PROBLEMS ON CATALAN AND RELATED NUMBERS

excerpted from *Enumerative Combinatorics*, vol. 2

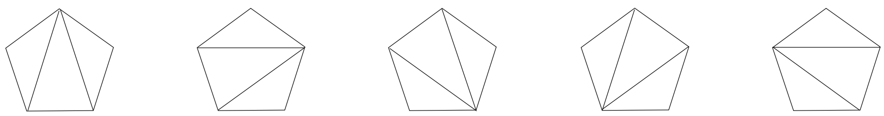
(to be published by Cambridge University Press)

by Richard P. Stanley

version of 1 February 1998

19. [1]–[3+] Show that the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$ count the number of elements of the 65 sets S_i, (a) $\leq i \leq$ (mmm) given below. We illustrate the elements of each S_i for $n = 3$, hoping that these illustrations will make any undefined terminology clear. (The terms used in (uu)–(xx) are defined in Chapter 7.) Ideally S_i and S_j should be proved to have the same cardinality by exhibiting a simple, elegant bijection $\phi_{ij} : S_i \to S_j$ (so 4160 bijections in all). In some cases the sets S_i and S_j will actually coincide, but their descriptions will differ.

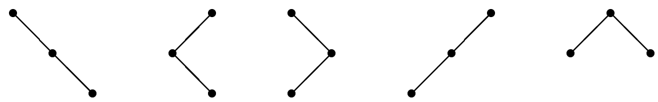
   (a) triangulations of a convex $(n+2)$-gon into n triangles by $n-1$ diagonals that do not intersect in their interiors

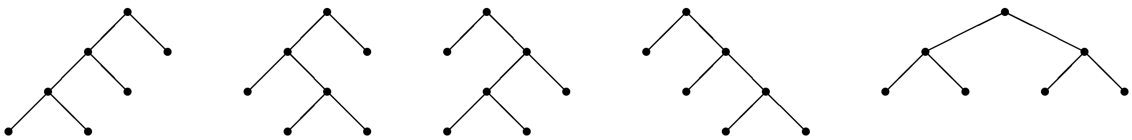
   (b) binary parenthesizations of a string of $n+1$ letters

   $$(x^2 \cdot x)x \quad x(x^2 \cdot x) \quad (x \cdot x^2)x \quad x(x \cdot x^2) \quad x^2 \cdot x^2$$

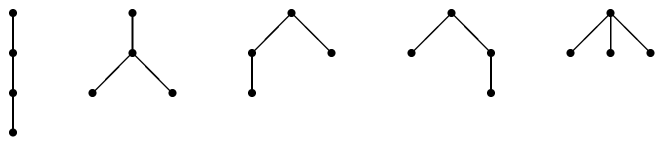
   (c) binary trees with n vertices

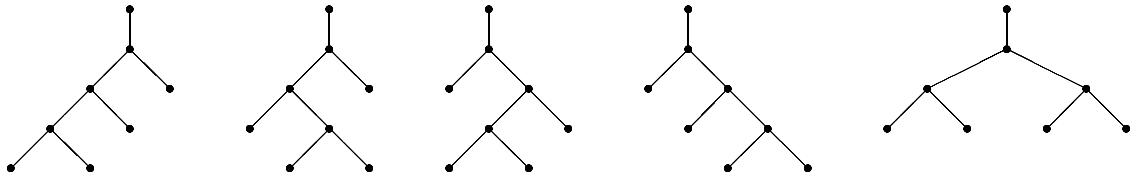
   (d) plane binary trees with $2n+1$ vertices (or $n+1$ endpoints)

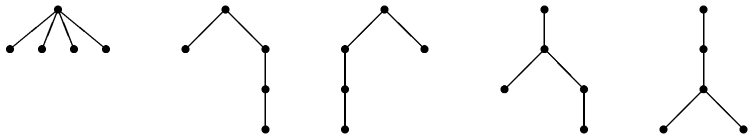
   (e) plane trees with $n+1$ vertices

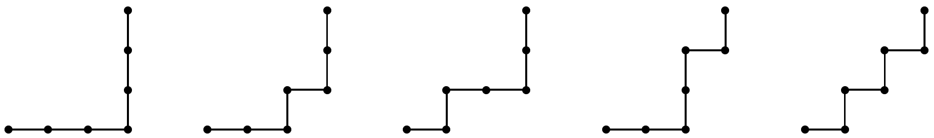
(f) planted (i.e., root has degree one) trivalent plane trees with $2n+2$ vertices



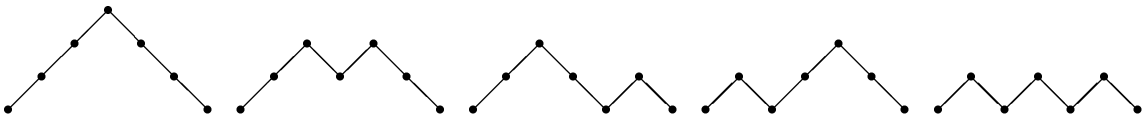
(g) plane trees with $n+2$ vertices such that the rightmost path of each subtree of the root has even length



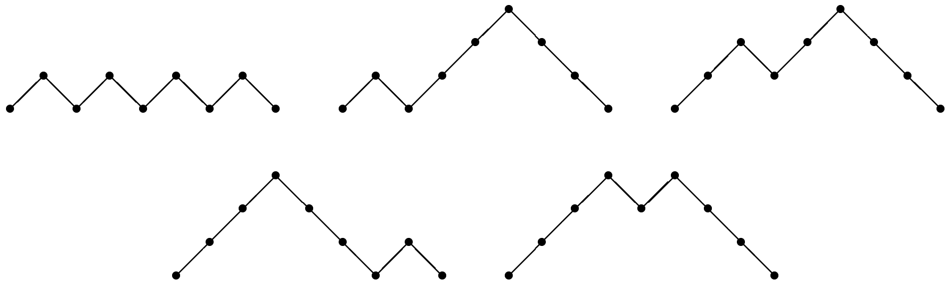
(h) lattice paths from $(0,0)$ to (n,n) with steps $(0,1)$ or $(1,0)$, never rising above the line $y=x$



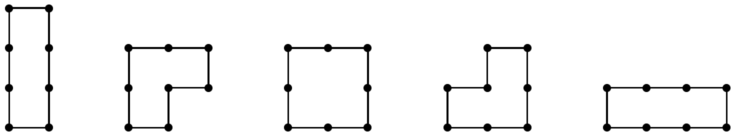
(i) Dyck paths from $(0,0)$ to $(2n,0)$, i.e., lattice paths with steps $(1,1)$ and $(1,-1)$, never falling below the x-axis



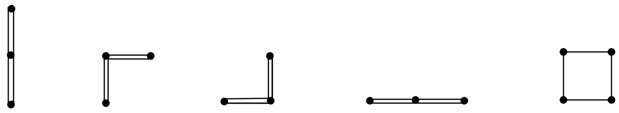
(j) Dyck paths (as defined in (i)) from $(0,0)$ to $(2n+2,0)$ such that any maximal sequence of consecutive steps $(1,-1)$ ending on the x-axis has odd length



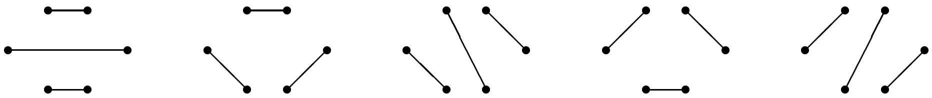
(k) (unordered) pairs of lattice paths with $n+1$ steps, starting at $(0,0)$, using steps $(1,0)$ or $(0,1)$, ending at the same point, and only intersecting at the beginning and end



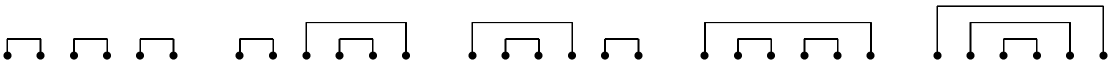
(l) (unordered) pairs of lattice paths with $n-1$ steps each, starting at $(0,0)$, using steps $(1,0)$ or $(0,1)$, ending at the same point, such that one path never arises above the other path



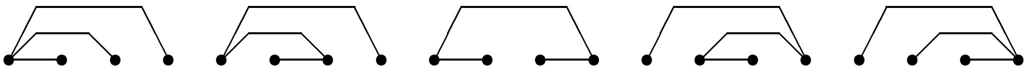
(m) n nonintersecting chords joining $2n$ points on the circumference of a circle



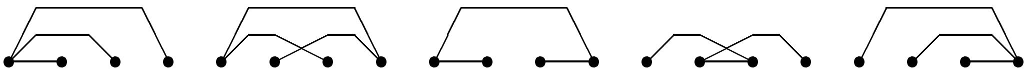
(n) ways of connecting $2n$ points in the plane lying on a horizontal line by n nonintersecting arcs, each arc connecting two of the points and lying above the points



(o) ways of drawing in the plane $n+1$ points lying on a horizontal line L and n arcs connecting them such that (α) the arcs do not pass below L, (β) the graph thus formed is a tree, (γ) no two arcs intersect in their interiors, and (δ) at every vertex, all the arcs exit in the same direction (left or right)



(p) ways of drawing in the plane $n+1$ points lying on a horizontal line L and n arcs connecting them such that (α) the arcs do not pass below L, (β) the graph thus formed is a tree, (γ) no arc (including its endpoints) lies strictly below another arc, and (δ) at every vertex, all the arcs exit in the same direction (left or right)



(q) sequences of n 1's and n -1's such that every partial sum is nonnegative (with -1 denoted simply as $-$ below)

111−−− 11−1−− 11−−1− 1−11−− 1−1−1−

(r) sequences $1 \leq a_1 \leq \cdots \leq a_n$ of integers with $a_i \leq i$

$$111 \quad 112 \quad 113 \quad 122 \quad 123$$

(s) sequences $a_1 < a_2 < \cdots < a_{n-1}$ of integers satisfying $1 \leq a_i \leq 2i$

$$12 \quad 13 \quad 14 \quad 23 \quad 24$$

(t) sequences $a_1, a_2, \ldots, a_n$ of integers such that $a_1 = 0$ and $0 \leq a_{i+1} \leq a_i + 1$

$$000 \quad 001 \quad 010 \quad 011 \quad 012$$

(u) sequences $a_1, a_2, \ldots, a_{n-1}$ of integers such that $a_i \leq 1$ and all partial sums are nonnegative

$$0,0 \quad 0,1 \quad 1,-1 \quad 1,0 \quad 1,1$$

(v) sequences $a_1, a_2, \ldots, a_n$ of integers such that $a_i \geq -1$, all partial sums are nonnegative, and $a_1 + a_2 + \cdots + a_n = 0$

$$0,0,0 \quad 0,1,-1 \quad 1,0,-1 \quad 1,-1,0 \quad 2,-1,-1$$

(w) sequences $a_1, a_2, \ldots, a_n$ of integers such that $0 \leq a_i \leq n-i$, and such that if $i < j$, $a_i > 0$, $a_j > 0$, and $a_{i+1} = a_{i+2} = \cdots = a_{j-1} = 0$, then $j - i > a_i - a_j$

$$000 \quad 010 \quad 100 \quad 200 \quad 110$$

(x) sequences $a_1, a_2, \ldots, a_n$ of integers such that $i \leq a_i \leq n$ and such that if $i \leq j \leq a_i$, then $a_j \leq a_i$

$$123 \quad 133 \quad 223 \quad 323 \quad 333$$

(y) sequences $a_1, a_2, \ldots, a_n$ of integers such that $1 \leq a_i \leq i$ and such that if $a_i = j$, then $a_{i-r} \leq j - r$ for $1 \leq r \leq j-1$

$$111 \quad 112 \quad 113 \quad 121 \quad 123$$

(z) equivalence classes B of words in the alphabet $[n-1]$ such that any three consecutive letters of any word in B are distinct, under the equivalence relation $uijv \sim ujiv$ for any words u, v and any $i, j \in [n-1]$ satisfying $|i-j| \geq 2$

$$\{\emptyset\} \quad \{1\} \quad \{2\} \quad \{12\} \quad \{21\}$$

(For $n = 4$ a representative of each class is given by $\emptyset$, 1, 2, 3, 12, 21, 13, 23, 32, 123, 132, 213, 321, 2132.)

(aa) partitions $\lambda = (\lambda_1, \ldots, \lambda_{n-1})$ with $\lambda_1 \leq n-1$ (so the diagram of λ is contained in an $(n-1) \times (n-1)$ square), such that if $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$ denotes the conjugate partition to λ then $\lambda'_i \geq \lambda_i$ whenever $\lambda_i \geq i$

$$(0,0) \quad (1,0) \quad (1,1) \quad (2,1) \quad (2,2)$$

(bb) permutations $a_1 a_2 \cdots a_{2n}$ of the multiset $\{1^2, 2^2, \ldots, n^2\}$ such that: (i) the first occurrences of $1, 2, \ldots, n$ appear in increasing order, and (ii) there is no subsequence of the form $\alpha\beta\alpha\beta$

112233 112332 122331 123321 122133

(cc) permutations $a_1 a_2 \cdots a_{2n}$ of the set $[2n]$ such that: (i) $1, 3, \ldots, 2n-1$ appear in increasing order, (ii) $2, 4, \ldots, 2n$ appear in increasing order, and (iii) $2i-1$ appears before $2i$, $1 \leq i \leq n$

123456 123546 132456 132546 135246

(dd) permutations $a_1 a_2 \cdots a_n$ of $[n]$ with longest decreasing subsequence of length at most two (i.e., there does not exist $i < j < k$, $a_i > a_j > a_k$), called 321-*avoiding* permutations

123 213 132 312 231

(ee) permutations $a_1 a_2 \cdots a_n$ of $[n]$ for which there does not exist $i < j < k$ and $a_j < a_k < a_i$ (called 312-*avoiding* permutations)

123 132 213 231 321

(ff) permutations w of $[2n]$ with n cycles of length two, such that the product $(1, 2, \ldots, 2n) \cdot w$ has $n+1$ cycles

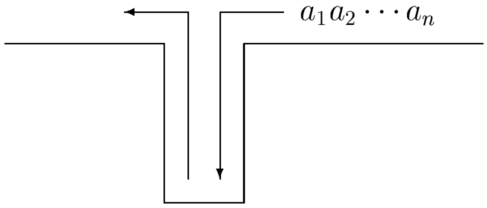
$$\begin{aligned}
(1,2,3,4,5,6)(1,2)(3,4)(5,6) &= (1)(2,4,6)(3)(5) \\
(1,2,3,4,5,6)(1,2)(3,6)(4,5) &= (1)(2,6)(3,5)(4) \\
(1,2,3,4,5,6)(1,4)(2,3)(5,6) &= (1,3)(2)(4,6)(5) \\
(1,2,3,4,5,6)(1,6)(2,3)(4,5) &= (1,3,5)(2)(4)(6) \\
(1,2,3,4,5,6)(1,6)(2,5)(3,4) &= (1,5)(2,4)(3)(6)
\end{aligned}$$

(gg) pairs (u, v) of permutations of $[n]$ such that u and v have a total of $n+1$ cycles, and $uv = (1, 2, \ldots, n)$

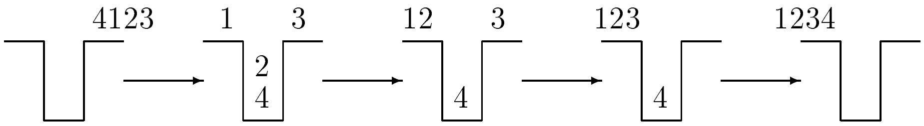
$$(1)(2)(3) \cdot (1,2,3) \quad (1,2,3) \cdot (1)(2)(3) \quad (1,2)(3) \cdot (1,3)(2)$$

$$(1,3)(2) \cdot (1)(2,3) \quad (1)(2,3) \cdot (1,2)(3)$$

(hh) permutations $a_1a_2\cdots a_n$ of $[n]$ that can be put in increasing order on a single stack, defined recursively as follows: If $\emptyset$ is the empty sequence, then let $S(\emptyset)=\emptyset$. If $w=unv$ is a sequence of distinct integers with largest term n, then $S(w)=S(u)S(v)n$. A *stack-sortable* permutation w is one for which $S(w)=w$.

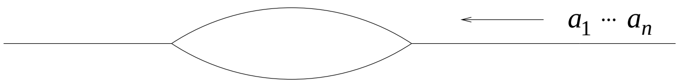


E.g.,



123 132 213 312 321

(ii) permutations $a_1a_2\cdots a_n$ of $[n]$ that can be put in increasing order on two parallel queues. Now the picture is



123 132 213 231 312

(jj) fixed-point free involutions w of $[2n]$ such that if $i<j<k<l$ and $w(i)=k$, then $w(j)\neq l$ (in other words, 3412-avoiding fixed-point free involutions)

$$(12)(34)(56) \quad (14)(23)(56) \quad (12)(36)(45) \quad (16)(23)(45) \quad (16)(25)(34)$$

(kk) cycles of length $2n+1$ in $\mathfrak{S}_{2n+1}$ with descent set $\{n\}$

2371456 2571346 3471256 3671245 5671234

(ll) Baxter permutations (as defined in Exercise **??**) of $[2n]$ or of $[2n+1]$ that are reverse alternating (as defined at the end of Chapter 3.16) and whose inverses are reverse alternating

132546 153426 354612 561324 563412

1325476 1327564 1534276 1735462 1756342

(mm) permutations w of $[n]$ such that if w has ℓ inversions then for all pairs of sequences $(a_1, a_2, \ldots, a_\ell), (b_1, b_2, \ldots, b_\ell) \in [n-1]^\ell$ satisfying

$$w = s_{a_1} s_{a_2} \cdots s_{a_\ell} = s_{b_1} s_{b_2} \cdots s_{b_\ell},$$

where s_j is the adjacent transposition $(j, j+1)$, we have that the ℓ-element *multisets* $\{a_1, a_2, \ldots, a_\ell\}$ and $\{b_1, b_2, \ldots, b_\ell\}$ are equal (thus, for example, $w = 321$ is not counted, since $w = s_1 s_2 s_1 = s_2 s_1 s_2$, and the multisets $\{1, 2, 1\}$ and $\{2, 1, 2\}$ are not equal)

123 132 213 231 312

(nn) permutations w of $[n]$ with the following property: Suppose that w has ℓ inversions, and let

$$R(w) = \{(a_1, \ldots, a_\ell) \in [n-1]^\ell : w = s_{a_1} s_{a_2} \cdots s_{a_\ell}\},$$

where s_j is as in (mm). Then

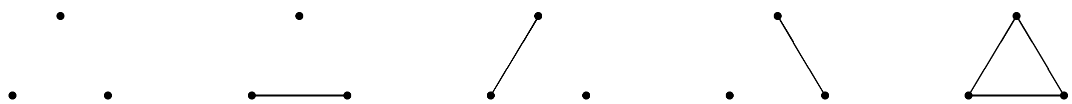
$$\sum_{(a_1, \ldots, a_\ell) \in R(w)} a_1 a_2 \cdots a_\ell = \ell!.$$

$$R(123) = \{\emptyset\}, \quad R(213) = \{(1)\}, \quad R(231) = \{(1, 2)\}$$
$$R(312) = \{(2, 1)\}, \quad R(321) = \{(1, 2, 1), (2, 1, 2)\}$$

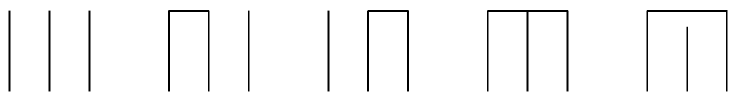
(oo) noncrossing partitions of $[n]$, ie., partitions $\pi = \{B_1, \ldots, B_k\} \in \Pi_n$ such that if $a < b < c < d$ and $a, c \in B_i$ and $b, d \in B_j$, then $i = j$

123 12–3 13–2 23–1 1–2–3

(pp) partitions $\{B_1, \ldots, B_k\}$ of $[n]$ such that if the numbers $1, 2, \ldots, n$ are arranged in order around a circle, then the convex hulls of the blocks $B_1, \ldots, B_k$ are pairwise disjoint



(qq) noncrossing Murasaki diagrams with n vertical lines



(rr) noncrossing partitions of some set $[k]$ with $n+1$ blocks, such that any two elements of the same block differ by at least three

1–2–3–4 14–2–3–5 15–2–3–4 25–1–3–4 16–25–3–4 (ss) noncrossing partitions of $[2n+1]$ into $n+1$ blocks, so that no block contains two consecutive integers

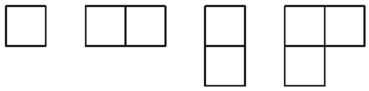
137–46–2–5 1357–2–4–6 157–24–3–6 17–246–3–5 17–26–35–4

(tt) *nonnesting partitions* of $[n]$, i.e., partitions of $[n]$ such that if a, e appear in a block B and b, d appear in a *different* block B' where $a < b < d < e$, then there is a $c \in B$ satisfying $b < c < d$

123 12–3 13–2 23–1 1–2–3

(The unique partition of [4] that isn't nonnesting is 14–23.)

(uu) Young diagrams that fit in the shape $(n-1, n-2, \ldots, 1)$

$\emptyset$ 

(vv) standard Young tableaux of shape (n, n) (or equivalently, of shape $(n, n-1)$)

$$\begin{array}{ccccc} 123 & 124 & 125 & 134 & 135 \\ 456 & 356 & 346 & 256 & 246 \end{array}$$

or

$$\begin{array}{ccccc} 123 & 124 & 125 & 134 & 135 \\ 45 & 35 & 34 & 25 & 24 \end{array}$$

(ww) pairs (P, Q) of standard Young tableaux of the same shape, each with n squares and at most two rows

$$(123, 123) \quad \left(\begin{array}{l} 12 \\ 3 \end{array}, \begin{array}{l} 12 \\ 3 \end{array}\right) \quad \left(\begin{array}{l} 12 \\ 3 \end{array}, \begin{array}{l} 13 \\ 2 \end{array}\right) \quad \left(\begin{array}{l} 13 \\ 2 \end{array}, \begin{array}{l} 12 \\ 3 \end{array}\right) \quad \left(\begin{array}{l} 13 \\ 2 \end{array}, \begin{array}{l} 13 \\ 2 \end{array}\right)$$

(xx) column-strict plane partitions of shape $(n-1, n-2, \ldots, 1)$, such that each entry in the ith row is equal to $n-i$ or $n-i+1$

$$\begin{array}{lllll} 3\,3 & 3\,3 & 3\,2 & 3\,2 & 2\,2 \\ 2 & 1 & 2 & 1 & 1 \end{array}$$

(yy) convex subsets S of the poset $\mathbb{Z} \times \mathbb{Z}$, up to translation by a diagonal vector (m, m), such that if $(i, j) \in S$ then $0 < i - j < n$.

$$\emptyset \quad \{(1,0)\} \quad \{(2,0)\} \quad \{(1,0),(2,0)\} \quad \{(2,0),(2,1)\}$$

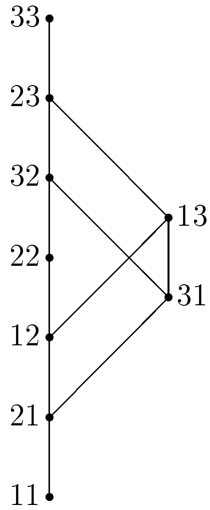
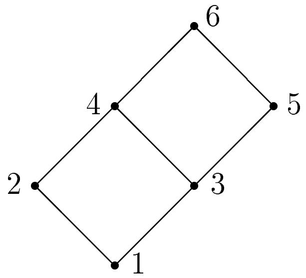


Figure 1: A poset with $C_4 = 14$ order ideals

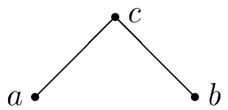
(zz) linear extensions of the poset $\mathbf{2} \times \mathbf{n}$



123456
123546
132456
132546
135246

(aaa) order ideals of $\text{Int}(\boldsymbol{n} - \mathbf{1})$, the poset of intervals of the chain $\mathbf{n} - \mathbf{1}$

$\mathbf{2}$



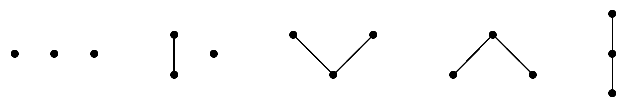
$\text{Int}(\mathbf{2})$

$\emptyset, \quad a, \quad b, \quad ab, \quad abc$

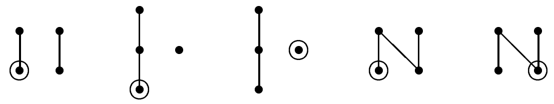
(bbb) order ideals of the poset A_{n-1} obtained from the poset $\mathbf{n} \times \mathbf{n}$ by adding the relations $(i, j) < (j, i)$ if $i > j$ (see Figure 1 for the Hasse diagram of A_3)

$$\emptyset \qquad \{11\} \qquad \{11, 21\} \qquad \{11, 21, 12\} \qquad \{11, 21, 12, 22\}$$

(ccc) nonisomorphic n-element posets with no induced subposet isomorphic to $\mathbf{2} + \mathbf{2}$ or $\mathbf{3} + \mathbf{1}$



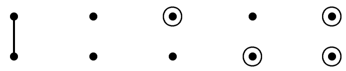
(ddd) nonisomorphic $(n+1)$-element posets that are a union of two chains and that are not a (nontrivial) ordinal sum, rooted at a minimal element



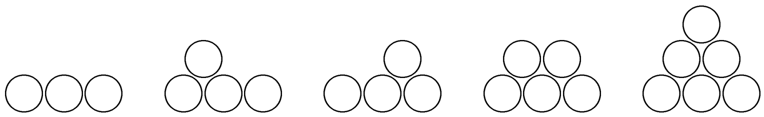
(eee) relations R on $[n]$ that are reflexive (iRi), symmetric ($iRj \Rightarrow jRi$), and such that if $1 \le i < j < k \le n$ and iRk, then iRj and jRk (in the example below we write ij for the pair (i,j), and we omit the pairs ii)

$$\emptyset \quad \{12, 21\} \quad \{23, 32\} \quad \{12, 21, 23, 32\} \quad \{12, 21, 13, 31, 23, 32\}$$

(fff) joining some of the vertices of a convex $(n-1)$-gon by disjoint line segments, and circling a subset of the remaining vertices



(ggg) ways to stack coins in the plane, the bottom row consisting of n consecutive coins



(hhh) n-tuples $(a_1, a_2, \dots, a_n)$ of integers $a_i \ge 2$ such that in the sequence $1a_1a_2\cdots a_n1$, each a_i divides the sum of its two neighbors

$$14321 \quad 13521 \quad 13231 \quad 12531 \quad 12341$$

(iii) n-element multisets on $\mathbb{Z}/(n+1)\mathbb{Z}$ whose elements sum to 0

$$000 \quad 013 \quad 022 \quad 112 \quad 233$$

(jjj) n-element subsets S of $\mathbb{N} \times \mathbb{N}$ such that if $(i,j) \in S$ then $i \ge j$ and there is a lattice path from $(0,0)$ to (i,j) with steps $(0,1)$, $(1,0)$, and $(1,1)$ that lies entirely inside S

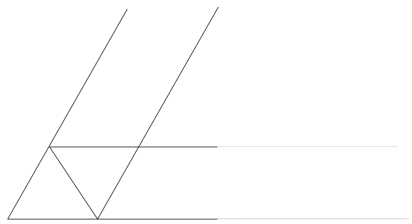
$$\{(0,0),(1,0),(2,0)\} \quad \{(0,0),(1,0),(1,1)\} \quad \{(0,0),(1,0),(2,1)\}$$

$$\{(0,0),(1,1),(2,1)\} \quad \{(0,0),(1,1),(2,2)\}$$

$$\begin{array}{ccccccccccccccccccccccccccc}
1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 \\
& 1 & & 3 & & 2 & & 1 & & 5 & & 1 & & 2 & & 3 & & 1 & & 3 & & 2 & & 1 & & 5 & \\
& & 2 & & 5 & & 1 & & 4 & & 4 & & 1 & & 5 & & 2 & & 2 & & 5 & & 1 & & 4 & & \\
& & & 3 & & 2 & & 3 & & 3 & & 3 & & 2 & & 3 & & 3 & & 3 & & 2 & & 3 & & & \\
& & & & 1 & & 5 & & 2 & & 2 & & 5 & & 1 & & 4 & & 4 & & 1 & & 5 & & & & \\
& & & & & 2 & & 3 & & 1 & & 3 & & 2 & & 1 & & 5 & & 1 & & 2 & & & & & \\
& & & & & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & & & & &
\end{array}$$

Figure 2: The frieze pattern corresponding to the sequence $(1,3,2,1,5,1,2,3)$

(kkk) regions into which the cone $x_1 \geq x_2 \geq \cdots \geq x_n$ in $\mathbb{R}^n$ is divided by the hyperplanes $x_i - x_j = 1$, for $1 \leq i < j \leq n$ (the diagram below shows the situation for $n = 3$, intersected with the hyperplane $x_1 + x_2 + x_3 = 0$)



(lll) positive integer sequences $a_1, a_2, \ldots, a_{n+2}$ for which there exists an integer array (necessarily with $n+1$ rows)

$$\begin{array}{ccccccccccccccccccc}
1 & & 1 & & 1 & & \cdots & & 1 & & 1 & & 1 & & \cdots & & 1 & & 1 \\
& a_1 & & a_2 & & a_3 & & \cdots & & a_{n+2} & & a_1 & & a_2 & & \cdots & & a_{n-1} & \\
& & b_1 & & b_2 & & b_3 & & \cdots & & b_{n+2} & & b_1 & & \cdots & & b_{n-2} & & \\
& & & & & & & & \vdots & & & & & & & & & & \\
& & & & r_1 & & r_2 & & r_3 & & \cdots & & r_{n+2} & & r_1 & & & & \\
& & & & & 1 & & 1 & & 1 & & \cdots & & 1 & & & & &
\end{array} \tag{1}$$

such that any four neighboring entries in the configuration $\begin{array}{cc} & r \\ s & t \\ & u \end{array}$ satisfy $st = ru+1$.

An example of such an array for $(a_1, \ldots, a_8) = (1,3,2,1,5,1,2,3)$ (necessarily unique) is given by Figure 2.

12213 22131 21312 13122 31221

(mmm) n-tuples $(a_1, \ldots a_n)$ of positive integers such that the tridiagonal matrix

$$\begin{bmatrix} a_1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & a_2 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & a_3 & 1 & \cdots & 0 & 0 \\ & & & & \vdots & & \\ 0 & 0 & 0 & 0 & \cdots & a_{n-1} & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & a_n \end{bmatrix}$$

is positive definite with determinant one

131   122   221   213   312

20. (a) [2+] Let m, n be integers satisfying $1 \leq n < m$. Show by a simple bijection that the number of lattice paths from $(1, 0)$ to (m, n) with steps $(0, 1)$ and $(1, 0)$ that intersect the line $y = x$ in at least one point is equal to the number of lattice paths from $(0, 1)$ to (m, n) with steps $(0, 1)$ and $(1, 0)$.

    (b) [2–] Deduce that the number of lattice paths from $(0, 0)$ to (m, n) with steps $(1, 0)$ and $(0, 1)$ that intersect the line $y = x$ only at $(0, 0)$ is given by $\frac{m-n}{m+n}\binom{m+n}{n}$.

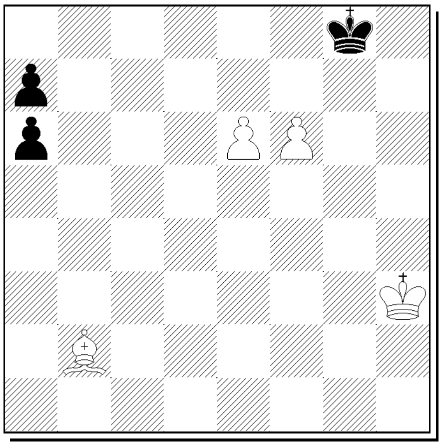
    (c) [1+] Show from (b) that the number of lattice paths from $(0, 0)$ to (n, n) with steps $(1, 0)$ and $(0, 1)$ that never rise above the line $y = x$ is given by the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. (This gives a direct combinatorial proof of interpretation (h) of C_n in Exercise 19.)

21. (a) [2+] Let X_n be the set of all $\binom{2n}{n}$ lattice paths from $(0, 0)$ to (n, n) with steps $(0, 1)$ and $(1, 0)$. Define the *excedance* (also spelled "exceedance") of a path $P \in X_n$ to be the number of i such that at least one point (i, i') of P lies above the line $y = x$ (i.e., $i' > i$). Show that the number of paths in X_n with excedance j is independent of j.

    (b) [1] Deduce that the number of $P \in X_n$ that never rise above the line $y = x$ is given by the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$ (a direct proof of interpretation (h) of C_n in Exercise 19). Compare with Example 5.**??**, which also gives a direct combinatorial interpretation of C_n when written in the form $\frac{1}{n+1}\binom{2n}{n}$ (as well as in the form $\frac{1}{2n+1}\binom{2n+1}{n}$).

22. [2+] Show (bijectively if possible) that the number of lattice paths from $(0, 0)$ to $(2n, 2n)$ with steps $(1, 0)$ and $(0, 1)$ that avoid the points $(2i - 1, 2i - 1)$, $1 \leq i \leq n$, is equal to the Catalan number C_{2n}.

23. [3–] Consider the following chess position.



Black is to make 19 consecutive moves, after which White checkmates Black in one move. Black may not move into check, and may not check White (except possibly on his last move). Black and White are *cooperating* to achieve the aim of checkmate. (In chess problem parlance, this problem is called a *serieshelpmate in 19.*) How many different solutions are there?

24. [?] Explain the significance of the following sequence:

$$\text{un, dos, tres, quatre, cinc, sis, set, vuit, nou, deu, } \ldots$$

25. [2]–[5] Show that the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$ has the algebraic interpretations given below.

(a) number of two-sided ideals of the algebra of all $(n-1)\times(n-1)$ upper triangular matrices over a field

(b) dimension of the space of invariants of $\mathrm{SL}(2,\mathbb{C})$ acting on the $2n$th tensor power $T^{2n}(V)$ of its "defining" two-dimensional representation V

(c) dimension of the irreducible representation of the symplectic group $\mathrm{Sp}(2(n-1),\mathbb{C})$ (or Lie algebra $\mathfrak{sp}(2(n-1),\mathbb{C})$) with highest weight λ_{n-1}, the $(n-1)$st fundamental weight

(d) dimension of the primitive intersection homology (say with real coefficients) of the toric variety associated with a (rationally embedded) n-dimensional cube

(e) the generic number of $\mathrm{PGL}(2,\mathbb{C})$ equivalence classes of degree n rational maps with a fixed branch set

(f) number of translation conjugacy classes of degree $n+1$ monic polynomials in one complex variable, all of whose critical points are fixed

(g) dimension of the algebra (over a field K) with generators $\epsilon_1, \ldots, \epsilon_{n-1}$ and relations

$$\begin{aligned} \epsilon_i^2 &= \epsilon_i \\ \beta\epsilon_i\epsilon_j\epsilon_i &= \epsilon_i, \ \text{ if } |i-j|=1 \\ \epsilon_i\epsilon_j &= \epsilon_j\epsilon_i, \ \text{ if } |i-j|\geq 2, \end{aligned}$$

where β is a nonzero element of K

(h) number of $\oplus$-sign types indexed by A_{n-1}^+ (the set of positive roots of the root system A_{n-1})

(i) Let the symmetric group $\mathfrak{S}_n$ act on the polynomial ring $A = \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ by $w \cdot f(x_1, \ldots, x_n, y_1, \ldots, y_n) = f(x_{w(1)}, \ldots, x_{w(n)}, y_{w(1)}, \ldots, y_{w(n)})$ for all $w \in \mathfrak{S}_n$. Let I be the ideal generated by all invariants of positive degree, i.e.,

$$I = \langle f \in A : w \cdot f = f \text{ for all } w \in \mathfrak{S}_n, \text{ and } f(0) = 0 \rangle.$$

Then (conjecturally) C_n is the dimension of the subspace of A/I affording the sign representation, i.e.,

$$C_n = \dim\{f \in A/I : w \cdot f = (\operatorname{sgn} w) f \text{ for all } f \in \mathfrak{S}_n\}.$$

26. (a) [3–] Let D be a Young diagram of a partition λ, as defined in Chapter 1.3. Given a square s of D let t be the lowest square in the same column as s, and let u be the rightmost square in the same row as s. Let $f(s)$ be the number of paths from t to u that stay within D, and such that each step is one square to the north or one square to the east. Insert the number $f(s)$ in square s, obtaining an array A. For instance, if $\lambda = (5,4,3,3)$ then A is given by

16	7	2	1	1
6	3	1	1	
3	2	1		
1	1	1		

Let M be the largest square subarray (using consecutive rows and columns) of A containing the upper left-hand corner. Regard M as a matrix. For the above example we have

$$M = \begin{bmatrix} 16 & 7 & 2 \\ 6 & 3 & 1 \\ 3 & 2 & 1 \end{bmatrix}.$$

Show that $\det M = 1$.

(b) [2] Find the unique sequence $a_0, a_1, \dots$ of real numbers such that for all $n \geq 0$ we have

$$\det \begin{bmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_n & a_{n+1} & \cdots & a_{2n} \end{bmatrix} = \det \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ a_2 & a_3 & \cdots & a_{n+1} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ a_n & a_{n+1} & \cdots & a_{2n-1} \end{bmatrix} = 1.$$

(When $n = 0$ the second matrix is empty and by convention has determinant one.)

27. (a) [3–] Let V_n be a real vector space with basis $x_0, x_1, \dots, x_n$ and scalar product defined by $\langle x_i, x_j \rangle = C_{i+j}$, the $(i+j)$-th Catalan number. It follows from Exercise 26(b) that this scalar product is positive definite, and therefore V has an orthonormal basis. Is there an orthonormal basis for V_n whose elements are *integral* linear combinations of the x_i's ?

(b) [3–] Same as (a), except now $\langle x_i, x_j \rangle = C_{i+j+1}$.

(c) [5–] Investigate the same question for the matrices M of Exercise 26(a) (so $\langle x_i, x_j \rangle = M_{ij}$) when λ is self-conjugate (so M is symmetric).

28. (a) [3–] Suppose that real numbers $x_1, x_2, \dots, x_d$ are chosen uniformly and independently from the interval $[0, 1]$. Show that the probability that the sequence $x_1, x_2, \dots, x_d$ is convex (i.e., $x_i \leq \frac{1}{2}(x_{i-1}+x_{i+1})$ for $2 \leq i \leq d-1$) is $C_{d-1}/(d-1)!^2$, where C_{d-1} denotes a Catalan number.

(b) [3–] Let $\mathcal{C}_d$ denote the set of all points $(x_1, x_2, \dots, x_d) \in \mathbb{R}^d$ such that $0 \leq x_i \leq 1$ and the sequence $x_1, x_2, \dots, x_d$ is convex. It is easy to see that $\mathcal{C}_d$ is a d-dimensional convex polytope, called the *convexotope*. Show that the vertices of $\mathcal{C}_d$ consist of the points

$$\left(1, \tfrac{j-1}{j}, \tfrac{j-2}{j}, \dots, \tfrac{1}{j}, 0, 0, \dots, 0, \tfrac{1}{k}, \tfrac{2}{k}, \dots, 1\right) \qquad (2)$$

(with at least one 0 coordinate), together with $(1,1,\ldots,1)$ (so $\binom{d+1}{2}+1$ vertices in all). For instance, the vertices of $\mathcal{C}_3$ are $(0,0,0)$, $(0,0,1)$, $(0,\frac{1}{2},1)$, $(1,0,0)$, $(1,\frac{1}{2},0)$, $(1,0,1)$, $(1,1,1)$.

(c) [3] Show that the Ehrhart quasi-polynomial $i(\mathcal{C}_d,n)$ of $\mathcal{C}_d$ (as defined in Chapter 4.6) is given by

$$\begin{aligned} y_d &:= \sum_{n\geq 0} i(\mathcal{C}_d,n)x^n \\ &= \frac{1}{1-x}\left(\sum_{r=1}^{d}\frac{1}{[1][r-1]!}*\frac{1}{[1][d-r]!}-\sum_{r=1}^{d-1}\frac{1}{[1][r-1]!}*\frac{1}{[1][d-1-r]!}\right), \end{aligned} \tag{3}$$

where $[i]=1-x^i$, $[i]!=[1][2]\cdots[i]$, and $*$ denotes Hadamard product. For instance,

$$\begin{aligned} y_1 &= \frac{1}{(1-x)^2} \\ y_2 &= \frac{1+x}{(1-x)^3} \\ y_3 &= \frac{1+2x+3x^2}{(1-x)^3(1-x^2)} \\ y_4 &= \frac{1+3x+9x^2+12x^3+11x^4+3x^5+x^6}{(1-x)^2(1-x^2)^2(1-x^3)} \\ y_5 &= \frac{1+4x+14x^2+34x^3+63x^4+80x^5+87x^6+68x^7+42x^8+20x^9+7x^{10}}{(1-x)(1-x^2)^2(1-x^3)^2(1-x^4)}. \end{aligned}$$

Is there a simpler formula than (3) for $i(\mathcal{C}_d,n)$ or y_d?

29. [3] Suppose that $n+1$ points are chosen uniformly and independently from inside a square. Show that the probablility that the points are in convex position (i.e., each point is a vertex of the convex hull of all the points) is $(C_n/n!)^2$.

30. [3–] Let f_n be the number of partial orderings of the set $[n]$ that contain no induced subposets isomorphic to $\mathbf{3}+\mathbf{1}$ or $\mathbf{2}+\mathbf{2}$. (This exercise is the labelled analogue of Exercise 19(ccc). As mentioned in the solution to this exercise, such posets are called *semiorders*.) Let $C(x)=1+x+2x^2+5x^3+\cdots$ be the generating function for Catalan numbers. Show that

$$\sum_{n\geq 0} f_n\frac{x^n}{n!}=C(1-e^{-x}), \tag{4}$$

the composition of $C(x)$ with the series $1-e^{-x}=x-\frac{1}{2}x^2+\frac{1}{6}x^3-\cdots$.

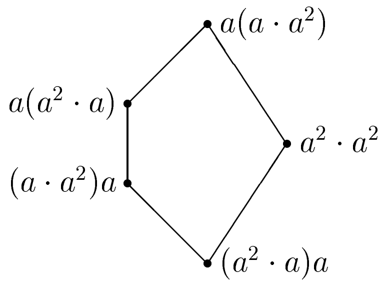


Figure 3: The Tamari lattice T_3

31. (a) [3–] Let $\mathcal{P}$ denote the convex hull in $\mathbb{R}^{d+1}$ of the origin together with all vectors $e_i - e_j$, where e_i is the ith unit coordinate vector and $i < j$. Thus $\mathcal{P}$ is a d-dimensional convex polytope. Show that the relative volume of $\mathcal{P}$ (as defined in Chapter 4.6) is equal to $C_d/d!$, where C_d denotes a Catalan number.

    (b) [3] Let $i(\mathcal{P}, n)$ denote the Ehrhart polynomial of $\mathcal{P}$. Find a combinatorial interpretation of the coefficients of the i-Eulerian polynomial (in the terminology of Chapter 4.3)
$$(1-x)^{d+1} \sum_{n \geq 0} i(\mathcal{P}, n) x^n.$$

32. (a) [3–] Define a partial order T_n on the set of all binary bracketings (parenthesizations) of a string of length $n+1$ as follows. We say that v covers u if u contains a subexpression $(xy)z$ (where x, y, z are bracketed strings) and v is obtained from u by replacing $(xy)z$ with $x(yz)$. For instance, $((a^2 \cdot a)a^2)(a^2 \cdot a^2)$ is covered by $((a \cdot a^2)a^2)(a^2 \cdot a^2)$, $(a^2(a \cdot a^2))(a^2 \cdot a^2)$, $((a^2 \cdot a)a^2)(a(a \cdot a^2))$, and $(a^2 \cdot a)(a^2(a^2 \cdot a^2))$. Figures 3 and 4 show the Hasse diagrams of T_3 and T_4. (In Figure 4, we have encoded the binary bracketing by a string of four $+$'s and four $-$'s, where a $+$ stands for a left parenthesis and a $-$ for the letter a, with the last a omitted.) Let U_n be the poset of all integer vectors $(a_1, a_2, ..., a_n)$ such that $i \leq a_i \leq n$ and such that if $i \leq j \leq a_i$ then $a_j \leq a_i$, ordered coordinatewise. Show that T_n and U_n are isomorphic posets.

    (b) [2] Deduce from (a) that T_n is a lattice (called the *Tamari lattice*).

33. Let C be a convex n-gon. Let $\mathcal{S}$ be the set of all sets of diagonals of C that do not intersect in the interior of C. Partially order the element of $\mathcal{S}$ by inclusion, and add a $\hat{1}$. Call the resulting poset A_n.

    (a) [3–] Show that A_n is a simplicial Eulerian lattice of rank $n-2$, as defined in Chapter 3.14.

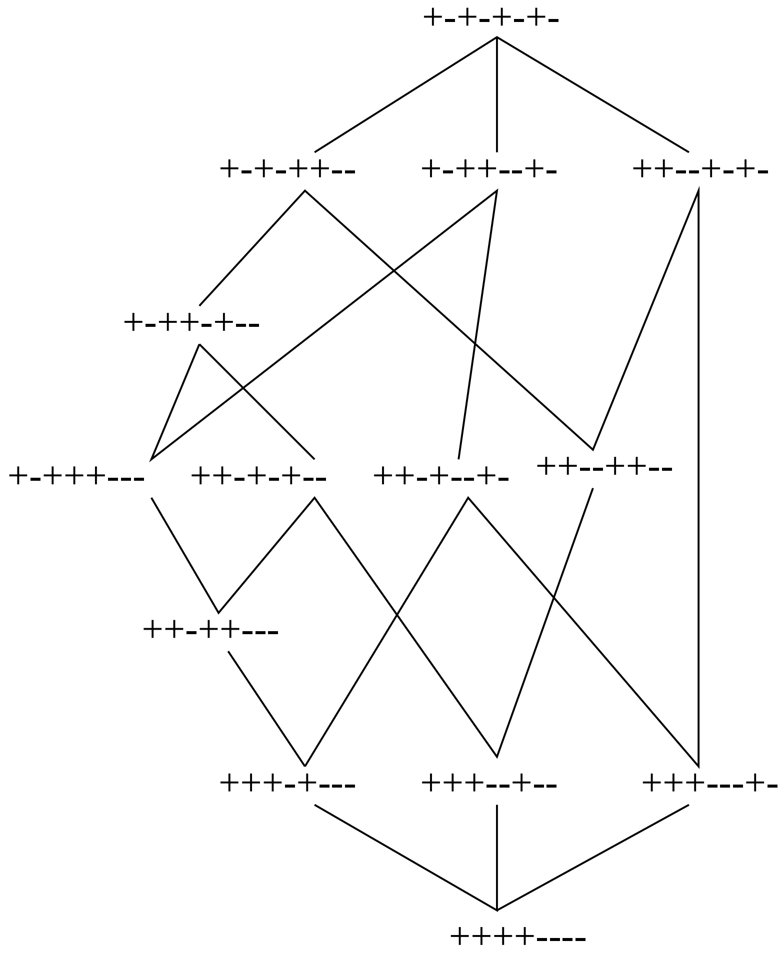


Figure 4: The Tamari lattice T_4

(b) [3] Show in fact that A_n is the lattice of faces of an $(n-3)$-dimensional convex polytope $\mathcal{Q}_n$.

(c) [3–] Find the number $W_i = W_i(n)$ of elements of A_n of rank i. Equivalently, W_i is the number of ways to draw i diagonals of C that do not intersect in their interiors. Note that by Proposition **??**, $W_i(n)$ is also the number of plane trees with $n+i$ vertices and $n-1$ endpoints, such that no vertex has exactly one successor.

(d) [3–] Define

$$\sum_{i=0}^{n-3} W_i(x-1)^{n-i-3} = \sum_{i=0}^{n-3} h_i x^{n-3-i}, \tag{5}$$

as in equation (3.44). The vector $(h_0, \ldots, h_{n-3})$ is called the *h-vector* of A_n (or of the polytope $\mathcal{Q}_n$). Find an explicit formula for each h_i.

34. There are many possible q-analogues of Catalan numbers. In (a) we give what is perhaps the most natural "combinatorial" q-analogue, while in (b) we give the most natural "explicit formula" q-analogue. In (c) we give an interesting extension of (b), while (d) and (e) are concerned with another special case of (c).

(a) [2+] Let

$$C_n(q) = \sum_P q^{A(P)},$$

where the sum is over all lattice paths P from $(0,0)$ to (n,n) with steps $(1,0)$ and $(0,1)$, such that P never rises above the line $y=x$, and where $A(P)$ is the area under the path (and above the x-axis). Note that by Exercise 19(h), we have $C_n(1) = C_n$. (It is interesting to see what statistic corresponds to $A(P)$ for many of the other combinatorial interpretations of C_n given in Exercise 19.) For instance, $C_0(q) = C_1(q) = 1$, $C_2(q) = 1+q$, $C_3(q) = 1+q+2q^2+q^3$, $C_4(q) = 1+q+2q^2+3q^3+3q^4+3q^5+q^6$. Show that

$$C_{n+1}(q) = \sum_{i=0}^{n} C_i(q) C_{n-i}(q) q^{(i+1)(n-i)}.$$

Deduce that if $\tilde{C}_n(q) = q^{\binom{n}{2}} C_n(1/q)$, then the generating function

$$F(x) = \sum_{n \geq 0} \tilde{C}_n(q) x^n$$

satisfies

$$xF(x)F(qx) - F(x) + 1 = 0.$$

From this we get the continued fraction expansion

$$F(x) = \cfrac{1}{1 - \cfrac{x}{1 - \cfrac{qx}{1 - \cfrac{q^2 x}{1 - \cdots}}}}.$$

(b) [2+] Define

$$c_n(q) = \frac{1}{(\boldsymbol{n+1})} \binom{\boldsymbol{2n}}{\boldsymbol{n}}.$$

For instance, $c_0(q) = c_1(q) = 1$, $c_2(q) = 1 + q^2$, $c_3(q) = 1 + q^2 + q^3 + q^4 + q^6$, $c_4(q) = 1 + q^2 + q^3 + 2q^4 + q^5 + 2q^6 + q^7 + 2q^8 + q^9 + q^{10} + q^{12}$. Show that

$$c_n(q) = \sum_w q^{\mathrm{maj}(w)},$$

where w ranges over all sequences $a_1 a_2 \cdots a_{2n}$ of n 1's and n -1's such that each partial sum is nonnegative, and where

$$\mathrm{maj}(w) = \sum_{\{i\,:\,a_i > a_{i+1}\}} i,$$

the major index of w.

(c) [3–] Let t be a parameter, and define

$$c_n(t;q) = \frac{1}{(\boldsymbol{n+1})} \sum_{i=0}^{n} \binom{\boldsymbol{n}}{\boldsymbol{i}} \binom{\boldsymbol{n}}{\boldsymbol{i+1}} q^{i^2+it}.$$

Show that

$$c_n(t;q) = \sum_w q^{\mathrm{maj}(w)+(t-1)\mathrm{des}(w)},$$

where w ranges over the same set as in (b), and where

$$\mathrm{des}(w) = \#\{i : a_i > a_{i+1}\},$$

the number of descents of w. (Hence $c_n(1;q) = c_n(q)$.)

(d) [3–] Show that

$$c_n(0;q) = \frac{1+q}{1+q^n} c_n(q).$$

For instance, $c_0(0;q) = c_1(0;q) = 1$, $c_2(0;q) = 1+q$, $c_3(0;q) = 1+q+q^2+q^3+q^4$, $c_4(0;q) = 1 + q + q^2 + 2q^3 + 2q^4 + 2q^5 + 2q^6 + q^7 + q^8 + q^9$.

(e) [3+] Show that the coefficients of $c_n(0;q)$ are *unimodal*, i.e., if $c_n(0;q) = \sum b_i q^i$, then for some j we have $b_0 \leq b_1 \leq \cdots \leq b_j \geq b_{j+1} \geq b_{j+2} \geq \cdots$. (In fact, we can take $j = \lfloor \frac{1}{2} \deg c_n(0;q) \rfloor = \lfloor \frac{1}{2}(n-1)^2 \rfloor$.)

35. Let Q_n be the poset of direct sum decompositions of an n-dimensional vector space V_n over the field $\mathbb{F}_q$, as defined in Example 5.5.2(b). Let $\bar{Q}_n$ denote Q_n with a $\hat{0}$ adjoined, and let $\mu_n(q) = \mu_{\bar{Q}_n}(\hat{0}, \hat{1})$. Hence by (5.74) we have

$$-\sum_{n\geq 1} \mu_n(q) \frac{x^n}{q^{\binom{n}{2}} \boldsymbol{(n)!}} = \log \left(\sum_{n \geq 0} \frac{x^n}{q^{\binom{n}{2}} \boldsymbol{(n)!}} \right).$$

(a) [3–] Show that

$$\mu_n(q) = \frac{1}{n}(-1)^n (q-1)(q^2-1)\cdots(q^{n-1}-1)P_n(q),$$

where $P_n(q)$ is a polynomial in q of degree $\binom{n}{2}$ with nonnegative integral coefficients, satisfying $P_n(1) = \binom{2n-1}{n}$. For instance,

$$\begin{aligned} P_1(q) &= 1 \\ P_2(q) &= 2+q \\ P_3(q) &= 3+3q+3q^2+q^3 \\ P_4(q) &= (2+2q^2+q^3)(2+2q+2q^2+q^3). \end{aligned}$$

(b) Show that

$$\exp \sum_{n \geq 1} q^{\binom{n}{2}} P_n(1/q) \frac{x^n}{n} = \sum_{n\geq 1} q^{\binom{n}{2}} C_n(1/q) x^n,$$

where $C_n(q)$ is the q-Catalan polynomial defined in Exercise 34(a).

36. (a) [2+] The *Narayana numbers* $N(n,k)$ are defined by $N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$. Let X_{nk} be the set of all sequences $w = w_1 w_2 \cdots w_{2n}$ of n 1's and n -1's with all partial sums nonnegative, such that

$$k = \#\{j : w_j = 1, w_{j+1} = -1\}.$$

Give a combinatorial proof that $N(n,k) = \#X_{nk}$. Hence by Exercise 19(q), there follows

$$\sum_{k=1}^{n} N(n,k) = C_n.$$

(It is interesting to find for each of the combinatorial interpretations of C_n given by Exercise 19 a corresponding decomposition into subsets counted by Narayana numbers.)

(b) [2+] Let $F(x,t) = \sum_{n\geq 1}\sum_{k\geq 1} N(n,k)x^n t^k$. Using the combinatorial interpretation of $N(n,k)$ given in (a), show that

$$xF^2 + (xt + x - 1)F + xt = 0, \tag{6}$$

so

$$F(x,t) = \frac{1 - x - xt - \sqrt{(1-x-xt)^2 - 4x^2t}}{2x}.$$

37. [2+] The *Motzkin numbers* M_n are defined by

$$\begin{aligned}\sum_{n\geq 0} M_n x^n &= \frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x^2} \\ &= 1 + x + 2x^2 + 4x^3 + 9x^4 + 21x^5 + 51x^6 + 127x^7 + 323x^8 \\ &\quad +835x^9 + 2188x^{10} + \cdots.\end{aligned}$$

Show that $M_n = \Delta^n C_1$ and $C_n = \Delta^{2n} M_0$, where C_n denotes a Catalan number.

38. [3–] Show that the Motzkin number M_n has the following combinatorial interpretations. (See Exercise **??**(b) for an additional interpretation.)

(a) Number of ways of drawing any number of nonintersecting chords among n points on a circle.

(b) Number of walks on $\mathbb{N}$ with n steps, with steps -1, 0, or 1, starting and ending at 0.

(c) Number of lattice paths from $(0,0)$ to (n,n), with steps $(0,2)$, $(2,0)$, and $(1,1)$, never rising above the line $y = x$.

(d) Number of pairs $1 \leq a_1 < \cdots < a_k \leq n$ and $1 \leq b_1 < \cdots < b_k \leq n$ of integer sequences such that $a_i \leq b_i$ and every integer in the set $[n]$ appears at least once among the a_i's and b_i's.

(e) Number of ballot sequences (as defined in Corollary **??**(**??**)) $(a_1, \ldots, a_{2n+2})$ such that we never have $(a_{i-1}, a_i, a_{i+1}) = (1, -1, 1)$.

(f) Number of plane trees with $n/2$ edges, allowing "half-edges" that have no successors and count as half an edge.

(g) Number of plane trees with $n+1$ edges in which no vertex, the root excepted, has exactly one successor.

(h) Number of plane trees with n edges in which every vertex has at most two successors.

(i) Number of binary trees with $n-1$ edges such that no two consecutive edges slant to the right.

(j) Number of plane trees with $n+1$ vertices such that every vertex of odd height (with the root having height 0) has at most one successor.

(k) Number of noncrossing partitions $\pi = \{B_1, \dots, B_k\}$ of $[n]$ (as defined in Exercise 3.68) such that if $B_i = \{b\}$ and $a < b < c$, then a and c appear in different blocks of π.

(l) Number of noncrossing partitions π of $[n+1]$ such that no block of π contains two consecutive integers.

39. [3–] The Schröder numbers r_n and s_n were defined in Section 2. Show that they have the following combinatorial interpretations.

(a) s_{n-1} is the total number of bracketings (parenthesizations) of a string of n letters.

(b) s_{n-1} is the number of plane trees with no vertex of degree one and with n endpoints.

(c) s_n is the number of lattice paths in the (x, y)-plane from $(0, 0)$ to the x-axis using steps $(1, k)$, where $k \in \mathbb{P}$ or $k = -1$, never passing below the x-axis, and with n steps of the form $(1, -1)$.

(d) s_n is the number of lattice paths in the (x, y)-plane from $(0, 0)$ to (n, n) using steps $(k, 0)$ or $(0, 1)$ with $k \in \mathbb{P}$, and never passing above the line $y = x$.

(e) s_n is the number of ways to draw any number of diagonals of a convex $(n+2)$-gon that do not intersect in their interiors

(f) s_n is the number of sequences $i_1 i_2 \cdots i_k$, where $i_j \in \mathbb{P}$ or $i_j = -1$ (and k can be arbitrary), such that $n = \#\{j : i_j = -1\}$, $i_1 + i_2 + \cdots + i_j \geq 0$ for all j, and $i_1 + i_2 + \cdots + i_k = 0$.

(g) r_n is the number of lattice paths from $(0, 0)$ to (n, n), with steps $(1, 0)$, $(0, 1)$, and $(1, 1)$, that never rise above the line $y = x$.

(h) r_{n-1} is the number of $n \times n$ permutation matrices P with the following property: We can eventually reach the all 1's matrix by starting with P and continually replacing a 0 by a 1 if that 0 has at least two adjacent 1's, where an entry a_{ij} is defined to be adjacent to $a_{i\pm1,j}$ and $a_{i,j\pm1}$.

(i) Let $u = u_1 \cdots u_k \in \mathfrak{S}_k$. We say that a permutation $w = w_1 \cdots w_n \in \mathfrak{S}_n$ is *u-avoiding* if no subsequence $w_{a_1}, \dots, w_{a_k}$ (with $a_1 < \cdots < a_k$) is in the same relative order as u, i.e., $u_i < u_j$ if and only if $w_{a_i} < w_{a_j}$. Let $\mathfrak{S}_n(u, v)$ denote the set of permutations $w \in \mathfrak{S}_n$ avoiding both the permutations $u, v \in \mathfrak{S}_4$. There is a group G of order 16 that acts on the set of pairs (u, v) of unequal elements of $\mathfrak{S}_4$ such that if (u, v) and (u', v') are in the same G-orbit (in which case we say that they are *equivalent*), then there is a simple bijection between $\mathfrak{S}_n(u,v)$ and $\mathfrak{S}_n(u',v')$ (for all n). Namely, identifying a permutation with the corresponding permutation matrix, the orbit of (u,v) is obtained by possibly interchanging u and v, and then doing a simultaneous dihedral symmetry of the square matrices u and v. There are then ten inequivalent pairs $(u,v) \in \mathfrak{S}_4 \times \mathfrak{S}_4$ for which $\#\mathfrak{S}_n(u,v) = r_{n-1}$, namely, $(1234, 1243)$, $(1243, 1324)$, $(1243, 1342)$, $(1243, 2143)$, $(1324, 1342)$, $(1342, 1423)$, $(1342, 1432)$, $(1342, 2341)$, $(1342, 3142)$, and $(2413, 3142)$.

(j) r_{n-1} is the number of permutations $w = w_1 w_2 \cdots w_n$ of $[n]$ with the following property: It is possible to insert the numbers $w_1, \ldots, w_n$ in order into a string, and to remove the numbers from the string in the order $1, 2, \ldots, n$. Each insertion must be at the beginning or end of the string. At any time we may remove the first (leftmost) element of the string. (*Example:* $w = 2413$. Insert 2, insert 4 at the right, insert 1 at the left, remove 1, remove 2, insert 3 at the left, remove 3, remove 4.)

(k) r_n is the number of sequences of length $2n$ from the alphabet A, B, C such that: (i) for every $1 \leq i < 2n$, the number of A's and B's among the first i terms is not less than the number of C's, (ii) the total number of A's and B's is n (and hence the also the total number of C's), and (iii) no two consecutive terms are of the form CB.

(l) r_{n-1} is the number of noncrossing partitions (as defined in Exercise 3.68) of some set $[k]$ into n blocks, such that no block contains two consecutive integers.

(m) s_{n-1} is the number of graphs G (without loops and multiple edges) on the vertex set $[n+1]$ with the following two properties: (α) All of the edges $\{1, n+1\}$ and $\{i, i+1\}$ are edges of G, and (β) G is *noncrossing*, i.e., there are not both edges $\{a, c\}$ and $\{b, d\}$ with $a < b < c < d$. Note that an arbitrary noncrossing graph on $[n+1]$ can be obtained from those satisfying (α)–(β) by deleting any subset of the required edges in (α). Hence the total number of noncrossing graphs on $[n+1]$ is $2^{n+1} s_{n-1}$.

(n) r_{n-1} is the number of reflexive and symmetric relations R on the set $[n]$ such that if iRj with $i < j$, then we never have uRv for $i \leq u < j < v$.

(o) r_{n-1} is the number of reflexive and symmetric relations R on the set $[n]$ such that if iRj with $i < j$, then we never have uRv for $i < u \leq j < v$.

(p) r_{n-1} is the number of ways to cover with disjoint dominos (or dimers) the set of squares consisting of $2i$ squares in the ith row for $1 \leq i \leq n-1$, and with $2(n-1)$ squares in the nth row, such that the row centers lie on a vertical line. See Figure 5 for the case $n = 4$.

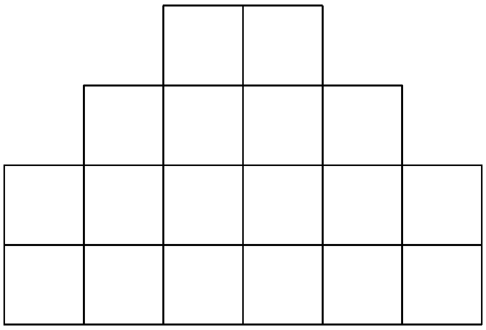


Figure 5: A board with $r_3 = 22$ domino tilings

40. [3–] Let a_n be the number of permutations $w = w_1 w_2 \cdots w_n \in \mathfrak{S}_n$ such that we never have $w_{i+1} = w_i \pm 1$. E.g., $a_4 = 2$, corresponding to 2413 and 3142. Equivalently, a_n is the number of ways to place n nonattacking kings on an $n \times n$ chessboard with one king in every row and column. Let

$$\begin{aligned} A(x) &= \sum_{n\geq 0} a_n x^n \\ &= 1 + x + 2x^4 + 14x^5 + 90x^6 + 646x^7 + 5242x^8 + \cdots. \end{aligned}$$

Show that $A(xR(x)) = \sum_{n\geq 0} n! x^n := E(x)$, where

$$R(x) = \sum_{n\geq 0} r_n x^n = \frac{1}{2x}\left(1 - x - \sqrt{1-6x+x^2}\right),$$

the generating function for Schröder numbers. Deduce that

$$A(x) = E\left(\frac{x(1-x)}{1+x}\right).$$

SOLUTIONS

19. It would require a treatise in itself to discuss thoroughly all the known interconnections among these problems. We will content ourselves with some brief hints and references that should serve as a means of further exposure to "Catalan disease" or "Catalania" (= Catalan mania). One interesting item omitted from the list because of its complicated description is the number of flexagons of order $n+1$; see C. O. Oakley and R. J. Wisner, *Amer. Math. Monthly* **64** (1957), 143–154 (esp. p. 152) for more information.

(a,b,c,h,i,q) These are covered by Corollary **??**.

(d) This is covered by Example **??**. Alternatively, do a depth-first search, ignoring the root edge, recording 1 when a left edge is first encountered, and recording -1 when a right edge is first encountered. This gives a bijection with (q). Note also that when all endpoints are removed (together with the incident edges), we obtain the trees of (c).

(e) This is covered by Example **??**. For a bijection with (q), do a depth-first (preorder) search through the tree. When going "down" an edge (away from the root) record a 1, and when going up an edge record a -1. For further information and references, see D. A. Klarner, *J. Combinatorial Theory* **9** (1970), 401–411.

(f) When the root is removed we obtain the trees of (d). See also Klarner, *op. cit.*

(g) The bijection between parts (**??**) and (**??**) of Proposition **??** gives a bijection between the present problem and (j). An elegant bijection with (e) was given by F. Bernhart (private communication, 1996).

(j) Let $A(x) = x + x^3 + 2x^4 + 6x^5 + \cdots$ (respectively, $B(x) = x^2 + x^3 + 3x^4 + 8x^5 + \cdots$) be the generating function for Dyck paths from $(0,0)$ to $(2n,0)$ $(n > 0)$ such that the path only touches the x-axis at the beginning and end, and the number of steps $(1,-1)$ at the end of the path is odd (respectively, even). Let $C(x) = 1 + x + 2x^2 + 5x^3 + \cdots$ be the generating function for all Dyck paths from $(0,0)$ to $(2n,0)$, so the coefficients are Catalan numbers by (i). It is easy to see that $A = x(1+CB)$ and $B = xCA$. (Also $C = 1/(1-A-B)$, though we don't need this fact here.) Solving for A gives $A = x/(1-x^2C^2)$. The generating function we want is $1/(1-A)$, which simplifies (using $1 + xC^2 = C$) to $1 + xC$, and the proof follows. This result is due to E. Deutsch (private communication, 1996).

(k) The region bounded by the two paths is called a *parallelogram polyomino*. It is an array of unit squares, say with k columns $C_1, \ldots, C_k$. Let a_i be the number of squares in column C_i, for $1 \le i \le k$, and let b_i be the number of rows in common

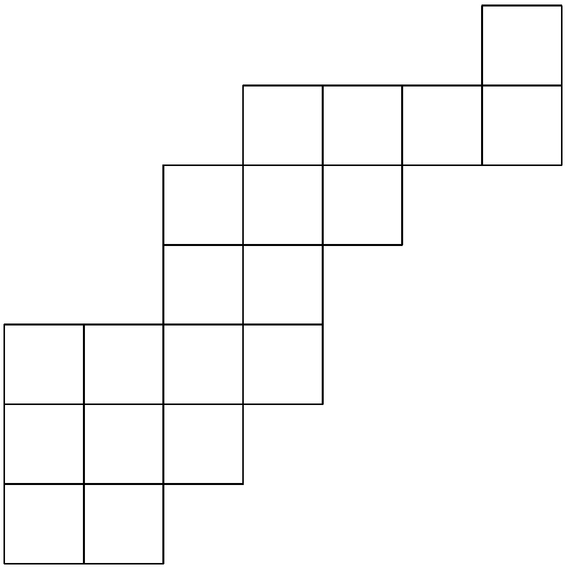


Figure 6: A parallelogram polyomino

to C_i and C_{i+1}, for $1 \le i \le k-1$. Define a sequence σ of 1's and -1's as follows (where exponentiation denotes repetition):

$$\sigma = 1^{a_1}(-1)^{a_1-b_1+1}1^{a_2-b_1+1}(-1)^{a_2-b_2+1}1^{a_3-b_2+1}\cdots 1^{a_k-b_{k-1}+1}(-1)^{a_k}.$$

This sets up a bijection with (q). For the parallelogram polyomino of Figure 6 we have $(a_1,\ldots,a_7) = (3,3,4,4,2,1,2)$ and $(b_1,\ldots,b_6) = (3,2,3,2,1,1)$. Hence (writing $-$ for -1)

$$\sigma = 111-1--111--11---1--1-11--.$$

The enumeration of parallelogram polyominoes is due to J. Levine, *Scripta Math.* **24** (1959), 335–338, and later G. Pólya, *J. Combinatorial Theory* **6** (1969) 102–105. See also L. W. Shapiro, *Discrete Math.* **14** (1976), 83–90. A further paper is W.-J. Woan, L. Shapiro, and D. G. Rogers, *Amer. Math. Monthly* **104** (1997), 926–931. For more information on the fascinating topic of polyomino enumeration, see M.-P. Delest and G. Viennot, *Theoretical Computer Science* **34** (1984), 169–206, and X. G. Viennot, in *Séries formelles et combinatoire algébrique* (P. Leroux and C. Reutenauer, eds.), Publications de Laboratoire de Combinatoire er d'Informatique Mathématique, vol. 11, Université du Québec à Montréal, 1992, pp. 399–420.

(l) Regarding a path as a sequence of steps, remove the first and last steps from the two paths in (k). This variation of (k) was suggested by L. W. Shapiro (private communication, 1998).

(m) Fix a vertex v. Starting clockwise from v, at each vertex write 1 if encountering a chord for the first time and -1 otherwise. This gives a bijection with (q). This result is apparently due to A. Errera, *Mém. Acad. Roy. Belgique Coll. 8º (2)* **11** (1931), 26 pp. See also J. Riordan, *Math. Comp.* **29** (1975), 215–222.

(n) Cut the circle in (m) between two fixed vertices and "straighten out."

(o,p) These results are due to I. M. Gelfand, M. I. Graev and A. Postnikov, Combinatorics of hypergeometric functions associated with positive roots, *Arnold-Gelfand Mathematical Seminars 1993–1995*, to appear (§6). For (o), note that there is always an arc from the leftmost to the rightmost vertex. When this arc is removed, we obtain two smaller trees satisfying the conditions of the problem. This leads to an easy bijection with (c). The trees of (o) are called *noncrossing alternating trees.*

An equivalent way of stating the above bijection is as follows. Let T be a noncrossing alternating tree on the vertex set $1, 2, \ldots, n+1$ (in that order from left to right). Suppose that vertex i has r_i neighbors that are larger than i. Let u_i be the word in the alphabet $\{1, -1\}$ consisting of r_i 1's followed by a -1. Let $u(T) = u_1 u_2 \cdots u_{n+1}$. Then u is a bijection between the objects counted by (o) and (q). It was shown by M. Schlosser that exactly the same definition of u gives a bijection between (p) and (q)! The proof, however, is considerably more difficult than in the case of (o). (A more complicated bijection was given earlier by C. Krattenthaler.)

For further information on trees satisfying conditions (α), (β), and (δ) (called *alternating trees*), see Exercise 5.**??**.

(r) Consider a lattice path P of the type (h). Let a_{i-1} be the area above the x-axis between $x = i-1$ and $x = i$, and below P. This sets up a bijection as desired.

(s) Subtract $i-1$ from a_i and append a one at the beginning to get (r). This result is closely related to Exercise 25(c). If we replace the alphabet $1, 2, \ldots, 2(n-1)$ with the alphabet $\overline{n-1}$, $n-1$, $\overline{n-2}$, $n-2, \ldots, \overline{1}, 1$ (in that order) and write the new sequence $b_1, b_2, \ldots, b_{n-1}$ in reverse order in a column, then we obtain the arrays of R. King, in *Lecture Notes in Physics*, vol. 50, Springer-Verlag, Berlin/Heidelberg/New York, 1975, pp. 490-499, (see also S. Sundaram, *J. Combinatorial Theory (A)* **53** (1990), 209–238 (Definition 1.1)) that index the weights of the $(n-1)$st fundamental representation of $\mathrm{Sp}(2(n-1), \mathbb{C})$.

(t) Let $b_i = a_i - a_{i+1} + 1$. Replace a_i with one 1 followed by b_i -1's for $1 \le i \le n$ (with $a_{n+1} = 0$) to get (q).

(u) Take the first differences of the sequences in (t).

(v) Do a depth-first search through a plane tree with $n+1$ vertices as in (e). When a vertex is encountered for the first time, record one less than its number of successors, except that the last vertex is ignored. This gives a bijection with (e).

(w) These sequences are just the inversion tables (as defined in Chapter 1.3) of the 321-avoiding permutations of (dd). For a proof see S. C. Billey, W. Jockusch, and R. P. Stanley, *J. Alg. Combinatorics* **2** (1993), 345–374 (Theorem 2.1). (The previous reference deals with the *code* $c(w)$ of a permutation w rather than the inversion table $I(w)$. They are related by $c(w) = I(w^{-1})$. Since w is 321-avoiding if and only if w^{-1} is 321-avoiding, it makes no difference whether we work with the code or with the inversion table.)

(x) If we replace a_i by $n-a_i$, then the resulting sequences are just the inversion tables of 213-avoiding permutations w (i.e., there does not exist $i<j<k$ such that $w_j < w_i < w_k$). Such permutations are in obvious bijection with the 312-avoiding permutations of (ee). For further aspects of this exercise, see Exercise 32.

(y) Given a sequence $a_1, \dots, a_n$ of the type being counted, define recursively a binary tree $T(a_1, \dots, a_n)$ as follows. Set $T(\emptyset) = \emptyset$. If $n > 0$, then let the left subtree of the root of $T(a_1, \dots, a_n)$ be $T(a_1, a_2, \dots, a_{n-a_n})$ and the right subtree of the root be $T(a_{n-a_n+1}, a_{n-a_n+2}, \dots, a_{n-1})$. This sets up a bijection with (c). Alternatively, the sequences $a_n - 1, a_{n-1} - 1, \dots, a_1 - 1$ are just the inversion tables of the 312-avoiding permutations of (ee). Let us also note that the sequences $a_1, a_2, \dots, a_n$ are precisely the sequences $\tau(u)$, $u \in \mathfrak{S}_n$, of Exercise 5.**??**(**??**).

(z) If $a = a_1 a_2 \dots a_k$ is a word in the alphabet $[n-1]$, then let $w(a) = s_{a_1} s_{a_2} \cdots s_{a_k} \in \mathfrak{S}_n$, where s_i denotes the adjacent transposition $(i, i+1)$. Then $w(a) = w(b)$ if $a \sim b$; and the permutations $w(a)$, as a ranges over a set of representatives for the classes B being counted, are just those enumerated by (dd). This statement follows from S. Billey, W. Jockusch, and R. Stanley, *J. Alg. Combinatorics* **2** (1993), 345–374 (Theorem 2.1).

(aa) Regard a partition whose diagram fits in an $(n-1) \times (n-1)$ square as an order ideal of the poset $\mathbf{n-1} \times \mathbf{n-1}$ in an obvious way. Then the partitions being counted correspond to the order ideals of (bbb). Bijections with other Catalan families were given by Knuth and A. Postnikov. Postnikov's bijection is the following. Let λ be a partition whose diagram is contained in an $(n-1) \times (n-1)$ square S. Let x be the lower-right corner of the Durfee square of λ. Let L_1 be the lattice path from the upper-right corner of S to x that follows the boundary of λ. Similarly let L_2 be the lattice path from the lower-left corner of S to x that follows the boundary of λ. Reflect L_2 about the main diagonal of S and translate

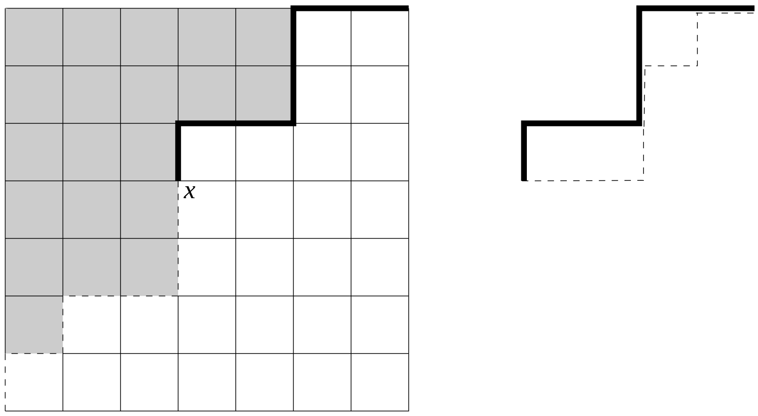


Figure 7: A bijection between (bbb) and (aa)

one unit to the right and one unit down. Adjoin a lower-left corner and upper-right corner joining L_1 with the reflection of L_2. We obtain a bijection with the parallelogram polyominoes of (k). Figure 7 illustrates this bijection for $n = 8$ and $\lambda = (5, 5, 3, 3, 3, 1)$. The path L_1 is shown in dark lines and L_2 and its reflection in dotted lines.

(bb) Remove the first occurrence of each number. What remains is a permutation w of $[n]$ that uniquely determines the original sequence. These permutations are precisely the ones in (ee). There is also an obvious bijection between the sequences being counted and the nonintersecting arcs of (n).

(bb) Remove the first occurrence of each number. What remains is a permutation w of $[n]$ that uniquely determines the original sequence. These permutations are precisely the ones in (ee).

(cc) Replace each odd number by 1 and even number by -1 to get a bijection with (q).

(dd) Corollary 7.**??** shows that the RSK-algorithm (Ch. 7.**??**)) establishes a bijection with (ww). See also D. Knuth, *The Art of Computer Programming*, vol. 3, *Sorting and Searching*, Addison-Wesley, Reading, MA, 1973 (p. 64).

The earliest explicit enumeration of 321-avoiding permutations seems to be a combinatorial proof due to D. G. Rogers, *Discrete Math.* **22** (1978), 35–40. Another direct combinatorial proof, based on an idea of Goodman, de la Harpe, and Jones, appears in S. Billey, W. Jockusch, and R. Stanley, *J. Alg. Combinatorics* **2** (1993), 345–374 (after the proof of Theorem 2.1). A sketch of this proof goes as follows. Given the 321-avoiding permutation $w = a_1 a_2 \cdots a_n$, define $c_i = \#\{j : j > i, w_j < w_i\}$. Let $\{j_1, \ldots, j_\ell\}_< = \{j : c_j > 0\}$. Define a lattice path from $(0,0)$ to (n,n) as follows. Walk horizontally from $(0,0)$ to $(c_{j_1}+j_1-1,0)$, then vertically to $(c_{j_1}+j_1-1,j_1)$, then horizontally to $(c_{j_2}+j_2-1,j_1)$, then vertically to $(c_{j_2}+j_2-1,j_2)$, etc. The last part of the path is a vertical line from $(c_{j_\ell}+j_\ell-1,j_{\ell-1})$ to $(c_{j_\ell}+j_\ell-1,j_\ell)$, then (if needed) a horizontal line to $(c_{j_\ell}+j_\ell-1,n)$, and finally a vertical line to (n,n). This establishes a bijection with (h).

For an elegant bijection with (ee), see R. Simion and F. W. Schmidt, *Europ. J. Combinatorics* **6** (1985), 383–406 (Prop. 19). Two other bijections with (ee) appear in D. Richards, *Ars Combinatoria* **25** (1988), 83–86, and J. West, *Discrete Math.* **146** (1995) 247–262 (Thm. 2.8).

(ee) There is an obvious bijection between 312-avoiding and 231-avoiding permutations, viz., $a_1a_2\ldots a_n \mapsto n+1-a_n,\ldots,n+1-a_2,n+1-a_1$. It is easily seen that the 231-avoiding permutations are the same as those of (hh), as first observed by D. Knuth [**?**, Exercise 2.2.1.5]$_5$. The enumeration *via* Catalan numbers appears in *ibid*, Exercise 2.2.1.4. References to bijections with (dd) are given in the solution to (dd). For the problem of counting permutations in $\mathfrak{S}_n$ according to the number of subsequences with the pattern 132 (equivalently, 213, 231, or 312), see M. Bona, in *Conference Proceedings*, vol. 1, Formal Power Series and Algebraic Combinatorics, July 14 – July 18, 1997, Universität Wien, pp. 107–118.

(ff) This result appears on p. 796 of D. M. Jackson, *Trans. Amer. Math. Soc.* **299** (1987), 785–801, but probably goes back much earlier. For a direct bijective proof, it is not hard to show that the involutions counted here are the same as those in (jj).

(gg) A coding of planar maps due to R. Cori, *Astérisque* **27** (1975), 169 pp., when restricted to plane trees, sets up a bijection with (e).

(hh) When an element a_i is put on the stack, record a 1. When it is taken off, record a -1. This sets up a bijection with (q). This result is due to D. Knuth [Kn$_2$, Exercise 2.2.1.4]$_5$. The permutations being counted are just the 231-avoiding permutations, which are in obvious bijection with the 312-avoiding permutations of (ee) (see Knuth, *ibid.*, Exercise 2.2.1.5).

(ii) Same set as (dd), as first observed by R. Tarjan, *J. Assoc. Comput. Mach.* **19** (1972), 341–346 (the case $m=2$ of Lemma 2). The concept of queue sorting is due to Knuth [**?**, Ch. 2.2.1]$_5$.

(jj) Obvious bijection with (n).

(kk) See I. M. Gessel and C. Reutenauer, *J. Combinatorial Theory (A)* **64** (1993), 189–215 (Theorem 9.4 and discussion following).

(ll) This result is due to O. Guibert and S. Linusson, in *Conference Proceedings*, vol. 2, Formal Power Series and Algebraic Combinatorics, July 14 – July 18, 1997, Universität Wien, pp. 243–252. Is there a nice formula for the number of alternating Baxter permutations of $[m]$?

(mm) This is the same set as (dd). See Theorem 2.1 of the reference given in (dd) to S. Billey, et al. For a generalization to other Coxeter groups, see J. Stembridge, *J. Alg. Combinatorics* **5** (1996), 353–385.

(nn) These are just the 132-avoiding permutations $w_1 \cdots w_n$ of $[n]$ (i.e., there does not exist $i < j < k$ such that $w_i < w_k < w_j$), which are in obvious bijection with the 312-avoiding permutations of (ee). This result is an immediate consequence of the following results: (i) I. G. Macdonald, *Notes on Schubert Polynomials*, Publications du LACIM, vol. 6, Univ. du Québec à Montréal, 1991, (4.7) and its converse stated on p. 46 (due to A. Lascoux and M.-P. Schützenberger), (ii) *ibid*, equation (6.11) (due to Macdonald), (iii) part (ee) of this exercise, and (iv) the easy characterization of dominant permutations (as defined in Macdonald, *ibid.*) as 132-avoiding permutations. For a simpler proof of the crucial (6.11) of Macdonald, see S. Fomin and R. Stanley, Schubert polynomials and the nilCoxeter algebra, *Advances in Math.* **103** (1994), 196–207 (Lemma 2.3).

(oo) See Exercise 3.68(b). Further results on noncrossing partitions are given by H. Prodinger, *Discrete Math.* **46** (1983), 205–206; N. Dershowitz and S. Zaks, *Discrete Math.* **62** (1986), 215–218; R. Simion and D. Ullman, *Discrete Math.* **98** (1991), 193–206; P. H. Edelman and R. Simion, *Discrete Math.* **126** (1994), 107–119; R. Simion, *J. Combinatorial Theory (A)* **65** (1994); R. Speicher, *Math. Ann.* **298** (1994), 611–628; A. Nica and R. Speicher, *J. Algebraic Combinatorics*, **6** (1997), 141–160; R. Stanley, *Elec. J. Combinatorics* **4**, R20 (1997), 14 pp. See also Exercise 5.**??**.

(pp) These partitions are clearly the same as the noncrossing partitions of (oo). This description of noncrossing partitions is due to R. Steinberg (private communication).

(qq) Obvious bijection with (oo). (Vertical lines are in the same block if they are connected by a horizontal line.) As mentioned in the Notes to Chapter 1, Murasaki diagrams were used in *The Tale of Genji* to represent the 52 partitions of a five-element set. The noncrossing Murasaki diagrams correspond exactly to the noncrossing partitions. The statement that noncrossing Murasaki diagrams are enumerated by Catalan numbers seems first to have been observed by H. Gould, who pointed it out to M. Gardner, leading to its mention in [**?**]. Murasaki diagrams were not actually used by Lady Murasaki herself. It wasn't until the Wasan period of old Japanese mathematics, from the late 1600's well into the 1700's, that the Wasanists started attaching the Murasaki diagrams (which were actually incense diagrams) to illustrated editions of *The Tale of Genji.*

(rr) See M. Klazar, *Europ. J. Combinatorics* **17** (1996), 53–68 (p. 56).

(ss) See R. C. Mullin and R. G. Stanton, *Pacific J. Math.* **40** (1972), 167-172 (p. 168). They set up a bijection with (e). They also show that $2n+1$ is the largest possible value of k for which there exists a noncrossing partition of $[k]$ with $n+1$ blocks such that no block contains two consecutive integers. A simple bijection with (a) was given by D. P. Roselle, *Utilitas Math.* **6** (1974), 91–93.

(tt) Let P_n denote the poset of intervals with at least two elements of the chain $\mathbf{n}$, ordered by inclusion. Let $\mathcal{A}_n$ denote the set of antichains of P_n. By the last paragraph of Chapter 3.1, $\#\mathcal{A}_n$ is equal to the number of order ideals of P_n. But P_n is isomorphic to the poset $\mathrm{Int}(\mathbf{n}-\mathbf{1})$ of *all* (nonempty) intervals of $\mathbf{n}-\mathbf{1}$, so by (aaa) we have $\#\mathcal{A}_n = C_n$. Given an antichain $A \in \mathcal{A}_n$, define a partition π of $[n]$ by the condition that i and j (with $i < j$) belong to the same block of π if $[i,j] \in A$ (and no other conditions not implied by these). This establishes a bijection between $\mathcal{A}_n$ and the nonnesting partitions of $[n]$. For a further result on nonnesting partitions, see the solution to Exercise 5.**??**. The present exercise was obtained in collaboration with A. Postnikov.

(uu) If $\lambda = (\lambda_1, \dots, \lambda_{n-1}) \subseteq (n-1, n-2, \dots, 1)$, then the sequences $(1, \lambda_{n-1}+1, \dots, \lambda_1+1)$ are in bijection with (r). Note also that the set of Young diagrams contained in $(n-1, n-2, \dots, 1)$, ordered by inclusion (i.e., the interval $[\emptyset, (n-1, n-2, \dots, 0)]$ in Young's lattice, as defined in Exercise 3.63), is isomorphic to $J(\mathrm{Int}(\mathbf{n}-\mathbf{1}))^*$, thereby setting up a bijection with (aaa).

(vv) Given a standard Young tableau T of shape (n,n), define $a_1 a_2 \cdots a_{2n}$ by $a_i = 1$ if i appears in row 1 of T, while $a_i = -1$ if i appears in row 2. This sets up a bijection with (q). See also [**?**, p. 63]$_7$ and our Proposition 7.**??**.

(ww) See the solution to (dd) (first paragraph) for a bijection with 321-avoiding permutations. An elegant bijection with (vv) appears in [15, vol. 1, p. 131]$_2$ (repeated in [**?**, p. 63]$_7$). Namely, given a standard Young tableau T of shape (n,n), let P consist of the part of T containing the entries $1, 2, \dots, n$; while Q consists of the complement in T of P, rotated 180°, with the entry i replaced by $2n+1-i$. See also Corollary 7.**??**.

(xx) Let b_i be the number of entries in row i that are equal to $n - i + 1$ (so $b_n = 0$). The sequences $b_n + 1, b_{n-1} + 1, \ldots, b_1 + 1$ obtained in this way are in bijection with (r).

(yy) This result is equivalent to Proposition 2.1 of S. C. Billey, W. Jockusch, and R. Stanley, *J. Algebraic Combinatorics* **2** (1993), 345–374. See also the last paragraph on page 363 of this reference.

(zz) Obvious bijection with (vv). This interpretation of Catalan numbers appears in [34, p. 222]$_3$. Note also that if we label the elements of $\mathbf{2} \times \mathbf{n}$ analogously to what was illustrated for $n = 3$, then the linear extensions coincide with the permutations of (cc).

(aaa) There is an obvious bijection with order ideals I of $\mathrm{Int}(\mathbf{n})$ that contain every one-element interval of $\mathbf{n}$. But the "upper boundary" of the Hasse diagram of I "looks like" the Dyck paths of (i). See [34, bottom of p. 222]$_3$.

(bbb) This result is equivalent to the $q = 1$ case of G. E. Andrews, *J. Stat. Plan. Inf.* **34** (1993), 19–22 (Corollary 1). For a more explicit statement and some generalizations, see R. G. Donnelly, Symplectic and odd orthogonal analogues of $L(m, n)$, preprint. For a bijective proof, see the solution to (aa). A sequence of posets interpolating between the poset $\mathrm{Int}(\boldsymbol{n-1})$ of (aaa) and A_{n-1}, and all having C_n order ideals, was given by D. E. Knuth (private communication dated December 9, 1997).

(ccc) Given a sequence $1 \leq a_1 \leq \cdots \leq a_n$ of integers with $a_i \leq i$, define a poset P on the set $\{x_1, \ldots, x_n\}$ by the condition that $x_i < x_j$ if and only if $j + a_{n+1-i} \geq n+1$. (Equivalently, if Z is the matrix of the zeta function of P, then the 1's in $Z - I$ form the shape of the Young diagram of a partition, rotated 90° clockwise and justified into the upper right-hand corner.) This yields a bijection with (r). This result is due to R. L. Wine and J. E. Freund, *Ann. Math. Stat.* **28** (1957), 256–259. See also R. A. Dean and G. Keller, *Can. J. Math.* **20** (1968), 535–554. Such posets are now called *semiorders*. For further information, see P. C. Fishburn, *Interval Orders and Interval Graphs*, Wiley-Interscience, New York, 1985, and W. T. Trotter, *Combinatorics and Partially Ordered Sets*, The Johns Hopkins University Press, Baltimore/London, 1992 (Chapter 8). For the labelled version of this exercise, see Exercise 30.

(ddd) The lattice $J(P)$ of order ideals of the poset P has a natural planar Hasse diagram. There will be two elements covering $\hat{0}$, corresponding to the two minimal elements of P. Draw the Hasse diagram of $J(P)$ so that the rooted minimal element of P goes to the left of $\hat{0}$ (so the other minimal element goes to the right). The "outside boundary" of the Hasse diagram then "looks like" the pair of paths in (k) (rotated 45° counterclockwise).

(eee) These relations are called *similarity relations.* See L. W. Shapiro, *Discrete Mathematics* **14** (1976), 83–90; V. Strehl, *Discrete Mathematics* **19** (1977), 99–101; D. G. Rogers, *J. Combinatorial Theory (A)* **23** (1977), 88–98; J. W. Moon, *Discrete Math.* **26** (1979), 251–260. Moon gives a bijection with (q). E. Deutsch (private communication) has pointed out an elegant bijection with (h), viz., the set enclosed by a path and its reflection in the diagonal *is* a similarity relation (as a subset of $[n]\times[n]$). The connectedness of the columns ensures the last requirement in the definition of a similarity relation.

(fff) A simple combinatorial proof was given by L. W. Shapiro, *J. Combinatorial Theory* **20** (1976), 375–376. Shapiro observes that this result is a combinatorial manifestation of the identity

$$\sum_{k\geq 0}\binom{n}{2k}2^{n-2k}C_k = C_{n+1},$$

due to J. Touchard, in *Proc. Int. Math. Congress, Toronto (1924)*, vol. 1, 1928 (p. 465).

(ggg) Obvious bijection with (aaa). This interpretation in terms of stacking coins is due to J. Propp. See A. M. Odlyzko and H. W. Wilf, *Amer. Math. Monthly* **95** (1988), 840–843 (Remark 1).

(hhh) See J. H. van Lint, *Combinatorial Theory Seminar, Eindhoven University of Technology*, Lecture Notes in Mathematics, no. 382, Springer-Verlag, Berlin/Heidelberg/ New York, 1974 (pp. 22 and 26–27).

(iii) The total number of n-element multisets on $\mathbb{Z}/(n+1)\mathbb{Z}$ is $\binom{2n}{n}$ (see Chapter 1.2). Call two such multisets M and N *equivalent* if for some $k \in \mathbb{Z}/(n+1)\mathbb{Z}$ we have $M = \{a_1,\ldots,a_n\}$ and $N = \{a_1+k,\ldots,a_n+k\}$. This defines an equivalence relation in which each equivalence class contains $n+1$ elements, exactly one of which has its elements summing to 0. Hence the number of multisets with elements summing to 0 (or to any other fixed element of $\mathbb{Z}/(n+1)\mathbb{Z}$) is $\frac{1}{n+1}\binom{2n}{n}$. This result appears in R. Guy, *Amer. Math. Monthly* **100** (1993), 287–289 (with a more complicated proof due to I. Gessel).

(jjj) This result is implicit in the paper G. X. Viennot, *Astérisque* **121–122** (1985), 225–246. Specifically, the bijection used to prove (12), when restricted to Dyck words, gives the desired bijection. A simpler bijection follows from the work of J.-G. Penaud, in *Séminaire Lotharingien de Combinatoire*, 22[e] Session, Université

Louis Pasteur, Strasbourg, 1990, pp. 93–130 (Cor. IV-2-8). Yet another proof follows from more general results of J. Bétréma and J. G. Penaud, *Theoret. Comput. Sci.* **117** (1993), 67–88. For some related problems, see Exercise **??**.

(kkk) Let I be an order ideal of the poset $\mathrm{Int}(\mathbf{n-1})$ defined in (aaa). Associate with I the set R_I of all points $(x_1, \dots, x_n) \in \mathbb{R}^n$ satisfying $x_1 > \cdots > x_n$ and $x_i - x_j < 1$ if $[i, j-1] \in I$. This sets up a bijection between (aaa) and the regions R_I being counted. This result is implicit in R. Stanley, *Proc. Natl. Acad. Sci. USA* **93** (1996), 2620–2625 (§2), and also appears (as part of more general results) in C. A. Athanasiadis, Ph.D. thesis, M.I.T., 1996 (Corollary 7.1.3) and A. Postnikov and R. Stanley, Deformations of Coxeter hyperplane arrangements, preprint, 1997 (Proposition 7.2).

(lll) Let P be a convex $(n+2)$-gon with vertices $v_1, v_2, \dots, v_{n+2}$ in clockwise order. Let T be a triangulation of T as in (a), and let a_i be the number of triangles incident to v_i. Then the map $T \mapsto (a_1, \dots, a_{n+2})$ establishes a bijection with (a). This remarkable result is due to J. H. Conway and H. S. M. Coxeter [**?**, problems (28) and (29)]. The arrays (1) are called *frieze patterns*.

(mmm) See F. T. Leighton and M. Newman, *Proc. Amer. Math. Soc.* **79** (1980), 177–180, and L. W. Shapiro, *Proc. Amer. Math. Soc.* **90** (1984), 488–496.

20. (a) Given a path P of the first type, let (i, i) be the first point on P that intersects $y = x$. Replace the portion of P from $(1, 0)$ to (i, i) by its reflection about $y = x$. This yields the desired bijection.

This argument is the famous "reflection principle" of D. André, *Comp. Rend. Acad. Sci. Paris* **105** (1887), 436–437. The application (b) below is also due to André. The importance of the reflection principle in combinatorics and probability theory was realized by W. Feller, *An Introduction to Probability Theory and Its Applications*, vol. 1, John Wiley and Sons, New York, 1950 (3rd ed., 1968). For a vast number of extensions and ramifications, see L. Takács, *Combinatorial Methods in the Theory of Stochastic Processes*, John Wiley and Sons, New York, 1967; T. V. Narayana, *Lattice Path Combinatorics with Statistical Applications*, Mathematical Expositions no. 23, University of Toronto Press, Toronto, 1979; and S. G. Mohanty, *Lattice Path Counting and Applications*, Academic Press, New York, 1979. For a profound generalization of the reflection principle based on the theory of Coxeter groups, as well as some additional references, see I. Gessel and D. Zeilberger, *Proc. Amer. Math. Soc.* **115** (1992), 27–31.

(b) The first step in such a lattice path must be from $(0, 0)$ to $(1, 0)$. Hence we must subtract from the total number of paths from $(1, 0)$ to (m, n) the number that intersect $y = x$, so by (a) we get $\binom{m+n-1}{n} - \binom{m+n-1}{m} = \frac{m-n}{m+n}\binom{m+n}{n}$.

(c) Move the path one unit to the right to obtain the case $m = n+1$ of (b).

21. (a) Given a path $P \in X_n$, define $c(P) = (c_0, c_1, \dots, c_n)$, where c_i is the number of horizontal steps of P at height $y = i$. It is not difficult to verify that the cyclic permutations $C_j = (c_j, c_{j+1}, \dots, c_n, c_1, \dots, c_{j-1})$ of $c(P)$ are all distinct, and for each such there is a unique $P_j \in X_n$ with $c(P_j) = C_j$. Moreover, the number of excedances of the paths $P = P_0, P_1, \dots, P_n$ are just the numbers $0, 1, \dots, n$ in some order. From these observations the proof is immediate.

This result is known as the *Chung-Feller theorem*, due to K. L. Chung and W. Feller, *Proc. Nat. Acad. Sci. U.S.A.* **35** (1949), 605–608. A refinement was given by T. V. Narayana, *Skand. Aktuarietidskr.* (1967), 23–30. For further information, see the books by Narayana (Ch. I.2) and Mohanty (Ch. 3.3) mentioned in the solution to Exercise 20(a).

(b) Immediate from (a).

22. This result was given by L. W. Shapiro, problem E2903, *Amer. Math. Monthly* **88** (1981), 619. An incorrect solution appeared in **90** (1983), 483–484. A correct but nonbijective solution was given by D. M. Bloom, **92** (1985), 430. The editors asked for a bijective proof in problem E3096, **92** (1985), 428, and such a proof was given by W. Nichols, **94** (1987), 465–466. Nichols' bijection is the following. Regard a lattice path as a sequence of E's (for the step $(1,0)$) and N's (for the step $(0,1)$). Given a path P of the type we are enumerating, define recursively a new path $\psi(P)$ as follows:

$$\psi(\emptyset) = \emptyset, \quad \psi(DX) = D\psi(X), \quad \psi(D'X) = E\psi(X)ND^*,$$

where (a) D is a path of positive length, with endpoints on the diagonal $x = y$ and all other points below the diagonal, (b) D' denotes the path obtained from D by interchanging E's and N's, and (c) $D = ED^*N$. Then ψ establishes a bijection between the paths we are enumerating and the paths of Exercise 19(h) with n replaced by $2n$. For an explicit description of ψ^{-1} and a proof that ψ is indeed a bijection, see the solution of Nichols cited above.

23. The Black pawn on a6 must promote to a knight and then move (in a unique way) to h7 in five additional moves. The Black pawn on a7 must also promote to a knight and then move (in a unique way) to f8 in four additional moves. White then plays Pf7 mate. The first move must be Pa5, after which the number of solutions is the same as if the pawn on a7 were on a6. Each pawn then makes nine moves (including moves after promotion). After the first move Pa5, denote a move by the pawn on a5 by $+1$

and a move by the pawn on a7 by -1. Since the pawn on a7 can never overtake the pawn on a5 (even after promotion), it follows that the number of solutions is just the number of sequences of nine 1's and nine -1's with all partial sums nonnegative. By Exercise 19(q), the number of solutions is therefore the Catalan number $C_9 = 4862$.

This problem is due to Kauko Väisänen, and appears in A. Puusa, *Queue Problems*, Finnish Chess Problem Society, Helsinki, 1992 (Problem 2). This booklet contains fifteen problems of a similar nature. See also Exercise 7.**??**. For more information on serieshelpmates in general, see A. Dickins, *A Guide to Fairy Chess*, Dover, New York, 1971, p. 10, and J. M. Rice and A. Dickins, *The Serieshelpmate*, second ed., Q Press, Kew Gardens, 1978.

24. These are just Catalan numbers! See for instance J. Gili, *Catalan Grammar*, Dolphin, Oxford, 1993, p. 39. A related question appears in *Amer. Math. Monthly* **103** (1996), 538 and 577.

25. (a) Follows from Exercise 3.29(b) and 19(aaa). See L. W. Shapiro, *American Math. Monthly* **82** (1975), 634–637.

    (b) We assume knowledge of Chapter 7. It follows from the results of Appendix 2 of Chapter 7 that we want the coefficient of the trivial Schur function $s_\emptyset$ in the Schur function expansion of $(x_1+x_2)^{2n}$ in the ring $\Xi_2 = \Lambda_2/(x_1x_2-1)$. Since $s_\emptyset = s_{(n,n)}$ in Ξ_2, the number we want is just $\langle s_1^{2n}, s_{(n,n)}\rangle = f^{(n,n)}$ (using Corollary 7.**??**), and the result follows from Exercise 19(vv).

    (c) See R. Stanley, *Ann. New York Acad. Sci.*, vol. 576, 1989, pp. 500–535 (Example 4 on p. 523).

    (d) See R. Stanley, in *Advanced Studies in Pure Math.*, vol. 11, Kinokuniya, Tokyo, and North-Holland, Amsterdam/New York, 1987, pp. 187–213 (bottom of p. 194). A simpler proof follows from R. Stanley, *J. Amer. Math. Soc.* **5** (1992), 805–851 (Prop. 8.6). For a related result, see C. Chan, *SIAM J. Disc. Math.* **4** (1991), 568–574.

    (e) See L. R. Goldberg, *Adv. Math.* **85** (1991), 129–144 (Theorem 1.7).

    (f) See D. Tischler, *J. Complexity* **5** (1989), 438–456.

    (g) This algebra is the *Temperley-Lieb algebra* $A_{\beta,n}$ (over K), with many interesting combinatorial properties. For its basic structure see F. M. Goodman, P. de la Harpe, and V. F. R. Jones, *Coxeter Graphs and Towers of Algebras*, Springer-Verlag, New York, 1989, p. 33 and §2.8. For a direct connection with 321-avoiding permutations (defined in Exercise 19(dd)), see S. C. Billey, W. Jockusch, and R. P. Stanley, *J. Algebraic Combinatorics* **2** (1993), 345–374 (pp. 360–361).

(h) See J.-Y. Shi, *Quart. J. Math.* **48** (1997), 93–105 (Theorem 3.2(a)).

(i) This remarkable conjecture is a small part of a vast conjectured edifice due to M. Haiman, *J. Alg. Combinatorics* **3** (1994), 17–76. See also A. M. Garsia and M. Haiman, *J. Algebraic Combin.* **5** (1996), 191–244; A. M. Garsia and M. Haiman, *Electron. J. Combin.* **3** (1996), no. 2, Paper 24; and M. Haiman, (T, Q)-Catalan numbers and the Hilbert scheme, *Discrete Math.*, to appear.

26. (a) This curious result can be proved by induction using suitable row and column operations. It arose from a problem posed by E. Berlekamp and was solved by L. Carlitz, D. P. Roselle, and R. A. Scoville, *J. Combinatorial Theory* **11** (1971), 258–271. A slightly different way of stating the result appears in [34, p. 223]$_3$.

(b) *Answer:* $a_n = C_n$, the nth Catalan number. One way (of many) to prove this result is to apply part (a) to the cases $\lambda = (2n+1, 2n, \dots, 2, 1)$ and $\lambda = (2n, 2n-1, \dots, 2, 1)$, and to use the interpretation of Catalan numbers given by Corollary **??**(v). Related work appears in A. Kellogg (proposer), Problem 10585, *Amer. Math. Monthly* **104** (1997), 361, and C. Radoux, *Bull. Belgian Math. Soc. (Simon Stevin)* **4** (1997), 289–292.

27. (a) The unique such basis $y_0, y_1, \dots, y_n$, up to sign and order, is given by

$$y_j = \sum_{i=0}^{j} (-1)^{j-i} \binom{i+j}{2i} x_i.$$

(b) Now

$$y_j = \sum_{i=0}^{j} (-1)^{j-i} \binom{i+j+1}{2i+1} x_i.$$

28. (a) The problem of computing the probability of convexity was raised by J. van de Lune and solved by R. B. Eggleton and R. K. Guy, *Math. Mag.* **61** (1988), 211–219, by a clever integration argument. The proof of Eggleton and Guy can be "combinatorialized" so that integration is avoided. The decomposition of $\mathcal{C}_d$ given below in the solution to (c) also yields a proof. For a more general result, see P. Valtr, Catalan numbers via random planar point sets, preprint.

(b) Suppose that $x = (x_1, x_2, \dots, x_d) \in \mathcal{C}_d$. We say that an index i is *slack* if $2 \le i \le d-1$ and $x_{i-1} + x_{i+1} > 2x_i$. If no index is slack, then either $x = (0, 0, \dots, 0)$, $x = (1, 1, \dots, 1)$, or $x = \lambda(1, 1, \dots, 1) + (1-\lambda)y$ for $y \in \mathcal{C}_d$ and sufficiently small $\lambda > 0$. Hence in this last case x is not a vertex. So suppose that x has a slack index. If for all slack indices i we have $x_i = 0$, then x is of the stated form (2). Otherwise, let i be a slack index such that $x_i > 0$. Let $j = i - p$ be the largest index such that $j < i$ and j is not slack. Similarly, let $k = i + q$ be the smallest index such that $k > i$ and k is not slack. Let

$$A(\epsilon) =$$

$$\left(x_1, \ldots, x_j, x_{j+1} + \tfrac{\epsilon}{p}, x_{j+2} + \tfrac{2\epsilon}{p}, \ldots, x_i + \epsilon, \ldots, x_{k-2} + \tfrac{2\epsilon}{q}, x_{k-1} + \tfrac{\epsilon}{q}, x_k, \ldots, x_n\right).$$

For small $\epsilon > 0$, both $A(\epsilon)$ and $A(-\epsilon)$ are in $\mathcal{C}_d$. Since $x = \frac{1}{2}\left(A(\epsilon) + \frac{1}{2}A(-\epsilon)\right)$, it follows that x is not a vertex. The main idea of this argument is due to A. Postnikov.

(c) For $1 \leq r \leq s \leq d$, let

$$\begin{aligned}
F_{rs} &= \{(x_1, \ldots, x_d) \in \mathcal{C}_d : x_r = x_{r+1} = \cdots = x_s = 0\} \\
F_r^- &= \{(x_1, \ldots, x_d) \in \mathcal{C}_d : x_r = x_{r+1} = \cdots = x_d = 0\} \\
F_s^+ &= \{(x_1, \ldots, x_d) \in \mathcal{C}_d : x_1 = x_2 = \cdots = x_s = 0\}.
\end{aligned}$$

Now F_r^- is a simplex with vertices $(1, \frac{k-1}{k}, \frac{k-2}{k}, \ldots, \frac{1}{k}, 0, 0, \ldots, 0)$, for $1 \leq k \leq r-1$, together with $(0, 0, \ldots, 0)$. These vertices have denominators (i.e., the smallest positive integer whose product with the vertex has integer coordinates) $1, 2, 3, \ldots, r-1, 1$, respectively. Hence

$$\sum_{n \geq 0} i(F_r^-, n) x^n = \frac{1}{[1][r-1]!}.$$

Similarly

$$\sum_{n \geq 0} i(F_s^+, n) x^n = \frac{1}{[1][d-s]!}.$$

Since $F_{rs} \cong F_r^- \times F_s^+$, we have $i(F_{rs}, n) = i(F_r^-, n) i(F_s^+, n)$ and

$$\sum_{n \geq 0} i(F_{rs}, n) x^n = \frac{1}{[1][r]!} * \frac{1}{[1][d-s]!}.$$

Let P be the poset of all F_{rs}'s, ordered by inclusion, and let μ denote the Möbius function of $P \cup \{\hat{1}\}$. Let $G = \bigcup_{r=1}^d F_{rr}$, a polyhedral complex in $\mathbb{R}^d$. By Möbius inversion we have

$$i(G, n) = -\sum_{F_{st} \in P} \mu(F_{st}, \hat{1}) i(F_{st}, n).$$

But $F_{tu} \subseteq F_{rs}$ if and only if $t \leq r \leq s \leq u$, from which it is immediate that

$$-\mu(F_{st}, \hat{1}) = \begin{cases} 1, & s = t \\ -1, & s = t-1 \\ 0, & \text{otherwise.} \end{cases}$$

Hence

$$\sum_{n\geq 0} i(G,n)x^n = \sum_{r=1}^{d} \frac{1}{[1][r-1]!} * \frac{1}{[1][d-r]!} - \sum_{r=1}^{d-1} \frac{1}{[1][r-1]!} * \frac{1}{[1][d-1-r]!}.$$

Now the entire polytope $\mathcal{C}_d$ is just a cone over G with apex $(1,1,\ldots,1)$. From this it is not hard to deduce that

$$\sum_{n\geq 0} i(\mathcal{C}_d,n)x^n = \frac{1}{1-x}\sum_{n\geq 0} i(G,n)x^n,$$

and the proof follows.

29. See P. Valtr, *Discrete Comput. Geom.* **13** (1995), 637–643.

30. Equation (4) is equivalent to

$$\sum_{n\geq 0} f_n \frac{1}{n!}\left(\log(1-x)^{-1}\right)^n x^n = C(x).$$

Hence by (5.**??**) we need to show that

$$n!C_n = \sum_{k=1}^{n} c(n,k)f_k,$$

where $c(n,k)$ is the number of permutations $w \in \mathfrak{S}_n$ with k cycles. Choose a permutation $w \in \mathfrak{S}_n$ with k cycles in $c(n,k)$ ways. Let the cycles of w be the elements of a semiorder P in f_k ways. For each cycle $(a_1,\ldots,a_i)$ of w, replace this element of P with an antichain whose elements are labelled $a_1,\ldots,a_i$. If $a=(a_1,\ldots,a_i)$ and $b=(b_1,\ldots,b_j)$ are two cycles of w, then define $a_r < b_s$ if and only if $a<b$ in P. In this way we get a poset $\rho(P,w)$ with vertices $1,2,\ldots,n$. It is not hard to see that $\rho(P,w)$ is a semiorder, and that every isomorphism class of n-element semiorders occurs exactly $n!$ times among the posets $\rho(P,w)$. Since by Exercise 19(ccc) there are C_n nonisomorphic n-element semiorders, the proof follows.

This result was first proved by J. L. Chandon, J. Lemaire, and J. Pouget, *Math. Sci. Hum.* **62** (1978), 61–80, 83. For a more general situation in which the number A_n of unlabelled objects is related to the number B_n of labelled objects by $\sum B_n \frac{x^n}{n!} = \sum A_n(1-e^{-x})^n$, see R. Stanley, *Proc. Natl. Acad. Sci. USA* **93** (1996), 2620–2625 (Theorem 2.3) and A. Postnikov and R. Stanley, Deformations of Coxeter hyperplane arrangements, preprint (§6).

31. (a) (sketch) We will triangulate $\mathcal{P}$ into d-dimensional simplices σ, all containing 0. Thus each σ will have d vertices of the form $e_i - e_j$, where $i<j$. Given a graph

G with d edges on the vertex set $[d+1]$, let σ_G be the convex hull of all vectors $e_i - e_j$ for which ij is an edge of G with $i < j$, and let $\tilde{\sigma}_G$ be the convex hull of σ_G and the origin. It is easy to see that $\tilde{\sigma}_G$ is a d-dimensional simplex if and only if G is a tree. Moreover, it can be shown that σ_G lies on the boundary of $\mathcal{P}$ (and hence can be part of a triangulation of the type we are looking for) if and only if G is an *alternating* tree, as defined in Exercise 5.??. We therefore want to choose a set $\mathcal{T}$ of alternating trees T on $[d+1]$ such that the $\tilde{\sigma}_T$'s are the facets of a triangulation of $\mathcal{P}$. One way to do this is to take $\mathcal{T}$ to consist of the *noncrossing* alternating trees on $[d+1]$, i.e., alternating trees such that if $i < j < k < l$, then not both ik and jl are edges. By Exercise 19(o) the number of such trees is C_d. (We can also take $\mathcal{T}$ to consist of alternating trees on $[d+1]$ such that if $i < j < k < l$ then not both il and jk are edges. By Exercise 19(p) the number of such trees is again C_d.) Moreover, it is easy to see that for any tree T on $[d+1]$ we have $V(\tilde{\sigma}_T) = 1/d!$, where V denotes relative volume. Hence $V(\mathcal{P}) = C_d/d!$. This result appears in M. I. Graev and A. Postnikov, Combinatorics of hypergeometric functions associated with positive roots, *Arnold-Gelfand Mathematical Seminars 1993–1995*, to appear (Theorem 2.3(2)).

(b) Order the $\binom{d+1}{2}$ edges ij, $1 \leq i < j \leq d+1$, lexicographically, e.g., $12 < 13 < 14 < 23 < 24 < 34$. Order the C_d noncrossing alternating trees $T_1, T_2, \dots, T_{C_d}$ lexicographically by edge set, i.e., $T_i < T_j$ if for some k the first (in lexicographic order) k edges of T_i and T_j coincide, while the $(k+1)$st edge of T_i precedes the $(k+1)$st edge of T_j. For instance, when $d = 3$ the ordering on the noncrossing alternating trees (denoted by their set of edges) is

$$\{12,13,14\},\ \{12,14,34\},\ \{13,14,23\},\ \{14,23,24\},\ \{14,23,34\}.$$

One can check that $\tilde{\sigma}_{T_i}$ intersects $\tilde{\sigma}_{T_1} \cup \cdots \cup \tilde{\sigma}_{T_{i-1}}$ in a union of $j-1$ $(d-1)$-dimensional faces of $\tilde{\sigma}_{T_i}$, where j is the number of vertices of T_i that are less than all their neighbors. It is not hard to see that the number of noncrossing alternating trees on $[d+1]$ for which exactly j vertices are less that all their neighbors is just the Narayana number $N(d,j)$ of Exercise 36. It follows from the techniques of R. Stanley, *Annals of Discrete Math.* **6** (1980), 333–342 (especially Theorem 1.6), that

$$(1-x)^{d+1} \sum_{n \geq 0} i(\mathcal{P}, n) x^n = \sum_{j=1}^{d} N(d,j) x^{j-1}.$$

32. The Tamari lattice was first considered by D. Tamari, *Nieuw Arch. Wisk.* **10** (1962), 131–146, who proved it to be a lattice. A simpler proof of this result was given by S. Huang and D. Tamari, *J. Combinatorial Theory (A)* **13** (1972), 7–13. The

proof sketched here follows J. M. Pallo, *Computer J.* **29** (1986), 171–175. For further properties of Tamari lattices and their generalizations, see P. H. Edelman and V. Reiner, *Mathematika* **43** (1996), 127–154; A. Björner and M. Wachs, Shellable nonpure complexes and posets. II, *Trans. Amer. Math. Soc.*, to appear (§9); and the references given there.

33. (a) Since $\mathcal{S}$ is a simplicial complex, A_n is a simplicial semilattice. It is easy to see that it is graded of rank $n-2$, the rank of an element being its cardinality (number of diagonals). To check the Eulerian property, it remains to show that $\mu(x,\hat{1}) = (-1)^{\ell(x,\hat{1})}$ for all $x \in A_n$. If $x \in A_n$ and $x \neq \hat{1}$, then x divides the polygon C into regions $C_1, \ldots, C_j$, where each C_i is a convex n_i-gon for some n_i. Let $\bar{A}_n = A_n - \{\hat{1}\}$. It follows that the interval $[x,\hat{1}]$ is isomorphic to the product $\bar{A}_{n_1} \times \cdots \times \bar{A}_{n_j}$, with a $\hat{1}$ adjoined. It follows from Exercise **??** (dualized) that it suffices to show that $\mu(\hat{0},\hat{1}) = (-1)^{n-2}$. Equivalently (since we have shown that every proper interval is Eulerian), we need to show that A_n has as many elements of even rank as of odd rank. One way to proceed is as follows. For any subset B of A_n, let $\eta(B)$ denote the number of elements of B of even rank minus the number of odd rank. Label the vertices of C as $1, 2, \ldots, n$ in cyclic order. For $3 \leq i \leq n-1$, let $\mathcal{S}^*$ be the set of all elements of $\mathcal{S}$ for which either there is a diagonal from vertex 1 to some other vertex, or else such a diagonal can be adjoined without introducing any interior crossings. Given $S \in \mathcal{S}^*$, let i be the least vertex that is either connected to 1 by a diagonal or for which we can connect it to vertex 1 by a diagonal without introducing any interior crossing. We can pair S with the set S' obtained by deleting or adjoining the diagonal from 1 to i. This pairing (or involution) shows that $\eta(\mathcal{S}^*) = 0$. But $A_n - \mathcal{S}^*$ is just the interval $[T,\hat{1}]$, where T contains the single diagonal connecting 2 and n. By induction (as mentioned above) we have $\eta([T,\hat{1}]) = 0$, so in fact $\eta(A_n) = 0$.

(b) See C. W. Lee, *Europ. J. Combinatorics* **10** (1989), 551–560. An independent proof was given by M. Haiman (unpublished). For a far-reaching generalization, see [**?**, Ch. 7] and the survey article C. W. Lee, in *DIMACS Series in Discrete Mathematics and Theoretical Computer Science*, vol. 4, 1991, pp. 443–456.

(c) Write

$$\begin{aligned} F(x,y) &= x + \sum_{n\geq 2}\sum_{i=1}^{n-1} W_{i-1}(n+1)x^n y^i \\ &= x + x^2y + x^3(y+2y^2) + x^4(y+5y^2+5y^3) + \cdots. \end{aligned}$$

By removing a fixed exterior edge from a dissected polygon and considering the edge-disjoint union of polygons thus formed, we get the functional equation

$$F = x + y\frac{F^2}{1-F}.$$

(Compare equation (**??**).) Hence by Exercise 5.**??** we have

$$\begin{aligned} F &= \sum_{m\geq 1} \frac{1}{m}[t^{m-1}] \left(x + y\frac{t^2}{1-t}\right)^m \\ &= \sum_{m\geq 1} \frac{1}{m}[t^{m-1}] \sum_{n=0}^{m} \binom{m}{n} x^n \left(y\frac{t^2}{1-t}\right)^{m-n}. \end{aligned}$$

From here it is a simple matter to obtain

$$F = x + \sum_{n\geq 2}\sum_{i=1}^{n-1} \frac{1}{n+i}\binom{n+i}{i}\binom{n-2}{i-1} x^n y^i,$$

whence

$$W_i(n) = \frac{1}{n+i}\binom{n+i}{i+1}\binom{n-3}{i}. \tag{7}$$

This formula goes back to T. P. Kirkman, *Phil. Trans. Royal Soc. London* **147** (1857), 217–272; E. Prouhet, *Nouvelles Annales Math.* **5** (1866), 384; and A. Cayley, *Proc. London Math. Soc. (1)* **22** (1890–1891), 237–262, who gave the first complete proof. For Cayley's proof see also [**?**, Ch. 7.3]. For a completely different proof, see Exercise 7.**??**.

(d) We have

$$h_i = \frac{1}{n-1}\binom{n-3}{i}\binom{n-1}{i+1}.$$

This result appears in C. W. Lee, *Europ. J. Combinatorics* **10** (1989), 551–560 (Thm. 3).

34. (a)–(d) See J. Fürlinger and J. Hofbauer, *J. Combinatorial Theory (A)* **40** (1985), 248–264, and the references given there.

(e) See R. Stanley, *Ann. New York Acad. Sci.*, vol. 574 (1989), 500–535 (Example 4 on p. 523). This result is closely related to Exercise 25(c).

35. These results are due to V. Welker, *J. Combinatorial Theory (B)* **63** (1995), 222–244 (§4).

36. (a) There are $\binom{n}{k-1}\binom{n-1}{k-1}$ pairs of compositions $A : a_1 + \cdots + a_k = n+1$ and $B : b_1 + \cdots + b_k = n$ of $n+1$ and n into k parts. Construct from these compositions a circular sequence $w = w(A, B)$ consisting of a_1 1's, then b_1 -1's, then a_2 1's,

then b_2 -1's, etc. Because n and $n+1$ are relatively prime, this circular sequence w could have arisen from exactly k pairs (A_i, B_i) of compositions of $n+1$ and n into k parts, viz., $A_i : a_i + a_{i+1} + \cdots + a_k + a_1 + \cdots + a_{i-1} = n+1$ and $B_i : b_i + b_{i+1} + \cdots + b_k + b_1 + \cdots + b_{i-1} = n$, $1 \leq i \leq k$. By the second proof of Theorem 5.**??** (or more specifically, the paragraph following it), there is exactly one way to break w into a linear sequence $\bar{w}$ such that $\bar{w}$ begins with a 1, and when this initial 1 is removed every partial sum is nonnegative. Clearly there are exactly k 1's in $\bar{w}$ (with or without its initial 1 removed) followed by a -1. This sets up a bijection between the set of all "circular equivalence classes" $\{(A_1, B_1), \ldots, (A_k, B_k)\}$ and X_{nk}. Hence

$$X_{nk} = \frac{1}{k}\binom{n}{k-1}\binom{n-1}{k-1} = \frac{1}{n}\binom{n}{k-1}\binom{n}{k}.$$

(b) For $k \geq 1$, let $X_k = X_{1k} \cup X_{2k} \cup \cdots$. Every $w \in X_k$ can be written uniquely in one of the forms (i) $1\ u\ {-1}\ v$, where $u \in X_j$ and $v \in X_{k-j}$ for some $1 \leq j \leq k-1$, (ii) $1\ {-1}\ u$, where $u \in X_{k-1}$, (iii) $1\ u\ {-1}$, where $u \in X_k$, and (iv) $1\ {-1}$ (when $k = 1$). Regarding X_k as a language as in Example **??**, and replacing for notational comprehensibility 1 by α and -1 by β, conditions (i)–(iv) are equivalent to the equation

$$X_k = \sum_{j=1}^{k-1} \alpha X_j \beta X_{k-j} + \alpha\beta X_{k-1} + \alpha X_k \beta + \delta_{1k}\alpha\beta.$$

Thus if $y_k = \sum_{n \geq 1} N(n,k) x^n$, it follows that (setting $y_0 = 0$)

$$y_k = x \sum_{j=0}^{k} y_j y_{k-j} + x y_{k-1} + x y_k + \delta_{1k} x.$$

Since $F(x,t) = \sum_{k \geq 1} y_k t^k$, we get (6).

Narayana numbers were introduced by T. V. Narayana, *C. R. Acad. Sci. Paris* **240** (1955), 1188–1189, and considered further by him in *Sankhya* **21** (1959), 91–98, and *Lattice Path Combinatorics with Statistical Applications*, Mathematical Expositions no. 23, University of Toronto Press, Toronto, 1979 (Chapter V.2). Further references include G. Kreweras and P. Moszkowski, *J. Statist. Plann. Inference* **14** (1986), 63–67; G. Kreweras and Y. Poupard, *European J. Combinatorics* **7** (1986), 141–149; R. A. Sulanke, *Bull. Inst. Combin. Anal.* **7** (1993), 60–66; and R. A. Sulanke, *J. Statist. Plann. Inference* **34** (1993), 291–303.

37. Equivalent to Exercise 1.37(c). See also the nice survey R. Donaghey and L. W. Shapiro, *J. Combinatorial Theory (A)* **23** (1977), 291–301.

38. All these results except (e), (j), (k), and (l) appear in Donaghey and Shapiro, *loc. cit.* Donaghey and Shapiro give several additional interpretations of Motzkin numbers and state that they have found a total of about 40 interpretations. For (e), see M. S. Jiang, in *Combinatorics and Graph Theory (Hefei, 1992)*, World Scientific Publishing, River Edge, NJ, 1993, pp. 31–35. For (j), see A. Kuznetsov, I. Pak, and A. Postnikov, *J. Combinatorial Theory (A)* **76** (1996), 145–147. For (k), see M. Aigner, Motzkin numbers, preprint. Aigner calls the partitions of (k) *strongly noncrossing*. Finally, for (l) see M. Klazar, *Europ. J. Combinatorics* **17** (1996), 53–68 (pp. 55–56) (and compare Exercise 1.29). Klazar's paper contains a number of further enumeration problems related to the present one that lead to algebraic generating functions; see Exercise 19(rr,ss) and Exercise 39(l) for three of them.

39. (a) This was the definition of Schröder numbers given in the discussion of Schröder's second problem in Section 2.

(b)–(f) These follow from (a) using the bijections of Proposition **??**.

(g,h) See L. W. Shapiro and A. B. Stephens, *SIAM J. Discrete Math.* **4** (1991), 275–280. For (g), see also Exercise **??**(b).

(i) L. Shaprio and S. Getu (unpublished) conjectured that the set $\mathfrak{S}_n(2413, 3142)$ and the set counted by (h) are identical (identifying a permutation matrix with the corresponding permutation). It was proved by J. West, *Discrete Math.* **146** (1995), 247–262, that $\#\mathfrak{S}_n(2413, 3142) = r_{n-1}$. Since it is easy to see that permutations counted by (h) are 2413-avoiding and 3142-avoiding, the conjecture of Shapiro and Getu follows from the fact that both sets have cardinality r_{n-1}. Presumably there is some direct proof that the set counted by (h) is identical to $\mathfrak{S}_n(2413, 3142)$.

West also showed in Theorem 5.2 of the above-mentioned paper that the sets $\mathfrak{S}_n(1342, 1324)$ and (j) are identical. The enumeration of $\mathfrak{S}_n(1342, 1432)$ was accomplished by S. Gire, Ph.D. thesis, Université Bordeaux, 1991. The remaining seven cases were enumerated by D. Kremer and K. Moneyhun, in preparation. Kremer and Moneybun also give proofs of the three previously known cases. They prove all ten cases using the method of "generating trees" introduced by F. R. K. Chung, R. L. Graham, V. E. Hoggatt, Jr., and M. Kleiman, *J. Combinatorial Theory (A)* **24** (1978), 382–394, and further developed by J. West, *Discrete Math.* **146** (1995), 247–262, and **157** (1996), 363–374. It has been verified by computer that there are no other pairs $(u, v) \in \mathfrak{S}_4 \times \mathfrak{S}_4$ for which $\#\mathfrak{S}_n(u, v) = r_{n-1}$ for all n.

(j) This is a result of Knuth [**?**, Exercises 2.2.1.10–2.2.1.11, pp. 239 and 533–534]$_5$; these permutations are now called *deque-sortable*. A combinatorial proof appears in D. G. Rogers and L. W. Shapiro, in *Lecture Notes in Math.*, no. 884, Springer-Verlag, Berlin, 1981, pp. 293–303. Some additional combinatorial interpretations of Schröder numbers and many additional references appear in the preceding reference. For q-analogues of Schröder numbers, see J. Bonin, L. W. Shapiro, and R. Simion, *J. Stat. Planning and Inference* **34** (1993), 35–55.

(k) See D. G. Rogers and L. W. Shapiro, *Lecture Notes in Mathematics*, no. 686, Springer-Verlag, Berlin, 1978, pp. 267–276 (§5) for simple bijections with (a) and other "Schröder structures."

(l) This result is due to R. C. Mullin and R. G. Stanton, *Pacific J. Math.* **40** (1972), 167–172 (§3), using the language of "Davenport-Schinzel sequences." It is also given by M. Klazar, *Europ. J. Combinatorics* **17** (1996), 53–68 (p. 55).

(m) Draw a noncrossing graph satisfying (α) by putting the vertices $1, 2, \dots, n+1$ on a horizontal line, and drawing the edges as nonintersecting arcs above the vertices. Place a vertex v_e in the center of each edge e, and draw an edge from v_e to v_f if edge f lies under e and there are no edges in between. This sets up a bijection with (b). This simple bijection was suggested by A. Postnikov.

(n,o) These results (which despite their similarity are not trivially equivalent) appear in D. G. Rogers, *Lecture Notes in Mathematics*, no. 622, Springer-Verlag, Berlin, 1977, pp. 175–196 (equations (38) and (39)), and are further developed in D. G. Rogers, *Quart. J. Math. (Oxford) (2)* **31** (1980), 491–506. In particular, a bijective proof that (n) and (o) are equinumerous appears in Section 3 of this latter reference. It is also easy to see that (m) and (o) are virtually identical problems. A further reference is D. G. Rogers and L. W. Shapiro, *Lecture Notes in Mathematics*, no. 686, Springer-Verlag, Berlin, 1978, pp. 267–276.

(p) See M. Ciucu, *J. Algebraic Combinatorics* **5** (1996), 87–103, Theorem 4.1.

40. Note that this exercise is the "opposite" of Exercise 39(h), i.e., here we are counting the permutation matrices P for which not even a single new 1 can be added (using the rules of Exercise 39(h)). The present exercise was solved by Shapiro and Stephens in Section 3 of the paper cited in the solution to Exercise 39(h). For a less elegant form of the answer and further references, see M. Abramson and M. O. J. Moser, *Ann. Math. Stat.* **38** (1967), 1245–1254.