# Unsolved Problems in Combinatorial Games

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We have retained the numbering from the list of unsolved problems given on pp. 183–189 of AMS *Proc. Sympos. Appl. Math.* **43**(1991). References in brackets, e.g., Loyd [1914], are listed in Fraenkel's Bibliography later in this book; WW refers to

Elwyn Berlekamp, John Conway & Richard Guy, *Winning Ways for your Mathematical Plays*, Academic Press, 1982.

and references in parentheses, e.g., Kraitchik (1941), are at the end.

**1.** Subtraction games are known to be periodic. Investigate the relationship between the subtraction set and the length and structure of the period. The same question can be asked about *partizan* subtraction games, in which each player is assigned an individual subtraction set. See Fraenkel & Kotzig [1987].

See also Subtraction Games in WW, 83–86, 487–498 and in the Impartial Games lecture of this workshop. A move in the game $S(s_1, s_2, s_3, \ldots)$ is to take a number of beans from a heap, provided that number is a member of the **subtraction-set**, $\{s_1, s_2, s_3, \ldots\}$. Analysis of such a game and of many other heap games is conveniently recorded by a **nim-sequence**,

$$n_0 n_1 n_2 n_3 \ldots,$$

meaning that the nim-value of a heap of $h$ beans is $n_h$, $h = 0, 1, 2, \ldots$, i.e., that the value of a heap of $h$ beans in this particular game is the **nimber** $*n_h$. To avoid having to print stars, we say that the nim-value of a position is $n$, meaning that its value is the nimber $*n$.

For examples see Table 2 in §**4** of the Impartial Games paper in this book.

In subtraction games the nim-values 0 and 1 are remarkably related by **Ferguson's Pairing Property** [Ferguson [1974]; WW, 86, 422]: if $s_1$ is the least member of the subtraction-set, then

$$\mathcal{G}(n) = 1 \qquad \text{just if} \qquad \mathcal{G}(n - s_1) = 0.$$

Here and later "$\mathcal{G}(n) = v$" means that the nim-value of a heap of $n$ beans is $v$.

Arbitrarily difficult subtraction games can be devised by using infinite subtraction-sets: the primes, for example.

**2.** Are all finite octal games ultimately periodic? Resolve any number of outstanding particular cases, e.g., **·6** (Officers), **·06**, **·14**, **·36**, **·64**, **·74**, **·76**, **·004**, **·005**, **·006**, **·007** (Onedimensional tic–tac–toe, Treblecross), **·016**, **·104**, **·106**, **·114**, **·135**, **·136**, **·142**, **·143**, **·146**, **·162**, **·163**, **·172**, **·324**, **·336**, **·342**, **·362**, **·371**, **·374**, **·404**, **·414**, **·416**, **·444**, **·454**, **·564**, **·604**, **·606**, **·644**, **·744**, **·764**, **·774**, **·776** and Grundy's Game (split a heap into two unequal heaps). Find a game with a period longer than 149459. Explain the structure of the periods of games known to be periodic.

[If the binary expansion of the $k$th code digit in the game with code $\mathbf{d}_0{\cdot}\mathbf{d}_1\mathbf{d}_2\mathbf{d}_3\ldots$ is

$$\mathbf{d}_k = 2^{a_k} + 2^{b_k} + 2^{c_k} + \ldots,$$

where $0 \le a_k < b_k < c_k < \ldots$, then it is legal to remove $k$ beans from a heap, provided that the rest of the heap is left in exactly $a_k$ or $b_k$ or $c_k$ or $\ldots$ non-empty heaps. See WW, 81–115. Some specimen games are exhibited in Table 3 of §**5** of the Impartial Games paper in this book.]

Gangolli & Plambeck [1989] established the ultimate periodicity of four octal games which were previously unknown: **·16** has period 149459 (a prime!), the last exceptional value being $\mathcal{G}(105350) = 16$. The game **·56** has period 144 and last exceptional value $\mathcal{G}(326639) = 26$. The games **·127** and **·376** each have period 4 (with cycles of values 4, 7, 2, 1 and 17, 33, 16, 32 respectively) and last exceptional values $\mathcal{G}(46577) = 11$ and $\mathcal{G}(2268247) = 42$.

In Problem 38 in *Discrete Math.*, **44**(1983) 331–334 Fraenkel raises questions concerning the computational complexity of octal games. In Problem 39, he & Kotzig define **partizan octal games** in which distinct octals are assigned to the two players. In Problem 40, Fraenkel introduces **poset games**, played on a partially ordered set of heaps, each player in turn selecting a heap and removing a positive number of beans from all heaps which are at least as high in the ordering as the selected heap.

**3.** Examine some hexadecimal games. Obtain conditions for arithmetic periodicity.

[**Hexadecimal games** are those with code digits $d_k$ in the interval from **0** to **15**. See WW, 115–116.]

The last three games in Table 2 are hexadecimal games with $\mathbf{d}_k \le \mathbf{15} = \mathbf{F}$. Such games may be arithmetically periodic, but most of the known ones are fairly trivial: **·8** is a first cousin of Triplicate Nim, **·A**, **·B**, **·D** and **·F** are essentially She-Loves-Me-She-Loves-Me-Not, while **·1A** and **·1B**, after a couple of exceptional values, display period 2 and saltus 1. Kenyon's Game, **·3F** in Table 3, whose saltus of 3 countered a conjecture of Guy & Smith that it should always be a power of two, is a little more interesting, but there must be many others such, which are possibly accessible to present-day computers.

**4.** Extend the analysis of **Domineering** to larger boards.

[Left & Right take turns to place dominoes on a checker-board. Left orients her dominoes North-South and Right orients his East-West. Each domino exactly covers two squares of the board and no two dominoes overlap. A player unable to play loses. See Berlekamp [1988]; WW, 495–498; and lectures at this workshop.]

In an earlier version of this paper it was proposed to find the values of $4 \times 4$ and $4 \times 5$ boards. David Wolfe has found these to be respectively $\pm\{0, \{2|0, 2 +_2 \,\|\, 2|0, -_2\} \,|||\, 0\}$ and 1. Berlekamp has shown that the value of a $4 \times n$ board, if $n$ is odd, is twice that of the corresponding $2 \times n$ board plus a correction term of lower temperature. For the value $g$ of the $4 \times 6$ board, Wolfe's program needed more than 200 characters to print it. $g$ is incomparable with both $+_2$ and $-_2$, but he can show that $-4(+_2) < g < 3(+_2)$. Dan Calistrate observes that the value of a $6 \times 6$ board is $\pm 1$ to within 'ish' (infinitesimally shifted).

Yonghoan Kim announced that there are snakes of value $2^{-n}$ for all positive integers $n$ (Figure 1). Various subsets of these snakes give a sequence of hot games (the penultimate row of the next table), another sequence of numbers, and several infinitesimals. Wolfe [1993] also has snakes with interesting values.

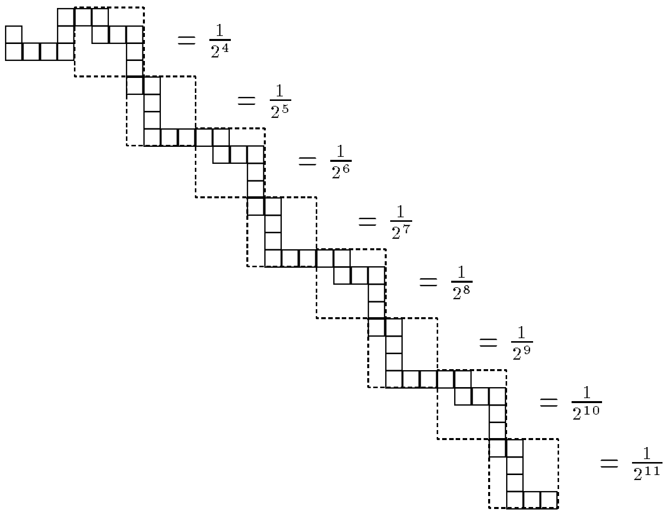
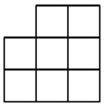



Figure 1. Yonghoan Kim's snakes of value $2^{-n}$.

Berlekamp gives the following table of known temperatures in Domineering. The rows are various overheating and warming operators associated with various (families of) positions, and the columns are their arguments. The entries are the temperatures of the resulting games (for references, see Problem 52 below).

Row 2 applies to $3 \times n$ boards with $n$ odd and an additional square appended at one corner; row 3 to $2 \times n$ boards with $n$ odd. The headings 1e, 1d refer to the argument 1 in the respective cases of *even* positions, which warm by overheating; and of *odd* positions, which warm by overheating and *then* adding $*$, distinguished by a star on the operator sign. If we omit the two entries in the 1d column, the rightmost temperature in each row (except the first) matches the leftmost temperature in the next row.

| | | $*$ | $\ldots,$ | $\frac{1}{32},$ | $\frac{1}{16}$ | $\frac{1}{8}$ | $\frac{1}{4}$ | $\frac{1}{2}$ | 1e | 1d |
|---|---|---|---|---|---|---|---|---|---|---|
|  | $\int^{3/2}$ | $1\frac{1}{2}$ | | | | | | | | |
| $3 \times \text{odd}\frac{1}{3}$ | $\int^{9/8}\int_{\frac{1}{4}}^{\frac{1}{4}*}$ | $1\frac{3}{8}$ | | | | $1\frac{11}{32}$ | $1\frac{5}{16}$ | $1\frac{1}{4}$ | | $1\frac{1}{8}$ |
| $2 \times \text{odd}$ | $\int^{3/4}\int_{\frac{1}{2}}^{\frac{1}{2}*}$ | $1\frac{1}{4}$ | | | | $1\frac{3}{16}$ | $1\frac{1}{8}$ | $1$ | | $\frac{3}{4}$ |
| Conway's ZigZags | $\int_1^1$ (ish) | $1$ | $\ldots,$ | $\frac{31}{32},$ | $\frac{15}{16}$ | $\frac{7}{8}$ | $\frac{3}{4}$ | $\frac{1}{2}$ | | |
| Wolfe's Snakes | $\int_{\frac{1}{2}*}^{\frac{1}{2}}$ | $\frac{1}{2}$ | $\ldots,$ | $\frac{31}{64},$ | $\frac{15}{32}$ | $\frac{7}{16}$ | $\frac{3}{8}$ | $\frac{1}{4}$ | | |
| Kim's Snakes | $\int_{\frac{1}{4}*}^{\frac{1}{4}}$ | $\frac{1}{4}$ | $\ldots,$ | $\frac{31}{128},$ | $\frac{15}{64}$ | $\frac{7}{32}$ | $\frac{3}{16}$ | $\frac{1}{8}$ | $0$ | |
| Kim's Numbers | Identity | $0$ | $\ldots,$ | $-\frac{1}{32},$ | $-\frac{1}{16}$ | $-\frac{1}{8}$ | $-\frac{1}{4}$ | $-\frac{1}{2}$ | $-1$ | |

Berlekamp asks, as a hard problem, to characterize all hot Domineering positions to within "ish" (i.e., infinitesimally shifted). As a possibly easier problem he asks for a Domineering position with a new temperature, i.e., one not occurring in the table above.

**5.** Analyze positions in the game of **Go**.

[Compare Berlekamp [1988], his book with Wolfe [1994], and their continuing discoveries, discussed in the present workshop.]

**6.** In an earlier edition of this paper I asked if **Go-Moku** (Five-in-a-row, Go-Bang, Pegotty) is a win for the first player. An affirmative answer for the unrestricted version and for the version where 6-in-a-row doesn't count as a win was given by Allis, van den Herik & Huntjens [1992]. Lustenberger (1967) showed that **Four-in-a-row** on a $4 \times n$ board is a first player win for $n$ sufficiently large. Selfridge states that $n = 28$ suffices. To find the least $n$ should now-a-days be a straightforward computer exercise.

**7.** Complete the analysis of impartial **Eatcakes** (WW, 269, 271, 276–277).

[Eatcakes is an example of a **join** or **selective compound** of games. Each player plays in **all** the component games. It is played with a number of rectangles, $m_i \times n_i$; a move is to remove a strip $1 \times n_i$ or $m_i \times 1$ from each rectangle, either splitting it into two rectangles, or reducing the length or breadth by one. Winner removes the last strip.]

For fixed breadth the remoteness becomes constant when the length is sufficiently large. But 'sufficiently large' seems to be an increasing function of the breadth and doesn't, in the hand calculations already made, settle into any clear pattern. Perhaps computer calculations will reveal something.

**8.** Complete the analysis of **Hotcakes** (WW, 279–282).

[Also played with integer-sided rectangles, but as a **union** or **selective compound** in which each player moves in **some** of the components. Left cuts as many rectangles vertically along an integer line as she wishes, and then rotates one from each pair of resulting rectangles through a right angle. Right cuts as many rectangles as he wishes, horizontally into pairs of integer-sided rectangles and rotates one rectangle from each pair through a right angle. The **tolls** for rectangles with one dimension small are understood, but much remains to be discovered.]

**9.** Develop a misère theory for unions of partizan games.

[In a union of two or more games, you move in as many component games as you wish. In misère play, the last player *loses*.]

**10.** Extend the analysis of **Squares Off** (WW, 299).

[Played with heaps of beans. A move is to take a perfect square ($> 1$) number of beans from any number of heaps. Heaps of 0, 1, 2 or 3 cannot be further reduced. A move leaving a heap of 0 is an overriding win for the player making it. A move leaving 1 is an overriding win for Right, and one leaving 2 is an overriding win for Left. A move leaving 3 doesn't end the game unless all other heaps are of size 3, in which case the last player wins.]

**11.** Extend the analysis of **Top Entails** (WW, 376–377).

[Played with stacks of coins. Either split a stack into two smaller ones, or remove the top coin from a stack. In the latter case your opponent's move must use the same stack. Last player wins. Don't leave a stack of 1 on the board, since your opponent must take it and win, since it's now your turn to move in an empty stack!]

Julian West wrote a program to check a student's work and calculated the first 38,000 values. He found loony positions at 2403 coins, 2505 coins, and 33,243 coins. The authors of *Winning Ways* did not know of a loony stack of more than 3 coins. These results are typical of the apparently quite unpredictable nature of combinatorial games, even when they have quite simple rules.

**12.** Extend the analysis of **All Square** (WW, 385).

[This game involves **complimenting moves** after which the same player has an extra **bonus move**. Note that this happens in Dots-and-Boxes when a box is completed. All Square is played with heaps of beans. A move splits a heap into two smaller ones. If both heap sizes are perfect squares, the player must move again: if he can't he loses!]

**13.** Extend the misère analysis of various octal games, e.g., **Officers**, **Dawson's Chess**, ..., and of **Grundy's Game**.

[See Allemang [1984], WW, 411–421. William L. Sibert made a breakthrough by completing the analysis of misère Kayles; see the post-script to Sibert & Conway [1992]. Plambeck [1992] has used their method to analyse a few other games, but there's a wide open field here.]

**14. Moebius**, when played on 18 coins has a remarkable pattern. Is there any trace of pattern for larger numbers of coins? Can any estimate be made for the rate of growth of the nim-values?

[See Coin-turning games in WW, 432–435; and Vera Pless's lecture and the Impartial Games lecture in PSAM **43** and in this workshop; and Curtis (1976, 1977, 1982). Moebius is played with a row of coins. A move turns 1, 2, 3, 4 or 5 coins, of which the rightmost must go from heads to tails (to make sure the game satisfies the Ending Condition). The winner is the player who makes all coins tails.]

**15. Mogul** has an even more striking pattern when played on 24 coins, which has some echoes when played on 40, 56 or 64 coins. Thereafter, is there complete chaos?

[See references for problem 14. A move turns 1, 2, ..., 7 coins.]

**16.** Find an analysis of **Antonim** with four or more coins (WW, 459–462).

[Played with coins on a strip of squares. A move moves a coin from one square to a smaller-numbered square. Only one coin to a square, except that square zero can have any number of coins. It is known that $(a, b, c)$ is a $\mathcal{P}$-position in Antonim just if $(a+1, b+1, c+1)$ is a $\mathcal{P}$-position in Nim, but for more than 3 coins much remains to be discovered.]

**17.** Extend the analysis of **Kotzig's Nim** (WW, 481–483). Is the game eventually periodic in terms of the length of the circle for every finite move set? Analyze the misère version of Kotzig's Nim.

[Players alternately place coins on a circular strip, at most one coin on a square. Each coin must be placed $m$ squares clockwise from the previously placed coin, provided $m$ is in the given **move set**, and provided the square is not already occupied. The complete analysis is known only for a few small move sets.]

There is a paper by Aviezri S. Fraenkel, Alan Jaffray, Anton Kotzig &Gert Sabidussi [1995], but a great deal remains to be discovered.

**18.** Obtain asymptotic estimates for the proportions of $\mathcal{N}$-, $\mathcal{O}$- and $\mathcal{P}$-positions in Epstein's **Put-or-Take-a-Square** game (WW, 484–486).

[Played with one heap of beans. At each turn there are just two options, to take away or add the largest perfect square number of beans that there is in the heap. 5 is a $\mathcal{P}$-position, because $5 \pm 4$ are both squares; 2 and 3 are $\mathcal{O}$-positions, a win for neither player, since the best play is to go from one to the other, and not to 1 or 4 which are $\mathcal{N}$-positions.]

**19.** Simon Norton's game of **Tribulations** is similar to Epstein's game, but squares are replaced by triangular numbers. Norton conjectures that there are no $\mathcal{O}$-positions, and that the $\mathcal{N}$-positions outnumber the $\mathcal{P}$-positions in golden ratio. True up to 5000 beans.

**20.** Complete the analysis of **D.U.D.E.N.E.Y.**

[Played with a single heap of beans. Either player may take any number of beans from 1 to $Y$, except that the immediately previous move mustn't be repeated. When you can't move you lose. Analysis easy for $Y$ even, and known (WW, 487–489) for 53/64 of the odd values of $Y$.]

Marc Wallace & Alan Jaffray have made a little progress here, but is the situation one in which there is always a small fraction of cases remaining, no matter how far the analysis is pursued?

**21.** **Schuhstrings** is the same as D.U.D.E.N.E.Y., except that a deduction of zero is also allowed, but cannot be immediately repeated (WW, 489–490).

**22.** Analyze Nim in which you are not allowed to repeat a move. There are at least five possible forms (assume that $b$ beans have been taken from heap $H$):

*medium local*: $b$ beans may not be taken from heap $H$ until some other move is made in heap $H$.

*short local*: $b$ beans may not be taken from heap $H$ on the next move.

*long local*: $b$ beans may never again be taken from heap $H$.

*short global*: $b$ beans may not be taken from any heap on the next move.

*long global*: $b$ beans may never again be taken from any heap.

**23. Burning-the-Candle-at-Both-Ends.** John Conway & Aviezri Fraenkel ask us to analyze Nim played with a row of heaps. A move may only be made in the leftmost or in the rightmost heap. When a heap becomes empty, then of course its neighbor becomes the end heap.

Conway has some recollection that the analysis was completed. For one or two heaps the game is the same as Nim. The $\mathcal{P}$-positions for three heaps are $\{n, m, n\}$ with $m \neq n$. If this game is indeed analyzed then there is **Hub-and-Spoke Nim**, proposed by Fraenkel. One heap is the hub and the others are arranged in rows forming spokes radiating from the hub.

There are several versions:

(a) beans may be taken only from a heap at the end of a spoke;

(b) beans may also be taken from the hub;

(c) beans may be taken from the hub only when all the heaps in a spoke are exhausted;

(d) beans may be taken from the hub only when just one spoke remains;

(e) in (b), (c) & (d), when the hub is exhausted, beans may be taken from a heap at either end of any remaining spoke; i.e. the game becomes the sum of a number of games of Burning-the-Candle-at-Both-Ends.

**24.** Continue the analysis of **The Princess and the Roses** (WW, 490–494).

[Played with heaps of beans. Take one bean, or two beans, one from each of two different heaps. The rules seem trivially simple, but the analysis takes on remarkable ramifications.]

**25.** Analyze the Conway-Paterson game of **Sprouts** with 7 or more spots, or the misère form with 5 or more spots (WW, 564–568).

[A move joins two spots, or a spot to itself by a curve which doesn't meet any other spot or previously drawn curve. When a curve is drawn, a new spot must be placed on it. The valence of any spot must not exceed three. Since this question was asked, Daniel Sleator of A.T.&T. Bell Laboratories has pushed the normal analysis to about 10 spots and the misère analysis to about 8.]

**26.** Extend the analysis of **Sylver Coinage** (WW, 575–597).

[Players alternately name different positive integers, but may not name a number which is the sum of previously named ones, with repetitions allowed. Whoever names 1 loses. See Section 3. of Richard Nowakowski's chapter in PSAM **43**.]

**27.** Extend the analysis of **Chomp** (WW, 598–599).

[Players alternately name divisors of $N$, which may not be multiples of previously named numbers. Whoever names 1 loses. David Gale offers a prize of US$100.00 for the first complete analysis of 3D-Chomp, i.e., where $N$ has three distinct prime divisors, raised to arbitrarily high powers.]

**28.** Extend Uléhla's or Berlekamp's analysis of **von Neumann's game** from diforests to directed acyclic graphs (WW, 570–572; Uléhla [1980]). Since Chomp and the superset game [Gale & Neyman [1982]] can be described by directed acyclic graphs but not by diforests, the proposed extension could presumably throw some light on these two unsolved games.

**29.** Prove that Black doesn't have a forced win in Chess.

**30.** A King and Rook versus King problem. Played on a quarter-infinite board, with initial position WKa1, WRb2 and BKc3. Can White win? If so, in how few moves? It may be better to ask, "what is the smallest board (if any) that White can win on if Black is given a win if he walks off the North or East edges of the board?" Is the answer $9 \times 11$? In an earlier edition of this paper I attributed this problem to Simon Norton, but it was proposed as a kriegsspiel problem, with unspecified position of the WK, and with W to win with probability 1, by Lloyd Shapley around 1960.

**31.** David Gale's version of Lion and Man. L and M are confined to the non-negative quadrant of the plane. They move alternately a distance of at most one unit. For which initial positions can L catch M?

*Conjecture*: L wins if he is northeast of M. This condition is clearly necessary, but the answer is not known even if M is at the origin and L is on the diagonal.

*Variation*. Replace quadrant by wedge-shaped region.

**32. Gale's Vingt-et-un**. Cards numbered 1 through 10 are laid on the table. L chooses a card. Then R chooses cards until his total of chosen cards exceeds the card chosen by L. Then L chooses until her cumulative total exceeds that of R, etc. The first player to get 21 wins. Who is it?

[As posed here it is not clear if the object is to get 21 exactly or 21-or-more. Jeffery Magnoli, a student of Julian West, thought that the latter rule was the more interesting and found a first-player win in 6-card Onze and in 8-card Dix-sept.]

**33. Subset Take-away**. Given a finite set, players alternately choose proper subsets subject to the rule that once a subset has been chosen, none of *its* subsets may be chosen subsequently by either player. Last player wins.

[David Gale conjectures that it's a second player win—this is true for sets of less than six elements.]

**34.** Eggleton & Fraenkel ask for a theory of **Cannibal Games** or an analysis of special families of positions. They are played on an arbitrary finite digraph. Place any numbers of "cannibals" on any vertices. A move is to select a cannibal and move it along a directed edge to a neighboring vertex. If this is occupied, the incoming cannibal eats the whole population (**Greedy Cannibals**) or just one cannibal (**Polite Cannibals**). A player unable to move loses. Draws are possible. A partizan version can be played with cannibals of two colors, each eating only the opposite color.

**35. Welter's Game** on an arbitrary digraph. Place a number of monochromatic tokens on distinct vertices of a directed acyclic graph. A token may be moved to any *unoccupied* immediate follower. Last player wins. Make a dictionary of $\mathcal{P}$-positions and formulate a winning strategy for other positions. See A. S. Fraenkel & Joseph Kahane, Problem 45, *Discrete Math.*, **45**(1983) 328–329, where a paper is said to be in preparation.

**36. Restricted Positrons & Electrons**. Fraenkel places a number of Positrons (Pink tokens) and Electrons (Ebony tokens) on distinct vertices of a Welter strip. Any particle can be moved by either player leftward to any square $u$ provided that $u$ is either unoccupied or occupied by a particle of the opposite type. In the latter case, of course, both particles become annihilated (i.e., they are removed from the strip), as physicists tell us positrons and electrons do. Play ends when the excess particles of one type over the other are jammed in the lowest positions of the strip. Last player wins. Formulate a winning strategy for those positions where one exists. Note that if the particles are of one type only, this is Welter's Game. As a strategy is known for Misère Welter [WW, 480–481] it may not be unreasonable to ask for a misère analysis as well. See Problem 47, *Discrete Math.*, **46**(1983) 215–216.

**37. General Positrons & Electrons**. As 36. but played on an arbitrary digraph. Last player wins.

**38. Fulves's Merger.** Karl Fulves, Box 433, Teaneck NJ 07666, sent the following game. Start with heaps of 1, 2, 3, 4, 5, 6 and 7 beans. Two players alternately transfer any number of beans from one heap to another, except that beans may not be transferred from a larger to a smaller heap. The player who makes all the heaps *even* in size is the winner.

The total number of beans remains constant, and is even (28 in this case, though one is interested in even numbers in general: a similar game can be played in which the total number is odd and the object is to make all the heaps odd in size). As the moves are nondisjunctive the Sprague-Grundy theory is not likely to be of help. Fulves's particular game can be solved using Steinhaus's **remoteness function** as follows.

The number of odd heaps is even. If the number of odd heaps is two, there is an immediate win: such a position is an $\mathcal{N}$-position of remoteness 1. In normal play, $\mathcal{P}$-positions have even remoteness, and $\mathcal{N}$-positions have odd remoteness.

To calculate the remoteness of a position:

If there are no options, the remoteness is 0.

If there is an option of even remoteness, add one to the *least* such remoteness,

Otherwise add one to the *greatest odd* remoteness,

WIN QUICKLY – LOSE SLOWLY.

If the number of odd heaps is zero, the position is terminal. All other $\mathcal{P}$-positions must contain at least four odd heaps. Some examples, with their even remotenesses, $r$, are:

| $\mathcal{P}$-position | $r$ |
|---|---|
| 1 1 1 25 | 2 |
| 1 3 3 21 | 4 |
| 1 5 5 17 | 6 |
| 1 7 7 13 | 8 |
| 1 9 9 9 | 10 |
| 3 5 9 11 | 10 |
| | |
| 1 1 2 5 19 | 6 |
| 1 1 3 6 17 | 8 |
| 1 1 4 7 15 | 10 |
| 1 2 3 9 13 | 12 |
| 1 3 5 5 14 | 12 |
| 1 5 6 7 9 | 14 |
| 2 3 3 3 17 | 8 |
| 2 5 5 5 11 | 14 |
| 2 5 7 7 7 | 14 |
| 3 3 4 5 13 | 14 |
| 3 3 7 7 8 | 16 |

| $\mathcal{P}$-position | $r$ |
|---|---|
| 1 1 1 1 1 23 | 4 |
| 1 1 1 2 2 21 | 6 |
| 1 1 1 3 3 19 | 8 |
| 1 1 1 4 4 17 | 10 |
| 1 1 1 5 5 15 | 12 |
| 1 1 1 6 6 13 | 14 |
| 1 1 1 7 7 11 | 16 |
| 1 1 1 8 8 9 | 18 |
| 1 1 2 3 7 14 | 12 |
| 1 1 2 4 9 11 | 14 |
| 1 1 3 5 8 10 | 18 |
| 1 2 2 3 5 15 | 12 |
| 1 2 2 7 7 9 | 14 |
| 1 2 3 3 8 11 | 14 |
| 1 2 5 5 7 8 | 16 |
| 1 3 3 3 3 15 | 14 |
| 1 3 3 4 7 10 | 16 |
| 1 3 3 6 6 9 | 18 |
| 1 3 4 4 5 11 | 18 |
| 2 2 3 3 9 9 | 16 |
| 2 3 3 5 5 10 | 18 |
| 3 3 3 5 6 8 | 20 |

| $\mathcal{P}$-position | $r$ |
|---|---|
| 1 1 1 1 2 4 18 | 10 |
| 1 1 1 1 3 5 16 | 12 |
| 1 1 1 1 6 8 10 | 18 |
| 1 1 1 2 3 8 12 | 14 |
| 1 1 1 2 5 9 9 | 20 |
| 1 1 1 3 6 7 9 | 22 |
| 1 1 2 2 2 3 17 | 10 |
| 1 1 2 2 4 5 13 | 14 |
| 1 1 2 3 5 5 11 | 18 |
| 1 1 2 4 6 7 7 | 20 |
| 1 1 3 3 3 4 13 | 16 |
| 1 1 3 4 4 7 8 | 24 |
| 1 1 4 4 4 5 9 | 24 |
| 1 2 2 3 3 3 14 | 16 |
| 1 2 3 3 5 7 7 | 22 |

Here is a winning strategy from Fulves's starting position:

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Rem                    Position
21                   1 2 3 4 5 6 7
20                   1 1 2 4 6 7 7       The only other winning move is to
                       /      \          1 2 3 3 5 7 7, and takes 2 moves longer.
19             1 1 1 4 6 7 8    1 1 1 2 6 7 10  ··· ∗
                       \      /          In the lefthand variation, 1 1 1 3 6 7 9
18                   1 1 1 1 6 8 10      also wins, but 4 moves more slowly.
17                   1 1 1 7 8 10             ··· ∗
16                   1 1 1 7 7 11
15                   1 1 1 6 7 12             ··· ∗
14                   1 1 1 6 6 13
13                   1 1 1 5 6 14             ··· ∗
12                   1 1 1 5 5 15
11                   1 1 1 4 5 16             ··· ∗
10                   1 1 1 4 4 17
9                    1 1 1 3 4 18             ··· ∗
8                    1 1 1 3 3 19
                   /       |       \
7        1 1 3 4 19    1 1 1 2 3 20    1 1 1 2 4 19
             |                  \      /
6        1 1 2 5 19           1 1 1 2 2 21
             |            /          |          \
5        1 3 5 19   1 1 2 3 21   1 1 1 1 2 22   1 1 1 1 3 21
            \     /                     \      /
4          1 3 3 21                  1 1 1 1 1 23
            /     \                     /      \
3        1 1 5 21   1 1 3 23   1 1 1 2 23     1 1 1 1 24
            \            \      /            /
2                    1 1 1 25
                     /      \               ∗ other options
1                1 2 25    1 1 26           lose more quickly.
                     \      /
0                    2 26   wins.
```

Richard Nowakowski has investigated some variants of Fulves's game:

1. Take a row of consecutive coins. A move is to take one or more coins from a heap and put them on an adjacent heap of coins provided that the second heap is at least as large as the first. Assume that we start with a row of adjacent coins — each heap consisting of one coin. The nim values are periodic with period 7 and the values are 0 1 2 0 3 1 0 — no exceptional values. To prove this one needs to know the nim values of the game with the first heap having two coins and also the game with both end heaps having two coins. The latter is periodic, period 7 — 2 1 0 0 1 3 0 — no exceptions. The former also has period 7 — 5 5 4 2 5 6 4 — but this does not start until the game $2.1^{15}$

2. Take a row of three consecutive heaps — sizes $i, j, k$. It is not too hard to see that

a. if $j > i$ and $j > k$ then the nim-value of the position $i.j.k$ is $i \overset{*}{+} k$.

b. if $i = j$ and $j \geq k$, then the nim-value is $i + k$

c. if either $i > j$ or $k > j$, then the nim-value is $j$

The proof follows easily by considering the tables of nim-values for $j = 0, 1, 2, 3, \ldots$.

The misère version of 1. might be interesting.

**39. Sowing** or **Mancala Games**

Kraitchik (1941, p. 282) describes Ruma, which he attributes to Mr. Punga. Bell & Cornelius (1988, pp. 22–38) list Congklak, from Indonesia; Mankal'ah L'ib Al-Ghashim (the game of the unlearned); Leab El-Akil (the game of the wise or intelligent); Wari; Kalah, from Sumeria; Gabata, from Ethiopia; Kiarabu, from Africa; as well as some solitaire games based on a similar idea. Botermans et al. (1989, pp. 174–179) describe Mefuhva from Malawi and Kiuthi from Kenya. Many of these games go back for thousands of years, but several should be susceptible to present day methods of analysis. See Jeff Erickson's article in these Proceedings.

Conway starts with a **line** of heaps of beans. A typical move is to take (some of) a heap of size $N$ and do something with it that depends on the game and on $N$. He regards the nicest move as what he calls the **African move** in which **all** the beans in a heap are picked up and 'sowed' onto successive heaps, **and** subject to the condition that the last bean must land on a nonempty heap. Beans are sowed to the right if you are Left, to the left if you are Right, or either way if you're playing impartially.

In the partizan version, the position 1 (a single bean) has value 0, of course; the position $1.1 = \{0.2 \,|\, 2.0\}$ has value $\{0 \,|\, 0\} = *$; and so does $1.1.1 = \{1.0.2, 2.1 \,|\, 1.2, 2.0.1\}$, since $2.1 = \{\ \,|\, 3.0\}$, $3.0 = \{\ \,|\ \} = 0$ and $1.0.2 = \{\ \,|\, 2.1\}$, so that '3' has value 0, 2.1 has value $-1$, 1.0.2 value $-2$, 1.1.1 value $\{-2, -1 \,|\, 1, 2\} = 0$, 1.1.1.1 value 0, and 1.1.1.1.1 value $\pm\frac{1}{2}$.

**40. Chess.** Noam Elkies has found endgames with values 0, 1, $\frac{1}{2}$, $*$, $\uparrow$, $\uparrow*$, $\Uparrow*$, etc.; see his paper in this book; and see also problems **29**, **30** and **45**.

**41. Sequential compounds** of games have been studied by Stromquist & Ullman. They mention a more general compound. Let $(P, <)$ be a finite poset and for each $x \in P$ let $G_x$ be a game. Consider a game $G(P)$ played as follows. Moves are allowed in any single component $G_x$ provided that no legal moves remain in any component $G_y$ with $y > x$. A player unable to move loses. The sequential compound is the special case when $(P, <)$ is a chain (or linear order). The **sum** or disjunctive compound is the case where $(P, <)$ is an antichain. They have no coherent theory of games $G(P)$ for arbitrary posets. They list some more specific problems which may be more tractable. Compare Problem **23** above.

**42. Beanstalk** and **Beans-Don't-Talk** are games invented by John Isbell and by John Conway. See Guy [1986]. Beanstalk is played between Jack and the Giant. The Giant chooses a positive integer, $n_0$. Then J. and G. play alternately $n_1$, $n_2$, $n_3$, ... according to the rule $n_{i+1} = n_i/2$ if $n_i$ is even, $= 3n_i \pm 1$ if $n_i$ is odd; i.e. if $n_i$ is even, there's only one option, while if $n_i$ is odd there are just two. The winner is the person moving to 1. If the Giant chooses an odd number $> 1$, can Jack always win? Not by using the Greedy Strategy (always descend when it's safe to do so, e.g., play the moves that are <u>underlined</u> in the game below), as may be seen from $n_0 = 7$ (J.'s moves in parentheses).

7 (22) 11 (34) 17 (50) 25 (74) 37 (110) 55 (166) 83 (<u>248</u>) 124 (62) 31 (94) 47 (142 71 (214) 107 (322) 161 (482) 241 (722) 361 (1082) 541 (<u>1624</u>) 812 (406) 203 (<u>608</u>) 304 (152) 76 (38) 19 (<u>56</u>) 28 (14) 7 ...

In Beans-Don't-Talk, the move is from $n$ to $(3n \pm 1)/2^*$ where $2^*$ is the highest power of two dividing the numerator; the winner is still the person moving to 1. Are there any drawn positions? There are certainly drawn **plays**, e.g., 7 (5) 7 (5) ... , but 5 is an $\mathcal{N}$-position because there is the immediate winning option $(5 \times 3 + 1)/2^4 = 1$, and 7 is a $\mathcal{P}$-position since the other option $(7 \times 3 + 1)/2 = 11$ is met by $(11 \times 3 - 1)/2^5 = 1$. What we want to know is: *are there any* $\mathcal{O}$-**positions** (positions of infinite remoteness)?

[For a definition of **remoteness** see **38.** above. There are several unanswered questions about the remotenesses of positions in these two games. Remoteness may also be the best tool we have for problems **18** and **19** above.]

**43. Inverting Hackenbush.**

John Conway turns Hackenbush, played on strings, into a hot game by amending the move to 'remove an edge of your color and everything thus disconnected from the ground, and then turn the remaining string upside-down and replant it'. The analysis replaces the 'number tree' (WW, p. 25) by a similar tree, but with the smaller binary fractions replaced by increasingly hot games. The game can be generalized to play on trees: a move which prunes the tree at a vertex $V$ includes replanting the tree with $V$ as its root.

**44. Konane.** See the paper by Ernst & Berlekamp in this volume. There is much to be discovered about this fascinating and eminently playable game, which exhibits the values $0$, $*$, $*2$, $\uparrow$, $2^{-n}$, and many other infinitesimals and also hot values of arbitrarily high temperature.

**45.** Elwyn Berlekamp asks: 'What is the habitat of $*2$ ?' [$*2 = \{0, *|0, *\}$.] It does not occur in Blockbusting, Hackenbush, Col or Go. It does occur in Konane and $6 \times 6$ Chess. What about Chilled Go, Domineering and $8 \times 8$ Chess?

**46.** There are various ways of playing **two-dimensional Nim**. One form is discussed on p. 313 of WW. Another is proposed by Berman, Fraenkel & Kahane in Problem 41, *Discrete Math.*, **45**(1983) 137–138. Start with a rectangular array of heaps of beans. At each move a row or column is selected and a positive number of beans is taken from some of the heaps in that row or column; see Fremlin [1973]. A variant is where beans may be taken only from contiguous heaps. Other variants are played on triangular or hexagonal boards; a special case of this last is Piet Hein's Nimbi, solved by Fraenkel & Herda [1980].

**47.** Many results are known concerning tiling rectangles with polyominoes. One can extend such problems to disconnected polyominoes. E.g., can a rectangle be tiled by

□□ □□ or □□ □□ □ ?

If so, what are the rectangles of least size that can be so tiled?

**48.** Find all words which can be reduced to 1 peg in 1-dimensional Peg Solitaire. E.g., 1, 11, 1101, 110101, $1(10)^k1$. Here 1 represents a peg and 0 an empty space. A move is for a peg to jump over an adjacent peg into an empty adjacent space, and remove the jumped-over peg. E.g., 1101 $\rightarrow$ 0011 $\rightarrow$ 0100. Georg Gunther, Bert Hartnell & Richard Nowakowski found that for an $n \times 1$ board with one empty space, $n$ must be even and the space next but one to the end. If the board is cyclic, the condition is simply $n$ even.

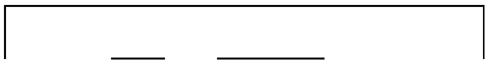
**49.** Elwyn Berlekamp asks if there is a game which has

1. simple, playable rules,
2. an intricate explicit solution, and
3. is provably NP or harder.

**50.** John Selfridge asks: is **Four-File** a draw? Four-File is played on a chess-board with the chess pieces in their usual starting posiions, but only on the a-, c-, e- and g-files. I.e. a rook, a bishop, a king, a knight and four pawns on each side. The moves are normal chess moves except that play takes place only on these four files. I.e., each move ends on one of the files a, c, e or g; pawns cannot capture and there is no castling. The aim is to checkmate your opponent's king.

**51.** Elwyn Berlekamp asks for a complete theory of closed $1 \times n$ Dots-and-Boxes. I.e., with starting position

a sample position being



See WW, Chapter 16. Are there more nimber decomposition theorems? Compile a datebase of nim-values.

**52.** Berlekamp notes that overheating operators provide a very concise way of expressing closed-form solutions to many games, and David Moews observes that monotonicity and linearity depend on the parameters and the domain. How does one play **sums** of games with varied overheating operators? Find a simple, elegant way of relating the operator parameters to the game. See WW, pp. 163–175, Berlekamp [1988], Berlekamp & Wolfe [1994] and Calistrate's paper in these Proceedings.

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