BILINEAR FORMS OF DIMENSION ≤ 5 AND FUNCTION FIELDS OF QUADRICS IN CHARACTERISTIC 2

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ABSTRACT. The aim of this paper is to give a complete answer to the isotropy of bilinear forms of dimension ≤ 5 over the function field of a quadric in characteristic 2.

1. INTRODUCTION

Let F be a commutative field. We denote by $F(\psi)$ the function field of the affine quadric given by an F-quadratic form ψ. The isotropy problem over function fields of quadrics consists in classifying F-quadratic forms ψ for which a given anisotropic F-quadratic form φ becomes isotropic over $F(\psi)$. When F is of characteristic 2, a similar problem can be formulated for bilinear forms. This problem was considered first in the case where F is of characteristic not 2 and φ is of dimension ≤ 8. Many persons contributed in this case: Shapiro [31], Leep [28], Merkurjev [26], Hoffmann[3, 4, 6], the first author [15, 16, 17], Karpenko and Izhboldin [10, 11, 12, 13] (probably our list is not exhaustive neither for the results proved about this problem in characteristic not 2, nor for the authors contributed to this problem). It is the work of Merkurjev in the nineties on the u-invariant [26, 29] which gave a revival to this problem. In characteristic 2, this problem was considered by the first author [18] and Faivre [2] for some quadratic forms φ of dimension ≤ 8. Also, Hoffmann in characteristic not 2, and Hoffmann jointly with the first author in characteristic 2, proved an important result on the isotropy problem: If φ and ψ are anisotropic quadratic forms such that $\dim\varphi \leq 2^n < \dim\psi$ for some integer $n \geq 1$, then φ remains anisotropic over $F(\psi)$ [5, 9].

¿From now on, we suppose that F is of characteristic 2. In this paper we are interested in the isotropy problem for bilinear forms. Recall that, to a given bilinear form B with underlying vector space V, we associate the quadratic form $\widetilde{B}$ defined over V by: $\widetilde{B}(v) = B(v,v)$ for any $v \in V$. This quadratic form is uniquely determined by B, and it is diagonal, which

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means that $\widetilde{B}$ is given by a polynomial $\sum_{i=1}^{\dim B} a_i x_i^2$ for some $a_1, \cdots, a_n \in F$, where $\dim B$ denotes the dimension of B. We say that B is isotropic when $\widetilde{B}$ is isotropic. The function field of B, denoted by $F(B)$, is defined to be the function field of the affine quadric given by $\widetilde{B}$. It is well-known that an anisotropic diagonal quadratic form remains anisotropic over the function field of a non-diagonal quadratic form [18, Cor. 3.3]. Hence, we will restrict our attention to the isotropy of bilinear forms over function fields of diagonal quadratic forms. Recently, the first author gave a complete answer to the isotropy of an Albert bilinear form over the function field of a quadric [24, Th. 1.1] (an Albert bilinear form is a bilinear form of dimension 6 and trivial determinant). Before this, the two authors studied the isotropy of an Albert bilinear form over the function field of a conic [25, Prop. 1.5].

Our aim in this paper is to treat the isotropy of bilinear forms of dimension ≤ 5 over the function field of a quadric. An important ingredient that we will use is the norm degree introduced in [8, Section 8]. This notion will be of great interest in the formulation of our results. Recall that the norm field of a nonzero diagonal quadratic form φ, denoted by $N_F(\varphi)$, is the field $F^2(\alpha\beta \mid \alpha, \beta \in D_F(\varphi))$, where $D_F(\varphi)$ is the set of nonzero scalars represented by φ. The norm degree of φ, denoted by $\operatorname{ndeg}_F(\varphi)$, is the integer $[N_F(\varphi) : F^2]$. The norm degree gives a criterion for an anisotropic diagonal quadratic form being a quasi-Pfister neighbor (see below). Recall that a quasi-Pfister form is a quadratic form isometric to $\widetilde{\pi}$ for some bilinear Pfister form π. An anisotropic diagonal quadratic form φ is called quasi-Pfister neighbor if it is similar to a subform of a quasi-Pfister form δ and $2\dim\varphi > \dim\delta$, in which case the form δ is uniquely determined by φ. If φ is an anisotropic diagonal quadratic form, then φ is a quasi-Pfister neighbor iff $2\dim\varphi > \operatorname{ndeg}_F(\varphi)$ [8, Prop. 8.9(ii)]. In our case, if B is an anisotropic bilinear form of dimension ≤ 5, then we have:

$$\operatorname{ndeg}_F(\widetilde{B}) = \begin{cases} 2 & \text{if } \dim B = 2 \\ 4 & \text{if } \dim B = 3 \\ 4 \text{ or } 8 & \text{if } \dim B = 4 \\ 8 \text{ or } 16 & \text{if } \dim B = 5. \end{cases}$$

Hence, by the criterion above, $\widetilde{B}$ is a quasi-Pfister form in the following cases: $\dim B = 2$ or 3, or $\dim B = 4$ and $\operatorname{ndeg}_F(\widetilde{B}) = 4$, or $\dim B = 5$ and $\operatorname{ndeg}_F(\widetilde{B}) = 8$. The isotropy in all these cases is given by the following general result:

Proposition 1.1. *(We combine* [8, Prop. 8.9(iii)] *and* [19, Prop. 2.4]*) Let φ be a quasi-Pfister neighbor of a quasi-Pfister form δ, and let ψ be an* *anisotropic diagonal quadratic form. Then, $\varphi_{F(\psi)}$ is isotropic iff ψ is similar to a subform of δ.*

For the other cases, where $\widetilde{B}$ is not a quasi-Pfister neighbor, we will prove the following theorem which is our main result:

Theorem 1.2. *Let B be an anisotropic bilinear form of dimension 3, 4 or 5. Let ψ be an anisotropic diagonal quadratic form of dimension ≥ 2. Assume that the following conditions hold:*
– $\mathrm{ndeg}_F(\widetilde{B}) = 8$ *if* $\dim B = 4$.
– $\mathrm{ndeg}_F(\widetilde{B}) = 16$ *if* $\dim B = 5$.
(1) Suppose that $\mathrm{ndeg}_F(\psi) = 8$ *if* $\dim \psi = 4$. *Then, B is isotropic over $F(\psi)$ iff ψ is similar to a subform of $\widetilde{B}$.*
(2) Suppose that $\dim \psi = 4$ *and* $\mathrm{ndeg}_F(\psi) = 4$, *and let ψ' be any subform of ψ of dimension 3. Then, B is isotropic over $F(\psi)$ iff ψ' is similar to a subform of $\widetilde{B}$.*

Note that in this theorem we include the case $\dim B = 3$ even if $\widetilde{B}$ is a quasi-Pfister neighbor. The reason is that, in this case, the answer given by the theorem refines that given by Proposition 1.1.

The proof of Theorem 1.2 will be given case by case as follows. In Section 3, we study the isotropy of 5-dimensional bilinear forms. To this end, our arguments will be based on the result [24, Th. 1.1] about the isotropy of Albert bilinear forms. Moreover, in this case, we will be inspired by the method used by Hoffmann in characteristic not 2 for the isotropy of 5-dimensional quadratic forms [3]. However, because we work with bilinear forms in characteristic 2, many details in our proof will differ from those used by Hoffmann in [3]. One of them is our use of the norm field which will play the rôle of the Clifford algebra in characteristic not 2. In Section 4, we study the isotropy of 4-dimensional bilinear forms. This will be again based on [24, Th. 1.1]. More precisely, starting from our anisotropic bilinear form B of dimension 4, we will consider the Albert bilinear form $B' = B \perp t\,\langle 1, d\rangle$ over the rational function field $F(t)$, where d is the determinant of B. Clearly, if ψ is a diagonal quadratic form (as in statement (1) of Theorem 1.2) such that $B_{F(\psi)}$ is isotropic, then $B'_{F(t)(\psi)}$ is also isotropic. By combining a specialization result (Proposition 2.5) with others, we will be able to derive from the isotropy of B' over $F(t)(\psi)$ that ψ is similar to a subform of $\widetilde{B}$. Finally, in Section 5, we study the isotropy of 3-dimensional bilinear forms. The idea is to introduce the bilinear form $B' = B \perp \langle t\rangle$ over $F(t)$, and then proceed as for the isotropy of 4-dimensional bilinear forms.

2. Preliminary results

All bilinear forms considered in this paper are supposed to be regular of finite dimension.

For two quadratic (or bilinear) forms φ and ψ, we say that ψ is a subform of φ, denoted by $\psi \subset \varphi$, if there exists a form ψ' such that $\varphi \simeq \psi \perp \psi'$, where $\simeq$ and $\perp$ denote the isometry and the orthogonal sum of forms. We write $\psi \prec \varphi$ when $\alpha\psi \subset \varphi$ for some scalar $\alpha \in F^* := F \setminus \{0\}$.

For $a_1, \cdots, a_n \in F$, let $\langle a_1, \cdots, a_n \rangle$ denote the diagonal quadratic form $a_1x_1^2 + \cdots + a_nx_n^2$. If moreover, $a_i \neq 0$ for $1 \leq i \leq n$, we denote by $\langle a_1, \cdots, a_n \rangle_b$ the bilinear form $a_1x_1y_1 + \cdots + a_nx_ny_n$.

A metabolic plane is a 2-dimensional isotropic bilinear form. An orthogonal sum of metabolic planes is called a metabolic form. Any bilinear form B decomposes as follows: $B \simeq B_{an} \perp M$, where M is a metabolic form, and B_{an} is an anisotropic form which is unique [14, 30]. We call B_{an} the anisotropic part of B. In general, the form M is not unique. For two bilinear forms B and C, we write $B \sim C$ if $B_{\mathrm{an}} \simeq C_{an}$. In particular, $B \sim 0$ means that B is metabolic. Any diagonal quadratic form φ decomposes as follows: $\varphi \simeq \varphi_{an} \perp r \times \langle 0 \rangle$, where φ_{an} is an anisotropic quadratic form which is unique [8, Prop. 2.4]. Let $i_d(\varphi)$ denote the integer r.

For $a_1, \cdots, a_n \in F^*$, the bilinear form $\pi = \langle 1, a_1 \rangle_b \otimes \cdots \otimes \langle 1, a_n \rangle_b$ is called an n-fold bilinear Pfister form, we denote it by $\langle\langle a_1, \cdots, a_n \rangle\rangle_b$. In this case, the quasi-Pfister form $\widetilde{\pi}$ is denoted by $\langle\langle a_1, \cdots, a_n \rangle\rangle$. The bilinear form π' satisfying $\pi \simeq \langle 1 \rangle_b \perp \pi'$ is unique [1, Cor. 2.18, page 101], called the pure part of π.

Recall that a bilinear Pfister form B is isotropic iff it is metabolic [20, Prop. 3.3], and B is multiplicative, i.e., $B \simeq \alpha B$ for any $\alpha \in D_F(B)$ [1, Cor. 2.16, page 101].

Let IF denote the ideal of the Witt ring $W(F)$ of even dimensional bilinear forms. For any integer $n \geq 1$, let I^nF denote the n-th power of IF. The ideal I^nF is additively generated by n-fold bilinear Pfister forms. An important property about the forms in I^nF, called the Arason-Pfister Hauptsatz, asserts the following: Any anisotropic bilinear form B in I^nF satisfies $\dim B \geq 2^n$ [22, Lem. 4.8].

Lemma 2.1. *Let π_1, π_2 be anisotropic n-fold bilinear Pfister forms, and $\alpha_1, \alpha_2 \in F^*$, such that $\alpha_1\pi_1 \perp \alpha_2\pi_2 \in I^{n+1}F$. Then, $\pi_1 \simeq \pi_2$.*

Proof. Since $\pi_i \equiv \alpha_i\pi_i \pmod{I^{n+1}F}$, $i = 1, 2$, the condition $\alpha_1\pi_1 \perp \alpha_2\pi_2 \in I^{n+1}F$ implies that $\pi_1 \perp \pi_2 \in I^{n+1}F$. Since $\dim(\pi_1 \perp \pi_2)_{\mathrm{an}} < 2^{n+1}$, it follows, from the Arason-Pfister Hauptsatz, that $\pi_1 \sim \pi_2$. By the uniqueness of the anisotropic part, we conclude that $\pi_1 \simeq \pi_2$. □

The following two lemmas are well-known and easy to prove:

Lemma 2.2. *For any $a, b \in F$, the quadratic form $\langle a, b\rangle$ is isometric to $\langle a, a+b\rangle$.*

Lemma 2.3. *Let B_1, B_2 be anisotropic bilinear forms over F, and t a variable over F. Then, the bilinear form $B_1 \perp tB_2$ is anisotropic over $F(t)$.*

We recall an important result on Witt kernels for bilinear forms:

Theorem 2.4. *(*[20, Th. 1.2]*) Let B be an anisotropic bilinear form and φ be an anisotropic diagonal quadratic form. Suppose that $N_F(\varphi) = F^2(a_1, \cdots, a_n)$ and $\mathrm{ndeg}_F(\varphi) = 2^n$. Let $\pi = \langle\langle a_1, \cdots, a_n\rangle\rangle_b$. Then, B becomes metabolic over $F(\varphi)$ iff $B \simeq \alpha_1 B_1 \perp \cdots \perp \alpha_r B_r$ for some scalars $\alpha_1, \cdots, \alpha_r \in F^*$, and n-fold bilinear Pfister forms $B_1, \cdots, B_r$ satisfying $\widetilde{B_i} \simeq \widetilde{\pi}$.*

2.1. **Specialization and substitution results.** The following specialization result will be of great interest in our proofs:

Proposition 2.5. *(*[24, Cor. 3.2]*) Let $R = F[t_1, \cdots, t_m]$ be the polynomial ring in the variables $t_1, \cdots, t_m$ over F. Let φ be a diagonal quadratic form over F, and $p \in R$ an irreducible polynomial. Let K and F_p denote the quotient fields of R and R/pR, respectively. Let $u_1, \cdots, u_n \in R$ be polynomials not divisible by p such that:*

– The K-quadratic form φ_K represents $pu_1, \cdots, pu_n$.

– The quadratic form $\langle \overline{u_1}, \cdots, \overline{u_n}\rangle$ is anisotropic over F_p, where $\overline{u_i}$ denotes the class of u_i in F_p, $1 \leq i \leq n$.

Then, $i_d(\varphi_{F_p}) \geq n$.

We will also need the following representation result:

Proposition 2.6. *(*[24, Cor. 3.5]*) Let φ and $\psi = \langle a_1, \cdots, a_n\rangle$ be diagonal quadratic forms over F, and $b \in F^*$ ($n \geq 2$). Suppose that, over $F(t)$, we have $b\,\langle a_1t^2 + a_2, a_3, \cdots, a_n\rangle \subset \varphi_{F(t)}$. Then, $b\,\langle a_1, a_2, a_3, \cdots, a_n\rangle \subset \varphi$.*

As in [24], Proposition 2.6 is a corollary of the following substitution result:

Proposition 2.7. *(*[21, Cor. 2.4]*) Let φ be an anisotropic diagonal quadratic form over F. Let $p \in F[t_1, \cdots, t_n]$ be a polynomial which is represented by φ over the rational function field $F(t_1, \cdots, t_n)$. If $c = (c_1, \cdots, c_n) \in F^n$ satisfies $p(c) \neq 0$, then $p(c) \in D_F(\varphi)$.*

Sometimes we need to lift an isometry between two diagonal quadratic forms to an isometry between two bilinear forms associated to them. To do so, we use the following result:

Lemma 2.8. *(*[24, Lem. 3.7]*) Let B be a bilinear form over F, and $\psi = \langle a_1, \cdots, a_n\rangle$ an anisotropic diagonal quadratic form over F. Then, the following statements are equivalent:*
(1) $\psi \subset \widetilde{B}$.
(2) $\langle b_1, \cdots, b_n\rangle_b \subset B$, where, for any $1 \le i \le n$,

$$b_i \equiv a_i \mod D_F(\langle a_1, \cdots, a_{i-1}\rangle)$$

(read $b_1 = a_1$).

As a corollary, we get:

Corollary 2.9. *Let B be a bilinear form over F, and ψ an anisotropic diagonal quadratic form over F. Then, the following statements are equivalent:*
(1) $\psi \subset \widetilde{B}$.
(2) There exists a bilinear form C satisfying: $\widetilde{C} \simeq \psi$ and $C \subset B$.

Proof. Put $\psi = \langle a_1, \cdots, a_n\rangle$, and let $C = \langle b_1, \cdots, b_n\rangle_b$ with the scalars $b_1, \cdots, b_n$ as given in Lemma 2.8. By Lemma 2.2, we have $\langle b_1, b_2\rangle \simeq \langle a_1, a_2\rangle$, which gives $\langle b_1, b_2, b_3\rangle \simeq \langle a_1, a_2, b_3\rangle \simeq \langle a_1, a_2, a_3\rangle$, and so on, we conclude that $\widetilde{C} \simeq \psi$. □

2.2. **Results on the norm degree.** One of the important facts needed is the following result, describing the behavior of the norm degree after scalar extension to the function field of an affine hypersurface:

Proposition 2.10. *(*[24, Prop. 3.6]*) Let $p \in F[t_1, \cdots, t_n]$ be an irreducible polynomial, and φ a diagonal F-quadratic form. Let $F(p)$ be the quotient field of $F[t_1, \cdots, t_n]/(p)$. If $\mathrm{ndeg}_{F(p)}(\varphi_{F(p)}) < \mathrm{ndeg}_F(\varphi)$, then p is inseparable, and $\mathrm{ndeg}_{F(p)}(\varphi_{F(p)}) = \frac{1}{2}\mathrm{ndeg}_F(\varphi)$. ($p$ inseparable means that $\partial p/\partial t_i = 0$ for any i.)*

We deduce a general corollary:

Corollary 2.11. *Let $n \ge 1$ be an integer, and $\psi = \langle 1, a_1, \cdots, a_m\rangle$ an anisotropic diagonal quadratic form over F with $m > 2^n$. Let $p \in F[t]$ be an irreducible polynomial. Then, there exist $n+1$ elements $c_1, \cdots, c_{n+1}$ of the set $\{t^2 + a_1, a_2, a_3, \cdots, a_m\}$ such that the diagonal quadratic form $\langle \overline{c_1}, \cdots, \overline{c_{n+1}}\rangle$ is anisotropic over $F(p)$, where $\bar{c}$ denotes the class of c in $F(p)$.*

Proof. Let $\psi' = \langle t^2 + a_1, a_2, a_3, \cdots, a_m\rangle$ and $\psi'' = \langle a_2, \cdots, a_m\rangle$. These forms are anisotropic. Since $\dim \psi'' = m - 1 \ge 2^n$, we get $\mathrm{ndeg}_F(\psi'') \ge 2^n$. It follows from Proposition 2.10 that $\mathrm{ndeg}_{F(p)}(\psi''_{F(p)}) \ge 2^{n-1}$. Hence, there exist n elements $\epsilon_1, \cdots, \epsilon_n \in \{a_2, \cdots, a_m\}$ such that the quadratic form $\langle \overline{\epsilon_1}, \cdots, \overline{\epsilon_n}\rangle$ is anisotropic over $F(p)$.

Now we are able to prove the corollary. Suppose that for any $n+1$ elements $c_1, \cdots, c_{n+1} \in \{t^2+a_1, a_2, a_3, \cdots, a_m\}$, the quadratic form $\langle \overline{c_1}, \cdots, \overline{c_{n+1}} \rangle$ is isotropic over $F(p)$. Hence, $\dim(\psi'_{F(p)})_{\text{an}} \leq n$. Moreover, we have found elements $\epsilon_1, \cdots, \epsilon_n \in \{a_2, \cdots, a_m\}$ such that $\langle \overline{\epsilon_1}, \cdots, \overline{\epsilon_n} \rangle$ is anisotropic over $F(p)$. Consequently, we have $(\psi'_{F(p)})_{\text{an}} \simeq \langle \overline{\epsilon_1}, \cdots, \overline{\epsilon_n} \rangle$. This implies that any element $\alpha \in \{t^2+a_1, a_2, a_3, \cdots, a_m\}$ satisfies $\overline{\alpha} \in D_{F(p)}(\langle \overline{\epsilon_1}, \cdots, \overline{\epsilon_n} \rangle)$. Hence, over $F(t)$, this implies that any $\alpha \in \{t^2+a_1, a_2, a_3, \cdots, a_m\}$ is represented by $\langle \epsilon_1, \cdots, \epsilon_n, p \rangle$. Then, $N_{F(t)}(\psi') \subset N_{F(t)}(\langle \epsilon_1, \cdots, \epsilon_n, p \rangle)$, and thus, $\mathrm{ndeg}_{F(t)}(\psi') \leq 2^n$, which yields a contradiction because $\dim \psi' = m > 2^n$. □

2.3. **On the similarity of 4-dimensional bilinear forms.** It is well-known in characteristic not 2 that two 4-dimensional quadratic forms having the same determinant are similar iff they are similar over the quadratic extension given by their determinant. In characteristic 2 for bilinear forms, the situation is different as it is shown by the following proposition:

Proposition 2.12. *([23, Prop. 2.5]) Let $B = \langle k, l, kl, d \rangle_b$ and $C = \langle p, q, pq, d \rangle_b$ be two anisotropic bilinear forms over F of dimension 4 having the same determinant d. Suppose that $\mathrm{ndeg}_F(\widetilde{B}) = 8$ and $\langle k, l, kl, p, q, pq \rangle_b$ is isotropic. Then, B and C are similar over $F(\sqrt{d})$ if and only if there exists $x \in F$ such that $\langle k, l, kl, d+x^2 \rangle_b$ is similar to $\langle p, q, pq, d+x^2 \rangle_b$.*

As a corollary, we get:

Corollary 2.13. *Let $B = \langle k, l, kl, d \rangle_b$ and $C = \langle p, q, pq, d \rangle_b$ be two anisotropic bilinear forms over F of dimension 4 having the same determinant d. Suppose that $\mathrm{ndeg}_F(\widetilde{B}) = 8$ and $F^2(k, l) = F^2(p, q)$. If B and C are similar over $F(\sqrt{d})$, then $\langle\langle k, l \rangle\rangle_b \simeq \langle\langle p, q \rangle\rangle_b$. In particular, $\langle k, l, kl \rangle_b \simeq \langle p, q, pq \rangle_b$.*

Proof. The assumption $F^2(k, l) = F^2(p, q)$ implies that $\langle\langle k, l \rangle\rangle \simeq \langle\langle p, q \rangle\rangle$. Also, it implies that $N_F(\langle k, l, kl, p, q, pq \rangle) = F^2(k, l)$, and thus $\langle k, l, kl, p, q, pq \rangle_b$ is isotropic. By Proposition 2.12, there exists $x, y \in F$, $y \neq 0$, such that $\langle k, l, kl, d+x^2 \rangle_b \simeq y \langle p, q, pq, d+x^2 \rangle_b$. In particular,

$$\langle\langle k, l \rangle\rangle_b \sim y \langle\langle p, q \rangle\rangle_b \perp \left\langle\left\langle y, d+x^2 \right\rangle\right\rangle_b. \tag{1}$$

Consequently, $\langle\langle y, d+x^2 \rangle\rangle_b$ is metabolic over $F(\langle\langle k, l \rangle\rangle)$. Since $d \notin F^2(k, l)$ because $\mathrm{ndeg}_F(\widetilde{B}) = 8$, it follows that $\langle\langle y, d+x^2 \rangle\rangle_b \sim 0$. Hence, we conclude by (1) that $\langle\langle k, l \rangle\rangle_b \simeq \langle\langle p, q \rangle\rangle_b$. By the uniqueness of the pure part of a bilinear Pfister form, we get the isometry $\langle k, l, kl \rangle_b \simeq \langle p, q, pq \rangle_b$. □

2.4. **Other results needed in the proofs.** In this subsection, we give two results that we will need for the isotropy of 5-dimensional bilinear forms over function fields of 4-dimensional quadratic forms.

Lemma 2.14. *Let $\psi = \langle k, l, kl, d\rangle$ be an anisotropic quadratic form of norm degree 8. Let $\eta, \beta \in F^*$ be such that $\eta \in F^2(k,l) - F^2$, and the quadratic form $\langle\langle k, l\rangle\rangle \perp d\langle 1, \eta\rangle \perp \langle\beta\rangle$ is isotropic over F. Then, $\beta \in F^2(k, l, d)$.*

Proof. The assumption $\mathrm{ndeg}_F(\langle k, l, kl, d\rangle) = 8$ implies that the quasi-Pfister form $\langle\langle k, l, d\rangle\rangle$ is anisotropic. Since $\eta \notin F^{*2}$, the form $\langle 1, \eta\rangle$ is anisotropic, and thus $\langle 1, \eta\rangle \subset \langle\langle k, l\rangle\rangle$ because $\eta \in D_F(\langle\langle k, l\rangle\rangle)$. Hence, $\langle\langle k, l\rangle\rangle \perp d\langle 1, \eta\rangle \subset \langle\langle k, l, d\rangle\rangle$, in particular, $\langle\langle k, l\rangle\rangle \perp d\langle 1, \eta\rangle$ is anisotropic. Hence, the isotropy of $\langle\langle k, l\rangle\rangle \perp d\langle 1, \eta\rangle \perp \langle\beta\rangle$ implies that $\beta \in D_F(d\langle 1, \eta\rangle \perp \langle\langle k, l\rangle\rangle) \subset F^2(k, l, d)$. ☐

Proposition 2.15. *Let us consider an isometry between two anisotropic Albert bilinear form*

$$e\alpha\langle k, l, kl, d\rangle_b \perp f\langle 1, d\rangle_b \simeq e'\alpha'\langle p, q, pq, e'r, e's, rs\rangle_b. \qquad (\star)$$

Moreover, assume the following conditions:

- $F^2(k, l) = F^2(p, q)$.
- $\mathrm{ndeg}_F(\langle k, l, kl, d\rangle) = 8$.
- *The diagonal quadratic form of the Albert bilinear form in $(\star)$ has norm degree 16.*

Then, the following statements hold:
(1) $\langle\langle k, l\rangle\rangle_b \simeq \langle\langle p, q\rangle\rangle_b$ and $\langle k, l, kl\rangle_b \simeq \langle p, q, pq\rangle_b$.
(2) There exists $u_0, v_0, w_0 \in F^*$ such that:

- $\langle d, ef\alpha\rangle_b \simeq \langle u_0, v_0\rangle_b$ *and* $\langle e'r, e's\rangle_b \simeq \langle u_0, w_0\rangle_b$.
- $\langle 1, u_0\rangle_b \simeq \langle v_0w_0, u_0v_0w_0\rangle_b$.

Proof. Let B' denote the Albert bilinear form in $(\star)$. We have

$$N_F(\widetilde{B'}) = F^2(k, l, d, ef\alpha) = F^2(p, q, e'r, e's) \qquad (2)$$

(1) Clearly, from the isometry in $(\star)$, we get the relation:

$$e\alpha\langle\langle k, l\rangle\rangle_b \perp e\alpha\langle\langle d, ef\alpha\rangle\rangle_b \sim e'\alpha'\langle\langle p, q\rangle\rangle_b \perp e'\alpha'\langle\langle e'r, e's\rangle\rangle_b. \qquad (3)$$

Claim 1. $\langle\langle e'r, e's\rangle\rangle_b$ is metabolic over $F(\sqrt{d})$.

In fact, the assumption $F^2(k, l) = F^2(p, q)$ implies $\langle\langle k, l\rangle\rangle \simeq \langle\langle p, q\rangle\rangle$. Hence, after extending (3) to $F(\sqrt{d})(\langle\langle k, l\rangle\rangle)$, we deduce that $\langle\langle e'r, e's\rangle\rangle_b$ is metabolic over $F(\sqrt{d})(\langle\langle k, l\rangle\rangle)$. Since $\mathrm{ndeg}_F(\langle k, l, kl, d\rangle) = 8$, it follows that $\langle\langle k, l\rangle\rangle$ is anisotropic over $F(\sqrt{d})$. Thus, if $\langle\langle e'r, e's\rangle\rangle_b$ is anisotropic over $F(\sqrt{d})$, then $e'r, e's \in F^2(k, l, d) = F^2(p, q, d)$, which implies, by

(2), that $N_F(\widetilde{B'}) \subset F^2(p,q,d)$, a contradiction to $\mathrm{ndeg}_F(\widetilde{B'}) = 16$. Hence, $\langle\langle e'r, e's\rangle\rangle_b$ is metabolic over $F(\sqrt{d})$.

Claim 2. $\langle\langle k,l\rangle\rangle_b \simeq \langle\langle p,q\rangle\rangle_b$ and $\langle k,l,kl\rangle_b \simeq \langle p,q,pq\rangle_b$.
We extend (3) to $F(\sqrt{d})$, and we use Claim 1 to get $(\langle\langle k,l\rangle\rangle_b)_{F(\sqrt{d})} \simeq (\langle\langle p,q\rangle\rangle_b)_{F(\sqrt{d})}$. In particular, $(\langle k,l,kl,d\rangle_b)_{F(\sqrt{d})} \simeq (\langle p,q,pq,d\rangle_b)_{F(\sqrt{d})}$. Since $\mathrm{ndeg}_F(\langle k,l,kl,d\rangle) = 8$ and $F^2(k,l) = F^2(p,q)$, we conclude, by Corollary 2.13, that $\langle\langle k,l\rangle\rangle_b \simeq \langle\langle p,q\rangle\rangle_b$ and $\langle k,l,kl\rangle_b \simeq \langle p,q,pq\rangle_b$. This proves statement (1).

(2) By Claim 2 and relation (3), it is clear that

$$e\alpha\,\langle\langle d, ef\alpha\rangle\rangle_b \perp e'\alpha'\,\langle\langle e'r, e's\rangle\rangle_b \in I^3F.$$

Hence, by Lemma 2.1, $\langle\langle d, ef\alpha\rangle\rangle_b \simeq \langle\langle e'r, e's\rangle\rangle_b$. In particular, $\langle d, ef\alpha, e'r, e's\rangle_b$ is isotropic. Then there exists $u_0, v_0, w_0 \in F^*$ such that

$$\langle d, ef\alpha\rangle_b \simeq \langle u_0, v_0\rangle_b \ \langle e'r, e's\rangle_b \simeq \langle u_0, w_0\rangle_b\,.$$

Cleary, we have

$$\begin{aligned} 0 &\sim \langle\langle d, ef\alpha\rangle\rangle_b \perp \langle\langle e'r, e's\rangle\rangle_b \\ &\sim \langle\langle u_0, v_0\rangle\rangle_b \perp \langle\langle u_0, w_0\rangle\rangle_b \\ &\sim \langle v_0, u_0v_0, w_0, u_0w_0\rangle_b \\ &\sim v_0\,\langle 1, u_0, v_0w_0, u_0v_0w_0\rangle_b. \end{aligned}$$

Hence, $\langle 1, u_0, v_0w_0, u_0v_0w_0\rangle_b$ is metabolic, which implies $\langle 1, u_0\rangle_b \simeq \langle v_0w_0, u_0v_0w_0\rangle_b$. This proves statement (2). □

3. Proof of Theorem 1.2: Case $\dim B = 5$

Let B be an anisotropic bilinear form of dimension 5 and determinant d such that $\mathrm{ndeg}_F(\widetilde{B}) = 16$. We introduce the Albert bilinear form $B' = B \perp \langle d\rangle_b$. Since $N_F(\widetilde{B}) = N_F(\widetilde{B'})$, it follows that $\mathrm{ndeg}_F(\widetilde{B'}) = 16$, and thus B' is anisotropic. Let ψ be an anisotropic diagonal quadratic form of dimension ≥ 3.

(1) Suppose that $\mathrm{ndeg}_F(\psi) = 8$ **if** $\dim\psi = 4$**.** It is clear that $B_{F(\psi)}$ is isotropic if ψ is similar to a subform of $\widetilde{B}$. Conversely, suppose that $B_{F(\psi)}$ is isotropic. Then, $B'_{F(\psi)}$ is also isotropic. By [24, Th. 1.1], $\alpha\psi \subset \widetilde{B'}$ for a suitable $\alpha \in F^*$. By Corollary 2.9, there exists a bilinear form C such that $\widetilde{C} \simeq \psi$ and $\alpha C \subset B'$. We continue our proof case by case.

(a) Case $\dim\psi = 3$**.** There exist scalars $e,k,l,x,y \in F^*$ such that $C \simeq e\,\langle k,l,kl\rangle_b$ and $B' \simeq e\alpha\,\langle k,l,kl,x,y,xy\rangle_b$. Then

$$B \perp d\,\langle x,y,xy\rangle_b \sim e\alpha\,\langle\langle k,l\rangle\rangle_b \perp e\alpha\,\langle\langle x,y,de\alpha\rangle\rangle_b\,. \tag{4}$$

Since $\widetilde{C} \simeq \psi$, the form $\langle\langle k, l\rangle\rangle_b$ is isotropic over $F(\psi)$, and $N_F(\psi) = F^2(k, l)$. If we extend (4) to $F(\psi)$, we get

$$(B \perp d\langle x, y, xy\rangle_b)_{F(\psi)} \in I^3F(\psi).$$

Since $B_{F(\psi)}$ is isotropic, it follows from the Arason-Pfister Hauptsatz that

$$(B \perp d\langle x, y, xy\rangle_b)_{F(\psi)} \sim 0.$$

By using Theorem 2.4, we get two possibilities:

• Either $\dim(B \perp d\langle x, y, xy\rangle_b)_{\text{an}} = 8$: In this case, there exist two 2-fold bilinear Pfister forms π_1, π_2, and $\alpha_1, \alpha_2 \in F^*$ such that $\widetilde{\pi_1} \simeq \widetilde{\pi_2} \simeq \langle\langle k, l\rangle\rangle$, and

$$B \perp d\langle x, y, xy\rangle_b \simeq \alpha_1\pi_1 \perp \alpha_2\pi_2.$$

Hence,

$$N_F(\widetilde{B}) \subset N_F(\alpha_1\widetilde{\pi_1} \perp \alpha_2\widetilde{\pi_2}) = F^2(k, l, \alpha_1\alpha_2),$$

and thus, $\operatorname{ndeg}_F(\widetilde{B}) \leq 8$, a contradiction.

• Or $\dim(B \perp d\langle x, y, xy\rangle_b)_{\text{an}} = 4$: In this case, there exists $\alpha \in F^*$, and a 2-fold bilinear Pfister form $\pi = \langle 1\rangle_b \perp \pi'$ such that $\widetilde{\pi} \simeq \langle\langle k, l\rangle\rangle$, and

$$(B \perp d\langle x, y, xy\rangle_b)_{\text{an}} \simeq \alpha\pi.$$

Hence, $B \sim \alpha\pi \perp d\langle x, y, xy\rangle_b$. By the multiplicativity of π, we conclude that $\pi' \prec B$. In particular, $\widetilde{\pi}' \prec \widetilde{B}$. Moreover, $\pi_{F(\psi)} \sim 0$ because $\widetilde{\pi} \simeq \langle\langle k, l\rangle\rangle$. Then, $\pi'_{F(\psi)}$ is isotropic. By Theorem 1.2, in dimension 3, we conclude that $\widetilde{\pi}'$ is similar to ψ, hence $\psi \prec \widetilde{B}$.

(b) Case $\dim\psi = 4$**.** Let $e, k, l, d \in F^*$ be such that $C \simeq e\langle k, l, kl, d\rangle_b$. Hence, $B' \simeq e\alpha\langle k, l, kl, d\rangle_b \perp f\langle 1, d\rangle_b$ for a suitable $f \in F^*$. We have

$$N_F(\psi) = F^2(k, l, d).$$

Let $\psi' = \langle k, l, kl\rangle$. Since $B_{F(\psi)}$ and $\psi_{F(\psi')}$ are isotropic, it follows from [7, Cor. 7.20] that $B_{F(\psi')}$ is isotropic. By the isotropy in dimension 3 (treated in the case (a)), there exists $\alpha' \in F^*$ such that $\alpha'\psi' \subset \widetilde{B}$. By Corollary 2.9, there exists a 3-dimensional bilinear form C' such that $\widetilde{C'} \simeq \psi'$ and $\alpha'C' \subset B$. Set $C' = e'\langle p, q, pq\rangle_b$ for suitable $e', p, q \in F^*$. Hence, the condition $\widetilde{C'} \simeq \psi'$ implies

$$F^2(k, l) = F^2(p, q). \tag{5}$$

Let $r, s \in F^*$ be such that $B = e'\alpha'\langle p, q, pq, e'r, e's\rangle_b$. Then,

$$\begin{aligned} B' &\simeq e\alpha\langle k, l, kl, d\rangle_b \perp f\langle 1, d\rangle_b \\ &\simeq e'\alpha'\langle p, q, pq, e'r, e's, rs\rangle_b. \end{aligned} \tag{6}$$

Hence, using (5) and (6), we get

$$N_F(\widetilde{B'}) = F^2(k,l,d,ef\alpha) = F^2(k,l,e'r,e's). \tag{7}$$

Since all the hypotheses of Proposition 2.15 are satisfied, there exists $u_0, v_0, w_0 \in F^*$ such that

$$\begin{aligned} \langle d, ef\alpha\rangle_b &\simeq \langle u_0, v_0\rangle_b \\ \langle e'r, e's\rangle_b &\simeq \langle u_0, w_0\rangle_b, \end{aligned} \tag{8}$$

and

$$\langle 1, u_0\rangle_b \simeq \langle v_0w_0, u_0v_0w_0\rangle_b \tag{9}$$

Claim 1. $\langle\langle k,l,v_0w_0\rangle\rangle_b$ is metabolic over $F(\psi)$.
Let $\tau = \langle k,l,kl,d,ef\alpha\rangle_b = \langle k,l,kl,u_0,v_0\rangle_b$. Since $\langle k,l,kl\rangle_b \simeq \langle p,q,pq\rangle_b$ (Proposition 2.15), it follows from (8) that

$$B \simeq e'\alpha' \langle p,q,pq,e'r,e's\rangle_b \simeq e'\alpha' \langle k,l,kl,u_0,w_0\rangle_b . \tag{10}$$

By using (8) and (9), it is easy to verify that

$$\tau \perp e'\alpha' v_0w_0B \sim \langle\langle k,l,v_0w_0\rangle\rangle_b . \tag{11}$$

Since $\tau_{F(\psi)}$ is isotropic (because ψ is similar to $\langle k,l,kl,d\rangle$), and $B_{F(\psi)}$ is isotropic, we deduce, after comparing the dimensions of the anisotropic parts of both sides in (11), that $\langle\langle k,l,v_0w_0\rangle\rangle_b$ is metabolic over $F(\psi)$. □

Claim 2. $\psi \prec \widetilde{B}$.
To prove the claim, we distinguish between two cases:

(A) Suppose that $\langle\langle k,l,v_0w_0\rangle\rangle_b$ is isotropic Then, this form is metabolic, and by (11), τ is similar to B. Hence, $\psi \prec \widetilde{B}$ since $C \prec \tau$ and $\widetilde{C}$ is similar to ψ.

(B) Suppose that $\langle\langle k,l,v_0w_0\rangle\rangle_b$ is anisotropic Since $N_F(\psi) = F^2(k,l,d)$, it follows from Theorem 2.4, that

$$\langle\langle k,l,v_0w_0\rangle\rangle \simeq \langle\langle k,l,d\rangle\rangle . \tag{12}$$

Hence, there exists $m,n \in D_F(\langle\langle k,l\rangle\rangle) \cup \{0\}$ such that $v_0w_0 = m + dn$. Since $\langle\langle k,l,v_0w_0\rangle\rangle_b$ is anisotropic, we have $n \neq 0$. Let $\eta = mn^{-1}$ and $x = \eta + d$. It is clear that $\eta \in F^2(k,l)$, $x \in F^2(k,l,d)$, and

$$v_0w_0x \in D_F(\langle\langle k,l\rangle\rangle). \tag{13}$$

(B.1) Suppose that $\eta \notin F^{*2}$. We have the following isometries of diagonal quadratic forms:

$$
\begin{array}{rcl}
v_0w_0x\langle 1,1\rangle \perp e'\alpha' v_0w_0x\widetilde{B} & \simeq & \\
& \overset{(10)}{\simeq} & v_0w_0x\langle 1,1\rangle \perp v_0w_0x\langle k,l,kl,u_0,w_0\rangle \\
& \simeq & x\langle u_0v_0w_0, v_0, v_0w_0\rangle \perp v_0w_0x\langle\langle k,l\rangle\rangle \\
& \overset{(9)}{\simeq} & x\langle 1,u_0,v_0\rangle \perp v_0w_0x\langle\langle k,l\rangle\rangle \\
& \overset{(8),(13)}{\simeq} & x\langle 1,d,ef\alpha\rangle \perp \langle\langle k,l\rangle\rangle \\
& \simeq & \langle \eta+d, d\eta+d^2\rangle \perp \langle xef\alpha\rangle \perp \langle\langle k,l\rangle\rangle \\
& \overset{Lem.2.2}{\simeq} & \langle d, d\eta\rangle \perp \langle xef\alpha\rangle \perp \langle\langle k,l\rangle\rangle.
\end{array} \tag{14}
$$

Since the form in the first line in (14) is isotropic, we get by Lemma 2.14 that $xef\alpha \in F^2(k,l,d)$. In particular, $ef\alpha \in F^2(k,l,d)$. Then, by (7), $N_F(\widetilde{B}') \subset F^2(k,l,d)$, a contradiction to $\mathrm{ndeg}_F(\widetilde{B}') = 16$.

(B.2) Suppose that $\eta \in F^{*2}$. Then, $x \in D_F(\langle 1,d\rangle_b)$, i.e., $x\langle 1,d\rangle_b \simeq \langle 1,d\rangle_b$. We reproduce some isometries in (14) for bilinear forms instead of diagonal quadratic forms. We get:

$$
\begin{array}{rcl}
v_0w_0x\langle 1,1\rangle_b \perp e'\alpha v_0w_0xB & \simeq & \\
& \overset{(10)}{\simeq} & v_0w_0x\langle 1,1\rangle_b \perp v_0w_0x\langle k,l,kl,u_0,w_0\rangle_b \\
& \simeq & x\langle u_0v_0w_0, v_0, v_0w_0\rangle_b \perp v_0w_0x\langle\langle k,l\rangle\rangle_b \\
& \overset{(9)}{\simeq} & x\langle 1,u_0,v_0\rangle_b \perp v_0w_0x\langle\langle k,l\rangle\rangle_b \\
& \overset{(8),(13)}{\simeq} & x\langle 1,d,ef\alpha\rangle_b \perp \langle\langle k,l\rangle\rangle_b \\
& \simeq & \langle 1,d,xef\alpha\rangle \perp \langle\langle k,l\rangle\rangle_b \\
& \simeq & \langle 1,1\rangle_b \perp \langle k,l,kl,d,xef\alpha\rangle_b.
\end{array} \tag{15}
$$

By the uniqueness of the anisotropic part, we deduce that B is similar to $\langle k,l,kl,d,xef\alpha\rangle_b$, which implies that $\psi \prec \widetilde{B}$.

(c) Case $\dim\psi = 5$**.** We can write $C = e\langle k,l,kl,d,ef\alpha\rangle_b$ for suitable scalars $e,f,k,l,d \in F^*$. Hence, $B' \simeq e\alpha\langle k,l,kl,d\rangle_b \perp f\langle 1,d\rangle_b$. Let $\psi' = \langle k,l,kl\rangle$.

Since $B_{F(\psi')}$ is isotropic, there exists $\alpha' \in F^*$ such that $\alpha'\psi' \subset \widetilde{B}$. Let C' be a bilinear form of dimension 3 such that $\alpha' C' \subset \widetilde{B}$ and $\psi' \simeq \widetilde{C}'$. Set $C' = e'\langle p,q,pq\rangle_b$. Hence,

$$F^2(k,l) = F^2(p,q). \tag{16}$$

Let $r,s \in F^*$ be such that $B = e'\alpha'\langle p,q,pq,e'r,e's\rangle$. Then,

$$
\begin{array}{rcl}
B' & \simeq & e\alpha\langle k,l,kl,d\rangle_b \perp f\langle 1,d\rangle \\
& \simeq & e'\alpha'\langle p,q,pq,e'r,e's,rs\rangle.
\end{array} \tag{17}
$$

We have

$$N_F(\widetilde{B}') = F^2(k, l, d, ef\alpha) = F^2(k, l, e'r, e's).$$

Now, we continue with the same arguments as in the case **(b)** to find scalars $v_0, w_0 \in F^*$ such that $\pi := \langle\langle k, l, v_0w_0\rangle\rangle_b$ is isotropic over $F(\psi)$, and

$$\tau \perp e'\alpha' v_0 w_0 B \sim \pi \tag{18}$$

where $\tau = \langle k, l, kl, d, ef\alpha\rangle_b$. If π is anisotropic, then ψ is a quasi-Pfister neighbor of $\widetilde{\pi}$ (because $\pi_{F(\psi)}$ is isotropic), which implies that $B_{F(\pi)}$ is isotropic. In particular, $B'_{F(\pi)}$ is isotropic, this is not possible by [24, Th. 1.1]. Hence, π is isotropic, and by (18), B is similar to τ. It follows that $\psi \prec \widetilde{B}$ because $\psi \simeq \widetilde{C}$. □

(d) Case $\dim\psi = 6$**.** We may suppose $1 \in D_F(\psi)$. Set $\psi = \langle 1, a_1, a_2, a_3, a_4, a_5\rangle$, and let $\psi' := \langle t^2 + a_1, a_2, a_3, a_4, a_5\rangle$. The field $F(\psi)$ is isomorphic to $F(t)(\psi')$. Hence, $B_{F(t)}$ is isotropic over $F(t)(\psi')$. Thus, $\psi' \prec \widetilde{B}_{F(t)}$. Let $f \in F[t]$ be a nonzero, square free, polynomial such that $\widetilde{B}_{F(t)} \simeq f\,\langle t^2 + a_1, a_2, a_3, a_4, a_5\rangle$.

Claim. f is a constant polynomial.

Suppose that f is not constant, and let p be an irreducible factor of f. By using Corollary 2.11 in the case $m = 5$ and $n = 2$, we get three elements $c_1, c_2, c_3 \in \{t^2 + a_1, a_2, a_3, a_4, a_5\}$ such that $\langle \overline{c_1}, \overline{c_2}, \overline{c_3}\rangle$ is anisotropic over $F(p)$. But, since $f\,\langle c_1, c_2, c_3\rangle \subset \widetilde{B}_{F(t)}$, it follows from Proposition 2.5, that $i_d(\widetilde{B}_{F(p)}) \geq 3$. Consequently, $\mathrm{ndeg}_{F(p)}(\widetilde{B}_{F(p)}) \leq 2$, a contradiction to Proposition 2.10. Hence, f is constant, and thus Proposition 2.6 implies that $\psi \prec \widetilde{B}$, which is not possible by dimension account. Then, B remains anisotropic over $F(\psi)$.

(e) Case $\dim\psi > 6$**.** In this case, the form $B'_{F(\psi)}$ is anisotropic [24, Th. 1.1], and thus, $B_{F(\psi)}$ is also anisotropic.

(2) Suppose that $\dim\psi = 4$ **and** $\mathrm{ndeg}_F(\psi) = 4$**.** Let ψ' be any subform of ψ of dimension 3. If B is isotropic over $F(\psi)$, then B is also isotropic over $F(\psi')$ [7, Cor. 7.20]. It follows, from the case (a), that $\psi' \prec \widetilde{B}$. Conversely, suppose that $\psi' \prec \widetilde{B}$. Since ψ is similar to a quasi-Pfister form (because $\mathrm{ndeg}_F(\psi) = 4$), the form ψ' is isotropic over $F(\psi)$. Hence, $B_{F(\psi)}$ is also isotropic.

4. Proof of Theorem 1.2: Case $\dim B = 4$

Let B be an anisotropic bilinear form of dimension 4 and determinant d. Suppose that $\mathrm{ndeg}_F(\widetilde{B}) = 8$. Let $B' = B \perp t\,\langle 1, d\rangle_b$ over $F(t)$. Since $d \neq 1$ (because $\mathrm{ndeg}_F(\widetilde{B}) = 8$), the Albert bilinear form B' is anisotropic (Lemma 2.3). Moreover, $\mathrm{ndeg}_{F(t)}(\widetilde{B}') = 16$.

Let ψ be an anisotropic diagonal quadratic form of dimension $n \geq 3$ such that $B_{F(\psi)}$ is isotropic, and $\mathrm{ndeg}_F(\psi) = 8$ when $n = 4$.

Since $B'_{F(t)(\psi)}$ is isotropic, there exists $f \in F[t]$, a nonzero square free polynomial, and a diagonal quadratic form ψ' over $F(t)$ such that $\widetilde{B}' \simeq f\psi \perp \psi'$ [24, Th. 1.1].

Claim. f is not divisible by t.
Suppose that t divides f. We consider the t-adic valuation of $F(t)$. By Corollary 2.9, there exists a bilinear form C over $F(t)$ such that $B' \simeq fC \perp C'$ for some bilinear form C' over $F(t)$, and $\widetilde{C} \simeq \psi_{F(t)}$. Set $C = \langle a_1, \cdots, a_n \rangle_b$ for some $a_1, \cdots, a_n \in F[t]^*$. Since ψ is anisotropic over F, we deduce that all the polynomials $a_1, \cdots, a_n$ are units (for the t-adic valuation of $F(t)$). If we take the first residue form of B', we deduce that B is isotropic over F because $\dim C' \leq 3$, a contradiction.

Now, using the fact that f is a unit and taking the first residue form of B', we get that $f(0) \langle a_1(0), \cdots, a_n(0) \rangle_b \subset B$. In particular, $\langle a_1(0), \cdots, a_n(0) \rangle_b$ is anisotropic. Moreover, since a_i is represented by $\psi_{F(t)}$, $1 \leq i \leq n$, we deduce from Proposition 2.7 that $a_i(0) \in D_F(\psi)$, $1 \leq i \leq n$. Because the form $\langle a_1(0), \cdots, a_n(0) \rangle_b$ is anisotropic, we get that

$$\langle a_1(0), \cdots, a_n(0) \rangle \simeq \psi.$$

Hence, $\psi \prec \widetilde{B}$.

When $\dim \psi = 4$ and $\mathrm{ndeg}_F(\psi) = 4$, the same proof as for the case $\dim B = 5$ works. □

5. Proof of Theorem 1.2: Case $\dim B = 3$

Let B be an anisotropic bilinear forms of dimension 3. We consider the bilinear form $B' = B \perp \langle t \rangle$ over $F(t)$. It is clear that $\mathrm{ndeg}_{F(t)}(\widetilde{B}') = 8$. Let ψ be an anisotropic diagonal quadratic form of dimension ≥ 3 such that $B_{F(\psi)}$ is isotropic, and $\mathrm{ndeg}_F(\psi) = 8$ when $\dim \psi = 4$. Since $B'_{F(t)(\psi)}$ is isotropic, it follows from the isotropy in dimension 4 that $\widetilde{B}' \simeq f\psi \perp \psi'$ for some $f \in F[t]$, a nonzero square free polynomial, and a diagonal quadratic form ψ' over $F(t)$. We proceed as for the isotropy of 4-dimensional bilinear form to conclude that $\psi \prec \widetilde{B}$. □

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