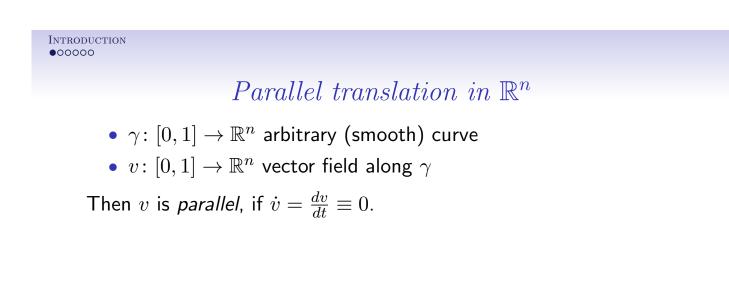
Holonomy groups in Riemannian geometry

## Lecture 1

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October 14, 2011



### Parallel translation on curved spaces

- $S \subset \mathbb{R}^n$  oriented hypersurface (e.g.  $S^2 \subset \mathbb{R}^3$ )
- n unit normal vector along S
- $\gamma \colon [0,1] \to S$  curve
- $v \colon [0,1] \to \mathbb{R}^n$  vector field along  $\gamma$  s.t.

$$v(t) \in T_{\gamma(t)}S \quad \Leftrightarrow \quad \left\langle v(t), n(\gamma(t)) \right\rangle = 0 \qquad \forall t$$
 (1)

(1)  $\Rightarrow v$  can not be constant in t. The eqn  $\dot{v} = 0$  is replaced by

$$\operatorname{proj}_{TS} \dot{v} = 0 \quad \Leftrightarrow \quad \dot{v} - \langle \dot{v}, n(\gamma) \rangle n(\gamma) = 0.$$

Differentiating (1) we obtain a first order ODE for *parallel* v:

$$\dot{v} + \langle v, \frac{d}{dt}n(\gamma)\rangle n(\gamma) = 0$$

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# Parallel transport

Parallel transport is a linear isomorphism

$$P_{\gamma} \colon T_{\gamma(0)}S \to T_{\gamma(1)}S, \qquad v_0 \mapsto v(1)$$

where  $\boldsymbol{v}$  is the solution of the problem

$$\dot{v} + \langle v, \frac{d}{dt}n(\gamma)\rangle n(\gamma) = 0, \qquad v(0) = v_0.$$

 $P_{\gamma}$  is an isometry, since

$$v, w$$
 are parallel  $\Rightarrow \langle v(t), w(t) \rangle$  is constant in t

#### Holonomy group

- $s \in S$  basepoint
- $Hol_s := \{P_\gamma \mid \gamma(0) = s = \gamma(1)\} \subset SO(T_sS)$  based holonomy group
- $Hol_{s'}$  is conjugated to  $Hol_s$  ("Holonomy group does not depend on the choice of the basepoint")
- Holonomy group is intrinsic to S, i.e. depends on the Riemannian metric on S but not on the embedding  $S\subset \mathbb{R}^n$
- Ex:  $Hol(S^2) = SO(2)$

Properties:

- $\diamond\,$  definition generalises to any Riemannian manifold (M,g)
- o encodes both local and global features of the metric
- ◊ "knows" about additional structures compatible with metric

RODUCTION OO⊕O		
Classifie	cation of holonor	ny groups
Berger's list, 1955		
Assume $M$ is a simple	oly–connected irreduci	ible nonsymmetric Rie-
mannian mfld of dime	nsion $n$ . Then $Hol(M$	) is one of the following:
Holonomy	Geometry	Extra structure
$\bullet$ SO(n)		
• $U(n/2)$	Kähler	complex
• $SU(n/2)$	Calabi–Yau	complex + hol. vol.
• $Sp(n/4)$	hyperKähler	quaternionic
• $Sp(1)Sp(n/4)$	quaternionic Kähler	"twisted" quaternionic
• $G_2$ (n=7)	exceptional	"octonionic"
	exceptional	"octonionic"

# Plan

- General theory (torsion, Levi–Civita connection, Riemannian curvature, holonomy)
- Proof of Berger's theorem (Olmos 2005)
- Properties of manifolds with non-generic holonomies (some constructions, examples, curvature tensors...)