A FUNCTORIAL APPROACH TO REGULAR HOMOMORPHISMS

JEFFREY D. ACHTER, SEBASTIAN CASALAINA-MARTIN, AND CHARLES VIAL

ABSTRACT. Classically, regular homomorphisms have been defined as a replacement for Abel–Jacobi maps for smooth varieties over an algebraically closed field. In this work, we interpret regular homomorphisms as morphisms from the functor of families of algebraically trivial cycles to abelian varieties and thereby define regular homomorphisms in the relative setting, *e.g.*, families of schemes parameterized by a smooth variety over a given field. In that general setting, we establish the existence of an initial regular homomorphism, going by the name of algebraic representative, for codimension-2 cycles on a smooth proper scheme over the base. This extends a result of Murre for codimension-2 cycles on a smooth projective scheme over an algebraically closed field. In addition, we prove base change results for algebraic representatives as well as descent properties for algebraic representatives along separable field extensions. In the case where the base is a smooth variety over a subfield of the complex numbers we identify the algebraic representative for relative codimension-2 cycles with a subtorus of the intermediate Jacobian fibration which was constructed in previous work.

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INTRODUCTION

Abel–Jacobi maps provide a fundamental tool in the study of smooth complex projective varieties. For algebraic curves, the Abel–Jacobi map on divisors essentially encodes all of the data of the curve, while for higher dimensional varieties, the Abel–Jacobi map provides a basic instrument for studying higher codimension algebraic cycles. The most well-studied examples are the Abel map, taking algebraically trivial divisors to the connected component of the Picard scheme, and the Albanese map, taking algebraically trivial 0-cycles to the Albanese variety. In fact, in these instances, the theory has been developed for smooth projective varieties over an arbitrary field. One of the main goals of this paper is to develop a framework for defining generalizations of the remaining intermediate Abel–Jacobi maps for smooth projective varieties over arbitrary fields. At the same time, studying Abel–Jacobi maps for families of smooth complex projective varieties has

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also been extremely fruitful, leading for instance to the theory of motivated normal functions. The theory of the Abel and Albanese maps has been generalized to the relative setting in the category of schemes, and another of our goals is to develop a framework for defining generalizations of the remaining Abel–Jacobi maps in the relative setting of morphisms of schemes. In what follows, we recall the classical definition of a regular homomorphism, which has been used as a replacement for Abel–Jacobi maps over algebraically closed fields, explain the relationship with our new functorial definition of a regular homomorphism, and then discuss our main results on existence, and on base change in the relative setting.

Let X be a scheme of finite type over a field K. A codimension-i cycle class $a \in \mathrm{CH}^i(X)$ is said to be *algebraically trivial* if it is "parameterized" by a smooth integral scheme over K. Precisely, the cycle class a is algebraically trivial if and only if there exist a smooth integral scheme T of finite type over K, a pair of K-points $t_0, t_1 \in T(K)$ and a cycle $Z \in \mathrm{CH}^i(T \times_K X)$, which is referred to as a family of codimension-i cycle classes on X parameterized by T, such that $a = Z_{t_1} - Z_{t_0}$, where Z_{t_i} denote the Gysin fibers. It is in fact equivalent to require T to be a smooth quasi-projective curve over K; see *e.g.*, [ACMV19b, Prop. 3.10]. Henceforth, the subgroup of $\mathrm{CH}^i(X)$ consisting of algebraically trivial cycles will be denoted $\mathrm{A}^i(X)$.

Let k be an algebraically closed field and let X be a scheme of finite type over k. Classically, a *regular homomorphism* consists of a group homomorphism $\phi : \mathrm{A}^i(X) \to A(k)$ to the closed points of an abelian variety A over k satisfying the property: for a smooth integral scheme T of finite type over k, a point $t_0 \in T(k)$ and a cycle class $Z \in \mathrm{CH}^i(T \times_k X)$, the map defined on k-points by

$$T(k) \to A(k), \quad t \mapsto \phi(Z_t - Z_{t_0})$$

is induced by a k-morphism $T \to A$. The standard example of a regular homomorphism is the Abel–Jacobi map for a smooth complex projective variety. Observe that replacing Z with $Z - (Z_{t_0} \times T)$, one can equivalently define a regular homomorphism to consist of a group homomorphism $\phi : \mathrm{A}^i(X) \to A(k)$ to the closed points of an abelian variety A over k satisfying the property: for a smooth scheme T of finite type over k and a cycle class $Z \in \mathrm{CH}^i(T \times_k X)$ that is fiberwise algebraically trivial, the map defined on k-points by

$$T(k) \to A(k), \quad t \mapsto \phi(Z_t)$$

is induced by a k-morphism $T \to A$. Our starting point is that a regular homomorphism can thus be seen as a morphism of functors

$$\Phi : \mathscr{A}^i_{X/k} \to A$$

by the recipe $\Phi(k) = \phi$, where $\mathscr{A}^i_{X/k}$ is the functor of "families of algebraically trivial cycle classes", *i.e.*, the functor assigning to each scheme T smooth and of finite type over k the subgroup of $\mathrm{CH}^i(T \times_k X)$ consisting of cycle classes that are fiberwise (relative to T) algebraically trivial. The advantage of this functorial approach is that now it makes sense to define regular homomorphisms without the restriction that k be algebraically closed. With the functorial language, an *algebraic representative* for a scheme X of finite type over a field K, if it exists, is a morphism

$$\Phi^i_X : \mathscr{A}^i_{X/K} \to \mathrm{Ab}^i_{X/K}$$

that is initial among all morphisms from $\mathscr{A}^i_{X/K}$ to abelian varieties over K. In particular, it is unique up to unique isomorphism. We refer to §1 for definitions. When X is a smooth projective variety over K, the algebraic representative for divisors is given by the Abel map to $(\mathrm{Pic}^0_{X/K})_{\mathrm{red}}$, and the algebraic representative for 0-cycles is given by the Albanese variety.

In the end, our goal is to extend the notion of regular homomorphism to the relative setting. However, as it turns out, many of the arguments in the relative setting can be reduced to the case of schemes over a field, and to a base change of field statement. Moreover, even in the case of schemes over fields, we can improve some of our previous results over fields [ACMV17] using our new framework. In this spirit, our first main result is a base-change/descent result for algebraic representatives along separable field extensions:

Theorem 1. *Let X be a scheme of finite type over a field K and let Ω/K be a (not necessarily algebraic) separable field extension. Then an algebraic representative $\Phi^i_{X_\Omega} : \mathscr{A}^i_{X_\Omega/\Omega} \to \mathrm{Ab}^i_{X_\Omega/\Omega}$ exists if and only if an algebraic representative $\Phi^i_X : \mathscr{A}^i_{X/K} \to \mathrm{Ab}^i_{X/K}$ exists. If this is the case, we have in addition:*

(i) There is a canonical isomorphism $\mathrm{Ab}^i_{X_\Omega/\Omega} \xrightarrow{\sim} (\mathrm{Ab}^i_{X/K})_\Omega$;

(ii) $\Phi^i_{X_\Omega}(\Omega) : \mathrm{A}^i(X_\Omega) \to \mathrm{Ab}^i_{X_\Omega/\Omega}(\Omega)$ is $\mathrm{Aut}(\Omega/K)$-equivariant, relative to the above identification.

Theorem 1 generalizes and strengthens our previous result [ACMV17, Thms. 3.7 & 4.4] not only by eliminating the assumption that K be perfect, but also by showing that the natural homomorphism $\mathrm{Ab}^i_{X_\Omega/\Omega} \to (\mathrm{Ab}^i_{X/K})_\Omega$ is an isomorphism rather than merely a purely inseparable isogeny. Theorem 1 is proven in Theorem 5.10.

We now turn our attention to regular homomorphisms in the relative setting. The natural framework for our definition (see Definition 1.8 and §1.3.1) is to consider a morphism $X \to S$ of finite type where S is a smooth separated scheme of finite type over a regular Noetherian scheme Λ. Then, as in the case of schemes over fields discussed above, one can define the sheaf $\mathscr{A}^i_{X/S/\Lambda}$ of "families of algebraically trivial cycle classes" (Definition 1.3), and with this one can then define a regular homomorphism $\Phi : \mathscr{A}^i_{X/S/\Lambda} \to A$ as a morphism of functors to an abelian S-scheme A (Definition 1.7). An algebraic representative is then defined as an initial regular homomorphism (Definition 1.8).

A natural starting point is to establish existence of algebraic representatives. In general, even for smooth projective varieties over an algebraically closed field it is still an open problem to decide whether algebraic representatives exist. However, there are three cases where existence is known for smooth projective varieties over an algebraically closed field: for cycles of dimension-0 and for cycles of codimension-1 or 2. Indeed, as mentioned above, for a smooth projective variety over an algebraically closed field, the algebraic representative in codimension-1 is given by the reduced connected component of the Picard scheme, while the algebraic representative in dimension-0 is given by the Albanese variety. Murre established the existence of algebraic representatives for codimension-2 cycles for smooth projective varieties over an algebraically closed field in [Mur85]. In [ACMV17], we showed that such algebraic representatives admit models over any perfect field. The following theorem provides a uniform statement for the case $i = 1, 2, \dim X$, generalizing the results to the relative setting. Note that even specializing to the case where $\Lambda = S = \operatorname{Spec} K$ for some field K, the following theorem in particular provides a new result, namely the existence of algebraic representatives for $i = 2$ for smooth proper varieties over any field.

Theorem 2. *Let S be a scheme that is smooth separated and of finite type over a regular Noetherian scheme Λ. Suppose X is a smooth proper scheme of finite type over S. Fix $i = 1, 2$, or $\dim_S X$. Then $X/S/\Lambda$ admits an algebraic representative*

$$\Phi^i_{X/S} : \mathscr{A}^i_{X/S/\Lambda} \to \mathrm{Ab}^i_{X/S}$$

for codimension-i cycles.
Moreover, if $\Lambda = \operatorname{Spec} K$ for some subfield $K \subseteq \mathbb{C}$, then

(i) For any dominant Λ-morphism $S' \to S$ of smooth separated schemes of finite type over Λ, there is a canonical isomorphism of S'-abelian schemes

$$\mathrm{Ab}^i_{X_{S'}/S'} \xrightarrow{\sim} (\mathrm{Ab}^i_{X/S})_{S'}.$$

(ii) *The homomorphism* $\Phi^i_{X/S}(\Omega) : \mathrm{A}^i(X_\Omega) \to \mathrm{Ab}^i_{X/S}(\Omega)$ *is an isomorphism on torsion for* $s : \operatorname{Spec}\Omega = \operatorname{Spec}\kappa(S)^{sep} \to S$ *a separable closure of a generic point of* S.

The most interesting case is the case $i = 2$. The existence statement of Theorem 2 is proven in Theorem 6.1, item (i) is proven in Theorem 8.7 and item (ii) is proven in Theorem 9.7. The latter is obtained by using the fact proved by Murre [Mur85, §10] that $\Phi^2_{X_\Omega/\Omega}$ coincides with the Abel–Jacobi map when $\Omega = \mathbb{C}$, which explains why we have to restrict to characteristic zero.

We note nonetheless that even in the cases $i = 1$ or $\dim_S X$, Theorem 6.1 provides some interesting new results. First, consider the case $i = 1$. Then the theorem does not require the existence of the connected component of the Picard scheme $\mathrm{Pic}^0_{X/S}$ to get the existence of the algebraic representative (we discuss cases where $\mathrm{Pic}^0_{X/S}$ and the algebraic representative agree in Theorem 7.1 and Remark 7.2). In other words, in this case the theorem does not just reduce to the theory of Picard schemes, and we get a more general relative existence theorem for the algebraic representative. Next consider the case $i = \dim_S X$. Then the theorem does not require the existence of Grothendieck's relative Albanese scheme [Gro62], which is predicated on the existence of $\mathrm{Pic}^0_{X/S}$, to get the existence of the algebraic representative. Again, we discuss cases where $\mathrm{Alb}_{X/S}$ and the algebraic representative agree in Theorem 7.9 and Lemma 7.6. In other words, in this case, the theorem does not just reduce to the theory of the Albanese scheme, and we get a more general relative existence theorem for the algebraic representative.

Having established existence results for the algebraic representative, we mention that nowhere in the above discussion did we assert that the algebraic representative is nontrivial. For $i = 1$ and $i = \dim_S X$, there are well-established results on the dimensions of the Picard scheme and the Albanese scheme that can be used. However, for codimension-2 cycles, the question is much more subtle. In particular, one can see that the main issue is to establish the existence of a single non-trivial regular homomorphism. In positive characteristic, this is quite subtle, and we refer the reader to [ACMV21] where we discuss this *via* the geometric coniveau filtration (for the case of fibrations in conics over the projective plane see also [Bea77]). However, in characteristic 0, due to the base change result in Theorem 2, the existence of nontrivial regular homomorphisms in the relative setting is ensured, *via* base-changing along a dominant morphism $\operatorname{Spec}\mathbb{C} \to S$ and then by using the fact that the Abel–Jacobi map induces a regular homomorphism (Example 1.14, Theorem 9.3). In fact, for $i = 1, 2, \dim X$, we can show that the Abel-Jacobi map provides an algebraic representative in the relative setting. Precisely, thanks to our functorial approach to regular homomorphisms, we can extend [Mur85, §10] to the relative setting and compare the base-change of the algebraic representative $\Phi^i_{X/S}$ to $\mathbb{C}$ with normal functions attached to families of fiberwise algebraically trivial codimension-i cycles. The following theorem, which can be read off Theorems 9.3 and 9.5, is obtained by combining our previous work on normal functions attached to cycles that are fiberwise algebraically trivial [ACMV19a] with Theorems 1 and 2.

Theorem 3. *Let* S *be a smooth separated scheme of finite type over a field* $K \subseteq \mathbb{C}$ *and let* $f : X \to S$ *be a smooth projective morphism. Denote* $\mathrm{J}^{2i-1}(X_\mathbb{C}/S_\mathbb{C})$ *the relative intermediate Jacobian attached to the variation of Hodge structures* $\mathrm{R}^{2i-1}(f_\mathbb{C})_*\mathbb{Z}$. *Then*

(i) (The relative AJ map is a regular homomorphism) *There exist a surjective regular homomorphism* $\Phi^i_{AJ_{X/S}} : \mathscr{A}^i_{X/S} \to \mathrm{J}^{2i-1}_{a,X/S}$ *and a canonical inclusion* $\iota : (\mathrm{J}^{2i-1}_{a,X/S})_\mathbb{C} \hookrightarrow \mathrm{J}^{2i-1}(X_\mathbb{C}/S_\mathbb{C})$ *of relative complex tori. The inclusion* ι *has the property that for any* $Z \in \mathscr{A}^i_{X_\mathbb{C}/S_\mathbb{C}}(S_\mathbb{C})$ *the composition* $\iota \circ \Phi^2_{X_\mathbb{C}/S_\mathbb{C}}(S_\mathbb{C})(Z)$ *coincides with the normal function* ν_Z *attached to* Z. *In particular, restricting the co-domain of the normal function to* $(\mathrm{J}^{2i-1}_{a,X/S})_\mathbb{C}$, *the normal function* ν_Z *is algebraic and if* Z *is defined over* K *then* ν_Z *is defined over* K.

(ii) (The relative AJ map is an algebraic representative for $i = 1, 2, \dim_S X$) *For $i = 1, 2, \dim_S X$, the natural homomorphism $\mathrm{Ab}^i_{X/S} \to \mathrm{J}^{2i-1}_{a,X/S}$, provided by Theorem 2, is an isomorphism.*

Note that for $i = 1$ or $\dim_S X$, Theorem 3 is classical and the inclusion ι is an isomorphism.

Regarding item (i) of Theorem 2, we have to restrict to characteristic zero in order to use the Faltings–Chai extension theorem for abelian schemes. However, a first step consists in using the Néron–Ogg–Shafarevich criterion, which holds without any restrictions on the characteristic. Consequently, we can show the following base-change result concerning the algebraic representative.

Theorem 4. *Let S/Λ be either the spectrum of a DVR (with $S = \Lambda$), or a smooth quasi-projective variety over a field $K \subseteq \mathbb{C}$, and denote η its generic point. Suppose X is a smooth and proper scheme over S. Fix $i = 1, 2$, or $\dim_S X$. Then the natural homomorphism of η-abelian varieties*

$$\mathrm{Ab}^i_{X_\eta/\eta} \longrightarrow (\mathrm{Ab}^i_{X/S})_\eta$$

is an isomorphism.

In the case where S is a DVR, this is Theorem 8.3, while in the other case this is Theorem 2(i) (by going to the limit over all open subsets of S) or can be seen as a combination of Theorems 9.3 and 9.5. In the first case, the key point is to show that $\mathrm{Ab}^2_{X_\eta/\eta}$ admits a model over S; this is achieved *via* the Néron–Ogg–Shafarevich criterion by showing (Proposition 8.2) that $T_\ell \mathrm{Ab}^2_{X_\eta/\eta}$ admits a $\mathrm{Gal}(\eta)$-equivariant embedding as a $\mathbb{Z}_\ell$-module in the torsion-free quotient of $H^3(X_{\eta^{\mathrm{sep}}}, \mathbb{Z}_\ell(2))$. In the second case, we can extend $\mathrm{Ab}^2_{X_\eta/\eta}$ to an abelian scheme over S thanks to the Faltings–Chai extension theorem [FC90, Cor. 6.8, p.185].

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Notation and Conventions. Given a field K, K^{sep} will denote an algebraic separable closure of K. Since we will deal with separable extensions of fields that are not necessarily algebraic, we make it clear that a field K is said to be *separably closed* if the only separable and algebraic extension of K is K itself. Given a scheme X of finite type over a field K, we denote $\mathrm{A}^i(X)$ the subgroup of $\mathrm{CH}^i(X)$ consisting of cycle classes that are algebraically trivial.

Unless stated otherwise, Λ denotes an integral regular Noetherian scheme, S denotes an integral scheme that is smooth separated and of finite type over Λ, and X denotes a scheme of finite type over S. Given an integral scheme T with function field $\kappa(T)$, we denote $\eta_T : \operatorname{Spec} \kappa(T) \to T$ its generic point and we denote $\eta_T^{\mathrm{sep}} : \operatorname{Spec} \kappa(T)^{\mathrm{sep}} \to T$ the corresponding separably closed point over η_T.

1. Regular homomorphisms as functors

1.1. **The functor of algebraically trivial cycle classes.** We fix an integral regular Noetherian scheme Λ, and an integral scheme S that is smooth separated and of finite type over Λ. We will denote the generic points of S and Λ, respectively, as η_S and η_Λ.

Remark 1.1. The assumption that S and Λ be irreducible (and hence integral, since Λ is regular and since S is smooth over the regular scheme Λ) is not essential; all statements in the paper hold without this integrality assumption simply by working component-wise.

We denote by

$$\mathsf{Sm}_\Lambda/S$$

the category whose objects are separated S-schemes that are smooth and of finite type over Λ with structure map to S being *dominant*, and whose morphisms are given by morphisms of S-schemes. For later reference, we recall that any morphism $t : T' \to T$ in Sm_Λ/S – in fact any morphism of smooth separated schemes of finite type over Λ – factors as

$$T' \xrightarrow{\gamma_t} T' \times_\Lambda T \xrightarrow{pr_T} T$$

where γ_t, the graph of t, is a regular embedding [Ful98, B.7.3], and pr_T is the second projection, and is smooth by base change. In particular, t is locally complete intersection (lci) in the sense of [Ful98, B.7.6], extended to the the situation of schemes over Λ, and if both T and T' are equidimensional over Λ, then t is of relative dimension $d_T - d_{T'}$, where $d_T = \dim_\Lambda T$ and $d_{T'} = \dim_\Lambda T'$ are the relative dimensions of T and T' over Λ.

Let $X \to S$ be a morphism of finite type over S. For each $T \to S$ in Sm_Λ/S, denote by X_T the base-change $X \times_S T$. Note that X_T is of finite type over Λ. Following [Ful98, §20], we have $\mathrm{CH}^i(X_T/\Lambda)$, which, abusing notation, we will henceforth simply write as $\mathrm{CH}^i(X_T)$.

Remark 1.2 (The case $S = \Lambda = \operatorname{Spec} K$). An important special case is when $S = \Lambda = \operatorname{Spec} K$ for some field K, and $f : X \to \operatorname{Spec} K$ is smooth projective. The classical case is the case where we assume in addition that $K = k$ is algebraically closed (see §1.4).

Recall that for any morphism $t : T' \to T$ in Sm_Λ/S, being lci as described above, there is a refined Gysin pull-back [Ful98, p.395]

$$t^! : \mathrm{CH}^i(X_T) \to \mathrm{CH}^i(X_{T'}),$$

which, as pointed out on [Ful98, p.395], satisfies all the fundamental properties of [Ful98, §3, §6].

Definition 1.3 (Functor of algebraically trivial cycles)**.** In the notation above, for a nonnegative integer i, the *functor of codimension-i algebraically trivial cycles on X over S* is the contravariant functor

$$\mathscr{A}^i_{X/S} : \mathsf{Sm}_\Lambda/S \longrightarrow \mathsf{AbGp},$$

sometimes denoted $\mathscr{A}^i_{X/S/\Lambda}$ for clarity, to the category of abelian groups AbGp given by families of algebraically trivial cycles on X/S. Precisely, given T in Sm_Λ/S, we take $\mathscr{A}^i_{X/S}(T)$ to be the group of cycles classes $Z \in \mathrm{CH}^i(X_T)$ such that for every separably closed field Ω and every Ω-point $t : \operatorname{Spec}(\Omega) \to T$ obtained as an inverse limit of dominant morphisms $(t_n : T_n \to T)_{n\in\mathbb{N}}$ in Sm_Λ/T, *i.e.*, $\operatorname{Spec}\Omega = \varprojlim_n T_n$, the cycle $Z_t := t^! Z := \varinjlim_n t_n^! Z \in \varinjlim_n \mathrm{CH}^i(X_{t_n}) = \mathrm{CH}^i(X_t)$ is algebraically trivial. The functor is defined on morphisms $t : T' \to T$ in Sm_Λ/S *via* the refined Gysin pullback $t^!$ for lci morphisms [Ful98, §6.6] ; we will sometimes use the notation $Z_{T'} = t^! Z$.

We now establish a result that helps to clarify the meaning of Definition 1.3. The first observation is that if Ω is a separably closed field, and $t : \operatorname{Spec}(\Omega) \to T$ is obtained as an inverse limit of morphisms $(t_n : T_n \to T)_{n\in\mathbb{N}}$ in Sm_Λ/T, and if T is integral, then the image of $\operatorname{Spec}\Omega$ is the generic point of T. Moreover, under the composition $\operatorname{Spec}\Omega \xrightarrow{t} T \to \Lambda$, the image of $\operatorname{Spec}\Omega$ is the generic point η_Λ, and $\Omega/\kappa(\eta_\Lambda)$ is separable ; this is an immediate consequence of the fact [Sta22, Tag 037X] that a field extension L/K is separable if and only if L is a direct limit of rings that are smooth and of finite type over K. Note also that if $P \in |T|$ is the image of t in the underlying topological space, then $\kappa(P)/\kappa(\eta_\Lambda)$ is separable. This is a consequence of the fact that if $L/E/K$ is a tower of field extensions, with L/K separable, then E/K is separable (*e.g.*, [Bou81, Prop. 8, p.V.116]).

Remark 1.4. Given any field extension $L/\kappa(\Lambda)$, in [ACMV19b, Rem. 3.2] we recalled the pullback of $Z \in \mathrm{CH}^i(X_T)$ over the morphism $t : \operatorname{Spec} L \to T$, defining $Z_L = Z_{\bar{t}} \in \mathrm{CH}^i(X_{\bar{t}})$. In particular, if L'/L is a further field extension then, if Z_L is algebraically trivial, then so is $Z_{L'}$.

The following lemma describes Definition 1.3 in terms of more easily identified fibers :

Lemma 1.5. *Let T be an integral scheme in Sm_Λ/S with generic point $\eta_T : \kappa(T) \to T$ and let Z be a cycle class in $\mathrm{CH}^i(X_T)$. The separable algebraic closure η_T^{sep} of the generic point η_T can be obtained as an inverse limit of morphisms $(t_n : T_n \to T)_{n\in\mathbb{N}}$ in Sm_Λ/T.*
Moreover, the following conditions are equivalent :

- *(i)* $Z \in \mathscr{A}^i_{X/S}(T)$ *;*
- *(ii)* $Z_{\eta_T^{sep}} \in \mathrm{CH}^i(X_{\eta_T^{sep}})$ *is algebraically trivial ;*
- *(iii)* $Z_t \in \mathrm{CH}^i(X_t)$ *is algebraically trivial for all points* $t : \operatorname{Spec} \kappa(\Lambda)^{sep} \to T$ *;*
- *(iii)′* $Z_t \in \mathrm{CH}^i(X_t)$ *is algebraically trivial for some point* $t : \operatorname{Spec} \kappa(\Lambda)^{sep} \to T$ *;*
- *(iv)* $Z_t \in \mathrm{CH}^i(X_t)$ *is algebraically trivial for all dominant points* $t : \operatorname{Spec} \Omega \to T$ *with* Ω *separably closed.*

Proof. The separable algebraic closure η_T^{sep} of the generic point η_T can be obtained as an inverse limit of morphisms $(t_n : T_n \to T)_{n\in\mathbb{N}}$ in Sm_Λ/T (see *e.g.* [Sta22, Tag 037X]), yielding the first assertion, as well as the implication $(i) \Rightarrow (ii)$. The implication $(iv) \Rightarrow (i)$ is obvious, while the equivalence $(ii) \Leftrightarrow (iv)$ follows from the fact that any point t as in (iv) factors through η_T^{sep}. Hence items (i), (ii) and (iv) are equivalent. Items (iii) and $(iii)'$ are equivalent : Since T is smooth over Λ, its base-change $T_{\eta_\Lambda^{\mathrm{sep}}}$ is smooth over $\operatorname{Spec} \kappa(\Lambda)^{\mathrm{sep}}$ and it follows by definition of algebraic equivalence that Z_t and Z_s are algebraically equivalent for any two points $t : \operatorname{Spec} \kappa(\Lambda)^{\mathrm{sep}} \to T$ and $s : \operatorname{Spec} \kappa(\Lambda)^{\mathrm{sep}} \to T$.

We now prove $(ii) \Rightarrow (iii)'$. By definition of algebraic equivalence, there exist a smooth variety V of finite type over $\operatorname{Spec} \kappa(T)^{\mathrm{sep}}$, points $v_0, v_1 \in V(\kappa(T)^{\mathrm{sep}})$, and a correspondence $Y \in \mathrm{CH}^i(X_{\eta_T^{\mathrm{sep}}} \times_{\kappa(T)^{\mathrm{sep}}} V)$ such that $Z_{\eta_T^{\mathrm{sep}}} = Y_{v_1} - Y_{v_0}$. The data consisting of V, v_0, v_1 and Y is defined over a finite (separable) extension L of $\kappa(T)$. Spreading out, we find an étale morphism $\widetilde{T} \to T$, a smooth family $\widetilde{V} \to \widetilde{T}$ with two sections $\tilde{v}_0$ and $\tilde{v}_1$, and a cycle class $\widetilde{Y} \in \mathrm{CH}^i(X_{\widetilde{T}} \times_{\widetilde{T}} \widetilde{V})$, whose generic fibers are the models of V, v_0, v_1 and Y over L. Consider now a point $t : \operatorname{Spec} \kappa(\Lambda)^{\mathrm{sep}} \to T$ in the image of $\widetilde{T} \to T$, and consider any point $\tilde{t} : \operatorname{Spec} \kappa(\Lambda)^{\mathrm{sep}} \to \widetilde{T}$ in the pre-image. Then, under the natural identifications, Z_t coincides with $Z_{\tilde{t}} = (\widetilde{Y}_{\tilde{v}_1})_{\tilde{t}} - (\widetilde{Y}_{\tilde{v}_0})_{\tilde{t}} = (\widetilde{Y}_{\tilde{t}})_{(\tilde{v}_1)_{\tilde{t}}} - (\widetilde{Y}_{\tilde{t}})_{(\tilde{v}_0)_{\tilde{t}}}$, and so Z_t is algebraically trivial.

To conclude, we prove $(iii)' \Rightarrow (ii)$. This follows from the simple observation that η_T^{sep} provides a point $\operatorname{Spec} \kappa(T)^{\mathrm{sep}} \to T_{\kappa(T)^{\mathrm{sep}}}$ and that the base-change of a point $t : \operatorname{Spec} \kappa(\Lambda)^{\mathrm{sep}} \to T$ along the field extension $\kappa(T)^{\mathrm{sep}}/\kappa(\Lambda)^{\mathrm{sep}}$ provides a point $\operatorname{Spec} \kappa(T)^{\mathrm{sep}} \to T_{\eta_T^{\mathrm{sep}}}$; hence $Z_{\eta_T^{\mathrm{sep}}}$ is algebraically equivalent to $(Z_{t_{\kappa(T)^{\mathrm{sep}}}}) = (Z_t)_{\kappa(T)^{\mathrm{sep}}}$, which is by assumption itself algebraically trivial. □

Corollary 1.6. *Suppose that* $\Lambda = \operatorname{Spec} K$ *for a field* K*, that* T *is an integral scheme in* Sm_K/S*, and* $Z \in \mathscr{A}^i_{X/S}(T)$*. Then, for any morphism* $t : \operatorname{Spec} \Omega \to T$ *with* Ω *separably closed, we have* $Z_t \in \mathrm{CH}^i(X_t)$ *is algebraically trivial.*

Proof. This is immediate from the lemma. □

1.2. **Regular homomorphisms and algebraic representatives.** We keep the notation and conventions of the previous subsection.

Definition 1.7 (Regular homomorphism)**.** Let A/S be an abelian scheme, viewed also as the contravariant functor $\mathrm{Hom}_S(-, A) : \mathsf{Sm}_\Lambda/S \to \mathsf{AbGp}$. A *regular homomorphism from $\mathscr{A}^i_{X/S}$ to A/S* is a natural transformation of functors $\Phi : \mathscr{A}^i_{X/S} \to A$.

Let us parse the definition. Given T in Sm_Λ/S, we obtain $\Phi(T) : \mathscr{A}^i_{X/S}(T) \to A(T)$; in other words, given a cycle class $Z \in \mathscr{A}^i_{X/S}(T)$, *i.e.*, a family of algebraically trivial cycle classes on X parameterized by T, we obtain a S-morphism $\Phi(T)(Z) : T \to A$.

Definition 1.8 (Algebraic representative)**.** An *algebraic representative in codimension-i* consists of an abelian S-scheme $\mathrm{Ab}^i_{X/S}$ together with a natural transformation of functors

$$\Phi^i_{X/S} : \mathscr{A}^i_{X/S} \to \mathrm{Ab}^i_{X/S}$$

over Sm_Λ/S that is initial among all natural transformations of functors $\Phi : \mathscr{A}^i_{X/S} \to A$ to abelian schemes A/S:

$$\begin{tikzcd} \mathscr{A}^i_{X/S} \arrow[r, "\Phi^i_{X/S}"] \arrow[dr, "\Phi"'] & \mathrm{Ab}^i_{X/S} \arrow[d, dashed, "\exists !"'] \\ & A \end{tikzcd} \tag{1.1}$$

In particular, if an algebraic representative exists, it is unique up to unique isomorphism.

Remark 1.9. Since abelian S-schemes, and morphisms of abelian S-schemes as S-schemes, form a full subcategory of Sm_Λ/S, by Yoneda a natural transformation of functors $\mathrm{Ab}^i_{X/S} \to A$ is equivalent to a morphism of S-schemes; from the commutativity of (1.1), this must send the zero section to the zero section. In particular, the morphism $\mathrm{Ab}^i_{X/S} \to A$ in (1.1) is induced by a homomorphism of abelian S-schemes.

1.3. **Remarks on the definition of regular homomorphisms.** We feel it would be useful here to provide some discussion of how we arrived at Definition 1.7. In fact, there are many possible variations on the definition, which lead to slight variations on the theory, and we find it to be useful to mention some pathologies that led to our choices here. An underlying theme is our desire to use Fulton's foundations for cycles and intersection theory [Ful98], and to ensure that the Abel–Jacobi map defines a regular homomorphism in the relative setting.

1.3.1. *Defining the category* Sm_Λ/S*.* In order to be able to manipulate fibers of cycles, we wanted to work in a setting where we could use the refined Gysin fibers [Ful98, §20.1]. For this we need points to be regularly embedded in the ambient space, and so insisting on parameter spaces T that are smooth separated and of finite type over Λ is expedient. This leads to considering the category of S-schemes that are smooth separated and of finite type over Λ (i.e., without the assumption that the structure map to S be dominant). For the abstract theory, the functor $\mathscr{A}^i_{X/S/\Lambda}$ extends naturally to this larger category and in fact most of the structural results (precisely all results up to Theorem 8.7 excluded, except for Corollary 4.15) in this paper go through unchanged in that setting. However, without the requirement that the structure morphisms to S be *dominant*, showing the existence of interesting (*i.e.*, nonzero) regular homomorphisms, which is a central step in showing the theory is nontrivial, becomes much harder. For instance, one can see that any time one has $f : X \to S$ a smooth projective morphism of complex algebraic manifolds, then if the geometric coniveau in a fiber jumps, the Abel–Jacobi map (see §9) would fail to be a regular homomorphism; see [ACMV17, Ex. 6.5] and compare with Theorem 3. This is the essential reason we require that objects in Sm_Λ/S have dominant structure morphism to S.

1.3.2. *Sheafification of* $\mathscr{A}^i_{X/S}$. Even in the setting of codimension-1 cycle classes, our functor $\mathscr{A}^1_{X/S}$ differs from the Picard functor. Namely, assuming $f : X \to S$ is flat, it could be natural to define a functor

$$\mathscr{P}^i_{X/S/\Lambda} := \mathscr{A}^i_{X/S/\Lambda} / f^* \mathscr{A}^i_{S/S/\Lambda},$$

where we use flat pull back of cycle classes. For instance, $\mathscr{P}^1_{X/S/\Lambda}$ agrees with the relative Picard functor $\mathscr{P}ic^0_{X/S}$ of algebraically trivial line bundles. Even in the case $i = 1$, this need not be a sheaf, and so one might prefer further to sheafify this functor, in the hope that its sheafification $(\mathscr{P}^i_{X/S/\Lambda})^\dagger$ might be representable by an abelian scheme in Sm_Λ/S. Unfortunately, for $i > 1$, there are examples where this sheaf is far from being representable by an abelian scheme (*e.g.*, Mumford's theorem that for any complex projective surface S with $h^{2,0}(S) \neq 0$, the group $\mathrm{A}^2(S)$ is infinite-dimensional ; see also Remark 7.10). Since any regular homomorphism $\Phi : \mathscr{A}^i_{X/S} \to A$ factors uniquely through a natural transformation

$$\begin{tikzcd} \mathscr{A}^i_{X/S} \arrow[r] \arrow[d, "\Phi"'] & \mathscr{P}^i_{X/S} \arrow[r] & (\mathscr{P}^i_{X/S})^\dagger \arrow[dll, dashed] \\ A & & \end{tikzcd}$$

we prefer to work with the functor $\mathscr{A}^i_{X/S}$, which although (again) far from being representable in general, is easier to work with. In light of this discussion, we view the algebraic representative as being something of a categorical moduli space for $\mathscr{A}^i_{X/S}$ (and indeed also for $\mathscr{P}^i_{X/S}$ and its sheafification).

Remark 1.10. While preparing this manuscript, we became aware of the recent work of Benoist–Wittenberg [BW19]. In the case where $S = \Lambda = \operatorname{Spec} K$ and where X is a *geometrically rational smooth projective threefold*, they define in [BW19, §2.3.2] a functor for codimension-2 cycle classes and show [BW19, Thm. 3.1] that this functor is a sheaf representable by a group scheme $\mathbf{CH}^2_{X/K}$ over K whose connected component of the identity is an abelian variety. In this setting, it would be interesting to compare their sheaf with the sheaf we work with here, namely $(\mathscr{P}^2_{X/K})^\dagger$, and to compare their abelian variety $(\mathbf{CH}^2_{X/K})^\circ$ with the algebraic representative we construct in Theorem 2 for codimension-2 cycles classes. For instance, if K is perfect, they have shown [BW19, Thm. 3.1(vi)] that the two algebraic representatives agree. On the other hand, as is mentioned in [BW19, Rmk. 3.2(i)], defining an algebraic representative in the sense of Benoist–Wittenberg for codimension-2 cycles under weaker hypotheses on the variety X could prove useful in extending Theorem 1 in the case of codimension-2 cycles to non-separable field extensions.

1.4. **The classical definition of regular homomorphisms and algebraic representatives.** We now show that over algebraically closed fields our notion of regular homomorphisms and algebraic representative agrees with the usual one (*i.e.*, [Mur85, Def. 1.6.1] or [Sam60, 2.5]).

Let k be an algebraically closed field, and take $\Lambda = S = \operatorname{Spec} k$. In that case, Lemma 1.5 takes the simple form :

Lemma 1.11. *Let T be a smooth scheme of finite type over k and let $Z \in \mathrm{CH}^i(T \times_k X)$. Then $Z \in \mathscr{A}^i_{X/k}(T)$ if and only if $Z_t \in \mathrm{A}^i(X)$ for all $t \in T(k)$.* □

For a smooth integral separated scheme T of finite type over k and a cycle class $Z \in \mathscr{A}^i_{X/k}(T)$, we therefore have a map

$$w_Z : T(k) \to \mathrm{A}^i(X)$$

$$w_Z(t) = Z_t.$$

On the other hand, for a smooth integral separated k-pointed scheme (T, t_0) of finite type over k and a cycle class $Z' \in \mathrm{CH}^i(T \times_k X)$, we have a map

$$w_{Z',t_0} : T(k) \to \mathrm{A}^i(X)$$
$$w_{Z',t_0}(t) = Z'_t - Z'_{t_0}.$$

Note that if $Z \in \mathscr{A}^i_{X/k}(T)$, then viewing $Z \in \mathrm{CH}^i(T \times_k X)$, we have $w_{Z,t_0}(t) = w_Z(t) - Z_{t_0}$. Conversely :

Lemma 1.12 (Cycle maps with and without base points)**.** *Let A/k be an abelian variety, and let $\phi : \mathrm{A}^i(X) \to A(k)$ be a homomorphism of groups. The following are equivalent :*

(i) For every smooth integral separated scheme T of finite type over k and every cycle class $Z \in \mathscr{A}^i_{X/k}(T)$, the induced map of sets $T(k) \xrightarrow{w_Z} \mathrm{A}^i(k) \xrightarrow{\phi} A(k)$ is induced by a morphism of k-schemes $\psi_Z : T \to A$.

(ii) For every smooth integral separated k-pointed scheme (T, t_0) over k and every cycle class $Z \in \mathrm{CH}^i(X \times T)$, the induced map of sets $T(k) \xrightarrow{w_{Z,t_0}} \mathrm{A}^i(k) \xrightarrow{\phi} A(k)$ is induced by a morphism of k-schemes $\psi_{Z,t_0} : T \to A$.

Proof. Assume (i). If $Z \in \mathrm{CH}^i(T \times_k X)$, set $Z' := Z - (Z_{t_0} \times_k T)$. The cycle Z' belongs to $\mathscr{A}^i_{X/k}(T)$ by Lemma 1.5 and on sets we have $\psi_{Z,t_0} = \phi \circ w_{Z,t_0} = \phi \circ w_{Z'} = \psi_{Z'}$. Thus since $\psi_{Z'}$ is induced by a morphism of schemes, so is ψ_{Z,t_0}. Conversely, given $Z \in \mathscr{A}^i_{X/k}(T)$, we have in particular $Z \in \mathrm{CH}^i(T \times_k X)$. From the definitions, we have $\psi_Z = \psi_{Z,t_0} + \phi(Z_{t_0})$, and we are done. □

The classical definition of a regular homomorphism is a homomorphism ϕ as in (ii) of the lemma above. The following lemma shows that our definition of regular homomorphisms, and hence of algebraic representatives, agrees with the usual one when working over an algebraically closed field.

Lemma 1.13 (Regular homomorphisms and cycle maps)**.** *Let A/k be an abelian variety.*

(i) Given a natural transformation of contravariant functors $\Phi : \mathscr{A}^i_{X/k} \to A$, the group homomorphism $\phi_\Phi := \Phi(\operatorname{Spec} k) : \mathscr{A}^i_{X/k}(\operatorname{Spec} k) = \mathrm{A}^i(X) \to A(k)$ has the property that for every smooth integral separated scheme T over k and every cycle class $Z \in \mathscr{A}^i_X(T)$, the induced map of sets $T(k) \xrightarrow{w_Z} \mathrm{A}^i(k) \xrightarrow{\phi_\Phi} A(k)$ is induced by a morphism of k- schemes $\psi_Z : T \to A$.

(ii) Conversely, given a group homomorphism $\phi : \mathrm{A}^i(X) \to A(k)$ with the property that for every smooth integral variety T over k and every cycle class $Z \in \mathscr{A}^i_X(T)$, the induced map of sets $T(k) \xrightarrow{w_Z} \mathrm{A}^i(k) \xrightarrow{\phi} A(k)$ is induced by a morphism of k-schemes $\psi_Z : T \to A$, there is a natural transformation of functors $\Phi_\phi : \mathscr{A}^i_{X/k} \to A$ defined by $\Phi_\phi(T)(Z) := \psi_Z$. The functor on morphisms is defined via *the refined Gysin pull-back.*

Moreover, given Φ as in (i), we have $\Phi_{\phi_\Phi} = \Phi$, and given ϕ as in (ii), we have $\phi_{\Phi_\phi} = \phi$.

Proof. This follows directly from the definitions. □

Example 1.14 (Abel–Jacobi maps are regular homomorphisms)**.** In the classical case where X is smooth projective over $S = \Lambda = \operatorname{Spec} \mathbb{C}$, the image of Griffiths' Abel–Jacobi map $AJ : \mathrm{A}^i(X) \to \mathrm{J}^{2i-1}(X)$ restricted to algebraically trivial cycles defines a subtorus $\mathrm{J}^{2i-1}_a(X)$ naturally endowed with a polarization and hence defines a complex abelian variety. It is a classical result of Griffiths that the induced Abel–Jacobi map $AJ : \mathrm{A}^i(X) \to \mathrm{J}^{2i-1}_a(X)$ defines a regular homomorphism. Additionally, in the case where $i = 1, 2$, or $\dim X$, then AJ is in fact the algebraic representative ; see [Mur85] and in particular [Mur85, Thm. C], but also Theorem 9.5 below. However, for $2 < i < \dim X$, the Abel–Jacobi map is not in general an algebraic representative ; *cf.* [OS20, Cor. 4.2]. We refer to §9.1 below for more details on intermediate Jacobians.

2. Regular homomorphisms and base change

2.1. **Base change in Sm_Λ/S.** In the notation above, let $\Phi : \mathscr{A}^i_{X/S} \to A$ be a regular homomorphism and let $S' \to S$ be a morphism in Sm_Λ/S. Then the regular homomorphism $\Phi : \mathscr{A}^i_{X/S} \to A$ induces naturally a regular homomorphism

$$\Phi_{S'} : \mathscr{A}^i_{X_{S'}/S'} \to A_{S'}. \tag{2.1}$$

Indeed, the forgetful functor $(\mathsf{Sm}_\Lambda/S') \to (\mathsf{Sm}_\Lambda/S)$ induces, *via* fibered products of functors, a functor

$$\begin{tikzcd}
\left(\mathscr{A}^i_{X/S}\right)_{S'} \arrow[r] \arrow[d] & \mathscr{A}^i_{X/S} \arrow[d] \\
(\mathsf{Sm}_\Lambda/S') \arrow[r] & (\mathsf{Sm}_\Lambda/S).
\end{tikzcd}$$

The first claim is that

$$\left(\mathscr{A}^i_{X/S}\right)_{S'} = \mathscr{A}^i_{X_{S'}/S'} \tag{2.2}$$

as functors over Sm_Λ/S'. To begin, let us show that for every $T' \to S'$ in Sm_Λ/S' we have

$$\left(\mathscr{A}^i_{X/S}\right)_{S'}(T') = \mathscr{A}^i_{X/S}(T') = \mathscr{A}^i_{X_{S'}/S'}(T').$$

The first equality follows immediately from the definition of the fibered product of functors. For the second equality, by definition, if $Z \in \mathscr{A}^i_{X/S}(T')$, then $Z \in \mathrm{CH}^i(X_{T'})$ has the property that for every separably closed field Ω and every Ω-point $t' : \mathrm{Spec}(\Omega) \to T'$ obtained as an inverse limit of morphisms $(t_n : T'_n \to T')_{n\in\mathbb{N}}$ in Sm_Λ/T', the cycle class $Z_{t'} \in \mathrm{CH}^i(X_{t'})$ is algebraically trivial.Now since $X_{T'} = (X_{S'})_{T'}$, unraveling the definition, we see that the previous sentence is also the condition that $Z \in \mathscr{A}^i_{X_{S'}/S'}(T')$. The functors also agree on morphisms since they are both defined *via* the refined Gysin pullback.

Now for $T' \to S'$, the diagram

$$\begin{tikzcd}
T' \arrow[dr, dashed] \arrow[drr, "\Phi(T')(Z)"] \arrow[ddr] & & \\
& A_{S'} \arrow[r] \arrow[d] & A \arrow[d] \\
& S' \arrow[r] & S
\end{tikzcd}$$

provides a natural transformation

$$\Phi_{S'} : \left(\mathscr{A}^i_{X/S}\right)_{S'} \to A_{S'}. \tag{2.3}$$

Combining (2.2) with (2.3) defines the natural transformation (2.1).

Lemma 2.1. *Let $\Phi : \mathscr{A}^i_{X/S} \to A$ be a regular homomorphism and let $S'' \to S' \to S$ be morphisms in Sm_Λ/S. Then $(\Phi_{S'})_{S''} = \Phi_{S''}$.*

Proof. This follows directly from the definitions and from the base-change construction. □

2.2. **Inverse limits and base change.** Both of the functors $\mathscr{A}^i_{X/S}$ and A send inverse limits of varieties to direct limits of abelian groups. Therefore, a natural transformation of functors

$$\Phi : \mathscr{A}^i_{X/S} \to A$$

extends in a canonical way to the category of schemes over S that can be obtained as inverse limits of schemes in Sm_Λ/S. For example, if $\Lambda = S = \operatorname{Spec} K$ for some field K, and if L/K is a separable extension of fields, using [Sta22, Tag 037X], we obtain a map $\Phi(L) : \mathscr{A}^i_{X/K}(L) \to A(L)$.

In fact, since in this case any scheme in Sm_L/L is an inverse limit of schemes in Sm_K/K, we obtain an induced regular homomorphism

$$\Phi_L : \mathscr{A}^i_{X_L/L} \to A_L \tag{2.4}$$

over Sm_L/L. This is a special case of the following lemma :

Lemma 2.2. *Let $\Phi : \mathscr{A}^i_{X/S/\Lambda} \to A$ be a regular homomorphism, and suppose we have a cartesian diagram*

$$\begin{array}{ccc} S' & \longrightarrow & S \\ \downarrow & & \downarrow \\ \Lambda' & \longrightarrow & \Lambda \end{array}$$

where Λ' is an integral regular Noetherian scheme, and S' is an integral scheme that is smooth separated and of finite type over Λ', which is obtained as the inverse limit of cartesian diagrams

$$\begin{array}{ccc} S'_n & \longrightarrow & S \\ \downarrow & & \downarrow \\ \Lambda'_n & \longrightarrow & \Lambda \end{array}$$

where Λ'_n is smooth separated of finite type over Λ, and $S'_n \to \Lambda'_n$ is smooth separated of finite type, so that $S'_n \to S$ is smooth separated of finite type, and so in particular in Sm_Λ/S. Every scheme in $\mathsf{Sm}_{\Lambda'}/S'$ is an inverse limit of schemes in Sm_Λ/S, and therefore Φ defines a regular homomorphism

$$\Phi_{S'} : \mathscr{A}^i_{X_{S'}/S'/\Lambda'} \to A_{S'}$$

over $\mathsf{Sm}_{\Lambda'}/S'$.

Proof. We only need to show that every scheme in $\mathsf{Sm}_{\Lambda'}/S'$ is an inverse limit of schemes in Sm_Λ/S. We may as well work with affine schemes. So, let $\Lambda = \operatorname{Spec} Q$, $\Lambda'_n = \operatorname{Spec} Q'$, $\Lambda' = \operatorname{Spec} Q$, $S = \operatorname{Spec} R$, $S'_n = \operatorname{Spec} R'_n$, and $S' = \operatorname{Spec} R'$. We have that R' is obtained as the direct limit of the rings R'_n, which are smooth and finite type over the Noetherian ring Q_n. Then any ring of finite type over R', say of the form $R'[x_1, \ldots, x_r]/I$ for some ideal I, which is smooth over Q', is the direct limit of the R-algebras $R'_n[x_1, \ldots, x_r]/I_n$, $I_n = I \cap R'_n[x_1, \ldots, x_n]$, which are smooth over Q'_n for sufficiently large n, say using the Jacobian criterion. Indeed, we may apply the direct limit to the short exact sequence $0 \to I_n \to R'_n[x_1, \ldots, x_r] \to R'_n[x_1, \ldots, x_r]/I_n \to 0$, and use exactness of the direct limit. □

We will often use this in the following form :

Corollary 2.3. *Let $\Phi : \mathscr{A}^i_{X/S} \to A$ be a regular homomorphism, let Ω be a separably closed field, and let $s : \operatorname{Spec}(\Omega) \to S$ be obtained as an inverse limit of morphisms $(s_n : S_n \to S)_{n \in \mathbb{N}}$ in Sm_Λ/S. Every scheme in $\mathsf{Sm}_\Omega/\Omega$ is an inverse limit of schemes in Sm_Λ/S, and therefore Φ defines a regular homomorphism*

$$\Phi_\Omega : \mathscr{A}^i_{X_\Omega/\Omega/\Omega} \to A_\Omega$$

over $\mathsf{Sm}_\Omega/\Omega$.

Proof. This follows immediately from the lemma. □

Corollary 2.4. *Given two cartesian diagrams*

$$\begin{array}{ccccc} S'' & \longrightarrow & S' & \longrightarrow & S \\ \downarrow & & \downarrow & & \downarrow \\ \Lambda'' & \longrightarrow & \Lambda' & \longrightarrow & \Lambda \end{array}$$

as in Lemma 2.2, we have $(\Phi_{S'})_{S''} = \Phi_{S''}$.

Proof. This follows immediately from the construction. □

2.3. **Base-change and equivariant morphisms.** Let $X \to S$ and $S' \to S$ be morphisms of schemes. Given the action of a group Γ' on S' over S, this can be extended to an action of Γ on $X_{S'}$ as follows. For each $\sigma' \in \Gamma'$, the commutativity of the diagram $\begin{tikzcd} S' \arrow[r, "\sigma'"'] \arrow[rr, bend left] & S' \arrow[r] & S \end{tikzcd}$ gives an isomorphism between $X_{S'}$ and the pull-back $(X_{S'})_{\sigma'}$ of $X_{S'}$ along σ'. In other words, for each $\sigma' \in \Gamma'$, we obtain a cartesian diagram

$$\begin{array}{ccc} X_{S'} & \xrightarrow{\sigma'} & X_{S'} \\ \downarrow & & \downarrow \\ S' & \xrightarrow{\sigma'} & S' \end{array} \tag{2.5}$$

giving the action of Γ' on $X_{S'}$.

In what follows, we will want to consider a filtered system of S-schemes S_n/S (indexed by an arbitrary filtered set), together with an S-scheme S' which is the cofiltered limit (*i.e.*, inverse limit) of the S_n. We will then assume that a group Γ' acts on the cofiltered system S_n/S, inducing an action on S'. Note that we do not assume that Γ' acts on each S_n individually. This induces an action of Γ' on the cofiltered system X_{S_n}/S_n, *via* the commutativity of the diagrams of the form $\begin{tikzcd} S_n \arrow[r, "\sigma'"'] \arrow[rr, bend left] & S_m \arrow[r] & S \end{tikzcd}$, and the same construction as given above for the action of Γ' on S'.

Example 2.5. The main example we have in mind is where $S = \operatorname{Spec} K$ for some field K, $S' = \operatorname{Spec} L$ for some separable field extension L/K, $S_n = \operatorname{Spec} R_n$ is the filtered system of smooth K-algebras R_n of finite type contained in L, and $\Gamma' = \operatorname{Aut}(L/K)$. To see that the R_n form a filtered system, we can argue as follows. Suppose we have R and R' smooth K-algebras of finite type contained in L. Consider the fraction fields $K(R), K(R') \subseteq L$. The field $K(R)K(R') \subseteq L$ is separable over K [Sta22, Lem. 030P]. Thus there is some smooth K-algebra of finite type $R'' \subseteq K(R)K(R')$ with $K(R'') = K(R)K(R')$ [Sta22, Lem. 037X]. The inclusion $K(R) \subseteq K(R'')$ induces a map of rings $R \to R''_f$ for some localization at $f \in R''$, and similarly for R'. Thus replacing R'' with the localization $R_{ff'}$, we see that the system R_n is filtered. Now since L is the localization at (0) of a smooth K-algebra of finite type over K [Sta22, Lem. 07ND, 037X], it is clear that $L = \varinjlim R_n$. Next we explain how $\Gamma' = \operatorname{Aut}(L/K)$ acts on the filtered system. For this, we simply observe that for $\sigma' \in \operatorname{Aut}(L/K)$, then for any smooth K-algebra R of finite type contained in L, we have that $\sigma'(R)$ is again a smooth K-algebra of finite type contained in L.

Definition 2.6 (Equivariant regular homomorphisms)**.** Suppose we are given a cartesian diagram

$$\begin{array}{ccc} S' & \longrightarrow & S \\ \downarrow & & \downarrow \\ \Lambda' & \longrightarrow & \Lambda \end{array}$$

as in Lemma 2.2, such that the filtered systems $S_n \to S$ and $\Lambda_n \to \Lambda$ from Lemma 2.2 admit compatible (in the obvious way) actions of a group Γ'. For each $\sigma' \in \Gamma'$, the action of the group on the filtered system X_{S_n} induces in the limit natural transformations $\sigma'^! : \mathscr{A}^i_{X_{S'}/S'} \to \mathscr{A}^i_{X_{S'}/S'}$ and

$\sigma' : A_{S'} \to A_{S'}$, defined in the obvious way. We say *a regular homomorphism*

$$\Phi' : \mathscr{A}^i_{X_{S'}/S'/\Lambda'} \to A_{S'}$$

is Γ'*-equivariant* if for all $\sigma' \in \Gamma'$ the following diagram is commutative:

$$\begin{array}{ccc} \mathscr{A}^i_{X_{S'}/S'} & \xrightarrow{\Phi'} & A_{S'} \\ \downarrow{\scriptstyle \sigma'!} & & \uparrow{\scriptstyle \sigma'} \\ \mathscr{A}^i_{X_{S'}/S'} & \xrightarrow{\Phi'} & A_{S'} \end{array}$$

Remark 2.7. Note the vertical arrow σ' in the diagram above. One could replace this isomorphism with the isomorphism $\sigma'^* = \sigma'^{-1}$ pointing down in the diagram, but we prefer to leave the diagram as it is above.

Lemma 2.8. *In the notation of Definition 2.6 and Lemma 2.2, if $\Phi : \mathscr{A}^i_{X/S/\Lambda} \to A$ is a regular homomorphism, then the regular homomorphism $\Phi_{S'} : \mathscr{A}^i_{X_{S'}/S'/\Lambda'} \to A_{S'}$ is Γ'-equivariant.*

Proof. This follows directly from the definitions and from the base-change construction of §2.1. □

Remark 2.9 (Galois descent data). Later we will discuss descent of regular homomorphisms along Galois field extensions. To clarify how this interacts with this discussion here, recall that a finite faithfully flat morphism of schemes $p : S' \to S$ is called a Galois covering if there is a finite group Γ of S-automorphisms of S' such that the morphism

$$\Gamma \times S' \longrightarrow S' \times_S S'$$

$$(\sigma, x) \mapsto (\sigma x, x)$$

is an isomorphism, where $\Gamma \times S'$ is the disjoint union of copies of S' parameterized by Γ. The standard example is where K'/K is a finite Galois extension of fields, $p : \operatorname{Spec} K' \to \operatorname{Spec} K$ is the associated map of schemes, and $\Gamma = \operatorname{Gal}(K'/K)$. Given an S'-scheme X', a descent datum for X' over S is equivalent to an action $\Gamma \times X' \to X'$ compatible with the action of Γ on S; *i.e.*, for each $\sigma \in \Gamma$, we have a cartesian diagram (2.5) [BLR90, Exa. B, p.139]. Recall that if X is a scheme over S, then $X_{S'}$ is equipped with a descent datum over S, as described above.

3. Base change and descent along field extensions

Here we assume that $S = \Lambda = \operatorname{Spec}(K)$ for some field K. Recall from §2.2 that if L/K is a separable extension of fields, then a regular homomorphism $\Phi : \mathscr{A}^i_{X/K/K} \to A$ induces in a canonical way a regular homomorphism $\Phi_L : \mathscr{A}^i_{X_L/L/L} \to A_L$. Combining Lemmas 2.2 and 2.8, we have in this setting:

Lemma 3.1. *Let $\Phi : \mathscr{A}^i_{X/K/K} \to A$ be a regular homomorphism. Let L/K be a separable extension of fields. Then*

(i) $(\Phi_L)_{L'} = \Phi_{L'}$ for any tower of separable field extensions $L'/L/K$.
(ii) Φ_L is $\operatorname{Aut}(L/K)$-equivariant.

Proof. (i) is a consequence of Lemma 2.1, and an elementary modification of the proof of Lemma 2.2. (ii) is an immediate consequence of Lemma 2.8 and Example 2.5. □

3.1. **Separable field extensions of separably closed fields.** We assume here that Ω/k is a regular (*i.e.*, primary and separable) field extension, and that k is separably closed. Equivalently, Ω/k is a separable extension of a separably closed field k. For instance, if k is algebraically closed, Ω is any field extension of k. We are interested in understanding the relationship between regular homomorphisms over k and over Ω.

If L/K is any field extension, recall that an L/K-*trace* of an abelian variety B over L is a final object $(\underline{\underline{B}}, \tau)$ (sometimes also denoted $(\mathrm{tr}_{L/K}(B), \tau)$) in the category of pairs (A, f) where A is an abelian variety over K and $f : A_L \to B$ is homomorphism of abelian varieties over L. If Ω/k is any *primary* extension, then an Ω/k-trace exists [Con06, Thm. 6.2] and is unique up to unique isomorphism. Concretely, if B is an abelian variety over Ω and A an abelian variety over k, any homomorphism $A_\Omega \to B$ factors uniquely as $A_\Omega \to \underline{\underline{B}}_\Omega \xrightarrow{\tau} B$ and there are thus canonical bijections

$$\mathrm{Hom}_\Omega(A_\Omega, B) = \mathrm{Hom}_\Omega(A_\Omega, \underline{\underline{B}}_\Omega) = \mathrm{Hom}_k(A, \underline{\underline{B}}).$$

We direct the reader to [ACMV17, §§3.3.1-3.3.2] for a review of the Ω/k-trace and Chow rigidity for primary extensions as exposited in [Con06].

Let now X be a scheme of finite type over a separably closed field k and let Ω/k be a separable extension. We start by observing as in [ACMV17, Step 2, Thm. 3.7] that if $\Psi : \mathscr{A}^i_{X_\Omega/\Omega} \to B$ is a regular homomorphism (over $\mathsf{Sm}_\Omega/\Omega$), and $(\underline{\underline{B}}, \tau : \underline{\underline{B}}_\Omega \to B)$ is the Ω/k-trace of B, then there is an induced regular homomorphism

$$\underline{\underline{\Psi}} : \mathscr{A}^i_{X/k} \to \underline{\underline{B}} \tag{3.1}$$

(over Sm_k/k) defined in the following way. Take T in Sm_k/k which we may assume to be integral (since k is separably closed, T is a disjoint union of integral schemes) and $Z \in \mathscr{A}^i_{X/k}(T)$. Since Ω/k is regular and in particular separable, Ω is an inverse limit of rings that are smooth and of finite type over k; we thereby obtain a morphism $\Psi(T_\Omega)(Z_\Omega) : T_\Omega \to B$. Let $t_0 \in T(k)$ be a base point (which exists because k is separably closed and T is smooth over k) and let $\mathrm{alb}_{t_0} : T_\Omega \to \mathrm{Alb}_{T_\Omega/\Omega}$ be the associated Albanese morphism over Ω (which classically exists; see, *e.g.*, the discussion in [ACMV22]). Since by definition alb_{t_0} is initial among morphisms from T_Ω to abelian varieties over Ω sending t_0 to 0, we obtain a commutative diagram

$$\begin{tikzcd} T_\Omega \arrow[r, "\Psi(T_\Omega)(Z_\Omega)"] \arrow[d, "\mathrm{alb}_{t_0}"'] & B \\ \mathrm{Alb}_{T_\Omega/\Omega} \arrow[ur, "\eta_{t_0}"'] & \end{tikzcd}$$

From [ACMV22, Thm. A], since Ω/k is separable, the canonical homomorphism $\mathrm{Alb}_{T_\Omega/\Omega} \to (\mathrm{Alb}_{T_k/k})_\Omega$ is an isomorphism. By the definition of the Ω/k-trace together with Chow rigidity for primary extensions, we thereby obtain a commutative diagram

$$\begin{tikzcd} T \arrow[r, dashed] \arrow[d, "\mathrm{alb}_{t_0}"'] & \underline{\underline{B}} \\ \mathrm{Alb}_{T/k} \arrow[ur, "\eta_{t_0}"'] & \end{tikzcd}$$

where the dashed arrow is the composition of the other arrows. It is easy to see that while alb_{t_0} and η_{t_0} depend on $t_0 \in T(k)$, their composition, the dashed arrow, does not. This is the morphism we define as $\underline{\underline{\Psi}}(T)(Z) : T \to \underline{\underline{B}}$, and this in turn defines the natural transformation $\underline{\underline{\Phi}} : \mathscr{A}^i_{X/k} \to \underline{\underline{B}}$ on objects. The definition on morphisms is similar.

Lemma 3.2 (Trace for regular homomorphisms)**.** *In the notation above:*

(i) Let $\Phi : \mathscr{A}^i_{X/k} \to A$ be a regular homomorphism, and let

$$\Phi_\Omega : \mathscr{A}^i_{X_\Omega/\Omega} \to A_\Omega$$

be the regular homomorphism defined in (2.4) above. Then we have $\underline{\underline{\Phi_\Omega}} = \Phi$.

(ii) Let $\Psi : \mathscr{A}^i_{X_\Omega/\Omega} \to B$ be a regular homomorphism, and let

$$\underline{\underline{\Psi}} : \mathscr{A}^i_{X_k/k} \to \underline{\underline{B}}$$

be the regular homomorphism defined in (3.1) above. Then $(\underline{\underline{\Psi}})_\Omega$ fits into a commutative diagram

$$\begin{array}{ccc} \mathscr{A}^i_{X_\Omega/\Omega} & \xrightarrow{(\underline{\underline{\Psi}})_\Omega} & \underline{\underline{B}}_\Omega \\ \| & & \downarrow{\scriptstyle \tau} \\ \mathscr{A}^i_{X_\Omega/\Omega} & \xrightarrow{\Psi} & B \end{array}$$

Proof. This is clear from the construction of the trace of a regular homomorphism and from the universal property of the trace of an abelian variety. □

3.2. **Galois extensions of fields.** Let L/K be a Galois extension of fields.

Definition 3.3 (Galois-equivariant regular homomorphism)**.** Let A be an abelian variety over K. We say *a regular homomorphism $\Psi : \mathscr{A}^i_{X_L/L} \to A_L$ is Galois-equivariant* if it is $\mathrm{Aut}(L/K)$-equivariant in the sense of Definition 2.6.

If A is an abelian variety over K, and $\Psi : \mathscr{A}^i_{X_L/L} \to A_L$ is a Galois-equivariant regular homomorphism (of functors over Sm_L/L), then there is a regular homomorphism

$$\underline{\Psi} : \mathscr{A}^i_{X/K} \to A \tag{3.2}$$

(over Sm_K/K) defined as follows. If T is in Sm_K/K and $Z \in \mathscr{A}^i_{X/K}(T)$, then we obtain an induced morphism $\Psi(T_L)(Z_L) : T_L \to A_L$ that satisfies Galois descent (see [ACMV17, Rmk. 4.3]). Thus there is an induced morphism $T \to A$, which we define to be $\underline{\Psi}(T)(Z)$. This defines $\underline{\Psi} : \mathscr{A}^i_{X/K} \to A$ on objects. The definition on morphisms is similar.

Lemma 3.4 (Regular homomorphisms and base change from K to L)**.** *Suppose X is a scheme of finite type over K, L/K is a Galois field extension, and A is an abelian variety over K.*

(i) If $\Phi : \mathscr{A}^i_{X/K} \to A$ is a regular homomorphism, then $\Phi_L : \mathscr{A}^i_{X_L/L} \to A_L$ is a Galois-equivariant regular homomorphism and we have $\underline{\Phi_L} = \Phi$.

(ii) Suppose $\Psi : \mathscr{A}^i_{X_L/L} \to A_L$ is a Galois-equivariant regular homomorphism (over Sm_L/L), and $\underline{\Psi} : \mathscr{A}^i_{X/K} \to A$ is the induced regular homomorphism defined in (3.2) above. Then we have $(\underline{\Psi})_L = \Psi$.

Proof. This is clear from the construction of $\underline{\Psi}$ for a Galois-equivariant regular homomorphism Ψ. □

4. Surjective regular homomorphisms and miniversal cycles

4.1. **Surjective regular homomorphisms.** Note that since AbGp is an abelian category, the category of contravariant functors $\mathsf{Sm}_\Lambda/S \to \mathsf{AbGp}$ is also an abelian category. Thus there is a natural notion of kernel, image, etc., for regular homomorphisms. This is *not* the definition we want to use for surjectivity. Rather, we would like to define surjectivity as meaning "surjective when restricted to algebraically closed fields". Since our results focus on certain separable extensions, we adopt the following definition:

Definition 4.1 (Surjective regular homomorphism)**.** A regular homomorphism $\Phi : \mathscr{A}^i_{X/S} \to A$ is called *surjective* if $\Phi_\Omega(\Omega) : \mathscr{A}^i_{X/S}(\Omega) \to A(\Omega)$ is surjective for all separably closed points $s : \operatorname{Spec}\Omega \to S$ obtained as inverse limits of morphisms to S in Sm_Λ/S (see Lemma 1.5).

In Proposition 4.13, we will see that the surjectivity of a regular homomorphism can be tested at the separable closure of the generic point of S. A regular homomorphism that is an epimorphism of functors is a surjective regular homomorphism (*e.g.*, Proposition 4.3 and Corollary 4.8). However, there are examples of surjective regular homomorphisms that are not epimorphisms of functors (Remark 5.2).

4.2. **Miniversal cycles for surjective regular homomorphisms.** The following notion of *miniversal cycle* for regular homomorphism will prove to be crucial to our understanding of surjectivity.

Definition 4.2 (Universal and miniversal cycle classes)**.** Let $\Phi : \mathscr{A}^i_{X/S} \to A$ be a regular homomorphism. Then a cycle class $Z \in \mathscr{A}^i_{X/S}(A)$ is called *universal* (resp. *miniversal*) for Φ if the induced morphism $\Phi(A)(Z) : A \to A$ is the identity (resp. an isogeny). If $\Phi = \Phi^i_{X/S}$ is an algebraic representative, we will often drop the reference to Φ, and simply call Z a universal (resp. miniversal) codimension-i cycle class.

The following proposition shows that the existence of a universal cycle characterizes regular homomorphisms that are epimorphisms of functors.

Proposition 4.3. *A regular homomorphism is an epimorphism of functors if and only if admits a universal cycle class.*

Proof. Let $\Phi : \mathscr{A}^i_{X/S} \to A$ be the regular homomorphism. First assume there is a universal cycle class Z. Then given any morphism $g : T \to A$, we have that $\Phi(T)(g^!Z) = g$. Thus Φ is an epimorphism of functors. Conversely, if Φ is an epimorphism of functors, then any cycle $Z \in \Phi(A)^{-1}(\mathrm{Id}_A)$ is a universal cycle class. ☐

4.3. **Existence of miniversal cycles.** Essential to the proof of Lemma 4.7 is the following result, extracted from [ACMV19b], regarding the field of definition of schemes parameterizing algebraically trivial cycles :

Proposition 4.4 ([ACMV19b])**.** *Let X be a scheme of finite type over a field K, let K^{sep} be a separable closure of K and let $\alpha \in \mathrm{A}^i(X_{K^{sep}})$ be an algebraically trivial cycle class. Then there exist a smooth quasi-projective curve C over K, a cycle $Z \in \mathrm{CH}^i(C \times_K X)$, and points $t_0, t_1 \in C(K^{sep})$, such that*

$$\alpha = Z_{t_1} - Z_{t_0} \quad \in \mathrm{A}^i(X_{K^{sep}}).$$

Moreover, if C can be chosen to be smooth projective over K, then there exist an abelian variety A over K, a cycle $Z' \in \mathrm{CH}^i(A \times_K X)$, and a point $t \in A(K^{sep})$ such that

$$\alpha = Z'_t - Z'_0 \quad \in \mathrm{A}^i(X_{K^{sep}}).$$

Proof. The existence of a smooth quasi-projective curve C over K is given by the equivalence of (i) and (ii) for separable extensions in [ACMV19b, Thm. 4.11]. The "moreover statement" can be proved along the lines of the proof of [ACMV19b, Prop. 3.14]. ☐

For the record, we have the following classical results concerning extensions of morphisms to abelian schemes.

Proposition 4.5 ([BLR90, Cor. 6, §8.4], [FC90, I.2.7])**.** *Let S be a Noetherian scheme and let A be an abelian S-scheme.*

(i) [BLR90, Cor. 6, §8.4] *Any rational S-morphism $T \dashrightarrow A$ from an S-scheme T to A is defined everywhere if T is regular.*

(ii) [FC90, I.2.7] *Suppose S is normal and let B be another abelian S-scheme. Any homomorphism $B_U \to A_U$ defined on some dense open subset $U \subseteq S$ extends to a homomorphism $B \to A$.* □

Remark 4.6. We will in fact only use Proposition 4.5(ii) [FC90, I.2.7] in the case where all schemes are taken over a fixed base field K, and S is assumed to be smooth over K; in this special case Proposition 4.5(ii) follows immediately from Proposition 4.5(i) [BLR90, Cor. 6, §8.4] and rigidity [MFK94, Prop. 6.1].

The following lemma is crucial.

Lemma 4.7 (Existence of miniversal cycle classes)**.** *Suppose that $\Phi : \mathscr{A}^i_{X/S} \to A$ is a regular homomorphism. Then there is an S-abelian subscheme $i : A' \hookrightarrow A$ such that Φ factors uniquely as*

$$\Phi : \mathscr{A}^i_{X/S} \xrightarrow{\Phi'} A' \xhookrightarrow{i} A$$

where Φ' is a surjective regular homomorphism. Moreover, for any surjective regular homomorphism $\Phi : \mathscr{A}^i_{X/S} \twoheadrightarrow A$ there exists $Z \in \mathscr{A}^i_{X/S}(A)$ such that the induced S-morphism $\Phi(A)(Z) : A \to A$ is given by $r \cdot \mathrm{Id}_A$ for some natural number r.

Proof. In case $\Lambda = S = \operatorname{Spec} K$ with K an algebraically closed field, this is [Mur85, Lem. 1.6.2 and Cor. 1.6.3]. This was generalized to the case $\Lambda = S = \operatorname{Spec} K$ with K a perfect field in [ACMV17, Lem. 4.9]. We note that in [Mur85] and in [ACMV17, Lem. 4.9], X is assumed further to be smooth and projective over K. This however is not necessary as is apparent in the proof given below for $S = \operatorname{Spec} K$ with K any field.

We now treat the case $S = \operatorname{Spec} K$ with K any field. Let A' be the abelian sub-variety of A which is obtained as the abelian sub-variety generated by all images $\Phi(D)(Z) : D \to A$, where D runs through all abelian varieties over K and $Z \in \mathscr{A}^i_{X/K/\Lambda}(D)$ runs through all cycles such that $Z_0 = 0 \in \mathrm{CH}^i(X)$. We immediately note that there exists an abelian variety D over K and a cycle class $Z \in \mathscr{A}^i_{X/K}(D)$ with the property that $\Phi(D)(Z) : D \to A$ has image A'. Better, there exists such an abelian variety D such that $\Phi(D)(Z) : D \to A$ defines an isogeny onto its image A'. Considering then a K-homomorphism $A' \to D$ such that the composition $A' \to D \to A'$ is multiplication by a non-zero integer r, we obtain by pull-back a cycle $Z' \in \mathscr{A}^i_{X/K}(A')$ with the property that $\Phi(A')(Z') : A' \to A$ factors as $A' \xrightarrow{\cdot r} A' \hookrightarrow A$. It is important to note (using Proposition 4.4) that, if Φ is surjective, then $A' = A$; this will provide the uniqueness statement regarding the factorization of Φ, and also the statement regarding the existence of miniversal cycles for surjective regular homomorphisms.

We now claim that Φ factors through $A' \hookrightarrow A$, thereby defining $\Phi' : \mathscr{A}^i_{X/K/\Lambda} \to A'$. Let $\gamma \in A(K^{\mathrm{sep}})$ be in the image of $\Phi(K^{\mathrm{sep}})$. Let $\xi \in \mathscr{A}^i_{X/K}(K^{\mathrm{sep}}) = \mathrm{A}^i(X_{K^{\mathrm{sep}}})$ be some algebraically trivial cycle class with $\Phi(K^{\mathrm{sep}})(\xi) = \gamma$. Then by Proposition 4.4 there exist a smooth quasi-projective (but not necessarily complete) curve C/K, a cycle $Z \in \mathrm{CH}^i(C \times X)$, and points $t_0, t_1 \in C(K^{\mathrm{sep}})$ with $\Phi(Z_{t_1} - Z_{t_0}) = \gamma$ and $\Phi(Z_{t_0} - Z_{t_0}) = 0_A$, the origin of A. To ease notation slightly, we let $\psi_Z = \Phi(C)(Z) : C \to A$. If C admits a *smooth* projective model, then by Proposition 4.4 the cycle ξ is parameterized by an abelian variety over K and we therefore already have $\psi_Z(C) \subset A'$ by construction of A'. Otherwise, we continue as follows. Find n sufficiently large that the Frobenius twist $C^{(p^n)}$ admits a smooth projective model (*e.g.* [Sch09, Lemma 1.2]). The iterated relative Frobenius $F^n_{C/K} : C \to C^{(p^n)}$ is a universal homeomorphism. Let $\widetilde{Z} := (F^n_{C/K} \times \mathrm{id})_* Z \in \mathrm{CH}^i(C^{(p^n)} \times X)$, and let $\psi_{\widetilde{Z}} := \Phi(C^{(p^n)})(Z)$. If Q is a point of C and $\widetilde{Q} = F^n_{C/K}(Q)$, then the projection formula shows that there is an equality $\widetilde{Z}_{\widetilde{Q}} = p^n Z_Q$. Consequently, we have a commuting diagram

$$\begin{tikzcd}
C \arrow[r, "\psi_Z"] \arrow[rr, bend left, "\overline{\psi}_Z"] \arrow[d, "F^n_{C/K}"] & A \arrow[r] \arrow[d, "{[p^n]_A}"] & \overline{A} \arrow[d, "{[p^n]_{\overline{A}}}"] \\
C^{(p^n)} \arrow[r, "\psi_{\widetilde{Z}}"] \arrow[rr, bend right, "\overline{\psi}_{\widetilde{Z}}"'] & A \arrow[r] & \overline{A}
\end{tikzcd}$$

where $\overline{A}$ is the quotient abelian variety A/A'. Since $C^{(p^n)}$ admits a smooth projective model, $\psi_{\widetilde{Z}}(C^{(p^n)}) \subset A'$, and thus $\overline{\psi}_{\widetilde{Z}}(C^{(p^n)}) = 0_{\overline{A}}$. Since $[p^n]_{\overline{A}}$ is a finite morphism, $\overline{\psi}_Z(C)$ is zero-dimensional, and $\psi_Z(C)$ is contained in a unique translate of A'. Because $\psi_Z(C)$ passes through $0_A \in A'$, we must have $\psi_Z(C) \subset A'$. In particular, $\gamma \in A'(K^{\text{sep}})$.

Having completed the case $S = \operatorname{Spec} K$, we consider the general case. This is achieved by spreading from the generic fiber. First we recall two general facts about abelian schemes. First, denoting η_S the generic point of S, we recall that if A is an abelian scheme over S and if $(A')^\circ$ is an η_S-abelian subvariety of A_{η_S}, then $(A')^\circ$ is the generic fiber of an abelian subscheme A' of A. Indeed, let $(A')^\circ$ be an abelian subvariety of A_{η_S}, where A/S is an abelian scheme over S. Then there is a homomorphism $f : A_{\eta_S} \to A_{\eta_S}$ with image $(A')^\circ$. Since S is normal, being smooth over the regular scheme Λ, f extends uniquely by Proposition 4.5(ii) [FC90, I.2.7] to a homomorphism $A \to A$. Its image defines an abelian subscheme A' over S whose generic fiber is $(A')^\circ$. Second we recall the elementary fact that if $f, g : A \to B$ are two morphisms of separated S-schemes that agree on the generic fiber, then they agree over S.

Proceeding with the proof of the general case, we start with the regular homomorphism $\Phi : \mathscr{A}^i_{X/S/\Lambda} \to A$. Using Lemma 2.2, we obtain a regular homomorphism $\Phi'_{\eta_S} : \mathscr{A}^i_{X_{\eta_S}/\eta_S/\eta_\Lambda} \to A_{\eta_S}$. From the previous case, we see that there exist an abelian subvariety $A'_{\eta_S} \subseteq A_{\eta_S}$ and a cycle $Z_{\eta_S} \in \mathscr{A}^i_{X_{\eta_S}/\eta_S/\eta_\Lambda}(A')$ that is miniversal for $\Phi'_{\eta_S} : \mathscr{A}^i_{X_{\eta_S}/\eta_S/\eta_\Lambda} \to A'_{\eta_S}$; *i.e.*, $\Phi'_{\eta_S}(A'_{\eta_S})(Z_{\eta_S}) : A'_{\eta_S} \to A'_{\eta_S}$ is multiplication by some natural number r.

Using the first observation on abelian varieties above, we find that A'_{η_S} is the generic point of a sub-abelian S-scheme $A' \subseteq A$. We next claim that Φ factors as

$$\Phi : \mathscr{A}^i_{X/S} \xrightarrow{\Phi'} A' \xhookrightarrow{i} A.$$

We define Φ' by restricting to the generic point, and then use Proposition 4.5(i) [BLR90, Cor. 6, §8.4] to get extension. More precisely, given $T \to S$ in Sm_Λ/S, and $\zeta \in \mathscr{A}^i_{X/S/\Lambda}(T)$, we define $\Phi'(T)(\zeta) : T \to A'$ by extending $\Phi'_{\eta_S}(T_{\eta_S})(\zeta_{\eta_S}) : T_{\eta_S} \to A'_{\eta_S}$. By construction, using our second observation about abelian varieties above, we have $\Phi(T)(\zeta) : T \to A$ factors through $\Phi'(T)(\zeta)$. This also provides the uniqueness of A' and the factorization.

Finally, any spread of Z_{η_S} to a cycle $Z \in \mathscr{A}^i_{X/S}(A')$ that restricts to Z_{η_S} induces an S-morphism $\Phi'(A')(Z) : A' \to A'$ whose restriction to η_S is multiplication by r. In particular, from our second observation on abelian varieties, the S-morphism $\Phi'(A')(Z)$ is multiplication by r. The surjectivity of Φ' can then be easily established using the miniversal cycle, since one can simply take appropriate Ω-points of A' to prove surjectivity. □

Corollary 4.8. *A regular homomorphism is surjective if and only if it admits a miniversal cycle class.*

Proof. If a regular homomorphism is surjective, it follows from Lemma 4.7 that it admits a miniversal cycle class. Conversely, if a regular homomorphism admits a miniversal cycle class, it is immediate that it is surjective. □

We include the following example for clarity.

Example 4.9. There are examples of surjective regular homomorphisms $\Phi : \mathscr{A}^n_{X/S} \to A$ that are epimorphisms of functors, and isomorphisms on points, but are not isomorphisms of functors. For instance, take $X = \mathbb{P}^2_k$, for an algebraically closed field k. Then $\mathrm{Ab}^2_{X/k} = 0$, the trivial abelian variety. The morphism $\Phi^2_{X/k} : \mathscr{A}^2_{X/k} \to \mathrm{Ab}^2_{X/k}$ is clearly a surjective regular homomorphism, which is an epimorphism of functors, and an isomorphism on k-points. However, let T be a smooth projective curve of positive genus, let $\alpha \in \mathrm{A}_0(T) \cong \mathrm{Alb}_{T/k}$ be any nontrivial cycle class, and take $Z = \mathrm{pr}_1^*\alpha \in \mathrm{A}^2(T \times_k X)$, where pr_1 is the first projection. We obtain a morphism $\Phi^2_{X/k}(T) : \mathscr{A}^2_{X/k}(T) \to \mathrm{Ab}^2_{X/k}(T) = 0$. Since $Z \in \mathscr{A}^2_{X/k}(T)$ is non-trivial (*e.g.*, [Ful98, Thm. 3.3]), we see that $\Phi^2_{X/k}(T)$ is not an isomorphism, and therefore $\Phi^2_{X/k}$ is not an isomorphism of functors. We note that in this special case of $X = \mathbb{P}^2_k$, there is an obvious functor that is representable by $\mathrm{Ab}^2_{X/k}$. Namely, if one considers the functor $\mathscr{P}^2_{X/k}$ given on T by taking the quotient of $\mathscr{A}^2_{X/k}(T)$ by the classes pulled back from T, then $\mathscr{P}^2_{X/k} = \mathrm{Ab}^2_{X/k}$. More generally, for a rational variety X over a field K, Benoist–Wittenberg have defined a functor of algebraically trivial cycle classes that is representable by $\mathrm{Ab}^2_{X/K}$ [BW19]; see §1.3.2 for more discussion of these topics.

4.4. **Surjective regular homomorphisms and base change.** We have the general fact that the notion of surjectivity for regular homomorphisms is invariant under base-change :

Proposition 4.10. *In the notation of Lemma 2.2, if $\Phi : \mathscr{A}^i_{X/S/\Lambda} \to A$ is a surjective regular homomorphism, then the regular homomorphism $\Phi_{S'} : \mathscr{A}^i_{X_{S'}/S'/\Lambda'} \to A_{S'}$ is surjective.*

Proof. Taking limits, we reduce to the case where $S' \to S$ is a morphism in Sm_Λ/S. By Corollary 4.8, a regular homomorphism is surjective if and only if it admits a miniversal cycle. A miniversal cycle $Z \in \mathscr{A}^i_{X/S}(A)$ for Φ provides by base-change along $S' \to S$ a miniversal cycle $Z_{S'} \in \mathscr{A}^i_{X_{S'}/S'}(A_{S'})$ for $\Phi_{S'}$. □

4.5. **Surjective regular homomorphisms, revisited.** In the proof of Lemma 4.7, we only used that the regular homomorphism $\Phi : \mathscr{A}^i_{X/S} \to A$ is surjective on the separable closure of the generic point of S. The aim of this section is to show Proposition 4.13 (see also Corollary 4.15 in the next subsection) saying not only that in Definition 4.1 we could have required surjectivity only at those points, but in fact only at ℓ-primary torsion points for some prime ℓ different from the characteristic exponent of the generic point of S.

First we have the general lemma :

Lemma 4.11 (Chow groups and purely inseparable extensions)**.** *Let X be a scheme of finite type over a field K of characteristic exponent p and let L/K be a purely inseparable extension. Denote $f : X_L \to X$ the natural projection. Then the proper push-forward $f_* : \mathrm{CH}^i(X_L) \to \mathrm{CH}^i(X)$ and the flat pull-back $f^* : \mathrm{CH}^i(X) \to \mathrm{CH}^i(X_L)$ are isomorphisms on prime-to-p torsion.*

Proof. Indeed, if L/K is finite of degree, say, p^r, then on Chow groups we have $f_*f^* = p^r$ and $f^*f_* = p^r$. □

Second we have the following lemma, which uses the existence of miniversal cycles (Lemma 4.7) :

Lemma 4.12. *Let $\Phi : \mathscr{A}^i_{X/K} \to A$ be a surjective regular homomorphism. Then there exists a natural number r such that for any separable extension Ω/K that is separably closed, and for any N invertible in K, the induced homomorphism*

$$\Phi_{\Omega,r}[N] : \mathrm{A}^i(X_\Omega)[(r,N)N] \xrightarrow{\Phi_\Omega[(r,N)N]} A(\Omega)[(r,N)N] \longrightarrow A(\Omega)[N]$$

is surjective, where $(r,N)=\gcd(r,N)$.

Proof. The lemma is proved in the case where Ω is algebraically closed in [ACMV20b, Lem. 3.2]. Let p be the characteristic exponent of K. First we claim that if $Z\in \mathrm{CH}^i(A_\Omega\times_\Omega X_\Omega)$, then the map $w_Z : A(\Omega)\to \mathrm{A}^i(X_\Omega)$, $a\mapsto Z_a - Z_0$ is a homomorphism on prime-to-p torsion ; more precisely, for each natural number N coprime to p, w_Z restricted to $A(\Omega)[N]$ gives a homomorphism $w_Z[N] : A(\Omega)[N]\to \mathrm{A}^i(X_\Omega)[N]$. Indeed, the map w_Z factors as $A(\Omega)\xrightarrow{\tau}\mathrm{A}_0(A_\Omega)\xrightarrow{Z_*}\mathrm{A}^i(X_\Omega)$, where $\tau(a)=[a]-[0]$ and Z_* is the group homomorphism induced by the action of the correspondence Z. The claim then follows from the fact that $\tau_{\overline{\Omega}} : A(\overline{\Omega})\to \mathrm{A}_0(A_{\overline{\Omega}}), a\mapsto [a]-[0]$ is an isomorphism on torsion by [Bea83, Prop. 11], where $\overline{\Omega}$ denotes an algebraic closure of Ω, from the fact that $A(\Omega)\hookrightarrow A(\Omega')$ is an isomorphism on prime-to-p torsion for any purely inseparable extension Ω'/Ω, and from the fact that the base change homomorphism $\mathrm{A}_0(A_\Omega)\to \mathrm{A}_0(A_{\Omega'})$ is an isomorphism on prime-to-p torsion for any purely inseparable extension Ω'/Ω by Lemma 4.11.

Second, let $Z\in \mathscr{A}^i_{X/K}(A)$ be a miniversal cycle provided by Lemma 4.7 such that the induced morphism $\Phi(A)(Z) : A\to A$ is given by $r\cdot \mathrm{Id}_A$ for some natural number r. Let N be a natural number relatively prime to p. By the claim, the composition of group homomorphisms

$$A(\Omega)[N]\xrightarrow{w_Z[N]}\mathrm{A}^i(X_\Omega)[N]\xrightarrow{\Phi_\Omega[N]}A(\Omega)[N]$$

is multiplication by r. In particular, $\ker w_Z[N]\subset A(\Omega)[(r,N)]$, and the composition $\Phi_{\Omega,r}[N]$ is surjective. $\square$

Proposition 4.13. *Let $\Phi : \mathscr{A}^i_{X/S}\to A$ be a regular homomorphism. Denote η_S and η_Λ the respective generic points of S and Λ. Let p be the characteristic exponent of η_S, and let Ω be a separable closure of $\kappa(\eta_S)$. The following conditions are equivalent :*

- *(i)* $\Phi : \mathscr{A}^i_{X/S}\to A$ *is surjective.*
- *(ii)* $\Phi_{\Omega'}(\Omega') : \mathrm{A}^i(X_{\Omega'})\to A_{\Omega'}(\Omega')$ *is surjective for all separably closed points $s : \operatorname{Spec}\Omega'\to S$ obtained as inverse limits of morphisms to S in Sm_Λ/S.*
- *(iii)* $\Phi_\Omega(\Omega) : \mathrm{A}^i(X_\Omega)\to A_\Omega(\Omega)$ *is surjective.*
- *(iv)* $\Phi_\Omega(\Omega) : \mathrm{A}^i(X_\Omega)\to A_\Omega(\Omega)$ *is surjective on ℓ-primary torsion for some prime $\ell\neq p$.*
- *(v)* $\Phi_\Omega(\Omega) : \mathrm{A}^i(X_\Omega)\to A_\Omega(\Omega)$ *contains the ℓ-torsion in its image for some prime $\ell\neq p$.*

Proof. The equivalence of (i) and (ii) is simply Definition 4.1, while the implication $(ii)\Rightarrow(iii)$ is obvious (since S is smooth over Λ, η_S is separable over η_Λ ; see Lemma 1.5). The implication $(iii)\Rightarrow(ii)$ follows from Lemma 3.2 and Proposition 4.10. The implication $(iii)\Rightarrow(iv)$ is Lemma 4.12, while the implication $(iv)\Rightarrow(iii)$ follows from the density of ℓ-power torsion points in A and from the fact implied by Lemma 4.7 that the image of $\Phi_\Omega(\Omega)$ consists of the Ω-points of an abelian subvariety of A. Finally, the implication $(iv)\Rightarrow(v)$ is trivial, while (v) implies (iii) because the dimension of the abelian subvariety of A which contains the image of $\Phi_\Omega(\Omega)$ can be read off from the rank of its ℓ-torsion. $\square$

4.6. **Surjective regular homomorphisms and descent.** Propositions 4.14 and 4.16 below show that surjectivity for regular homomorphisms descends in the situations described in §§3.1 and 3.2.

First we show that the trace (3.1) of a surjective regular homomorphism is surjective.

Proposition 4.14. *Let k be a separably closed field and let Ω/k be a separable field extension. Suppose $\Psi : \mathscr{A}^i_{X_\Omega/\Omega}\to B$ is a surjective regular homomorphism. Then the regular homomorphism*

$$\underline{\underline{\Psi}} : \mathscr{A}^i_{X/k}\to \underline{\underline{B}}$$

defined in (3.1) *is surjective.*

Proof. Let p be the characteristic exponent of k. First we claim that $\mathrm{A}^i(X) \to \mathrm{A}^i(X_K)$ is an isomorphism on prime-to-p torsion for all extensions K/k of separably closed fields. Lecomte's rigidity theorem [Lec86] states indeed that for an extension K/k of algebraically closed fields and for any separated scheme X of finite type over k, the base change homomorphism $\mathrm{CH}^i(X) \to \mathrm{CH}^i(X_K)$ is an isomorphism on torsion. In fact, the inverse is given by the specialization homomorphism (*cf.* [ACMV17, proof of Thm. 3.8]) and so the base change homomorphism $\mathrm{A}^i(X) \to \mathrm{A}^i(X_K)$ is also an isomorphism on torsion. Now one passes from algebraically closed fields to separably closed fields by utilizing Lemma 4.11. The proposition then follows from the characterization of surjectivity given in item (iv) of Proposition 4.13 applied to the diagram of Lemma 3.2(ii), together with the above rigidity result for prime-to-p torsion and the rigidity for prime-to-p torsion points on abelian varieties for extensions of separably closed fields. Note that a key point we are using when we employ Lemma 3.2(ii) is that for a regular extension of fields, the kernel of the trace is zero-dimensional, and supported at the identity [Con06, Thm. 6.12]. ☐

As a corollary of Proposition 4.14, we get yet another characterization of surjective regular homomorphisms :

Corollary 4.15. *A regular homomorphism $\Phi : \mathscr{A}^i_{X/S} \to A$ is surjective if and only if one of the following conditions holds :*

(v) *$\Phi_\Omega(\Omega) : \mathrm{A}^i(X_\Omega) \to A_\Omega(\Omega)$ is surjective for some separably closed point $s : \operatorname{Spec}\Omega \to S$ obtained as inverse limits of morphisms to S in Sm_Λ/S.*

(vi) *$\Phi_\Omega(\Omega) : \mathrm{A}^i(X_\Omega) \to A_\Omega(\Omega)$ is surjective on ℓ-primary torsion for some prime $\ell \neq p$ and for some separably closed point $s : \operatorname{Spec}\Omega \to S$ obtained as inverse limits of morphisms to S in Sm_Λ/S.*

Proof. Clearly if Φ is surjective, then it satisfies (v). Conversely, since $s : \operatorname{Spec}\Omega \to S$ in fact factors through the separable closure η_S^{sep} of the generic point η_S and since $\Omega/\kappa(S)^{\mathrm{sep}}$ is separable (e.g., [ACMV22, Lem. 3.1]), we can apply Proposition 4.14 to get that $\Phi_{\eta_S^{\mathrm{sep}}}(\eta_S^{\mathrm{sep}}) : \mathrm{A}^i(X_{\eta_S^{\mathrm{sep}}}) \to A_{\eta_S^{\mathrm{sep}}}(\eta_S^{\mathrm{sep}})$ is surjective. We conclude with Proposition 4.13 that Φ is surjective. Finally, the equivalence of (v) and (vi) is proven in exactly the same way as the equivalence of (iii) and (iv) in Proposition 4.13. ☐

Likewise, surjectivity for Galois-equivariant regular homomorphisms descends :

Proposition 4.16. *Suppose L/K is a Galois field extension, and A is an abelian variety over K. Suppose $\Psi : \mathscr{A}^i_{X_L/L} \to A_L$ is a surjective Galois-equivariant regular homomorphism (over Sm_L/L). Then the induced regular homomorphism*

$$\underline{\Psi} : \mathscr{A}^i_{X/K} \to A$$

defined in (3.2) *is surjective.*

Proof. Since $(\underline{\Psi})_L = \Psi$ by Lemma 3.4(ii), we have $(\underline{\Psi})(K^{\mathrm{sep}}) = \Psi(K^{\mathrm{sep}}) : \mathrm{A}^i(X_{K^{\mathrm{sep}}}) \to A(K^{\mathrm{sep}})$. Hence, by Proposition 4.13, if Ψ is surjective, then $\underline{\Psi}$ is also surjective. ☐

5. Algebraic representatives

5.1. **Algebraic representatives are surjective regular homomorphisms.** Recall from Definition 1.8 that an *algebraic representative* is a regular homomorphism that is initial among all regular homomorphisms.

Proposition 5.1. *If $\Phi^i_{X/S} : \mathscr{A}^i_{X/S} \to \mathrm{Ab}^i_{X/S}$ is an algebraic representative, then $\Phi^i_{X/S}$ is a surjective regular homomorphism.*

Proof. This follows immediately from the definition and Lemma 4.7. ☐

Remark 5.2. A result of Voisin [Voi15] shows that there exist algebraic representatives, and therefore surjective regular homomorphisms, that do not admit universal cycles. Therefore, using Proposition 4.3, Voisin's example also shows that there are surjective regular homomorphisms that are not epimorphisms of functors.

5.2. **Saito's criterion for the existence of an algebraic representative.** We can also use Lemma 4.7 to give a generalization of [Sai79, Thm. 2.2] and [Mur85, Prop. 2.1] to the relative setting.

Proposition 5.3 (Saito's criterion)**.** *An algebraic representative $\Phi^i_{X/S} : \mathscr{A}^i_{X/S} \to \mathrm{Ab}^i_{X/S}$ exists if and only if there exists a natural number M such that for every surjective regular homomorphism $\Phi : \mathscr{A}^i_{X/S} \to A$, we have $\dim_S A \leq M$.*

Proof. The proof is formally the same as the argument in [Mur85, Prop. 2.1], except we use Lemma 4.7 instead of [Mur85, Lem. 1.6.2, Cor. 1.6.3]. We include the argument for completeness.

We consider the category of surjective regular homomorphisms $\Phi : \mathscr{A}^i_{X/S} \to A$; we will denote these by (A, Φ). We set $\dim(A, \Phi) = \dim A$. A morphism from (A, Φ) to (A', Φ') is a morphism $\eta : A \to A'$ of abelian varieties such that $\Phi' = \eta \circ \Phi$.

Let M be the maximal dimension of the (A, Φ), and fix (A_0, Φ_0) of dimension M. Considering products, and using the fact that the image of a regular homomorphism is an abelian variety (Lemma 4.7), it is easy to see that for any (A, Φ) there exists some (A', Φ') of dimension M, which surjects onto both (A, Φ) and (A_0, Φ_0) (set A' to be the image of the regular homomorphism $\Phi_0 \times \Phi : \mathscr{A}^i_{X/K} \to A_0 \times A$).

Therefore, if (A_0, Φ_0) is not an algebraic representative, there is an isogeny $\eta_1 : (A_1, \Phi_1) \to (A_0, \Phi_0)$ of degree > 1. Thus, inductively, if there is no algebraic representative, we can construct an isogeny $(A_N, \Phi_N) \to (A_0, \Phi_0)$ of arbitrarily large degree. But this would then contradict the existence of the cycle $Z_0 \in \mathscr{A}^i_{X/K}(A_0)$ such that $\Phi_0(A_0)(Z) : A_0 \to A_0$ is $r_0 \cdot \mathrm{Id}_{A_0}$ for some natural number r_0, established in Lemma 4.7. ☐

5.3. **Algebraic representatives and base change.**

Proposition 5.4. *In the notation of Lemma 2.2, if there exists an algebraic representative*

$$\Phi^i_{X_{S'}/S'} : \mathscr{A}^i_{X_{S'}/S'/\Lambda'} \to \mathrm{Ab}^i_{X_{S'}/S'},$$

then there exists an algebraic representative $\Phi^i_{X/S} : \mathscr{A}^i_{X/S/\Lambda} \to \mathrm{Ab}^i_{X/S}$.

Proof. This follows immediately from Proposition 4.10 and Proposition 5.3. ☐

Corollary 5.5. *Let $s : \operatorname{Spec} \Omega \to S$ be a separably closed point obtained as an inverse limit of morphisms to S in Sm_Λ/S. If there exists an algebraic representative $\Phi^i_{X_\Omega/\Omega} : \mathscr{A}^i_{X_\Omega/\Omega/\Omega} \to \mathrm{Ab}^i_{X_\Omega/\Omega}$, then there exists an algebraic representative $\Phi^i_{X/S} : \mathscr{A}^i_{X/S/\Lambda} \to \mathrm{Ab}^i_{X/S}$.*

Proof. This follows immediately from the proposition. ☐

5.4. **Algebraic representatives and descent along separable field extensions.**

5.4.1. *Algebraic representatives and separable extensions of separably closed fields.* With Lemma 3.2 and Proposition 4.14, we can easily prove :

Theorem 5.6 (Algebraic representatives and separable extensions of separably closed fields)**.** *Let k be a separably closed field, and Ω/k be any separable field extension. There is an algebraic representative*

$$\Phi^i_{X/k} : \mathscr{A}^i_{X/k} \to \mathrm{Ab}^i_{X/k}$$

(over Sm_k/k) if and only if there is an algebraic representative

$$\Phi^i_{X_\Omega/\Omega} : \mathscr{A}^i_{X_\Omega/\Omega} \to \mathrm{Ab}^i_{X_\Omega/\Omega}$$

(over $\mathsf{Sm}_\Omega/\Omega$). In addition, if the algebraic representatives exist, then the vertical arrows in the diagrams below

$$\begin{tikzcd}
\mathscr{A}^i_{X/k} \arrow[r, "\Phi^i_{X/k}"] \arrow[d, equal] & \mathrm{Ab}^i_{X/k} \arrow[d, dashed] \\
\mathscr{A}^i_{X/k} \arrow[r, "\underline{\underline{\Phi}}^i_{X_\Omega/\Omega}"] & \underline{\underline{\mathrm{Ab}}}^i_{X_\Omega/\Omega}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{A}^i_{X_\Omega/\Omega} \arrow[r, "\Phi^i_{X_\Omega/\Omega}"] \arrow[d, equal] & \mathrm{Ab}^i_{X_\Omega/\Omega} \arrow[d, dashed] \\
\mathscr{A}^i_{X_\Omega/\Omega} \arrow[r, "(\Phi^i_{X/k})_\Omega"] & (\mathrm{Ab}^i_{X/k})_\Omega
\end{tikzcd}
\tag{5.1}$$

defined by the universal property of the algebraic representative with respect to the regular homomorphisms in the bottom row are isomorphisms. Moreover, denoting τ the trace morphism, the natural morphism obtained as the composition $(\mathrm{Ab}^i_{X/k})_\Omega \longrightarrow (\underline{\underline{\mathrm{Ab}}}^i_{X_\Omega/\Omega})_\Omega \xrightarrow{\tau} \mathrm{Ab}^i_{X_\Omega/\Omega}$ is an isomorphism. In particular, the trace for an algebraic representative is an isomorphism.

Proof. The first statement of the theorem regarding existence follows from Proposition 5.4, together with the conjunction of Saito's criterion (Proposition 5.3) with Proposition 4.14.

The proof that the vertical arrows of (5.1) are isomorphisms is *via* the commutative diagrams :

$$\begin{tikzcd}
\mathscr{A}^i_{X/k} \arrow[r, "\Phi^i_{X/k}"] \arrow[d, equal] & \mathrm{Ab}^i_{X/k} \arrow[d, dashed] \\
\mathscr{A}^i_{X/k} \arrow[r, "\underline{\underline{\Phi}}^i_{X_\Omega/\Omega}"] \arrow[d, equal] & \underline{\underline{\mathrm{Ab}}}^i_{X_\Omega/\Omega} \arrow[d, dashed] \\
\mathscr{A}^i_{X/k} \arrow[r, "\Phi^i_{X/k}"] & \mathrm{Ab}^i_{X/k}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{A}^i_{X_\Omega/\Omega} \arrow[r, "\Phi^i_{X_\Omega/\Omega}"] \arrow[d, equal] & \mathrm{Ab}^i_{X_\Omega/\Omega} \arrow[d, dashed] \\
\mathscr{A}^i_{X_\Omega/\Omega} \arrow[r, "(\Phi^i_{X/k})_\Omega"] \arrow[d, equal] & (\mathrm{Ab}^i_{X/k})_\Omega \arrow[d, dashed] \\
\mathscr{A}^i_{X_\Omega/\Omega} \arrow[r, "\left(\underline{\underline{\Phi}}^i_{X_\Omega/\Omega}\right)_\Omega"] \arrow[d, equal] & \left(\underline{\underline{\mathrm{Ab}}}^i_{X_\Omega/\Omega}\right)_\Omega \arrow[d, "\tau"] \\
\mathscr{A}^i_{X_\Omega/\Omega} \arrow[r, "\Phi^i_{X_\Omega/\Omega}"] & \mathrm{Ab}^i_{X_\Omega/\Omega}
\end{tikzcd}
\tag{5.2}$$

where on the left-hand side of (5.2) we have applied the Ω/k-trace to the right-hand side of (5.1) (and used Lemma 3.2(i)) and addended it to the bottom of the left-hand side of (5.1), and on the right-hand side of (5.2) we have applied the pull back to the left-hand side of (5.1), addended it to the bottom of the right hand side of (5.1), and then finally addended the Ω/k-trace from Lemma 3.2(ii).

The theorem then follows by the universal property of the algebraic representative (the composition of the right vertical arrows of both diagrams is the identity) and the fact that the trace morphism τ has finite kernel [Con06, Thm. 6.4(4)]. Indeed, from the former, one finds that the top vertical dashed arrows of both diagrams are injective homomorphisms. By base-change, the injectivity of the top vertical arrow on the left-hand side diagram implies that the middle vertical dashed arrow of the right-hand side diagram is also injective. From the latter, one finds that $\left(\underline{\underline{\mathrm{Ab}}}^i_{X_\Omega/\Omega}\right)_\Omega$ and $\mathrm{Ab}^i_{X_\Omega/\Omega}$ have the same dimension. It follows that the injective vertical dashed arrows of the right-hand side diagram are isomorphisms, and one also concludes that τ is an isomorphism. Finally, we also get that $\mathrm{Ab}^i_{X/k}$ and $\underline{\underline{\mathrm{Ab}}}^i_{X_\Omega/\Omega}$ have the same dimension (since their base-changes to Ω have the same dimension), and we conclude that the vertical dashed arrows of the left-hand side diagram are isomorphisms. □

Remark 5.7. This strengthens the results in [ACMV17, Thm. 3.7] in two ways. First, the vertical arrows of both diagrams of (5.2) and the natural morphism $(\mathrm{Ab}^i_{X/k})_\Omega \to \mathrm{Ab}^i_{X_\Omega/\Omega}$ were only shown to be purely inseparable isogenies ; they are in fact isomorphisms. Second, in [ACMV17, Thm. 3.7] we did not make precise the relationship among the regular homomorphisms $\Phi^i_{X/K}$, $(\Phi^i_{X,k})_\Omega$, $\Phi^i_{X_\Omega/\Omega}$, and $\underline{\underline{\Phi}}^i_{X_\Omega/\Omega}$.

5.5. **Algebraic representatives and Galois field extensions.** We now consider base change of field where L/K is an algebraic Galois field extension. First recall from (2.4) and Lemma 2.2 that if $\Phi : \mathscr{A}^i_{X/K} \to A$ is a regular homomorphism (of functors over Sm_K/K), then the induced regular homomorphism $\Phi_L : \mathscr{A}^i_{X_L/L} \to A_L$ (over Sm_L/L) is Galois-equivariant in the sense of Definition 3.3. Conversely, if A is an abelian variety over K, and $\Psi : \mathscr{A}^i_{X_L/L} \to A_L$ is a Galois-equivariant regular homomorphism (of functors over Sm_L/L), then there is a regular homomorphism

$$\underline{\Psi} : \mathscr{A}^i_{X/K} \to A$$

(over Sm_K/K) defined in (3.2).

Proposition 5.8. *Let L/K be an algebraic Galois field extension. An algebraic representative $\Psi^i_{X_L/L} : \mathscr{A}^i_{X_L/L} \to \mathrm{Ab}^i_{X_L/L}$, if it exists, is necessarily Galois-equivariant. Moreover, it is initial among all Galois-equivariant regular homomorphisms.*

Proof. This is [ACMV17, Thm. 4.4]. □

With Lemma 3.4 and Proposition 4.16 we can prove the following theorem :

Theorem 5.9 (Algebraic representatives and Galois field extensions)**.** *Let L/K be a Galois field extension. There is an algebraic representative*

$$\Phi^i_{X/K} : \mathscr{A}^i_{X/K} \to \mathrm{Ab}^i_{X/K}$$

(over Sm_K/K) if and only if there is a (Galois-equivariant) algebraic representative

$$\Psi^i_{X_L/L} : \mathscr{A}^i_{X_L/L} \to \mathrm{Ab}^i_{X_L/L}$$

(over Sm_L/L). In addition, if both exist, then the vertical arrows in the diagram below,

$$\begin{tikzcd} \mathscr{A}^i_{X/K} \arrow[r, "\Phi^i_{X/k}"] \arrow[d, equal] & \mathrm{Ab}^i_{X/K} \arrow[d, dashed] \\ \mathscr{A}^i_{X/k} \arrow[r, "\underline{\Psi}^i_{X_L/L}"] & \underline{\mathrm{Ab}}^i_{X_L/L} \end{tikzcd} \qquad \begin{tikzcd} \mathscr{A}^i_{X_L/L} \arrow[r, "\Psi^i_{X_L/L}"] \arrow[d, equal] & \mathrm{Ab}^i_{X_L/L} \arrow[d, dashed] \\ \mathscr{A}^i_{X_L/L} \arrow[r, "(\Phi^i_{X/K})_L"] & (\mathrm{Ab}^i_{X/K})_L, \end{tikzcd} \tag{5.3}$$

defined by the universal property of the (Galois-equivariant) algebraic representative with respect to the (Galois-equivariant) regular homomorphism in the bottom row, are isomorphisms.

Proof. The first statement of the theorem regarding existence follows from Proposition 5.8, Proposition 5.4, together with the combination of Saito's criterion (Proposition 5.3) with Proposition 4.16. One then establishes (5.3) exactly as in the proof of Theorem 5.6, using a modified version of Diagram (5.2). □

5.6. **Algebraic representatives and separable field extensions.** Theorems 5.6 and 5.9 can be combined to provide a proof of Theorem 1, which we recall below, concerning descent of algebraic representatives along separable extensions of fields.

Theorem 5.10 (Theorem 1)**.** *Let Ω/K be a separable field extension. Then an algebraic representative $\Phi^i_{X_\Omega} : \mathscr{A}^i_{X_\Omega/\Omega/\Omega} \to \mathrm{Ab}^i_{X_\Omega/\Omega}$ exists if and only if an algebraic representative $\Phi^i_X : \mathscr{A}^i_{X/K/K} \to \mathrm{Ab}^i_{X/K}$ exists. If this is the case, we have in addition :*

(i) $\mathrm{Ab}^i_{X_\Omega/\Omega}$ *identifies with* $(\mathrm{Ab}^i_{X/K})_\Omega$ via *the natural homomorphism* $\mathrm{Ab}^i_{X_\Omega/\Omega} \to (\mathrm{Ab}^i_{X/K})_\Omega$ *induced by the regular homomorphism* $(\Phi^i_X)_\Omega : \mathscr{A}^i_{X_\Omega/\Omega/\Omega} \to (\mathrm{Ab}^i_{X/K})_\Omega$ *and the universal property of* $\Phi^i_{X_\Omega}$ *;*

(ii) $\Phi^i_{X_\Omega}(\Omega) : \mathrm{A}^i(X_\Omega) \to \mathrm{Ab}^i_{X_\Omega/\Omega}(\Omega)$ *is* $\mathrm{Aut}(\Omega/K)$*-equivariant, relative to the above identification.*

Proof. Let $\Omega^{\mathrm{sep}} \supseteq K^{\mathrm{sep}}$ be separable closures of Ω and K; note that $\Omega^{\mathrm{sep}} \supseteq K^{\mathrm{sep}}$ is separable [ACMV22, Lemma 3.1]. That an algebraic representative $\Phi^i_{X_\Omega}$ exists if and only if an algebraic representative Φ^i_X exists is obtained by applying Theorem 5.9 to the extension $\Omega^{\mathrm{sep}}/\Omega$, Theorem 5.6 to the extension $\Omega^{\mathrm{sep}}/K^{\mathrm{sep}}$ and Theorem 5.9 to the extension K^{sep}/K. The identification $\mathrm{Ab}^i_{X_\Omega/\Omega} \xrightarrow{=} (\mathrm{Ab}^i_{X/K})_\Omega$ comes from the isomorphisms provided by the vertical dotted arrows of the right-hand squares of (5.1) and (5.3). The $\mathrm{Aut}(\Omega/K)$-equivariance of $\Phi^i_{X_\Omega}(\Omega)$ is then given by Lemma 3.1. ☐

Remark 5.11. Slightly more is true : Let $\sigma : \Omega_1 \simeq \Omega_2$ be an isomorphism of separable fields over K Assume that $\mathrm{Ab}^i_{X_{\Omega_1}}/\Omega_1$ exists. Then $\mathrm{Ab}^i_{X_{\Omega_2}}/\Omega_2$ exists, $\mathrm{Ab}^i_{X_{\Omega_j}}/\Omega_j$ identifies with $(\mathrm{Ab}^i_{X/K})_{\Omega_j}$ for $j = 1, 2$, and relative to these identifications $\mathrm{A}^i(X_{\Omega_1}) \to \mathrm{Ab}^i_{X_{\Omega_1}/\Omega_1}(\Omega_1)$ is mapped to $\mathrm{A}^i(X_{\Omega_2}) \to \mathrm{Ab}^i_{X_{\Omega_2}/\Omega_2}(\Omega_2)$ *via* σ.

6. Existence results for algebraic representatives

We now prove the existence statement of Theorem 2 concerning algebraic representatives for cycles of dimension 0, codimension 1 and codimension 2. In particular, in the codimension-2 case, this generalizes Murre's result concerning the existence of algebraic representatives for codimension-2 cycles [Mur85, Thm. A] on smooth projective varieties defined over an algebraically closed field.

Theorem 6.1. *Suppose X is a scheme of finite type over S with geometric generic fiber $X_{\overline{\eta}_S}$, and let i be a non-negative integer. Suppose there exists some ℓ, invertible in $\kappa(\eta_S)$, such that the ℓ-torsion group $\mathrm{A}^i(X_{\overline{\eta}_S})[\ell]$ is finite. Then there exists an algebraic representative*

$$\Phi^i_{X/S} : \mathscr{A}^i_{X/S/\Lambda} \longrightarrow \mathrm{Ab}^i_{X/S}\,.$$

In particular, if $X \to S$ is smooth and proper, then an algebraic representative $\Phi^i_{X/S} : \mathscr{A}^i_{X/S/\Lambda} \longrightarrow \mathrm{Ab}^i_{X/S}$ exists for $i = 1, 2$ and $\dim_S X$.

Proof. In the case where X is a smooth projective scheme over $S = \operatorname{Spec} K$ for some algebraically closed field K, the theorem is classical and is due to Murre [Mur85, Proof of Thm. A]. We have suitably generalized Saito's criterion and the existence of miniversal cycles in order to generalize Murre's argument to the general setting where S is any separated smooth scheme of finite type over a regular Noetherian scheme (rather than the spectrum of an algebraically closed field) and where X is any scheme of finite type over S (rather than smooth and projective). Via Corollary 5.5, the theorem reduces to S being the spectrum of a separably closed field. By Saito's criterion of Proposition 5.3, we have to show the existence of a natural number M such that for every surjective regular homomorphism $\Phi : \mathscr{A}^i_{X_{\eta_S}/\eta_S} \to A$, we have $\dim_{\eta_S} A \le M$. Let R be the rank of $\mathrm{A}^i(X_{\overline{\eta}_S})[\ell]$. It suffices to show that, for each such Φ and A, $\#A(\eta_S^{\mathrm{sep}})[\ell]$ has rank at most R. Note that our hypothesis implies that, for every $e \ge 1$, $\#A(\eta_S^{\mathrm{sep}})[\ell^e]$ has rank R.

To this end, let $\Phi : \mathscr{A}^i_{X_{\eta_S}/\eta_S} \to A$ be a surjective regular homomorphism. Let $Z \in \mathscr{A}^i_{X_{\eta_S}/\eta_S}(A)$ be a miniversal cycle provided by Lemma 4.7 such that the induced morphism $\Phi(A)(Z) : A \to A$ is given by $r \cdot \mathrm{Id}_A$ for some natural number r. By Lemma 4.12, there is a surjective homomorphism $\mathrm{A}^i(X_{\eta_S^{\mathrm{sep}}})[\ell^{\mathrm{ord}_\ell(r)+1}] \twoheadrightarrow A(\eta_S^{\mathrm{sep}})[\ell]$, which shows that the latter indeed has rank at most R.

Now assume $X \to S$ is smooth and proper; we need to show that for $i = 1, 2$ and $\dim_S X$, $\mathrm{A}^i(X_{\overline{\eta}_S})[\ell]$ is finite. The case $i = 1$ follows from the identification of $\mathrm{A}^1(X_{\overline{\eta}_S})[\ell]$ with $\mathrm{Pic}^0_{X_{\overline{\eta}_S}}[\ell]$, the case $i = 2$ from the finiteness of $\mathrm{CH}^2(X_{\overline{\eta}_S})[\ell]$ due to Merkurjev and Suslin [MS82] (see also Theorem 8.1 below), and the case $i = \dim_S X$ from Roitman's theorem [Blo79]. (In the projective case, the latter can be found in [Blo79, Thm. 4.1] or [ACMV21, Prop. A.26], combined with the fact, *e.g.* [ACMV19b, Prop. 3.11], that numerical and algebraic equivalence agree on for 0-cycles on X_Ω. The proper case follows from Chow's lemma and the facts that CH_0 and Alb are birational invariants for smooth proper varieties.) $\square$

Remark 6.2. In Theorem 7.7, we will establish the existence of the algebraic representative $\Phi^d_{X/S}$ under the assumptions that $X \to S$ is proper with reduced and connected geometric fiber and that the composition $X \to S \to \Lambda$ is smooth. We also mention that, in the codimension-1 case, one can further establish the existence of the algebraic representative $\Phi^1_{X/S}$ under the weaker assumption that the geometric generic fiber of $X \to S$ admits an open embedding in a smooth proper scheme over $\overline{\eta}_S$. In that case, it can indeed be established, *via* the localization exact sequence for Chow groups, that, for ℓ invertible in $\kappa(\eta_S)$, the ℓ-torsion group $\mathrm{A}^i(X_{\overline{\eta}_S})[\ell]$ is finite.

7. Relation to the Picard scheme and the Albanese scheme

The aim of this section is to compare the algebraic representatives for codimension-1 cycles and for dimension-0 cycles to the relative Picard scheme and to the relative Albanese scheme, respectively (Theorems 7.1 and 7.9). In light of the importance of the (non-)existence of universal cycle classes (*e.g.*, [Voi15, ACMV20a]), we also discuss the existence of universal codimension-1 cycle classes in Corollary 7.3. In Theorem 7.7 we generalize the existence result for algebraic representatives for dimension-0 cycles of Theorem 6.1 to S-schemes X that are separated, geometrically reduced, geometrically connected and of finite type over S. In relation to §1.3.1, we note that all results of this section hold in the setting where the structure morphisms to S of parameter spaces are not necessarily dominant.

7.1. Algebraic representatives for codimension-1 cycles.

7.1.1. *Picard schemes and regular homomorphisms.* Recall from Theorem 6.1 that if the morphism $f : X \to S$ is smooth and proper, then there exists an algebraic representative $\Phi^1_{X/S} : \mathscr{A}^1_{X/S/\Lambda} \to \mathrm{Ab}^1_{X/S}$. We now compare this to the relative Picard scheme:

Theorem 7.1 (Picard schemes and algebraic representatives for divisors)**.** *Suppose $f : X \to S$ is smooth and projective with geometrically integral fibers, and $R^2 f_*\mathcal{O}_X = 0$.*

(i) The Abel–Jacobi map $AJ^1 : \mathrm{Div}^0_{X/S} \to \mathrm{Pic}^0_{X/S}$ for divisors induces a regular homomorphism

$$\Phi^1_{X/S} : \mathscr{A}^1_{X/S} \to \mathrm{Pic}^0_{X/S},$$

which is an algebraic representative in codimension-1.

(ii) For any morphism $S' \to S$ obtained as an inverse limit of morphisms in Sm_Λ/S, the natural homomorphism of S'-abelian schemes

$$\mathrm{Ab}^1_{X_{S'}/S'} \to (\mathrm{Ab}^1_{X/S})_{S'}$$

is an isomorphism.

(iii) For any separably closed point $s : \operatorname{Spec}\Omega \to S$ obtained as an inverse limit of morphisms to S in Sm_Λ/S, the homomorphism $\Phi^1_{X/S}(\Omega)$ is an isomorphism; in particular, the map

$$T_l\,\mathrm{A}^1(X_\Omega) \xrightarrow[\cong]{T_l\Phi^1_{X/S}(\Omega)} T_l\,\mathrm{Ab}^1_{X/S}(\Omega).$$

is an isomorphism for all primes l.

Proof. The proof follows readily from the standard results in say [Kle05]. □

Remark 7.2. If $\Lambda = S = \operatorname{Spec} K$, and X/K is proper and geometrically normal, then $(\mathrm{Pic}^0_{X/K})_{\mathrm{red}}$ is an abelian variety [Gro62, Cor. VI.3.2] (see also [Kle05, Rem. 9.5.6, Thm. 9.5.3, Cor. 9.4.18.3] and [Gro62, Prop. VI.3.1]), and the Abel–Jacobi map $AJ^1 : \mathrm{Div}^0_{X/S} \to \mathrm{Pic}^0_{X/K}$ for divisors still induces a regular homomorphism

$$\Phi^1_{X/K} : \mathscr{A}^1_{X/K} \to (\mathrm{Pic}^0_{X/K})_{\mathrm{red}},$$

which is an algebraic representative in codimension-1, as our parameter spaces are reduced, and therefore every morphism from a reduced scheme to $\mathrm{Pic}^0_{X/K}$ factors uniquely through $(\mathrm{Pic}^0_{X/K})_{\mathrm{red}}$.

7.1.2. *Some remarks on universal codimension-1 cycle classes.* Bearing in mind that the (non-)existence of universal cycle classes is a subtle and interesting invariant (*e.g.*, [Voi15, ACMV20a]), we note :

Corollary 7.3 (Universal codimension-1 cycle classes)**.** *Assume $f : X \to S$ is smooth and projective with geometrically integral fibers, and $R^2 f_* \mathcal{O}_X = 0$. If f admits a section, then $\mathrm{Ab}^1_{X/S}$ admits a universal codimension-1 cycle class.*

Proof. The proof follows readily from the standard results in say [Kle05]. □

Remark 7.4. Assume $\Lambda = S = \operatorname{Spec} K$, and X/K is proper and geometrically normal. If X admits a K-point, then $\mathrm{Ab}^1_{X/K} = (\mathrm{Pic}^0_{X/K})_{\mathrm{red}}$ admits a universal codimension-1 cycle class.

Remark 7.5. Even when working with a smooth projective geometrically integral variety X over a perfect field K there are some subtleties regarding universal codimension-1 cycle classes. For instance, if $K = \mathbb{Q}_p$ and X/K is any curve of genus one with $X(K) = \emptyset$, then $\mathrm{Ab}^1_{X/K}$ does not admit a universal codimension-1 cycle class. At the same time, we note that in general, the existence of a K-point on a smooth projective geometrically integral variety is not a necessary condition for the existence of a universal codimension-1 cycle class ; *e.g.*, any smooth projective geometrically integral variety over a finite field admits a universal codimension-1 cycle class.

7.2. **Algebraic representatives for dimension-0 cycles.** We denote by $d = d_{X/S}$ the relative dimension of X over S.

7.2.1. *Albanese schemes and regular homomorphisms.* Let $X \to S$ be a proper morphism. An S-morphism $g : X \to A$ to an abelian scheme induces a regular homomorphism

$$\Phi_g : \mathscr{A}^d_{X/S/\Lambda} \to A.$$

Conversely, suppose in addition that X belongs to Sm_Λ/S, *i.e.*, X is smooth over Λ and $X \to S$ is dominant and proper, and we assume the generic fiber of X/Λ is geometrically connected, and X/S admits a section $\sigma : S \to X$. Then the cycle $Z = \Delta - (X \times_S \sigma(S))$ is in $\mathscr{A}^d_{X/S/\Lambda}(X)$, and for any regular homomorphism $\Phi : \mathscr{A}^d_{X/S/\Lambda} \to A$, we obtain a morphism $\Phi(X)(Z) : X \to A$. These constructions are inverses of one another, in the sense that if g sends σ to 0, then $\Phi_g(X)(Z) = g$, and $\Phi_{\Phi(X)(Z)} = \Phi$. This gives the following lemma. Recall that if X/S is a scheme over S, with section $\sigma : S \to X$, then a *pointed S-Albanese morphism* is an initial S-morphism $X \to A$ to an abelian S-scheme sending σ to 0 ; *i.e.*, $S \to X \to A$ is the zero section ; and the abelian scheme is called the pointed S-Albanese scheme of X.

Lemma 7.6. *Let X be in Sm_Λ/S and assume that X/Λ has generic fiber that is geometrically connected. Assume that X/S is proper and admits a section σ, and set $Z = \Delta - (X \times_S \sigma(S))$ where Δ is the diagonal on $X \times_S X$. If*

$$\Phi^d_{X/S} : \mathscr{A}^d_{X/S/\Lambda} \longrightarrow \mathrm{Ab}^d_{X/S}$$

is an algebraic representative, then the associated map $\Phi^d_{X/S}(X)(Z) : X \to \mathrm{Ab}^d_{X/S}$ *is a pointed S-Albanese morphism sending* σ *to zero, with* $\Phi_{\Phi^d_{X/S}(X)(Z)} = \Phi^d_{X/S}$*. Conversely, if*

$$\alpha_\sigma : X \longrightarrow \mathrm{Alb}_{X/S}$$

is a pointed S-Albanese morphism sending σ *to zero, then* $\Phi_{\alpha_\sigma} : \mathscr{A}^d_{X/S/\Lambda} \to \mathrm{Alb}_{X/S}$ *is an algebraic representative with* $\Phi_{\alpha_\sigma}(X)(Z) = \alpha_\sigma$*.*

Proof. This follows easily from the discussion above. □

We have the following consequence, generalizing results of Serre–Grothendieck–Conrad on the existence of Albanese varieties over fields ([Ser60]; see also the discussion in [Wit08, Appendix] and [ACMV22, §1]), and also Grothendieck's relative existence result (which requires that $\mathscr{P}ic^0_{X/S}$ be represented by an abelian scheme ; see Theorem 7.9, below).

Theorem 7.7 (Albanese schemes and algebraic representatives for dimension-0 cycles)**.** *Let* Λ *be a regular Noetherian scheme, let* S *be a scheme that is smooth separated and of finite type over* Λ*, and let* $f : X \to S$ *be a proper surjective morphism such that the composition* $X \to S \to \Lambda$ *is smooth. If the generic fiber of* X/Λ *is geometrically connected, then there is an algebraic representative* $\Phi^d_{X/S} : \mathscr{A}^d_{X/S/\Lambda} \to \mathrm{Ab}^d_{X/S}$*. If moreover* X/S *admits a section, then there exists a pointed S-Albanese morphism* $X \to \mathrm{Alb}_{X/S}$*.*

Proof. This is a straightforward application of Lemma 7.6. □

Remark 7.8. The existence of the algebraic representative in Theorem 7.7 follows from Theorem 6.1 in the case where $f : X \to S$ is smooth and proper.

7.2.2. *Grothendieck's Albanese schemes and algebraic representatives for dimension-*0 *cycles.* Recall from Theorem 6.1 that if the morphism $f : X \to S$ is smooth and proper, then there exists an algebraic representative $\Phi^d_{X/S} : \mathscr{A}^d_{X/S/\Lambda} \to \mathrm{Ab}^d_{X/S}$. We also have Theorem 7.7 above, comparing this with a relative Albanese scheme. We now compare this to Grothendieck's construction :

Theorem 7.9 (Grothendieck's Albanese schemes and algebraic representatives for points)**.** *Suppose that* $f : X \to S$ *is smooth and projective with geometrically integral fibers,* $R^2 f_* \mathcal{O}_X = 0$*, and* f *admits a section.*

(i) The universal line bundle $\mathcal{P}$ *on* $\mathrm{Pic}^0_{X/S} \times_S X$*, distinguished by being trivialized along the section, determines a pointed S-Albanese morphism* $X \to \mathrm{Alb}_{X/S} := (\mathrm{Pic}^0_{X/S})^\vee$*,* $(t : T \to X) \mapsto (\mathrm{Id} \times t)^* \mathcal{P}$*, that induces a regular homomorphism*

$$\Phi^d_{X/S} : \mathscr{A}^d_{X/S} \to \mathrm{Alb}_{X/S},$$

*which is an algebraic representative for dimension-*0 *cycles.*

(ii) For any morphism $S' \to S$ *obtained as an inverse limit of morphisms in* Sm_Λ/S*, the natural homomorphism of* S'*-abelian schemes*

$$\mathrm{Ab}^d_{X_{S'}/S'} \to (\mathrm{Ab}^d_{X/S})_{S'}$$

is an isomorphism.

(iii) For any separably closed point $s : \operatorname{Spec}\Omega \to S$ *obtained as an inverse limit of morphisms to* S *in* Sm_Λ/S*, the homomorphism* $\Phi^d_{X/S}(\Omega)$ *is an isomorphism on torsion ; in particular, the map*

$$T_l\, \mathrm{A}^d(X_\Omega) \xrightarrow[\cong]{T_l \Phi^d_{X/S}(\Omega)} T_l\, \mathrm{Ab}^d_{X/S}(\Omega).$$

is an isomorphism for all primes l*.*

Proof. This follows readily from the standard arguments in [Gro62, VI, Thm. 3.3(iii)]. □

Remark 7.10. Mumford's examples (e.g., a smooth K3 surface over the complex numbers) show that $\Phi^d_{X/S}(\Omega)$ need not be an isomorphism. Precisely, if the generic fiber X_{η_S} of the smooth projective morphism $f : X \to S$ is such that some ℓ-adic cohomology group $H^i(X_{\eta_S^{\mathrm{sep}}}, \mathbb{Q}_\ell)$ is not supported on a divisor for some $i > 1$, then $\Phi^d_{X/S}(\Omega)$ is not injective for any separable field extension Ω/η_S that is separably closed and of infinite transcendence degree over its prime subfield. However, if Ω is $\overline{\mathbb{Q}}$ or a separably closed subfield of $\overline{\mathbb{F}_p(T)}$, then the Beilinson conjecture predicts that $\Phi^d_{X/S}(\Omega)$ should be an isomorphism.

Remark 7.11. We are unaware if $\Phi^d_{X/S}$ is an epimorphism of functors, *i.e.*, in view of Proposition 4.3, if it admits a universal codimension-d cycle class.

Remark 7.12. If $\Lambda = S = \operatorname{Spec} K$, and X/K is proper and geometrically normal, then the Abel–Jacobi map induces an algebraic representative $\Phi^1_{X/K} : \mathscr{A}^1_{X/K} \to (\mathrm{Pic}^0_{X/K})_{\mathrm{red}} = \mathrm{Ab}^1_{X/K}$ (Remark 7.2). If in addition X admits a zero-cycle α_0 of degree-one (but not necessarily a K-point), then X still admits a universal divisor $\mathcal{P}$, which induces, as in Theorem 7.9(i), a regular homomorphism $\Phi : \mathscr{A}^d_{X/K} \to (\mathrm{Pic}^0_{X/K})^\vee_{\mathrm{red}}$ that on $\overline{K}$ points is given by $\alpha \mapsto \mathcal{P}_\alpha$, where if $\alpha = \sum n_i p_i$, then $\mathcal{P}_\alpha = \bigotimes \mathcal{P}_{p_i}^{\otimes n_i}$, viewed as a line bundle on $(\mathrm{Pic}^0_{X/K})_{\mathrm{red}}$. The same arguments one uses to prove Theorem 7.9(i) (using the map $X \to \mathscr{A}^d_{X/K}$, $x \mapsto x - \alpha_0$) show that Φ is an algebraic representative in codimension-d ; *i.e.*, under these hypotheses

$$\mathrm{Ab}^d_{X/K} = ((\mathrm{Pic}^0_{X/K})_{\mathrm{red}})^\vee = (\mathrm{Ab}^1_{X/K})^\vee.$$

Note that if in addition X admits a K-point (in which case X is also geometrically connected) then the morphism $X \to \mathrm{Ab}^d_{X/K}$ is an Albanese morphism.

In §8 below, we study the behavior of algebraic representatives under base change. Granting Theorems 8.3 and 8.7 for now – their proofs are independent of the present section – we can secure a duality like that of Theorem 7.9(i) under slightly weaker hypotheses.

Proposition 7.13. *Assume S is the spectrum of a DVR, or $\Lambda = \operatorname{Spec} K$ for a field $K \subseteq \mathbb{C}$. Suppose that $f : X \to S$ is smooth and proper and that $\Phi^1_{X/S} : \mathscr{A}^1_{X/S} \to \mathrm{Ab}^1_{X/S}$ and $\Phi^d_{X/S} : \mathscr{A}^d_{X/S} \to \mathrm{Ab}^d_{X/S}$ are algebraic representatives (Theorem 6.1).*

- *(i) If the generic fiber admits a zero-cycle α_0 of degree-one, then it induces a regular homomorphism $\Phi^d_{X/S} : \mathscr{A}^d_{X/S} \to ((\mathrm{Ab}^1_{X/S}))^\vee$ that is an algebraic representative; i.e., $((\mathrm{Ab}^1_{X/S}))^\vee \cong \mathrm{Ab}^d_{X/S}$.*
- *(ii) If X/S admits a section and the generic fiber is geometrically connected, then there is an induced map $X \to \mathrm{Ab}^d_{X/S} = ((\mathrm{Ab}^1_{X/S}))^\vee$ making $((\mathrm{Ab}^1_{X/S}))^\vee$ a pointed Albanese for X/S (Lemma 7.6).*
- *(iii) If, moreover, there exists a universal codimension-1 cycle class Z (see Corollary 7.3), then it induces a morphism $X \to ((\mathrm{Ab}^1_{X/S}))^\vee$ that agrees with the pointed Albanese map in (ii).*

Proof. The proofs are straightforward applications of Theorems 8.3 and 8.7. □

8. Surjective regular homomorphisms, algebraic representatives and cohomology

8.1. **The ℓ-adic Bloch map.** Recall that, for X a smooth projective variety over an algebraically closed field K, Bloch [Blo79] defined for ℓ a prime different from the characteristic of K and for all nonnegative integers i a functorial map

$$\lambda^i : \mathrm{CH}^i(X)[\ell^\infty] \to H^{2i-1}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(i)),$$

which we will refer to as the *Bloch map*. Here, for an abelian group M, $M[\ell^\infty]$ denotes the ℓ-primary torsion subgroup $\varinjlim M[\ell^n]$. In fact, Bloch's map is defined for smooth projective varieties defined over a separably closed field (*e.g.*, [ACMV21, Rem. A.7]), and one may define an *ℓ-adic Bloch map*

$$\lambda^i : T_\ell \operatorname{CH}^i(X)[\ell^\infty] \to H^{2i-1}(X, \mathbb{Z}_\ell(i))_\tau$$

for all smooth projective varieties over a separably closed field. Here, for an abelian group M, $T_\ell M$ denotes the Tate module $\varprojlim M[\ell^n]$ and M_τ denotes the quotient by the torsion subgroup. Here we use the observation, originally due to Suwa [Suw88] that by applying T_ℓ to the usual Bloch map one obtains the ℓ-adic Bloch map. Classically, the usual Bloch map is bijective for codimension-1 cycles and for dimension-0 cycles ; *cf.* [Blo79, Prop. 3.6, Prop. 3.9, Thm. 4.1]. The corresponding statement for the ℓ-adic Bloch map follows by applying T_ℓ ; *cf.* [ACMV21, Prop. A.25, Prop. A.26].

Theorem 8.1 ([MS82], [CTR85]). *Let X be a smooth proper variety over a separably closed field K. Then the ℓ-adic Bloch map*

$$T_\ell \operatorname{CH}^2(X) \xrightarrow{\lambda^2} H^3(X, \mathbb{Z}_\ell(2))_\tau$$

is injective.

Proof. The injectivity of the usual second Bloch map $\lambda^2 : \operatorname{CH}^2(X)[\ell^\infty] \to H^3(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(2))$ for X smooth projective is due to Merkurjev and Suslin [MS82], and the corresponding statement for the second ℓ-adic Bloch map follows by applying T_ℓ ; *cf.* [ACMV21, Prop. A.27]. In fact, the usual second Bloch map is shown to be injective for X smooth and proper over a separably closed field in [CTR85, Prop. 3.1 and Rmk. 3.2]. Applying T_ℓ gives the theorem. $\square$

8.2. **Surjective regular homomorphisms and cohomology.** Let X be a smooth proper variety over a field K of characteristic exponent p. Recall from the proof of Lemma 4.12 that any cycle $Z \in \mathscr{A}^i_{X/K}(A)$ induces a homomorphism of $\operatorname{Gal}(K)$-representations on N-torsion

$$w_Z : A(K^{\mathrm{sep}})[N] \to \mathrm{A}^i(X_{K^{\mathrm{sep}}})[N], \quad a \mapsto Z_*([a] - [0])$$

for all N coprime to p ; and hence a homomorphism of $\operatorname{Gal}(K)$-representations

$$w_{Z,\ell} : T_\ell A \to T_\ell \mathrm{A}^i(X_{K^{\mathrm{sep}}}) \tag{8.1}$$

for all primes $\ell \neq p$.

Let now $\Phi : \mathscr{A}^i_{X/K} \to A$ be a surjective regular homomorphism and assume $Z \in \mathscr{A}^i_{X/K}(A)$ is a miniversal cycle, provided by Lemma 4.7, with $\Phi(A)(Z) : A \to A$ given by $r \cdot \mathrm{Id}_A$ for some natural number r. Then, still by the arguments of the proof of Lemma 4.12, the homomorphism of $\operatorname{Gal}(K)$-representations $w_{Z,\ell}$ of (8.1) is injective for all primes $\ell \neq p$ coprime to r. The following proposition is then an immediate consequence of Theorem 8.1 (and the discussion preceding it concerning the cases $i = 1, \dim X$).

Proposition 8.2. *With the notation above, let ℓ be a prime number different from p and coprime to r. Then, for $i = 1, 2, \dim X$, the morphism of* $\operatorname{Gal}(K)$*-representations $\iota_{Z,\ell} : T_\ell A \to H^{2i-1}(X_{K^{sep}}, \mathbb{Z}_\ell(i))$ given as the composition*

$$\iota_{Z,\ell} : T_\ell A \xhookrightarrow{w_{Z,\ell}} T_\ell \mathrm{A}^i(X_{K^{sep}}) \hookrightarrow T_\ell \operatorname{CH}^i(X_{K^{sep}}) \xrightarrow{\lambda^i} H^{2i-1}(X_{K^{sep}}, \mathbb{Z}_\ell(i))_\tau$$

is injective. Moreover, if Φ is an algebraic representative, then $\iota_{Z,\ell}$ is an isomorphism for $i = 1, \dim X$. $\square$

8.3. **Algebraic representatives over a DVR.**

Theorem 8.3. *Let $S = \Lambda$ be the spectrum of a DVR and denote η its generic point. Suppose X is a smooth and proper scheme over S. Fix $i = 1, 2$ or $\dim_S X$. Then the natural homomorphism of η-abelian varieties*

$$\mathrm{Ab}^i_{X_\eta/\eta} \longrightarrow (\mathrm{Ab}^i_{X/S})_\eta$$

is an isomorphism.

Proof. We start by observing that, due to Proposition 8.2 and the Néron–Ogg–Shafarevich criterion, $\mathrm{Ab}^i_{X_\eta/\eta}$ admits a model A over S. Moreover, there is a regular homomorphism $\Phi : \mathscr{A}^i_{X/S} \to A$ whose base-change to η is $\Phi^i_{X_\eta/\eta}$, defined in the following way. If T is a separated smooth scheme of finite type over $S = \Lambda$ and if $Z \in \mathscr{A}^i_{X/S}(T)$, then $\Phi(T)(Z)$ is the unique morphism $T \to A$ lifting $\Phi^i_{X_\eta/\eta}(T_\eta)(Z_\eta) : T_\eta \to A_\eta = \mathrm{Ab}^i_{X_\eta/\eta}$. Such a morphism exists due to the fact that any rational map defined over a base S from a regular scheme to an abelian S-scheme extends to a morphism (Proposition 4.5(i), [BLR90, Cor. 6, §8.4]). (Note that the above argument in fact establishes that any surjective regular homomorphism $\mathscr{A}^i_{X_\eta/\eta} \to B$ extends to a surjective regular homomorphism $\mathscr{A}^i_{X/S} \to \mathcal{B}$.)

By the universal property of the algebraic representatives, there exist a unique homomorphism $f : \mathrm{Ab}^i_{X_\eta/\eta} \to (\mathrm{Ab}^i_{X/S})_\eta$ such that $(\Phi^i_{X/S})_\eta = f \circ \Phi^i_{X_\eta/\eta}$ as well as a unique homomorphism $g : \mathrm{Ab}^i_{X/S} \to A$ such that $\Phi = g \circ \Phi^i_{X/S}$. By base-changing the latter to η, one obtains $\Phi^i_{X_\eta/\eta} = g_\eta \circ (\Phi^i_{X/S})_\eta$. In other words, we have a commutative diagram

$$\begin{tikzcd}
\mathscr{A}^i_{X_\eta/\eta} \arrow[rr, "\Phi_{X_\eta/\eta}"] \arrow[dr, "(\Phi^i_{X/S})_\eta"'] & & \mathrm{Ab}^i_{X_\eta/\eta} \arrow[dl, dashed, bend left, "f"] \\
& (\mathrm{Ab}^i_{X/S})_\eta \arrow[ur, dashed, "g_\eta"] &
\end{tikzcd}$$

This yields the identity $\Phi^i_{X_\eta/\eta} = g_\eta \circ f \circ \Phi^i_{X_\eta/\eta}$, and hence by the universal property of $\Phi^i_{X_\eta/\eta}$ that $g_\eta \circ f = \mathrm{id}_{\mathrm{Ab}^i_{X_\eta/\eta}}$. In particular, f is an injective homomorphism. On the other hand, we also obtain the identity $f \circ g_\eta \circ (\Phi_{X/S})_\eta = (\Phi_{X/S})_\eta$. The surjectivity of $\Phi_{X/S}$ (Proposition 5.1) implies then that $f \circ g_\eta : (\mathrm{Ab}^i_{X/S})_\eta \to (\mathrm{Ab}^i_{X/S})_\eta$ is the identity on η^{sep}-points, in particular that f is surjective on η^{sep}-points. In conclusion f is an isomorphism. □

Remark 8.4. We remind the reader that, in contrast, the *special* fiber of the algebraic representative is typically *not* the algebraic representative of the special fiber (*e.g.*, [ACMV17, Ex. 6.5]).

On the other hand, we have the following, which gives a condition for the special fiber of the algebraic representative to be isogenous to a sub-abelian variety of the algebraic representative of the special fiber :

Corollary 8.5. *Let $S = \Lambda$ be the spectrum of a discrete valuation ring with generic point $\eta = \operatorname{Spec} K$ and closed point $\circ = \operatorname{Spec} \kappa$. Let X/S be a smooth projective scheme, and let $\Gamma \in \mathscr{A}^2_{X_\eta/K}(\mathrm{Ab}^2_{X_\eta/K})$ be a miniversal cycle of minimal degree r. Suppose that for some prime $\ell \nmid r \cdot \mathrm{char}(K)$ we have $\phi^2_{X_\circ/\kappa}[\ell^\infty]$ is injective. Then the morphism $(\mathrm{Ab}^2_{X/S})_\circ \to \mathrm{Ab}^2_{X_\circ/\kappa}$ induced by Γ is an isogeny onto its image.*

Proof. By Theorem 8.3 we have $(\mathrm{Ab}^2_{X/S})_\eta \cong \mathrm{Ab}^2_{X_\eta/\eta}$. Let $\Gamma_{X/S} \in \mathcal{A}^2_{X/S}(\mathrm{Ab}^2_{X/S})$ be a miniversal cycle of minimal degree r induced by the one in the assumption of the lemma. Its specialization induces a group homomorphism $w_{\Gamma_{X/S,\circ}} : \mathrm{Ab}^2_{X/S}(\overline{\kappa}) \to \mathrm{A}^2(X_{\circ,\overline{\kappa}})$, and thus a homomorphism $\psi_{\Gamma_{X/S,\circ}} : (\mathrm{Ab}^2_{X/S})_\circ \to \mathrm{Ab}^2_{X_\circ/\circ}$. On ℓ-primary torsion, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Ab}^2_{X/S}[\ell^\infty](\overline{K}) \arrow[r, "\sim"] \arrow[d, "w_{\Gamma_{X/S}}"] & (\mathrm{Ab}^2_{X/S})_\circ[\ell^\infty](\overline{\kappa}) \arrow[d, "w_{\Gamma_{X/S,\circ}}"] \arrow[dd, bend left=60, "\psi_{\Gamma_{X/S,\circ}}"] \\
\mathrm{A}^2(X_{\overline{K}})[\ell^\infty] \arrow[r, hook] & \mathrm{A}^2(X_0)[\ell^\infty] \arrow[d, hook, "\phi^2_{X_\circ/\overline{\kappa}}[\ell^\infty]"] \\
 & \mathrm{Ab}^2_{X_\circ/\kappa}[\ell^\infty](\overline{\kappa})
\end{tikzcd} \tag{8.2}$$

Both the top and bottom horizontal arrows are the specialization maps; the fact that the specialization map on torsion cycle classes is injective in codimension-2 follows from the fact that the second Bloch map is an inclusion. Choose a prime $\ell \neq \mathrm{char}(K)$ relatively prime to r. Then $w_{\Gamma_{X/S}}$ and is injective, and by commutativity, this implies $w_{\Gamma_{X/S,\circ}}$ is injective. Then since we assume that $\phi^2_{X_\circ/\kappa}[\ell^\infty]$ is injective, it follows that all arrows in (8.2) are injective. In particular, it follows that $\ker \psi_{\Gamma_{X/S,\circ}}$ is trivial, and therefore has no nontrivial ℓ-torsion, so that the morphism of abelian varieties is an isogeny onto its image. $\square$

Remark 8.6. The condition that $\phi^2_{X_\circ/\kappa}[l^\infty]$ be injective holds for all primes l if $\mathrm{char}(\kappa) = 0$ [Mur85, Thm. 10.3], or if X is geometrically rationally chain connected with κ perfect [ACMV21, Prop. 3.8]. In other words, under these hypotheses, the morphism $(\mathrm{Ab}^2_{X/S})_\circ \to \mathrm{Ab}^2_{X_\circ/\kappa}$ induced by Γ is an isogeny onto its image. We are unaware of an example of a smooth projective variety $X_\circ$ for which $\phi^2_{X_\circ/\kappa}[l^\infty]$ is not an isomorphism ; in other words, we are unaware of an example where the morphism $(\mathrm{Ab}^2_{X/S})_\circ \to \mathrm{Ab}^2_{X_\circ/\kappa}$ induced by Γ is *not* an isogeny onto its image.

8.4. **Algebraic representatives and base change in characteristic zero.** Our aim is to prove Theorem 2(i) :

Theorem 8.7. *Suppose that $\Lambda = \mathrm{Spec}\, K$ for a field $K \subseteq \mathbb{C}$, and X is a smooth proper scheme over S. Fix $i = 1, 2$ or $\dim_S X$. If $S' \to S$ is a morphism obtained as an inverse limit of morphisms in Sm_K / S, then the natural homomorphism of S'-abelian schemes*

$$\mathrm{Ab}^i_{X_{S'}/S'} \to (\mathrm{Ab}^i_{X/S})_{S'}$$

is an isomorphism.

Proof. By rigidity (*e.g.*, [MFK94, Prop. 6.1]) the natural homomorphism $\mathrm{Ab}^i_{X_{S'}/S'} \to (\mathrm{Ab}^i_{X/S})_{S'}$ is an isomorphism if and only if it is an isomorphism when restricted to the generic point $\eta_{S'}$. By Theorem 5.10(i) applied to X_{η_S} together with the field extension $\kappa(S')/\kappa(S)$, we are reduced to showing that the natural homomorphism

$$\mathrm{Ab}^i_{X_{\eta_S}/\eta_S} \to (\mathrm{Ab}^i_{X/S})_{\eta_S}$$

is an isomorphism for any smooth proper scheme over a smooth separated scheme S of finite type over K. This is achieved exactly as in the proof of Theorem 8.3 : the crux consists in showing that any surjective regular homomorphism $\Phi : \mathscr{A}^i_{X_{\eta_S}/\eta_S} \to B$ extends to a surjective regular homomorphism $\widetilde{\Phi} : \mathscr{A}^i_{X/S} \to \mathcal{B}$. For that purpose, we first use, as in [ACMV19a, p. 19], both the Néron–Ogg–Shafarevich criterion *and* the Faltings–Chai criterion [FC90, Cor. 6.8 p. 185], to extend B to an abelian scheme $\mathcal{B}$ over S. Second, if T is a smooth separated scheme of finite type over K

with a *dominant* morphism to S and if $Z \in \mathscr{A}^i_{X/S}(T)$, then we use Proposition 4.5(i), [BLR90, Cor. 6, §8.4] to define $\widetilde{\Phi}(T)(Z)$ as the unique morphism $T \to A$ lifting $\Phi(T_{\eta_S})(Z_{\eta_S}) : T_{\eta_S} \to \mathcal{B}_{\eta_S} = B$. $\square$

Remark 8.8. Vasiu and Zink [VZ10] have investigated the structure of mixed-characteristic rings over which a suitable version of the Faltings–Chai extension theorem holds. If $S \to \mathbb{Z}_{(p)}$ satisfies the hypotheses of [VZ10, Cor. 5] (for example, if S is smooth over an unramified discrete valuation ring of mixed characteristic $(0,p)$), and if $X \to S$ is smooth and proper, then $\mathrm{Ab}^i_{X_{\eta_S}}$ spreads out to an abelian scheme over S, and we again have $\mathrm{Ab}^i_{X_{\eta_S}} \cong (\mathrm{Ab}^i_{X/S})_{\eta_S}$ for $i = 1, 2$ or $\dim_S X$.

Let X/K be a smooth projective variety. In [ACMV20a, §4] we investigate geometric conditions, such as a decomposition of the diagonal in Chow, which imply that there exists a correspondence $\Gamma \in \mathrm{CH}^{d_{\mathrm{Ab}^2_{X/K}}+1}(\widehat{\mathrm{Ab}}^2_{X/K} \times X)$ which, in any Weil cohomology theory $\mathcal{H}$, induces an isomorphism $\Gamma_{\mathcal{H},*} : \mathcal{H}^1(\mathrm{Ab}^2_{X/K}) \xrightarrow{\sim} \mathcal{H}^3(X)(1)$. In this setting, even in positive characteristic, we can obtain an analogue of Theorem 8.7.

In the following statement, we denote by $W = W(K)$ the ring of Witt vectors of a field K, and let $B = B(K) = \operatorname{Frac} W$ be its fraction field. The hypothesis that the crystalline cohomology be locally free is satisfied by, for example, families of complete intersections [Mor19, Ex. 3.13], or families which admit a decomposition of the diagonal [ACMV20a, § 7.2]. In such a case, inducing an inclusion of crystals is equivalent to inducing an inclusion of isocrystals.

Lemma 8.9. *Let $\Lambda = \operatorname{Spec} K$ be the spectrum of a perfect field of characteristic $p > 0$, let S/Λ be smooth, and let $f : X \to S$ be a smooth proper morphism. Suppose that, over the generic point η_S of S, there exists a correspondence $\Gamma \in \mathrm{CH}^{d_{\mathrm{Ab}^2_{X_{\eta_S}/\eta_S}}+2}(\mathrm{Ab}^2_{X_{\eta_S}/\eta_S} \times X_{\eta_S})$ such that $\Gamma_{cris,*} : H^1_{cris}(\mathrm{Ab}^2_{X_{\eta_S}/\eta_S}) \to H^3_{cris}(X_{\eta_S})(1)$ is an inclusion of crystals. Suppose further that the crystal $R^3 f_{cris,*}\mathcal{O}_{S/W(K)}$ is locally free. Then $\mathrm{Ab}^2_{X_{\eta_S}/\eta_S}$ extends to an abelian scheme $\underline{\mathrm{Ab}}^2_{X_{\eta_S}/\eta_S} \to S$.*

Proof. Take a spread $g : \underline{\mathrm{Ab}}^2_{X_{\eta_S}/\eta_S} \to U$ of the generic algebraic representative to an abelian scheme over a nonempty open subscheme U of S. Using the Néron–Ogg–Shafarevich criterion, we may and do assume that the complement of U in S has codimension at least two. To slightly ease notation, let $\mathcal{M} = R^1 g_{\mathrm{cris},*}\mathcal{O}_{\underline{\mathrm{Ab}}^2_{X_{\eta_S}/\eta_S}/W}$ and $\mathcal{N} = R^3 f_{\mathrm{cris},*}\mathcal{O}_{X/W}(1)$; they are locally free crystals of $\mathcal{O}_{U/W}$- and $\mathcal{O}_{S/W}$-modules, respectively.

Since S is smooth and thus regular, $\Gamma_{\mathrm{cris},*}$ extends to an inclusion of crystals $\gamma : \mathcal{M} \hookrightarrow \mathcal{N}_U$ [dJ98, Thm. 1.1]. Using the elementary divisors of the image of γ, we choose an endomorphism $\beta \in \mathrm{End}(\mathcal{N}_U)$ such that $\beta(\mathcal{N}_U) = \gamma(\mathcal{M})$. Restriction of F-isocrystals from S to U is fully faithful (e.g, [Ked22, Thm. 5.1]); since by hypothesis $\mathcal{N}$ is locally free, the restriction map of (integral) endomorphisms from $\mathrm{End}(\mathcal{N})$ to $\mathrm{End}(\mathcal{N}_U)$ is a bijection. Let $\widetilde{\beta}$ be the extension of β to S. Then $\widetilde{\beta}(\mathcal{N})$ is a crystal on S whose restriction to U is isomorphic to $\mathcal{M}$. Thus, $\mathcal{M} := \widetilde{\beta}(\mathcal{N})$ is a locally free F-crystal on S. Since the Dieudonné functor over S is an equivalence of categories [dJM99, Thm. 4.6], there is a p-divisible group $\mathcal{G} \to S$ with $\mathcal{G}_U$ isomorphic to $\underline{\mathrm{Ab}}^2_{X_{\eta_S}/\eta_S}[p^\infty]$. Using the proof of [Vas04, Prop. 4.1] (the argument given there is valid over arbitrary regular Noetherian rings), the abelian scheme $\underline{\mathrm{Ab}}^2_{X_{\eta_S}/\eta_S}$ extends to an abelian scheme over all of S. $\square$

9. Regular homomorphisms and intermediate Jacobians

The point of this section is to show that Abel–Jacobi maps in the relative setting induce regular homomorphisms in the sense we define here; indeed, this was one of our main motivations in proving [ACMV19a, Thm. 1]. To explain briefly, recall that given a smooth projective morphism

$f : X \to S$ of smooth complex varieties, a result of Griffiths states that the Abel–Jacobi map induces a regular homomorphism in the analytic category. In other words, given any dominant morphism of complex manifolds $T \to S$ and any cycle class $Z \in \mathrm{CH}^i(X_T)$ such that every refined Gysin fiber Z_t is homologically trivial, the associated map of sets $\nu_Z : T \to \mathrm{J}^{2i-1}(X_T/T) = \mathrm{J}^{2i-1}(X/S)_T$, $t \mapsto AJ_{X_t}(Z_t)$, is induced by a holomorphic map of analytic spaces; ν_Z is called the motivated normal function associated to Z.

Our recent result [ACMV19a, Thm. 1] shows that assuming f, X, and S are algebraic, the Abel–Jacobi map actually induces a regular homomorphism (in the algebraic sense we define here) if we restrict to algebraically trivial cycle classes. More precisely, there is an algebraic relative subtorus $\mathrm{J}^{2i-1}_a(X/S) \subseteq \mathrm{J}^{2i-1}(X/S)$ such that if we assume further that T and Z are algebraic with every refined Gysin fiber Z_t algebraically trivial, then the normal function factors as $\nu_Z : T \to \mathrm{J}^{2i-1}_a(X/S)_T \subseteq \mathrm{J}^{2i-1}(X/S)_T$, and is algebraic (not just holomorphic). In particular, there is a regular homomorphism $\Phi_{AJ} : \mathscr{A}^i_{X/S/\mathbb{C}} \to \mathrm{J}^{2i-1}_a(X/S)$, with $\Phi_{AJ}(T)(Z) = \nu_Z : T \to \mathrm{J}^{2i-1}_a(X/S)_T$ given by the associated normal function. We now explain this in more detail, and recall some of the arithmetic properties of these normal functions.

9.1. **The Abel–Jacobi map over subfields of $\mathbb{C}$.** We reformulate the main result of [ACMV20b] within our functorial setting. We recall the definition of the *algebraic intermediate Jacobian*. Let X be a smooth projective variety over $\mathbb{C}$. Griffiths defined an Abel–Jacobi map $AJ : \mathrm{CH}^i(X)_{\mathrm{hom}} \to \mathrm{J}^{2i-1}(X)$ on homologically trivial cycles. Here $\mathrm{J}^{2i-1}(X)$ is the so-called *intermediate Jacobian*; it is the complex torus defined by

$$\mathrm{J}^{2i-1}(X) := \mathrm{F}^i\mathrm{H}^{2i-1}(X,\mathbb{C})\backslash \mathrm{H}^{2i-1}(X,\mathbb{C})/\mathrm{H}^{2i-1}(X,\mathbb{Z})_\tau,$$

with $\mathrm{F}^\bullet$ being the Hodge filtration and with the subscript "τ" referring to the quotient by the maximal torsion subgroup. Given a homologically trivial cycle class $\gamma = \partial\Gamma$, the Abel–Jacobi map assigns, *via* Poincaré duality, the linear form $\int_\Gamma(-)$ to γ. It is a theorem of Griffiths that this assignment is well-defined. Moreover, the image of the restriction of the Abel–Jacobi map to algebraically trivial cycles defines a subtorus $\mathrm{J}^{2i-1}_a(X)$ which is naturally endowed with a polarization and hence defines a complex abelian variety. This complex abelian variety will be called the *algebraic intermediate Jacobian*. A basic result of Griffiths shows that the induced map $AJ : \mathrm{A}^i(X) \to \mathrm{J}^{2i-1}_a(X)(\mathbb{C})$ (which we will henceforth refer to by abuse as the Abel–Jacobi map) is a regular homomorphism in the classical sense, and hence by §1.4 provides a regular homomorphism $AJ : \mathscr{A}^i_{X/\mathbb{C}} \to \mathrm{J}^{2i-1}_a(X)$ in the sense of Definition 1.7 whose evaluation at $\operatorname{Spec}\mathbb{C}$ gives the classical Abel–Jacobi map $AJ : \mathrm{A}^i(X_\mathbb{C}) \to \mathrm{J}^{2i-1}_a(X_\mathbb{C})(\mathbb{C})$.

Theorem 9.1 ([ACMV20b])**.** *Let X be a smooth projective variety over a field $K \subseteq \mathbb{C}$. Then the algebraic intermediate Jacobian $\mathrm{J}^{2i-1}_a(X_\mathbb{C})$ admits a distinguished model $\mathrm{J}^{2i-1}_{a,X/K}$ over K and there exists a surjective regular homomorphism*

$$\Phi_{AJ_{X/K}} : \mathscr{A}^i_{X/K} \longrightarrow \mathrm{J}^{2i-1}_{a,X/K}$$

whose evaluation at $\operatorname{Spec}\mathbb{C}$ *is the Abel–Jacobi map $AJ : \mathrm{A}^i(X_\mathbb{C}) \to \mathrm{J}^{2i-1}_a(X_\mathbb{C})(\mathbb{C})$. In particular, the Abel–Jacobi map is* $\operatorname{Aut}(\mathbb{C}/K)$*-equivariant.*

Remark 9.2. Since $AJ : \mathrm{A}^i(X_\mathbb{C}) \to \mathrm{J}^{2i-1}_a(X_\mathbb{C})$ is surjective, the abelian variety $\mathrm{J}^{2i-1}_a(X_\mathbb{C})$ admits at most one structure of a scheme over K such that AJ is $\operatorname{Aut}(\mathbb{C}/K)$-equivariant. This is the sense in which $\mathrm{J}^{2i-1}_a(X_\mathbb{C})$ admits a *distinguished model* over K. See also [HT21, Thm. 4.1].

Proof. The following proof fixes a gap in the proof of the $\operatorname{Aut}(\mathbb{C}/K)$-equivariance of the Abel–Jacobi map given in [ACMV20b]. The starting point is [ACMV20b, Prop. 1.1] which provides a smooth projective curve C over K and a surjective homomorphism $(\mathrm{Pic}^0_{C/K})_\mathbb{C} = J(C_\mathbb{C}) \to$

$J^{2i-1}_a(X_{\mathbb{C}})$, and hence shows that the $\mathbb{C}/\overline{K}$-trace homomorphism $\tau : \mathrm{tr}_{\mathbb{C}/\overline{K}}(J^{2i-1}_a(X_{\mathbb{C}}))_{\mathbb{C}} \to J^{2i-1}_a(X_{\mathbb{C}})$ is an isomorphism (*cf.* [ACMV20b, §2.1]). We set $J^{2i-1}_{a,X_{\overline{K}}/\overline{K}} = \mathrm{tr}_{\mathbb{C}/\overline{K}}(J^{2i-1}_a(X_{\mathbb{C}}))$. By Lemma 3.2, we obtain a regular homomorphism $\underline{\underline{AJ}} : \mathscr{A}^i_{X_{\overline{K}}/\overline{K}} \to J^{2i-1}_{a,X_{\overline{K}}/\overline{K}}$, which satisfies $AJ = \tau \circ (\underline{\underline{AJ}})_{\mathbb{C}}$ and which is surjective by Proposition 4.14.

Next we claim that the surjective regular homomorphism $\underline{\underline{AJ}}$ is $\mathrm{Gal}(\overline{K}/K)$-equivariant. This is achieved in two steps in [ACMV20b] : first one shows [ACMV20b, §2.2] that $J^{2i-1}_{a,X_{\overline{K}}/\overline{K}}$ descends to an abelian variety $J^{2i-1}_{a,X/K}$ over K and that $\underline{\underline{AJ}}(\overline{K}) : \mathrm{A}^i(X_{\overline{K}}) \to (J^{2i-1}_{a,X/K})(\overline{K})$ is $\mathrm{Gal}(\overline{K}/K)$-equivariant on torsion ; then one establishes the general fact [ACMV20b, Prop. 3.8] that a regular homomorphism over $\overline{K}$ that is $\mathrm{Gal}(\overline{K}/K)$-equivariant on torsion is in fact $\mathrm{Gal}(\overline{K}/K)$-equivariant. By Lemma 3.4, we obtain a regular homomorphism $\Phi : \mathscr{A}^i_{X/K} \to J^{2i-1}_{a,X/K}$, which satisfies $\Phi_{\overline{K}} = \underline{\underline{AJ}}$ and which is surjective by Proposition 4.16.

Combining the above, Lemma 3.1(i) gives $\Phi_{\mathbb{C}} = AJ$, where $(J^{2i-1}_{X/K})_{\mathbb{C}}$ is identified with $J^{2i-1}_a(X_{\mathbb{C}})$ *via* τ. We conclude with Lemma 3.1(ii) that relative to the above identification AJ is $\mathrm{Aut}(\mathbb{C}/K)$-equivariant. □

9.2. **The Abel–Jacobi map in families.** In this section, we assume that $\Lambda = \operatorname{Spec} K$ for some subfield K of $\mathbb{C}$ and that S is an irreducible smooth quasi-projective variety over K. Theorem 9.3 below is a reformulation of the main result of [ACMV19a] within our functorial setting. This result, which in fact was motivated by the present work, shows the existence of non-trivial regular homomorphisms in families. Before we state the theorem, recall that Griffiths' Abel–Jacobi map can be constructed family-wise : given a smooth projective family X of complex varieties over a smooth quasi-projective complex variety S, there is a family of complex tori $J^{2i-1}(X/S)$ over S which consists fiberwise of the intermediate Jacobians. Moreover, a classical result of Griffiths says that, given a cycle class $Z \in \mathrm{CH}^i(X)$ which is fiberwise homologically trivial, the *normal function* $\nu_Z : S \to J^{2i-1}(X/S)$ defined by $t \mapsto AJ(Z_t)$ is holomorphic.

Theorem 9.3 ([ACMV19a]). *Suppose $f : X \to S$ is a smooth projective morphism of smooth varieties over a field $K \subseteq \mathbb{C}$. Then the Abel–Jacobi map $\Phi_{AJ_{X_{\eta_S}/\eta_S}} : \mathscr{A}^i_{X_{\eta_S}/\eta_S} \to J^{2i-1}_{a,X_{\eta_S}/\eta_S}$ of Theorem 9.1 extends uniquely to a surjective regular homomorphism*

$$\Phi_{AJ_{X/S}} : \mathscr{A}^i_{X/S} \to J^{2i-1}_{a,X/S}$$

such that the base change $(J^{2i-1}_{a,X/S})_{\mathbb{C}}$ is canonically identified with an algebraic subtorus $J^{2i-1}_a(X_{\mathbb{C}}/S_{\mathbb{C}}) \subseteq J^{2i-1}(X_{\mathbb{C}}/S_{\mathbb{C}})$ of the intermediate Jacobian. Moreover, for each T in Sm_K/S and each $Z \in \mathscr{A}^i_{X_{X/S}}(T)$, we have $\nu_{Z_{\mathbb{C}}} = (\Phi_{AJ_{X/S}}(T)(Z))_{\mathbb{C}} : T_{\mathbb{C}} \to (J^{2i-1}_{a,X/S})_{\mathbb{C}} \subseteq J^{2i-1}(X_{\mathbb{C}}/S_{\mathbb{C}})$ is the associated normal function.

Proof. The content of [ACMV19a, Thm. 1] and its proof is to show that there is an abelian scheme $J^{2i-1}_{a,X/S}$ over S such that the Abel–Jacobi map $\Phi_{AJ_{X_{\eta_S}/\eta_S}} : \mathscr{A}^i_{X_{\eta_S}/\eta_S} \to J^{2i-1}_{a,X_{\eta_S}/\eta_S}$ of Theorem 9.1 defines a map $\delta : \mathscr{A}^i_{X/S}(S) \to J^{2i-1}_{a,X/S}(S)$ such that the base change $(J^{2i-1}_{a,X/S})_{\mathbb{C}}$ is canonically identified with an algebraic subtorus of the intermediate Jacobian $J^{2i-1}(X_{\mathbb{C}}/S_{\mathbb{C}})$, and such that for each $Z \in \mathscr{A}^i_{X_{X/S}}(S)$ we have $\nu_{Z_{\mathbb{C}}} = (\delta(Z))_{\mathbb{C}} : S_{\mathbb{C}} \to (J^{2i-1}_{a,X/S})_{\mathbb{C}} \subseteq J^{2i-1}(X_{\mathbb{C}}/S_{\mathbb{C}})$. Since this holds also for $X_T \to T$ for any $T \to S$ in Sm_K/S, in order to define the regular homomorphism

$$\Phi_{AJ_{X/S}} : \mathscr{A}^i_{X/S} \to J^{2i-1}_{a,X/S},$$

it suffices to show $\mathrm{J}^{2i-1}_{a,X_T/T} = \mathrm{J}^{2i-1}_{a,X/S} \times_S T$. Working on connected components, we may as well assume S and T are integral, and the claim is that in fact we have a fibered product diagram

$$\begin{tikzcd}
\mathrm{J}^{2i-1}_a(X_{\mathbb{C}}) \arrow[rr] \arrow[dd] \arrow[dr] & & \mathrm{J}^{2i-1}_{a,X_{\eta_S}/\eta_S} \arrow[rr] \arrow[dd] & & \mathrm{J}^{2i-1}_{a,X/S} \arrow[dd] \\
& \mathrm{J}^{2i-1}_{a,X_{\eta_T}/\eta_T} \arrow[ur] \arrow[rr] \arrow[dd] & & \mathrm{J}^{2i-1}_{a,X_T/T} \arrow[ur] \arrow[dd] & \\
\operatorname{Spec}\mathbb{C} \arrow[rr] \arrow[dr] & & \eta_S \arrow[rr] & & S \\
& \eta_T \arrow[ur] \arrow[rr] & & T \arrow[ur] &
\end{tikzcd} \tag{9.1}$$

where $\operatorname{Spec}\mathbb{C} \to T \to S$ is any very general point of T; *i.e.*, maps to the generic point of T. That the left-hand triangular prism is a fibered product diagram is [ACMV19a, Rem. 1.6]. To show the right-hand cube is a fibered product diagram we argue as follows. The only thing to show is that the far right-hand vertical face of the cube is cartesian. Using that the rest of the diagram is cartesian, we see that $J^{2i-1}_{a,X/S}$ pulls back to $J^{2i-1}_{a,X_{\eta_T}/\eta_T}$ over η_T. Thus both $J^{2i-1}_{a,X_S/S} \times_S T$ and $J^{2i-1}_{a,X_T/T}$ are abelian schemes over T that pull back along $\eta_T \to T$ to give $J^{2i-1}_{a,X_{\eta_T}/\eta_T}$; *i.e.*, they have the same generic fiber. Thus they agree (Proposition 4.5(i), [BLR90, Cor. 6, §8.4]), and therefore we have defined the regular homomorphism $\Phi_{AJ_{X/S}}$.

That $\Phi_{AJ_{X/S}}$ is surjective is simply due to Proposition 4.13 and to the fact that $(\Phi_{AJ_{X/S}})_{\eta_S}$ is the Abel–Jacobi map $\Phi_{AJ_{X_{\eta_S}/\eta_S}} : \mathscr{A}^i_{X_{\eta_S}/\eta_S} \to \mathrm{J}^{2i-1}_{a,X_{\eta_S}/\eta_S}$, which is surjective by Theorem 9.1. That $\Phi_{AJ_{X/S}}$ is uniquely determined by its restriction $\Phi_{AJ_{X_{\eta_S}/\eta_S}}$ is simply due to the fact that if T is an element of Sm_K/S, then a η_S-morphism $T_{\eta_S} \to (\mathrm{J}^{2i-1}_{a,X/S})_{\eta_S} = \mathrm{J}^{2i-1}_{X_{\eta_S}/\eta_S}$ is the restriction of at most one S-morphism $T \to \mathrm{J}^{2i-1}_{a,X/S}$. $\square$

For clarity we extract the following assertion, which was established in the proof above.

Corollary 9.4. *Suppose $T \to S$ is in Sm_K/S. Then we have a fibered product diagram:*

$$\begin{tikzcd}
\mathscr{A}^i_{X_T/T} \arrow[r, "\Phi_{AJ_{X_T/T}}"] \arrow[d] & \mathrm{J}^{2i-1}_{a,X_T/T} \arrow[r] \arrow[d] & T \arrow[d] \\
\mathscr{A}^i_{X/S} \arrow[r, "\Phi_{AJ_{X/S}}"] & \mathrm{J}^{2i-1}_{a,X_S/S} \arrow[r] & S
\end{tikzcd}$$

Proof. That the right-hand square is cartesian comes from (9.1). That the outer rectangle is cartesian is (2.2). $\square$

9.3. **Algebraic representatives and intermediate Jacobians.** We show that $\mathrm{J}^{2i-1}_{a,X/S}$ is an algebraic representative for codimension-i cycles with $i = 1, 2, \dim_S X$.

Theorem 9.5 (Algebraic representatives over $\mathbb{C}$)**.** *Suppose $\Lambda = \operatorname{Spec} K$ for some field $K \subseteq \mathbb{C}$, and that $f : X \to S$ is a smooth projective. Fix $i = 1, 2, \dim_S X$. Then the homomorphism $\mathrm{Ab}^i_{X/S} \to \mathrm{J}^{2i-1}_{a,X/S}$ induced by the universal property of the algebraic representative is an isomorphism. In other words, $\mathrm{J}^{2i-1}_{a,X/S}$ is an algebraic representative for codimension-i cycles.*

Proof. The cases $i = 1, \dim_S X$ can be proven in the same way as the case $i = 2$, and so we focus on this last case. In the classical situation where $S = \operatorname{Spec}\mathbb{C}$, this is [Mur85, Thm. C]. Our strategy is to reduce to that known case. By base-changing to the generic point of S and by the universal property of $\mathrm{Ab}^2_{X_{\eta_S}/\eta_S}$ we obtain a composition of homomorphism of abelian varieties over η_S:

$$\mathrm{Ab}^2_{X_{\eta_S}/\eta_S} \twoheadrightarrow (\mathrm{Ab}^2_{X/S})_{\eta_S} \to (\mathrm{J}^3_{a,X/S})_{\eta_S} = \mathrm{J}^3_{a,X_{\eta_S}/\eta_S}, \tag{9.2}$$

where the identification on the right is provided by Theorem 9.3 and the surjectivity of the arrow on the left is provided by Proposition 4.10. In order to show that $\mathrm{Ab}^2_{X/S} \to \mathrm{J}^3_{a,X/S}$ is an isomorphism, it is enough to show that it is generically an isomorphism by Proposition 4.5(i) ([BLR90, Cor. 6, §8.4]). Hence, in view of (9.2), it is enough to see that the natural homomorphism $\mathrm{Ab}^2_{X_{\eta_S}/\eta_S} \to \mathrm{J}^3_{a,X_{\eta_S}/\eta_S}$ is an isomorphism. This is achieved as follows. Fix an embedding of $\kappa(S)$ in $\mathbb{C}$. Thanks to Theorems 5.6 and 5.9, we obtain after base-changing the latter homomorphism along the inclusion $\kappa(X) \subset \mathbb{C}$ a homomorphism

$$\mathrm{Ab}^2_{X_{\mathbb{C}}/\mathbb{C}} \to \mathrm{J}^3_{a,X_{\mathbb{C}}},$$

which by [Mur85, Thm. C] is an isomorphism. □

Remark 9.6. As already mentioned in Example 1.14, for $2 < i < \dim X_{\mathbb{C}}$, even if an algebraic representative exists, the canonical morphism $\mathrm{Ab}^i_{X_{\mathbb{C}}/\mathbb{C}} \twoheadrightarrow \mathrm{J}^{2i-1}_a(X_{\mathbb{C}})$ need not be injective ; *cf.* [OS20, Cor. 4.2].

Building on the $K = \mathbb{C}$ version of Theorem 9.5, Murre established the following theorem in the special case when $S = \operatorname{Spec}\mathbb{C}$:

Theorem 9.7. *Suppose* $\mathrm{char}(\kappa(\Lambda)) = 0$ *and let* $X \to S$ *be a smooth projective morphism. Fix* $i = 1, 2, \dim_S X$. *Let* $\Phi^i_{X/S} : \mathscr{A}^i_{X/S} \to \mathrm{Ab}^i_{X/S}$ *be the algebraic representative for codimension-i cycles (whose existence is provided by Theorem 6.1). Then for any separably closed point* $s : \operatorname{Spec}\Omega \to S$ *obtained as an inverse limit of morphisms in* Sm_Λ/S*, the homomorphism* $\Phi^i_{X/S}(\Omega)$ *is an isomorphism on torsion. In particular, for all primes* ℓ*, the map*

$$T_\ell\,\mathrm{A}^i(X_\Omega) \xrightarrow{T_\ell \Phi^i_{X/S}(\Omega)} T_\ell\,\mathrm{Ab}^i_{X/S}(\Omega) \tag{9.3}$$

is an isomorphism.

Proof. This is standard over $\mathbb{C}$ for $i = 1, \dim_S X$. For $i = 2$, this is [Mur85, §10] in the case $S = \operatorname{Spec}\mathbb{C}$: in that case the natural morphism $\mathrm{Ab}^2_{X/\mathbb{C}} \to \mathrm{J}^3_a(X)$ is an isomorphism (Theorem 9.5) and the corresponding statement with $\mathrm{J}^3_a(X)$ in place of $\mathrm{Ab}^2_{X/\mathbb{C}}$ is [Mur85, Thm. 10.3]. One reduces the general case to the previous case *via* Theorem 8.7 and Theorem 5.6. □

References

[ACMV17] Jeffrey D. Achter, Sebastian Casalaina-Martin, and Charles Vial, *On descending cohomology geometrically*, Compos. Math. **153** (2017), no. 7, 1446–1478. MR 3705264

[ACMV19a] ______, *Normal functions for algebraically trivial cycles are algebraic for arithmetic reasons*, Forum Math. Sigma **7** (2019), Paper No. e36, 22. MR 4024934

[ACMV19b] ______, *Parameter Spaces for Algebraic Equivalence*, Int. Math. Res. Not. IMRN (2019), no. 6, 1863–1893. MR 3932597

[ACMV20a] ______, *Decomposition of the diagonal, algebraic representatives, and universal codimension-2 cycles over perfect fields*, preprint, arXiv:2007.07470, 2020.

[ACMV20b] ______, *Distinguished models of intermediate Jacobians*, Journal of the Institute of Mathematics of Jussieu **19** (2020), no. 3, 891–918.

[ACMV21] ______, *On the image of the second l-adic Bloch map*, Rationality of Algebraic Varieties, Progr. Math., vol. 342, Birkhäuser, 2021.

[ACMV22] ______, *A complete answer to Albanese base change for incomplete varieties*, 2022.

[Bea77] Arnaud Beauville, *Variétés de Prym et jacobiennes intermédiaires*, Ann. Sci. École Norm. Sup. (4) **10** (1977), no. 3, 309–391. MR 0472843 (57 #12532)

[Bea83] A. Beauville, *Quelques remarques sur la transformation de Fourier dans l'anneau de Chow d'une variété abélienne*, Algebraic geometry (Tokyo/Kyoto, 1982), Lecture Notes in Math., vol. 1016, Springer, Berlin, 1983, pp. 238–260. MR 726428

[Blo79] S. Bloch, *Torsion algebraic cycles and a theorem of Roitman*, Compositio Math. **39** (1979), no. 1, 107–127. MR 539002 (80k:14012)

[BLR90] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud, *Néron models*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 21, Springer-Verlag, Berlin, 1990. MR 1045822 (91i:14034)

[Bou81] Nicolas Bourbaki, *Éléments de mathématique*, Masson, Paris, 1981, Algèbre. Chapitres 4 à 7. [Algebra. Chapters 4–7]. MR 643362

[BW19] Olivier Benoist and Olivier Wittenberg, *Intermediate Jacobians and rationality over arbitrary fields*, 2019, 1909.12668.

[Con06] Brian Conrad, *Chow's K/k-image and K/k-trace, and the Lang-Néron theorem*, Enseign. Math. (2) **52** (2006), no. 1-2, 37–108. MR 2255529 (2007e:14068)

[CTR85] Jean-Louis Colliot-Thélène and Wayne Raskind, *$\mathscr{K}_2$-cohomology and the second Chow group*, Math. Ann. **270** (1985), no. 2, 165–199. MR 771978

[dJ98] A. J. de Jong, *Homomorphisms of Barsotti-Tate groups and crystals in positive characteristic*, Invent. Math. **134** (1998), no. 2, 301–333. MR 1650324

[dJM99] A. J. de Jong and W. Messing, *Crystalline Dieudonné theory over excellent schemes*, Bull. Soc. Math. France **127** (1999), no. 2, 333–348. MR 1708635

[FC90] Gerd Faltings and Ching-Li Chai, *Degeneration of abelian varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 22, Springer-Verlag, Berlin, 1990, With an appendix by David Mumford. MR 1083353

[Ful98] William Fulton, *Intersection theory*, second ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 2, Springer-Verlag, Berlin, 1998. MR 1644323 (99d:14003)

[Gro62] Alexander Grothendieck, *Fondements de la géométrie algébrique. [Extraits du Séminaire Bourbaki, 1957–1962.]*, Secrétariat mathématique, Paris, 1962. MR 0146040

[HT21] Brendan Hassett and Yuri Tschinkel, *Cycle class maps and birational invariants*, Comm. Pure Appl. Math. **74** (2021), no. 12, 2675–2698. MR 4373165

[Ked22] Kiran S. Kedlaya, *Notes on isocrystals*, J. Number Theory **237** (2022), 353–394. MR 4410030

[Kle05] S. L. Kleiman, *The Picard scheme*, Fundamental Algebraic Geometry, Math. Surveys Monogr., vol. 123, Amer. Math. Soc., Providence, RI, 2005, pp. 235–321. MR 2223410

[Lec86] Florence Lecomte, *Rigidité des groupes de Chow*, Duke Math. J. **53** (1986), no. 2, 405–426. MR 850543 (88c:14013)

[MFK94] D. Mumford, J. Fogarty, and F. Kirwan, *Geometric invariant theory*, third ed., Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)], vol. 34, Springer-Verlag, Berlin, 1994. MR 1304906

[Mor19] Matthew Morrow, *A variational Tate conjecture in crystalline cohomology*, J. Eur. Math. Soc. (JEMS) **21** (2019), no. 11, 3467–3511. MR 4012343

[MS82] A. S. Merkurjev and A. A. Suslin, *K-cohomology of Severi-Brauer varieties and the norm residue homomorphism*, Izv. Akad. Nauk SSSR Ser. Mat. **46** (1982), no. 5, 1011–1046, 1135–1136. MR 675529

[Mur85] J. P. Murre, *Applications of algebraic K-theory to the theory of algebraic cycles*, Algebraic geometry, Sitges (Barcelona), 1983, Lecture Notes in Math., vol. 1124, Springer, Berlin, 1985, pp. 216–261. MR 805336 (87a:14006)

[OS20] John Christian Ottem and Fumiaki Suzuki, *A pencil of Enriques surfaces with non-algebraic integral Hodge classes*, Math. Ann. **377** (2020), no. 1-2, 183–197. MR 4099632

[Sai79] Hiroshi Saito, *Abelian varieties attached to cycles of intermediate dimension*, Nagoya Math. J. **75** (1979), 95–119. MR 542191 (81a:14021)

[Sam60] Pierre Samuel, *Relations d'équivalence en géométrie algébrique*, Proc. Internat. Congress Math. 1958, Cambridge Univ. Press, New York, 1960, pp. 470–487. MR 0116010

[Sch09] Stefan Schröer, *On genus change in algebraic curves over imperfect fields*, Proc. Amer. Math. Soc. **137** (2009), no. 4, 1239–1243. MR 2465645

[Ser60] J.-P. Serre, *Morphismes universels et variétés d'Albanese*, Séminaire C. Chevalley, 3ième année: 1958/59, vol. 4, École Normale Supérieure, Paris, 1960. MR 0157865

[Sta22] The Stacks Project Authors, *Stacks Project*, `http://stacks.math.columbia.edu`, 2022.

[Suw88] Noriyuki Suwa, *Sur l'image de l'application d'Abel-Jacobi de Bloch*, Bull. Soc. Math. France **116** (1988), no. 1, 69–101. MR 946279

[Vas04] Adrian Vasiu, *A purity theorem for abelian schemes*, Michigan Math. J. **52** (2004), no. 1, 71–81. MR 2043397

[Voi15] Claire Voisin, *Unirational threefolds with no universal codimension* 2 *cycle*, Invent. Math. **201** (2015), no. 1, 207–237. MR 3359052

[VZ10] Adrian Vasiu and Thomas Zink, *Purity results for p-divisible groups and abelian schemes over regular bases of mixed characteristic*, Doc. Math. **15** (2010), 571–599. MR 2679067

[Wit08] Olivier Wittenberg, *On Albanese torsors and the elementary obstruction*, Math. Ann. **340** (2008), no. 4, 805–838. MR 2372739

COLORADO STATE UNIVERSITY, DEPARTMENT OF MATHEMATICS, FORT COLLINS, CO 80523, USA
E-mail address: j.achter@colostate.edu

UNIVERSITY OF COLORADO, DEPARTMENT OF MATHEMATICS, BOULDER, CO 80309, USA
E-mail address: casa@math.colorado.edu

UNIVERSITÄT BIELEFELD, FAKULTÄT FÜR MATHEMATIK, GERMANY
E-mail address: vial@math.uni-bielefeld.de