Holonomy groups

in Riemannian geometry

Lecture 4

November 17, 2011

Equivalent formulation of the Berger theorem

By inspection, each group in Berger's list acts transitively on the unit sphere. On the other hand, all groups acting transitively on spheres were classified by Montgomery and Samelson in 1943. The list consists of

$$U(1) \cdot Sp(m), \qquad Spin(9),$$

and the groups from Berger's list. The first group never occurs as a holonomy group (follows from the Bianchi identity). Alekseevsky proved in 1968 that $Spin(9)$ can occur as holonomy group of a symmetric space only. Hence, the following theorem is equivalent to Berger's classification theorem.

Theorem (Berger)

Assume that the holonomy group of an irreducible Riemannian manifold does not act transitively on spheres. Then M is locally symmetric.

Second fundamental form

Let $\bar{M}$ be a Riemannian mfld, $M \subset \bar{M}$.
Write $T\bar{M} = TM \oplus \nu M$ along M.

$\bar{\nabla}_v w = \big(\bar{\nabla}_v w\big)^T + \big(\bar{\nabla}_v w\big)^\perp = \nabla_v w + \alpha(v, w),$ where $v, w \in \mathfrak{X}(M)$.

Prop.

- *∇ is the Levi–Civita connection on M wrt the induced metric;*
- *$\alpha \in \Gamma\big(S^2(TM) \otimes \nu M\big)$.*

α is called the *second fundamental form* of M.

M is called *totally geodesic*, if geodesic in M $\Rightarrow$ geodesic in $\bar{M}$.

Let γ be a geodesic in M. Then $\bar{\nabla}_{\dot\gamma}\dot\gamma = 0 + \alpha(\dot\gamma, \dot\gamma)$. Hence,

$$M \text{ is totally geodesic} \iff \alpha = 0.$$

Shape operator

Similarly, if $v \in \mathfrak{X}(M)$, $\xi \in \Gamma(\nu M)$, then

$$\bar{\nabla}_v \xi = \big(\bar{\nabla}_v \xi\big)^T + \big(\bar{\nabla}_v \xi\big)^\perp = -A_\xi v + \nabla^\perp_v \xi.$$

A_ξ is called the *shape operator*.

Let $w \in \mathfrak{X}(M)$. Then, differentiating equality $\bar{g}(w, \xi) = 0$ in the direction of v, we obtain

$$\bar{g}\big(\alpha(v, w), \xi\big) = \bar{g}(A_\xi v, w).$$

$M \subset \bar{M}$, $\bar{\Pi}_\gamma$ parallel transport of $\bar{M}$.

Prop. *M is totally geodesic if and only if* $\forall\gamma\colon [0,1] \to M$ *and* $\forall v \in T_{\gamma(0)}M \quad \bar{\Pi}_\gamma v \in T_{\gamma(1)}M$.

Proof. ($\Leftarrow$) Let $\gamma = \gamma_v$ be a geodesic in M through m. Denote by $\bar{\Pi}^t_\gamma$ the parallel transport in $\bar{M}$ along $\gamma(\tau), \tau \in [0,t]$. Then

$$\bar{\Pi}^t_\gamma v = \mathrm{proj}_{TM}\, \bar{\Pi}^t_\gamma v = \Pi^t_\gamma v = \dot{\gamma}(t),$$

i.e. γ is a geodesic in $\bar{M}$.

($\Rightarrow$) [KN, Thm VII.8.4] □

Let M be a smooth G-mfld, where G is a Lie gp acting properly. $G_m := \{g \mid gm = m\}$ isotropy subgroup.

Theorem

Let G be cmpt. For $m \in M$ and $H = G_m$ there exist a unique H–representation V and a G–equivariant diffeomorphism $\varphi\colon G \times_H V \to M$ onto an open neighbourhood of Gm s.t. $\varphi([g,0]) = gm$.

V is called the *slice representation* of M at m.

Observe: $G \to G/H$ is a principal H–bundle. Moreover, $G/H = G/G_m \cong Gm$. Since the zero–section of $G \times_H V \to G/H$ is identified with the orbit Gm, we obtain $\nu(Gm) \cong G \times_H V$. In particular, $\nu_m(Gm) \cong V$.

On the other hand, H preserves Gm. The induced representation of H on $T_m(Gm)$ is called the *isotropy representation*.

For subgroups $H, K \subset G$ we write $H \sim K$ if H is conjugate to K.
(H) conjugacy class of H.
$(H) \le (K)$ if H is conjugate to a subgroup of K.
$M_{(H)} = \{m \mid G_m \sim H\}$.

Theorem

Let G be a compact group. Assume M/G is connected. Then there exists a unique isotropy type (H) of M such that $M_{(H)}$ is open and dense in M. Each other isotropy type (K) satisfies $(H) \le (K)$.

Proof. [tom Dieck. Transformation groups. Thm. 5.14] □

Strategy of the proof of the Berger thm

Step 1. $H = \mathrm{Hol}_m$ is not transitive on the sphere $\Rightarrow$ for any principal v there exists a family $\mathcal{F}_v$ of normal subspaces to non–trivial orbits of H, which generates T_mM.

Step 2. For any $v \in T_mM$, $v \neq 0$, the submfld $N^v = \exp_m\big(\nu_v(Hv)\big)$ is totally geodesic.

Step 3. The normal holonomy group $H^\perp$ of $Hv \subset T_mM$ acts by isometries on N^v. Moreover, $H^\perp \supset \mathrm{Hol}(N^v)$.

Step 4. $\mathrm{Hol}(N^v)$ acts by isometries on $N^v \Rightarrow N^v$ is locally symmetric.

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Let M be a Riemannian mfld, $m \in M$, ρ injectivity radius at m.

Gluing Lemma

$\forall v \in T_m M$ let $\mathcal{F}_v$ be a family of subspaces of $T_m M$ s.t.

(i) $v \in W$ *for any* $W \in \mathcal{F}_v$*;*

(ii) $\exp_m(W_\rho)$ *is a totally geodesic and (intrinsically) loc. symm.*

Assume that for any v in some dense $\Omega \subset B_\rho(0)$ the family $\mathcal{F}_v$ spans $T_m M$, where $B_\rho(0) \subset T_m M$ is the ball of radius ρ. Then the local geodesic symmetry s_m is an isometry.

Proof. Let $v \in \Omega$, $\gamma = \gamma_v$ is the geodesic through m. Choose a frame $(e_1, \ldots, e_n)$ of $T_m M$ s.t. e_i belongs to some $W_i \in \mathcal{F}_v$. Let $(E_1, \ldots, E_n)$ be parallel vector fields along γ with $E_i(0) = e_i$.

Then $r_{ij} = \langle R(E_i, \dot\gamma)\dot\gamma, E_j \rangle$ is constant in t. Indeed, $\exists W \in \mathcal{F}_v$ s.t. $e_i \in W$. Hence, E_i is tangent to $\exp_m(W)$ and $\gamma(t) \in \exp_m(W)$. $\exp_m(W)$ is loc. symmetric $\Rightarrow (\nabla_{\dot\gamma} R)(E_i, \dot\gamma) = 0 \Rightarrow \ \dot r_{ij} = 0$.

Thus, in the frame E_i, Jacobi fields correspond to solutions of $\ddot a + ra = 0$, where $r = const$. Hence the statement. □

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Lemma A

Assume a compact subgroup $G \subset SO(n)$ does not act transitively on S^{n-1}. Let v be a principal vector of G. Then there exists $\xi \in \nu_v(Gv),\ \xi \neq \lambda v$, s.t. the family of normal spaces $\nu_{\gamma(t)}(G\gamma(t))$ spans $\mathbb{R}^n$, where $\gamma(t) = v + t\xi,\ t \in \mathbb{R}$.

Proof. [Olmos, A geometric proof..., Lemma 2.2] □

Lemma B

(i) N^v is a totally geodesic submanifold of M;

(ii) N^v is (intrinsically) locally symmetric.

Proof. Will be sketched below. □

Theorem (Berger)

Assume that the holonomy group of an irreducible Riemannian manifold does not act transitively on spheres. Then M is locally symmetric.

Proof. Pick $m \in M$. Let $\mathcal{O} \subset T_m M$ be subset of principal vectors. Then $\mathcal{O}$ is open and dense. Pick $v \in \mathcal{O}$.

Lemma A $\Rightarrow$ $\exists \gamma(t) = v + t\xi$ s.t. the family $\mathcal{F}_v = \{\nu_{\gamma(t)}(G\gamma(t)) \mid t \in \mathbb{R}\}$ spans $T_m M$.

Observe: $\xi \in \nu_v(Gv) \;\Rightarrow v \in \nu_{v+\xi}(G(v+\xi))$. Indeed, $G \subset SO(T_m M) \Rightarrow \mathfrak{g} \subset \mathfrak{so}(T_m M)$. Hence, for any $A \in \mathfrak{g}$ we have

$$0 = \langle Av, v+\xi\rangle = -\langle v, A(v+\xi)\rangle.$$

The first equality follows from $T_v(Gv) = \{Av \mid A \in \mathfrak{g}\}$.

Therefore, $v \in \nu_{\gamma(t)}(G\gamma(t))$ for any t. Lemma B $\Rightarrow$ assumptions of the Gluing Lemma are satisfied. Then Gluing Lemma implies that M is locally symmetric. □

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Theorem (Cartan)

Let $V \subset T_mM$. Then $\exp_m(V_\rho)$ is totally geodesic submanifold if and only if the curvature tensor of M preserves the parallel transport of V along geodesics γ_v with $\gamma_v(0) = m, v \in V$.

$U := \Pi_\gamma V$. Then "R preserves U" means: $R_{\gamma(1)}(U, U)U \subset U$.

Proof. [Berndt–Olmos–Console, Submflds and hol., Thm 8.3.1] □

$N^v := \exp_m\big(\nu_v(Hv) \cap B_\rho(0)\big)$, where $v \in T_mM \setminus \{0\}$.

Lemma B

(i) N^v is a totally geodesic submanifold of M.

Proof. Denote

$$\mathcal{R} = \operatorname{span}\{\, \bar{R}(x, y) = \Pi_\gamma^{-1} R\big(\Pi_\gamma x, \Pi_\gamma y\big)\Pi_\gamma \,\}.$$

Then the Ambrose–Singer thm states that $\mathcal{R} = \mathfrak{h} \subset \mathfrak{so}(T_mM)$.

$$\xi \in \nu_v(Hv) \iff 0 = \langle \bar{R}(x, y)v, \xi\rangle = \langle \bar{R}(v, \xi)x, y\rangle,$$

where $x, y \in T_mM$, and $\bar{R} \in \mathcal{R}$ are arbitrary. Hence, $\bar{R}(v, \xi) = 0$. Then, for any $\eta \in \nu_v(Hv)$, the Bianchi identity yields: $\bar{R}(\xi, \eta)v = -\bar{R}(\eta, v)\xi - \bar{R}(v, \xi)\eta = 0$. Thus $\bar{R}(\xi, \eta)$ belongs to the isotropy subalgebra and $\bar{R}(\xi, \eta)\nu_v(Hv) \subset \nu_v(Hv) \;\Rightarrow$

$$\bar{R}\big(\nu_v(Hv), \nu_v(Hv)\big)\nu_v(Hv) \subset \nu_v(Hv). \tag{1}$$

Since (1) holds at any pt (after parallel transport), the hypotheses of the Cartan Thm are satisfied. Hence the statement. □

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Lem. *Let $\varphi_t\colon S \to M$ be a smooth family of totally geodesic submanifolds of M. If $\xi_t = \partial_t\varphi_t \perp \varphi_t(S)$, then $id\colon (S,\varphi_0^*g) \to (S,\varphi_t^*g)$ is an isometry.*

Proof. Put $S_t = \varphi_t(S) \subset M$ with its induced metric. Let γ_w be a geodesic of S_0 through m, $w \in T_mM$. Then

$$\begin{aligned}
\tfrac{d}{dt} g\big((\varphi_t)_*w,(\varphi_t)_*w\big) &= \tfrac{\partial}{\partial t} g\Big(\tfrac{\partial}{\partial s}\big|_{s=0}\varphi_t(\gamma_w(s)),\tfrac{\partial}{\partial s}\big|_{s=0}\varphi_t(\gamma_w(s))\Big)\\
&= 2g\Big(\nabla_t \tfrac{\partial}{\partial s}\big|_{s=0}\varphi_t(\gamma_w(s)),\tfrac{\partial}{\partial s}\big|_{s=0}\varphi_t(\gamma_w(s))\Big)\\
&= 2g\Big(\nabla_s\big|_{s=0}\tfrac{\partial}{\partial t}\varphi_t(\gamma_w(s)),(\varphi_t)_*w\Big)\\
&= -2g\big(A_{\xi_t}(\varphi_t)_*w,(\varphi_t)_*w\big)\\
&= 0.
\end{aligned}$$

Therefore, $g\big((\varphi_t)_*w,(\varphi_t)_*w\big)$ does not depend on t. □

Lem. *The normal holonomy group $H^\perp$ of $Hv \subset T_mM$ acts by isometries on N^v.*

Proof. Let $c\colon [0,1] \to Hv,\ c(0) = v$. Denote by $\Pi_t^\perp$ the normal parallel transport along $c|_{[0,t]}$. By Lemma B, (i)

$$\varphi_t\colon \nu_v(Hv) \to M, \qquad \varphi_t = \exp_m \circ \Pi_t^\perp$$

is a one–parameter family of totally geodesic submanifolds.
Put $\xi_t = \partial_t \varphi_t$. Want to show $\xi_t \perp \operatorname{Im} \varphi_t = \exp_m\big(\Pi_t^\perp(\nu_v(Hv))\big)$.
It suffices to show that $\xi_0 \perp \exp_m(\nu_v(Hv)) = N^v$, since for $t > 0$ the proof is obtained by replacing v by $c(t)$.
For an arbitrary $\eta \in \nu_v(Hv)$, $J(s) = \xi_0(s\eta) = \frac{\partial}{\partial t}|_{t=0} \exp_m(s\Pi_t^\perp \eta)$ is the Jacobi v.f. along $\gamma_\eta(s)$. Initial conditions: 0 and $\frac{d}{dt}|_{t=0}\Pi_t^\perp \eta = -A_\eta \dot c(0) + \nabla^\perp \Pi_t^\perp \eta = -A_\eta \dot c(0) \perp T_m N^v = \nu_v(Hv)$.
Hence, $\xi_0(s\eta) \perp N^v$ for all s. Hence, $\xi_0 \perp N^v$.

Therefore, φ_t induces an isometry $N^v \to N^{c(t)}$. If c is a loop, we obtain an isometry $N^v \to N^v$. □

Theorem

Assume a connected Lie gp $H \subset SO(n)$ acts irreducibly on $\mathbb{R}^n$. Then the image of the connected component of the isotropy gp $(H_v)_0$ is contained in $H^\perp$.

Proof. [Berndt–Console–Olmos, Cor. 6.2.6] □

Prop. *The holonomy gp H^v of N^v is contained in the image of $(H_v)_0$ under the slice representation.*

Proof. The proof is similar to the proof of the fact that N^v is totally geodesic. For details see [Olmos, p.586] □

Cor. $H^v \subset H^\perp$.

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Lem. *Let M be a Riemannian mfld with the following property: for any $m \in M$ each restricted holonomy transformation of T_mM extends via the exponential map to a local isometry. Then M is locally symmetric.*

Sketch of the proof. Can assume that $H = Hol(M)$ acts irreducibly. Denote $\mathcal{K} = \{K \mid \mathcal{L}_K g = 0,\ K \in \mathfrak{X}(U_m)\}$. Then $\mathcal{K}_m = \{K(m) \mid K \in \mathcal{K}\}$ is a non-trivial H–invariant subspace of T_mM. Hence, $\mathcal{K}_m = T_mM$.

Then, for each $v \in T_mM$ there exists a unique $K \in \mathcal{K}$ s.t. $K(m) = v$ and $(\nabla K)_m = 0$. For such K, the integral curve $t \mapsto \varphi_t^K(m)$ through m is a geodesic. Moreover, the parallel transport along this geodesic is given by $(\varphi_t^K)_*$. This implies the local symmetry. □

Lemma B

(ii) N^v *is (intrinsically) locally symmetric.*

Hodge theory in a nutshell

Let V be an oriented Euclidean vector space, $\dim V = n$. Then the Hodge operator $*\colon \Lambda^k V^* \to \Lambda^{n-k} V^*$ is defined by the relation

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle vol, \qquad \text{for all } \alpha \in \Lambda^k V^*.$$

$*$ is an $SO(V)$–equivariant isomorphism, $*^{-1} = (-1)^{k(n-k)} *$. Hence, for any oriented Riemannian manifold (M, g) we have a well defined map $*\colon \Lambda^k T^*M \to \Lambda^{n-k} T^*M$.

Define $d^*\colon \Omega^k(M) \to \Omega^{k-1}(M)$ by $d^* = (-1)^{n(k+1)+1} * d *$. Then, if M is compact, Stokes' theorem implies that

$$\langle d\alpha, \beta \rangle_{L_2} = \langle \alpha, d^*\beta \rangle_{L_2}, \qquad \text{for any } \alpha \in \Omega^{k-1},\ \beta \in \Omega^k.$$

$\Delta = dd^* + d^*d\colon \Omega^k \to \Omega^k$ is called the Laplace operator. It is second order elliptic PDO. Denote $\mathscr{H}^k = \operatorname{Ker}(\Delta\colon \Omega^k \to \Omega^k)$.

Theorem (Hodge)

Every de Rham cohomology class contains a unique harmonic representative and $H^k_{dR} \cong \mathscr{H}^k$.

It is known, that all $\Lambda^k(\mathbb{R}^n)^*$ are irreducible as $O(n)$–representations. However, if $G \subset O(n)$, then $\Lambda^k(\mathbb{R}^n)^*$ does not need to be irreducible as G–representation.

Model example: $G = SO(4) \subset O(4)$

$*^2 = id$ on $\Lambda^2(\mathbb{R}^4)^* \Rightarrow \Lambda^2(\mathbb{R}^4)^* \cong \Lambda^2_+ \oplus \Lambda^2_-$ as $SO(4)$–representation. Hence, for any oriented Riemannian four–manifold we have $\Lambda^n T^*M \cong \Lambda^n_+ T^*M \oplus \Lambda^n_- T^*M$. Since $\Delta * = *\Delta$, we have $\mathscr{H}^2 \cong \mathscr{H}^2_+ \oplus \mathscr{H}^2_-$, $b_2 = b_+ + b_-$.

Let $H = \mathrm{Hol}$ and P be the holonomy bundle. Consider $\Lambda^k(\mathbb{R}^n)^*$ as H–representation. Let

$$\Lambda^k(\mathbb{R}^n)^* \cong \bigoplus_{i\in I_k} \Lambda^k_i(\mathbb{R}^n)^*$$

be the decomposition into irreducible components. Then

$$\Lambda^k T^*M \cong \bigoplus_{i\in I_k} \Lambda^k_i T^*M, \qquad \text{where } \Lambda^k_i T^*M = P \times_H \Lambda^k_i(\mathbb{R}^n)^*.$$

Lem. *Denote $\Omega^k_i(M) = \Gamma(\Lambda^k_i T^*M)$. Then $\Delta(\Omega^k_i) \subset \Omega^k_i$. Hence,*

$$\mathscr{H}^k \cong \bigoplus \mathscr{H}^k_i, \qquad b_k = \sum_{i\in I_k} b^i_k.$$

This statement follows from the Weitzenböck formula for the Laplacian [Besse. 1I, Lawson–Michelson. II.8]

The refined Betti numbers b^i_k carry both topological and geometrical information. They give obstructions to existence of metrics with non–generic holonomy.

Ex. If M admits a Kähler metric, then odd Betti numbers of M are even.

Another example of connection between holonomy groups and cohomology gives the following consideration. If for some i $\Lambda^k_i(\mathbb{R}^n)^*$ is a trivial H–representation, then $b^k_i = \dim \Lambda^k_i(\mathbb{R}^n)^*$. Indeed, each $\xi_0 \in \Lambda^k_i(\mathbb{R}^n)^*$ corresponds to a parallel $\xi \in \Omega^k_i$. Then $\nabla\xi = 0 \;\Rightarrow\; d\xi = 0 = d^*\xi$. Hence, $\Delta\xi = 0$. On the other hand, from the Weitzenböck formula one obtains $\Delta\xi = 0 \;\Rightarrow\; \nabla\xi = 0$. Therefore,

$$\mathscr{H}^k_i \cong \{\xi \mid \nabla\xi = 0\}.$$