In Search of New "Homology" Functors Having a Close Relationship to K-theory

by

R. Keith Dennis*

For any group G and G-module M one would like to obtain a sequence of functors $\tilde{H}_i(G,M)$ which are intimately related to K-theory. They should satisfy the following axioms:

I. $\tilde{H}_i(G,M)$ is a covariant functor which commutes with direct limits. There are natural surjective maps

$$\tilde{H}_i(G,M) \longrightarrow H_i(G,M) .$$

II. For any ring R there is a natural map

$$\tilde{H}_i(R^*,Z) \longrightarrow K_i(R) \qquad \text{(trivial action on Z)}$$

such that for all $n \geq 1$ the diagrams

$$\begin{array}{ccccc}
\tilde{H}_i(GL_n(R),Z) & \longrightarrow & K_i(M_n(R)) & \searrow^{\approx} & \\
\downarrow & & & & K_i(R) \\
\tilde{H}_i(GL_{n+1}(R),Z) & \longrightarrow & K_i(M_{n+1}(R)) & \nearrow_{\approx} &
\end{array}$$

commute.

III. For G an abelian group $\tilde{H}_*(G,Z)$ is equipped with a product making it into an anti-commutative graded ring such that the maps

$$\tilde{H}_*(G,Z) \longrightarrow H_*(G,Z)$$

and

$$\tilde{H}_*(R^*,Z) \longrightarrow K_*(R)$$

are homomorphisms of graded rings. Here R denotes a commutative ring.

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It is clear that one can take $\tilde{H}_i(G,M) = H_i(G,M)$ for $i = 0, 1$ in the above requirements (and meet no immediate conflicts). However, for $i = 2$ there is an obvious obstruction to using $\tilde{H}_2 = H_2$. Namely, the product in homology is strictly anti-commutative ($a \circ a = 0$) whereas in K-theory the product is only anti-commutative ($a \circ b = -b \circ a$). Thus no such map $H_2(R^*, Z) \longrightarrow K_2(R)$ exists (e.g., $-1 \star -1 \neq 1$ in $K_2(Z)$, but $-1 \cdot -1$ is trivial in $H_2(Z^*, Z)$). In case $i = 2$ we will show that this is the only obstruction and exhibit such a functor $\tilde{H}_2(G,Z)$.

In analogy with our result for $i = 2$ one might hope to strengthen I by requiring in addition

I'. There is an exact sequence

$$0 \longrightarrow t_i(G) \longrightarrow \tilde{H}_i(G,Z) \longrightarrow H_i(G,Z) \longrightarrow 0 .$$

Here $t_i(G)$ denotes a certain group of elements of order 2 which is defined in §1 below.

In general we do not know how to define functors $\tilde{H}_i(G,M)$ nor do we even know how to define $\tilde{H}_2(G,M)$ for arbitrary coefficients M. This note is written in the hope that it will inspire someone to do so. We now continue with some remarks that may or may not be relevant. Independent work of S. M. Gersten (unpublished) and J.-L. Loday (lecture at the K-theory conference in Oberwolfach, July 15, 1976) established the existence of sequences of the form

$$1 \longrightarrow H_3(G,Z) \longrightarrow X \longrightarrow F \longrightarrow G \longrightarrow 1$$

for groups G with $H_1G = H_2G = 0$. It is conceivable that one might be able to modify this to obtain a definition of $\tilde{H}_3$. In fact, the construction of Loday [5] or [6] for the product

$$K_1(R) \times K_1(R) \times K_1(R) \longrightarrow K_3(R)$$

is reminiscent of the construction of the pairing

$GL(R) \times GL(R) \longrightarrow St(R)$ used to define our map $\tilde{H}_2(GL(R),Z) \longrightarrow K_2(R)$. One might hope that his construction could be used to define a map $\tilde{H}_3(GL(R),Z) \longrightarrow K_3(R)$.

If it is possible to define such groups, one should also try to generalize to obtain groups $\tilde{H}_i(R,M)$ analogous to the Hochschild homology of a ring R with coefficients in an R-R-bimodule M. In this case one would presumably expect an exact sequence

$$1 \longrightarrow \tilde{H}_2(R,R) \longrightarrow K_2(R[\in]) \longrightarrow K_2(R) \longrightarrow 1$$

where $R[\in]$ denotes the dual numbers over the (not necessarily commutative) ring R.

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1. Preliminaries.

Let R denote a commutative ring and M an R-module. As usual $\bigwedge^* M$ denotes the exterior algebra on M : $\bigwedge^* M = T(M)/I(M)$ where $T(M)$ is the tensor algebra on M and $I(M)$ denotes the two-sided ideal of $T(M)$ generated by the elements $m \otimes m$ for all $m \in M$. In an analogous manner we define $\tilde{\bigwedge}^* M = T(M)/J(M)$ where $J(M)$ is the two-sided ideal of $T(M)$ generated by all elements of the form $m \otimes n + n \otimes m$ for $m,n \in M$. Clearly $J(M) \subset I(M)$ and there is a surjective homómorphism of graded algebras $\tilde{\bigwedge}^* M \longrightarrow \bigwedge^* M$. We define $t_*(M)$ to be the kernel. Note that $t_0(M) = t_1(M) = 0$.

If M is a free module with ordered basis $\{m_i\}$, then a basis for $\bigwedge^n M$ can be obtained as the set of all

$$m_{i_1} \wedge m_{i_2} \wedge \cdots \wedge m_{i_n}$$

where $(i_1, i_2, \ldots, i_n)$ ranges over all strictly increasing sets of indices of length n. If 2 is a unit, then $t_*(M) = 0$. If $\operatorname{char} R = 2$, then in a similar manner one can obtain a basis for $\tilde{\bigwedge}^n M$ as the set of all

$$m_{i_1} \tilde{\wedge} m_{i_2} \tilde{\wedge} \cdots \tilde{\wedge} m_{i_n}$$

$$(= \text{the coset of } m_{i_1} \otimes m_{i_2} \otimes \cdots \otimes m_{i_n})$$

where $(i_1, i_2, \ldots, i_n)$ ranges over all non-decreasing sets of n-tuples of indices.

LEMMA 1. <u>For any R-module M and any integer n, the sequence</u>

$$0 \longrightarrow t_n(M) \longrightarrow \tilde{\bigwedge}^n M \longrightarrow \bigwedge^n M \longrightarrow 0$$

<u>is exact. If $M/2M$ is a free $R/2R$-module, then the sequence splits (non-canonically) as a sequence of R-modules. Further, $t_n(M)$ is annihilated by 2. For $n = 2$ and $R = Z$ there is a canonical isomorphism</u>

$$M \otimes Z_2 \approx M/2M \approx t_2(M)$$

<u>induced by</u> $m \longmapsto m \tilde{\wedge} m$.

The sequence is clearly exact and as $2I(M) \subset J(M)$, we have $2t_n(M) = 0$. For $n \geq 1$ considering the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & t_n(M) & \longrightarrow & \tilde{\bigwedge}^n M & \longrightarrow & \bigwedge^n M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & t_n(M/2M) & \longrightarrow & \tilde{\bigwedge}^n (M/2M) & \longrightarrow & \bigwedge^n (M/2M) & \longrightarrow & 0 \end{array}$$

and noting that $t_n(M) \longrightarrow t_n(M/2M)$ is an isomorphism, we see that the bottom sequence (and hence the top sequence) splits as each module is free over $R/2R$. $t_n(M) \longrightarrow t_n(M/2M)$ is an isomorphism as each is a module over $R/2R$ and as one can find a spanning set of $t_n(M)$ which is mapped to a basis of the free $R/2R$-module $t_n(M/2M)$. The last statement is easy.

REMARK. Let R have characteristic 2. If M is a free module of rank r, then $\bigwedge^n M$ is free of rank $\binom{r}{n}$ and $\tilde{\bigwedge}^n M$ is free of rank $\binom{r+n-1}{n}$. Thus if $R = Z$ and M is a finitely generated abelian group, $t_n(M)$ is an $\mathbb{F}_2$ vector space of dimension $\binom{r+n-1}{n} - \binom{r}{n}$ where r is the $\mathbb{F}_2$ dimension of $M/2M$.

Finally for G any group, we define $t_i(G) = t_i(G^{ab})$ where G^{ab} is the abelianization of G considered as a Z-module. In particular, note that

$$t_2(G) = t_2(G^{ab}) \approx G^{ab} \otimes Z_2 .$$

2. H_2 and $\tilde{H}_2$.

We first recall C. Miller's definition of $H_2(G,Z)$ [8]. Let G be a group and let $F(G)$ be the free group on the set $G \times G$ whose elements will be denoted by $\langle x,y \rangle$, $x,y \in G$. $B(G)$ denotes the normal subgroup of $F(G)$ generated by the "universal commutator relations"

(1) $\langle x,y \rangle \langle y,x \rangle$

(2) $\langle {}^x y, {}^x z \rangle \langle x,z \rangle \langle xy,z \rangle^{-1}$

(3) $\langle {}^x y, {}^x z \rangle \langle z,y \rangle \langle x,[y,z] \rangle^{-1}$

(4) $\langle x,x \rangle$

for all $x,y,z \in G$. There is a homomorphism $F(G) \longrightarrow G$ defined by $\langle x,y \rangle \longmapsto [x,y] = {}^x y \cdot y^{-1} = xyx^{-1}y^{-1}$ which vanishes on $B(G)$. We denote its kernel by $Z(G)$.

THEOREM 2 (Miller [8]). *There is a canonical isomorphism* $H_2(G,Z) \approx Z(G)/B(G)$. *More precisely, if* $1 \longrightarrow R \longrightarrow F \longrightarrow G \longrightarrow 1$ *is exact with* F *free, then there is a commutative diagram with exact rows*

$$\begin{array}{cccccc} 1 \longrightarrow & Z(G)/B(G) & \longrightarrow & F(G)/B(G) & \longrightarrow & G \\ & \downarrow & & \downarrow & & \downarrow = \\ 1 \longrightarrow & R \cap F'/[R,F] & \longrightarrow & F'/[R,F] & \longrightarrow & G \end{array}$$

and vertical isomorphisms given by $\langle x,y \rangle \longmapsto [\tilde{x},\tilde{y}]$ *where* $\tilde{x}$ *and* $\tilde{y}$ *are any liftings of* x *and* y *to* F.

REMARKS. 1. $R \cap F'/[R,F]$ is Hopf's definition of H_2 which is canonically isomorphic to the definition given via the bar resolution. An explicit formula for the isomorphism can be found in [8, p. 594].

2. Miller proves directly from this definition that H_2 of a free group is trivial. It thus follows that any commutator relation that is universally true is a consequence of the above four.

For simplicity we will now use congruence notation to denote computation modulo certain of the above relations. First we have

$$(2') \quad \langle x,yz\rangle \equiv \langle x,y\rangle \langle {}^{y}x, {}^{y}z \rangle \quad \text{mod } (1), (2) .$$

Next expanding $\langle ax,by\rangle$ in two different ways yields

$$(5) \quad \langle ax,by\rangle \equiv \langle {}^{a}x, {}^{a}b \rangle \langle a,b\rangle \langle {}^{ba}x, {}^{ba}y \rangle \langle {}^{b}a, {}^{b}y \rangle \quad \text{mod } (2), (2')$$

and

$$(5') \quad \langle ax,by\rangle \equiv \langle {}^{a}x, {}^{a}b \rangle \langle {}^{ab}x, {}^{ab}y \rangle \langle a,b\rangle \langle {}^{b}a, {}^{b}y \rangle \quad \text{mod } (2), (2').$$

Combining these two and replacing x by $(ba)^{-1}xba$ and y by $(ba)^{-1}yba$ yields

$$(6) \quad \langle a,b\rangle \langle x,y\rangle \langle a,b\rangle^{-1} \equiv \langle {}^{[a,b]}x, {}^{[a,b]}y \rangle \quad \text{mod } (2), (2').$$

Thus if $C \in F(G)$ maps to $c \in [G,G]$, then we have

$$C \langle x,y\rangle C^{-1} \equiv \langle {}^{c}x, {}^{c}y \rangle \quad \text{mod } (2), (2') .$$

Note as a consequence that $Z(G)/N$ will be a central subgroup of $F(G)/N$ for any normal subgroup N which contains all elements of the form (2) and (2') .

We note that relation (3) can be replaced by either of

$$(7) \quad \langle x,yz\rangle \langle y,zx\rangle \langle z,xy\rangle$$

$$(7') \quad \langle zy,x\rangle \langle xz,y\rangle \langle yx,z\rangle .$$

It is easy to check that (7) and (7') are equivalent in the presence of (2) and (2') . We now show that (3) and (7) are equivalent in the presence of (1) and (2) . It already follows from Remark 2 above that (7) is a consequence of (1) - (4) . Explicitly we have

$$\begin{aligned}
\langle x,yz\rangle \langle y,zx\rangle &\equiv \langle x,y\rangle \langle {}^{y}x, {}^{y}z\rangle \langle y,z\rangle \langle {}^{z}y, {}^{z}x \rangle && \text{mod } (2') \\
&\equiv \langle {}^{x}y_{x}, {}^{x}y_{z} \rangle \langle {}^{[x,y]}y, {}^{[x,y]}z \rangle \langle x,y\rangle \langle {}^{z}y, {}^{z}x \rangle && \text{mod } (6) \\
&\equiv \langle {}^{x}y_{x}, {}^{x}y_{z} \rangle \langle {}^{[x,y]}y, {}^{[x,y]}z \rangle \langle [x,y],z \rangle && \text{mod } (3), (1) \\
&\equiv \langle {}^{x}y_{x}, {}^{x}y_{z} \rangle \langle [x,y]y,z \rangle && \text{mod } (2) \\
&\equiv \langle xy,z\rangle && \text{mod } (2)
\end{aligned}$$

and we are done via (1). Next

$$< [y,z],x > \equiv < {}^{yz}(y^{-1}z^{-1}), {}^{yz}x > <yz,x> \quad \text{mod } (2)$$

$$\equiv <y,z> < {}^{zy}(y^{-1}z^{-1}), {}^{zy}x > <y,z>^{-1} <y,zx> <z,xy> \text{ mod } (6),(7),(1)$$

$$\equiv <y,z> <x,zy> <y,z>^{-1} <y,zx> <z,xy> \quad \text{mod } (1),\ (2)$$

$$\equiv <y,z> <x,zy> < {}^{z}y, {}^{z}x > <z,x> < {}^{x}z, {}^{x}y > \quad \text{mod } (2'),\ (1)$$

$$\equiv <y,z> < {}^{x}z, {}^{x}y > \quad \text{mod } (2'),\ (1)$$

and we are done via (1).

Taking $x = y = 1$ in (2) yields $< 1,z >$ and then taking $z = 1$ in (7) yields (1). Thus (1) is a consequence of (2) and (7). We thus find that (1), (2), (3) are equivalent to (2) and (7).

We now define $B_0(G)$ to be the normal subgroup of $F(G)$ generated by (1), (2), (3) or equivalently (2), (7). We let $(G,G) = F(G)/B_0(G)$, $\tilde{H}_2G = Z(G)/B_0(G)$ and denote the class of $<x,y>$ in (G,G) by (x,y) . There is then an exact sequence

$$1 \longrightarrow \tilde{H}_2G \longrightarrow (G,G) \longrightarrow [G,G] \longrightarrow 1$$

with $\tilde{H}_2G$ a central subgroup of (G,G) .

By the definition of $\tilde{H}_2G$ there is a natural way of constructing homomorphisms.

MAPPING PROPERTY 3. <u>Given any group</u> H <u>and a function</u>

$$* : H \times H \longrightarrow (G,G)$$

<u>satisfying</u>

(i) $(x * y)\ (y * x) = 1$

(ii) $(xy) * z = ({}^{x}y * {}^{x}z)\ (x * z)$

(iii) $x * [y,z] = ({}^{x}y * {}^{x}z)\ (z * y)$

<u>or equivalently</u> (ii) <u>and</u>

(iv) $(x * yz)\ (y * zx)\ (z * xy) = 1$,

<u>then there exists a homomorphism</u>

$$(H,H) \longrightarrow (G,G) .$$

If there exists a homomorphism $[H,H] \longrightarrow [G,G]$ *making the diagram*

$$\begin{array}{ccc} (H,H) & \longrightarrow & (G,G) \\ \downarrow & & \downarrow \\ [H,H] & \longrightarrow & [G,G] \end{array}$$

commute, then there is an induced homomorphism

$$\tilde{H}_2H \longrightarrow \tilde{H}_2G \; .$$

REMARK. The proof is trivial. Note in particular that $\tilde{H}_2$ is thus a covariant functor from the category of groups to the category of abelian groups. It is also clear from the definition that $\tilde{H}_2$ preserves direct limits. An analogous mapping property can be given for H_2.

LEMMA 4. *If* A *is an abelian group, there is a canonical isomorphism* $\tilde{\bigwedge}^2 A \longrightarrow \tilde{H}_2A$ *given by* $a \tilde{\wedge} b \longmapsto (a,b)$.

As A is abelian, $Z(A) = F(A)$ and hence $\tilde{H}_2A = F(A)/B_0(A)$. By (2) and (2')

$$(ab,c) = (b,c)\;(a,c)$$

$$(a,bc) = (a,b)\;(a,c)$$

and (1) yields

$$(a,b)\;(b,a) = 1 \; .$$

(3) is easily seen to be a consequence of (1) and (2) in this case. We thus have a presentation for $\tilde{H}_2A$ which is precisely that of $\tilde{\bigwedge}^2A$.

REMARK. A similar argument gives the well-known $\bigwedge^2 A \approx H_2A$.

PROPOSITION 5. *For any group* G *there is an exact sequence*

$$1 \longrightarrow G^{ab} \otimes Z_2 \longrightarrow \tilde{H}_2G \longrightarrow H_2G \longrightarrow 1$$

which splits (non-canonically).

The addition of relation (4) gives the surjective homomorphism $\tilde{H}_2G \longrightarrow H_2G$ whose kernel is generated by the elements (x,x). We define a homorphism $G \longrightarrow \tilde{H}_2G$ by sending x to (x,x). Now

$$(xy,xy) = ({}^xy,{}^xx)\,(x,x)\,({}^{x^2}y,{}^{x^2}y)\,({}^xx,{}^xy)$$

by (5). As (x,x), $({}^{x^2}y,{}^{x^2}y)$ are in the center of (G,G), we have

$$(xy,xy) = (x,x)\,({}^{x^2}y,{}^{x^2}y)\,.$$

since $({}^xy,x)\,(x,{}^xy) = 1$ by (1). By (3) and (1)

$$({}^zy,{}^zy) = (z,[y,y])\,(y,y) = (y,y)$$

as $(z,1) = 1$ by (2'). Thus

$$(xy,xy) = (x,x)\,(y,y)$$

as asserted. Since $\tilde{H}_2G$ is abelian and $(x^2,x^2) = (x,x)\,(x,x) = 1$ by (1), we obtain a homomorphism

$$G^{ab} \otimes Z_2 \longrightarrow \tilde{H}_2G\,.$$

The commutative diagram with exact rows

$$\begin{array}{ccccccccc}
 & & G^{ab} \otimes Z_2 & \longrightarrow & \tilde{H}_2G & \longrightarrow & H_2G & \longrightarrow & 1 \\
 & & = \downarrow & & \downarrow & & \downarrow & & \\
 & & G^{ab} \otimes Z_2 & \longrightarrow & \tilde{H}_2G^{ab} & \longrightarrow & H_2G^{ab} & \longrightarrow & 1 \\
 & & = \downarrow & & \downarrow \cong & & \downarrow \cong & & \\
1 & \longrightarrow & G^{ab} \otimes Z_2 & \underset{\longleftarrow\,\text{-}\,\text{-}}{\longrightarrow} & \tilde{\Lambda}^2G^{ab} & \longrightarrow & \Lambda^2G^{ab} & \longrightarrow & 1
\end{array}$$

reduces the injectivity of the top left-hand map to the abelian case where it is known via Lemmas 1 and 4. The top sequence splits since the bottom one does.

COROLLARY 6. <u>If G is perfect or if G^{ab} is divisible by 2, then $\tilde{H}_2G \longrightarrow H_2G$ is an isomorphism.</u>

In particular note that

$$1 \longrightarrow \tilde{H}_2G \longrightarrow (G,G) \longrightarrow [G,G] \longrightarrow 1$$

is the universal central extension of G if G is perfect.

3. $\tilde{H}_2$ and K-theory.

Let R be any associative ring with 1. As usual GL(R) and E(R) will denote the infinite general linear and elementary groups over R. St(R) will denote the usual Steinberg group. Rather than construct maps $\tilde{H}_2R^* \longrightarrow K_2(R)$ and obtain a map $\tilde{H}_2GL(R) \longrightarrow K_2(R)$ by taking direct limits, we construct the latter map directly and obtain the former by composition.

THEOREM 7. <u>There is a canonical isomorphism</u>

$$(GL(R),GL(R)) \approx (E(R),E(R)) \times (GL(R)^{ab},GL(R)^{ab})$$

<u>and hence there is a canonical split exact sequence</u>

$$1 \longrightarrow \tilde{H}_2E(R) \longrightarrow \tilde{H}_2GL(R) \longrightarrow \tilde{H}_2GL(R)^{ab} \longrightarrow 1 \quad .$$

We abbreviate GL(R) by G and E(R) by E. There is a well-defined pairing

$$\star : G \times G \longrightarrow (E,E) \approx St(R)$$

defined as follows (see [4]). Let $a \in GL(m,R)$, $b \in GL(n,R)$, $a' \in GL(m',R)$, $b' \in GL(n',R)$ be such that

$$\bar{a} = a \oplus a' \oplus 1_{n'} , \quad \bar{b} = b \oplus 1_{m'} \oplus b' \in E.$$

Choose liftings $\tilde{a}, \tilde{b} \in (E,E) \approx St(R)$ and let $a \star b = [\tilde{a},\tilde{b}] = (\bar{a},\bar{b})$ (that the last two are equal is immediate from (6); see Miller [8, equation 7] for a proof). It is easy to check that $a \star b$ is independent of m, n, m', n', a', b' and the liftings chosen (see [4]). Equations (ii) and (iv) follow immediately from the usual commutator relations. Say $\tilde{a}, \tilde{b}, \tilde{c}$ are liftings of $a \oplus a' \oplus 1 \oplus 1$, $b \oplus 1 \oplus b' \oplus 1$, $c \oplus 1 \oplus 1 \oplus c'$ in E. Then

$$[\tilde{a}\tilde{b},\tilde{c}] = [\,{}^{\tilde{a}}\tilde{b}, {}^{\tilde{a}}\tilde{c}\,]\,[\tilde{a},\tilde{c}]$$

and using the independence statement above yields

$$(ab) \star c = (\,{}^{a}b \star {}^{a}c\,)\,(a \star c) \quad .$$

A similar argument works for (iv) (and also for (i) and (iii) but not for (4) !). Thus we obtain a homomorphism $(G,G) \longrightarrow (E,E)$. Note that if $a, b \in E$, then (by independence) one can take $a' = b' = 1$ and hence the composition

$$(E,E) \longrightarrow (G,G) \longrightarrow (E,E)$$

is the identity.

Recall that $(G^{ab},G^{ab}) = \tilde{H}_2 G^{ab}$ since G^{ab} is abelian. Let $a \in GL(m,R)$, $b \in GL(n,R)$ and define

$$a\cdot b = (a \oplus 1, 1 \oplus b) \in \tilde{H}_2 G \subset (G,G) .$$

If $Z \in (G,G)$ maps to $z \in [G,G]$, then as $a\cdot b$ is central we have

$$\begin{aligned} a\cdot b &= Z(a\cdot b)Z^{-1} \\ &= ({}^{z}a \oplus 1 , {}^{z}1 \oplus b) \end{aligned}$$

from (6). By choosing appropriate z it follows that

$$\begin{aligned} a\cdot b &= (1 \oplus a, b \oplus 1) \\ &= (b \oplus 1, 1 \oplus a)^{-1} \\ &= (b\cdot a)^{-1} . \end{aligned}$$

Now

$$\begin{aligned} (ab)\cdot c &= ((b \oplus 1)(b^{-1}ab \oplus 1), 1 \oplus c) \\ &= ({}^{b \oplus 1} b^{-1}ab \oplus 1, {}^{b \oplus 1} 1 \oplus c) (b \oplus 1, 1 \oplus c) \\ &= (a\cdot c) (b\cdot c) . \end{aligned}$$

Similarly $a\cdot(bc) = (a\cdot b)(a\cdot c)$. Finally as $\tilde{H}_2 G$ is abelian there is a map

$$G^{ab} \times G^{ab} \longrightarrow \tilde{H}_2 G \subset (G,G) .$$

Clearly $a\cdot b \in \tilde{H}_2 G$ maps to (class a, class b) $\in \tilde{H}_2 G^{ab}$ since b and $1 \oplus b$ represent the same class in G^{ab}. Thus the composition

$$(G^{ab},G^{ab}) = \tilde{H}_2 G^{ab} \longrightarrow (G,G) \longrightarrow (G^{ab},G^{ab})$$

is the identity.

As the image of (G^{ab},G^{ab}) in (G,G) is central and as the composition

$$(E,E) \longrightarrow (G,G) \longrightarrow (G^{ab},G^{ab})$$

is clearly trivial (since $[G,G] = E$), to complete the proof that (G,G) is canonically the direct product of (E,E) and

(G^{ab},G^{ab}) , it will suffice to show that every generator of (G,G) is a product of an element from (E,E) and one from the image of (G^{ab},G^{ab}). By equation (5)

$$(a \oplus a^{-1} \oplus 1, b \oplus 1 \oplus b^{-1})$$

$$= (1 \oplus a^{-1} \oplus 1, {}^{a}b \oplus 1 \oplus 1)\ (1 \oplus a^{-1} \oplus 1, 1 \oplus 1 \oplus b^{-1})\ (a \oplus 1 \oplus 1, b \oplus 1 \oplus 1)\ ({}^{b}a \oplus 1 \oplus 1, 1 \oplus 1 \oplus b^{-1}) .$$

Thus

$$\begin{aligned} (a,b) &= (a \oplus 1 \oplus 1,\ b \oplus 1 \oplus 1) \\ &= (a \star b)\ ({}^{b}a \cdot b^{-1})^{-1}\ (a^{-1} \cdot b^{-1})^{-1}\ (a^{-1} \cdot {}^{a}b)^{-1} \\ &= (a \star b)\ (a \cdot b) . \end{aligned}$$

The last statement of the theorem is immediate from the first.

COROLLARY 8. _There is an exact sequence_

$$1 \longrightarrow H_2E(R) \longrightarrow H_2GL(R) \longrightarrow H_2GL(R)^{ab} \longrightarrow 1$$

which splits (non-canonically). The splitting is canonical up to certain elements of order 2.

The exactness can be seen from the commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 1 & \longrightarrow & G^{ab} \otimes Z_2 & \xrightarrow{\approx} & G^{ab} \otimes Z_2 & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \tilde{H}_2E & \longrightarrow & \tilde{H}_2G & \longrightarrow & \tilde{H}_2G^{ab} & \longrightarrow & 1 \\
 & & \downarrow\approx & & \downarrow & & \downarrow & & \\
 & & H_2E & \longrightarrow & H_2G & \longrightarrow & H_2G^{ab} & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array} .$$

The non-canonical splittings of the vertical exact sequences and the canonical splitting of the middle exact sequence give the non-canonical splitting of the lower sequence.

Upon more careful examination we see that the equation

$$(a,a) = (a \star a)\ (a \cdot a)$$

gives the only obstruction to a canonical splitting. For in H_2G we have $a \cdot a = a \star a$ which is of order 2 and not necessarily trivial. However the corresponding element of H_2G^{ab} is trivial.

Let $\tilde{H}_i = H_i$ for $i = 0, 1$. Then for G abelian and $i + j \leq 2$, there is an anti-commutative pairing

$$\tilde{H}_i G \times \tilde{H}_j G \longrightarrow \tilde{H}_{i+j} G \quad .$$

The one interesting case is for $i = j = 1$. Then

$$G \times G = H_1 G \times H_1 G \longrightarrow H_2 G \approx \tilde{\bigwedge}^2 G$$

is defined by $(x,y) \longmapsto x \tilde{\wedge} y$.

COROLLARY 9. <u>The functors $\tilde{H}_i$, $i = 0, 1, 2$ satisfy the relevant portions of axioms I, II, and III.</u>

There is a homomorphism $R^* \longrightarrow GL(R)$ given by sending a unit to the diagonal matrix with the unit in the (1,1) position and 1's elsewhere on the diagonal. This gives the map $\tilde{H}_2 R^* \longrightarrow K_2 R$ as the composition

$$\tilde{H}_2 R^* \longrightarrow \tilde{H}_2 GL(R) \longrightarrow \tilde{H}_2 E(R) = H_2 E(R) = K_2 R \ .$$

More explicitly, the map is induced by the pairing

$$R^* \times R^* \longrightarrow (E,E) \approx St(R)$$

defined by $(u,v) \longmapsto [h_{12}(u), h_{13}(v)]$ (notation of [9]).

It is now clear that the relevant portions of I, II, and III are satisfied.

We now interpret some K-theoretic results in this language.

PROPOSITION 10. 1. <u>$\tilde{H}_2 R^* \longrightarrow K_2 R$ is surjective if and only if $K_2(R)$ is contained in the subgroup of $St(R)$ generated by the $h_{ij}(u)$, $u \in R^*$ (equivalently, contained in the subgroup generated by the $w_{ij}(u)$, $u \in R^*$).</u>

2. <u>If R satisfies Bass' stable range condition SR_m, then</u>

$$\tilde{H}_2 GL(n,R) \longrightarrow K_2 R$$

<u>is surjective for all $n \geq m$.</u>

These are immediate from our definition and known results (see [9], [1], [2], [3]).

THEOREM 11 (Matsumoto-Rehmann). <u>Let D be a division ring (skew field). Then the kernel of the surjective map $\tilde{H}_2 D^* \longrightarrow K_2 D$ is generated by the elements $(u,1-u)$ for $u \in D^*$, $u \neq 1$.</u>

This follows immediately from a theorem of Rehmann [10] which we now state in a convenient form. Let $c(u,v) = [h_{12}(u), h_{13}(v)]$ for $u, v \in D^*$. The subgroup U_D of St(D) generated by the $c(u,v)$ is just the set of elements $z \in St(D)$ satisfying

$$\varphi(z) = \mathrm{diag}(a,1,\ldots,1)$$

where necessarily $a \in [D^*, D^*]$. Then there is an exact sequence

$$1 \longrightarrow K_2D \longrightarrow U_D \longrightarrow [D^*, D^*] \longrightarrow 1 .$$

THEOREM 12 (Rehmann [10]). *If D is a division ring, U_D is presented by generators $c(u,v)$, $u, v \in D^*$, subject only to the relations*

Uo $\quad c(u,1-u) = 1, \quad u \neq 1$

U1 $\quad c(uv,w) = c({}^u v, {}^u w)\, c(u,w)$

U2 $\quad c(u,vw)\, c(v,wu)\, c(w,uv) = 1$

U3 $\quad c(u,v)\, c(u',v') = c({}^{[u,v]}u', {}^{[u,v]}v')\, c(u,v)$.

Note that U1 is (2), U2 is (7) and U3 (which can be omitted as it is a consequence of U1 and U2) is (6). Thus the set of relations U1 - U3 is equivalent to (1) - (3). The result is now clear.

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