Holonomy groups in Riemannian geometry

Lecture 2

October 27, 2011



Smooth manifold comes equipped with a collection of charts $(U_{\alpha}, \varphi_{\alpha})$, where $\{U_{\alpha}\}$ is an open covering and the maps $\varphi_{\beta} \circ \varphi_{\alpha}^{-1} \colon \mathbb{R}^n \to \mathbb{R}^n$ are smooth.

A Lie group G is a group which has a structure of a smooth mfld such that the structure maps, i.e. $m: G \times G \to G, \ \cdot^{-1}: G \to G$, are smooth.

 $\mathfrak{g} := T_e G$ is a *Lie algebra*, i.e. a vector space endowed with a map $[\cdot, \cdot] \colon \Lambda^2 \mathfrak{g} \to \mathfrak{g}$ satisfying the Jacobi identity:

$$\left[\xi, \left[\eta, \zeta\right]\right] + \left[\eta, \left[\zeta, \xi\right]\right] + \left[\zeta, \left[\xi, \eta\right]\right] = 0.$$

Ex.	G	$GL_n(\mathbb{R})$	$GL_n(\mathbb{C})$	SO(n)	U(n)	
LX.	g	$\operatorname{End} \mathbb{R}^n$	$\operatorname{End} \mathbb{C}^n$	$\{A^t = -A\}$	$\{\bar{A}^t = -A\}$	

Identification: $\mathfrak{g} \cong \{$ left-invariant vector fields on $G \}$

- ξ_1, \ldots, ξ_n a basis of \mathfrak{g}
- $\omega_1, \ldots, \omega_n$ dual basis

 $\omega := \sum \omega_i \otimes \xi_i \in \Omega^1(G; \mathfrak{g})$ canonical 1-form with values in \mathfrak{g} , which satisfies the Maurer–Cartan equation

$$d\omega + \frac{1}{2}[\omega \wedge \omega] = \sum_{i} d\omega_{i} \otimes \xi_{i} + \frac{1}{2} \sum_{i,j} \omega_{i} \wedge \omega_{j} \otimes [\xi_{i}, \xi_{j}] = 0.$$

LIE GROUPS	VECTOR BUNDLES	Principal bundles	Connections on G -bundles	Holonomy	TORSION

Vector bundles

A vector bundle E over M satisfies:

- E is a manifold endowed with a submersion $\pi \colon E \to M$
- $\forall m \in M \ E_m := \pi^{-1}(m)$ has the structure of a vector space
- $\forall m \in M \quad \exists U \ni m \text{ s.t. } \pi^{-1}(U) \cong U \times E_m$

 $\Gamma(E) = \{s \colon M \to E \mid \pi \circ s = id_M\}$ space of sections of E

Ex.

<i>E</i>	$\Gamma(E)$	
TM		vector fields
$\Lambda^k T^*M$	$\Omega^k(M)$	differential k-forms
$T^p_q(M) := \bigotimes^p TM \otimes \bigotimes^q T^*M$?	tensors of type $\left(p,q\right)$

de Rham complex

Exterior derivative $d: \Omega^k \to \Omega^{k+1}$ is the unique map with the properties:

- df is the differential of f for $f \in \Omega^0(M) = C^\infty(M)$
- $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$, if $\alpha \in \Omega^p$

•
$$d^2 = 0$$

Thus, we have the de Rham complex:

$$0 \to \Omega^0 \to \Omega^1 \to \dots \to \Omega^n \to 0, \qquad n = \dim M.$$

Betti numbers:

$$b_k = \dim H^k(M; \mathbb{R}) = \dim \frac{\operatorname{Ker} d \colon \Omega^k \to \Omega^{k+1}}{\operatorname{im} d \colon \Omega^{k-1} \to \Omega^k}$$

LIE GROUPS	VECTOR BUNDLES	Principal bundles	Connections on G -bundles	Holonomy	Torsion

Lie bracket of vector fields

A vector field can be viewed as an \mathbb{R} -linear derivation of the algebra $C^{\infty}(M)$. Then $\mathfrak{X}(M)$ is a Lie algebra:

$$[v,w] \cdot f = v \cdot (w \cdot f) - w \cdot (v \cdot f).$$

The exterior derivative and the Lie bracket are related by

$$2d\omega(v,w) = v \cdot \omega(w) - w \cdot \omega(v) - \omega([v,w])$$

Rem. "2" is optional in the above formula.

Lie derivative

For $v \in \mathfrak{X}(M)$ let φ_t be the corresponding 1-parameter (semi)group of diffeomorphisms of M, i.e.

$$\frac{d}{dt}\varphi_t(m) = v(\varphi_t(m)), \qquad \varphi_0 = id_M.$$

The *Lie derivative* of a tensor S is defined by

$$\mathcal{L}_v S = \frac{d}{dt} \Big|_{t=0} \varphi_t^* S$$

In particular, this means:

$$\mathcal{L}_{v}f(m) = \frac{d}{dt}\Big|_{t=0} f(\varphi_{t}(m)) = df_{m}(v(m)), \quad \text{if } f \in C^{\infty}(M),$$
$$\mathcal{L}_{v}w(m) = \frac{d}{dt}\Big|_{t=0} (d\varphi_{t})_{m}^{-1}w(\varphi_{t}(m)), \quad \text{if } w \in \mathfrak{X}(M)$$

LIE GROUPS	VECTOR BUNDLES	Principal bundles	Connections on G -bundles	Holonomy	Torsion
	Prop	perties of th	ne Lie derivative		
•	$\mathcal{L}_v(S\otimes T) =$	$(\mathcal{L}_v S) \otimes T + S$	$S\otimes (\mathcal{L}_v T)$		
•	$\mathcal{L}_v w = [v, w]$	for $w \in \mathfrak{X}(M)$			
•	$[\mathcal{L}_v, \mathcal{L}_w] = \mathcal{L}_{ }$	[v,w]			
•	Cartan formu	la			
	$\mathcal{L}_v\omega$	$= \imath_v d\omega + d(\imath_v \omega)$	$\omega)$ where $\omega \in \Omega($	M).	
•	$[\mathcal{L}_v,d]=0$ or	ו $\Omega(M)$			

Connections on vector bundles

Def. A connection on E is a linear map $\nabla \colon \Gamma(E) \to \Gamma(T^*M \otimes E)$ satisfying the Leibnitz rule:

$$\nabla(fs) = df \otimes s + f \nabla s, \qquad \forall f \in C^{\infty}(M) \quad \text{and} \ \forall s \in \Gamma(E)$$

For $v \in \mathfrak{X}(M)$ we write

 $\nabla_{\!v} s = v \cdot \nabla s$, where " \cdot " is a contraction.

Then

$$\nabla_{\!\alpha v}(\beta s) = \alpha \nabla_{\!v}(\beta s) = \alpha (v \cdot \beta) \nabla_{\!v} s + \alpha \beta \nabla_{\!v} s.$$



Prop. For
$$v, w \in \mathfrak{X}(M)$$
 and $s \in \Gamma(E)$ the expression

$$abla_{\!v}(
abla_{\!w}s) -
abla_{\!w}(
abla_{\!v}s) -
abla_{\![v,w]}s$$

is $C^{\infty}(M)$ -linear in v, w, and s.

Def. The unique section $R = R(\nabla)$ of $\Lambda^2 T^* M \otimes \operatorname{End}(E)$ satisfying

$$R(
abla)(v \wedge w \otimes s) =
abla_v(
abla_w s) -
abla_w(
abla_v s) -
abla_{[v,w]}s$$

is called the *curvature* of the connection ∇ .

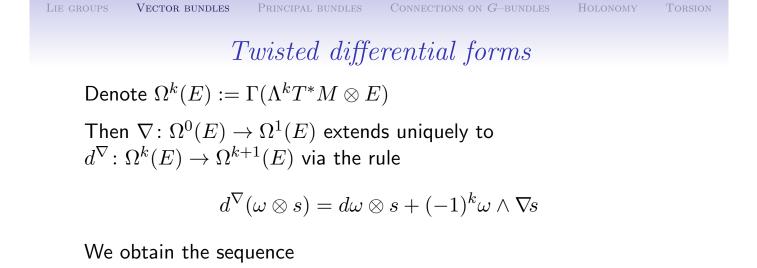
Choose local coordinates (x_1, \ldots, x_n) on M

$$v_i := \frac{\partial}{\partial x_i} \qquad \Rightarrow \quad [v_i, v_j] = 0$$

Then
$$R(v_i, v_j)s = \nabla_{\!\! v}(\nabla_{\!\! w}s) - \nabla_{\!\! w}(\nabla_{\!\! v}s)$$

Think of $\nabla_{\!\! v_i} s$ as "partial derivative" of s

Curvature measures how much "partial derivatives" of sections of E fail to commute.



$$\Omega^{0}(E) \xrightarrow{\nabla = d^{\nabla}} \Omega^{1}(E) \xrightarrow{d^{\nabla}} \Omega^{2}(E) \xrightarrow{d^{\nabla}} \dots \xrightarrow{d^{\nabla}} \Omega^{n}(E) \quad (1)$$

Then
$$\boxed{\left(d^{\nabla} \circ d^{\nabla}\right)\sigma = R(\nabla) \cdot \sigma}$$

Curvature measures the extend to which sequence (1) fails to be a complex.

Principal bundles

Let G be a Lie group

A principal bundle P over M satisfies:

- P is a manifold endowed with a submersion $\pi: P \to M$
- G acts on P on the right and $\pi(p \cdot g) = \pi(p)$
- $\forall m \in M$ the group G acts freely and transitively on $P_m := \pi^{-1}(m)$. Hence $P_m \cong G$
- Local triviality: $\forall m \in M \quad \exists U \ni m \text{ s.t. } \pi^{-1}(U) \cong U \times G$

LIE GROUPS	Vector bundles	Principal bundles	Connections on G -bundles	Holonomy	Torsion

Example: Frame bundle

Let $E \to M$ be a vector bundle. A *frame* at a point m is a linear isomorphism $p \colon \mathbb{R}^k \to E_m$.

$$Fr(E) := \bigcup_{m,p} \{(m,p) \mid p \text{ is a frame at } m \}$$

(i) $GL(k;\mathbb{R}) = Aut(\mathbb{R}^k)$ acts freely and transitively on $Fr_m(E)$:

$$p \cdot g = p \circ g.$$

(ii) A moving frame on $U \subset M$ is a set $\{s_1, \ldots, s_k\}$ of pointwise linearly independent sections of E over U. This gives rise to a section s of Fr(E) over U:

$$s(m)x = \sum x_i s_i(m), \qquad x \in \mathbb{R}^k.$$

By (i) this defines a trivialization of Fr(E) over U.

Frame bundle: variations

If in addition E is

- oriented, i.e. Λ^{top}E is trivial, Fr⁺(E) is a principal GL⁺(k; ℝ)−bundle
- Euclidean Fr_O is a principal O(k)-bundle
- Hermitian Fr_U is a principal U(k)-bundle
- quaternion–Hermitian is a principal Sp(k)–bundle
-

Def. Let G be a subgroup of $GL(n; \mathbb{R})$, $n = \dim M$. A G-structure on M is a principal G-subbundle of $Fr_M = Fr(TM)$.

- orientation $\Leftrightarrow GL^+(n; \mathbb{R})$ -structure
- Riemannian metric $\Leftrightarrow O(n)$ -structure
-

LIE GROUPS	VECTOR BUNDLES	Principal bundles	Connections on G -bundles	Holonomy	TORSION

Associated bundle

 $P \rightarrow M$ principal G-bundle

V G-representation, i.e. a homomorphism $\rho: G \to GL(V)$ is given

$$P \times_G V := (P \times V)/G,$$
 action: $(p, v) \cdot g = (pg, \rho(g^{-1})v)$

is called the bundle associated to P with fibre V.

- **Ex.** For $P = Fr_M$, $G = GL(n; \mathbb{R})$, and $E = P \times_G V$ we have
 - E = TM for $V = \mathbb{R}^n$ (tautological representation)
 - $E = T^*M$ for $V = (\mathbb{R}^n)^*$
 - $E = \Lambda^k T^* M$ for $V = \Lambda^k (\mathbb{R}^n)^*$

Sections of associated bundles correspond to equivariant maps:

$$\{f \colon P \to V \mid f(pg) = \rho(g^{-1})f(p)\} \equiv \Gamma(E)$$

$$f \mapsto s_f, \qquad s_f(m) = [p, f(p)], \quad p \in P_m$$

Connection as horizontal distribution

For $\xi \in \mathfrak{g}$ the Killing vector at $p \in P$ is given by

$$K_{\xi}(p) := \frac{d}{dt} \Big|_{t=0} \left(p \cdot \exp t\xi \right)$$

 $\mathcal{V}_p = \left\{ K_{\xi}(p) \mid \xi \in \mathfrak{g} \right\} \cong \mathfrak{g} \text{ is called } vertical space at } p$

Def. A connection on P is a subbundle \mathcal{H} of TP satisfying

(i) \mathcal{H} is G-invariant, i.e. $\mathcal{H}_{pg} = (R_g)_* \mathcal{H}_p$

(*ii*)
$$TP = \mathcal{V} \oplus \mathcal{H}$$

 \mathcal{H} is called a *horizontal* bundle.



$$T_p P \to \mathcal{V}_p \cong \mathfrak{g}$$

 ω is called the *connection form* and satisfies:

 $\begin{array}{l} (a) \ \ \omega(K_{\xi}) = \xi \\ (b) \ \ R_g^* \omega = a d_{g^{-1}} \, \omega \text{, where } ad \text{ denotes the adjoint representation} \end{array}$

Prop. Every $\omega \in \Omega^1(P; \mathfrak{g})$ satisfying (a) and (b) defines a connection via

 $\mathcal{H} = \operatorname{Ker} \omega.$

TORSION

Horizontal lift

 $\operatorname{Ker}(\pi_*)_p = \mathcal{V}_p$. Hence $(\pi_*)_p \colon \mathcal{H}_p \to T_{\pi(p)}M$ is an isomorphism. In particular, $\mathcal{H} \cong \pi^*TM$. Hence, we have

Prop. For any $w \in \mathfrak{X}(M)$ there exists $\tilde{w} \in \mathfrak{X}(P)$ s.t. (i) \tilde{w} is G-invariant and horizontal (ii) $(\pi_*)_p \tilde{w} = w(\pi(p))$ Vice versa, if $\tilde{w} \in \mathfrak{X}(P)$ is G-invariant and horizontal, then $\exists ! w \in \mathfrak{X}(M)$ s.t. $\pi_* \tilde{w} = w$.



Invariant and equivariant forms

 $\tilde{\alpha} \in \Omega^k(P)$ is called *basic* if $\imath_v \tilde{\alpha} = 0$ for any vertical vector field v.

Then $\forall \alpha \in \Omega^k(M)$ the form $\tilde{\alpha} = \pi^* \alpha$ is *G*-invariant and basic. On the other hand, any *G*-invariant and basic *k*-form $\tilde{\alpha}$ on *P* induces a *k*-form on *M*. **Notice:** no connection required here.

V is a representation of G $\tilde{\alpha} \in \Omega^k(P; V)$ is G-equivariant if $R_q^* \tilde{\alpha} = \rho(g^{-1}) \tilde{\alpha}$.

Ex. Connection 1-form is an equivariant form for $V = \mathfrak{g}$.

For basic and equivariant forms we have the identification

$$\Omega^k_{G,bas}(P,V) \cong \Omega^k(M;E), \qquad \pi^* \alpha \leftrightarrow \alpha$$

PRINCIPAL BUNDLES

Curvature tensor

Prop. Let ω be a connection form. The 2-form $\tilde{F}_{\omega} = d\omega + \frac{1}{2}[\omega \wedge \omega]$ is basic and *G*-equivariant, i.e. $R_g^* \tilde{F} = ad_{g^{-1}} \tilde{F}$.

Cor. Denote $ad P := P \times_{G,ad} \mathfrak{g}$. Then there exists $F \in \Omega^2(M; ad P)$ s.t. $\pi^*F = \tilde{F}$.

The 2-form F is called the *curvature form* of the connection ω . The defining equation for F is often written as

$$d\omega = -\frac{1}{2}[\omega \wedge \omega] + F$$

and is called the structural equation.



 $P \to M$ G-bundle, $\rho: G \to GL(V)$, $E := P \times_G V$, $f: P \to V$ equivariant map, i.e. section of E.

Def. $\nabla f = d^h f = df |_{\mathcal{H}}$ is called the covariant derivative of f.

Rem. Denote $\tau = d\rho_e \colon \mathfrak{g} \to \mathfrak{gl}(V) = \operatorname{End} V$. Then for a vertical vector $K_{\xi}(p)$ we have: $df(K_{\xi}(p)) = -\tau(\xi)f(p)$, that is all information about df is contained in $d^h f$.

Prop.

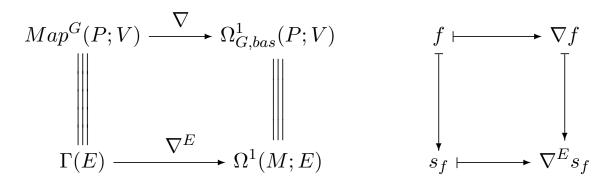
$$\nabla f = df + \omega \cdot f$$

Here " \cdot " means the action of \mathfrak{g} on V via the map τ .

RINCIPAL BUNDLES

Prop. $\nabla f \in \Omega^1(P; V)$ is G-equivariant and basic form.

Thus ∇f can be interpreted as an element of $\Omega^1(M; E)$ and we have a diagram



Prop. ∇^E is a connection on E.

LIE GROUPS	Vector bundles	Principal bundles	Connections on G -bundles	Holonomy	TORSION
		Bianchi	identity		
ω co	nnection on <i>I</i>	^D . <i>F</i> curvature			

adP has an induced connection ∇

Theorem (Bianchi identity)

 $d^{\nabla}F = 0$

Proof. For $\tilde{\varphi} \in \Omega^k(P; \mathfrak{g})$ denote $D\tilde{\varphi} = d\tilde{\varphi} + [\omega \wedge \tilde{\varphi}]$ **Step 1.** For any $\varphi \in \Omega^k(M; ad P)$ we have $\widetilde{d^{\nabla}\varphi} = D\tilde{\varphi}$. Can assume $\varphi = s \cdot \varphi_0$, where $\varphi_0 \in \Omega^k(M)$ and $\Gamma(ad P) \ni s \iff f \in Map^G(P; \mathfrak{g}).$

Then

$$\widetilde{d^{\nabla}\varphi} = \widetilde{\nabla s} \wedge \widetilde{\varphi}_0 + \widetilde{s} \cdot d\widetilde{\varphi}_0$$
$$= (df + [\omega, f]) \wedge \widetilde{\varphi}_0 + f \, d\widetilde{\varphi}_0$$
$$= d(f\widetilde{\varphi}_0) + [\omega \wedge f\widetilde{\varphi}_0]$$
$$= D\varphi$$

Proof of the Bianchi identity (continued) **Step 2.** $D\tilde{F} = 0$, where $\tilde{F} = d\omega + \frac{1}{2}[\omega \wedge \omega]$.

$$d\tilde{F} = \frac{1}{2} \left([d\omega \wedge \omega] - [\omega \wedge d\omega] \right)$$
$$= [d\omega \wedge \omega]$$
$$= [\tilde{F} \wedge \omega] - \frac{1}{2} [[\omega \wedge \omega] \wedge \omega]$$

lacobi identity
$$\implies [[\omega \wedge \omega] \wedge \omega] = 0$$

Thus, $D\tilde{F} = 0 \iff d^{\nabla}F = 0.$



 $\gamma \colon [0,1] \to M$ (piecewise) smooth curve, $p_0 \in P_{\gamma(0)}$.

Prop. [KN, Prop. II.3.1] For any γ there exists a unique horizontal lift of γ through p_0 , i.e. a curve $\Gamma \colon [0,1] \to P$ with the following properties: (i) $\dot{\Gamma}(t) \in \mathcal{H}_{\Gamma(t)}$ for any $t \in [0,1]$ (" Γ is horizontal") *(ii)* $\Gamma(0) = p_0$ (*iii*) $\pi \circ \Gamma = \gamma$

Sketch of the proof. Let Γ_0 be an arbitrary lift of γ , $\Gamma_0(0) = p_0$. Then $\Gamma = \Gamma_0 \cdot g$ for some curve $g \colon [0,1] \to G$. Hence,

$$\dot{\Gamma} = \dot{\Gamma}_0 \cdot g + \Gamma_0 \cdot \dot{g} \implies \omega(\dot{\Gamma}) = ad_{g^{-1}}\omega(\dot{\Gamma}_0) + g^{-1}\dot{g}.$$

Then there exists a unique curve g, g(0) = e, such that $g^{-1}\dot{g} + ad_{q^{-1}}\omega(\dot{\Gamma}_0) = 0 \quad \Longleftrightarrow \quad \omega(\dot{\Gamma}) = 0.$

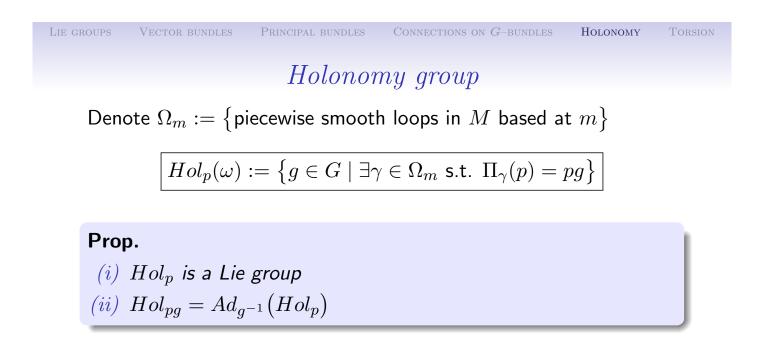
Parallel transport

 $\gamma \colon [0,1] \to M, \ \gamma(0) = m, \ \gamma(1) = n$ Parallel transport $\Pi_{\gamma} \colon P_m \to P_n$ is defined by

 $\Pi_{\gamma}(p) = \Gamma(1),$

where Γ is the horizontal lift of γ satisfying $\Gamma(0) = p$.

Prop. (i) Π_{γ} commutes with the action of G for any curve γ (ii) Π_{γ} is bijective (iii) $\Pi_{\gamma_1*\gamma_2} = \Pi_{\gamma_1} \circ \Pi_{\gamma_2}, \quad \Pi_{\gamma^{-1}} = \Pi_{\gamma}^{-1}$



Proof. Group structure follows from *(iii)* of the previous Prop. For the structure of Lie group see [Kobayashi–Nomizu, Thm 4.2]. Statement *(ii)* follows from the observation

 Γ is horizontal \implies $R_g \circ \Gamma$ is also horizontal.

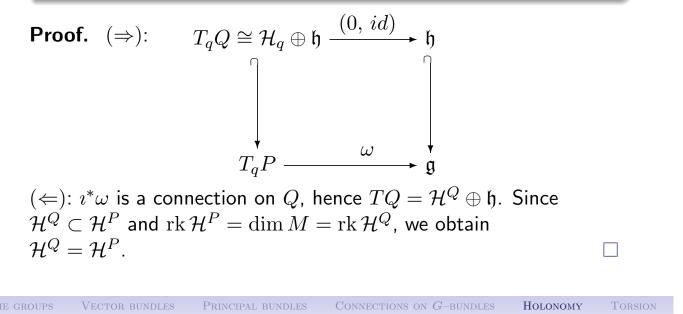
RINCIPAL BUNDLES

Reduction of connections

Let $H \subset G$ be a Lie subgroup and $Q \subset P$ be a principal H-bundle ("structure group reduces to H").

Def. A connection \mathcal{H} on P reduces to Q if $\mathcal{H}_q \subset T_q Q \quad \forall q \in Q$.

Prop. A connection reduces to $Q \iff i^*\omega$ takes values in \mathfrak{h} , where $i: Q \hookrightarrow P$.



Reduction theorem

For $p_0 \in P$ define the *holonomy bundle* through p_0 as follows: $Q(p_0) := \{ p \in P \mid \exists \text{ a horizontal curve } \Gamma \text{ s.t. } \Gamma(0) = p_0, \ \Gamma(1) = p \}.$

Theorem ("Reduction theorem")

Put $H = Hol_{p_0}(P, \omega)$. Then the following holds: (i) Q is a principal H-bundle (ii) connection ω reduces to Q

Proof. (i): $p \in Q$, $g \in H \Rightarrow pg \in Q$ (by the def of H). *Exercise:* Show that $Hol_p(\omega) = H \quad \forall p \in Q$. From the def of Q follows, that H acts transitively on fibres. Local triviality: Use parallel transport over coordinate chart U wrt segments to obtain a local section of Q (see [KN, Thm II.7.1] for details).

(ii): Follows immediately from the def of Q.

RINCIPAL BUNDLES

Connections on G^{-1}

Parallel transport and covariant derivative

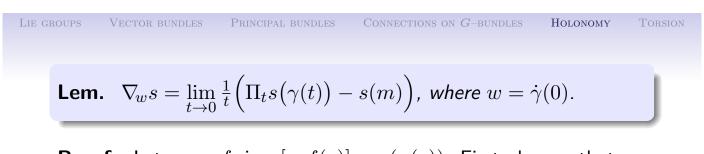
Let $\Gamma: [0,1] \to P$ be a horizontal lift of γ $\Gamma_E(t) := [\Gamma(t), v], \quad v \in V, \ E = P \times_G V$ $\Gamma_E: [0,1] \to E$ is called the horizontal lift of γ to E $\Pi_t: E_{\gamma(t)} \to E_m$ parallel transport in $E, \ m = \gamma(0)$

Lem.
$$\nabla_w s = \lim_{t \to 0} \frac{1}{t} \left(\Pi_t s(\gamma(t)) - s(m) \right)$$
, where $w = \dot{\gamma}(0)$.

Proof. Let
$$s \nleftrightarrow f$$
, i.e. $[p, f(p)] = s(\pi(p))$. First observe that

$$\Pi_{\gamma}^{E}[p, v] = [\Pi_{\gamma}p, v].$$
Since $[\Gamma(t), f(\Gamma(t))] = s(\gamma(t))$, we obtain

$$\Pi_{t}s = [p, f(\Gamma(t))].$$
 \downarrow to be continued \downarrow



Proof. Let $s \nleftrightarrow f$, i.e. $[p, f(p)] = s(\pi(p))$. First observe that $\Pi^E_{\gamma}[p, v] = [\Pi_{\gamma}p, v].$ Since $[\Gamma(t), f(\Gamma(t))] = s(\gamma(t))$, we obtain $\Pi_t s = [p, f(\Gamma(t))].$

Then

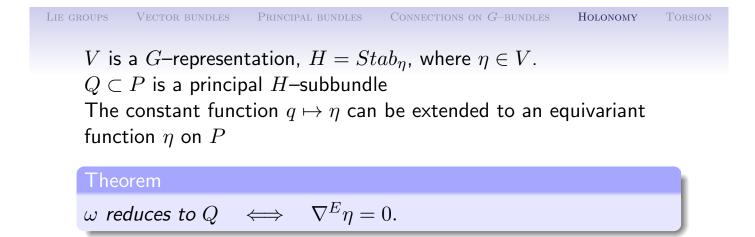
$$\nabla_{w}s = [p, df(\tilde{w})]$$

= $[p, \frac{d}{dt}|_{t=0} f \circ \Gamma(t)]$
= $\lim_{t \to 0} \frac{1}{t} \left([p, f(\Gamma(t))] - [p, f(p)] \right)$
= $\lim_{t \to 0} \frac{1}{t} \left(\Pi_{t}s(\gamma(t)) - s(m) \right).$

Rem. Let $w \in \mathfrak{X}(M)$. If $s \iff f$, then $\nabla_w s \iff df(\tilde{w})$.

Lem. Let $s \in \Gamma(E)$, $s_0 = s(m)$. Assume $\nabla s = 0$. Then for any loop γ based at m we have $\Pi^E_{\gamma} s_0 = s_0$.

Proof. Let Γ be a horizontal lift of γ . Then $f \circ \Gamma = const$. Hence $\Pi_t s(\gamma(t)) = [p, f \circ \Gamma]$ does not depend on t.



Proof. (\Rightarrow) : $\forall q \in Q \ d\eta |_{\mathcal{H}_q} = 0$, since η is constant on Q and $\mathcal{H} \subset TQ$.

 (\Leftarrow) : For any $q \in Q$ we have

$$[q,\eta] = \Pi_{\gamma}^{E}[q,\eta] = [\Pi_{\gamma}q,\eta] = [qg,\eta] = [q,\rho(g^{-1})\eta].$$

Hence $Hol_q(\omega) \subset H$. Then the holonomy bundle through q is contained in Q. Therefore, ω reduces to Q.

PRINCIPAL BUNDLES

Ambrose-Singer theorem

Theorem (Ambrose–Singer)

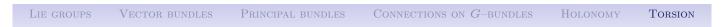
Let Q be the holonomy bundle through p_0 , $\tilde{F} \in \Omega^2(P; \mathfrak{g})$ curvature of ω . Then

$$\mathfrak{hol}_{p_0} = \operatorname{span} \{ \tilde{F}_q(w_1, w_2) \mid q \in Q, \ w_1, w_2 \in \mathcal{H}_q \}.$$

Sketch of the proof. Can assume Q = P. Denote

$$\mathfrak{g}' = \operatorname{span}\left\{\tilde{F}_q(w_1, w_2) \mid q \in Q, \ w_1, w_2 \in \mathcal{H}_q\right\} \subset \mathfrak{g}.$$

Further, $S_p := \mathcal{H}_p \oplus \{K_{\xi}(p) \mid \xi \in \mathfrak{g}'\}$. Then the distribution S is integrable. If $P_0 \ni p_0$ is a maximal integral submanifold, then $P_0 = P$, since each horizontal curve must lie in P_0 . Then $\dim \mathfrak{g} = \dim P - \dim M = \dim P_0 - \dim M = \dim \mathfrak{g}'$. Hence $\mathfrak{g} = \mathfrak{g}'$.



From now on P = Fr(M) is the principal $G = GL_n(\mathbb{R})$ -bundle of linear frames

Def. A canonical 1-form $\theta \in \Omega^1(P; \mathbb{R}^n)$ is given by

$$\theta(v) = p^{-1}(d\pi(v)), \qquad v \in T_p P.$$

Rem. θ is defined for bundles of linear frames only.

 θ is G-equivariant in the following sense: $R_g^*\theta=g^{-1}\theta.$ Indeed, for any $v\in T_pP$ we have

$$R_g^*\theta(v) = (pg)^{-1} (d\pi(R_g v)) = g^{-1} p^{-1} (d\pi(v)) = g^{-1} \theta(v).$$

Torsion

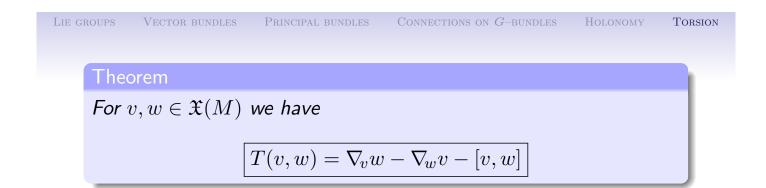
 ω is a connection on Fr(M). In particular, ω is $\mathfrak{gl}_n(\mathbb{R})$ -valued. Thus, we have induced connections on TM, T^*M , $\Lambda^k T^*M \dots$

Def. $\Theta = d\theta + \frac{1}{2}[\omega, \theta] \in \Omega^2(Fr(M); \mathbb{R}^n)$ is called the *torsion* form of ω .

Rem. $[\omega, \theta](v, w) = \omega(v)\theta(w) - \omega(w)\theta(v).$

Prop. Θ is horizontal and equivariant. Hence there exists $T \in \Omega^2(M;TM)$ s.t. $2\Theta = \pi^*T$.

T can be viewed as a skew–symmetric linear map $TM \otimes TM \rightarrow TM$ and is called the *torsion tensor*.



Proof. Represent v, w by equivariant functions $f_v, f_w \colon Fr \to \mathbb{R}^n$. Then $\nabla_v w$ is represented by $df_w(\tilde{v})$.

For the bundle of frames, $f_w = \theta(\tilde{w})$. Hence $\nabla_v w = p(\tilde{v} \cdot \theta(\tilde{w}))$. Therefore we obtain

$$T(v,w) = p(2\Theta(\tilde{v},\tilde{w}))$$

= $p(\tilde{v} \cdot \theta(\tilde{w}) - \tilde{w} \cdot \theta(\tilde{v}) - \theta([\tilde{v},\tilde{w}]))$
= $\nabla_{v}w - \nabla_{w}v - [v,w].$

The last equality follows from $[\tilde{v}, \tilde{w}]^h = \widetilde{[v, w]}$ (exercise).

Denote

$$\Gamma(T^*M) \xrightarrow{\nabla} \Gamma(T^*M \otimes T^*M) \xrightarrow{\operatorname{Alt}} \Omega^2(M), \quad \alpha \mapsto \operatorname{Alt}(\nabla \alpha).$$

Theorem

$$\operatorname{Alt}(\nabla \alpha) = d\alpha - \alpha \circ T$$

In particular, for torsion-free connections $Alt(\nabla \alpha) = d\alpha$.

Proof. This follows from the previous Thm with the help of the formulae $v \cdot \alpha(w) = \nabla_v(\alpha(w)) = (\nabla_v \alpha)(w) + \alpha(\nabla_v w)$.