Holonomy groups

in Riemannian geometry

Lecture 8

Spin Geometry

December 15, 2011

Clifford algebras

Recall: For $n \geq 3$, $\mathrm{Spin}\,(n)$ is a connected simply connected group fitting into the short exact sequence

$$0 \to \{\pm 1\} \to \mathrm{Spin}\,(n) \to SO(n) \to 0,$$

Aim: Construct spinor groups explicitly.

Let V be a (real) finite dimensional vector space. Denote by TV the tensor algebra of V: $TV = \mathbb{R} \oplus V \oplus V \otimes V \oplus \dots$

Def. Let q be a quadratic form on V. Then the Clifford algebra is defined by

$$Cl(V, q) = TV / \langle v \cdot v + q(v) \rangle.$$

In other words, the algebra $Cl(V, q)$ is generated by elements of V and 1 subject to relations

$$v \cdot v = -q(v) \quad \Longleftrightarrow \quad v \cdot w + w \cdot v = -2q(v, w).$$

Rem. $Cl(V,q)$ is $\mathbb{Z}/2\mathbb{Z}$-graded: $Cl(V,q) = Cl^0(V,q) \oplus Cl^1(V,q)$.

From now on we assume that q is positive definite for the sake of simplicity.

Prop. *There is a (canonical) vector space isomorphism $\Lambda V \longrightarrow Cl(V,q)$.*

Proof. Choose an orthogonal basis $(e_1, \ldots, e_n)$ of V. Then $e_i \cdot e_j = -e_j \cdot e_i$ for all i, j. Hence, the map

$$\varphi\colon \Lambda V \longrightarrow Cl(V,q)$$
$$e_{i_1} \wedge \cdots \wedge e_{i_k} \mapsto e_{i_1} \ldots e_{i_k}$$

is well-defined and surjective. This map is also injective (excercise). □

Cor. $\dim Cl(V,q) = 2^n$, *where* $n = \dim V$.

Rem. ΛV and $Cl(V,q)$ are not isomorphic as algebras (unless $q = 0$).

In fact we have

Prop. *With respect to the isomorphism $Cl(\mathbb{R}^n, q_{st}) \cong \Lambda(\mathbb{R}^n)^*$, Clifford multiplication between $v \in \mathbb{R}^n$ and $\varphi \in \Lambda(\mathbb{R}^n)^*$ can be written as*

$$v \cdot \varphi = q_{st}(v, \cdot) \wedge \varphi - i_v \varphi$$

Proof. [Lawson, Michelsohn. Prop. I.3.9] □

Let x be a unit in $Cl(V,q)$. Define

$$Ad_x : Cl(V,q) \longrightarrow Cl(V,q), \quad Ad_x y = xyx^{-1}$$

Observe that each non-zero $v \in V \hookrightarrow Cl(V,q)$ is a unit:

$$v^{-1} = -\frac{1}{q(v)} v.$$

Prop. *For any non-zero $v \in V$ the map Ad_v preserves V and the following equality holds:*

$$-Ad_v w = w - 2\frac{q(v,w)}{q(v,v)}v$$

(i.e. $-Ad_v$ is the reflection in $v^\perp$).

Proof.

$$Ad_v w = -\frac{1}{q(v,v)} v \cdot w \cdot v = \frac{1}{q(v,v)} v \cdot (v \cdot w + 2q(v,w))$$
$$= -w + 2\frac{q(v,w)}{q(v,v)}v.$$

□

Rem. Ad_v preserves q but not orientation (in general).

Spin groups

Def. $Spin(V,q)$ is the group generated by

$$\{v \cdot w \mid q(v) = 1 = q(w)\} \subset Cl^\times(V,q).$$

It is well-known that the group $O(V,q)$ is generated by reflections (recall the normal form for orthogonal matrices and observe that each rotation of the plane is a product of two reflections). Then $SO(V,q)$ is generated by compositions of even numbers of reflections. In other words, the map

$$Ad\colon Spin(V,q) \longrightarrow SO(V,q)$$

is surjective.

Prop. $\mathrm{Ker}\, Ad \cong \{\pm 1\}$, *i.e. we have the short exact sequence*

$$0 \longrightarrow \{\pm 1\} \longrightarrow Spin(V,q) \longrightarrow SO(V,q) \longrightarrow 0$$

Proof. Denote by $\tilde{\cdot}$ the automorphism of Cl generated by $\tilde{\cdot} : TV \to TV$, $\tilde{v} = -v$. Let

$$\widetilde{Ad}_v w = \tilde{v} \cdot w \cdot v, \quad w \in Cl(V,q).$$

This induces a homomorphism

$$\widetilde{Ad} : Cl^{\times}(V,q) \longrightarrow GL(Cl(V,q)).$$

Choose an ONB $(e_1, \ldots, e_n)$ of V. Suppose $\varphi \in Cl^{\times}(V,q)$ belongs to $\mathrm{Ker}\, \widetilde{Ad} : Cl^{\times} \to GL(V)$, i.e. $\tilde{\varphi} \cdot w = w \cdot \varphi$ for all $w \in V$. Write $\varphi = \varphi_0 + \varphi_1$, where $\varphi_i \in Cl^i(V,q)$. Then

$$(\varphi_0 - \varphi_1)w = w(\varphi_0 + \varphi_1) \iff \begin{cases} \varphi_0 \cdot w = w \cdot \varphi_0 \\ -\varphi_1 \cdot w = w \cdot \varphi_1 \end{cases} \quad (1)$$

Proof of $\mathrm{Ker}\, Ad = \{\pm 1\}$ *continued*

Further, write $\varphi_0 = \psi_0 + e_1\psi_1$, where ψ_0, ψ_1 are expressions in $e_2, \ldots, e_n$ only. We have

$$\begin{aligned} e_1(\psi_0 + e_1\psi_1) &= (\psi_0 + e_1\psi_1)e_1 && \text{(by (1) with } w = e_1) \\ &= \psi_0 e_1 + e_1\psi_1 e_1 \\ &= e_1\psi_0 - e_1^2\psi_1 && (\text{since } \psi_i \in Cl^i) \end{aligned}$$

Hence, $\psi_1 = 0 \;\Rightarrow\; \varphi_0$ does not involve $e_1 \;\Rightarrow \varphi_0 = \lambda \cdot 1$.

A similar argument shows that φ_1 does not involve any $e_j \;\Rightarrow\; \varphi_1 = 0$.

Thus, $\mathrm{Ker}\big(\widetilde{Ad} : Cl^{\times} \to GL(V)\big) \cong \mathbb{R}^*$. Therefore, $\mathrm{Ker}\big(\widetilde{Ad} : Spin(V,q) \to SO(V)\big) \cong \{\pm 1\}$. Finally, $\widetilde{Ad} = Ad$ on $Spin(V,q)$. □

Prop. *$Spin(n) := Spin(\mathbb{R}^n, q_{st})$ is a nontrivial double covering of $SO(n)$.*

Proof. It suffices to show that 1 and -1 can be joined by a path in $Spin(n)$. The path

$$\begin{aligned}\gamma(t) &= (e_1 \cos t + e_2 \sin t)(e_2 \sin t - e_1 \cos t) = \\ &= \cos 2t \cdot 1 + \sin 2t \cdot e_1 e_2\end{aligned}$$

does the job. □

Cor. *$Spin(n)$ is connected and simply connected provided $n \geq 3$.*

Proof. Follows from the facts that $SO(n)$ is connected and $\pi_1(SO(n)) \cong \{\pm 1\}$. □

Ex. ("accidental isomorphisms in low dimensions")

1) $Spin(2) := U(1) \cong S^1$
2) $Spin(3) \cong Sp(1) \cong SU(2)$
3) $Spin(4) \cong Sp(1) \times Sp(1)$
4) $Spin(5) \cong Sp(2)$
   To see this, consider the action of $Sp(2)$ on $M_2(\mathbb{H})$ by conjugation. Then $\mathbb{R}^5$ can be identified with the subspace of traceless, quaternion-Hermitian matrices. Hence, $Sp(2)/\pm 1 \cong SO(5)$.
5) $Spin(6) \cong SU(4)$

Some facts from representation theory of Clifford algebras and Spin groups

Theorem

Let ν_n and $\nu_n^{\mathbb{C}}$ denote the number of inequivalent irreducible real and complex representations of $Cl_n := Cl(\mathbb{R}^n, q_{st})$ and $Cl_n \otimes \mathbb{C}$ respectively. Then

$$\nu_n = \begin{cases} 2 & n \equiv 1 (\mathrm{mod}\, 4), \\ 1 & \text{otherwise} \end{cases} \quad \text{and} \quad \nu_n^{\mathbb{C}} = \begin{cases} 2 & n \text{ is odd}, \\ 1 & n \text{ is even}. \end{cases}$$

Proof. [Lawson, Michelsohn. Thm I.5.7]. □

Def. The real (complex) spinor representation of $Spin(n)$ is the homomorphism

$$\begin{aligned} \Delta_n &\colon Spin(n) \to \mathrm{End}_{\mathbb{R}}(S), && \text{if real} \\ \Delta_n^{\mathbb{C}} &\colon Spin(n) \to \mathrm{End}_{\mathbb{C}}(S), && \text{if complex} \end{aligned}$$

given by restricting an irreducible real (complex) representation of Cl_n ($Cl_n \otimes \mathbb{C}$) to $Spin(n)$.

Theorem

Let W be a real Cl_n-representation. Then there exists a scalar product on W s.t. $\langle v \cdot w, v \cdot w' \rangle = \langle w, w' \rangle$ $\forall v \in V$ s.t. $\|v\| = 1$.

Cor. $\langle v \cdot w, w' \rangle = -\langle w, v \cdot w' \rangle$.

Spin structures

Let $P \to M$ be a principal $SO(n)$-bundle, $n \geq 3$.

Def. The Spin-structure on P (equivalently, on $E = P \times_{SO(n)} \mathbb{R}^n$) is a principal $Spin(n)$-bundle $\widetilde{P} \to M$ together with a $Spin(n)$-equivariant map $\xi : \widetilde{P} \to P$, which is (fiberwise) a 2-sheeted covering.

Thus, we have a commutative diagram

$$\begin{array}{ccc} \widetilde{P} & \xrightarrow{\ \xi\ } & P \\ \Big\downarrow{\scriptstyle \tilde{\pi}} & & \Big\downarrow{\scriptstyle \pi} \\ M & = & M \end{array}$$

From the short exact sequence

$$1 \to \{\pm 1\} \to Spin(n) \to SO(n) \to 1$$

we obtain

$$H^0(M; SO(n)) \to H^1(M; \mathbb{Z}_2) \to H^1(M; Spin(n)) \to$$
$$\to H^1(M; SO(n)) \xrightarrow{\delta} H^2(M; \mathbb{Z}_2).$$

Then $\delta[P]$ equals the second Stiefel-Whitney class, $w_2(P)$. Hence, P admits a spin structure iff $w_2(P) = 0$. If this is the case, all spin structures are classified by $H^1(M, \mathbb{Z}_2)$ (assuming M is connected).

Def. A spin mfld is an oriented Riemannian mfld with a spin structure on its tangent bundle.

Rem. Thus, M admits a spin structure iff $w_2(M) = 0$. This is a topological condition on M, not on the Riemannian metric.

Rem. Since $\xi : \widetilde{P} \to P$ is a covering, $\xi^* \varphi_{LC}$ is a (distinguished) connection on $\widetilde{P}$.

For the spinor representation $\Delta\colon Spin(n) \to End(S)$ the associated spinor bundle

$$S := \widetilde{P} \times_{Spin(n)} S$$

is equipped with a connection and Euclidean scalar product.

Rem. For any $m \in M$, the fibre S_m is a module over $Cl(T_m M)$.

Denote by $R^S \in \Omega^2(M; End(S))$ the induced curvature form.

Prop. *Let $e = (e_1, \ldots, e_n)$ be a local section of $P = P_{SO}$. Then*

$$R^S(v,w)\sigma = \sum_{i,j} \langle R(v,w)e_i, e_j\rangle e_i e_j \cdot \sigma. \tag{2}$$

Proof. [Lawson, Michelson. Thm I.4.15] □

Parallel spinors and holonomy groups

Theorem

Assume M admits a nontrivial parallel spinor. Then M is Ricci-flat.

Proof. Assume $\psi \in \Gamma(S)$ is parallel. Then $d^\nabla(\nabla\psi) = d^\nabla \cdot d^\nabla \psi = 0 \iff R^S(v,w)\cdot\psi = 0$ for any $v, w \in \mathfrak{X}(M)$. With the help of (2) with $v = e_k$ we obtain

$$\begin{aligned}
0 =& \sum_{i,j,k} \langle R(e_k,w)e_i, e_j\rangle e_k e_i e_j \cdot \psi = \sum_{i,j,k} \langle R(e_i,e_j)e_k, w\rangle e_i e_j e_k \cdot \psi \\
=& \frac{1}{3} \sum_{i\neq j\neq k\neq i} \langle R(e_i,e_j)e_k + R(e_j,e_k)e_i + R(e_k,e_i)e_j, w\rangle e_i e_j e_k \cdot \psi \\
&+ \sum_{i,j} \langle R(e_i,e_j)e_i, w\rangle e_i e_j e_i \cdot \psi + \sum_{i,j} \langle R(e_i,e_j)e_j, w\rangle e_i e_j e_j \cdot \psi \\
=& 0 + \sum_{i,j,} \langle R(e_i,w)e_i, e_j\rangle e_j \cdot \psi - \sum_{i,j,} \langle R(e_j,w)e_i, e_j\rangle e_i \cdot \psi \\
=& 2Ric(w)\cdot\psi.
\end{aligned}$$

Proof of $\nabla\psi = 0,\ \psi \neq 0 \ \Rightarrow\ Ric = 0$ *continued*

Here Ric is viewed as a linear map $TM \to TM$, namely $Ric(w) = \sum_{j=1}^{n} R(e_j, w)e_j$. Hence

$$Ric(w) \cdot \psi = 0 \quad \Longrightarrow \quad Ric(w)^2 \cdot \psi = -\|Ric(w)\|^2 \psi = 0.$$

Hence, $Ric(w) = 0$ for all w. □

Clearly, if M admits a parallel spinor then M must have a non-generic holonomy. Only metrics with the following holonomies

$$SU(\tfrac{n}{2}),\ Sp(\tfrac{n}{4}),\ G_2,\ Spin(7) \tag{3}$$

are Ricci-flat.

Theorem

Let M be a complete, simply-connected, and irreducible Riemannian spin mfld. Then M admits a not-trivial parallel spinor iff $\mathrm{Hol}(M)$ is one of the four groups listed in (3).

Dirac bundles

Let $P \to M$ be the principal $SO(n)$-bundle of orthonormal oriented frames. Then $Cl(M) := P \times_{SO(n)} Cl(\mathbb{R}^n)$ is called the Clifford bundle of M. Notice: $Cl_m(M) = Cl(T_m M)$.

Def. A *Dirac bundle* is a bundle S of left modules over $Cl(M)$ equipped with an Euclidean scalar product and a connection s.t. the following holds:

$$\langle v \cdot \sigma_1, v \cdot \sigma_2 \rangle = \|v\|^2 \langle \sigma_1, \sigma_2 \rangle$$
$$\nabla(\varphi \cdot \sigma) = (\nabla^{LC}\varphi) \cdot \sigma + \varphi \cdot (\nabla\sigma).$$

Here σ, $\sigma_i \in \Gamma(S)$, $v \in \mathfrak{X}(M)$, and $\varphi \in \Gamma(Cl(M))$.

- Spinor bundle S is a Dirac bundle [See LM. II.4 for details].
- $\Lambda T^*M \cong Cl(M)$ is a Dirac bundle (with the Levi-Civita connection). Hence, the existence of Dirac bundles does not require M to be spin.

Dirac operators

Let S be a Dirac bundle.

Def. The map

$$D\colon \Gamma(S) \xrightarrow{\nabla} \Gamma(T^*M \otimes S) \xrightarrow{Cl} \Gamma(S)$$

is called the *Dirac operator*.

In terms of a local frame $(e_1, \ldots, e_n)$ of TM the Dirac operator is given by

$$D\sigma = \sum_{i=1}^{n} e_i \cdot (\nabla_{e_i}\sigma).$$

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Prop. *D is elliptic and formally self-adjoint operator (wrt the L_2-scalar product).*

Proof. Ellipticity: $\sigma_\xi(D) = i\xi\cdot\colon S \to S$ is clearly invertible for any $\xi \neq 0$.

To prove that D is formally self-adjoint, choose a local orthonormal basis $e = (e_1, \ldots, e_n)$ of TM s.t. $(\nabla e_i)_m = 0$ for all i. Then

$$\begin{aligned}\langle D\sigma_1, \sigma_2\rangle_m &= \sum_j \langle e_j \cdot \nabla_{e_j}\sigma_1, \sigma_2\rangle_m = \\ &= -\sum_j \langle \nabla_{e_j}\sigma_1, e_j \cdot \sigma_2\rangle_m = \\ &= -\sum_j \big(e_j \cdot \langle \sigma_1, e_j \cdot \sigma_2\rangle - \langle \sigma_1, e_j \cdot \nabla_{e_j}\sigma_2\rangle\big)_m.\end{aligned}$$

Proof continued

Further, define $v \in \mathfrak{X}(M)$ by the condition

$$\langle v, w\rangle = -\langle \sigma_1, w \cdot \sigma_2\rangle \quad \text{for all} \quad w \in \mathfrak{X}(M).$$

Then

$$\begin{aligned} \operatorname{div}_m(v) &= \sum_j \langle \nabla_{e_j} v, e_j\rangle_m \\ &= \sum_j (e_j \cdot \langle v, e_j\rangle)_m \\ &= -\sum_j (e_j \cdot \langle \sigma_1, e_j \cdot \sigma_2\rangle)_m \end{aligned}$$

Hence, $\langle D\sigma_1, \sigma_2\rangle = \operatorname{div}(v) + \langle \sigma_1, D\sigma_2\rangle$ pointwise. Hence, D is formally self-adjoint. □

Examples of Dirac operators

1) $M = \mathbb{R}^2$. Then $Cl(\mathbb{R}^2)$ has a basis $(1, e_1, e_2, e_1 \cdot e_2)$. Then we have the isomorphism of vector spaces

$$Cl(\mathbb{R}^2) = Cl^0(\mathbb{R}^2) \oplus Cl^1(\mathbb{R}^2) \cong \mathbb{C} \oplus \mathbb{C}.$$

Notice that the Clifford multiplication by $v \in \mathbb{R}^2$ is an antidiagonal operator. Then

$$D = \begin{pmatrix} 0 & -\frac{\partial}{\partial z} \\ \frac{\partial}{\partial \bar{z}} & 0 \end{pmatrix}.$$

2) Similarly, for $M = \mathbb{R}^4$ one obtains

$$D = \begin{pmatrix} 0 & -\frac{\partial}{\partial q} \\ \frac{\partial}{\partial \bar{q}} & 0 \end{pmatrix},$$

where $\frac{\partial}{\partial \bar{q}} : C^\infty(\mathbb{R}^4; \mathbb{H}) \to C^\infty(\mathbb{R}^4; \mathbb{H})$,
$\frac{\partial f}{\partial \bar{q}} = \frac{\partial f}{\partial x_0} + i\frac{\partial f}{\partial x_1} + j\frac{\partial f}{\partial x_2} + k\frac{\partial f}{\partial x_3}$ is the Fueter operator.

Examples of Dirac operators: continued

3) M is a Riemannian mfld, $S = Cl(M)$. Then

$$D = d + d^* : \quad \Omega(M) \to \Omega(M).$$

This follows from the following two observations:

$$a) \quad v \cdot \varphi = q_{st}(v, \cdot) \wedge \varphi - i_v \varphi \quad \text{if} \quad v \in \mathbb{R}^n, \quad \varphi \in \Lambda(\mathbb{R}^n)^*$$

$$b) \quad d = \sum_j e_j^* \wedge \nabla_{e_j}, \qquad d^* = -\sum_j \imath_{e_j} \nabla_{e_j}$$

This is just a restatement of the facts that the sequences

$$\Gamma(\Lambda^k T^*M) \xrightarrow{\nabla^{LC}} \Gamma(T^*M \otimes \Lambda^k T^*M) \xrightarrow{Alt} \Gamma(\Lambda^{k+1} T^*M)$$

$$\Gamma(\Lambda^k T^*M) \xrightarrow{\nabla^{LC}} \Gamma(T^*M \otimes \Lambda^k T^*M) \xrightarrow{-contr.} \Gamma(\Lambda^{k-1} T^*M)$$

represent d and d^* respectively. Details concerning d^* can be found in [LM. Lemma II.5.13].

Weitzenböck formulae and Bochner technique

Assume M is a compact Riemannian mfld. Let $E \to M$ be an Euclidean vector bundle equipped with a connection ∇. Define

$$\nabla^2_{v,w} s = \nabla_v(\nabla_w s) - \nabla_{\nabla_v w} s,$$

where $s \in \Gamma(E)$, v, $w \in \mathfrak{X}(M)$. Notice that

$$\nabla^2_{v,w} - \nabla^2_{w,v} = R(v, w).$$

Hence, $\quad \nabla^2_{\cdot,\cdot} \in \Gamma(T^*M \otimes T^*M \otimes S)$.

Def. The map

$$\nabla^*\nabla \colon \Gamma(S) \xrightarrow{\nabla^2} \Gamma(T^*M \otimes T^*M \otimes S) \xrightarrow{-tr} \Gamma(S)$$

is called the *connection Laplacian*.

In terms of local orthonormal frames we have

$$\nabla^*\nabla s = -\sum_j \nabla^2_{e_j,e_j} s.$$

Prop. *The operator* $\nabla^*\nabla$ *is formally self-adjoint and satisfies*

$$\langle \nabla^*\nabla s_1, s_2 \rangle_{L_2} = \langle \nabla s_1, \nabla s_2 \rangle_{L_2}.$$

In particular, $\nabla^*\nabla$ *is non-negative.*

Proof. Similar to the proof of the fact that D is formally self-adjoint. For details see [LM. Prop. II.2.1.]. □

Let S be a Dirac bundle. If $R \in \Omega^2(M; \mathrm{End}(S))$ is the curvature form, define $\mathcal{R} \in \Gamma(\mathrm{End}(S))$ by

$$\mathcal{R}(s) = \tfrac{1}{2} \sum_{j,k} e_j e_k \cdot R(e_j, e_k)(s).$$

Theorem (general Bochner identity)

$$\boxed{D^2 = \nabla^*\nabla + \mathcal{R}}$$

Proof. Choose a local frame $(e_1, \ldots, e_n)$ of TM s.t. $(\nabla e_j)_m = 0$. Then

$$\begin{aligned} D^2 &= \sum_{j,k} e_j \cdot \nabla_{e_j}(e_k \cdot \nabla_{e_k} \cdot) \\ &= \sum_{j,k} e_j e_k \cdot \nabla_{e_j}(\nabla_{e_k} \cdot) \\ &= \sum_{j,k} e_j e_k \cdot \nabla^2_{e_j, e_k} \\ &= -\sum_j \nabla^2_{e_j, e_j} + \sum_{j<k} e_j e_k \cdot (\nabla^2_{e_j, e_k} - \nabla^2_{e_k, e_j}) \\ &= \nabla^*\nabla + \mathcal{R}. \end{aligned}$$

□

Cor. *Let $\Delta = dd^* + d^*d$ be the Hodge Laplacian and $\nabla^*\nabla$ be the connection Laplacian on T^*M. Then*

$$\Delta = \nabla^*\nabla + Ric$$

This follows from the previous thm for $D = d + d^*$, which acts on $Cl(M) \cong \Lambda T^*M$. The computation of $\mathcal{R}$ in this case follows the same lines as the proof of the implication

$$\nabla\psi = 0 \quad \Longrightarrow \quad Ric(w) \cdot \psi = 0.$$

[LM. Cor. II.8.3].

Theorem (Bochner)

$$Ric > 0 \quad \Longrightarrow \quad b_1(M) = 0.$$

Theorem (Lichnerowicz)

Let M be spin and suppose S is a spinor bundle. Then

$$D^2 = \nabla^*\nabla + \tfrac{s}{4}\,,$$

where s is the scalar curvature.

Proof. [LM. Thm. II.8.8]. □

Cor.

$$s > 0 \quad \Longrightarrow \quad \operatorname{Ker} D = 0.$$

Theorem (Hitchin)

In every dimension $n > 8$, $n \equiv 1 \pmod 8$ or $n \equiv 2 \pmod 8$, there exist compact mflds, which are homeomorpic to S^n, but which do not admit any Riemannian metric with $s > 0$.