Moreover, $\bar a_{\mathrm{v}}(0) = 0 = a_{\mathrm{v}}(0)$ and $\dot{\bar a}_{\mathrm{v}}(0) = \dot a_{\mathrm{v}}(0)$. Hence $\bar J_{\mathrm{v}}(1) = J_{\mathrm{v}}(1)$. Then

$$\begin{aligned}\langle J_{\mathrm{v}}(1), J_{\mathrm{v}}(1)\rangle = \langle \mathrm{v}, \mathrm{v}\rangle &= \langle \bar J_{\mathrm{v}}(1), \bar J_{\mathrm{v}}(1)\rangle \\ &= \langle (s_m)_* J_{\mathrm{v}}(1), (s_m)_* J_{\mathrm{v}}(1)\rangle.\end{aligned}$$

□

Berger theorem revisited

Theorem (Berger thm)

Assume M is a simply–connected irreducible not locally symmetric Riemannian mfld of dimension n. Then Hol is one of the following:

	Holonomy	Geometry	Extra structure
•	$SO(n)$		
•	$U(n/2)$	Kähler	complex
•	$SU(n/2)$	Calabi–Yau	complex + hol. vol.
•	$Sp(n/4)$	hyperKähler	quaternionic
•	$Sp(1)Sp(n/4)$	quaternionic Kähler	"twisted" quaternionic
•	G_2 (n=7)	exceptional	"octonionic"
•	$Spin(7)$ (n=8)	exceptional	"octonionic"

Comments to the Berger theorem

- The assumption $\pi_1(M) = 0$ could be dropped by restricting attention to Hol^0.
- M is locally symmetric $\Rightarrow$ M is locally isometric to a symmetric space. Holonomies of simply connected symmetric spaces are known.
- Irreducibility could be dropped by taking all possible products of the entries of the Berger list.
- In the theorem, Hol is not just an abstract group, but rather a subgroup of $SO(n)$, or, equivalently, comes together with an irreducible n–dimensional representation.

Ex. For instance,

$$SO(m) = \left\{ \left(\begin{array}{c|c} A & 0 \\ \hline 0 & A \end{array} \right) \right\} \subset SO(2m)$$

is never a holonomy representation of an irreducible manifold (in fact, this is never a holonomy representation of any Riemannian manifold).

Holonomy groups

in Riemannian geometry

Lecture 4

November 17, 2011

Equivalent formulation of the Berger theorem

By inspection, each group in Berger's list acts transitively on the unit sphere. On the other hand, all groups acting transitively on spheres were classified by Montgomery and Samelson in 1943. The list consists of

$$U(1) \cdot Sp(m), \qquad Spin(9),$$

and the groups from Berger's list. The first group never occurs as a holonomy group (follows from the Bianchi identity). Alekseevsky proved in 1968 that $Spin(9)$ can occur as holonomy group of a symmetric space only. Hence, the following theorem is equivalent to Berger's classification theorem.

Theorem (Berger)

Assume that the holonomy group of an irreducible Riemannian manifold does not act transitively on spheres. Then M is locally symmetric.

Second fundamental form

Let $\bar{M}$ be a Riemannian mfld, $M \subset \bar{M}$.
Write $T\bar{M} = TM \oplus \nu M$ along M.

$$\bar{\nabla}_v w = \big(\bar{\nabla}_v w\big)^T + \big(\bar{\nabla}_v w\big)^{\perp} = \nabla_v w + \alpha(v,w), \qquad \text{where } v, w \in \mathfrak{X}(M).$$

Prop.

- *∇ is the Levi–Civita connection on M wrt the induced metric;*
- *$\alpha \in \Gamma\big(S^2(TM) \otimes \nu M\big)$.*

α is called the *second fundamental form* of M.

M is called *totally geodesic*, if geodesic in M $\Rightarrow$ geodesic in $\bar{M}$.

Let γ be a geodesic in M. Then $\bar{\nabla}_{\dot\gamma}\dot\gamma = 0 + \alpha(\dot\gamma, \dot\gamma)$. Hence,

$$M \text{ is totally geodesic} \iff \alpha = 0.$$

Shape operator

Similarly, if $v \in \mathfrak{X}(M)$, $\xi \in \Gamma(\nu M)$, then

$$\bar{\nabla}_v \xi = \big(\bar{\nabla}_v \xi\big)^T + \big(\bar{\nabla}_v \xi\big)^{\perp} = -A_\xi v + \nabla_v^{\perp} \xi.$$

A_ξ is called the *shape operator*.

Let $w \in \mathfrak{X}(M)$. Then, differentiating equality $\bar{g}(w, \xi) = 0$ in the direction of v, we obtain

$$\bar{g}\big(\alpha(v,w), \xi\big) = \bar{g}(A_\xi v, w).$$

$M \subset \bar{M}$, $\bar{\Pi}_\gamma$ parallel transport of $\bar{M}$.

Prop. *M is totally geodesic if and only if* $\forall \gamma \colon [0,1] \to M$ *and* $\forall v \in T_{\gamma(0)}M \quad \bar{\Pi}_\gamma v \in T_{\gamma(1)}M$.

Proof. $(\Leftarrow)$ Let $\gamma = \gamma_v$ be a geodesic in M through m. Denote by $\bar{\Pi}^t_\gamma$ the parallel transport in $\bar{M}$ along $\gamma(\tau), \tau \in [0,t]$. Then

$$\bar{\Pi}^t_\gamma v = \mathrm{proj}_{TM}\, \bar{\Pi}^t_\gamma v = \Pi^t_\gamma v = \dot{\gamma}(t),$$

i.e. γ is a geodesic in $\bar{M}$.

$(\Rightarrow)$ [KN, Thm VII.8.4] □

Let M be a smooth G-mfld, where G is a Lie gp acting properly. $G_m := \{g \mid gm = m\}$ isotropy subgroup.

Theorem

Let G be cmpt. For $m \in M$ and $H = G_m$ there exist a unique H–representation V and a G–equivariant diffeomorphism $\varphi \colon G \times_H V \to M$ onto an open neighbourhood of Gm s.t. $\varphi([g, 0]) = gm$.

V is called the *slice representation* of M at m.

Observe: $G \to G/H$ is a principal H–bundle. Moreover, $G/H = G/G_m \cong Gm$. Since the zero–section of $G \times_H V \to G/H$ is identified with the orbit Gm, we obtain $\nu(Gm) \cong G \times_H V$. In particular, $\nu_m(Gm) \cong V$.

On the other hand, H preserves Gm. The induced representation of H on $T_m(Gm)$ is called the *isotropy representation*.

For subgroups $H, K \subset G$ we write $H \sim K$ if H is conjugate to K.
(H) conjugacy class of H.
$(H) \leq (K)$ if H is conjugate to a subgroup of K.
$M_{(H)} = \{m \mid G_m \sim H\}$.

Theorem

Let G be a compact group. Assume M/G is connected. Then there exists a unique isotropy type (H) of M such that $M_{(H)}$ is open and dense in M. Each other isotropy type (K) satisfies $(H) \leq (K)$.

Proof. [tom Dieck. Transformation groups. Thm. 5.14] □

Strategy of the proof of the Berger thm

Step 1. $H = \mathrm{Hol}_m$ is not transitive on the sphere $\Rightarrow$ for any principal v there exists a family $\mathcal{F}_v$ of normal subspaces to non–trivial orbits of H, which generates T_mM.

Step 2. For any $v \in T_mM$, $v \neq 0$, the submfld $N^v = \exp_m\big(\nu_v(Hv)\big)$ is totally geodesic.

Step 3. The normal holonomy group $H^\perp$ of $Hv \subset T_mM$ acts by isometries on N^v. Moreover, $H^\perp \supset \mathrm{Hol}(N^v)$.

Step 4. $\mathrm{Hol}(N^v)$ acts by isometries on $N^v \Rightarrow N^v$ is locally symmetric.

Step 5. Almost all geodesics through m are contained in a family of loc. symmetric and totally geodesic submflds $\Rightarrow M$ is locally symmetric at m.

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Let M be a Riemannian mfld, $m \in M$, ρ injectivity radius at m.

Gluing Lemma

$\forall v \in T_mM$ let $\mathcal{F}_v$ be a family of subspaces of T_mM s.t.

(i) $v \in W$ for any $W \in \mathcal{F}_v$;

(ii) $\exp_m(W_\rho)$ is a totally geodesic and (intrinsically) loc. symm.

Assume that for any v in some dense $\Omega \subset B_\rho(0)$ the family $\mathcal{F}_v$ spans T_mM, where $B_\rho(0) \subset T_mM$ is the ball of radius ρ. Then the local geodesic symmetry s_m is an isometry.

Proof. Let $v \in \Omega$, $\gamma = \gamma_v$ is the geodesic through m. Choose a frame $(e_1, \ldots, e_n)$ of T_mM s.t. e_i belongs to some $W_i \in \mathcal{F}_v$. Let $(E_1, \ldots, E_n)$ be parallel vector fields along γ with $E_i(0) = e_i$.

Then $r_{ij} = \langle R(E_i, \dot\gamma)\dot\gamma, E_j\rangle$ is constant in t. Indeed, $\exists W \in \mathcal{F}_v$ s.t. $e_i \in W$. Hence, E_i is tangent to $\exp_m(W)$ and $\gamma(t) \in \exp_m(W)$. $\exp_m(W)$ is loc. symmetric $\Rightarrow (\nabla_{\dot\gamma}R)(E_i, \dot\gamma) = 0 \Rightarrow\ \dot r_{ij} = 0$.

Thus, in the frame E_i, Jacobi fields correspond to solutions of $\ddot a + ra = 0$, where $r = const$. Hence the statement. □

Strategy of the proof of the Berger thm

Step 1. $H = \mathrm{Hol}_m$ is not transitive on the sphere $\Rightarrow$ for any principal v there exists a family $\mathcal{F}_v$ of normal subspaces to non–trivial orbits of H, which generates T_mM.

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Lemma A

Assume a compact subgroup $G \subset SO(n)$ does not act transitively on S^{n-1}. Let v be a principal vector of G. Then there exists $\xi \in \nu_v(Gv)$, $\xi \neq \lambda v$, s.t. the family of normal spaces $\nu_{\gamma(t)}(G\gamma(t))$ spans $\mathbb{R}^n$, where $\gamma(t) = v + t\xi$, $t \in \mathbb{R}$.

Proof. [Olmos, A geometric proof..., Lemma 2.2] □

Lemma B

(i) N^v is a totally geodesic submanifold of M;

(ii) N^v is (intrinsically) locally symmetric.

Proof. Will be sketched below. □

Theorem (Berger)

Assume that the holonomy group of an irreducible Riemannian manifold does not act transitively on spheres. Then M is locally symmetric.

Proof. Pick $m \in M$. Let $\mathcal{O} \subset T_m M$ be subset of principal vectors. Then $\mathcal{O}$ is open and dense. Pick $v \in \mathcal{O}$.

Lemma A $\Rightarrow$ $\exists \gamma(t) = v + t\xi$ s.t. the family $\mathcal{F}_v = \{\nu_{\gamma(t)}\big(G\gamma(t)\big) \mid t \in \mathbb{R}\}$ spans $T_m M$.

Observe: $\xi \in \nu_v\big(Gv\big) \;\Rightarrow v \in \nu_{v+\xi}\big(G(v+\xi)\big)$. Indeed, $G \subset SO(T_m M) \Rightarrow \mathfrak{g} \subset \mathfrak{so}(T_m M)$. Hence, for any $A \in \mathfrak{g}$ we have

$$0 = \langle Av, v + \xi \rangle = -\langle v, A(v+\xi)\rangle.$$

The first equality follows from $T_v(Gv) = \{Av \mid A \in \mathfrak{g}\}$.

Therefore, $v \in \nu_{\gamma(t)}\big(G\gamma(t)\big)$ for any t. Lemma B $\Rightarrow$ assumptions of the Gluing Lemma are satisfied. Then Gluing Lemma implies that M is locally symmetric. □

Strategy of the proof of the Berger thm

Step 1. $H = \mathrm{Hol}_m$ is not transitive on the sphere $\Rightarrow$ for any principal v there exists a family $\mathcal{F}_v$ of normal subspaces to non–trivial orbits of H, which generates $T_m M$.

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Theorem (Cartan)

Let $V \subset T_m M$. Then $\exp_m(V_\rho)$ is totally geodesic submanifold if and only if the curvature tensor of M preserves the parallel transport of V along geodesics γ_v with $\gamma_v(0) = m, v \in V$.

$U := \Pi_\gamma V$. Then "R preserves U" means: $R_{\gamma(1)}(U,U)U \subset U$.

Proof. [Berndt–Olmos–Console, Submflds and hol., Thm 8.3.1] □

Submanifolds G-actions Berger Thm Holonomy and cohomology

$N^v := \exp_m\big(\nu_v(Hv) \cap B_\rho(0)\big)$, where $v \in T_m M \setminus \{0\}$.

Lemma B

(i) N^v is a totally geodesic submanifold of M.

Proof. Denote

$$\mathcal{R} = \operatorname{span}\{\, \bar{R}(x,y) = \Pi_\gamma^{-1} R\big(\Pi_\gamma x, \Pi_\gamma y\big)\Pi_\gamma \,\}.$$

Then the Ambrose–Singer thm states that $\mathcal{R} = \mathfrak{h} \subset \mathfrak{so}(T_m M)$.

$$\xi \in \nu_v(Hv) \iff 0 = \langle \bar{R}(x,y)v, \xi\rangle = \langle \bar{R}(v,\xi)x, y\rangle,$$

where $x, y \in T_m M$, and $\bar{R} \in \mathcal{R}$ are arbitrary. Hence, $\bar{R}(v,\xi) = 0$. Then, for any $\eta \in \nu_v(Hv)$, the Bianchi identity yields: $\bar{R}(\xi,\eta)v = -\bar{R}(\eta,v)\xi - \bar{R}(v,\xi)\eta = 0$. Thus $\bar{R}(\xi,\eta)$ belongs to the isotropy subalgebra and $\bar{R}(\xi,\eta)\nu_v(Hv) \subset \nu_v(Hv) \;\Rightarrow$

$$\bar{R}\big(\nu_v(Hv), \nu_v(Hv)\big)\nu_v(Hv) \subset \nu_v(Hv). \tag{1}$$

Since (1) holds at any pt (after parallel transport), the hypotheses of the Cartan Thm are satisfied. Hence the statement. □

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Lem. *Let $\varphi_t \colon S \to M$ be a smooth family of totally geodesic submanifolds of M. If $\xi_t = \partial_t \varphi_t \perp \varphi_t(S)$, then $id \colon (S, \varphi_0^* g) \to (S, \varphi_t^* g)$ is an isometry.*

Proof. Put $S_t = \varphi_t(S) \subset M$ with its induced metric. Let γ_w be a geodesic of S_0 through m, $w \in T_m M$. Then

$$
\begin{aligned}
\tfrac{d}{dt} g\big((\varphi_t)_* w, (\varphi_t)_* w\big) &= \tfrac{\partial}{\partial t} g\Big(\tfrac{\partial}{\partial s}\big|_{s=0} \varphi_t(\gamma_w(s)), \tfrac{\partial}{\partial s}\big|_{s=0} \varphi_t(\gamma_w(s))\Big) \\
&= 2g\Big(\nabla_t \tfrac{\partial}{\partial s}\big|_{s=0} \varphi_t(\gamma_w(s)), \tfrac{\partial}{\partial s}\big|_{s=0} \varphi_t(\gamma_w(s))\Big) \\
&= 2g\Big(\nabla_s\big|_{s=0} \tfrac{\partial}{\partial t} \varphi_t(\gamma_w(s)), (\varphi_t)_* w\Big) \\
&= -2g\big(A_{\xi_t} (\varphi_t)_* w, (\varphi_t)_* w\big) \\
&= 0.
\end{aligned}
$$

Therefore, $g\big((\varphi_t)_* w, (\varphi_t)_* w\big)$ does not depend on t. □

Lem. *The normal holonomy group $H^\perp$ of $Hv \subset T_mM$ acts by isometries on N^v.*

Proof. Let $c\colon [0,1] \to Hv,\ c(0) = v$. Denote by $\Pi_t^\perp$ the normal parallel transport along $c|_{[0,t]}$. By Lemma B, (i)

$$\varphi_t\colon \nu_v(Hv) \to M, \qquad \varphi_t = \exp_m \circ \Pi_t^\perp$$

is a one–parameter family of totally geodesic submanifolds.
Put $\xi_t = \partial_t \varphi_t$. Want to show $\xi_t \perp \operatorname{Im} \varphi_t = \exp_m\big(\Pi_t^\perp(\nu_v(Hv))\big)$.
It suffices to show that $\xi_0 \perp \exp_m(\nu_v(Hv)) = N^v$, since for $t > 0$ the proof is obtained by replacing v by $c(t)$.
For an arbitrary $\eta \in \nu_v(Hv)$, $J(s) = \xi_0(s\eta) = \frac{\partial}{\partial t}|_{t=0} \exp_m(s\Pi_t^\perp \eta)$ is the Jacobi v.f. along $\gamma_\eta(s)$. Initial conditions: 0 and $\frac{d}{dt}|_{t=0}\Pi_t^\perp \eta = -A_\eta \dot{c}(0) + \nabla^\perp \Pi_t^\perp \eta = -A_\eta \dot{c}(0) \perp T_m N^v = \nu_v(Hv)$.
Hence, $\xi_0(s\eta) \perp N^v$ for all s. Hence, $\xi_0 \perp N^v$.

Therefore, φ_t induces an isometry $N^v \to N^{c(t)}$. If c is a loop, we obtain an isometry $N^v \to N^v$. □

Theorem

Assume a connected Lie gp $H \subset SO(n)$ acts irreducibly on $\mathbb{R}^n$. Then the image of the connected component of the isotropy gp $(H_v)_0$ is contained in $H^\perp$.

Proof. [Berndt–Console–Olmos, Cor. 6.2.6] □

Prop. *The holonomy gp H^v of N^v is contained in the image of $(H_v)_0$ under the slice representation.*

Proof. The proof is similar to the proof of the fact that N^v is totally geodesic. For details see [Olmos, p.586] □

Cor. $H^v \subset H^\perp$.

Strategy of the proof of the Berger thm

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Step 5. Almost all geodesics through m are contained in a family of loc. symmetric and totally geodesic submflds $\Rightarrow M$ is locally symmetric at m.

Lem. *Let M be a Riemannian mfld with the following property: for any $m \in M$ each restricted holonomy transformation of $T_m M$ extends via the exponential map to a local isometry. Then M is locally symmetric.*

Sketch of the proof. Can assume that $H = Hol(M)$ acts irreducibly. Denote $\mathcal{K} = \{K \mid \mathcal{L}_K g = 0,\ K \in \mathfrak{X}(U_m)\}$. Then $\mathcal{K}_m = \{K(m) \mid K \in \mathcal{K}\}$ is a non-trivial H–invariant subspace of $T_m M$. Hence, $\mathcal{K}_m = T_m M$.

Then, for each $v \in T_m M$ there exists a unique $K \in \mathcal{K}$ s.t. $K(m) = v$ and $(\nabla K)_m = 0$. For such K, the integral curve $t \mapsto \varphi_t^K(m)$ through m is a geodesic. Moreover, the parallel transport along this geodesic is given by $(\varphi_t^K)_*$. This implies the local symmetry. □

Lemma B

(ii) N^v *is (intrinsically) locally symmetric.*

Hodge theory in a nutshell

Let V be an oriented Euclidean vector space, $\dim V = n$. Then the Hodge operator $*\colon \Lambda^k V^* \to \Lambda^{n-k} V^*$ is defined by the relation

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle vol, \qquad \text{for all } \alpha \in \Lambda^k V^*.$$

$*$ is an $SO(V)$–equivariant isomorphism, $*^{-1} = (-1)^{k(n-k)}*$. Hence, for any oriented Riemannian manifold (M, g) we have a well defined map $*\colon \Lambda^k T^*M \to \Lambda^{n-k} T^*M$.

Define $d^*\colon \Omega^k(M) \to \Omega^{k-1}(M)$ by $d^* = (-1)^{n(k+1)+1} * d *$. Then, if M is compact, Stokes' theorem implies that

$$\langle d\alpha, \beta \rangle_{L_2} = \langle \alpha, d^*\beta \rangle_{L_2}, \qquad \text{for any } \alpha \in \Omega^{k-1},\ \beta \in \Omega^k.$$

$\Delta = dd^* + d^*d\colon \Omega^k \to \Omega^k$ is called the Laplace operator. It is second order elliptic PDO. Denote $\mathscr{H}^k = \mathrm{Ker}(\Delta\colon \Omega^k \to \Omega^k)$.

Theorem (Hodge)

Every de Rham cohomology class contains a unique harmonic representative and $H^k_{dR} \cong \mathscr{H}^k$.

It is known, that all $\Lambda^k(\mathbb{R}^n)^*$ are irreducible as $O(n)$–representations. However, if $G \subset O(n)$, then $\Lambda^k(\mathbb{R}^n)^*$ does not need to be irreducible as G–representation.

Model example: $G = SO(4) \subset O(4)$

$*^2 = id$ on $\Lambda^2(\mathbb{R}^4)^* \Rightarrow \Lambda^2(\mathbb{R}^4)^* \cong \Lambda^2_+ \oplus \Lambda^2_-$ as $SO(4)$–representation. Hence, for any oriented Riemannian four–manifold we have $\Lambda^n T^*M \cong \Lambda^n_+ T^*M \oplus \Lambda^n_- T^*M$. Since $\Delta * = *\Delta$, we have $\mathscr{H}^2 \cong \mathscr{H}^2_+ \oplus \mathscr{H}^2_-$, $b_2 = b_+ + b_-$.

Let $H = \mathrm{Hol}$ and P be the holonomy bundle. Consider $\Lambda^k(\mathbb{R}^n)^*$ as H–representation. Let

$$\Lambda^k(\mathbb{R}^n)^* \cong \bigoplus_{i\in I_k} \Lambda^k_i(\mathbb{R}^n)^*$$

be the decomposition into irreducible components. Then

$$\Lambda^k T^*M \cong \bigoplus_{i\in I_k} \Lambda^k_i T^*M, \qquad \text{where } \Lambda^k_i T^*M = P\times_H \Lambda^k_i(\mathbb{R}^n)^*.$$

Lem. *Denote $\Omega^k_i(M) = \Gamma(\Lambda^k_i T^*M)$. Then $\Delta(\Omega^k_i) \subset \Omega^k_i$. Hence,*

$$\mathscr{H}^k \cong \bigoplus \mathscr{H}^k_i, \qquad b_k = \sum_{i\in I_k} b^i_k.$$

This statement follows from the Weitzenböck formula for the Laplacian [Besse. 1I, Lawson–Michelson. II.8]

The refined Betti numbers b^i_k carry both topological and geometrical information. They give obstructions to existence of metrics with non–generic holonomy.

Ex. If M admits a Kähler metric, then odd Betti numbers of M are even.

Another example of connection between holonomy groups and cohomology gives the following consideration. If for some i $\Lambda^k_i(\mathbb{R}^n)^*$ is a trivial H–representation, then $b^k_i = \dim \Lambda^k_i(\mathbb{R}^n)^*$. Indeed, each $\xi_0 \in \Lambda^k_i(\mathbb{R}^n)^*$ corresponds to a parallel $\xi \in \Omega^k_i$. Then $\nabla\xi = 0 \;\Rightarrow\; d\xi = 0 = d^*\xi$. Hence, $\Delta\xi = 0$. On the other hand, from the Weitzenböck formula one obtains $\Delta\xi = 0 \;\Rightarrow\; \nabla\xi = 0$. Therefore,

$$\mathscr{H}^k_i \cong \{\xi \mid \nabla\xi = 0\}.$$

Holonomy groups

in Riemannian geometry

Lecture 5

November 24, 2011

Algebraic preliminaries Complex Mflds Structure Function Kähler metrics

A complex structure on a real vector space V (necessarily of even dimension) is an endomorphism J s.t. $J^2 = -1$. This establishes the correspondence

$$\{\text{real vector spaces equipped with } J\} \cong \{\text{complex vector spaces}\}$$

Notice: J^* is a complex structure on V^*.

Let V be a real vector space. Then $V_{\mathbb{C}} = V \otimes \mathbb{C}$ is a complex vector space endowed with an antilinear map $\bar{\cdot}\colon V_{\mathbb{C}} \to V_{\mathbb{C}}$, $v \otimes z \mapsto v \otimes \bar{z}$.

Prop. *Let V be a real vector space equpped with a complex structure. Then*

- $V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$, *where $V^{1,0}$ and $V^{0,1}$ are eigenspaces of J corresponding to eigenvalues $+i$ and $-i$ respectively;*
- $V^{1,0} = \{v \otimes 1 - Jv \otimes i \mid v \in V\}$, $V^{0,1} = \{v \otimes 1 + Jv \otimes i\}$;
- $\bar{\cdot}\colon V^{1,0} \to V^{0,1}$ *is an (antilinear) isomorphism.*
- $V^{1,0} \cong (V, J)$, $V^{0,1} \cong (V, -J)$.

Similarly, $V_{\mathbb{C}}^* \cong (V^*)^{1,0} \oplus (V^*)^{0,1}$ and therefore

$$\Lambda^k V_{\mathbb{C}}^* \cong \bigoplus_{p+q=k} \Lambda^{p,q} V^*, \qquad \text{where } \Lambda^{p,q} V^* = \Lambda^p (V^*)^{1,0} \otimes \Lambda^q (V^*)^{0,1}.$$

A *Hermitian scalar product* on (V,J) is a scalar product h on V s.t. $h(Jv,Jw) = h(v,w)$. Then $\omega(v,w) = h(Jv,w)$ is skew–symmetric. Since $\omega(Jv,Jw) = \omega(v,w)$ we obtain $\omega \in \Lambda^{1,1}$.

Consider the case $(V,J) = (\mathbb{R}^{2m}, J_0)$, where

$$J_0 = \left(\begin{array}{c|c} 0 & -\mathbf{1}_m \\ \hline \mathbf{1}_m & 0 \end{array}\right)$$

Thus, $(\mathbb{R}^{2m}, J_0)$ can be identified with $\mathbb{C}^m$. Then the standard Euclidean scalar product is Hermitian and $\omega_0 = 2\sum_{j=1}^m dx_j \wedge dy_j$.

Denote

$$Sp(2m;\mathbb{R}) = \{A \in GL_{2m}(\mathbb{R}) \mid \omega_0(A\cdot, A\cdot) = \omega_0(\cdot,\cdot) \;\Leftrightarrow\; AJ_0A^T = J_0\},$$
$$GL_m(\mathbb{C}) = \{A \in GL_{2m}(\mathbb{R}) \mid A \circ J_0 = J_0 \circ A\}.$$

Then we have

$$\begin{aligned} U(m) &= SO(2m) \cap Sp(2m;\mathbb{R}) \\ &= SO(2m) \cap GL_m(\mathbb{C}) \\ &= GL_m(\mathbb{C}) \cap Sp(2m;\mathbb{R}). \end{aligned}$$

Representations of $U(m)$

Observe that $\Lambda^{p,p}$ is invariant subspace wrt the conjugation. Hence, $\Lambda^{p,p}$ is the complexification of some real vector space:

$$\Lambda^{p,p} \cong [\Lambda^{p,p}]_r \otimes \mathbb{C}.$$

Namely, $[\Lambda^{p,p}]_r = \{\alpha \mid \bar{\alpha} = \alpha\}$. Similarly, if $p \neq q$

$$\Lambda^{p,q} \oplus \Lambda^{q,p} = [\Lambda^{p,q}]_r \otimes \mathbb{C}.$$

In particular, we have

$$(\mathbb{R}^{2m})^* \cong [\Lambda^{1,0}]_r, \qquad \Lambda^2(\mathbb{R}^{2m})^* \cong [\Lambda^{1,1}]_r \oplus [\Lambda^{2,0}]_r.$$

Since $U(m) \subset SO(2m)$, we also have

$$\Lambda^2(\mathbb{R}^{2m})^* \cong \mathfrak{so}(2m) = \mathfrak{u}(m) \oplus \mathfrak{u}(m)^\perp.$$

Prop. $\mathfrak{u}(m) = [\Lambda^{1,1}]_r, \quad \mathfrak{u}(m)^\perp \cong [\Lambda^{2,0}]_r.$

Proof. Exercise. □

Let (V, J, h) be a Hermitian vector space, $\omega = h(J\cdot, \cdot)$. Consider the map $L\colon \Lambda V^*_{\mathbb{C}} \to \Lambda V^*_{\mathbb{C}}$, $L(\alpha) = \omega \wedge \alpha$, which is $U(V)$–equivariant. Denote $\Lambda = L^*$, $B = [\Lambda, L]$. Then

$$[B, L] = -2L \quad \text{and} \quad [B, \Lambda] = 2\Lambda,$$

i.e. $\Lambda V^*_{\mathbb{C}}$ is an $\mathfrak{sl}_2(\mathbb{C})$–representation. This leads to the following decomposition of $\Lambda^{p,q}$ into irreducible components.
For $p + q \leq m$, denote $\Lambda_0^{p,q} = L(\Lambda^{p-1,q-1})^\perp$. It is called the space of primitive (p, q)–forms.

Theorem (Lefschetz decomposition)

For $p \geq q$ and $p + q \leq m$ there is a $U(V)$–invariant decomposition

$$\Lambda^{p,q} \cong \Lambda_0^{p,q} \oplus \Lambda_0^{p-1,q-1} \oplus \cdots \oplus \Lambda_0^{p-q+1,1} \oplus \Lambda^{p-q,0}.$$

See [Wells. Differential analysis on cx mflds. 5.1] for details.

Complex manifolds

For a real mfld M, a section I of $\mathrm{End}(TM)$ s.t. $I^2 = -id$ is called an *almost complex structure*. If M admits an almost complex structure, then M is necessarily orientable mfld of even dimension. To each I, we associate the *Nijenhuis tensor*:

$$N_I(v,w) = [Iv, Iw] - I[Iv, w] - I[v, Iw] - [v, w], \qquad v, w \in (M).$$

Denote $\Omega^{p,q}(M) = \Gamma(\Lambda^{p,q}T^*M)$.

Theorem

For an almost complex mfld the following statements are equivalent:

(i) $v, w \in \Gamma(T^{1,0}M) \;\Rightarrow\; [v,w] \in \Gamma(T^{1,0}M)$;

(ii) $d\Omega^{1,0} \subset \Omega^{2,0} + \Omega^{1,1}$;

(iii) $d\Omega^{p,q} \subset \Omega^{p+1,q} + \Omega^{p,q+1}$;

(iv) $N_I \equiv 0$.

Proof. $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$: Exercise.
To prove $(i) \Leftrightarrow (iv)$ observe that $v \in \Gamma(T^{1,0}M) \;\Leftrightarrow\; v = v_0 - iIv_0$, $v_0 \in \mathfrak{X}(M)$, and similarly for w. Denote $x = [v,w]$. Then

$$2(x + iIx) = -N(v_0, w_0) - iIN(v_0, w_0).$$

Hence, $x^{0,1} = 0 \;\Leftrightarrow\; N(v_0, w_0) = 0$. □

Exercise. Let $\alpha \in \Omega^{1,0}(M)$. Show that $(d\alpha)^{0,2}$ can be identified with $\alpha \circ N_I$.

Newlander–Nirenberg Theorem

$\alpha_1, \dots, \alpha_m \in \Omega^{1,0}(U),\ m = \dim_{\mathbb{R}} M/2, M \supset U$ is open

Assume α_j are closed and pointwise linearly independent. Then $N \equiv 0$, since $(d\alpha_j)^{0,2} \cong 0$ for all j. After restricting to a possibly smaller domain, all α_j can be assumed to be exact: $\alpha_j = df_j,\ f_j = x_j + y_j i \colon U \to \mathbb{C}$. Then each f_j is I–holomorphic, i.e.

$$df_j \circ I = i df_j \quad \Longleftrightarrow \quad df_j \in \Omega^{1,0}.$$

Hence we obtain local holomorphic coordinates on M.

Rem. This reasoning shows that if $N_I \neq 0$ usually there are no holomorphic functions on M (even locally).

Theorem (Newlander–Nirenberg)

$N_I \equiv 0$ iff M is a complex mfld, i.e. admits an atlas whose transition functions are holomorphic.

Algebraic preliminaries | Complex Mflds | Structure Function | Kähler metrics

Write

$$\partial = d^{1,0} \colon \Omega^{p,q} \to \Omega^{p+1,q}, \qquad \bar\partial = d^{0,1} \colon \Omega^{p,q} \to \Omega^{p,q+1}.$$

For complex mflds, $d = \partial + \bar\partial$. Hence,

$$d^2 = 0 \quad \Longleftrightarrow \quad \partial^2 = 0,\ \bar\partial^2 = 0,\ \partial\bar\partial + \bar\partial\partial = 0. \qquad (1)$$

Any $\alpha \in \Omega^{p,q}$ can be written locally as a sum of the following forms: $\beta = f dz_{j_1} \wedge \dots \wedge dz_{j_p} \wedge d\bar z_{k_1} \wedge \dots \wedge d\bar z_{k_q}$. Then

$$\partial\beta = \sum_{j=1}^n \frac{\partial f}{\partial z_j} dz_j \wedge \dots, \qquad \partial\beta = \sum_{j=1}^n \frac{\partial f}{\partial \bar z_j} d\bar z_j \wedge \dots$$

From (1) we obtain that

$$\Omega^{p,0} \xrightarrow{\bar\partial} \Omega^{p,1} \xrightarrow{\bar\partial} \dots \xrightarrow{\bar\partial} \Omega^{p,n}$$

is a complex for any p. It is called *Dolbeault complex*.

$$H^{p,q} = \frac{\mathrm{Ker}\big(\bar\partial \colon \Omega^{p,q} \to \Omega^{p,q+1}\big)}{\mathrm{Im}\big(\bar\partial \colon \Omega^{p,q-1} \to \Omega^{p,q}\big)}$$

are called *Dolbeault cohomology groups*.

Structure function of an H–structure

Recall: Let $P \subset Fr_M$ be an H–structure endowed with two connections ω and $\omega' = \omega - \xi$. Then $T' - T = \delta\xi$. Here $T, T' \colon P \to \Lambda^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n$, $\xi \colon P \to (\mathbb{R}^n)^* \otimes \mathfrak{h}$ are regarded as H–equivariant maps and

$$\delta \colon (\mathbb{R}^n)^* \otimes \mathfrak{h} \hookrightarrow (\mathbb{R}^n)^* \otimes (\mathbb{R}^n)^* \otimes \mathbb{R}^n \longrightarrow \Lambda^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n.$$

For $H = SO(n)$ the map δ is an isomorphism.

Consider

$$T_0 \colon P \xrightarrow{T} \Lambda^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n \to \operatorname{Coker}\delta = \Lambda^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n / \operatorname{Im}\delta.$$

By construction, T_0 does not depend on the choice of connection and is called the *structure function* of P. It is the obstruction to the existence of a torsion–free connection on P.

Structure function of a $GL_m(\mathbb{C})$–structure

Theorem

Let $P \subset Fr$ be a $GL_m(\mathbb{C})$–structure, i.e. M is an almost cx mfld. Then P admits a connection, whose torsion is given by $T = \frac{1}{8}N$.

Proof. [KN, Thm IX.3.4]. □

Cor. *The structure function of a $GL_m(\mathbb{C})$–structure can be identified with the Nijenhuis tensor.*

Assume that V is an $SO(n)$–representation and $H = \mathrm{Stab}_\eta,\ \eta \in V$. Then

$$\Lambda^2(\mathbb{R}^n)^* \cong \mathfrak{so}(n) = \mathfrak{h} \oplus \mathfrak{h}^\perp.$$

Since $\delta_{\mathfrak{so}(n)}$ is an isomorphism, we have

- $\delta\colon (\mathbb{R}^n)^* \otimes \mathfrak{h} \to \Lambda^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n$ is injective;
- $\mathrm{Coker}\,\delta \cong (\mathrm{Im}\,\delta)^\perp \cong (\mathbb{R}^n)^* \otimes \mathfrak{h}^\perp$.

Recall that η defines an equivariant map $\tilde\eta\colon Fr_{SO} \to V$.

Prop. *The obstruction $T_0(p)$ to the existence of a torsion–free H–connection can be identified with $(\nabla\tilde\eta)(p)$, and has values in the space $(\mathbb{R}^n)^* \otimes \mathfrak{h}^\perp$.*

Prop. *The obstruction $T_0(p)$ to the existence of a torsion–free H–connection can be identified with $(\nabla\tilde\eta)(p)$, and has values in the space $(\mathbb{R}^n)^* \otimes \mathfrak{h}^\perp$.*

Proof. The obstruction $T_0(p)$ is a component of the torsion of any H–connection ω' on $P \subset Fr_{SO}$. Extend ω' to a connection on P and denote $\xi = \omega - \omega'\colon P \to (\mathbb{R}^n)^* \otimes \mathfrak{so}(n)$, where ω is the Levi–Civita connection. Since $T \equiv 0$, T' is identified with ξ. Observe

$$\nabla'\tilde\eta = 0 \quad\Rightarrow\quad \nabla\tilde\eta(p) = -\xi(p)\tilde\eta. \tag{2}$$

Consider the map $\nu\colon \mathfrak{so}(n) \to \mathrm{End}\,V \xrightarrow{ev_\eta} V$, where the first arrow is the infinitesimal $SO(n)$–action. Then $\mathrm{Ker}\,\nu = \mathfrak{h}$ and $\nu\colon \mathfrak{h}^\perp \to V$ is an embedding. From (2), $\xi(p)\tilde\eta \equiv T_0(p)$ has values in $(\mathbb{R}^n)^* \otimes \mathfrak{h}^\perp$ and can be identified with $\nabla\tilde\eta$. $\square$

Recall:
$$U(m) = SO(2m) \cap Sp(2m;\mathbb{R}) = SO(2m) \cap GL_m(\mathbb{C}) = GL_m(\mathbb{C}) \cap Sp(2m;\mathbb{R}).$$

Hence, a $U(m)$–structure on M is given by one of the following piece of data

(i) A Riemannian metric g and an "almost symplectic form" ω s.t. $TM \xrightarrow{\hat{g}} T^*M \xrightarrow{\hat{\omega}^{-1}} TM$ is an almost cx structure;

(ii) A Riemannian metric g and an orthogonal almost cx str. I;

(iii) An almost complex structure I and an "almost symplectic form" ω s.t. $\omega(\cdot, I\cdot)$ is positive–definite.

Recalling that $\mathfrak{u}(m)^\perp \cong [\Lambda^{0,2}]_r$ we obtain

Prop. *The structure function T_0 of a $U(m)$–structure can be identified with $\nabla\omega$ and takes values in*
$$(\mathbb{R}^{2m})^* \otimes [\Lambda^{0,2}]_r \cong [\Lambda^{0,1} \otimes \Lambda^{0,2}]_r \oplus [\Lambda^{1,2}]_r.$$

Kähler metrics

A manifold M equipped with a $U(m)$–structure P is called *Kähler* if the Levi–Civita connection reduces to P. This is equivalent to any of the following conditions

(i) $\nabla\omega = 0$;

(ii) $\nabla J = 0$;

(iii) $\mathrm{Hol}(M) \subset U(m)$;

(iv) P admits a torsion–free connection.

Prop. *Let (M,g) be a Riemannian mfld equipped with an orthogonal integrable complex structure I. Denote $\omega(I\cdot,\cdot)$. Then g is Kähler iff*
$$d\omega = 0 \quad \Leftrightarrow \quad \bar{\partial}\omega = 0.$$

Cor. *Let M be Kähler and $Z \subset M$ be a complex submanifold. Then the induces metric on Z is also Kähler.*

Prop. *Let (M, g) be a Riemannian mfld equipped with an orthogonal integrable complex structure I. Denote $\omega(\cdot,\cdot) = g(I\cdot,\cdot)$. Then g is Kähler iff*

$$d\omega = 0 \quad \Leftrightarrow \quad \bar\partial\omega = 0.$$

Proof. First observe that $d\omega = 0 \Leftrightarrow \bar\partial\omega = 0$, since ω is a real $(1,1)$–form and $(d\omega)^{0,3} = 0 = (d\omega)^{3,0}$ by the integrability of the complex structure.

If g is Kähler, then $\nabla\omega = 0 \Rightarrow d\omega = 0$.

Assume now $d\omega = 0$. First observe that the component of $\nabla\omega$ lying in $[\Lambda^{0,1} \otimes \Lambda^{0,2}]_r$ can be identified with the structure function of the corresponding $GL_m(\mathbb{C})$–structure and therefore vanishes. $d\omega$ is the image of $\nabla\omega$ under the antisymmetrisation map:

$$[\Lambda^{1,2}]_r \cong [\Lambda^{1,2}_0]_r \oplus [\Lambda^{0,1}]_r \longrightarrow \Lambda^3 \cong [\Lambda^{0,3}]_r \oplus [\Lambda^{2,1}_0] \oplus [\Lambda^{0,1}]_r.$$

Hence, the component of $\nabla\omega$ in $[\Lambda^{1,2}]_r$ is determined by $(d\omega)^{1,2}$ and therefore vanishes. □

Kähler potentials

Let $f\colon \mathbb{C}^m \to \mathbb{R}$. The *Levi form* of f

$$-i\partial\bar\partial f = -i\sum_{j,k}^{m} \frac{\partial^2 f}{\partial z_j \partial \bar z_k} dz_j \wedge d\bar z_k$$

is of type $(1,1)$, real, and closed, since $\partial\bar\partial = \frac{1}{2}d(\bar\partial - \partial)$. The Levi form defines a Kähler metric iff it is positive definite. Conversely, a real closed $(1,1)$–form ω is locally expressible as $-i\partial\bar\partial f$ for some real function f. If ω is a Kähler form, the function f is called a *Kähler potential*.

Ex.

(i) $f = \sum_{j=1}^m |z_j|^2$ is a Kähler potential of the flat metric on $\mathbb{C}^m$;

(ii) $-\log f\colon \mathbb{C}^m \setminus 0 \to \mathbb{R}$ determines a Kähler potential on $\mathbb{CP}^{m-1}$. This metric is called the *Fubini–Study* metric.

Cor. *Any complex submanifold of $\mathbb{CP}^m$ is Kähler.*

Cohomology of Kähler manifolds

Let (M, I, g, ω) be an almost Kähler mfld. Then $H(v, \omega) = g(v, \bar{w})$ is a Hermitian scalar product on $T_{\mathbb{C}}M$, i.e. H is a sesquilinear and positive-definite. The Hodge operator for complexified forms is defined similarly to the real case:

$$\alpha \wedge *\beta = H(\alpha, \beta) vol.$$

Hence, $*$ is antilinear. Moreover, $* : \Omega^{p,q} \to \Omega^{m-q,m-p}$. By analogy with the real case, define

$$\bar{\partial}^* = - * \bar{\partial} * \quad \text{and} \quad \Delta_{\bar{\partial}} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}.$$

Then, just like for the de Rham cohomology, we have

Theorem

Every Dolbeault cohomology class on a compact Hermitian mfld has a unique $\Delta_{\bar{\partial}}$–harmonic representative and $H^{p,q} \cong \mathcal{H}^{p,q} = \operatorname{Ker}\big(\Delta_{\bar{\partial}} \colon \Omega^{p,q} \to \Omega^{p,q}\big)$.

Prop. *If M is Kähler, then $2\Delta_{\bar{\partial}} = \Delta$.*

Hence, we obtain

Theorem

Let M be a compact Kähler mfld. Then

$$H^k(M; \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(M).$$

Moreover, $\overline{H^{p,q}} = H^{q,p}$ and $H^{p,q} \cong \left(H^{m-p,m-q}\right)^*$ (Serre duality).

Serre duality: If $\alpha \in \mathcal{H}^{p,q}$, then $*\alpha \in \mathcal{H}^{m-q,m-p}$. Since $\int_M \alpha \wedge *\alpha = \int_M \|\alpha\|^2 vol$, the pairing

$\mathcal{H}^{p,q} \times \mathcal{H}^{n-p,n-q} \to \mathbb{C}, \quad (\alpha, \beta) \mapsto \int_M \alpha \wedge \beta$ is nondegenerate.

Hence, $\mathcal{H}^{p,q} \cong \left(\mathcal{H}^{n-p,n-q}\right)^*$.

Define the Hodge numbers $h^{p,q}$ by $h^{p,q} = \dim H^{p,q}(M)$. Then for compact Kähler mflds we have

$$b_k = \sum_{j=0}^{k} h^{j,k-j} \quad \text{and} \quad h^{p,q} = h^{q,p} = h^{m-p,m-q} = h^{m-q,m-p}.$$

Cor. *If M is compact Kähler mfld, then odd Betti numbers of M are even.*

Theorem (Hard Lefschetz theorem)

On a compact Kähler mfld M^{2m}, there is a decomposition

$$H^k(M,\mathbb{R}) = \bigoplus_{p+q=k} \bigoplus_{r=0}^{\min(p,q)} H_0^{p-r,q-r}(M), \quad 0 \le k \le m.$$

Idea of the proof: The $\mathfrak{sl}_2(\mathbb{C})$-action on $\Omega^\bullet(M,\mathbb{C})$ descends to $H^\bullet(M;\mathbb{C})$ and respects bidegree and real structure. See [Wells] or [Huybrechts, Complex geometry] for details. □

Curvature of Kähler mflds

Recall: $\mathfrak{R} = \mathrm{Ker}\big(b : S^2(\Lambda^2(\mathbb{R}^n)) \to S^2(\Lambda^2\mathbb{R}^n)\big)$ is the space of algebraic curvature tensors, where $b : S^2(\Lambda^2\mathbb{R}^n) \to \Lambda^4\mathbb{R}^n$ is the Bianchi map (full antisymmetrization).
Let $P \subset Fr_{SO}$ be a principal H-bundle equipped with a connection φ. then the curvature tensor takes values in $\mathfrak{h}$. Hence, we obtain

Prop. *For any $p \in P$ the curvature $R(p)$ belongs to the space*

$$\mathfrak{R}^H = \mathrm{Ker}(b : S^2\mathfrak{h} \to S^2\mathfrak{h})$$

and we have the commutative diagram

$$\begin{array}{ccccc}
\mathfrak{R} & \hookrightarrow & S^2(\Lambda^2\mathbb{R}^n) & \xrightarrow{\;b\;} & \Lambda^4\mathbb{R}^n \\
\uparrow & & \uparrow & & \| \\
\mathfrak{R}^H & \hookrightarrow & S^2\mathfrak{h} & \longrightarrow & \Lambda^4\mathbb{R}^n
\end{array}$$

Consider now the case $H = U(m)$ and recall that $\mathfrak{u}(m) \cong [\Lambda^{1,1}]_r$. Hence,

$$\begin{aligned} S^2(\mathfrak{u}(m)_{\mathbb{C}}) &\cong S^2(\Lambda^{1,1}) \\ &\cong S^2(\Lambda^{1,0}) \otimes S^2(\Lambda^{0,1}) \oplus \Lambda^2(\Lambda^{1,0}) \otimes \Lambda^2(\Lambda^{0,1}) \\ &\cong S^{2,2} \oplus \Lambda^{2,2}. \end{aligned}$$

In analogy to the decomposition

$$\Lambda^{2,2} \cong \Lambda^{2,2}_0 \oplus \Lambda^{1,1}_0 \oplus \mathbb{C}$$

we may write

$$S^{2,2} \cong \mathfrak{B}_{\mathbb{C}} \oplus \Lambda^{1,1}_0 \oplus \mathbb{C},$$

where $\mathfrak{B}_{\mathbb{C}}$ denotes the primitive component.

Prop. $\mathfrak{R}^{U(m)} \cong \mathfrak{B} \oplus [\Lambda^{1,1}_0]_r \oplus \mathbb{R}, \quad \mathfrak{R}^{SU(m)} \cong \mathfrak{B}.$

Proof. [Salamon, Prop. 4.7]. □

Ricci form

Observe: $\mathfrak{R}^{U(m)} \subset End(\Lambda^{1,1})$.

Prop. *For $R \in \mathfrak{R}^{U(m)}$ denote $r = c(R)$, where c is the Ricci contraction. Then $R(\omega_0) = r(I\cdot, \cdot) =: \rho$.*

Proof. Let $(e_1, I_0e_1, \dots, e_m, I_0e_m)$ be an orthonormal basis of $\mathbb{R}^{2m}$. Then

$$\begin{aligned} r(x,y) &= \sum_j \langle R(e_j,x)e_j, y\rangle + \sum_j \langle R(I_0e_j,x)I_0e_j, y\rangle \\ &= \sum_j \langle R(e_j,x)I_0e_j, I_0y\rangle - \sum_j \langle R(e_j,x)e_j, I_0y\rangle \\ &= \sum_j \langle R(e_j, I_0e_j)x, I_0y\rangle, \end{aligned}$$

where $1 \le j \le m$ and the last equality follows from the Bianchi identity. The statement follows since ω_0 is identified with $\sum e_j \wedge I_0e_j$. □

If M is Kähler with curvature tensor R, then the associated $(1,1)$-form ρ is called the Ricci form.

Prop. *The Ricci form is closed.*

Proof. The Ricci form is obtained as contraction of R and ω. Then $d\rho = 0$ follows from $d^{\nabla}R = 0$ and $d\omega = 0$. □

Any $\beta \in [\Lambda^{1,1}]_r \cong \mathfrak{u}(m)$ can be viewed as a $\mathbb{C}$-linear endomorphism of $\mathbb{C}^m$. Then $\mathrm{tr}_{\mathbb{C}}\beta$ is purely imaginary.

Rem. If β is viewed as $\mathbb{R}$-linear map of $\mathbb{R}^{2m}$, then $\mathrm{tr}_{\mathbb{R}}\beta = 0$.

The proof of the previous Proposition shows that $i\rho = \mathrm{tr}_{\mathbb{C}}R$, where R is viewed as a $(1,1)$-form with values in $\mathrm{End}_{\mathbb{C}}(TM)$. Hence,

Prop. *The first Chern class $c_1(M)$ is represented by $\frac{1}{2\pi}\rho$*

Cor. *The curvature tensor of the canonical line bundle $\Lambda^{m,0}T^*M = \Lambda^m(T^*M)^{1,0}$ equals $i\rho$.*

Theorem

Let M^{2m} be a Kähler mfld. Then $\mathrm{Hol}^0(M) \subset SU(m)$ iff $Ric \equiv 0$.

Proof. Let P be the holonomy bundle. Then $\mathrm{Hol}^0(M) \subset SU(m)$ iff for any $p \in P$ $R(p)$ takes values in $\mathfrak{su}(m)$. Observe that

$$\mathfrak{su}(m) = \{A \in \mathfrak{u}(m) \mid \mathrm{tr}_{\mathbb{C}} A = 0\}.$$

Hence, $R(p) \in \mathfrak{su}(m)$ iff $i\rho_{\pi(p)} = \mathrm{tr}_{\mathbb{C}} R(p) = 0 \Leftrightarrow Ric(p) = 0$. □

Theorem

$\mathrm{Hol}(M) \subset SU(M)$ *iff M admits a parallel $(m,0)$–form.*

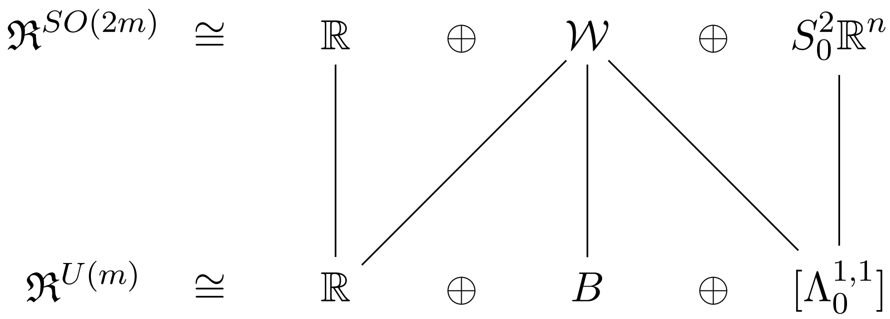
Recall:

$$\mathfrak{R} \cong \mathbb{R} \oplus S_0^2\mathbb{R}^n \oplus \mathcal{W},$$
$$R = \tfrac{s}{2n(n-1)}\, q \owedge q + \tfrac{1}{n-2}\, Ric_0 \owedge q + W.$$

Tracing the identifications for Kähler mflds we can write

$$\mathfrak{R}^{U(m)} \cong \mathbb{R} \oplus [\Lambda_0^{1,1}]_r \oplus \mathfrak{B},$$
$$R = \tfrac{s}{2m^2}\omega \otimes \omega + \tfrac{1}{m}\omega \otimes \rho_0 + \tfrac{1}{m}\rho_0 \otimes \omega + B,$$

where ρ_0 is the primitive component of ρ. In particular, we have the diagram ($m \geq 3$):



Holonomy groups

in Riemannian geometry

Lecture 6

December 1, 2011

Some results from the previous lecture

Prop. *The first Chern class $c_1(M)$ is represented by $\frac{1}{2\pi}\rho$, where ρ is the Ricci form.*

Cor. *The curvature tensor of the canonical line bundle $K_M = \Lambda^{m,0}T^*M = \Lambda^m(T^*M)^{1,0}$ equals $i\rho$.*

Theorem

Let M^{2m} be a Kähler mfld. Then $\mathrm{Hol}^0(M) \subset SU(m)$ iff $Ric \equiv 0$.

Theorem

$\mathrm{Hol}(M) \subset SU(M)$ *iff M admits a parallel $(m, 0)$–form.*

Calabi-Yau and Kähler-Einstein metrics

Let (M, I) be be a closed connected complex mfld.

Def. A Kähler metric g is said to be Kähler-Einstein if it is Einstein, i.e. if there exists a constant λ such that

$$\rho = \lambda\omega. \tag{1}$$

Rem.

(i) $\lambda \colon M \to \mathbb{R}$ in (1) $\implies$ $\lambda = const.$

(ii) (1) $\iff$ $R(\omega) = \lambda\omega.$

Def. A class $c \in H^2(M; \mathbb{R})$ is said to be

- positive, if $\exists \beta \in c \cap \Omega^{1,1}$ s.t. $\beta(\cdot, I\cdot) > 0$;
- negative, if $\exists \beta \in c \cap \Omega^{1,1}$ s.t. $\beta(\cdot, I\cdot) < 0$.

Main Theorems

Theorem (Calabi-Yau)

Let $\rho' \in 2\pi\, c_1(M)$ be a closed real $(1,1)$-form. Then there exists a unique Kähler metric g' on M with Kähler form ω' cohomologous to ω and with Ricci form ρ'.

Cor. *If $c_1(M) = 0$, then M has a unique Ricci-flat Kähler metric g' with $[\omega'] = [\omega]$.*

Theorem (Aubin-Calabi-Yau)

Assume $c_1(M) < 0$. Then, up to a scaling constant, M has a unique Kähler-Einstein metric (with negative Einstein constant) .

On the proof of Calabi-Yau and Aubin-Calabi-Yau theorems

Let $\Omega \in \Omega^{m,0}(U)$, where $U \subset M$ is open. Write

$$\nabla\Omega = \psi \otimes \Omega,$$

where ψ is a local connection form of $\Lambda^{m,0}T^*M$.

Observe: $\Omega \in \Omega^{m,0} \;\Rightarrow\; \partial\Omega = 0 \;\Rightarrow\; \bar\partial\Omega = d\Omega = \psi \wedge \Omega$. By definition, Ω is holomorphic, if $\bar\partial\Omega = 0$. Since Ω is a complex volume form,

$$\bar\partial\Omega = 0 \quad\Longleftrightarrow\quad \psi^{0,1} \wedge \Omega = 0 \quad\Longleftrightarrow\quad \psi \in \Omega^{1,0}.$$

We have

$$\begin{aligned} d(\log\|\Omega\|^2) &= \frac{1}{\|\Omega\|^2}\, d\langle\Omega,\Omega\rangle \\ &= \frac{1}{\|\Omega\|^2}\,(\psi\|\Omega\|^2 + \bar\psi\|\Omega\|^2) \\ &= \psi + \bar\psi. \end{aligned}$$

Ω is holomorphic $\Longrightarrow \quad \psi = (d(\log\|\Omega\|^2))^{1,0} = \partial(\log\|\Omega\|^2)$.

Hence, the curvature of $\Lambda^{m,0}T^*M$ is represented by $d\psi = \bar\partial\partial\log\|\Omega\|^2$. In particular, $d\psi$ is purely imaginary $(1,1)$-form. Hence,

$$\boxed{\rho = i\,d\psi = -i\,\partial\bar\partial\log\|\Omega\|^2.}$$

Further, observe that

$$*\Omega = a \cdot \bar{\Omega},$$

where $a \in \mathbb{C}^*$. Hence, $a \cdot m!\,\Omega \wedge \bar{\Omega} = \|\Omega\|^2 \omega^m$. If g' is another Kähler metric s.t. $[\omega'] = [\omega]$, then

$$(\omega')^m = e^f \cdot \omega^m$$

for some $f : M \to \mathbb{R}$. Therefore,

$$\|\Omega\|^2_{g'} = e^{-f}\|\Omega\|^2_g \quad \Longrightarrow \rho' = \rho - i\partial\bar{\partial} f.$$

Vice versa, by the $\partial\bar{\partial}$-Lemma, for any real closed $(1,1)$-form ρ' cohomologous to ρ, there exists $f\colon M \to \mathbb{R}$ s.t.

$$\rho' - \rho = -i\,\partial\bar{\partial} f.$$

Moreover, f is unique up to an additive constant. Similarly,

$$\omega' - \omega = i\,\partial\bar{\partial}\varphi, \qquad \varphi\colon M \to \mathbb{R}.$$

Thus, in the setting of the CY thm, we are looking for φ s.t.

(i) $(\omega + i\,\partial\bar{\partial}\varphi)^m = e^f \cdot \omega^m$, $(*)$
(ii) $\omega + i\,\partial\bar{\partial}\varphi > 0$,
where f is a fixed function.

Claim. $(i) \Rightarrow (ii)$
Proof. [Ballmann. Lectures on Kähler mflds, p.90]. □

Rem. For Kähler mflds, eqn $Ric(g) = 0$ is therefore equivalent to $(*)$. Notice that

- $(*)$ is an eqn for a *function* rather than for a metric tensor,
- $(*)$ is highly nonlinear (nonlinear in derivatives of the highest order).

Claim. The Kähler-Einstein condition (under the setup of Aubin-Calabi-Yau thm) is equivalent to the eqn

$$(\omega + i\,\partial\bar{\partial}\varphi)^m = e^{f-\lambda\varphi} \cdot \omega^m,$$

where ω is a suitably chosen Kähler metric on M.
Proof. [see Ballmann, p.91 for details]. □

Idea of the proof of the Calabi-Yau thm

Uniqueness: Let φ_1, φ_2 be solutions of the eqn

$$(\omega + i\,\partial\bar{\partial}\varphi)^m = e^{F(p,\varphi)}\omega^m.$$

It can be shown that

$$\frac{1}{m}\int |\mathrm{grad}(\varphi_1 - \varphi_2)|^2_{g_1}\omega_1^m + \\ + \int (\varphi_1 - \varphi_2)(e^{F(p,\varphi_1)} - (e^{F(p,\varphi_2)})\omega^m \le 0.$$

Hence, uniqueness follows from the (weak) monotonicity of F in φ (for each fixed $p \in M$).

Existence (by the continuity method): Consider the eqn

$$(\omega + i\,\partial\bar{\partial}\varphi)^m = e^{tf}\omega^m,$$

where $t \in [0, 1]$ is a parameter. Denote by $\mathcal{T}$ the set of those t, for which there exists a solution. Then $\mathcal{T} \ni 0$, hence $\mathcal{T} \neq \emptyset$. Moreover, $\mathcal{T}$ is open and closed. Hence, $1 \in \mathcal{T}$.

Examples of Calabi–Yau manifolds

A compact (simply connected) Riemannian mfld with $\mathrm{Hol}(M, g) \subset SU(m)$ is called *Calabi-Yau*. If $\pi_1(M) = \{1\}$ this is equivalent to $c_1(M) = 0$.

Ex.

1) Let M be a degree d hypersurface in $\mathbb{C}P^N$. From the adjunction formula we have

$$K_M = \big(K_{\mathbb{CP}^N} \otimes \mathcal{O}(d)\big)\big|_M \cong \mathcal{O}(-N-1+d)\big|_M.$$

Therefore, $c_1(K_M) = 0 \;\Leftrightarrow\; d = N + 1$. Hence, the Fermat quartic $M = \{z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0\} \subset \mathbb{CP}^3$ admits a metric with holonomy $SU(2)$.

2) Let M be a complete intersection: $M = M_{d_1} \cap \cdots \cap M_{d_k} \subset \mathbb{C}P^N$. Then $c_1(M) = 0 \;\Leftrightarrow\; d_1 + \cdots + d_k = N + 1$.

A non-compact example: Calabi metric

Theorem (Calabi)

Let M be Kähler–Einstein with positive sc. curvature. Then there exists a metric on the total space of K_M with $\mathrm{Hol}^0 \subset SU(m+1)$.

Proof. Let $P \to M$ be the $U(m)$-structure. Since $\mathfrak{u}(m) \cong \mathfrak{su}(m) \oplus i\mathbb{R}$, the Levi-Civita connection on P decomposes: $\varphi_{LC} = \varphi_0 + \psi i$. Observe that ψi is essentially the connection of K_M. It follows that M is KE iff $d\psi = \lambda\pi^*\omega$, where $\pi : P \to M$. Consider $\beta = dz + z\psi i \in \Omega^1(P \times \mathbb{C}; \mathbb{C})$, where z is a coordinate on $\mathbb{C}$. Put $\rho = |z|^2 = z\bar{z}$. With the help of

$$d\beta = (\beta \wedge \psi + \lambda z \pi^*\omega)i, \qquad d\rho = dz \cdot \bar{z} + z d\bar{z} = \beta \cdot \bar{z} + z\beta,$$

one easily shows that the 2-form

$$\tilde{\omega} = u\pi^*\omega - \frac{1}{\lambda}u' \cdot i\beta \wedge \bar{\beta}$$

is closed, where $u = u(\rho)$.

Proof of the Calabi theorem (continued)

Moreover, $\tilde{\omega} = u\pi^*\omega - \frac{1}{\lambda}u' \cdot i\beta \wedge \bar{\beta}$ is $U(m)$-invariant and basic and therefore descends to a $(1,1)$-form $\tilde{\omega}$ on $(P \times \mathbb{C})/U(m) = K$. If both u and u' are positive, $\tilde{\omega}$ is also positive.
Recall that each $p \in P$ is a unitary basis of $T_{\pi(p)}M$, i.e. $p = (p_1, \dots, p_m)$. Then $\Omega = p_1^* \wedge \dots \wedge p_m^*$ is a global complex m-form on P. Consider

$$\tilde{\Omega} = \beta \wedge \Omega.$$

Just like $\tilde{\omega}$, $\tilde{\Omega}$ descends to an $(m+1, 0)$-form on K. Then $\tilde{\Omega}$ is parallel iff $\|\tilde{\Omega}\| = const \Rightarrow u^m u' = \lambda(m+1) \Rightarrow u(\rho) = (\lambda\rho + l)^{\frac{1}{m+1}}$. Hence we obtain an explicit metric on K with $\mathrm{Hol}^0 \subset SU(m+1)$, namely

$$g = u(p)\pi_K^* g_M \oplus u'(\rho)\mathrm{Re}(\beta \otimes \bar{\beta}). \qquad \square$$

Rem. If the scalar curvature of M is negative, the Calabi metric is defined on a neighbourhood of the zero section only.

HyperKähler manifolds

A quaternionic vector space is a real vector space V equipped with a triple (I_1, I_2, I_3) of endomorphisms s.t.

$$I_r^2 = -1, \quad I_1 I_2 = I_3 = -I_2 I_1.$$

In other words, V is an $\mathbb{H}$-module.

V is *quaternion-Hermitian*, if V is equipped with an Euclidean scalar product, which is Hermitian wrt each complex structure I_r. Denote $\omega_r(\cdot,\cdot) = \langle I_r \cdot, \cdot\rangle$, $\omega = \omega_1 i + \omega_2 j + \omega_3 k$.

Ex. $V = \mathbb{H}^m$, $I_1(h) = h\bar{i}$, $I_2(h) = h\bar{j}$, $I_3(h) = h\bar{k}$, $\langle h_1, h_2\rangle = \mathrm{Re}(\bar{h}_1 h_2)$. Then $\omega(h_1, h_2) = \mathrm{Im}(\bar{h}_1 h_2)$

Put $h = \langle\cdot,\cdot\rangle + i\omega_1$ and $\omega_c = \omega_2 + \omega_3 i$. Then h is an Hermitian scalar product and ω_c is a complex symplectic form. Hence,

$$\begin{aligned} Sp(m) &= \{A \in O(\mathbb{H}^n) | AI_r = I_r A, \quad r = 1,2,3\} \\ &= O(4n) \cap GL_n(\mathbb{H}) \\ &= U(2n) \cap Sp(2n;\mathbb{C}). \end{aligned}$$

CY and KE mflds HyperKähler mflds Quaternion–Kahler mflds

Assume M^{4m} is endowed with with an $Sp(m)$–structure. In other words, M is a Riemannian mfld equipped with a triple (I_1, I_2, I_3) of almost complex structures s.t. the metric is Hermitian wrt each I_r.

Alternatively, M can be seen as an almost Hermitian mfld equipped with a complex symplectic form $\omega_c \in \Omega^{2,0}(M)$.

M is called *hyperKähler*, if $\mathrm{Hol(M)} \subset Sp(m)$. This is equivalent to one of the following conditions:

(i) $\nabla I_1 = \nabla I_2 = \nabla I_3 = 0$;

(ii) $\nabla \omega_1 = \nabla \omega_2 = \nabla \omega_3 = 0$;

(iii) g is Kähler wrt each complex structure I_r.

Prop. *For an almost hyperKähler manifold the following holds:*

$$\nabla\omega_1 = \nabla\omega_2 = \nabla\omega_3 = 0 \iff d\omega_1 = d\omega_2 = d\omega_3 = 0.$$

Proof. Need to show that each almost complex structure is integrable. Observe: $v \in \mathfrak{X}^{1,0}_{I_1}(M) \Leftrightarrow \imath_v\omega_2 = i\,\imath_v\omega_3$. Indeed,

$$i_v\omega_2 = g(I_2v,\cdot) = g(I_3I_1v,\cdot) = \omega_3(I_1v,\cdot).$$

Then $\imath_v\omega_2 = i\imath_v\omega_3 \Leftrightarrow I_1v = iv$.
Assume now $v, w \in \mathfrak{X}^{1,0}_{I_1}(M)$. Then

$$\begin{aligned}\imath_{[v,w]}\omega_2 &= \mathcal{L}_v(\imath_w\omega_2) - \imath_w(\mathcal{L}_v\omega_2)\\ &= \mathcal{L}_v(\imath_w\omega_2) - \imath_w(\imath_v\omega_2) && \text{(Cartan)}\\ &= \mathcal{L}_v(i\imath_w\omega_3) - \imath_w(i\imath_w\omega_3)\\ &= i\,\imath_{[v,w]}\omega_3.\end{aligned}$$

□

Examples of hyperKähler manifolds

Ex.

(i) We have an exceptional isomorphism $Sp(1) \cong SU(2)$, since $\omega_c \in \lambda^{2,0}\mathbb{C}^2$ is a complex volume form. Hence, if $\dim_{\mathbb{R}} M = 4$

Calabi-Yau ≡ hyperKähler

Hence, there is a hK metric on the Fermat quartic.

(ii) Similar methods as in the proof of the fact that for KE M the total space of K_M has a Ricci-flat metric, also give that the total space of $T^*\mathbb{C}P^m$ has a complete metric with holonomy $Sp(m)$ for any m (this fact is also due to Calabi).

Let M^{4m} be a *compact* Kähler with a complex sympl. form ω_c. Then ω_c^m trivializes K_M and hence there exists a Ricci-flat Kähler metric on M.

Observe that any closed $(p,0)$–form on closed Ricci–flat Kähler mfld must be parallel. This follows from the fact that the Weitzenböck formula for $(p,0)$–forms involves Ricci–curvature only.

Hence, with respect to the new Ricci–flat metric $\nabla\omega_c = 0$. Thus if M is compact Kähler

$$\text{hyperKähler} \equiv \text{complex symplectic}$$

This is used to show that there are compact 8-mflds with holonomy $Sp(2)$ by blowing-up the diagonal in $M_4 \times M_4$ and quotening by the involution. Further generalization of this yields compact mflds with holonomy $Sp(m)$.

HyperKähler reduction

Let M be a hK mfld and assume G acts on M preserving hK structure. Then for any $\xi \in \mathfrak{g}$

$$0 = \mathcal{L}_{K_\xi}\omega_r = \imath_{K_\xi} d\omega_r + d\imath_{K_\xi}\omega_r = 0 + d\imath_{K_\xi}\omega_r,$$

where K_ξ is the Killing v.f.
Assume there exists $\mu_r(\xi) : M \to \mathbb{R}$ s.t. $i_{K_\xi}\omega_r = d\mu_r(\xi)$.
Construct a G-equivariant map

$$\mu = \mu_1 i + \mu_2 j + \mu_3 k : \quad M \to \mathfrak{g}^* \otimes \operatorname{Im}\mathbb{H},$$

which is called the hK moment map.

Theorem

If $M /\!/\!/_\tau G = \mu^{-1}(\tau)/G$ is a mfld, where $\tau \in \mathfrak{g}^*$ is central, then it is hyperKähler (with respect to the induces metric).

Proof. For $m \in \mu^{-1}(\tau)$ put $\mathcal{K}_m = \{K_\xi(m) \mid \xi \in \mathfrak{g}\}$. Since $d\mu_r(\xi) = g(I_r K_\xi, \cdot)$, the orthogonal complement to

$$\mathcal{K}_m \oplus I_1\mathcal{K}_m \oplus I_2\mathcal{K}_m \oplus I_3\mathcal{K}_m$$

can be identified with $T_{[m]}(M /\!/\!/_\tau G)$. Hence $M /\!/\!/_\tau G$ is almost hyperKähler. The corresponding 2-forms are closed, hence $M /\!/\!/_\tau G$ is hyperKähler. □

Further examples of hyperKähler manifolds

Ex.

1) S^1 acts on $\mathbb{H}^{n+1}$ by multiplication on the left. The moment map is

$$\mu(x) = -\sum_{p=1}^{n+1} \bar{x}_p i x_p = i\sum_{p=1}^{n+1}(|w_p|^2 - |z_p|^2) - 2k\sum_{p=1}^{n+1} z_p w_p,$$

where $x_p = z_p + jw_p,\ z_p, w_p \in \mathbb{C}$. Clearly,

$$\mathbb{H}^{n+1} /\!/\!/ S^1 = \mu^{-1}(-i)/S^1 \cong$$

$$\cong \{(z_p, w_p) \in \mathbb{C}^{2n+2} | \sum_{p=1}^{n+1} z_p w_p = 0,\ (z_1, \dots, z_{n+1}) \neq 0\}/\mathbb{C}^*$$

$$\cong T^*\mathbb{C}P^n.$$

Hence, the total space of $T^*\mathbb{CP}^n$ is hK and the metric obtained via the hK reduction coincides with the Calabi metric.

Ex.

2) $T^*Gr_p(\mathbb{C}^{p+q})$ is hK. This is also obtained as a hK reduction: $T^*Gr_p(\mathbb{C}^{p+q}) \cong \mathbb{H}^{p(p+q)} /\!/\!/ U(p)$.

3) Let X^4 be a hK mfld. Pick a G-bundle $P \to X$. Then the space $\mathcal{A}(P)$ inherits a hK structure. The action of the gauge gp $\mathcal{G} = AutP$ preserves this hK structure and the moment map is

$$\begin{aligned}\mu : A \longmapsto F_A^+ \in \Omega^2_+(X; \mathrm{ad}\, P) \cong \\ \cong \Gamma(\mathrm{ad}\, P) \otimes \mathrm{Im}\,\mathbb{H} \cong \\ \cong \mathrm{Lie}(\mathcal{G})^* \otimes \mathrm{Im}\,\mathbb{H}.\end{aligned}$$

Hence, the moduli space of asd instantons

$$\mu^{-1}(0)/\mathcal{G} \cong \{A \mid F_A^+ = 0\}/\mathcal{G}$$

is hyperKähler.

Quaternion-Kähler manifolds

Consider the action of $Sp(n) \times Sp(1)$ on $\mathbb{H}^n$:

$$(A, q) \cdot x = Ax\bar{q}.$$

Obviously, $(-1, -1)$ acts trivially and we define

$$Sp(n)Sp(1) = Sp(n) \times Sp(1)/\pm 1 \subset SO(4n).$$

Consider $\Lambda^1 = \mathbb{R}^{4n}$ as $Sp(n)Sp(1)$-representation. Then

$$\Lambda^1_{\mathbb{C}} \cong E \otimes_{\mathbb{C}} W,$$

where E denotes the complex tautological representation of $Sp(n) \subset SU(2n)$ of dimension $2n$ and W denotes the two dimensional complex representation of $Sp(1) \cong SU(2)$. Explicitly,

$$v \longmapsto v^{1,0} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + I_2 v^{0,1} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then
$$\mathfrak{so}(4n) \cong \Lambda^2(\mathbb{R}^{4n})^* \cong \Lambda^2[E\otimes W]_r$$
$$\cong [S^2E\otimes\Lambda^2W]_r \oplus [\Lambda^2E\otimes S^2W]_r$$
$$\cong \mathfrak{sp}(n)\oplus[\Lambda^2E\otimes W_2]_r$$
$$\cong \mathfrak{sp}(n)\oplus\mathfrak{sp}(1)\oplus[\Lambda^2_0E\otimes W_2]_r.$$

Here: $W_p = S^pW$ is the irreducible $(p+1)$-dimensional $Sp(1)$-representation. In particular, $W_1 = W$, $W_2 = \mathfrak{sp}(1)_{\mathbb{C}}$. Consider the 4-form

$$\Omega_0 = \omega_1\wedge\omega_1 + \omega_2\wedge\omega_2 + \omega_3\wedge\omega_3 \qquad \in \Lambda^4(\mathbb{R}^{4n})^*,$$

which is $Sp(n)Sp(1)$-invariant.

Lem. *For $n \geq 2$, the subgp of $GL_{4n}(\mathbb{R})$ preserving Ω_0 is equal to $Sp(n)Sp(1)$.*

Proof. [Salamon. Lemma 9.1] □

Rem. Hence, the 4-form Ω_0 determines the Euclidean scalar product.

An $Sp(n)Sp(1)$-structure on M^{4n}, $n \geq 2$ can be described by $\Omega \in \Omega^4(M)$, which is linearly equivalent to Ω_0 at each pt. Then M is quaternion-Kähler, i.e. $\mathrm{Hol}(M) \subset Sp(n)Sp(1)$, iff $\nabla\Omega = 0$. In particular, $d\Omega = 0$.

Theorem (Swann)

If $\dim M \geq 12$, then $\nabla\Omega = 0 \;\Leftrightarrow\; d\Omega = 0$.

In contrast to hK mflds, qK mflds do not have global almost complex structures but rather are endowed with rank 3 subbundle of $End(TM)$ admitting *local* trivialization (I_1, I_2, I_3) satisfying quaternionic relations. This is apparent from the decomposition

$$\mathfrak{so}(4n) \cong \mathfrak{sp}(n)\oplus\mathfrak{sp}(1)\oplus[\Lambda^2_0E\otimes W_2]_r.$$

Prop. *The spaces of algebraic curvature tensors for qK and hK mflds are given respectively by*

$$\mathcal{R}^{Sp(n)Sp(1)} \cong [S^4E]_r \oplus \mathbb{R},$$
$$\mathcal{R}^{Sp(n)} \cong [S^4E]_r.$$

Proof. Similar to the corresponding proof for Kähler mflds. For details see [Salamon. Prop. 9.3]. □

Cor. *Any qK mfld is Einstein, and its Ricci tensor vanishes iff it is locally hK, i.e.* $\mathrm{Hol}^0 \subset Sp(n)$.

Ex. $\mathbb{H}P^n = \mathbb{H}^{n+1} \backslash \{0\} / \mathbb{H}^* \cong \frac{Sp(n+1)}{Sp(n)\times Sp(1)}$ is a symmetric qK mfld. All qK symmetric spaces were classified by Woff.

CY and KE mflds HyperKähler mflds Quaternion–Kahler mflds

Theorem (Swann)

Let M^{4n} *be a positive qK mfld with the corresponding* $Sp(n)Sp(1)$*-structure* P. *Then the total space of the bundle* $\mathcal{U}(M) = P \times_{Sp(n)Sp(1)} \mathbb{H}^*/\pm 1$ *carries a hK metric.*

The construction of this hK metric is similar to the construction of the Calabi metrics (Ricci–flat on K_M and hK on $T^*\mathbb{CP}^n$).

Holonomy groups

in Riemannian geometry

Lecture 7

Exceptional holonomy groups

December 8, 2011

Groups $Spin(3)$, $Spin(4)$, and $Sp(1)$

Recall: For $n \geq 3$, $\mathrm{Spin}\,(n)$ is a connected simply connected group fitting into the short exact sequence

$$0 \to \{\pm 1\} \to \mathrm{Spin}\,(n) \to SO(n) \to 0,$$

In other words, $SO(n) \cong \mathrm{Spin}\,(n)/\pm 1$.

The group $Sp(1) = \{q \in \mathbb{H} \mid q\bar{q} = 1\}$ acts on $\mathrm{Im}\,\mathbb{H}$: $\quad q \cdot x = qx\bar{q}$. Hence, we have the short exact sequence

$$0 \to \{\pm 1\} \to Sp(1) \to SO(3) \to 0,$$

which establishes the isomorphism $\mathrm{Spin}\,(3) \cong Sp(1) \cong SU(2)$.

Consider also the action of $Sp_+(1) \times Sp_-(1)$ on $\mathbb{H}$: $(q_+, q_-) \cdot x = q_+ x \bar{q}_-$. This leads to the short exact sequence

$$0 \to \{\pm 1\} \to Sp_+(1) \times Sp_-(1) \to SO(4) \to 0.$$

Hence, $\mathrm{Spin}\,(4) \cong Sp_+(1) \times Sp_-(1)$.

The group G_2

Put $V = \mathrm{Im}\,\mathbb{H}_x \oplus \mathbb{H}_y \cong \mathbb{R}^7$, which is considered as oriented Euclidean vector space. $SO(4)$ acts on V:

$$[q_+, q_-] \cdot (x, y) = (q_- x \bar{q}_-, q_+ y \bar{q}_-).$$

Write

$$\begin{aligned}\tfrac{1}{2}\, d\bar{y} \wedge dy =& \omega_1 i + \omega_2 j + \omega_3 k \\ =& (dy_0 \wedge dy_1 - dy_2 \wedge dy_3) i + (dy_0 \wedge dy_2 + dy_1 \wedge dy_3) j + \\ &+ (dy_0 \wedge dy_3 - dy_1 \wedge dy_2) k.\end{aligned}$$

Notice that $(\omega_1, \omega_2, \omega_3)$ is the standard basis of $\Lambda^2_-(\mathbb{R}^4)^*$. Put

$$\begin{aligned}\varphi &= vol_x - \tfrac{1}{2}\mathrm{Re}\,(dx \wedge dy \wedge d\bar{y}) \\ &= dx_1 \wedge dx_2 \wedge dx_3 + dx_1 \wedge \omega_1 + dx_2 \wedge \omega_2 + dx_3 \wedge \omega_3.\end{aligned}$$

Def. The stabilizer of φ in $GL_7(\mathbb{R})$ is called G_2.

$$\varphi = vol_x - \tfrac{1}{2}\mathrm{Re}\,(dx \wedge dy \wedge d\bar{y}).$$

Observe the following:

- $L^*_{[q_+, q_-]} d\bar{y} \wedge dy = q_- d\bar{y} \wedge dy\, \bar{q}_- \;\Rightarrow\; \mathrm{Re}\,(dx \wedge dy \wedge d\bar{y})$ is $SO(4)$-invariant $\Rightarrow\; SO(4) \subset G_2$.
- Write $V = (\mathbb{R} \oplus \mathbb{C}_z) \oplus \mathbb{C}^2_{w_1, w_2}$, $(x_0, z, w_1, w_2) \mapsto x_0 i + zj + \bar{w}_1 + w_2 j$. Then

  $$\begin{aligned}\varphi = &\tfrac{1}{2} dx_0 \wedge \mathrm{Im}(dz \wedge d\bar{z} + dw_1 \wedge d\bar{w}_1 + dw_2 \wedge d\bar{w}_2) \\ &+ \mathrm{Re}\,(dz \wedge dw_1 \wedge dw_2)\end{aligned}$$

  Hence, $G_2 \supset SU(3)$.
- $SO(4) \subset G_2$, $SU(3) \subset G_2 \;\Rightarrow G_2 \cap SO(7)$ acts transitively on S^6.

- For $Q : V \to \Lambda^7 V$, $Q(v) = (i_v\varphi)^2 \wedge \varphi$ we have $Q(e_1) = \|e_1\|^2 vol_7 \Rightarrow Q(v) = \|v\|^2 vol_7$ for all $v \in V$.
- $g \in G_2 \Rightarrow g^*Q(gv) = Q(v) \Rightarrow (\det g) \cdot \|gv\|^2 = \|v\|^2$ $\Rightarrow \det g = 1$, i.e. $G_2 \subset SO(7)$
- $\{g \in G_2 \mid ge_1 = e_1\} \cong SU(3)$. Hence, we have that topologically G_2 is the fibre bundle

$$\begin{array}{ccc} SU(3) & \hookrightarrow & G_2 \\ & & \downarrow \\ & & S^6 \end{array}$$

In particular, $\dim G = 14$; G is connected and simply connected.

- $\Lambda^3 V^* \supset GL_7(\mathbb{R}) \cdot \varphi \cong GL_7(\mathbb{R})/G_2$ has dimension $35 = \dim \Lambda^3 V^*$. Hence, $GL_7(\mathbb{R}) \cdot \varphi$ is an open set in $\Lambda^3 V^*$.

Fact. G_2 is the automorphism group of octonions, i.e.

$$\{g \in GL_8(\mathbb{R}) \mid g(ab) = g(a) \cdot g(b)\} \cong G_2.$$

Some representation theory of G_2

Consider $V \cong \mathbb{R}^7$ as a G_2-representation via the embedding $G_2 \subset SO(7)$. Then V is irreducible.

Further $\Lambda^2 V^*$ contains the following G_2-invariant subspaces

- $\Lambda^2_{14} V^* \cong \mathfrak{g}_2$
- $\Lambda^2_7 V^* = \{i_v\varphi \mid v \in V\} \cong V$

which are irreducible. By dimension counting,

$$\Lambda^2 V^* \cong \Lambda^2_{14} V^* \oplus \Lambda^2_7 V^*.$$

Rem. The subspaces Λ^2_7 and Λ^2_{14} can be described equivalently as follows:

$$\Lambda^2_7 = \{\alpha \mid *(\varphi \wedge \alpha) = 2\alpha\}$$
$$\Lambda^2_{14} = \{\alpha \mid *(\varphi \wedge \alpha) = -\alpha\}$$

To decompose $\Lambda^3 V^*$, consider

$$\gamma\colon \mathrm{End}(V) \cong V \otimes V \mapsto \Lambda^3 V^*, \qquad \gamma(a) = a^*\varphi.$$

Then $\mathrm{Ker}\,\gamma = \mathfrak{g}_2$. Since $\dim \mathrm{Im}\,\gamma = 7 \times 7 - \dim \mathrm{Ker}\,\gamma = 35$ $= \dim \Lambda^3 V^*$, γ is surjective. Hence,

$$\Lambda^3 V^* \cong S^2 V^* \oplus \Lambda^2_7 V^* \cong \mathbb{R} \oplus S^2_0 V^* \oplus V^*$$

and $S^2_0 V^*$ is irreducible. We summarize,

Lem.

$$\Lambda^2 V^* \cong \mathfrak{g}_2 \oplus V,$$
$$\Lambda^3 V^* \cong \mathbb{R} \oplus V \oplus S^2_0 V^*$$

G_2 as a structure group

A G_2-structure on M^7 is determined by a 3-form φ, which is pointwise linearly equivalent to the 3–form $\varphi_0 \in \Lambda^3(\mathbb{R}^7)^*$. In particular, φ determines a Riemannian metric g_φ and an orientation.

The following Lemma is auxiliary and will be proved in the next lecture.

Lem. *Denote by $\sigma : \mathbb{R}^n \otimes \Lambda^k(\mathbb{R}^n)^* \to \Lambda^{k-1}(\mathbb{R}^n)^*$ the contraction map.Then, for any Riemannian mfld M, the map*

$$\Gamma(\Lambda^k T^*M) \xrightarrow{\nabla^{LC}} \Gamma(T^*M \otimes \Lambda^k T^*M) \xrightarrow{-\sigma} \Gamma(\Lambda^{k-1} T^*M)$$

coincides with $d^* : \Omega^k \to \Omega^{k-1}$.

Theorem

φ is parallel wrt the Levi-Vita connection of g_φ iff $d\varphi = 0 = d(*_\varphi \varphi)$.

Proof. Recall that the intrinsic torsion of the G_2–structure can be identified with $\nabla\varphi$. In particular, $\nabla\varphi$ takes values in $V^* \otimes \mathfrak{g}_2^\perp \cong V^* \otimes V \cong (S_0^2 V^* \oplus \mathbb{R}) \oplus (\mathfrak{g}_2 \oplus V)$. Observe that $d\varphi$ and $d(*\varphi)$ can be obtained from $\nabla\varphi$ by means of the algebraic maps

$$V^* \otimes V \hookrightarrow V^* \otimes \Lambda^3 V^* \longrightarrow \Lambda^4 V^* \cong \Lambda^3 V^* \cong \mathbb{R} \oplus V \oplus S_0^2 V^*.$$
$$V^* \otimes V \hookrightarrow V^* \otimes \Lambda^3 V^* \mapsto \Lambda^2 V^* \cong \mathfrak{g}_2 \oplus V.$$

One can show that both maps are surjective. Comparing components of target spaces with the components of

$$V^* \otimes V \cong S_0^2 V^* \oplus \mathbb{R} \oplus \mathfrak{g}_2 \oplus V$$

we obtain that $\nabla\varphi = 0 \iff d\varphi = 0 = d(*\varphi)$. □

Curvature of a G_2–manifold

Let $c : S^2\mathfrak{g}_2 \to S^2 V^*$ be the Ricci contraction. Denote $F = \operatorname{Ker} c$. This is an irreducible G_2-representation of dimension 77.

Recall that $\mathcal{R}^{G_2} \cong \operatorname{Ker} b \cap S^2\mathfrak{g}_2$, where

$$b : S^2(\Lambda^2 V^*) \to \Lambda^4 V^*$$

is the Bianchi map. Notice that

$$S^2\mathfrak{g}_2 \cong F \oplus S_0^2 V^* \oplus \mathbb{R},$$
$$\Lambda^4 V^* \cong \Lambda^3 V^* \cong V \oplus S_0^2 V^* \oplus \mathbb{R}$$

The Bianchi map is injective on $S_0^2 V^* \oplus \mathbb{R}$. Hence $\mathcal{R}^{G_2} \cong F$. We summarize

Prop. *$\mathcal{R}^{G_2} \cong F$. A 7-mfld with holonomy in G_2 is Ricci-flat.*

The group $Spin(7)$

Put $U = \mathbb{H}_x \oplus \mathbb{H}_y$. Let $Sp_0(1) \times Sp_+(1) \times Sp_-(1)$ act on U via

$$(q_0, q_+, q_-) \cdot (x, y) = (q_0 x \bar{q}_-, q_+ y \bar{q}_-).$$

Define the Cayley 4-form $\Omega_0 \in \Omega^4(V)$ by

$$\Omega_0 = vol_x + \omega_x^1 \wedge \omega_y^1 + \omega_x^2 \wedge \omega_y^2 + \omega_x^3 \wedge \omega_y^3 + vol_y =$$
$$= vol_x - \mathrm{Re}(d\bar{x} \wedge dx \wedge d\bar{y} \wedge dy) + vol_y.$$

Denote by K the stabilizer of Ω_0 in $GL_8(\mathbb{R})$. The following facts are obtained in a similar fashion as for the group G_2:

- $\Omega_0 = dx_0 \wedge \varphi_0 + *_4\varphi_0 \quad \Longrightarrow \quad G_2 = K \cap SO(7)$
- $SU(4) \subset K$
- $K \subset SO(8)$
- K is a compact, connected and simply connected Lie group of dimension 21 acting transitively on S^7

- Consider U as a G_2-representation. Then $U \cong \mathbb{R} \oplus V \;\Rightarrow\; \Lambda^2 U \cong \Lambda^2 V \oplus V \cong \mathfrak{g}_2 \oplus V \oplus V$. By dimension counting, $\mathfrak{K} \cong \mathfrak{g}_2 \oplus V$. Hence,

$$\Lambda^2 U \cong \mathfrak{K} \oplus \mathfrak{K}^\perp \quad \text{with} \quad \dim \mathfrak{K}^\perp = 7.$$

- Obviously, $-\mathbf{1}_U \in K$ acts trivially on $\Lambda^2 U$. One can show that the map

$$K/\pm 1 \to SO(\mathfrak{K}^\perp)$$

is an isomorphism. Hence,

$$K \cong Spin(7).$$

Rem. Unlike in the G_2 case, the orbit of Ω_0 in $\Lambda^4(\mathbb{R}^8)^*$ is not open.

$Spin(7)$ as a structure group

A $Spin(7)$-structure on M^8 is determined by $\Omega \in \Omega^4(M)$, which is pointwise linearly equivalent to the Cayley form.

Theorem

Ω is parallel wrt the Levi-Civita connection of g_Ω iff $d\Omega = 0$.

Proof. [Salamon, Prop. 12.4]. □

Prop. *$\mathcal{R}^{Spin(7)} \cong W$, where W is an irreducible $Spin(7)$-representation of dimension 168. In particular, an 8-mfld with holonomy in $Spin(7)$ is Ricci-flat.*

Proof. [Salamon, Cor. 12.6]. □

Examples

Ex.

- Since $SU(3) \subset G_2$, for any Z with $\mathrm{Hol}(Z) \subset SU(3)$, $M = Z \times \mathbb{R}$ can be considered as G_2-mfld
- First local examples were constructed by Bryant in 1987.

Theorem (Bryant-Salamon)

Let M be a positive self-dual Einstein four–manifold. Then there exists a metric with holonomy in G_2 on the total space of $\Lambda^2_- T^*M$.

Sketch of the proof. Let $P \to M$ be the principal $SO(4)$-bundle. Since $\mathfrak{so}(4) = \mathfrak{so}_+(3) \oplus \mathfrak{so}_-(3)$ we can decompose the Levi-Vita connection: $\tau = \tau_+ + \tau_-$. Further, since $Sp(1) \cong Spin(3)$ we have

$$\mathfrak{so}(3) = \mathfrak{spin}(3) \cong \mathfrak{sp}(1) = \mathrm{Im}\,\mathbb{H}.$$

Hence, $\tau_\pm \in \Omega^1(P; \mathrm{Im}\,\mathbb{H})$. Similarly, the canonical 1-form θ can be thought of as an element of $\Omega^1(P; \mathbb{H})$.

Consider the action of $SO(4) = Sp_+(1) \times Sp_-(1)/\pm 1$ on $P \times \mathrm{Im}\,\mathbb{H}_x$

$$[q_+, q_-] \cdot (p, x) = (p \cdot [q_+, q_-], q_- x \bar{q}_-).$$

Clearly, $P \times \mathrm{Im}\,\mathbb{H}/SO(4) \cong \Lambda^2_- T^*M$.

Put $\alpha = dx + \tau_- x - x\tau_- \in \Omega^1(P \times \mathrm{Im}\,\mathbb{H}, \mathrm{Im}\,\mathbb{H})$. It is easy to check that the following forms are $SO(4)$-equivariant:

$$\begin{aligned}
\gamma_1 =& \alpha_1 \wedge \alpha_2 \wedge \alpha_3, \\
\gamma_2 =& -\mathrm{Re}\,(\alpha \wedge \bar{\theta} \wedge \theta) = \alpha_1 \wedge \omega_1 + \alpha_2 \wedge \omega_2 + \alpha_3 \wedge \omega_3, \\
\varepsilon_1 =& \tfrac{1}{6}\mathrm{Re}\,(\bar{\theta} \wedge \theta \wedge \bar{\theta} \wedge \theta) = \pi^* vol_M, \\
\varepsilon_2 =& \tfrac{1}{4}\mathrm{Re}\,(\alpha \wedge \alpha \wedge \bar{\theta} \wedge \theta) = \\
=& \alpha_2 \wedge \alpha_3 \wedge \omega_1 + \alpha_3 \wedge \alpha_1 \wedge \omega_2 + \alpha_1 \wedge \alpha_2 \wedge \omega_3.
\end{aligned}$$

Moreover, for any functions $f = f(|x|^2)$, $h = h(|x|^2)$ without zeros the symmetric tensor

$$g = f^2(\alpha_1^2 + \alpha_2^2 + \alpha_3^2) + h^2(\theta_0^2 + \theta_1^2 + \theta_2^2 + \theta_4^2)$$

determines a metric on $\Lambda^2_- T^*M$. Then

$$\varphi = f^3\gamma_1 + fh^2\gamma_2$$

determines a G_2-structure on $\Lambda^2_- T^*M$. We have also

$$*\varphi = h^4\varepsilon_1 - f^2h^2\varepsilon_2.$$

With the help of the fact that M is positive, self-dual, and Einstein, equations $d\varphi = 0 = d*\varphi$ essentially imply that

$$f(r) = (1+r)^{-1/4} \qquad h(r) = \sqrt{2\varkappa}(1+r)^{1/4}.$$

Here $\varkappa = (sc.curv.)/12 > 0$. □

Rem. Hitchin showed that the only complete self-dual Einstein 4-mflds with positive sc. curvature are S^4 and $\mathbb{C}P^2$ with their standard metrics. For these 4-mflds the holonomy of the Bryant-Salamon metric *equals* G_2.

Using similar technique, Bryant and Salamon prove the following.

Theorem

Let M^3 be S^3 or its quotient by a finite group. Then there exists an explicite metric with holonomy G_2 on $M \times \mathbb{R}^4$ (total space of the spinor bundle).

Consider S^4 as $\mathbb{HP}^1$. Let $\mathbb{S}$ denote the tautological quaternionic line bundle (the spinor bundle).

Theorem

The total space of $\mathbb{S}$ carries an explicite metric with holonomy $Spin(7)$.

Calabi metric revisited

Recall: If S^1 acts on $\mathbb{C}^4 \cong \mathbb{H}^2$ via

$$\lambda \cdot (z_1, z_2, w_1, w_2) = (\lambda z_1, \lambda z_2, \bar{\lambda} w_1, \bar{\lambda} w_2),$$

then the hyperKähler moment map is given by

$$\mu = -(|z_1|^2 + |z_2|^2 - |w_1|^2 - |w_2|^2)i - 2k(z_1 w_1 + z_2 w_2).$$

In particular, the induced metric on $\mu^{-1}(i)/S^1 \cong T^*\mathbb{C}P^1$ has holonomy $Sp(1) \cong SU(2)$.

Want to study asymptotic properties of the Calabi metric. First consider

$$\left.\begin{array}{c}\mu = 0\\ z \neq 0\end{array}\right\} \Longleftrightarrow \left\{\begin{array}{c}(w_1, w_2) = a(z_2, -z_1)\\ |a| = 1\end{array}\right.$$

Hence, the map $\mathbb{C}^2 \to \mathbb{C}^4$

$$(t_1, t_2) \mapsto (t_1, t_2, t_2, -t_1)$$

induces a diffeomorphism $\mathbb{C}^2/\pm 1 \cong \mu^{-1}(0)/S^1$ (away from the singular pt). It is easy to see that in fact this is an isometry.

Observe also that we have a commutative diagram

$$\begin{array}{ccc}\mu^{-1}(-i) & \hookrightarrow & \mu_c^{-1}(0)\\ \downarrow /S^1 & & \downarrow /\mathbb{C}^*\\ T^*\mathbb{P}^1 & \xrightarrow{\chi} & \mathbb{C}^2/\pm 1\end{array}$$

where the map χ is induced by the inclusion in the top row. Moreover, χ is holomorphic and

$$\chi^{-1}(z) = \begin{cases} pt, & z \neq 0\\ \mathbb{P}^1, & z = 0\end{cases}$$

i.e. χ is a resolution of singularity.

Prop. *Let g denote the Calabi metric on $T^*\mathbb{CP}^1$. Then*

$$\chi^* g = g_{flat} + O(r^{-4}),$$

where r is the radial function on $\mathbb{C}^2/\pm 1$.

A metric with asymptotics as in the Prop. above is called ALE (asymptotically locally Euclidean).

The fact that the leading term is g_{flat} follows from the following observation. Denote by $M_\rho = \mu^{-1}(-i\rho)/S^1$, where $\rho \in \mathbb{R}$. Clearly, M_ρ is diffeomorphic to $T^*\mathbb{C}P^1$ for any ρ. As $\rho \to 0$, the metric g_ρ tends to the flat metric on $M_0 \cong \mathbb{C}^2/\pm 1$ (away from the singularity).

A sketch of the construction of a compact G_2-mfld

Consider $\mathbb{T}^7$ with its flat G_2-structure (g_0, φ_0). The group $\mathbb{Z}_2^3$ acts on $\mathbb{T}^7$ via

$$\alpha(x_1, \dots, x_7) = (x_1, x_2, x_3, -x_4, -x_5, -x_6, -x_7)$$
$$\beta(x_1, \dots, x_7) = (x_1, -x_2, -x_3, x_4, x_5, \frac{1}{2} - x_6, -x_7)$$
$$\gamma(x_1, \dots, x_7) = (-x_1, x_2, -x_3, x_4, \frac{1}{2} - x_5, x_6, \frac{1}{2} - x_7)$$

Lem. *The singular set S of $\mathbb{T}^7/\mathbb{Z}_2^3$ consists of 12 disjoint $\mathbb{T}^3$ with singularities modelled on $\mathbb{T}^3 \times \mathbb{C}^2/\pm 1$.*

Since $T^*\mathbb{P}^1$ is asymptotic to flat $\mathbb{C}^2/\pm 1$, we can cut out a small neihbourhood of each connected component of S and replace it with $\mathbb{T}^3 \times T^*\mathbb{P}^1$. The metric on the resulting mfld, as well as a G_2-structure, is obtained by glueing the flat metric on $\mathbb{T}^7$ to the product (non-flat) metric on $\mathbb{T}^3 \times T^*\mathbb{P}^1$. The 3-form φ is not parallel, but can be chosen so that $d\varphi = 0$ and $d * \varphi$ is small.

Then Joyce proves that such (g, φ) can be deformed into a metric with holonomy G_2.

Examples of compact $Spin(7)$-mflds can be constructed in a similar manner.

Holonomy groups

in Riemannian geometry

Lecture 8

Spin Geometry

December 15, 2011

Clifford algebras

Recall: For $n \geq 3$, $\mathrm{Spin}\,(n)$ is a connected simply connected group fitting into the short exact sequence

$$0 \to \{\pm 1\} \to \mathrm{Spin}\,(n) \to SO(n) \to 0,$$

Aim: Construct spinor groups explicitly.

Let V be a (real) finite dimensional vector space. Denote by TV the tensor algebra of V: $TV = \mathbb{R} \oplus V \;\oplus\; V \otimes V \;\oplus \dots$

Def. Let q be a quadratic form on V. Then the Clifford algebra is defined by

$$Cl(V, q) = TV / \langle v \cdot v + q(v) \rangle.$$

In other words, the algebra $Cl(V, q)$ is generated by elements of V and 1 subject to relations

$$v \cdot v = -q(v) \quad \Longleftrightarrow \quad v \cdot w + w \cdot v = -2q(v, w).$$

Rem. $Cl(V,q)$ is $\mathbb{Z}/2\mathbb{Z}$-graded: $Cl(V,q) = Cl^0(V,q) \oplus Cl^1(V,q)$.

From now on we assume that q is positive definite for the sake of simplicity.

Prop. *There is a (canonical) vector space isomorphism* $\Lambda V \longrightarrow Cl(V,q)$.

Proof. Choose an orthogonal basis $(e_1,\dots,e_n)$ of V. Then $e_i \cdot e_j = -e_j \cdot e_i$ for all i, j. Hence, the map

$$\varphi\colon \Lambda V \longrightarrow Cl(V,q)$$
$$e_{i_1} \wedge \dots \wedge e_{i_k} \mapsto e_{i_1} \dots e_{i_k}$$

is well-defined and surjective. This map is also injective (excercise). □

Cor. $\dim Cl(V,q) = 2^n$, *where* $n = \dim V$.

Rem. ΛV and $Cl(V,q)$ are not isomorphic as algebras (unless $q = 0$).

In fact we have

Prop. *With respect to the isomorphism* $Cl(\mathbb{R}^n, q_{st}) \cong \Lambda(\mathbb{R}^n)^*$, *Clifford multiplication between* $v \in \mathbb{R}^n$ *and* $\varphi \in \Lambda(\mathbb{R}^n)^*$ *can be written as*

$$v \cdot \varphi = q_{st}(v,\cdot) \wedge \varphi - i_v\varphi$$

Proof. [Lawson, Michelsohn. Prop. I.3.9] □

Let x be a unit in $Cl(V,q)$. Define

$$Ad_x : Cl(V,q) \longrightarrow Cl(V,q), \quad Ad_x y = xyx^{-1}$$

Observe that each non-zero $v \in V \hookrightarrow Cl(V,q)$ is a unit:

$$v^{-1} = -\frac{1}{q(v)}v.$$

Prop. *For any non-zero $v \in V$ the map Ad_v preserves V and the following equality holds:*

$$-Ad_v w = w - 2\frac{q(v,w)}{q(v,v)}v$$

(i.e. $-Ad_v$ is the reflection in $v^\perp$).

Proof.

$$Ad_v w = -\frac{1}{q(v,v)} v \cdot w \cdot v = \frac{1}{q(v,v)} v \cdot (v \cdot w + 2q(v,w))$$
$$= -w + 2\frac{q(v,w)}{q(v,v)}v.$$

□

Rem. Ad_v preserves q but not orientation (in general).

Spin groups

Def. $Spin(V,q)$ is the group generated by

$$\{v \cdot w \mid q(v) = 1 = q(w)\} \subset Cl^\times(V,q).$$

It is well-known that the group $O(V,q)$ is generated by reflections (recall the normal form for orthogonal matrices and observe that each rotation of the plane is a product of two reflections). Then $SO(V,q)$ is generated by compositions of even numbers of reflections. In other words, the map

$$Ad\colon Spin(V,q) \longrightarrow SO(V,q)$$

is surjective.

Prop. $\mathrm{Ker}\, Ad \cong \{\pm 1\}$, *i.e. we have the short exact sequence*

$$0 \longrightarrow \{\pm 1\} \longrightarrow Spin(V,q) \longrightarrow SO(V,q) \longrightarrow 0$$

Proof. Denote by $\tilde{\cdot}$ the automorphism of Cl generated by $\tilde{\cdot} : TV \to TV$, $\tilde{v} = -v$. Let

$$\widetilde{Ad}_v w = \tilde{v} \cdot w \cdot v, \quad w \in Cl(V,q).$$

This induces a homomorphism

$$\widetilde{Ad} : Cl^{\times}(V,q) \longrightarrow GL(Cl(V,q)).$$

Choose an ONB $(e_1, \ldots, e_n)$ of V. Suppose $\varphi \in Cl^{\times}(V,q)$ belongs to $\mathrm{Ker}\, \widetilde{Ad} : Cl^{\times} \to GL(V)$, i.e. $\tilde{\varphi} \cdot w = w \cdot \varphi$ for all $w \in V$. Write $\varphi = \varphi_0 + \varphi_1$, where $\varphi_i \in Cl^i(V,q)$. Then

$$(\varphi_0 - \varphi_1)w = w(\varphi_0 + \varphi_1) \quad \Longleftrightarrow \quad \begin{cases} \varphi_0 \cdot w = w \cdot \varphi_0 \\ -\varphi_1 \cdot w = w \cdot \varphi_1 \end{cases} \qquad (1)$$

Proof of $\mathrm{Ker}\, Ad = \{\pm 1\}$ *continued*

Further, write $\varphi_0 = \psi_0 + e_1\psi_1$, where ψ_0, ψ_1 are expressions in $e_2, \ldots, e_n$ only. We have

$$\begin{aligned} e_1(\psi_0 + e_1\psi_1) &= (\psi_0 + e_1\psi_1)e_1 \qquad &&(\text{by } (1) \text{ with } w = e_1) \\ &= \psi_0 e_1 + e_1\psi_1 e_1 \\ &= e_1\psi_0 - e_1^2\psi_1 &&(\text{since } \psi_i \in Cl^i) \end{aligned}$$

Hence, $\psi_1 = 0 \;\Rightarrow\; \varphi_0$ does not involve $e_1 \;\Rightarrow \varphi_0 = \lambda \cdot 1$.

A similar argument shows that φ_1 does not involve any $e_j \;\Rightarrow\; \varphi_1 = 0$.

Thus, $\mathrm{Ker}\big(\widetilde{Ad} : Cl^{\times} \to GL(V)\big) \cong \mathbb{R}^*$. Therefore, $\mathrm{Ker}\big(\widetilde{Ad} : Spin(V,q) \to SO(V)\big) \cong \{\pm 1\}$. Finally, $\widetilde{Ad} = Ad$ on $Spin(V,q)$. □

Prop. *$Spin(n) := Spin(\mathbb{R}^n, q_{st})$ is a nontrivial double covering of $SO(n)$.*

Proof. It suffices to show that 1 and -1 can be joined by a path in $Spin(n)$. The path

$$\begin{aligned}\gamma(t) &= (e_1 \cos t + e_2 \sin t)(e_2 \sin t - e_1 \cos t) = \\ &= \cos 2t \cdot 1 + \sin 2t \cdot e_1 e_2\end{aligned}$$

does the job. □

Cor. *$Spin(n)$ is connected and simply connected provided $n \geq 3$.*

Proof. Follows from the facts that $SO(n)$ is connected and $\pi_1(SO(n)) \cong \{\pm 1\}$. □

Ex. ("accidental isomorphisms in low dimensions")

1) $Spin(2) := U(1) \cong S^1$
2) $Spin(3) \cong Sp(1) \cong SU(2)$
3) $Spin(4) \cong Sp(1) \times Sp(1)$
4) $Spin(5) \cong Sp(2)$
   To see this, consider the action of $Sp(2)$ on $M_2(\mathbb{H})$ by conjugation. Then $\mathbb{R}^5$ can be identified with the subspace of traceless, quaternion-Hermitian matrices. Hence, $Sp(2)/\pm 1 \cong SO(5)$.
5) $Spin(6) \cong SU(4)$

Some facts from representation theory of Clifford algebras and Spin groups

Theorem

Let ν_n and $\nu_n^{\mathbb{C}}$ denote the number of inequivalent irreducible real and complex representations of $Cl_n := Cl(\mathbb{R}^n, q_{st})$ and $Cl_n \otimes \mathbb{C}$ respectively. Then

$$\nu_n = \begin{cases} 2 & n \equiv 1 (\operatorname{mod} 4), \\ 1 & \text{otherwise} \end{cases} \quad \text{and} \quad \nu_n^{\mathbb{C}} = \begin{cases} 2 & n \text{ is odd}, \\ 1 & n \text{ is even}. \end{cases}$$

Proof. [Lawson, Michelsohn. Thm I.5.7]. □

Def. The real (complex) spinor representation of $Spin(n)$ is the homomorphism

$$\Delta_n : Spin(n) \to \operatorname{End}_{\mathbb{R}}(S), \qquad \text{if real}$$
$$\Delta_n^{\mathbb{C}} : Spin(n) \to \operatorname{End}_{\mathbb{C}}(S), \qquad \text{if complex}$$

given by restricting an irreducible real (complex) representation of Cl_n ($Cl_n \otimes \mathbb{C}$) to $Spin(n)$.

Theorem

Let W be a real Cl_n-representation. Then there exists a scalar product on W s.t. $\langle v \cdot w, v \cdot w' \rangle = \langle w, w' \rangle$ $\forall v \in V$ s.t. $\|v\| = 1$.

Cor. $\langle v \cdot w, w' \rangle = -\langle w, v \cdot w' \rangle$.

Spin structures

Let $P \to M$ be a principal $SO(n)$-bundle, $n \geq 3$.

Def. The Spin-structure on P (equivalently, on $E = P \times_{SO(n)} \mathbb{R}^n$) is a principal $Spin(n)$-bundle $\widetilde{P} \to M$ together with a $Spin(n)$-equivariant map $\xi : \widetilde{P} \to P$, which is (fiberwise) a 2-sheeted covering.

Thus, we have a commutative diagram

$$\begin{array}{ccc} \widetilde{P} & \xrightarrow{\ \xi\ } & P \\ {\scriptstyle \tilde{\pi}}\downarrow & & \downarrow{\scriptstyle \pi} \\ M & = & M \end{array}$$

From the short exact sequence

$$1 \to \{\pm 1\} \to Spin(n) \to SO(n) \to 1$$

we obtain

$$H^0(M; SO(n)) \to H^1(M; \mathbb{Z}_2) \to H^1(M; Spin(n)) \to$$
$$\to H^1(M; SO(n)) \xrightarrow{\delta} H^2(M; \mathbb{Z}_2).$$

Then $\delta[P]$ equals the second Stiefel-Whitney class, $w_2(P)$. Hence, P admits a spin structure iff $w_2(P) = 0$. If this is the case, all spin structures are classified by $H^1(M, \mathbb{Z}_2)$ (assuming M is connected).

Def. A spin mfld is an oriented Riemannian mfld with a spin structure on its tangent bundle.

Rem. Thus, M admits a spin structure iff $w_2(M) = 0$. This is a topological condition on M, not on the Riemannian metric.

Rem. Since $\xi : \widetilde{P} \to P$ is a covering, $\xi^* \varphi_{LC}$ is a (distinguished) connection on $\widetilde{P}$.

For the spinor representation $\Delta\colon Spin(n) \to End(S)$ the associated spinor bundle

$$S := \widetilde{P} \times_{Spin(n)} S$$

is equipped with a connection and Euclidean scalar product.

Rem. For any $m \in M$, the fibre S_m is a module over $Cl(T_m M)$.

Denote by $R^S \in \Omega^2(M; End(S))$ the induced curvature form.

Prop. *Let $e = (e_1, \dots, e_n)$ be a local section of $P = P_{SO}$. Then*

$$R^S(v,w)\sigma = \sum_{i,j} \langle R(v,w)e_i, e_j\rangle e_i e_j \cdot \sigma. \qquad (2)$$

Proof. [Lawson, Michelson. Thm I.4.15] □

Parallel spinors and holonomy groups

Theorem

Assume M admits a nontrivial parallel spinor. Then M is Ricci-flat.

Proof. Assume $\psi \in \Gamma(S)$ is parallel. Then $d^\nabla(\nabla\psi) = d^\nabla \cdot d^\nabla \psi = 0 \iff R^S(v,w)\cdot\psi = 0$ for any $v, w \in \mathfrak{X}(M)$. With the help of (2) with $v = e_k$ we obtain

$$\begin{aligned}
0 =& \sum_{i,j,k} \langle R(e_k,w)e_i, e_j\rangle e_k e_i e_j \cdot \psi = \sum_{i,j,k} \langle R(e_i,e_j)e_k, w\rangle e_i e_j e_k \cdot \psi \\
=& \frac{1}{3} \sum_{i\neq j\neq k\neq i} \langle R(e_i,e_j)e_k + R(e_j,e_k)e_i + R(e_k,e_i)e_j, w\rangle e_i e_j e_k \cdot \psi \\
&+ \sum_{i,j} \langle R(e_i,e_j)e_i, w\rangle e_i e_j e_i \cdot \psi + \sum_{i,j} \langle R(e_i,e_j)e_j, w\rangle e_i e_j e_j \cdot \psi \\
=& 0 + \sum_{i,j,} \langle R(e_i,w)e_i, e_j\rangle e_j \cdot \psi - \sum_{i,j,} \langle R(e_j,w)e_i, e_j\rangle e_i \cdot \psi \\
=& 2Ric(w)\cdot\psi.
\end{aligned}$$

Proof of $\nabla\psi = 0,\ \psi \neq 0 \ \Rightarrow\ Ric = 0$ continued

Here Ric is viewed as a linear map $TM \to TM$, namely $Ric(w) = \sum_{j=1}^{n} R(e_j, w)e_j$. Hence

$$Ric(w)\cdot\psi = 0 \quad \Longrightarrow \quad Ric(w)^2\cdot\psi = -\|Ric(w)\|^2\psi = 0.$$

Hence, $Ric(w) = 0$ for all w. □

Clearly, if M admits a parallel spinor then M must have a non-generic holonomy. Only metrics with the following holonomies

$$SU(\tfrac{n}{2}),\ Sp(\tfrac{n}{4}),\ G_2,\ Spin(7) \qquad (3)$$

are Ricci-flat.

Theorem

Let M be a complete, simply-connected, and irreducible Riemannian spin mfld. Then M admits a not-trivial parallel spinor iff $\mathrm{Hol}(M)$ is one of the four groups listed in (3).

Dirac bundles

Let $P \to M$ be the principal $SO(n)$-bundle of orthonormal oriented frames. Then $Cl(M) := P \times_{SO(n)} Cl(\mathbb{R}^n)$ is called the Clifford bundle of M. Notice: $Cl_m(M) = Cl(T_mM)$.

Def. A *Dirac bundle* is a bundle S of left modules over $Cl(M)$ equipped with an Euclidean scalar product and a connection s.t. the following holds:

$$\langle v\cdot\sigma_1, v\cdot\sigma_2\rangle = \|v\|^2\langle\sigma_1,\sigma_2\rangle$$
$$\nabla(\varphi\cdot\sigma) = (\nabla^{LC}\varphi)\cdot\sigma + \varphi\cdot(\nabla\sigma).$$

Here σ, $\sigma_i \in \Gamma(S)$, $v \in \mathfrak{X}(M)$, and $\varphi \in \Gamma(Cl(M))$.

- Spinor bundle S is a Dirac bundle [See LM. II.4 for details].
- $\Lambda T^*M \cong Cl(M)$ is a Dirac bundle (with the Levi-Civita connection). Hence, the existence of Dirac bundles does not require M to be spin.

Dirac operators

Let S be a Dirac bundle.

Def. The map

$$D\colon \Gamma(S) \xrightarrow{\nabla} \Gamma(T^*M \otimes S) \xrightarrow{Cl} \Gamma(S)$$

is called the *Dirac operator*.

In terms of a local frame $(e_1, \ldots, e_n)$ of TM the Dirac operator is given by

$$D\sigma = \sum_{i=1}^{n} e_i \cdot (\nabla_{e_i}\sigma).$$

Prop. *D is elliptic and formally self-adjoint operator (wrt the L_2-scalar product).*

Proof. Ellipticity: $\sigma_\xi(D) = i\xi\cdot\colon S \to S$ is clearly invertible for any $\xi \neq 0$.

To prove that D is formally self-adjoint, choose a local orthonormal basis $e = (e_1, \ldots, e_n)$ of TM s.t. $(\nabla e_i)_m = 0$ for all i. Then

$$\begin{aligned}\langle D\sigma_1, \sigma_2\rangle_m &= \sum_j \langle e_j \cdot \nabla_{e_j}\sigma_1, \sigma_2\rangle_m = \\ &= -\sum_j \langle \nabla_{e_j}\sigma_1, e_j\cdot\sigma_2\rangle_m = \\ &= -\sum_j \big(e_j\cdot\langle\sigma_1, e_j\cdot\sigma_2\rangle - \langle\sigma_1, e_j\cdot\nabla_{e_j}\sigma_2\rangle\big)_m.\end{aligned}$$

Proof continued

Further, define $v \in \mathfrak{X}(M)$ by the condition

$$\langle v, w\rangle = -\langle \sigma_1, w \cdot \sigma_2\rangle \quad \text{for all} \quad w \in \mathfrak{X}(M).$$

Then

$$\begin{aligned}\mathrm{div}_m(v) &= \sum_j \langle \nabla_{e_j} v, e_j\rangle_m \\ &= \sum_j (e_j \cdot \langle v, e_j\rangle)_m \\ &= -\sum_j (e_j \cdot \langle \sigma_1, e_j \cdot \sigma_2\rangle)_m\end{aligned}$$

Hence, $\langle D\sigma_1, \sigma_2\rangle = \mathrm{div}(v) + \langle \sigma_1, D\sigma_2\rangle$ pointwise. Hence, D is formally self-adjoint. □

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Examples of Dirac operators

1) $M = \mathbb{R}^2$. Then $Cl(\mathbb{R}^2)$ has a basis $(1, e_1, e_2, e_1 \cdot e_2)$. Then we have the isomorphism of vector spaces

$$Cl(\mathbb{R}^2) = Cl^0(\mathbb{R}^2) \oplus Cl^1(\mathbb{R}^2) \cong \mathbb{C} \oplus \mathbb{C}.$$

Notice that the Clifford multiplication by $v \in \mathbb{R}^2$ is an antidiagonal operator. Then

$$D = \begin{pmatrix} 0 & -\frac{\partial}{\partial z} \\ \frac{\partial}{\partial \bar{z}} & 0 \end{pmatrix}.$$

2) Similarly, for $M = \mathbb{R}^4$ one obtains

$$D = \begin{pmatrix} 0 & -\frac{\partial}{\partial q} \\ \frac{\partial}{\partial \bar{q}} & 0 \end{pmatrix},$$

where $\frac{\partial}{\partial \bar{q}} : C^\infty(\mathbb{R}^4; \mathbb{H}) \to C^\infty(\mathbb{R}^4; \mathbb{H})$,
$\frac{\partial f}{\partial \bar{q}} = \frac{\partial f}{\partial x_0} + i\frac{\partial f}{\partial x_1} + j\frac{\partial f}{\partial x_2} + k\frac{\partial f}{\partial x_3}$ is the Fueter operator.

Examples of Dirac operators: continued

3) M is a Riemannian mfld, $S = Cl(M)$. Then

$$D = d + d^* : \quad \Omega(M) \to \Omega(M).$$

This follows from the following two observations:

$$a) \quad v \cdot \varphi = q_{st}(v, \cdot) \wedge \varphi - i_v \varphi \quad \text{if} \quad v \in \mathbb{R}^n, \quad \varphi \in \Lambda(\mathbb{R}^n)^*$$

$$b) \quad d = \sum_j e_j^* \wedge \nabla_{e_j}, \qquad d^* = -\sum_j \imath_{e_j} \nabla_{e_j}$$

This is just a restatement of the facts that the sequences

$$\Gamma(\Lambda^k T^* M) \xrightarrow{\nabla^{LC}} \Gamma(T^*M \otimes \Lambda^k T^* M) \xrightarrow{Alt} \Gamma(\Lambda^{k+1} T^* M)$$

$$\Gamma(\Lambda^k T^* M) \xrightarrow{\nabla^{LC}} \Gamma(T^*M \otimes \Lambda^k T^* M) \xrightarrow{-contr.} \Gamma(\Lambda^{k-1} T^* M)$$

represent d and d^* respectively. Details concerning d^* can be found in [LM. Lemma II.5.13].

Weitzenböck formulae and Bochner technique

Assume M is a compact Riemannian mfld. Let $E \to M$ be an Euclidean vector bundle equipped with a connection ∇. Define

$$\nabla^2_{v,w} s = \nabla_v(\nabla_w s) - \nabla_{\nabla_v w} s,$$

where $s \in \Gamma(E)$, v, $w \in \mathfrak{X}(M)$. Notice that

$$\nabla^2_{v,w} - \nabla^2_{w,v} = R(v, w).$$

Hence, $\quad \nabla^2_{\cdot,\cdot} \in \Gamma(T^*M \otimes T^*M \otimes S)$.

Def. The map

$$\nabla^*\nabla \colon \Gamma(S) \xrightarrow{\nabla^2} \Gamma(T^*M \otimes T^*M \otimes S) \xrightarrow{-tr} \Gamma(S)$$

is called the *connection Laplacian*.

In terms of local orthonormal frames we have

$$\nabla^*\nabla s = -\sum_j \nabla^2_{e_j, e_j} s.$$

Prop. *The operator $\nabla^*\nabla$ is formally self-adjoint and satisfies*

$$\langle \nabla^*\nabla s_1, s_2\rangle_{L_2} = \langle \nabla s_1, \nabla s_2\rangle_{L_2}.$$

In particular, $\nabla^*\nabla$ is non-negative.

Proof. Similar to the proof of the fact that D is formally self-adjoint. For details see [LM. Prop. II.2.1.]. □

Let S be a Dirac bundle. If $R \in \Omega^2(M; \mathrm{End}(S))$ is the curvature form, define $\mathcal{R} \in \Gamma(\mathrm{End}(S))$ by

$$\mathcal{R}(s) = \tfrac{1}{2}\sum_{j,k} e_j e_k \cdot R(e_j, e_k)(s).$$

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Theorem (general Bochner identity)

$$\boxed{D^2 = \nabla^*\nabla + \mathcal{R}}$$

Proof. Choose a local frame $(e_1, \ldots, e_n)$ of TM s.t. $(\nabla e_j)_m = 0$. Then

$$\begin{aligned} D^2 &= \sum_{j,k} e_j \cdot \nabla_{e_j}(e_k \cdot \nabla_{e_k} \cdot) \\ &= \sum_{j,k} e_j e_k \cdot \nabla_{e_j}(\nabla_{e_k} \cdot) \\ &= \sum_{j,k} e_j e_k \cdot \nabla^2_{e_j, e_k} \\ &= -\sum_j \nabla^2_{e_j, e_j} + \sum_{j<k} e_j e_k \cdot (\nabla^2_{e_j, e_k} - \nabla^2_{e_k, e_j}) \\ &= \nabla^*\nabla + \mathcal{R}. \end{aligned}$$

□

Cor. *Let $\Delta = dd^* + d^*d$ be the Hodge Laplacian and $\nabla^*\nabla$ be the connection Laplacian on T^*M. Then*

$$\Delta = \nabla^*\nabla + Ric$$

This follows from the previous thm for $D = d + d^*$, which acts on $Cl(M) \cong \Lambda T^*M$. The computation of $\mathcal{R}$ in this case follows the same lines as the proof of the implication

$$\nabla\psi = 0 \quad \Longrightarrow \quad Ric(w) \cdot \psi = 0.$$

[LM. Cor. II.8.3].

Theorem (Bochner)

$$Ric > 0 \quad \Longrightarrow \quad b_1(M) = 0.$$

Theorem (Lichnerowicz)

Let M be spin and suppose S is a spinor bundle. Then

$$D^2 = \nabla^*\nabla + \tfrac{s}{4}\,,$$

where s is the scalar curvature.

Proof. [LM. Thm. II.8.8]. □

Cor.

$$s > 0 \quad \Longrightarrow \quad \operatorname{Ker} D = 0.$$

Theorem (Hitchin)

In every dimension $n > 8$, $n \equiv 1 (\operatorname{mod} 8)$ or $n \equiv 2 (\operatorname{mod} 8)$, there exist compact mflds, which are homeomorpic to S^n, but which do not admit any Riemannian metric with $s > 0$.