Holonomy groups

in Riemannian geometry

Lecture 5

November 24, 2011

A complex structure on a real vector space V (necessarily of even dimension) is an endomorphism J s.t. $J^2 = -1$. This establishes the correspondence

$$\{\text{real vector spaces equipped with } J\} \cong \{\text{complex vector spaces}\}$$

Notice: J^* is a complex structure on V^*.

Let V be a real vector space. Then $V_{\mathbb{C}} = V \otimes \mathbb{C}$ is a complex vector space endowed with an antilinear map $\bar{\cdot}\colon V_{\mathbb{C}} \to V_{\mathbb{C}}$, $v \otimes z \mapsto v \otimes \bar{z}$.

Prop. *Let V be a real vector space equpped with a complex structure. Then*

- $V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$, *where $V^{1,0}$ and $V^{0,1}$ are eigenspaces of J corresponding to eigenvalues $+i$ and $-i$ respectively;*
- $V^{1,0} = \{v \otimes 1 - Jv \otimes i \mid v \in V\}$, $V^{0,1} = \{v \otimes 1 + Jv \otimes i\}$;
- $\bar{\cdot}\colon V^{1,0} \to V^{0,1}$ *is an (antilinear) isomorphism.*
- $V^{1,0} \cong (V, J)$, $V^{0,1} \cong (V, -J)$.

Similarly, $V^*_{\mathbb{C}} \cong (V^*)^{1,0} \oplus (V^*)^{0,1}$ and therefore

$$\Lambda^k V^*_{\mathbb{C}} \cong \bigoplus_{p+q=k} \Lambda^{p,q} V^*, \qquad \text{where } \Lambda^{p,q} V^* = \Lambda^p (V^*)^{1,0} \otimes \Lambda^q (V^*)^{0,1}.$$

A *Hermitian scalar product* on (V, J) is a scalar product h on V s.t. $h(Jv, Jw) = h(v, w)$. Then $\omega(v, w) = h(Jv, w)$ is skew–symmetric. Since $\omega(Jv, Jw) = \omega(v, w)$ we obtain $\omega \in \Lambda^{1,1}$.

Consider the case $(V, J) = (\mathbb{R}^{2m}, J_0)$, where

$$J_0 = \left(\begin{array}{c|c} 0 & -\mathbf{1}_m \\ \hline \mathbf{1}_m & 0 \end{array} \right)$$

Thus, $(\mathbb{R}^{2m}, J_0)$ can be identified with $\mathbb{C}^m$. Then the standard Euclidean scalar product is Hermitian and $\omega_0 = 2 \sum_{j=1}^m dx_j \wedge dy_j$.

Denote

$$Sp(2m; \mathbb{R}) = \{A \in GL_{2m}(\mathbb{R}) \mid \omega_0(A\cdot, A\cdot) = \omega_0(\cdot, \cdot) \;\Leftrightarrow\; AJ_0A^T = J_0\},$$
$$GL_m(\mathbb{C}) = \{A \in GL_{2m}(\mathbb{R}) \mid A \circ J_0 = J_0 \circ A\}.$$

Then we have

$$\begin{aligned} U(m) &= SO(2m) \cap Sp(2m; \mathbb{R}) \\ &= SO(2m) \cap GL_m(\mathbb{C}) \\ &= GL_m(\mathbb{C}) \cap Sp(2m; \mathbb{R}). \end{aligned}$$

Representations of $U(m)$

Observe that $\Lambda^{p,p}$ is invariant subspace wrt the conjugation. Hence, $\Lambda^{p,p}$ is the complexification of some real vector space:

$$\Lambda^{p,p} \cong [\Lambda^{p,p}]_r \otimes \mathbb{C}.$$

Namely, $[\Lambda^{p,p}]_r = \{\alpha \mid \bar{\alpha} = \alpha\}$. Similarly, if $p \neq q$

$$\Lambda^{p,q} \oplus \Lambda^{q,p} = [\Lambda^{p,q}]_r \otimes \mathbb{C}.$$

In particular, we have

$$(\mathbb{R}^{2m})^* \cong [\Lambda^{1,0}]_r, \qquad \Lambda^2(\mathbb{R}^{2m})^* \cong [\Lambda^{1,1}]_r \oplus [\Lambda^{2,0}]_r.$$

Since $U(m) \subset SO(2m)$, we also have

$$\Lambda^2(\mathbb{R}^{2m})^* \cong \mathfrak{so}(2m) = \mathfrak{u}(m) \oplus \mathfrak{u}(m)^\perp.$$

Prop. $\mathfrak{u}(m) = [\Lambda^{1,1}]_r, \quad \mathfrak{u}(m)^\perp \cong [\Lambda^{2,0}]_r.$

Proof. Exercise. □

Let (V, J, h) be a Hermitian vector space, $\omega = h(J\cdot, \cdot)$. Consider the map $L\colon \Lambda V^*_{\mathbb{C}} \to \Lambda V^*_{\mathbb{C}}$, $L(\alpha) = \omega \wedge \alpha$, which is $U(V)$–equivariant. Denote $\Lambda = L^*$, $B = [\Lambda, L]$. Then

$$[B, L] = -2L \quad \text{and} \quad [B, \Lambda] = 2\Lambda,$$

i.e. $\Lambda V^*_{\mathbb{C}}$ is an $\mathfrak{sl}_2(\mathbb{C})$–representation. This leads to the following decomposition of $\Lambda^{p,q}$ into irreducible components.
For $p + q \leq m$, denote $\Lambda_0^{p,q} = L(\Lambda^{p-1,q-1})^\perp$. It is called the space of primitive (p, q)–forms.

Theorem (Lefschetz decomposition)

For $p \geq q$ and $p + q \leq m$ there is a $U(V)$–invariant decomposition

$$\Lambda^{p,q} \cong \Lambda_0^{p,q} \oplus \Lambda_0^{p-1,q-1} \oplus \cdots \oplus \Lambda_0^{p-q+1,1} \oplus \Lambda^{p-q,0}.$$

See [Wells. Differential analysis on cx mflds. 5.1] for details.

Complex manifolds

For a real mfld M, a section I of $\mathrm{End}(TM)$ s.t. $I^2 = -id$ is called an *almost complex structure*. If M admits an almost complex structure, then M is necessarily orientable mfld of even dimension. To each I, we associate the *Nijenhuis tensor*:

$$N_I(v,w) = [Iv, Iw] - I[Iv, w] - I[v, Iw] - [v, w], \qquad v, w \in (M).$$

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Denote $\Omega^{p,q}(M) = \Gamma(\Lambda^{p,q}T^*M)$.

Theorem

For an almost complex mfld the following statements are equivalent:

(i) $v, w \in \Gamma(T^{1,0}M) \ \Rightarrow \ [v,w] \in \Gamma(T^{1,0}M)$;

(ii) $d\Omega^{1,0} \subset \Omega^{2,0} + \Omega^{1,1}$;

(iii) $d\Omega^{p,q} \subset \Omega^{p+1,q} + \Omega^{p,q+1}$;

(iv) $N_I \equiv 0$.

Proof. $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$: Exercise.
To prove $(i) \Leftrightarrow (iv)$ observe that $v \in \Gamma(T^{1,0}M) \ \Leftrightarrow \ v = v_0 - iIv_0$, $v_0 \in \mathfrak{X}(M)$, and similarly for w. Denote $x = [v,w]$. Then

$$2(x + iIx) = -N(v_0, w_0) - iIN(v_0, w_0).$$

Hence, $x^{0,1} = 0 \ \Leftrightarrow \ N(v_0, w_0) = 0$. □

Exercise. Let $\alpha \in \Omega^{1,0}(M)$. Show that $(d\alpha)^{0,2}$ can be identified with $\alpha \circ N_I$.

Newlander–Nirenberg Theorem

$\alpha_1, \dots, \alpha_m \in \Omega^{1,0}(U),\ m = \dim_{\mathbb{R}} M/2, M \supset U$ is open

Assume α_j are closed and pointwise linearly independent. Then $N \equiv 0$, since $(d\alpha_j)^{0,2} \cong 0$ for all j. After restricting to a possibly smaller domain, all α_j can be assumed to be exact: $\alpha_j = df_j,\ f_j = x_j + y_j i \colon U \to \mathbb{C}$. Then each f_j is I–holomorphic, i.e.

$$df_j \circ I = i df_j \quad \iff \quad df_j \in \Omega^{1,0}.$$

Hence we obtain local holomorphic coordinates on M.

Rem. This reasoning shows that if $N_I \neq 0$ usually there are no holomorphic functions on M (even locally).

Theorem (Newlander–Nirenberg)

$N_I \equiv 0$ iff M is a complex mfld, i.e. admits an atlas whose transition functions are holomorphic.

Write

$$\partial = d^{1,0} \colon \Omega^{p,q} \to \Omega^{p+1,q}, \qquad \bar\partial = d^{0,1} \colon \Omega^{p,q} \to \Omega^{p,q+1}.$$

For complex mflds, $d = \partial + \bar\partial$. Hence,

$$d^2 = 0 \quad \iff \quad \partial^2 = 0,\ \bar\partial^2 = 0,\ \partial\bar\partial + \bar\partial\partial = 0. \tag{1}$$

Any $\alpha \in \Omega^{p,q}$ can be written locally as a sum of the following forms: $\beta = f dz_{j_1} \wedge \dots \wedge dz_{j_p} \wedge d\bar z_{k_1} \wedge \dots \wedge d\bar z_{k_q}$. Then

$$\partial\beta = \sum_{j=1}^n \frac{\partial f}{\partial z_j} dz_j \wedge \dots, \qquad \partial\beta = \sum_{j=1}^n \frac{\partial f}{\partial \bar z_j} d\bar z_j \wedge \dots$$

From (1) we obtain that

$$\Omega^{p,0} \xrightarrow{\bar\partial} \Omega^{p,1} \xrightarrow{\bar\partial} \dots \xrightarrow{\bar\partial} \Omega^{p,n}$$

is a complex for any p. It is called *Dolbeault complex*.

$$H^{p,q} = \frac{\operatorname{Ker}\big(\bar\partial \colon \Omega^{p,q} \to \Omega^{p,q+1}\big)}{\operatorname{Im}\big(\bar\partial \colon \Omega^{p,q-1} \to \Omega^{p,q}\big)}$$

are called *Dolbeault cohomology groups*.

Structure function of an H–structure

Recall: Let $P \subset Fr_M$ be an H–structure endowed with two connections ω and $\omega' = \omega - \xi$. Then $T' - T = \delta\xi$. Here $T, T' \colon P \to \Lambda^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n$, $\xi \colon P \to (\mathbb{R}^n)^* \otimes \mathfrak{h}$ are regarded as H–equivariant maps and

$$\delta \colon (\mathbb{R}^n)^* \otimes \mathfrak{h} \hookrightarrow (\mathbb{R}^n)^* \otimes (\mathbb{R}^n)^* \otimes \mathbb{R}^n \longrightarrow \Lambda^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n.$$

For $H = SO(n)$ the map δ is an isomorphism.

Consider

$$T_0 \colon P \xrightarrow{T} \Lambda^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n \to \operatorname{Coker}\delta = \Lambda^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n / \operatorname{Im}\delta.$$

By construction, T_0 does not depend on the choice of connection and is called the *structure function* of P. It is the obstruction to the existence of a torsion–free connection on P.

Structure function of a $GL_m(\mathbb{C})$–structure

Theorem

Let $P \subset Fr$ be a $GL_m(\mathbb{C})$–structure, i.e. M is an almost cx mfld. Then P admits a connection, whose torsion is given by $T = \frac{1}{8}N$.

Proof. [KN, Thm IX.3.4]. □

Cor. *The structure function of a $GL_m(\mathbb{C})$–structure can be identified with the Nijenhuis tensor.*

Assume that V is an $SO(n)$–representation and $H = \mathrm{Stab}_\eta,\ \eta \in V$. Then

$$\Lambda^2(\mathbb{R}^n)^* \cong \mathfrak{so}(n) = \mathfrak{h} \oplus \mathfrak{h}^\perp.$$

Since $\delta_{\mathfrak{so}(n)}$ is an isomorphism, we have

- $\delta\colon (\mathbb{R}^n)^* \otimes \mathfrak{h} \to \Lambda^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n$ is injective;
- $\mathrm{Coker}\,\delta \cong (\mathrm{Im}\,\delta)^\perp \cong (\mathbb{R}^n)^* \otimes \mathfrak{h}^\perp$.

Recall that η defines an equivariant map $\tilde\eta\colon Fr_{SO} \to V$.

Prop. *The obstruction $T_0(p)$ to the existence of a torsion–free H–connection can be identified with $(\nabla\tilde\eta)(p)$, and has values in the space $(\mathbb{R}^n)^* \otimes \mathfrak{h}^\perp$.*

Prop. *The obstruction $T_0(p)$ to the existence of a torsion–free H–connection can be identified with $(\nabla\tilde\eta)(p)$, and has values in the space $(\mathbb{R}^n)^* \otimes \mathfrak{h}^\perp$.*

Proof. The obstruction $T_0(p)$ is a component of the torsion of any H–connection ω' on $P \subset Fr_{SO}$. Extend ω' to a connection on P and denote $\xi = \omega - \omega'\colon P \to (\mathbb{R}^n)^* \otimes \mathfrak{so}(n)$, where ω is the Levi–Civita connection. Since $T \equiv 0$, T' is identified with ξ. Observe

$$\nabla'\tilde\eta = 0 \quad \Rightarrow \quad \nabla\tilde\eta(p) = -\xi(p)\tilde\eta. \tag{2}$$

Consider the map $\nu\colon \mathfrak{so}(n) \to \mathrm{End}\,V \xrightarrow{ev_\eta} V$, where the first arrow is the infinitesimal $SO(n)$–action. Then $\mathrm{Ker}\,\nu = \mathfrak{h}$ and $\nu\colon \mathfrak{h}^\perp \to V$ is an embedding. From (2), $\xi(p)\tilde\eta \equiv T_0(p)$ has values in $(\mathbb{R}^n)^* \otimes \mathfrak{h}^\perp$ and can be identified with $\nabla\tilde\eta$. □

Recall:

$$\begin{aligned} U(m) &= SO(2m) \cap Sp(2m;\mathbb{R}) \\ &= SO(2m) \cap GL_m(\mathbb{C}) \\ &= GL_m(\mathbb{C}) \cap Sp(2m;\mathbb{R}). \end{aligned}$$

Hence, a $U(m)$–structure on M is given by one of the following piece of data

(i) A Riemannian metric g and an "almost symplectic form" ω s.t. $TM \xrightarrow{\hat{g}} T^*M \xrightarrow{\hat{\omega}^{-1}} TM$ is an almost cx structure;

(ii) A Riemannian metric g and an orthogonal almost cx str. I;

(iii) An almost complex structure I and an "almost symplectic form" ω s.t. $\omega(\cdot, I\cdot)$ is positive–definite.

Recalling that $\mathfrak{u}(m)^\perp \cong [\Lambda^{0,2}]_r$ we obtain

Prop. *The structure function T_0 of a $U(m)$–structure can be identified with $\nabla\omega$ and takes values in*

$$(\mathbb{R}^{2m})^* \otimes [\Lambda^{0,2}]_r \cong [\Lambda^{0,1} \otimes \Lambda^{0,2}]_r \oplus [\Lambda^{1,2}]_r.$$

Kähler metrics

A manifold M equipped with a $U(m)$–structure P is called *Kähler* if the Levi–Civita connection reduces to P. This is equivalent to any of the following conditions

(i) $\nabla\omega = 0$;

(ii) $\nabla J = 0$;

(iii) $\mathrm{Hol}(M) \subset U(m)$;

(iv) P admits a torsion–free connection.

Prop. *Let (M, g) be a Riemannian mfld equipped with an orthogonal integrable complex structure I. Denote $\omega(I\cdot, \cdot)$. Then g is Kähler iff*

$$d\omega = 0 \quad \Leftrightarrow \quad \bar{\partial}\omega = 0.$$

Cor. *Let M be Kähler and $Z \subset M$ be a complex submanifold. Then the induces metric on Z is also Kähler.*

Prop. *Let (M, g) be a Riemannian mfld equipped with an orthogonal integrable complex structure I. Denote $\omega(\cdot, \cdot) = g(I\cdot, \cdot)$. Then g is Kähler iff*

$$d\omega = 0 \quad \Leftrightarrow \quad \bar{\partial}\omega = 0.$$

Proof. First observe that $d\omega = 0 \;\Leftrightarrow\; \bar{\partial}\omega = 0$, since ω is a real $(1,1)$–form and $(d\omega)^{0,3} = 0 = (d\omega)^{3,0}$ by the integrability of the complex structure.

If g is Kähler, then $\nabla\omega = 0 \;\Rightarrow\; d\omega = 0$.

Assume now $d\omega = 0$. First observe that the component of $\nabla\omega$ lying in $[\Lambda^{0,1} \otimes \Lambda^{0,2}]_r$ can be identified with the structure function of the corresponding $GL_m(\mathbb{C})$–structure and therefore vanishes. $d\omega$ is the image of $\nabla\omega$ under the antisymmetrisation map:

$$[\Lambda^{1,2}]_r \cong [\Lambda^{1,2}_0]_r \oplus [\Lambda^{0,1}]_r \longrightarrow \Lambda^3 \cong [\Lambda^{0,3}]_r \oplus [\Lambda^{2,1}_0] \oplus [\Lambda^{0,1}]_r.$$

Hence, the component of $\nabla\omega$ in $[\Lambda^{1,2}]_r$ is determined by $(d\omega)^{1,2}$ and therefore vanishes. □

Kähler potentials

Let $f\colon \mathbb{C}^m \to \mathbb{R}$. The *Levi form* of f

$$-i\partial\bar{\partial} f = -i \sum_{j,k}^{m} \frac{\partial^2 f}{\partial z_j \partial \bar{z}_k} dz_j \wedge d\bar{z}_k$$

is of type $(1,1)$, real, and closed, since $\partial\bar{\partial} = \frac{1}{2} d(\bar{\partial} - \partial)$. The Levi form defines a Kähler metric iff it is positive definite. Conversely, a real closed $(1,1)$–form ω is locally expressible as $-i\partial\bar{\partial} f$ for some real function f. If ω is a Kähler form, the function f is called a *Kähler potential*.

Ex.

(i) $f = \sum_{j=1}^m |z_j|^2$ is a Kähler potential of the flat metric on $\mathbb{C}^m$;

(ii) $-\log f\colon \mathbb{C}^m \setminus 0 \to \mathbb{R}$ determines a Kähler potential on $\mathbb{CP}^{m-1}$. This metric is called the *Fubini–Study* metric.

Cor. *Any complex submanifold of $\mathbb{CP}^m$ is Kähler.*

Cohomology of Kähler manifolds

Let (M, I, g, ω) be an almost Kähler mfld. Then $H(v, \omega) = g(v, \bar{w})$ is a Hermitian scalar product on $T_{\mathbb{C}}M$, i.e. H is a sesquilinear and positive-definite. The Hodge operator for complexified forms is defined similarly to the real case:

$$\alpha \wedge *\beta = H(\alpha, \beta) vol.$$

Hence, $*$ is antilinear. Moreover, $* : \Omega^{p,q} \to \Omega^{m-q,m-p}$. By analogy with the real case, define

$$\bar{\partial}^* = - * \bar{\partial} * \quad \text{and} \quad \Delta_{\bar{\partial}} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}.$$

Then, just like for the de Rham cohomology, we have

Theorem

Every Dolbeault cohomology class on a compact Hermitian mfld has a unique $\Delta_{\bar{\partial}}$–harmonic representative and $H^{p,q} \cong \mathcal{H}^{p,q} = \mathrm{Ker}\big(\Delta_{\bar{\partial}} \colon \Omega^{p,q} \to \Omega^{p,q}\big)$.

Prop. *If M is Kähler, then $2\Delta_{\bar{\partial}} = \Delta$.*

Hence, we obtain

Theorem

Let M be a compact Kähler mfld. Then

$$H^k(M; \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(M).$$

Moreover, $\overline{H^{p,q}} = H^{q,p}$ and $H^{p,q} \cong \left(H^{m-p,m-q}\right)^*$ (Serre duality).

Serre duality: If $\alpha \in \mathcal{H}^{p,q}$, then $*\alpha \in \mathcal{H}^{m-q,m-p}$. Since $\int_M \alpha \wedge *\alpha = \int_M \|\alpha\|^2 vol$, the pairing

$\mathcal{H}^{p,q} \times \mathcal{H}^{n-p,n-q} \to \mathbb{C}, \quad (\alpha, \beta) \mapsto \int_M \alpha \wedge \beta$ is nondegenerate.

Hence, $\mathcal{H}^{p,q} \cong \left(\mathcal{H}^{n-p,n-q}\right)^*$.

Define the Hodge numbers $h^{p,q}$ by $h^{p,q} = \dim H^{p,q}(M)$. Then for compact Kähler mflds we have

$$b_k = \sum_{j=0}^{k} h^{j,k-j} \quad \text{and} \quad h^{p,q} = h^{q,p} = h^{m-p,m-q} = h^{m-q,m-p}.$$

Cor. *If M is compact Kähler mfld, then odd Betti numbers of M are even.*

Theorem (Hard Lefschetz theorem)

On a compact Kähler mfld M^{2m}, there is a decomposition

$$H^k(M,\mathbb{R}) = \bigoplus_{p+q=k} \bigoplus_{r=0}^{\min(p,q)} H_0^{p-r,q-r}(M), \quad 0 \le k \le m.$$

Idea of the proof: The $\mathfrak{sl}_2(\mathbb{C})$-action on $\Omega^\bullet(M,\mathbb{C})$ descends to $H^\bullet(M;\mathbb{C})$ and respects bidegree and real structure. See [Wells] or [Huybrechts, Complex geometry] for details. □

Curvature of Kähler mflds

Recall: $\mathfrak{R} = \mathrm{Ker}\big(b : S^2(\Lambda^2(\mathbb{R}^n)) \to S^2(\Lambda^2\mathbb{R}^n)\big)$ is the space of algebraic curvature tensors, where $b : S^2(\Lambda^2\mathbb{R}^n) \to \Lambda^4\mathbb{R}^n$ is the Bianchi map (full antisymmetrization).
Let $P \subset Fr_{SO}$ be a principal H-bundle equipped with a connection φ. then the curvature tensor takes values in $\mathfrak{h}$. Hence, we obtain

Prop. *For any $p \in P$ the curvature $R(p)$ belongs to the space*

$$\mathfrak{R}^H = \mathrm{Ker}(b : S^2\mathfrak{h} \to S^2\mathfrak{h})$$

and we have the commutative diagram

$$\begin{array}{ccccc}
\mathfrak{R} & \hookrightarrow & S^2(\Lambda^2\mathbb{R}^n) & \xrightarrow{\;b\;} & \Lambda^4\mathbb{R}^n \\
\uparrow & & \uparrow & & \| \\
\mathfrak{R}^H & \hookrightarrow & S^2\mathfrak{h} & \longrightarrow & \Lambda^4\mathbb{R}^n
\end{array}$$

Consider now the case $H = U(m)$ and recall that $\mathfrak{u}(m) \cong [\Lambda^{1,1}]_r$. Hence,

$$\begin{aligned} S^2(\mathfrak{u}(m)_{\mathbb{C}}) &\cong S^2(\Lambda^{1,1}) \\ &\cong S^2(\Lambda^{1,0}) \otimes S^2(\Lambda^{0,1}) \oplus \Lambda^2(\Lambda^{1,0}) \otimes \Lambda^2(\Lambda^{0,1}) \\ &\cong S^{2,2} \oplus \Lambda^{2,2}. \end{aligned}$$

In analogy to the decomposition

$$\Lambda^{2,2} \cong \Lambda_0^{2,2} \oplus \Lambda_0^{1,1} \oplus \mathbb{C}$$

we may write

$$S^{2,2} \cong \mathfrak{B}_{\mathbb{C}} \oplus \Lambda_0^{1,1} \oplus \mathbb{C},$$

where $\mathfrak{B}_{\mathbb{C}}$ denotes the primitive component.

Prop. $\mathfrak{R}^{U(m)} \cong \mathfrak{B} \oplus [\Lambda_0^{1,1}]_r \oplus \mathbb{R}, \quad \mathfrak{R}^{SU(m)} \cong \mathfrak{B}.$

Proof. [Salamon, Prop. 4.7]. □

Ricci form

Observe: $\mathfrak{R}^{U(m)} \subset End(\Lambda^{1,1})$.

Prop. *For $R \in \mathfrak{R}^{U(m)}$ denote $r = c(R)$, where c is the Ricci contraction. Then $R(\omega_0) = r(I\cdot, \cdot) =: \rho$.*

Proof. Let $(e_1, I_0e_1, \ldots, e_m, I_0e_m)$ be an orthonormal basis of $\mathbb{R}^{2m}$. Then

$$\begin{aligned} r(x,y) &= \sum_j \langle R(e_j,x)e_j, y\rangle + \sum_j \langle R(I_0e_j,x)I_0e_j, y\rangle \\ &= \sum_j \langle R(e_j,x)I_0e_j, I_0y\rangle - \sum_j \langle R(e_j,x)e_j, I_0y\rangle \\ &= \sum_j \langle R(e_j,I_0e_j)x, I_0y\rangle, \end{aligned}$$

where $1 \le j \le m$ and the last equality follows from the Bianchi identity. The statement follows since ω_0 is identified with $\sum e_j \wedge I_0e_j$. □

If M is Kähler with curvature tensor R, then the associated $(1,1)$-form ρ is called the Ricci form.

Prop. *The Ricci form is closed.*

Proof. The Ricci form is obtained as contraction of R and ω. Then $d\rho = 0$ follows from $d^\nabla R = 0$ and $d\omega = 0$. □

Any $\beta \in [\Lambda^{1,1}]_r \cong \mathfrak{u}(m)$ can be viewed as a $\mathbb{C}$-linear endomorphism of $\mathbb{C}^m$. Then $\mathrm{tr}_\mathbb{C}\beta$ is purely imaginary.

Rem. If β is viewed as $\mathbb{R}$-linear map of $\mathbb{R}^{2m}$, then $\mathrm{tr}_\mathbb{R}\beta = 0$.

The proof of the previous Proposition shows that $i\rho = \mathrm{tr}_\mathbb{C} R$, where R is viewed as a $(1,1)$-form with values in $\mathrm{End}_\mathbb{C}(TM)$. Hence,

Prop. *The first Chern class $c_1(M)$ is represented by $\frac{1}{2\pi}\rho$*

Cor. *The curvature tensor of the canonical line bundle $\Lambda^{m,0}T^*M = \Lambda^m(T^*M)^{1,0}$ equals $i\rho$.*

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Theorem

Let M^{2m} be a Kähler mfld. Then $\mathrm{Hol}^0(M) \subset SU(m)$ iff $Ric \equiv 0$.

Proof. Let P be the holonomy bundle. Then $\mathrm{Hol}^0(M) \subset SU(m)$ iff for any $p \in P$ $R(p)$ takes values in $\mathfrak{su}(m)$. Observe that

$$\mathfrak{su}(m) = \{A \in \mathfrak{u}(m) \mid \mathrm{tr}_\mathbb{C} A = 0\}.$$

Hence, $R(p) \in \mathfrak{su}(m)$ iff $i\rho_{\pi(p)} = \mathrm{tr}_\mathbb{C} R(p) = 0 \Leftrightarrow Ric(p) = 0$. □

Theorem

$\mathrm{Hol}(M) \subset SU(M)$ *iff M admits a parallel $(m,0)$–form.*

Recall:

$$\mathfrak{R} \cong \mathbb{R} \oplus S_0^2\mathbb{R}^n \oplus \mathcal{W},$$
$$R = \tfrac{s}{2n(n-1)}\, q \owedge q + \tfrac{1}{n-2}\, Ric_0 \owedge q + W.$$

Tracing the identifications for Kähler mflds we can write

$$\mathfrak{R}^{U(m)} \cong \mathbb{R} \oplus [\Lambda_0^{1,1}]_r \oplus \mathfrak{B},$$
$$R = \tfrac{s}{2m^2}\omega \otimes \omega + \tfrac{1}{m}\omega \otimes \rho_0 + \tfrac{1}{m}\rho_0 \otimes \omega + B,$$

where ρ_0 is the primitive component of ρ. In particular, we have the diagram ($m \geq 3$):