

ON THE IMAGE OF THE SECOND l -ADIC BLOCH MAP

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ABSTRACT. For a smooth projective geometrically uniruled threefold defined over a perfect field we show that there exists a canonical abelian variety over the field, namely the second algebraic representative, whose rational Tate modules model canonically the third l -adic cohomology groups of the variety for all primes l . In addition, there exists a rational correspondence inducing these identifications. In the case of a geometrically rationally chain connected variety, one obtains canonical identifications between the *integral* Tate modules of the second algebraic representative and the third l -adic cohomology groups of the variety, and if the variety is a geometrically stably rational threefold, these identifications are induced by an integral correspondence. Our overall strategy consists in studying – for arbitrary smooth projective varieties – the image of the second l -adic Bloch map restricted to the Tate module of algebraically trivial cycle classes in terms of the “correspondence (co)niveau filtration”. This complements results with rational coefficients due to Suwa. In the appendix, we review the construction of the Bloch map and its basic properties.

INTRODUCTION

0.1. Mazur’s question with \mathbb{Q}_ℓ -coefficients. In the context of the generalized Hodge conjecture, it is natural to ask the following question [Voi14, Que. 2.43]: *Let X be a smooth complex projective manifold, and let ν, i be natural numbers. Given a weight- i Hodge structure $L \subseteq H^i(X, \mathbb{Q})$ such that the Tate twist $L(\nu)$ is effective (i.e., $L^{p,q} = 0$ for $p, q > \nu$), does there exist a complex projective manifold Y and an inclusion of Hodge structures $L(\nu) \hookrightarrow H^{i-2\nu}(Y, \mathbb{Q})$?* Essentially by definition, one can rephrase this question, via the Hodge coniveau filtration $N_H^\bullet H^i(X, \mathbb{Q})$, as asking whether for a given ν , there exists a smooth projective manifold Y such that $N_H^\nu H^i(X, \mathbb{Q}(\nu)) \subseteq H^{i-2\nu}(Y, \mathbb{Q})$. The generalized Hodge conjecture predicts that the Hodge coniveau filtration coincides with the so-called geometric coniveau filtration, $N^\bullet H^i(X, \mathbb{Q})$; in this sense one can rephrase the question in terms of the geometric coniveau filtration, and it is this version we will focus on in this paper.

As a motivating example for this work, consider the case where $i = 2n - 1$ is odd and where $\nu = n$. Setting $J_a^{2n-1}(X)$ to be the algebraic intermediate Jacobian, i.e., the image of the Abel–Jacobi map $AJ : A^n(X) \rightarrow J^{2n-1}(X)$ restricted to algebraically trivial cycles, it is well-known that $N^{n-1} H^{2n-1}(X, \mathbb{Q}) = H^1(\widehat{J}_a^{2n-1}(X), \mathbb{Q}) \simeq H^1(J_a^{2n-1}(X), \mathbb{Q})$, answering the question in the case of the geometric coniveau filtration.

For smooth projective varieties over arbitrary fields, one can rephrase the Hodge theoretic question above by replacing Betti cohomology with ℓ -adic cohomology. Mazur [Maz14, Maz11] asked the following:

Question 1 (Mazur’s question with \mathbb{Q}_ℓ -coefficients). *Let X be a smooth projective variety over a field K with separable closure \overline{K} . Given a natural number n , does there exist an abelian variety A/K such that for all primes $\ell \neq \text{char}(K)$ there is an isomorphism of Galois modules*

$$V_\ell A \xrightarrow{\simeq} N^{n-1} H^{2n-1}(X_{\overline{K}}, \mathbb{Q}_\ell(n)) ? \tag{0.1}$$

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In fact, one can pose an analogue of Mazur's question for any Weil cohomology $\mathcal{H}(\cdot)$. In positive characteristic p , much of our work also extends to the case of cohomology $H(-, \mathbb{Q}_p)$, which can be recovered as the F -invariants of the crystalline cohomology. In this introduction, where possible, we phrase statements uniformly in a prime l , although we remind the reader that, for example, $\dim H^i(X, \mathbb{Q}_p)$ is typically smaller than $\dim H^i(X, \mathbb{Q}_\ell)$; we reserve l for primes distinct from the characteristic of the base field.

From the motivic perspective, it is natural to ask that the isomorphism (0.1) be induced by a correspondence. Note that given the isomorphism (0.1), the Tate conjecture provides for each l a correspondence $\Gamma_\ell \in \text{CH}^n(A \times_K X) \otimes \mathbb{Q}_\ell$ inducing the isomorphism for that l . One might expect to find a correspondence Γ with integral coefficients, and that it be independent of l :

Question 2 (Mazur's motivic question with \mathbb{Q}_l -coefficients). *Does there exist a correspondence $\Gamma \in \text{CH}^n(A \times_K X)$ inducing for all primes l the above isomorphisms (0.1)?*

Our first observation is that our results in [ACMV20] provide an affirmative answer to Questions 1 and 2 for any field K of characteristic zero, and for any n . Recall that in [ACMV20], it is shown that the algebraic intermediate Jacobian $J_a^{2n-1}(X_{\mathbb{C}})$ attached to a smooth projective variety X defined over $K \subseteq \mathbb{C}$ admits a *distinguished model* $J_{a, X/K}^{2n-1}$ over K in the sense that this model makes the Abel–Jacobi map $AJ : A^n(X_{\mathbb{C}}) \rightarrow J_a^{2n-1}(X_{\mathbb{C}})$ restricted to algebraically trivial cycles an $\text{Aut}(\mathbb{C}/K)$ -equivariant map.

Theorem 3 ([ACMV20, Thm. 2.1]). *Let X be a smooth projective variety over a field $K \subseteq \mathbb{C}$. Given a natural number n , let $J_{a, X/K}^{2n-1}$ denote the distinguished model of the intermediate Jacobian $J^{2n-1}(X_{\mathbb{C}})$. Then there exists a correspondence $\Gamma \in \text{CH}^n(J_{a, X/K}^{2n-1} \times_K X)$ inducing for all primes l an inclusion of Galois modules*

$$\Gamma_* : V_\ell J_{a, X/K}^{2n-1} \hookrightarrow H^{2n-1}(X_{\bar{K}}, \mathbb{Q}_\ell(n))$$

with image $\mathbb{N}^{n-1} H^{2n-1}(X_{\bar{K}}, \mathbb{Q}_\ell(n))$.

Thus we turn next to Questions 1 and 2 in the case where K has positive characteristic. As a first partial result, we establish in Proposition 6.1 a positive answer to Question 1 and 2 under the further assumptions that K is perfect, $2n - 1 \leq d_X := \dim X$, and $H^{2n-1}(X_{\bar{K}}, \mathbb{Q}_l(n))$ has geometric coniveau $n - 1$ (this was established in [ACMV17, Thm. 2.1(d)] for $K \subseteq \mathbb{C}$). While the condition that $H^{2n-1}(X_{\bar{K}}, \mathbb{Q}_l(n))$ have geometric coniveau $n - 1$ is in general quite restrictive, it does not impose a condition for $n = 1$ or $n = d_X$, and therefore Proposition 6.1 establishes an affirmative answer to Mazur's Questions 1 and 2 for $n = 1, d_X$ and K perfect (see Remark 6.2 for the case $n = d_X$). Thus, moving forward, we will essentially be focusing on the case $n = 2$ in positive characteristic.

0.2. Mazur's question with \mathbb{Q}_ℓ -coefficients in positive characteristic. We focus now on Question 1 in positive characteristic, and set aside the issue of the correspondence in Question 2. We attempt to use algebraic representatives and the Bloch map as replacements for intermediate Jacobians and Abel–Jacobi maps.

More precisely, recall that over an arbitrary field a replacement for $J_{a, X/K}^{2n-1}$ is the *algebraic representative* $\text{Ab}_{X/K}^n$. If it exists (as in the case $n = 1, 2, d_X$), it comes with a $\text{Gal}(K)$ -equivariant morphism

$$\phi_{X_{\bar{K}}/K}^n : A^n(X_{\bar{K}}) \longrightarrow \text{Ab}_{X/K}^n(\bar{K}),$$

where as in the case $\bar{K} = \mathbb{C}$ we set

$$A^n(X_{\bar{K}}) := \{\alpha \in \text{CH}^n(X_{\bar{K}}) \mid \alpha \text{ is algebraically trivial}\}.$$

For $n = 1, d_X$, the algebraic representative is the reduced Picard variety with the Abel–Jacobi map, and the Albanese variety with the Albanese map, respectively. For the case $n = 2$, the existence was proved for smooth projective varieties over an algebraically closed field in [Mur85]. This was extended to smooth projective varieties defined over a perfect field (e.g., a finite field) in [ACMV17], and to smooth projective varieties defined over any field in [ACMVb]. In characteristic 0, this agrees with the distinguished model of $J_a^3(X_C)$ of [ACMV20]. We refer to §3.2 for more details.

At the same time, recall that for a smooth projective variety X over a field K with separable closure \bar{K} , and a prime l , Bloch [Blo79] in the case $l \neq \text{char}(K)$ and later Gros–Suwa [GS88] in the case $l = \text{char}(K)$ defined a map

$$\lambda^n : \text{CH}^n(X_{\bar{K}})[l^\infty] \longrightarrow H^{2n-1}(X_{\bar{K}}, \mathbf{Q}_l/\mathbf{Z}_l(n)) \quad (0.2)$$

on l -primary torsion, extending the Abel–Jacobi map on homologically trivial l -torsion cycle classes (see (A.5)) in the case $K = \mathbf{C}$. Suwa [Suw88] in the case $l \neq \text{char}(K)$ and Gros–Suwa [GS88] in the case $l = \text{char}(K)$ then defined an l -adic Bloch map

$$T_l \lambda^n : T_l \text{CH}^n(X_{\bar{K}}) \longrightarrow H^{2n-1}(X_{\bar{K}}, \mathbf{Z}_l(n))_\tau \quad (0.3)$$

by taking the Tate module of the Bloch map λ^n . Here the subscript τ indicates the quotient by the torsion subgroup; recall that a result of Gabber states that for a given X , the cohomology groups are torsion-free for all but finitely many l . Tensoring by $- \otimes_{\mathbf{Z}_l} \mathbf{Q}_l$ defines a map

$$V_l \lambda^n : V_l \text{CH}^n(X_{\bar{K}}) \longrightarrow H^{2n-1}(X_{\bar{K}}, \mathbf{Q}_l(n)).$$

In the appendix, we give a more direct construction of the l -adic Bloch map for primes $l \neq \text{char}(K)$, following Bloch’s original construction, but taking an inverse limit rather than a direct limit. We show these two constructions agree, and that the l -adic Bloch map agrees with the l -adic Abel–Jacobi map in the case $K = \mathbf{C}$, when one restricts to homologically trivial cycle classes. We then review in §A.4 a few properties of the l -adic Bloch map that we use in the body of the paper.

It is well-known that one can use these maps to model $H^{2n-1}(X_{\bar{K}}, \mathbf{Q}_l(n))$ for $n = 1, d_X$ via the Picard and Albanese: in those cases, we have isomorphisms (see Propositions A.25 and A.26)

$$\begin{aligned} V_l(\text{Pic}_{X/K_{\text{red}}}^0) &\xrightarrow[\simeq]{(V_l \phi_{X_{\bar{K}}/\bar{K}}^1)^{-1}} V_l A^1(X_{\bar{K}}) \simeq V_l \text{CH}^1(X_{\bar{K}}) \xrightarrow[\simeq]{V_l \lambda^1} H^1(X_{\bar{K}}, \mathbf{Q}_l(1)) \\ V_l \text{Alb}_{X/K} &\xrightarrow[\simeq]{(V_l \phi_{X_{\bar{K}}/\bar{K}}^{d_X})^{-1}} V_l A^{d_X}(X_{\bar{K}}) \simeq V_l \text{CH}^{d_X}(X_{\bar{K}}) \xrightarrow[\simeq]{V_l \lambda^{d_X}} H^{2d_X-1}(X_{\bar{K}}, \mathbf{Q}_l(d_X)). \end{aligned}$$

Therefore, we focus here on the cases $n \neq 1, d_X$, and in particular, the case $n = 2$. While in positive characteristic, the relationship between the Tate module of the algebraic representative and the coniveau filtration is not known in general, a result of Suwa relates the coniveau filtration to the image of the l -adic Bloch map (restricted to algebraically trivial cycles).

Proposition 4 (Suwa [Suw88, Prop. 5.2]). *Let X be a smooth projective variety over a perfect field K and let n be a natural number. For all prime numbers l , the image of the composition*

$$V_l A^n(X_{\bar{K}}) \hookrightarrow V_l \text{CH}^n(X_{\bar{K}}) \xrightarrow{V_l \lambda^n} H^{2n-1}(X_{\bar{K}}, \mathbf{Q}_l(n)) \quad (0.4)$$

is equal to $\mathbf{N}^{n-1} H^{2n-1}(X_{\bar{K}}, \mathbf{Q}_l(n))$.

We refer to Proposition 2.1 for a more precise statement. As a minor technical point, we mention that Suwa proves this result for a slightly different coniveau filtration, which we call the correspondence coniveau filtration; he shows this filtration agrees with the usual coniveau filtration in characteristic 0, and we extend this result to perfect fields in Proposition 1.1.

Since $T_l\lambda^2$ is an inclusion (see Proposition A.27), as an immediate corollary of our results above, and Suwa's proposition, one obtains:

Corollary 5 (Modeling \mathbb{Q}_l -cohomology). *Let X be a smooth projective variety over a perfect field K and let l be a prime number. If $V_l\phi_{X_{\bar{K}}/\bar{K}}^2 : V_l A^2(X_{\bar{K}}) \rightarrow V_l \text{Ab}_{X/K}^2$ is an isomorphism, e.g., if X is a geometrically uniruled threefold (Proposition 3.8(3)), then the composition*

$$V_l \text{Ab}_{X/K}^2 \xrightarrow[\simeq]{(V_l\phi_{X_{\bar{K}}/\bar{K}}^2)^{-1}} V_l A^2(X_{\bar{K}}) \hookrightarrow V_l \text{CH}^2(X_{\bar{K}}) \xrightarrow{V_l\lambda^2} H^3(X_{\bar{K}}, \mathbb{Q}_l(2)) \quad (0.5)$$

is an inclusion of $\text{Gal}(K)$ -modules with image $N^1 H^3(X_{\bar{K}}, \mathbb{Q}_l(2))$.

The assumption that $V_l\phi_{X_{\bar{K}}/\bar{K}}^2$ be an isomorphism is implied (see Lemma 3.6) by the, possibly vacuous, assumption that $\phi_{X_{\bar{K}}/\bar{K}}^2 : A^2(X_{\bar{K}}) \rightarrow \text{Ab}_{X/K}^2$ be an isomorphism on l -primary torsion. It turns out that this assumption also plays a crucial role in our work [ACMVa], so we single it out:

Definition 6 (Standard assumption at l). Let X be a smooth projective variety over a field K and let l be a prime number. We say that $\phi_{X_{\bar{K}}/\bar{K}}^2$ (or by abuse, X) satisfies the *standard assumption at the prime l* if

$$\phi_{X_{\bar{K}}/\bar{K}}^2[l^\infty] : A^2(X_{\bar{K}})[l^\infty] \xrightarrow{\simeq} \text{Ab}_{X/K}^2[l^\infty] \quad (0.6)$$

is an isomorphism. We say that $\phi_{X_{\bar{K}}/\bar{K}}^2$ (or by abuse X) satisfies the standard assumption if it satisfies the standard assumption at l for all primes l .

In Proposition 3.8, we give sufficient conditions for the standard assumption to be satisfied; in particular, the standard assumption holds if $\text{char}(K) = 0$ [Mur85, Thm. 10.3] or if X is geometrically rationally connected with K perfect. We are unaware of an example of a smooth projective variety for which the standard assumption at a prime l fails.

We mention here that both Proposition 4 and Corollary 5 hold with $\mathbb{Q}_l/\mathbb{Z}_l$ -coefficients so long as one replaces the coniveau filtration (1.1) with the correspondence niveau filtration (1.4), thus providing an answer to Mazur's Question 1 with $\mathbb{Q}_l/\mathbb{Z}_l$ -coefficients and with the geometric coniveau filtration N^\bullet replaced with the correspondence niveau filtration \tilde{N}^\bullet .

As a final note, we mention that in principle, the technique used to prove Corollary 5 would work for any n , assuming that there exists an algebraic representative in codimension- n , that $V_l\phi_{X_{\bar{K}}/\bar{K}}^n$ is an isomorphism, and that $V_l\lambda^n$ is an inclusion; however, unlike the case $n = 1, 2, d_X$, in general, for $n \neq 1, 2, d_X$, one does not expect these conditions to hold. Nevertheless, in the body of the paper, we explain the general case, and indicate where special assumptions are needed.

0.3. Mazur's question with \mathbb{Z}_ℓ -coefficients. Next we consider Mazur's question in the case of \mathbb{Z}_l -coefficients (while acknowledging that, in positive characteristic p and with $l = p$, it might be more natural to seek an isomorphism of F -crystals than an isomorphism of cohomology groups $H^\bullet(-, \mathbb{Z}_p)$). To start with, as in the case of \mathbb{Q}_l -coefficients, it is well-known that one can model

$H^{2n-1}(X_{\bar{K}}, \mathbb{Z}_l(n))_\tau$ for $n = 1, d_X$ via the Picard and Albanese: in those cases, we have isomorphisms (see Propositions A.25 and A.26)

$$T_l(\text{Pic}_{X/K}^0)_{\text{red}} \xrightarrow[\simeq]{(T_l\phi_{X_{\bar{K}}/\bar{K}}^1)^{-1}} T_l A^1(X_{\bar{K}}) = T_l \text{CH}^1(X_{\bar{K}}) \xrightarrow[\simeq]{T_l\lambda^1} H^1(X_{\bar{K}}, \mathbb{Z}_l(1))$$

$$T_l \text{Alb}_{X/K} \xrightarrow[\simeq]{(T_l\phi_{X_{\bar{K}}/\bar{K}}^{d_X})^{-1}} T_l A^{d_X}(X_{\bar{K}}) = T_l \text{CH}^{d_X}(X_{\bar{K}}) \xrightarrow[\simeq]{T_l\lambda^{d_X}} H^{2d_X-1}(X_{\bar{K}}, \mathbb{Z}_l(d_X))_\tau$$

Motivated by the discussion in §0.1 leading to Corollary 5, we proceed in a similar way, focusing on the case $n = 2$. The starting point is again to assume that $\phi_{X_{\bar{K}}/\bar{K}}^2[l^\infty] : A^2(X_{\bar{K}})[l^\infty] \rightarrow \text{Ab}_{X/K}^2[l^\infty]$ is an isomorphism; *i.e.*, we assume the so-called standard assumption for $n = 2$ (Definition 6); *e.g.*, we assume $\text{char}(K) = 0$ or X is geometrically rationally chain connected.

By taking Tate modules one has (Lemma 3.6) that

$$T_l\phi_{X_{\bar{K}}/\bar{K}}^2 : T_l A^2(X_{\bar{K}}) \longrightarrow T_l \text{Ab}_{X/K}^2 \quad (0.7)$$

is an isomorphism as well. Consequently, one can consider the composition

$$T_l \text{Ab}_{X/K}^2 \xrightarrow[\simeq]{(T_l\phi_{X_{\bar{K}}/\bar{K}}^2)^{-1}} T_l A^2(X_{\bar{K}}) \hookrightarrow T_l \text{CH}^2(X_{\bar{K}}) \xrightarrow{T_l\lambda^2} H^3(X_{\bar{K}}, \mathbb{Z}_l(2))_\tau. \quad (0.8)$$

That $T_l\lambda^2$ is an inclusion is reviewed in Proposition A.27. In Proposition 2.1, we show that the image of the map $T_l\lambda^2 : T_l A^2(X_{\bar{K}}) \rightarrow H^3(X_{\bar{K}}, \mathbb{Z}_l(2))_\tau$ contains $\tilde{N}^{n-1} H^{2n-1}(X_{\bar{K}}, \mathbb{Z}_l(n))_\tau$; here \tilde{N}^\bullet is the *correspondence niveau filtration* defined in (1.4). Combined with Suwa's Proposition 4 (together with Proposition 1.1 comparing \tilde{N}^\bullet with N^\bullet), we find that the image contains $\tilde{N}^{n-1} H^{2n-1}(X_{\bar{K}}, \mathbb{Z}_l(n))_\tau$ as a finite index subgroup. With the expectation that the standard assumption should be true in general, we are thus led to ask:

Question 7 (Mazur's question with \mathbb{Z}_l -coefficients). *Let X be a smooth projective variety over a field K . Does there exist an abelian variety A/K such that for almost all primes l there is an isomorphism of Galois modules*

$$T_l A \xrightarrow{\simeq} \tilde{N}^{n-1} H^{2n-1}(X_{\bar{K}}, \mathbb{Z}_l(n))_\tau ? \quad (0.9)$$

The main technical result of this paper is Theorem 4.2, a particular instance of which takes the form below.

Theorem 8 (Image of the l -adic Bloch map). *Let X be a smooth projective variety over a perfect field K . Assume that $\phi_{X_{\bar{K}}/\bar{K}}^2[l^\infty] : A^2(X_{\bar{K}})[l^\infty] \rightarrow \text{Ab}_{X/K}^2[l^\infty]$ is an isomorphism for all but finitely many primes l ; *i.e.*, X satisfies the standard assumption at all but finitely many primes l (Definition 6). Then, for all but finitely many prime numbers l , the image of the composition*

$$T_l A^2(X_{\bar{K}}) \hookrightarrow T_l \text{CH}^2(X_{\bar{K}}) \xrightarrow{T_l\lambda^2} H^3(X_{\bar{K}}, \mathbb{Z}_l(2))_\tau$$

is equal to $\tilde{N}^1 H^3(X_{\bar{K}}, \mathbb{Z}_l(2))_\tau$.

We obtain the following immediate corollary providing a partial answer to Question 7:

Corollary 9 (Modeling \mathbb{Z}_l -cohomology). *Under the hypotheses of Theorem 8, the inclusion (0.8) induces an isomorphism of Galois modules*

$$T_l \text{Ab}_{X/K}^2 \xrightarrow{\simeq} \tilde{N}^1 H^3(X_{\bar{K}}, \mathbb{Z}_l(2))_\tau$$

for all but finitely many prime numbers l .

Theorem 8 is proved in §4. In fact, we can control the primes for which Theorem 8 and Corollary 9 might fail; this is related to miniversal cycle classes, as well as decomposition of the diagonal (see Theorem 4.2). Moreover, in a way we make precise in Lemma 4.3 and Proposition 4.4, to prove Theorem 8 and Corollary 9, if one knows the standard assumption holds for all varieties over finite fields, then one does not need to assume the standard assumption for X .

We note again that in principle, the technique used to prove the Corollary 9 would work for any n , assuming that there exists an algebraic representative in codimension n , that $\phi_{X_{\bar{K}}/\bar{K}}^n[l^\infty]$ is an isomorphism, and that $T_l \lambda^n$ is an inclusion; but in general, for $n \neq 1, 2, d_X$, one does not expect these conditions to hold. Nevertheless, in the body of the paper, we explain the general case, and indicate where special assumptions are needed.

0.4. Universal cycles and the image of the second l -adic Bloch map. Still under the standard assumption that

$$\phi_{X_{\bar{K}}/\bar{K}}^2[l^\infty] : A^2(X_{\bar{K}})[l^\infty] \xrightarrow{\simeq} \text{Ab}_{X/K}^2[l^\infty]$$

is an isomorphism for all primes l , we show that a sufficient condition for the composition (0.8) to have image equal to $\tilde{N}^1 H^3(X_{\bar{K}}, \mathbb{Z}_l(2))_\tau$ for all l is provided by the existence of a so-called *universal cycle* for $\phi_{X_{\bar{K}}/\bar{K}}^2$; see §3.3 for a definition. As before, we start by determining the image of the second l -adic Bloch map under such conditions:

Theorem 10 (Universal cycles and the image of the second l -adic Bloch map). *Let X be a smooth projective variety over a perfect field K . Assume that X satisfies the standard assumption for all primes l . If $\phi_{X_{\bar{K}}/\bar{K}}^2 : A^2(X_{\bar{K}}) \rightarrow \text{Ab}_{X/K}^2(\bar{K})$ admits a universal cycle, then for all prime numbers l the image of the composition*

$$T_l A^2(X_{\bar{K}}) \hookrightarrow T_l \text{CH}^2(X_{\bar{K}}) \xrightarrow{T_l \lambda^2} H^3(X_{\bar{K}}, \mathbb{Z}_l(2))_\tau$$

is equal to $\tilde{N}^1 H^3(X_{\bar{K}}, \mathbb{Z}_l(2))_\tau$.

Theorem 10 is a particular instance of our main Theorem 4.2. As an immediate consequence, we obtain:

Corollary 11 (Universal cycles and modeling \mathbb{Z}_l -cohomology). *Under the hypotheses of Theorem 10, the inclusion (0.8) induces for all prime numbers l an isomorphism of Galois modules*

$$T_l \text{Ab}_{X/K}^2 \xrightarrow{\simeq} \tilde{N}^1 H^3(X_{\bar{K}}, \mathbb{Z}_l(2))_\tau.$$

Due to the connection with universal cycles and decomposition of the diagonal (see Proposition 3.10), this is connected with the notion of rationality, as we discuss in the next section.

0.5. Decomposition of the diagonal and the image of the second l -adic Bloch map. We next turn our focus to the case of smooth projective varieties X over a perfect field K with $\text{CH}_0(X_{\bar{K}}) \otimes \mathbb{Q}$ universally supported in dimension 2 (see Definition 3.1). It is well-known, via a decomposition of the diagonal argument [BS83], that in this case we have $N^1 H^3(X_{\bar{K}}, \mathbb{Q}_l(2)) = H^3(X_{\bar{K}}, \mathbb{Q}_l(2))$, but also that $V_l \phi_{X_{\bar{K}}/\bar{K}}^2$ is an isomorphism (Proposition 3.8(3)), for all primes l . As a consequence, we see that in this case (0.5) induces an isomorphism $V_l \text{Ab}_{X/K}^2 \simeq H^3(X_{\bar{K}}, \mathbb{Q}_l(2))$. We show the following result for cohomology with \mathbb{Z}_l -coefficients:

Theorem 12 (Decomposition of the diagonal and the image of the second l -adic Bloch map). *Let X be a smooth projective variety over a perfect field K of characteristic exponent p .*

- (1) Assume $\mathrm{CH}_0(X_{\bar{K}}) \otimes \mathbb{Z}[\frac{1}{N}]$ is universally supported in dimension 2 for some $N > 0$, e.g. X is a geometrically uniruled threefold. Then, for all primes $\ell \nmid Np$, the inclusion $\tilde{N}^1 H^3(X_{\bar{K}}, \mathbb{Z}_\ell(2)) \subseteq H^3(X_{\bar{K}}, \mathbb{Z}_\ell(2))$ is an equality and the second ℓ -adic Bloch map restricted to algebraically trivial cycles

$$T_\ell A^2(X_{\bar{K}}) \hookrightarrow T_\ell \mathrm{CH}^2(X_{\bar{K}}) \xrightarrow{T_\ell \lambda^2} H^3(X_{\bar{K}}, \mathbb{Z}_\ell(2))_\tau$$

is an isomorphism of $\mathrm{Gal}(K)$ -modules.

- (2) Assume $\mathrm{CH}_0(X_{\bar{K}}) \otimes \mathbb{Z}[\frac{1}{N}]$ is universally supported in dimension 1 for some $N > 0$, e.g. X is a geometrically rationally chain connected. Then, for all primes l , the second l -adic Bloch map restricted to algebraically trivial cycles

$$T_l A^2(X_{\bar{K}}) \hookrightarrow T_l \mathrm{CH}^2(X_{\bar{K}}) \xrightarrow{T_l \lambda^2} H^3(X_{\bar{K}}, \mathbb{Z}_l(2))_\tau$$

is an isomorphism of $\mathrm{Gal}(K)$ -modules. Moreover, for all primes $\ell \nmid Np$, $H^3(X_{\bar{K}}, \mathbb{Z}_\ell(2))$ is torsion-free.

A slight generalization of Theorem 12 that deals with the prime p in case resolution of singularities holds over K in dimensions $< \dim X$ is proved in Proposition 5.2. See §3.1, and in particular Remark 3.4, for the classical link between stable rationality, rational connectedness, and decomposition of the diagonal.

Again, from Theorem 12 (combined with Proposition 3.8), we have the following corollary.

Corollary 13 (Decomposition of the diagonal and modeling \mathbb{Z}_ℓ -cohomology). *Let X be a smooth projective variety over a perfect field K of characteristic exponent p .*

- (1) Assume $\mathrm{CH}_0(X_{\bar{K}}) \otimes \mathbb{Z}[\frac{1}{N}]$ is universally supported in dimension 2 for some $N > 0$, e.g. X is a geometrically uniruled threefold. Then, for all primes $\ell \nmid Np$, $T_\ell \phi_{X_{\bar{K}}/\bar{K}}^2 : T_\ell A^2(X_{\bar{K}}) \rightarrow T_\ell \mathrm{Ab}_{X/K}^2$ is an isomorphism and the canonical inclusion (0.8) induces an isomorphism of Galois modules

$$T_\ell \mathrm{Ab}_{X/K}^2 \xrightarrow{\cong} H^3(X_{\bar{K}}, \mathbb{Z}_\ell(2))_\tau.$$

- (2) Assume $\mathrm{CH}_0(X_{\bar{K}}) \otimes \mathbb{Q}$ is universally supported in dimension 1, e.g. X is a geometrically rationally chain connected. Then, for all primes l , $T_l \phi_{X_{\bar{K}}/\bar{K}}^2 : T_l A^2(X_{\bar{K}}) \rightarrow T_l \mathrm{Ab}_{X/K}^2$ is an isomorphism and the canonical inclusion (0.8) induces an isomorphism of Galois modules

$$T_l \mathrm{Ab}_{X/K}^2 \xrightarrow{\cong} H^3(X_{\bar{K}}, \mathbb{Z}_l(2))_\tau.$$

0.6. Stably rational vs. geometrically stably rational varieties over finite fields. We now turn to the motivic question :

Question 14 (Mazur's motivic question with \mathbb{Z}_ℓ -coefficients). *For which smooth projective varieties X over a field K do there exist an abelian variety A/K and a correspondence $\Gamma \in \mathrm{CH}^2(A \times_K X)$ such that for all primes $\ell \neq \mathrm{char}(K)$*

$$\Gamma_* : T_\ell A \xrightarrow{\cong} \tilde{N}^1 H^3(X_{\bar{K}}, \mathbb{Z}_\ell(2))_\tau$$

is an isomorphism of $\mathrm{Gal}(K)$ -modules ?

In other words, we turn now to the issue of addressing the existence of a correspondence $\Gamma \in \mathrm{CH}^2(\mathrm{Ab}_{X/K}^2 \times_K X)$ inducing the isomorphisms (0.9). As already mentioned, it is easy to establish a positive answer under the further assumption that $2n - 1 \leq \dim X$ and $H^{2n-1}(X_{\bar{K}}, \mathbb{Q}_\ell(n))$ has geometric coniveau $n - 1$; see Proposition 6.1. In case X is a smooth projective geometrically uniruled threefold, then Proposition 6.3 establishes more precisely the existence of a correspondence $\Gamma \in \mathrm{CH}^2(\mathrm{Ab}_{X/K}^2 \times_K X) \otimes \mathbb{Q}$ such that the induced morphism of Galois modules

$\Gamma_* : V_l \text{Ab}_{X/K}^2 \xrightarrow{\cong} H^3(X_{\bar{K}}, \mathbf{Q}_l(2))$ coincides with the canonical map (0.5) and is an isomorphism for all primes l .

On the other hand, due to the failure of the integral Tate conjecture over finite fields [Ant16, Kam15, PY15], an isomorphism as in (0.9) might not be induced by some correspondence $\Gamma \in \text{CH}^2(\text{Ab}_{X/K}^2 \times_K X)$. However, using the ℓ -adic Bloch map, we provide a positive answer for the third ℓ -adic cohomology groups of smooth projective stably rational varieties over finite or algebraically closed fields, thereby addressing Question 14:

Theorem 15 (Modeling \mathbb{Z}_ℓ -cohomology via correspondences). *Let X be a smooth projective stably rational variety over a field K that is either finite or algebraically closed. Then there exists a correspondence $\Gamma \in \text{CH}^2(\text{Ab}_{X/K}^2 \times_K X)$ inducing for all primes $\ell \neq \text{char } K$ the isomorphisms (0.8)*

$$\Gamma_* : T_\ell \text{Ab}_{X/K}^2 \xrightarrow{\cong} H^3(X_{\bar{K}}, \mathbb{Z}_\ell(2)) \quad (0.10)$$

of $\text{Gal}(K)$ -modules. Moreover, if $\text{char}(K) = 0$, the correspondence Γ induces an isomorphism

$$\Gamma_* : H_1(J^3(X_C), \mathbb{Z}) \xrightarrow{\cong} H^3(X_C, \mathbb{Z}(2)). \quad (0.11)$$

The proof of Theorem 15 is given in Theorem 6.4, via a decomposition of the diagonal argument. There we also explain how the conclusion of Theorem 15 holds at $l = p$ in case $\dim X \leq 4$, due to the existence resolution of singularities in dimensions ≤ 3 . There are two reasons for restricting to algebraically closed fields or finite fields in Theorem 15. First, in order to use alterations, we restrict to the case of perfect fields. Second, in order to obtain the existence of the universal line-bundle, we use that K is finite or separably closed (see [ACMVb, §7.1.2]).

0.7. Notation and conventions. For a field K , we will denote by \bar{K} a separable closure, and by \bar{K}^a an algebraic closure. If X is a scheme of finite type over a field K , we denote by $\text{CH}^i(X)$ its Chow group of codimension- i cycle classes, and by $A^i(X) \subset \text{CH}^i(X)$ the subgroup consisting of algebraically trivial cycle classes (see [Fu198, §10.3]). If X is pure-dimensional, we denote d_X its dimension. In case X is smooth over K , we still denote d_X its dimension, which should be thought of as a locally constant function on X . A *variety* over K is a separated geometrically reduced scheme of finite type over K . The symbol l is allowed to denote an arbitrary prime, whereas ℓ is always assumed invertible in the base field K . The phrase “for almost all” means “for all but finitely many”.

Let M be an abelian group, let l be a prime, and let ν be an integer. We denote:

$$\begin{aligned} M_{\text{tors}} &:= \text{Tor}_1^{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, M) = \text{the torsion subgroup of } M; \\ M_{\text{cotors}} &:= \text{the quotient of } M \text{ by its largest divisible subgroup}; \\ M_\tau &:= M/M_{\text{tors}} = \text{the quotient of } M \text{ by its torsion subgroup}; \\ M[l^\nu] &:= \text{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/l^\nu\mathbb{Z}, M) = \text{the } l^\nu\text{-torsion subgroup of } M; \\ M[l^\infty] &:= \varinjlim_{\nu} M[l^\nu] = \text{the } l\text{-primary torsion subgroup of } M; \\ T_l M &:= \varprojlim_{\nu} M[l^\nu] = \text{Hom}_{\mathbb{Z}}(\mathbb{Q}_l/\mathbb{Z}_l, M) = \text{the Tate module of } M; \\ V_l M &:= T_l M \otimes_{\mathbb{Z}_l} \mathbb{Q}_l. \end{aligned}$$

In the definition of $T_l M$, the transition maps are given by the multiplication by l morphisms $M[l^{\nu+1}] \xrightarrow{l} M[l^\nu]$ and the equality $\varprojlim_{\nu} M[l^\nu] = \text{Hom}_{\mathbb{Z}}(\mathbb{Q}_l/\mathbb{Z}_l, M)$ can be found, e.g., in [Mil06, Prop. 0.19]. Note that $T_l M = T_l(M[l^\infty])$. Note also that for a \mathbb{Z}_l -module M , we have $M_{\text{tors}} = \text{Tor}_1^{\mathbb{Z}_l}(\mathbb{Q}_l/\mathbb{Z}_l, M)$.

We denote the l -adic valuation by v_l , so that for a natural number r , we have $r = \prod_l l^{v_l(r)}$.

For a smooth projective variety X over an algebraically closed field $K = \bar{K}^a = \bar{K}$, we denote by $H_i(X_{\bar{K}}, \mathbb{Z}_\ell)$ the ℓ -adic homology. This will be primarily an indexing convention that is useful

from the motivic perspective, since in the case where X is smooth and projective of pure dimension d_X , the cap product with the fundamental class of X induces for all i (e.g., [Lau76, p.173]) an isomorphism $-\cap [X] : H^{2d_X-i}(X_{\bar{K}}, \mathbb{Z}_\ell(d_X)) \xrightarrow{\sim} H_i(X_{\bar{K}}, \mathbb{Z}_\ell)$.

1. ON VARIOUS NOTIONS OF CONIVEAU FILTRATIONS

Given a smooth projective variety X over a field K , one obtains coniveau filtrations on cohomology with various coefficients. In (1.1), (1.3) and (1.4) below we recall the definitions of the (classical) geometric coniveau filtration N^\bullet , of the (less classical) correspondence coniveau filtration N'^\bullet and of the (still less classical) correspondence niveau filtration \tilde{N}^\bullet .

Although the filtrations might not agree in general, in Proposition 1.1 below we recall that they are related by

$$\tilde{N}^\bullet \subseteq N'^\bullet \subseteq N^\bullet,$$

where, over a perfect field K and with \mathbb{Q}_ℓ -coefficients, the second inclusion is an equality while the first is conjecturally an equality.

In this section, the ring of coefficients Λ denotes either \mathbb{Z} , \mathbb{Q} , $\mathbb{Z}/\ell^r\mathbb{Z}$, \mathbb{Z}_ℓ , \mathbb{Q}_ℓ , or $\mathbb{Q}_\ell/\mathbb{Z}_\ell$. Cohomology groups $H^\bullet(-, \Lambda)$ are computed in the corresponding topology: e.g., $H^\bullet(-, \mathbb{Z})$ is computed in the analytic topology, while $H^\bullet(-, \mathbb{Z}/\ell^r\mathbb{Z})$ is computed in the étale topology.

1.1. Recalling the geometric coniveau filtrations. Let X be a smooth projective variety over a field K with separable closure \bar{K} . We recall the ν -th piece of the geometric coniveau filtration

$$N^\nu H^i(X_{\bar{K}}, \Lambda) := \sum_{Z \subset X_{\bar{K}}} \text{im} \left(H_Z^i(X_{\bar{K}}, \Lambda) \rightarrow H^i(X_{\bar{K}}, \Lambda) \right) \quad (1.1)$$

$$= \sum_{Z \subset X_{\bar{K}}} \text{ker} \left(H^i(X_{\bar{K}}, \Lambda) \rightarrow H^i(X_{\bar{K}} \setminus Z, \Lambda) \right) \quad (1.2)$$

where the sum runs through all Zariski closed subsets Z of $X_{\bar{K}}$ of codimension $\geq \nu$. (The equivalence of (1.1) and (1.2) comes from the long exact sequence of a pair.)

For our purpose, we will have to work with the following variant of the coniveau filtration. We recall the ν -th piece of the correspondence coniveau filtration

$$N'^\nu H^i(X_{\bar{K}}, \Lambda) := \sum_{\Gamma: Z \dashrightarrow X_{\bar{K}}} \text{im} \left(\Gamma_* : H^{i-2\nu}(Z, \Lambda(-\nu)) \rightarrow H^i(X_{\bar{K}}, \Lambda) \right), \quad (1.3)$$

where the sum is over all smooth projective varieties Z over \bar{K} and all correspondences $\Gamma \in \text{CH}_{d_X-\nu}(Z \times_{\bar{K}} X_{\bar{K}})$.

Let us also introduce a related filtration, which was considered in [FM94] and in [Via13]. The ν -th piece of the correspondence niveau filtration is defined as

$$\tilde{N}^\nu H^i(X_{\bar{K}}, \Lambda) := \sum_{\Gamma: Z \dashrightarrow X_{\bar{K}}} \text{im} \left(\Gamma_* : H_{i-2\nu}(Z, \Lambda(\nu-i)) \rightarrow H^i(X_{\bar{K}}, \Lambda) \right), \quad (1.4)$$

where the sum is over all smooth projective varieties Z over \bar{K} and all correspondences $\Gamma \in \text{CH}^{d_X-\nu}(Z \times_{\bar{K}} X_{\bar{K}})$. As we will see in the proof of Proposition 1.1 below, one may restrict the sum in (1.4) to smooth projective varieties Z over \bar{K} of pure dimension $i-2\nu$. Here, as outlined in the Notation and Conventions §0.7, we use homology as a convenience; for Z of pure dimension d_Z , we set $H_{i-2\nu}(Z, \Lambda(\nu-i)) := H^{2d_Z-(i-2\nu)}(Z, \Lambda(d_Z-\nu-i))$.

By considering fields of definitions of Z (and Γ) that are finite Galois over K and by considering Galois orbits, we note that in the definitions of all three filtrations above, we could have restricted the sums to those Z (and Γ) defined over K .

The above three filtrations admit the following already-known containment relation :

Proposition 1.1. *Let X be a smooth projective variety over a field K . Suppose that Λ is one of $\mathbb{Z}/\ell^r\mathbb{Z}$, \mathbb{Z}_ℓ , \mathbb{Q}_ℓ , or $\mathbb{Q}_\ell/\mathbb{Z}_\ell$ or that $K = \mathbb{C}$ and Λ is one of \mathbb{Z} , \mathbb{Q} or \mathbb{Q}/\mathbb{Z} . Then there are natural inclusions*

$$\tilde{N}^\nu H^i(X_{\bar{K}}, \Lambda) \subseteq N^{\nu} H^i(X_{\bar{K}}, \Lambda) \subseteq N^\nu H^i(X_{\bar{K}}, \Lambda).$$

In case $(\nu, i) = (n, 2n)$ or $(n-1, 2n-1)$, the first inclusion is an equality. Moreover, assuming Λ is either \mathbb{Q}_ℓ or \mathbb{Q} , if K is perfect then the second inclusion is an equality, and if Grothendieck's Lefschetz standard conjecture holds then the first inclusion is an equality.

Proof. The containment $\tilde{N}^\nu H^i(X_{\bar{K}}, \Lambda) \subseteq N^{\nu} H^i(X_{\bar{K}}, \Lambda)$ is explained in [Via13, §1.1] in the case $K \subseteq \mathbb{C}$ and $\Lambda = \mathbb{Q}$. The same argument applies here and we spell it out for the sake of completeness. First, note that up to replacing Z with $Z \times_{\bar{K}} \mathbb{P}_{\bar{K}}^n$, we can assume $\dim Z \geq i - 2\nu$. Second, if $\iota : Y \hookrightarrow Z$ is a smooth linear intersection of Z of dimension $i - 2\nu$, then the push-forward $\iota_* : H_{i-2\nu}(Y, \Lambda) \rightarrow H_{i-2\nu}(Z, \Lambda)$ is surjective by the Lefschetz hyperplane theorem (e.g., [SGA72, Exp. XIV, Cor. 3.3]), and the image of $\Gamma_* : H_{i-2\nu}(Z, \Lambda(i-\nu)) \rightarrow H^i(X_{\bar{K}}, \Lambda)$ coincides thus with the image of $\Gamma_* \circ \iota_*$. Therefore, one may restrict the sum in (1.4) to those smooth projective varieties Z over \bar{K} of pure dimension $i - 2\nu$. In particular, since $H_{i-2\nu}(Z, \Lambda) = H^{i-2\nu}(Z, \Lambda(i-2\nu))$ when $\dim Z = i - 2\nu$, we get the asserted containment.

As outlined in the argument in [Via13, Prop. 1.1], a sufficient condition for the first inclusion to be an equality is the following : if for all smooth projective varieties Z over \bar{K} there exists a smooth projective variety Z' over \bar{K} and a correspondence $L \in \text{CH}^{i-2\nu}(Z' \times_{\bar{K}} Z)$ inducing an isomorphism $L_* : H_{i-2\nu}(Z', \Lambda(i-\nu)) \xrightarrow{\sim} H^{i-2\nu}(Z, \Lambda(\nu))$, then the image of $\Gamma_* : H^{i-2\nu}(Z, \Lambda(-\nu)) \rightarrow H^i(X_{\bar{K}}, \Lambda)$ coincides with the image of $\Gamma_* \circ L_*$ and it follows that the containment $\tilde{N}^\nu H^i(X_{\bar{K}}, \Lambda) \subseteq N^{\nu} H^i(X_{\bar{K}}, \Lambda)$ is an equality. Now, in case $(\nu, i) = (n, 2n)$, the class of $Z \times_{\bar{K}} Z$ induces an isomorphism $H_0(Z, \Lambda) \xrightarrow{\sim} H^0(Z, \Lambda)$, while in case $(\nu, i) = (n-1, 2n-1)$, using the identification of torsion line bundles and étale covers, the universal line bundle \mathcal{L} on $\text{Pic}_{Z/\bar{K}}^0 \times_{\bar{K}} Z$ induces natural identifications $H_1(\text{Pic}_{Z/\bar{K}}^0, \mathbb{Z}_\ell) = H^{2g-1}(\text{Pic}_{Z/\bar{K}}^0, \mathbb{Z}_\ell(g)) = T_\ell \text{Pic}_{Z/\bar{K}}^0 = H^1(Z, \mathbb{Z}_\ell(1))$. In case $\Lambda = \mathbb{Q}$ or \mathbb{Q}_ℓ , a correspondence L as above exists with $Z' = Z$ for all Z and all (ν, i) provided Grothendieck's standard conjecture holds.

We now turn to the containment $N^{\nu} H^i(X_{\bar{K}}, \Lambda) \subseteq N^\nu H^i(X_{\bar{K}}, \Lambda)$. Let $\Gamma \in \text{CH}_{d_X-\nu}(Z \times_{\bar{K}} X_{\bar{K}})$ be a correspondence with Z a smooth projective variety over \bar{K} . By refined intersection, the image of Γ_* is supported on the closed subscheme $\bar{Z} := p_{X_{\bar{K}}}(\Gamma)$ of dimension $\leq d_X - \nu$, where $p_{X_{\bar{K}}} : \Gamma \rightarrow X_{\bar{K}}$ is the natural projection. In other words, the composition

$$H^{i-2\nu}(Z, \Lambda(-\nu)) \xrightarrow{\Gamma_*} H^i(X_{\bar{K}}, \Lambda) \longrightarrow H^i(X_{\bar{K}} \setminus \bar{Z}, \Lambda) \quad (1.5)$$

vanishes, thereby giving the asserted containment.

For the statement of equality when Λ denotes \mathbb{Q} or \mathbb{Q}_ℓ and when K is perfect, since Jannsen [Jan94] only asserts this for fields of characteristic 0, as de Jong's results were not available at the time, here we reproduce the argument of [Jan94, p.265–6] to show how the argument can be extended to fields of positive characteristic. Consider a closed embedding $\iota : Z \hookrightarrow X_{\bar{K}}$ with $\dim Z = d_X - \nu$ and use the theory of alterations to produce a diagram

$$f : Z' \xrightarrow{\pi} Z \xrightarrow{\iota} X_{\bar{K}}$$

with Z' smooth of pure dimension $d_X - v$. We get a commutative diagram (using ℓ -adic homology)

$$\begin{array}{ccccc}
& & \xrightarrow{f_*} & & \\
H^{i-2v}(Z', \mathbb{Q}_\ell(-v)) & \longrightarrow & H_Z^i(X_{\bar{K}}, \mathbb{Q}_\ell) & \longrightarrow & H^i(X_{\bar{K}}, \mathbb{Q}_\ell) \\
\downarrow \simeq & & \downarrow & & \downarrow \simeq \\
H_{2d_X-i}(Z', \mathbb{Q}_\ell(d_X)) & \xrightarrow{\pi_*} & H_{2d_X-i}(Z, \mathbb{Q}_\ell(d_X)) & \xrightarrow{\iota_*} & H_{2d_X-i}(X_{\bar{K}}, \mathbb{Q}_\ell(d_X))
\end{array}$$

and we then argue using weights (see [Jan90, §6]). To utilize weights, we must assume that K is finitely generated over its prime field; however, since all the varieties are of finite type, they are all defined over a base field $K' \subseteq \bar{K}$ that is finitely generated over its prime field, and we may work over K' , and then base change to \bar{K} . In other words, we may assume that K is finitely generated over its prime field and that X, Z, Γ and π are defined over K . Now, since $H^i(X_{\bar{K}}, \mathbb{Q}_\ell)$ is pure of weight i , the image of ι_* equals the image of $W_i H_{2d_X-i}(Z_{\bar{K}}, \mathbb{Q}_\ell(d_X))$. On the other hand, it is shown in [Jan90, Rem. 7.7] that $H_{2d_X-i}(Z'_{\bar{K}}, \mathbb{Q}_\ell(d_X))$ surjects onto this space *via* π_* . \square

1.2. p -adic coniveau filtrations. If $\text{char}(K) = p > 0$ and if K is perfect, we also allow Λ to variously denote $\mathbb{W}_n(K), \mathbb{W}(K)$ or $\mathbb{K}(K)$, in which case $H^\bullet(-, \Lambda)$ denotes a crystalline cohomology group. (See §A.1.2 for our notations concerning p -adic cohomology theories in characteristic p .) One defines

$$N^v H^i(X_{\bar{K}}, \Lambda) \quad \text{and} \quad N'^v H^i(X_{\bar{K}}, \Lambda)$$

exactly as in (1.1) (note that in this setting (1.2) is not well-behaved) and (1.3), respectively. Since we will only need homology for smooth projective varieties X , we simply define $H_i(X, \Lambda) = H^{2d_X-i}(X, \Lambda(d_X))$. With this notation we then define the correspondence niveau filtration \tilde{N} exactly as in (1.4).

In contrast to the crystalline cohomology groups, the groups $H^i(X_{\bar{K}}, \mathbb{Q}_p)$ no longer have a useful theory of weights or Poincaré duality. Since $H^i(X_{\bar{K}}, \mathbb{Z}_p)$ can be recovered as $(H^i(X/\mathbb{W}))^F$, the suitably-defined F -invariants of $H^i(X/\mathbb{W})$ [Gro85, §I.3], we simply *define* the p -adic coniveau filtrations $\mathcal{N} = \tilde{N}, N, N'$ by

$$\begin{aligned}
\mathcal{N}^v H^i(X_{\bar{K}}, \mathbb{Z}_p) &= (\mathcal{N}^v H^i(X/\mathbb{W}))^F \\
\mathcal{N}^v H^i(X_{\bar{K}}, \mathbb{Z}_p/p^r) &= (\mathcal{N}^v H^i(X/\mathbb{W}_r))^F \\
\mathcal{N}^v H^i(X_{\bar{K}}, \mathbb{Q}_p/\mathbb{Z}_p) &= \varinjlim \mathcal{N}^v H^i(X_{\bar{K}}, \mathbb{Z}_p/p^r).
\end{aligned}$$

Then Proposition 1.1 holds in this context, too:

Proposition 1.1(bis). *Let X be a smooth projective variety over a perfect field K . Suppose that Λ is one of $\mathbb{Z}/l^r\mathbb{Z}, \mathbb{Z}_l, \mathbb{Q}_l$, or $\mathbb{Q}_l/\mathbb{Z}_l$, or that $\text{char}(K) > p$ and Λ is one of $\mathbb{W}_r(K), \mathbb{W}(K)$ or $\mathbb{K}(K)$, or that $K = \mathbb{C}$ and Λ is one of \mathbb{Z}, \mathbb{Q} or \mathbb{Q}/\mathbb{Z} . Then there are natural inclusions*

$$\tilde{N}^v H^i(X_{\bar{K}}, \Lambda) \subseteq N'^v H^i(X_{\bar{K}}, \Lambda) \subseteq N^v H^i(X_{\bar{K}}, \Lambda).$$

In case $(v, i) = (n, 2n)$ or $(n-1, 2n-1)$, the first inclusion is an equality. If Λ is a field of characteristic zero, then the second inclusion is an equality, and if in addition Grothendieck's Lefschetz standard conjecture holds, then the first inclusion is an equality.

Proof. It only remains to prove the assertions for the various p -adic coefficients in characteristic $p > 0$. If $\Lambda = \mathbb{W}_n(K), \mathbb{W}(K)$ or $\mathbb{K}(K)$, the argument is identical; the key point in the case $\Lambda = \mathbb{K}(K)$ is that like étale cohomology, rigid cohomology has a good theory of weights and Poincaré duality. (The second map in (1.5) simply needs to be replaced with $H^i(X, \Lambda) \rightarrow$

$H^i(X, \Lambda)/H_Z^i(X, \Lambda)$, which makes sense in crystalline cohomology.) The results for $\Lambda = \mathbb{Z}_p/p^n$, \mathbb{Z}_p , \mathbb{Q}_p or $\mathbb{Q}_p/\mathbb{Z}_p$ follow by taking F -invariants. \square

2. THE IMAGE OF THE l -ADIC BLOCH MAP AND THE CONIVEAU FILTRATION

Again, for brevity, in this section, the ring of coefficients Λ denotes either \mathbb{Z}_l , \mathbb{Q}_l , or $\mathbb{Q}_l/\mathbb{Z}_l$. The subscript τ_Λ on $H^\bullet(-, \Lambda)_{\tau_\Lambda}$ indicates that when $\Lambda = \mathbb{Z}_l$, we take the quotient by the torsion subgroup. For each of these Λ and for $M = \text{CH}^n(X_{\bar{K}})$ or $A^n(X_{\bar{K}})$, we have the corresponding groups $M_\Lambda : M_{\mathbb{Z}_l} := T_l M$, $M_{\mathbb{Q}_l} := V_l M$ and $M_{\mathbb{Q}_l/\mathbb{Z}_l} := M[l^\infty]$.

We now consider the l -adic Bloch map constructed by Suwa [Suw88] in the case $l \neq \text{char}(K)$ and by Gros–Suwa [GS88] in the case $l = \text{char}(K)$; see the appendix for a review of these maps, especially §A.3.3 and §A.3.5. Our main result in this section extends a result of Suwa [Suw88, Prop. 5.2], originally stated for $\Lambda = \mathbb{Q}_\ell$ with $\ell \neq \text{char}(K)$, and gives a preliminary description of the image of the l -adic Bloch map restricted to the Tate module of algebraically trivial cycles in terms of the correspondence coniveau filtration (1.3):

Proposition 2.1 ([Suw88, Prop. 5.2]). *Let X be a smooth projective variety over a perfect field K , and let l be a prime number. The image of the composition*

$$A^n(X_{\bar{K}})_\Lambda \hookrightarrow \text{CH}^n(X_{\bar{K}})_\Lambda \xrightarrow{\lambda^n} H^{2n-1}(X_{\bar{K}}, \Lambda(n))_{\tau_\Lambda} \quad (2.1)$$

contains $\tilde{N}^{n-1} H^{2n-1}(X_{\bar{K}}, \Lambda(n))_{\tau_\Lambda}$, with equality if $\Lambda = \mathbb{Q}_l/\mathbb{Z}_l$ or \mathbb{Q}_l . Recall that the subscript τ_Λ indicates that when $\Lambda = \mathbb{Z}_l$, we take the quotient by the torsion subgroup.

Remark 2.2. Note that in the case where $\Lambda = \mathbb{Q}_l/\mathbb{Z}_l$, this implies that $\tilde{N}^{n-1} H^{2n-1}(X_{\bar{K}}, \mathbb{Q}_l/\mathbb{Z}_l(n)) \subseteq H^{2n-1}(X_{\bar{K}}, \mathbb{Z}_l(n)) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l/\mathbb{Z}_l \subseteq H^{2n-1}(X_{\bar{K}}, \mathbb{Q}_l/\mathbb{Z}_l(n))$; see §A.5.

We split the proof of Proposition 2.1 in two, depending on whether the prime l is invertible in K . Key to the proof is the following lemma of Suwa [Suw88, Lem. 3.2] originally stated for primes $l \neq \text{char}(K)$; see Remark 2.5 below, where we verify that Suwa’s argument works even when $l = \text{char}(K)$.

Lemma 2.3 ([Suw88, Lem. 3.2]). *Let C be a smooth projective irreducible curve (resp. abelian variety) over a field K and let $\Gamma \in \text{CH}^n(C \times_K X)$. Let $\alpha \in A^n(X_{\bar{K}})[l^v]$, and assume there exists $\beta \in A_0(C_{\bar{K}})$ such that $\Gamma_* \beta = \alpha$. Then there exists $\gamma \in A_0(C_{\bar{K}})[l^\mu]$ for some $\mu \geq v$ such that $\lambda^n(\Gamma_* \gamma) = \lambda^n(\alpha)$. \square*

Example 2.4. In Lemma 2.3, it may be that one must take $\mu > v$. For instance, take $X = C = E$ to be an elliptic curve over K , let $\Gamma = \Gamma_f$ be the graph of the multiplication by ℓ map $f : C \rightarrow X$, and let $\alpha \in A^1(X_{\bar{K}})[\ell]$. Then there does not exist $\gamma \in A^1(C_{\bar{K}})[\ell]$ such that $\lambda^1(\Gamma_* \gamma) = \lambda^1(\alpha)$.

2.1. The image of the l -adic Bloch map. Here we review the argument of Suwa [Suw88] yielding the proof of Proposition 2.1 in the case $\Lambda = \mathbb{Q}_\ell$ and extend it to other rings of coefficients.

Proof of Proposition 2.1, prime-to-char(K). Suppose $l = \ell \neq \text{char}(K)$. Consider the diagram

$$\begin{array}{ccccc} \bigoplus_{\Gamma: Z \dashrightarrow X_{\bar{K}}} A_0(Z)_\Lambda & \xrightarrow{=} & \bigoplus_{\Gamma: Z \dashrightarrow X_{\bar{K}}} \text{CH}_0(Z)_\Lambda & \xrightarrow[\cong]{\lambda_0} & \bigoplus_{\Gamma: Z \dashrightarrow X_{\bar{K}}} H_1(Z, \Lambda)_{\tau_\Lambda} \\ \downarrow \Gamma_* & & \downarrow \Gamma_* & & \downarrow \Gamma_* \\ A^n(X_{\bar{K}})_\Lambda & \hookrightarrow & \text{CH}^n(X_{\bar{K}})_\Lambda & \xrightarrow{\lambda^n} & H^{2n-1}(X_{\bar{K}}, \Lambda(n))_{\tau_\Lambda} \end{array} \quad (2.2)$$

where the direct sums are over all smooth projective varieties Z over \bar{K} and all correspondences $\Gamma \in \text{CH}^n(Z \times_{\bar{K}} X_{\bar{K}})$. The image of the right vertical arrow is by definition $\tilde{N}^{n-1} H^{2n-1}(X_{\bar{K}}, \Lambda(n))_{\tau_\Lambda}$.

We conclude from commutativity of the diagram that the image of the bottom row of (2.2) contains $\tilde{N}^{n-1}H^{2n-1}(X_{\bar{K}}, \Lambda(n))$.

We now show that the inclusion is an equality in case $\Lambda = \mathbb{Q}_\ell/\mathbb{Z}_\ell$. By Lemma 2.3, for any $\alpha \in A^n(X_{\bar{K}})[\ell^\infty]$, there exists an element $\gamma \in \bigoplus_{\Gamma: Z \dashrightarrow X} A_0(Z_{\bar{K}})[\ell^\infty]$ such that $\lambda^n(\Gamma_*\gamma) = \lambda^n(\alpha)$. It readily follows from a diagram chase that the image of the bottom row of (2.2) is contained in $\tilde{N}^{n-1}H^{2n-1}(X_{\bar{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))$.

Finally we show equality in the case $\Lambda = \mathbb{Q}_\ell$. Taking Tate modules in (2.2) with $\mathbb{Q}_\ell/\mathbb{Z}_\ell$ -coefficients, and using the previous case, we have that the image of the bottom row of (2.2) with \mathbb{Z}_ℓ -coefficients is $T_\ell \tilde{N}^{n-1}H^{2n-1}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))$. Since $H^1(Z, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1))$ is a divisible group for any smooth projective variety Z over \bar{K} , and since an increasing chain of divisible abelian ℓ -torsion subgroups of a finite corank abelian ℓ -torsion group is stationary [Suuw88, Lem. 1.2], then by adding connected components to Z , one can conclude there exist a smooth projective curve Z over \bar{K} and a correspondence $\Gamma \in \text{CH}^n(Z \times_{\bar{K}} X_{\bar{K}})$ such that

$$\begin{aligned} \tilde{N}^{n-1}H^{2n-1}(X_{\bar{K}}, \mathbb{Z}_\ell(n)) &= \text{im} \left(\Gamma_* : H^1(Z, \mathbb{Z}_\ell(1)) \rightarrow H^{2n-1}(X_{\bar{K}}, \mathbb{Z}_\ell(n)) \right) \\ \tilde{N}^{n-1}H^{2n-1}(X_{\bar{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)) &= \text{im} \left(\Gamma_* : H^1(Z, \mathbb{Q}_\ell/\mathbb{Z}_\ell(1)) \rightarrow H^{2n-1}(X_{\bar{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)) \right). \end{aligned}$$

Then the image of the bottom row of (2.2) with \mathbb{Z}_ℓ -coefficients is

$$\begin{aligned} T_\ell \tilde{N}^{n-1}H^{2n-1}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n)) &= T_\ell \Gamma_* H_1(Z, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \\ &\supseteq \Gamma_* T_\ell H_1(Z, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \\ &= \Gamma_* H_1(Z, \mathbb{Z}_\ell) \\ &= \tilde{N}^{n-1}H^{2n-1}(X_{\bar{K}}, \mathbb{Z}_\ell(n))_\tau. \end{aligned}$$

From Lemma A.16, the inclusion above has torsion cokernel. This gives the assertion with \mathbb{Q}_ℓ -coefficients. \square

2.2. The image of the p -adic Bloch map. Now let K be a perfect field of characteristic $p > 0$. Here we secure Proposition 2.1 for p -adic cohomology groups.

Remark 2.5. Lemma 2.3 ([Suuw88, Lem. 3.2]) holds as well for p -power torsion. The proof is essentially identical. Briefly, suppose K is finite; then $A^1(C_{\bar{K}}) = \text{Pic}_{C/K}^0(\bar{K})$ is a torsion group. Write the order of β as $p^m M$ with M relatively prime to p , and choose N with $NM \equiv 1 \pmod{p^v}$. Then $\gamma := NM\beta$ has order p^m , and $\Gamma_*\gamma = NM\alpha = \alpha$. To account for arbitrary K , we may assume that K is the perfection of a field K_0 which is finitely generated over \mathbb{F}_p , and then spread out the data X, Γ, C, α and β to an irreducible scheme S of finite type over \mathbb{F}_p , with function field K_0 ; let k be the algebraic closure of \mathbb{F}_p in K_0 . Consider the finite flat group scheme $G := \text{Pic}_{C/S}^0[p^v] \rightarrow S$, and let $Q \in S$ be a closed point. Because the residue field of Q is finite, possibly after base-change by a finite extension of k , there exists some $\gamma_Q \in G_Q(k) = A^1(C_Q)$ such that $\Gamma_{Q,*}\gamma_Q = \alpha_Q$. Possibly after replacing S with an open neighborhood of Q , there exists a finite surjective $T \rightarrow S$ such that $G_T \rightarrow T$ admits a section $\gamma \in G(T)$ passing through some pre-image of γ_Q in $G_Q \times_S T$ [Gro66, 14.5.10]; the generic fiber of γ is the sought-for class.

Proof of Proposition 2.1, char(K)-torsion. The assertion of the inclusion in Proposition 2.1 with $\Lambda = \mathbb{Q}_p/\mathbb{Z}_p$ (for arbitrary n) follows immediately from diagram (2.2). For the opposite inclusion, the argument is identical. For the inclusion of Proposition 2.1 with \mathbb{Z}_p -coefficients, one uses diagram (2.2) with $\Lambda = \mathbb{W}$, and the fact that taking F -invariants commutes with push-forward [Gro85, Cor. I.3.2.7]. Tensoring with \mathbb{Q}_p gives the inclusion with \mathbb{Q}_p -coefficients. One then argues identically that the inclusion with \mathbb{Z}_p -coefficients has torsion quotient. Indeed, taking Tate

modules of (2.2) with $\mathbb{Q}_p/\mathbb{Z}_p$ -coefficients, one sees that the image of the bottom row of (2.2) with \mathbb{Z}_p -coefficients is $\tilde{N}^{n-1}H^{2n-1}(X_{\bar{K}}, \mathbb{Q}_p/\mathbb{Z}_p(n))$. As before, since $H^1(Z, \mathbb{Q}_p/\mathbb{Z}_p(1))$ is a divisible group for any smooth projective variety Z over \bar{K} , and since an increasing chain of divisible abelian p -torsion subgroups of a finite corank abelian p -torsion group is stationary [Suu88, Lem. 1.2], then by adding connected components to Z , one can conclude there is a smooth surface Z/\bar{K} , possibly disconnected but of finite type, equipped with a morphism $f : Z \rightarrow X_{\bar{K}}$ such that $f_*H^1(Z, \mathbb{Q}_p/\mathbb{Z}_p(1)) = \tilde{N}^{n-1}H^{2n-1}(X_{\bar{K}}, \mathbb{Q}_p/\mathbb{Z}_p(n))$ and $f_*H^1(Z/\mathbb{W}(1)) = \tilde{N}^{n-1}H^{2n-1}(X_{\bar{K}}/\mathbb{W}(n))$. We then have

$$\begin{aligned}
T_p \tilde{N}^{n-1} H^{2n-1}(X_{\bar{K}}, \mathbb{Q}_p/\mathbb{Z}_p(n)) &= T_p f_* H^1(Z, \mathbb{Q}_p/\mathbb{Z}_p(1)) \\
&\supseteq f_* T_p H^1(Z, \mathbb{Q}_p/\mathbb{Z}_p(1)) \\
&= f_* H^1(Z, \mathbb{Z}_p(1)) \\
&= f_*(H^1(Z/\mathbb{W}(1))^F) \\
&= (f_* H^1(Z/\mathbb{W}(1)))^F \text{ by [Gro85, Cor. I.3.2.7]} \\
&= (\tilde{N}^{n-1} H^{2n-1}(X_{\bar{K}}/\mathbb{W}(n)))^F = \tilde{N}^{n-1} H^{2n-1}(X_{\bar{K}}, \mathbb{Z}_p(n))_{\tau}.
\end{aligned}$$

Since the inclusion has torsion cokernel (see Lemma A.16 and Remark A.20), we are done. \square

3. DECOMPOSITION OF THE DIAGONAL, ALGEBRAIC REPRESENTATIVES, AND MINIVERSAL CYCLES

In [ACMVa] we consider decomposition of the diagonal and algebraic representatives in detail. These ideas also come into play in the proofs of Theorems 8, 12 and 15, and so we review these notions in this section. We refer the reader to [ACMVa] for more details.

3.1. Decomposition of the diagonal. The aim of this subsection is to fix the notation for decomposition of the diagonal, and to recall the existence of decompositions of the diagonal for various flavors of rational varieties.

Let K be a field and let ℓ be a prime not equal to $\text{char } K$. Let R be a commutative ring, let X be a smooth projective variety over a field K and let W_1 and W_2 be two closed subschemes of X not containing any component of X . A cycle class $Z \in \text{CH}^{d_X}(X \times_K X) \otimes_{\mathbb{Z}} R$ is said to admit a *decomposition of type* (W_1, W_2) if

$$Z = Z_1 + Z_2 \in \text{CH}^{d_X}(X \times_K X) \otimes_{\mathbb{Z}} R,$$

where $Z_1 \in \text{CH}^{d_X}(X \times_K X) \otimes_{\mathbb{Z}} R$ is supported on $W_1 \times_K X$ and $Z_2 \in \text{CH}^{d_X}(X \times_K X) \otimes_{\mathbb{Z}} R$ is supported on $X \times_K W_2$. If $Z = \Delta_X$ and if one can choose W_2 with $\dim W_2 = 0$, we simply say that $\Delta_X \in \text{CH}^d(X \times_K X) \otimes R$ has a Chow decomposition.

Definition 3.1 (Universal support of $\text{CH}_0 \otimes R$). Let X be a smooth projective variety over a field K . We say that $\text{CH}_0(X) \otimes R$ is *universally supported* in dimension d if there exists a closed subscheme $W_2 \subseteq X$ of dimension $\leq d$ such that $\text{CH}_0(X) \otimes R$ is universally supported on W_2 , i.e., if the push-forward map $\text{CH}_0((W_2)_L) \otimes_{\mathbb{Z}} R \rightarrow \text{CH}_0(X_L) \otimes_{\mathbb{Z}} R$ is surjective for all field extensions L/K .

The following proposition is classical and goes back to Bloch and Srinivas [BS83], and relates decomposition of the diagonal to the universal support of CH_0 :

Proposition 3.2 (Bloch–Srinivas [BS83]). *The diagonal $\Delta_X \in \text{CH}^{d_X}(X \times_K X) \otimes_{\mathbb{Z}} R$ of a smooth projective variety X over K admits a decomposition of type (W_1, W_2) if and only if $\text{CH}_0(X) \otimes_{\mathbb{Z}} R$ is universally*

supported on W_2 . In particular, taking $R = \mathbb{Z}$, the diagonal $\Delta_X \in \mathrm{CH}^{d_X}(X \times_K X)$ of a smooth projective, stably rational, variety X admits an decomposition of type (W_1, P) for any choice of K -point $P \in X(K)$. \square

Remark 3.3. The existence of a K -point on a stably rational variety over K is ensured by the Lang–Nishimura theorem [Nis55]; see [RY00, Prop. A.6] for a modern treatment.

Remark 3.4 (Varieties admitting decompositions of the diagonal). Let X and Y be smooth projective varieties over K of respective dimension $d_Y \leq d_X$. If X and Y are *stably rationally equivalent*, i.e., if there exists a nonnegative integer n such that $Y \times_K \mathbb{P}_K^n$ is birational to $X \times_K \mathbb{P}_K^{n+d_X-d_Y}$, then $\mathrm{CH}_0(X)$ is universally supported in dimension d_Y . In particular, if X is stably rational, then $\mathrm{CH}_0(X)$ is universally supported in dimension 0, and in fact on a point by the Lang–Nishimura theorem. Moreover, if X is only assumed to be geometrically rationally chain connected, then $\mathrm{CH}_0(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ is universally supported in dimension 0 (see e.g. [ACMVa, Rem. 2.8]).

Notation 3.5 (Decomposition of the diagonal and alterations). Let X be a pure-dimensional smooth projective variety over a perfect field K of characteristic exponent p . Suppose we have a cycle class $Z_1 + Z_2$ in $\mathrm{CH}^{d_X}(X \times_K X)$ with Z_1 supported on $W_1 \times_K X$ and Z_2 supported on $X \times_K W_2$ with W_1 and W_2 two closed subschemes of X not containing any component of X such that $\dim W_1 \leq n_1$ and $\dim W_2 \leq n_2$. By [Tem17], there exist alterations $\tilde{W}_1 \rightarrow W_1$ and $\tilde{W}_2 \rightarrow W_2$ of degree some power of p such that \tilde{W}_1 and \tilde{W}_2 are smooth projective over K . The cycle classes Z_1 and Z_2 , seen as self-correspondences on X , factor up to inverting p through \tilde{W}_1 and \tilde{W}_2 , respectively. Precisely, there exists a nonnegative integer e such that

$$p^e Z_1 = r_1 \circ s_1 \quad \text{and} \quad p^e Z_2 = r_2 \circ s_2 \quad \text{in } \mathrm{CH}^{d_X}(X \times_K X)$$

for some $s_1 \in \mathrm{CH}^{d_X}(X \times_K \tilde{W}_1)$, $r_1 \in \mathrm{CH}_{d_X}(\tilde{W}_1 \times_K X)$, $s_2 \in \mathrm{CH}_{d_X}(X \times_K \tilde{W}_2)$ and $r_2 \in \mathrm{CH}^{d_X}(\tilde{W}_2 \times_K X)$. Note that by replacing each component of \tilde{W}_1 and \tilde{W}_2 with a product with projective space of an appropriate dimension, we may assume that \tilde{W}_1 and \tilde{W}_2 are of pure dimension n_1 and n_2 , respectively. We refer to [ACMVa, §3.1] for more details. Since resolution of singularities exists for threefolds over a perfect field [CP09], if each $\dim W_i \leq 3$, then we may take $e = 0$.

3.2. Surjective regular homomorphisms and algebraic representatives. The aim of this subsection is to fix notation for algebraic representatives. We start by reviewing the definition of an algebraic representative (i.e., [Mur85, Def. 1.6.1] or [Sam60, 2.5]). Let X be a smooth projective variety over a perfect field K and let n be a nonnegative integer. For a smooth separated scheme T of finite type over K , we define $\mathcal{A}_{X/K}^n(T)$ to be the abelian group consisting of those cycle classes $Z \in \mathrm{CH}^n(T \times_K X)$ such that for every $t \in T(\bar{K})$ the Gysin fiber Z_t is algebraically trivial. For $Z \in \mathcal{A}_{X/\bar{K}}^n(T)$ denote by $w_Z : T(\bar{K}) \rightarrow A^n(X_{\bar{K}})$ the map defined by $w_Z(t) = Z_t$.

Given an abelian variety A/\bar{K} , a *regular homomorphism* (in codimension n)

$$\phi : A^n(X_{\bar{K}}) \longrightarrow A(\bar{K})$$

is a homomorphism of groups such that for every $Z \in \mathcal{A}_{X/K}^n(T)$ the composition

$$T(\bar{K}) \xrightarrow{w_Z} A^n(X_{\bar{K}}) \xrightarrow{\phi} A(\bar{K})$$

is induced by a morphism of varieties $\psi_Z : T_{\bar{K}} \rightarrow A$. An *algebraic representative* (in codimension n) is a regular homomorphism

$$\phi_{X_{\bar{K}}}^n : A^n(X_{\bar{K}}) \longrightarrow \mathrm{Ab}_{X_{\bar{K}}/\bar{K}}^n(\bar{K})$$

that is initial among all regular homomorphisms. For $n = 1$, an algebraic representative is given by $(\mathrm{Pic}_{X_{\bar{K}}/\bar{K}}^0)_{\mathrm{red}}$ together with the Abel–Jacobi map. For $n = d_X$, an algebraic representative is given

by the Albanese variety and the Albanese map. For $n = 2$, it is a result of Murre [Mur85, Thm. A] that there exists an algebraic representative for $X_{\bar{K}}$, which in the case $K = \mathbb{C}$ is the algebraic intermediate Jacobian $J_a^3(X)$; *i.e.*, the image of the Abel–Jacobi map restricted to algebraically trivial cycle classes.

The main result of [ACMV17] (see also [ACMVb]) is that if there exists an algebraic representative $\phi_{X_{\bar{K}}}^n : A^n(X_{\bar{K}}) \rightarrow \text{Ab}_{X_{\bar{K}}/\bar{K}}^n(\bar{K})$, then $\text{Ab}_{X_{\bar{K}}/\bar{K}}^n$ admits a canonical model over K , denoted $\text{Ab}_{X_K/K}^n$, such that $\phi_{X_{\bar{K}}}^n$ is $\text{Gal}(\bar{K}/K)$ -equivariant and such that for any $Z \in \mathcal{A}_{X_K/K}^n(T)$ the morphism $\psi_{Z_{\bar{K}}} : T_{\bar{K}} \rightarrow A_{\bar{K}}$ descends to a morphism $\psi_Z : T \rightarrow A$ of K -schemes. In particular, the algebraic representative $\text{Ab}_{X_{\bar{K}}/\bar{K}}^2$ of [Mur85] admits a canonical model over K , denoted $\text{Ab}_{X_K/K}^2$. In the case $K \subseteq \mathbb{C}$, the abelian variety $\text{Ab}_{X_K/K}^2$ is the distinguished model $J_{a,X/K}^3$ of the algebraic intermediate Jacobian, as defined in [ACMV20].

We include the following lemma for clarity; we also note that it is clear from the definitions that an algebraic representative $\phi_{X_{\bar{K}}}^n : A^n(X_{\bar{K}}) \rightarrow \text{Ab}_{X_{\bar{K}}/\bar{K}}^n(\bar{K})$ is a *surjective* regular homomorphism.

Lemma 3.6. *Let $\phi : A^n(X_{\bar{K}}) \rightarrow A(\bar{K})$ be a surjective regular homomorphism.*

- (1) *Let l be prime. Then :*
 - (a) $\phi[l^\infty] : A^n(X_{\bar{K}})[l^\infty] \rightarrow A[l^\infty]$ *is surjective.*
 - (b) *The following are equivalent :*
 - (i) $\phi[l^\infty] : A^n(X_{\bar{K}})[l^\infty] \rightarrow A[l^\infty]$ *is an isomorphism.*
 - (ii) $\phi[l^\infty] : A^n(X_{\bar{K}})[l^\infty] \rightarrow A[l^\infty]$ *is an inclusion.*
 - (iii) $\phi[l^\nu] : A^n(X_{\bar{K}})[l^\nu] \rightarrow A[l^\nu]$ *is an inclusion for all natural numbers ν .*
 - (iv) $\phi[l] : A^n(X_{\bar{K}})[l] \rightarrow A[l]$ *is an inclusion.*
 - (c) *If any of the equivalent conditions in (1)(b) hold, then $T_l\phi : T_l A^n(X_{\bar{K}}) \rightarrow T_l A$ is an isomorphism.*
- (2) *There exists a natural number e (independent of l) such that for all natural numbers ν the image of the map $\phi[l^{\nu+e}] : A^n(X_{\bar{K}})[l^{\nu+e}] \rightarrow A[l^{\nu+e}]$ contains $A[l^\nu] \subseteq A[l^{\nu+e}]$.*
- (3) *For all but finitely many primes l the map $\phi[l^\nu] : A^n(X_{\bar{K}})[l^\nu] \rightarrow A[l^\nu]$ is surjective.*

Proof. (1)(a) is [ACMV20, Rem. 3.3]. (Even though it is only claimed there for K of characteristic zero, the arguments of the references cited there are valid for arbitrary torsion in arbitrary characteristic.) The equivalence of (1)(b)(i) and (1)(b)(ii) is obvious. To show the equivalence of (1)(b)(ii) and (1)(b)(iii) we argue as follows. For a group G , there is an inclusion $G[l^\nu] \subseteq G[l^{\nu+1}]$, so that in our situation, we have $A^n(X_{\bar{K}})[l^\nu] \subseteq A^n(X_{\bar{K}})[l^\infty]$ and $A[l^\nu] \subseteq A[l^\infty]$. The equivalence of (1)(b)(ii) and (1)(b)(iii) then follows by a diagram chase; the equivalence of (1)(b)(iii) and (1)(b)(iv) is elementary. One obtains (1)(c) by applying the Tate module to the isomorphism (1)(b)(i). Item (3) follows from Item (2), which in turn is [ACMV20, Rem. 3.3]. \square

Remark 3.7. It is worth noting that if there exists a regular homomorphism $\phi : A^n(X_{\bar{K}}) \rightarrow A(\bar{K})$ such that $\phi[l^\infty]$ is injective, then there is an algebraic representative in codimension- n ; this follows directly from Saito’s criterion ([Mur85, Prop. 2.1]).

3.3. Miniversal cycles and miniversal cycles of minimal degree. Let X be a smooth projective variety over a field K and let $\phi : A^n(X_{\bar{K}}) \rightarrow A(\bar{K})$ be a regular homomorphism. A *miniversal cycle* for ϕ is a cycle $Z \in \mathcal{A}_{X_K/K}^n(A)$ such that the homomorphism $\psi_Z : A \rightarrow A$ is given by multiplication by r for some natural number r , which we call the *degree* of the cycle. A miniversal cycle is called *universal* if $\psi_Z : A \rightarrow A$ is given by the identity. In case ϕ is an algebraic representative for codimension- n cycles on X , we call a universal cycle for ϕ a universal cycle in codimension- n for X . Clearly there is a minimal r such that there exists a miniversal cycle of degree r ; taking

linear combinations of miniversal cycles, one can see this minimum is achieved by the GCD of all of the degrees of miniversal cycles.

If K is algebraically closed, it is a classical and crucial fact [Mur85, 1.6.2 & 1.6.3] that a miniversal cycle exists if and only if ϕ is surjective; this also holds without any restrictions on the field K by [ACMVb, Lem. 4.7]. In particular, since an algebraic representative is always a surjective regular homomorphism [ACMVb, Prop. 5.1], it always admits a miniversal cycle. However, the existence of a universal cycle is restrictive. For example, both over the complex numbers [Voi15] and over a field of characteristic at least three [ACMVa], the standard desingularization of the very general double quartic solid with 7 nodes does not admit a universal cycle in codimension-2.

3.4. Decomposition of the diagonal and algebraic representatives. We now recall a result due to [Mur85] and [BS83]:

Proposition 3.8 (Murre [Mur85], Bloch–Srinivas [BS83]). *Let X be a smooth projective variety over a perfect field K of characteristic exponent p .*

(1) *If $\text{char}(K) = 0$, then*

$$\phi_{X_{\bar{K}}}^2[\ell^\infty] : A^2(X_{\bar{K}})[\ell^\infty] \longrightarrow \text{Ab}_{X/K}^2[\ell^\infty](\bar{K}). \quad (3.1)$$

is an isomorphism of $\text{Gal}(K)$ -modules for all prime numbers ℓ .

(2) *Assume that the diagonal $\Delta_{X_{\bar{K}}} \in \text{CH}^{d_X}(X_{\bar{K}} \times_{\bar{K}} X_{\bar{K}}) \otimes \mathbb{Q}$ admits a decomposition of type (W_1, W_2) with $\dim W_2 \leq 1$. Then*

$$\phi_{X_{\bar{K}}}^2 : A^2(X_{\bar{K}}) \longrightarrow \text{Ab}_{X/K}^2(\bar{K})$$

is an isomorphism of $\text{Gal}(K)$ -modules.

(3) *Assume that the diagonal $N\Delta_{X_{\bar{K}}} \in \text{CH}^{d_X}(X_{\bar{K}} \times_{\bar{K}} X_{\bar{K}})$ admits a decomposition of type (W_1, W_2) with $\dim W_2 \leq 2$ for some positive integer N . Then*

$$V_l \phi_{X_{\bar{K}}}^2 : V_l A^2(X_{\bar{K}}) \longrightarrow V_l \text{Ab}_{X/K}^2$$

is an isomorphism of $\text{Gal}(K)$ -modules for all primes l , and

$$T_\ell \phi_{X_{\bar{K}}}^2 : T_\ell A^2(X_{\bar{K}}) \longrightarrow T_\ell \text{Ab}_{X/K}^2$$

is an isomorphism of $\text{Gal}(K)$ -modules for all prime numbers ℓ not dividing Np .

(4) *In the setting of (3), further assume that $p \geq 2$, resolution of singularities holds in dimensions $< d_X$, and $p \nmid N$. Then*

$$T_p \phi_{X_{\bar{K}}}^2 : T_p A^2(X_{\bar{K}}) \longrightarrow T_p \text{Ab}_{X/K}^2$$

is an isomorphism of $\text{Gal}(K)$ -modules.

Proof. First recall from [ACMV17] that $\phi_{X_{\bar{K}}}^2$ is $\text{Gal}(K)$ -equivariant. Item (2) is [BS83, Thm. 1(i)], while Item (1) reduces via [ACMV17] to the case $K = \mathbb{C}$, which is covered by [Mur85, Thm. 10.3].

We now prove Item (3) and assume that $\text{char}(K) = p > 0$. With Notation 3.5, we have a commutative diagram with composition of horizontal arrows being multiplication by Np^e :

$$\begin{array}{ccccc} A^2(X_{\bar{K}}) & \xrightarrow{r_1^* \oplus r_2^*} & A^1(\tilde{W}_1) \oplus A^2(\tilde{W}_2) & \xrightarrow{s_1^* + s_2^*} & A^2(X_{\bar{K}}) \\ \downarrow \phi_{X_{\bar{K}}}^2 & & \downarrow \phi_{\tilde{W}_1}^1 \oplus \phi_{\tilde{W}_2}^2 & & \downarrow \phi_{X_{\bar{K}}}^2 \\ \text{Ab}_{X/K}^2(\bar{K}) & \longrightarrow & \text{Pic}_{\tilde{W}_1}^0(\bar{K}) \oplus \text{Alb}_{\tilde{W}_2}(\bar{K}) & \longrightarrow & \text{Ab}_{X/K}^2(\bar{K}). \end{array} \quad (3.2)$$

where the bottom horizontal arrows are the \bar{K} -homomorphisms induced by the universal property of algebraic representatives. (Note that since $\dim \tilde{W}_2 = 2$ we have identified $\phi_{\tilde{W}_2}^2$ with the Albanese

map.) Since ϕ^1 is an isomorphism and since the Albanese morphism is an isomorphism on torsion by Rojzman [Blo79, GS88, Mil82], a simple diagram chase establishes that $\phi_X^2 : A^2(X_{\bar{K}}) \rightarrow \text{Ab}_X^2(\bar{K})$ is an isomorphism on prime-to- Np^e torsion. It follows that ϕ_X^2 is an isomorphism on l -primary torsion for all primes l not dividing Np^e . The statement about $T_l\phi_{X_{\bar{K}}}^2$ then ensues by passing to the inverse limit. Alternately, since the middle vertical arrow is an isomorphism on torsion, it is an isomorphism on Tate modules. By applying T_l to (3.2), and since $T_l A^2(X_{\bar{K}})$ and $T_l \text{Ab}_{X/K}^2$ are finite free \mathbb{Z}_l -modules ($T_l \lambda^2 : T_l A^2(X_{\bar{K}}) \hookrightarrow T_l \text{Ab}_{X/K}^2$ is injective, Proposition A.27), we directly see that $T_l \phi_{X_{\bar{K}}}^2$ is an isomorphism for $l \nmid Np$ and we also see, after tensoring with \mathbb{Q}_l that $V_l \phi_{X_{\bar{K}}}^2$ is an isomorphism for all primes l .

For (4), it suffices to observe that, if W_1 and W_2 admit resolutions of singularities (which is the case if $d_X \leq 4$ by [CP09]), then we may take $e = 0$ above. \square

Remark 3.9. By rigidity, the same results in Proposition 3.8 hold if the separable closure \bar{K} is replaced with the algebraic closure \bar{K}^a .

Proposition 3.10 (Decomposition of the diagonal and miniversal cycle classes). *Let X be a smooth projective variety over a field K that is either finite or algebraically closed, and let N be a natural number. Assume that $N\Delta_{X_{\bar{K}}} \in \text{CH}^{d_X}(X \times_K X)$ admits a decomposition of type (W_1, W_2) with $\dim W_2 \leq 1$. Then $\text{Ab}_{X/K}^2$ admits a miniversal cycle of degree $p^e N$ for some nonnegative integer e that may be chosen to be zero if $\dim X \leq 4$.*

In particular, if $\dim X \leq 4$ and if $\text{CH}_0(X)$ is universally supported in dimension 1, then $\text{Ab}_{X/K}^2$ admits a universal cycle.

Proof. Similarly to (3.2), we have with Notation 3.5 a commutative diagram with composition of horizontal arrows being multiplication by Np^e (where, due to resolution of singularities in dimensions < 4 , e can be chosen to be zero if $\dim X \leq 4$):

$$\begin{array}{ccccc} A^2(X_{\bar{K}}) & \xrightarrow{r_1^*} & A^1(\tilde{W}_1) & \xrightarrow{s_1^*} & A^2(X_{\bar{K}}) \\ \downarrow \phi_X^2 & & \downarrow \phi_{\tilde{W}_1}^1 & & \downarrow \phi_X^2 \\ \text{Ab}_{X/K}^2(\bar{K}) & \xrightarrow{r_1^*} & \text{Pic}_{\tilde{W}_1}^0(\bar{K}) & \xrightarrow{s_1^*} & \text{Ab}_{X/K}^2(\bar{K}), \end{array}$$

where $r_1^* : \text{Ab}_{X/K}^2 \rightarrow (\text{Pic}_{\tilde{W}_1/K}^0)_{\text{red}}$ and $s_1^* : (\text{Pic}_{\tilde{W}_1/K}^0)_{\text{red}} \rightarrow \text{Ab}_{X/K}^2$ denote the K -homomorphisms induced by the correspondences r_1 and s_1 . Since we are assuming K to be either finite or algebraically closed, the Abel–Jacobi map $\phi_{\tilde{W}_1}^1$ admits a universal divisor $\tilde{\mathcal{D}} \in \mathcal{A}_{\tilde{W}_1/K}^1((\text{Pic}_{\tilde{W}_1/K}^0)_{\text{red}})$ (see, e.g., [ACMVb, §7.1]), meaning that the induced morphism

$$\psi_{\tilde{\mathcal{D}}} : (\text{Pic}_{\tilde{W}_1/K}^0)_{\text{red}} \rightarrow (\text{Pic}_{\tilde{W}_1/K}^0)_{\text{red}}$$

is the identity. It is then clear that the homomorphism associated to the cycle class

$$Z := s_1^* \circ \tilde{\mathcal{D}} \circ r_1^* \in \mathcal{A}_{X/K}^n(\text{Ab}_{X/K}^n).$$

is given by $Np^e \text{Id}_{\text{Ab}_{X/K}^2}$. \square

4. MINIVERSAL CYCLES AND THE IMAGE OF THE SECOND l -ADIC BLOCH MAP

In this section, we prove our main Theorem 4.2. As an immediate consequence of the existence of an algebraic representative for codimension-2 cycles (see §3.2), we obtain proofs of Theorem 8 and of Theorem 10. We start with a lemma that parallels Lemma 2.3.

Lemma 4.1. *Let X be a smooth projective variety over an algebraically closed field $K = \bar{K}^a = \bar{K}$ and let l be a prime. Let $\phi : A^n(X_{\bar{K}}) \rightarrow A(\bar{K})$, be a surjective regular homomorphism, and let $\Gamma \in \mathcal{A}_{X/K}^n(A)$ be a miniversal cycle of degree r . If*

$$\phi[l^\infty] : A^n(X_{\bar{K}})[l^\infty] \rightarrow A[l^\infty]$$

is an inclusion, then

$$l^{v_l(r)} \cdot T_l A^n(X_{\bar{K}}) \subseteq \text{im}(\Gamma_* : T_l A_0(A) \rightarrow T_l A^n(X_{\bar{K}})),$$

where $v_l(r)$ is the l -adic valuation of r .

Proof. By the definition of a miniversal cycle and its degree, the composition

$$A(\bar{K}) \longrightarrow A_0(A) \xrightarrow{\Gamma_*} A^n(X_{\bar{K}}) \xrightarrow{\phi} A(\bar{K})$$

is multiplication by r . Here, the map $A(\bar{K}) \rightarrow A_0(A)$ is the map of sets $a \mapsto [a] - [0]$. Recall the general fact about abelian varieties, due to Beauville, that the map of sets $A(\bar{K}) \rightarrow A_0(A_{\bar{K}})$, $a \mapsto [a] - [0]$ is an isomorphism on torsion (see [ACMV20, Lem. 3.3] for references, and recall that Beauville's argument works for arbitrary torsion in arbitrary characteristic). Therefore, restricting to l -primary torsion, we get a composition of homomorphisms

$$A[l^\infty] \xrightarrow{\simeq} A_0(A)[l^\infty] \xrightarrow{\Gamma_*} A^n(X_{\bar{K}})[l^\infty] \xrightarrow{\phi[l^\infty]} A[l^\infty]$$

which is given by multiplication by r . Here $\phi[l^\infty]$ is an isomorphism due to Lemma 3.6. Passing to the inverse limit, we obtain a composition of homomorphisms

$$T_l A \xrightarrow{\simeq} T_l A_0(A) \xrightarrow{\Gamma_*} T_l A^n(X_{\bar{K}}) \xrightarrow{\simeq} T_l A$$

which is given by multiplication by r . It immediately follows that $l^{v_l(r)} T_l A^n(X_{\bar{K}})$ lies in the image of Γ_* . \square

Theorem 4.2. *Let X be a smooth projective variety over an algebraically closed field $K = \bar{K}^a = \bar{K}$ and let l be a prime. Let $\phi : A^n(X_{\bar{K}}) \rightarrow A(\bar{K})$, be a surjective regular homomorphism, and let $\Gamma \in \mathcal{A}_{X/K}^n(A)$ be a miniversal cycle of minimal degree r (see §3.3). Then the morphisms*

$$T_l A \xrightarrow{\Gamma_*} T_l A^n(X_{\bar{K}}) \xrightarrow{T_l \lambda^n} H^{2n-1}(X_{\bar{K}}, \mathbb{Z}_l(n))_\tau$$

induce inclusions

$$\text{im}(T_l \lambda^n \circ \Gamma_*) \subseteq \tilde{N}^{n-1} H^{2n-1}(X_{\bar{K}}, \mathbb{Z}_l(n))_\tau \subseteq \text{im}(T_l \lambda^n).$$

Moreover, if $\phi[l^\infty] : A^n(X_{\bar{K}})[l^\infty] \rightarrow A[l^\infty]$ is an inclusion, then in addition we have

$$l^{v_l(r)} \text{im}(T_l \lambda^n) \subseteq \text{im}(T_l \lambda^n \circ \Gamma_*).$$

In other words, if $\phi[l^\infty]$ is an inclusion, then $\text{im}(T_l \lambda^n)$ is an extension of $\tilde{N}^{n-1} H^{2n-1}(X_{\bar{K}}, \mathbb{Z}_l(n))_\tau$ by a finite l -primary torsion group killed by multiplication by $l^{v_l(r)}$. In particular, if l does not divide r , then

$$\tilde{N}^{n-1} H^{2n-1}(X_{\bar{K}}, \mathbb{Z}_l(n))_\tau = \text{im}(T_l \lambda^n).$$

Proof. The inclusion $\tilde{N}^{n-1} H^{2n-1}(X_{\bar{K}}, \mathbb{Z}_l(n))_\tau \subseteq \text{im}(T_l \lambda^n)$ is Proposition 2.1 (due to Suwa).

Let us now show $\text{im}(T_l \lambda^n \circ \Gamma_*) \subseteq \tilde{N}^{n-1} H^{2n-1}(X_{\bar{K}}, \mathbb{Z}_l(n))_\tau$. For that purpose, consider the commutative diagram

$$\begin{array}{ccccccc}
T_l A_0(A_{\bar{K}}) \hookrightarrow & \bigoplus_{\Gamma': Z \dashrightarrow X} T_l A_0(Z_{\bar{K}}) & \xrightarrow{=} & \bigoplus_{\Gamma': Z \dashrightarrow X} T_l \text{CH}_0(Z_{\bar{K}}) & \xrightarrow[\simeq]{T_l \lambda_0} & \bigoplus_{\Gamma': Z \dashrightarrow X} H_1(Z_{\bar{K}}, \mathbb{Z}_l)_\tau & (4.1) \\
& \searrow \Gamma_* & & \downarrow \Gamma'_* & & \downarrow \Gamma'_* & \\
& & & T_l A^n(X_{\bar{K}}) \hookrightarrow & \longrightarrow & T_l \text{CH}^n(X_{\bar{K}}) & \xrightarrow{T_l \lambda^n} & H^{2n-1}(X_{\bar{K}}, \mathbb{Z}_l(n))_\tau
\end{array}$$

where the direct sums run through all smooth projective varieties Z over K and all correspondences $\Gamma' \in \text{CH}^{d_X - n + 1}(Z \times_K X)$. By definition, the image of the right vertical arrow consists of $\tilde{N}^{n-1} H^{2n-1}(X_{\bar{K}}, \mathbb{Z}_l(n))_\tau$, completing the proof via a diagram chase.

Finally, under the assumption that $\phi[l^\infty] : A^n(X_{\bar{K}})[l^\infty] \rightarrow A[l^\infty]$ is an isomorphism, the assertion $l^{v_l(r)} \text{im}(T_l \lambda^n) \subseteq \text{im}(T_l \lambda \circ \Gamma_*)$ follows from Lemma 4.1. \square

Proof of Theorems 8 and 10. Recall from §3.2 that an algebraic representative for codimension-2 cycles $\phi_{X_{\bar{K}}}^2 : A^2(X_{\bar{K}}) \rightarrow \text{Ab}_{X_{\bar{K}}/\bar{K}}^2(\bar{K})$ always exists. Both Theorems 8 and 10 are then a special case of Theorem 4.2. \square

Since an algebraic representative always exists for codimension-2 cycles (see §3.2), in order to prove Theorem 4.2 unconditionally for the algebraic representative in codimension-2, it suffices to show the standard assumption holds. The following lemma allows us to reduce to assuming the standard assumption holds for varieties over finite fields:

Lemma 4.3 (Standard assumption and generization). *Let S be the spectrum of a discrete valuation ring with generic point $\eta = \text{Spec } K$ and closed point $\circ = \text{Spec } \kappa$. Let X/S be a smooth projective scheme, and let $\Gamma \in \mathcal{A}_{X_\eta/K}^2(\text{Ab}_{X_\eta/K}^2)$ be a miniversal cycle of minimal degree r . For all primes $\ell \nmid r \cdot \text{char}(K)$, if X_\circ satisfies the standard assumption at ℓ (i.e., $\phi_{X_\circ/\kappa}^2[l^\infty]$ is an isomorphism), then X_η satisfies the standard assumption at ℓ (i.e., $\phi_{X_\eta/K}^2[l^\infty]$ is an isomorphism).*

Proof. By [ACMVb, Thm. 8.3] we have $(\text{Ab}_{X/S}^2)_\eta \simeq \text{Ab}_{X_\eta/\eta}^2$. Let $\Gamma_{X/S} \in \mathcal{A}_{X/S}^2(\text{Ab}_{X/S}^2)$ be a miniversal cycle of minimal degree r induced by the one in the assumption of the lemma (see [ACMVb, Lem. 4.7]). Its specialization induces a group homomorphism $w_{\Gamma_{X/S}, \circ} : \text{Ab}_{X/S}^2(\bar{\kappa}) \rightarrow A^2(X_{\circ, \bar{\kappa}})$, and thus a homomorphism $\psi_{\Gamma_{X/S}, \circ} : (\text{Ab}_{X/S}^2)_\circ \rightarrow \text{Ab}_{X_\circ/\circ}^2$.

On ℓ -primary torsion, we have a commutative diagram

$$\begin{array}{ccc}
\text{Ab}_{X/S}^2[l^\infty](\bar{K}) & \xrightarrow{\simeq} & (\text{Ab}_{X/S}^2)_\circ[l^\infty](\bar{\kappa}) & (4.2) \\
\downarrow w_{\Gamma_{X/S}} & & \downarrow w_{\Gamma_{X/S}, \circ} & \\
A^2(X_{\bar{K}})[l^\infty] & \hookrightarrow & A^2(X_\circ)[l^\infty] & \\
& & \downarrow \phi_{X_\circ/\kappa}^2[l^\infty] & \\
& & \text{Ab}_{X_\circ/\kappa}^2[l^\infty](\bar{\kappa}) & \\
& & \swarrow \psi_{\Gamma_{X/S}, \circ} &
\end{array}$$

Both the top and bottom horizontal arrows are the specialization maps; the fact that the specialization map on torsion cycle classes is injective in codimension 2 follows from the fact, due to Merkurjev–Suslin [MS82] (see also Propositions A.27 and A.30), that the second Bloch map is an inclusion. Choose a prime $\ell \neq \text{char}(K)$ relatively prime to r . Then $w_{\Gamma_{X/S}}$ and is injective, and by

commutativity, this implies $w_{\Gamma_{X/S, \circ}}$ is injective. Then since we assume that $\phi_{X_0/\kappa}^2[\ell^\infty]$ is injective, it follows that all arrows in (4.2) are injective.

We now complete diagram (4.2) by introducing a second copy of the isogeny $\psi_{\Gamma_{X/S, \circ}}$. Note that the bottom square does *not* commute, and the outer rectangle fails to commute by a factor of r :

$$\begin{array}{ccc}
\text{Ab}_{X/S}^2[\ell^\infty](\bar{K}) & \xrightarrow{\simeq} & (\text{Ab}_{X/S}^2)_\circ[\ell^\infty](\bar{\kappa}) \\
\downarrow w_{\Gamma_{X/S}} & & \downarrow w_{\Gamma_{X/S, \circ}} \\
\text{A}^2(X_{\bar{K}})[\ell^\infty] & \hookrightarrow & \text{A}^2(X_0)[\ell^\infty] \\
\downarrow \phi_{X_\eta/\bar{K}}^2 & \xrightarrow{\emptyset} & \downarrow \phi_{X_0/\bar{\kappa}} \\
(\text{Ab}_{X/S}^2)[\ell^\infty](\bar{K}) & \longrightarrow & \text{Ab}_{X_0/\kappa}^2[\ell^\infty](\bar{\kappa}) \\
\downarrow \simeq & \nearrow \psi_{\Gamma_{X/S, \circ}} & \\
\text{Ab}_{X/S}^2[\ell^\infty](\bar{\kappa}) & &
\end{array} \tag{4.3}$$

If $\ell \nmid r$, then the injectivity of $\phi_{X_0/\bar{\kappa}}^2$ implies the injectivity of $\phi_{X_\eta/\bar{K}}^2$. \square

Proposition 4.4. *Assume that for any finite field \mathbb{F} and any smooth projective variety Y of dimension d over $\bar{\mathbb{F}}$ we have that for all primes $\ell \neq \text{char}(\mathbb{F})$,*

$$\phi_{Y_{\bar{\mathbb{F}}}}^2[\ell^\infty] : \text{A}^2(Y_{\bar{\mathbb{F}}})[\ell^\infty] \rightarrow \text{Ab}_{Y_{\bar{\mathbb{F}}}/\bar{\mathbb{F}}}^2(\bar{\mathbb{F}})[\ell^\infty]$$

is an isomorphism; i.e., assume the standard assumption holds for varieties of dimension d over finite fields.

Let X be a smooth projective variety of dimension d over an algebraically closed field $K = \bar{K}^a = \bar{K}$, and let $\Gamma \in \mathcal{A}_{X/\bar{K}}^2(\text{Ab}_{X_{\bar{K}}/\bar{K}}^2)$ be a miniversal cycle of minimal degree r . Then for all primes $\ell \nmid r \cdot \text{char}(K)$, we have

$$\tilde{N}^1 H^3(X_{\bar{K}}, \mathbb{Z}_\ell(1))_\tau = \text{im}(T_\ell \lambda^2 : T_\ell \text{A}^2(X_{\bar{K}}) \hookrightarrow H^3(X_{\bar{K}}, \mathbb{Z}_\ell(2))_\tau).$$

Proof. By Proposition 3.8, we need only treat the case where K has characteristic $p > 0$. From Theorem 4.2 we only need to establish that X satisfies the standard assumptions for $\ell \nmid r \cdot \text{char}(K)$. Since X is of finite type over K , we may and do replace K with a field of finite transcendence degree n over the prime field \mathbb{F}_p . By spreading out and then taking successive hypersurface sections, we may conclude from our hypothesis on finite fields and Lemma 4.3.

We provide details for the sake of completeness. Spread X to a smooth scheme over $\text{Spec } R$, where R is a smooth \mathbb{F}_p -algebra of Krull dimension n with $\text{Frac}(R) = K$. Let $D \subset \text{Spec}(R)$ be a prime divisor with generic point η_D . Then D defines a discrete valuation on K , whose valuation ring $R_{\eta_D} \supset R$ has residue field isomorphic to the function field of D . By iterating this construction we obtain a sequence of ring surjections $R = R_n \twoheadrightarrow R_{n-1} \cdots \twoheadrightarrow R_0$ where $\dim R_j = j$, and for $j \geq 1$ the field $K_j := \text{Frac}(R_j)$ admits a discrete valuation with valuation ring $S_j \supset R_j$ and residue field isomorphic to K_{j-1} . By hypothesis, $X \times_{\text{Spec } R} \text{Spec } R_0$ satisfies the standard assumption at ℓ . By repeated invocation of Lemma 4.3 for $X \times_{\text{Spec } R} \text{Spec } S_j$ with $j = 1, 2, \dots, n$, we find that X_K satisfies the standard assumption at ℓ . \square

5. DECOMPOSITION OF THE DIAGONAL AND THE IMAGE OF THE SECOND ℓ -ADIC BLOCH MAP

The aim of this section is to establish Theorem 12. First, we have the following proposition that extends [BW20, Prop. 2.3(ii)] to the $\ell = p$ case.

Proposition 5.1 ([BW20, Prop. 2.3(ii)]). *Let X be a smooth projective variety over a perfect field K . Assume that $N\Delta_{X_{\bar{K}}} \in \text{CH}^{d_X}(X_{\bar{K}} \times_{\bar{K}} X_{\bar{K}})$ admits a decomposition of type (W_1, W_2) with $\dim W_2 \leq 1$. Then, for all primes l , the second Bloch map*

$$\lambda^2 : A^2(X_{\bar{K}})[l^\infty] \longrightarrow H^3(X_{\bar{K}}, \mathbb{Z}_l(2)) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l / \mathbb{Z}_l$$

and the second l -adic Bloch map

$$T_l \lambda^2 : T_l A^2(X_{\bar{K}}) \longrightarrow H^3(X_{\bar{K}}, \mathbb{Z}_l(2))_\tau$$

are isomorphism of $\text{Gal}(K)$ -modules.

Proof. The following argument is due to Benoist–Wittenberg for $l \neq \text{char}(K)$. We check that it holds for $l = \text{char}(K)$ as well. For X smooth and projective, we have a diagram with exact row (A.27):

$$\begin{array}{ccccccc} & & & & \text{CH}^2(X_{\bar{K}})[l^\infty] & & \\ & & & & \downarrow \lambda^2 & \dashrightarrow & \\ 0 & \longrightarrow & H^3(X_{\bar{K}}, \mathbb{Z}_l(2)) \otimes \mathbb{Q}_l / \mathbb{Z}_l & \longrightarrow & H^3(X_{\bar{K}}, \mathbb{Q}_l / \mathbb{Z}_l(2)) & \longrightarrow & H^4(X_{\bar{K}}, \mathbb{Z}_l(2)) \end{array}$$

where the dashed arrow is, up to sign, the cycle class map ([CTSS83, Cor. 4], [GS88, Prop. III.1.16 and Prop. III.1.21]). Since algebraically trivial cycles are homologically trivial, it follows that the image of $A^2(X_{\bar{K}})[l^\infty]$ under λ^2 is contained in $H^3(X_{\bar{K}}, \mathbb{Z}_l) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l / \mathbb{Z}_l \subseteq H^3(X_{\bar{K}}, \mathbb{Q}_l / \mathbb{Z}_l)$. In particular, the cokernel of $\lambda^2 : A^2(X_{\bar{K}})[l^\infty] \rightarrow H^3(X_{\bar{K}}, \mathbb{Z}_l) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l / \mathbb{Z}_l$ is divisible.

Now suppose $N\Delta_{X_{\bar{K}}} \in \text{CH}^{d_X}(X_{\bar{K}} \times_{\bar{K}} X_{\bar{K}})$ admits a decomposition of type (W_1, W_2) with $\dim W_2 \leq 1$. With Notation 3.5 we obtain by the naturality of the Bloch map (Proposition A.23) a commutative diagram

$$\begin{array}{ccccccc} A^2(X_{\bar{K}})[l^\infty] & \xrightarrow{r_1^*} & A^1(\tilde{W}_1)[l^\infty] & \xrightarrow{s_1^*} & A^2(X_{\bar{K}})[l^\infty] & & (5.1) \\ \downarrow \lambda^2 & & \simeq \downarrow \lambda^1 & & \downarrow \lambda^2 & & \\ H^3(X_{\bar{K}}, \mathbb{Z}_l(2)) \otimes \mathbb{Q}_l / \mathbb{Z}_l & \xrightarrow{r_1^*} & H^1(\tilde{W}_1, \mathbb{Z}_l(1)) \otimes \mathbb{Q}_l / \mathbb{Z}_l & \xrightarrow{s_1^*} & H^3(X_{\bar{K}}, \mathbb{Z}_l(2)) \otimes \mathbb{Q}_l / \mathbb{Z}_l. & & \end{array}$$

Note there is no \tilde{W}_2 term above for reasons of codimension. The middle vertical arrow in (5.1) is an isomorphism by Proposition A.28, while the composition of the horizontal arrows in (5.1) is multiplication by Np^e . It follows that $\text{coker } \lambda^2$ is torsion, annihilated by Np^e , and consequently that this cokernel is trivial, i.e., $\lambda^2 : A^2(X_{\bar{K}})[l^\infty] \rightarrow H^3(X_{\bar{K}}, \mathbb{Z}_l(2)) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l / \mathbb{Z}_l$ is an isomorphism.

Taking the Tate module of this isomorphism gives the result for the l -adic Bloch map, since $H^3(X_{\bar{K}}, \mathbb{Z}_l)$ is a finitely generated \mathbb{Z}_l -module, and so it is elementary to check that there is an identification $T_l(H^3(X_{\bar{K}}, \mathbb{Z}_l) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l / \mathbb{Z}_l) \simeq H^3(X_{\bar{K}}, \mathbb{Z}_l)_\tau$. \square

The following proposition establishes Theorem 12.

Proposition 5.2 (Theorem 12). *Let X be a smooth projective variety over a perfect field K of characteristic exponent p and let N be a natural number. Assume that $N\Delta_{X_{\bar{K}}} \in \text{CH}^{d_X}(X_{\bar{K}} \times_{\bar{K}} X_{\bar{K}})$ admits a decomposition of type (W_1, W_2) .*

- (1) *Assume $\dim W_2 \leq 2$. Suppose l is a prime such that $l \nmid N$, and such that either $l \neq \text{char}(K)$ or resolution of singularities holds in dimensions $< d_X$. Then the inclusion $\tilde{N}^1 H^3(X_{\bar{K}}, \mathbb{Z}_l(2)) \subseteq$*

$H^3(X_{\bar{K}}, \mathbb{Z}_l(2))$ is an equality, the inclusion $T_l A^2(X_{\bar{K}}) \hookrightarrow T_l \text{CH}^2(X_{\bar{K}})$ is an equality, and the second l -adic Bloch map

$$T_l \lambda^2 : T_l \text{CH}^2(X_{\bar{K}}) \longrightarrow H^3(X_{\bar{K}}, \mathbb{Z}_l(2))_{\tau}$$

is an isomorphism of $\text{Gal}(K)$ -modules.

- (2) Assume $\dim W_2 \leq 1$. Let l be any prime. Then the inclusion $T_l A^2(X_{\bar{K}}) \hookrightarrow T_l \text{CH}^2(X_{\bar{K}})$ is an equality, and the second l -adic Bloch map

$$T_l \lambda^2 : T_l \text{CH}^2(X_{\bar{K}}) \longrightarrow H^3(X_{\bar{K}}, \mathbb{Z}_l(2))_{\tau}$$

is an isomorphism of $\text{Gal}(K)$ -modules. Moreover, if $l \nmid N$ and if either $l \neq \text{char}(K)$ or resolution of singularities holds in dimensions $< d_X$, then $H^3(X_{\bar{K}}, \mathbb{Z}_l)$ is torsion-free.

Proof. We first assume that $l = \ell \neq p$. That $T_{\ell} \lambda^2$ is a morphism of $\text{Gal}(K)$ -modules is Proposition A.22 and that $T_{\ell} \lambda^2$ is injective in general is Proposition A.27.

Concerning item (1), with Notation 3.5 we obtain by the naturality of the ℓ -adic Bloch map (Proposition A.23) a commutative diagram

$$\begin{array}{ccccc} T_{\ell} A^2(X_{\bar{K}}) & \xrightarrow{r_1^* \oplus r_2^*} & T_{\ell} A^1(\tilde{W}_1) \oplus T_{\ell} A^2(\tilde{W}_2) & \xrightarrow{s_1^* + s_2^*} & T_{\ell} A^2(X_{\bar{K}}) \\ \downarrow T_{\ell} \lambda_X^2 & & \simeq \downarrow T_{\ell} \lambda_{\tilde{W}_1}^1 \oplus T_{\ell} \lambda_{\tilde{W}_2}^2 & & \downarrow T_{\ell} \lambda_X^2 \\ H^3(X_{\bar{K}}, \mathbb{Z}_{\ell}(2))_{\tau} & \xrightarrow{r_1^* \oplus r_2^*} & H^1(\tilde{W}_1, \mathbb{Z}_{\ell}(1)) \oplus H^3(\tilde{W}_2, \mathbb{Z}_{\ell}(2))_{\tau} & \xrightarrow{s_1^* + s_2^*} & H^3(X_{\bar{K}}, \mathbb{Z}_{\ell}(2))_{\tau}. \end{array} \quad (5.2)$$

The middle vertical arrow in (5.2) is an isomorphism by Propositions A.25 and A.26, while the composition of the horizontal arrows in (5.2) is multiplication by Np^e . In particular, the latter are bijective if ℓ does not divide Np^e . A diagram chase then establishes the surjectivity of $T_{\ell} \lambda_X^2$ restricted to algebraically trivial cycles, and hence the bijectivity of $T_{\ell} A^2(X_{\bar{K}}) \hookrightarrow T_{\ell} \text{CH}^2(X_{\bar{K}})$ and of $T_{\ell} \lambda^2 : T_{\ell} \text{CH}^2(X_{\bar{K}}) \longrightarrow H^3(X_{\bar{K}}, \mathbb{Z}_{\ell}(2))_{\tau}$. Finally, since the composition

$$H^3(X_{\bar{K}}, \mathbb{Z}_{\ell}(2)) \xrightarrow{r_1^* \oplus r_2^*} H^1(\tilde{W}_1, \mathbb{Z}_{\ell}(1)) \oplus H^3(\tilde{W}_2, \mathbb{Z}_{\ell}(2)) \xrightarrow{s_1^* + s_2^*} H^3(X_{\bar{K}}, \mathbb{Z}_{\ell}(2))$$

is multiplication by Np^e and since $\dim \tilde{W}_2 \leq 2$, we obtain from the equality $\tilde{N}^1 H^3 = N^1 H^3$ of Proposition 1.1, for $\ell \nmid Np^e$, the inclusion $H^3(X_{\bar{K}}, \mathbb{Z}_{\ell}(2)) \subseteq \tilde{N}^1 H^3(X_{\bar{K}}, \mathbb{Z}_{\ell}(2))$.

In case (2), by Proposition 5.1, it suffices to see that $H^3(X_{\bar{K}}, \mathbb{Z}_{\ell}(2))$ is torsion-free for ℓ not dividing Np^e . This follows simply from the factorization of the multiplication by Np^e map as

$$H^3(X_{\bar{K}}, \mathbb{Z}_{\ell}(2)) \xrightarrow{r_1^*} H^1(\tilde{W}_1, \mathbb{Z}_{\ell}(1)) \xrightarrow{s_1^*} H^3(X_{\bar{K}}, \mathbb{Z}_{\ell}(2))$$

and the fact that $H^1(\tilde{W}_1, \mathbb{Z}_{\ell}(1))$ is torsion-free.

Now suppose $l = \text{char}(K) = p > 0$. Bearing in mind the properties of the p -adic Bloch map summarized in §A.4, we see that the composition of the horizontal arrows in (5.2) is again multiplication by Np^e . If resolution of singularities holds in dimension at most $d_X - 1$, then we may take $e = 0$. Under this hypothesis, if $p \nmid N$, we again see that $T_p \lambda^2$ is an isomorphism of $\text{Gal}(K)$ -modules. \square

Remark 5.3. Note that Proposition 5.2(2), together with Theorem 4.2, implies that $\text{CH}_0(X_{\bar{K}}) \otimes \mathbb{Q}$ is universally supported in dimension 1, then the primes ℓ for which $\tilde{N}^1 H^3(X_{\bar{K}}, \mathbb{Z}_{\ell}(2))_{\tau} \subseteq H^3(X_{\bar{K}}, \mathbb{Z}_{\ell}(2))_{\tau}$ might fail to be an equality are the primes dividing the minimal degree of a miniversal cycle. Due to Proposition 3.10, this is in this case *a priori* finer than the conclusion of Proposition 5.2(1).

6. MODELING COHOMOLOGY VIA CORRESPONDENCES

In this section, we prove Theorems 12 and 15. The starting point is that a geometrically rationally chain connected variety (resp. stably rational variety) has *universally trivial* Chow group of zero-cycles with \mathbb{Q} -coefficients (resp. \mathbb{Z} -coefficients); see §3.1 and specifically Remark 3.4. We then combine the existence of the ℓ -adic Bloch map with the existence of an algebraic representative for codimension-2 cycles to establish Proposition 5.2 (which implies Theorem 12) and the main Theorem 6.4 (which implies Theorem 15). Along the way we establish related results concerning the third ℓ -adic cohomology group of uniruled threefolds (Proposition 6.1).

6.1. Modeling \mathbb{Q}_ℓ -cohomology via correspondences. The aim of this section is to show Mazur's Questions 1 and 2, which are with \mathbb{Q}_ℓ -coefficients, can be easily answered positively under some assumption on the coniveau of $H^{2n-1}(X_{\bar{K}}, \mathbb{Q}_\ell(n))$. The following Proposition extends [ACMV17, Thm. 2.1(d)] in the positive characteristic case. Note that it applies to smooth projective geometrically uniruled threefolds.

Proposition 6.1. *Let X be a smooth projective variety over a perfect field K . Assume $H^{2n-1}(X_{\bar{K}}, \mathbb{Q}_{\ell_0}(n)) = \mathbb{N}^{n-1} H^{2n-1}(X_{\bar{K}}, \mathbb{Q}_{\ell_0}(n))$ for some prime $\ell_0 \neq \text{char}(K)$ and for some integer n such that $2n - 1 \leq d_X$. Then there exist an abelian variety A over K and a cycle class $\Gamma \in \text{CH}^n(A \times_K X)$ such that the induced morphism*

$$\Gamma_* : V_l A \longrightarrow H^{2n-1}(X_{\bar{K}}, \mathbb{Q}_l(n)) \quad (6.1)$$

is an isomorphism of $\text{Gal}(K)$ -modules for all primes l . (In particular, we have $H^{2n-1}(X_{\bar{K}}, \mathbb{Q}_l(n)) = \mathbb{N}^{n-1} H^{2n-1}(X_{\bar{K}}, \mathbb{Q}_l(n))$.)

Moreover, if K has positive characteristic, then Γ induces an isomorphism of F -isocrystals

$$\Gamma_* : H^{2d_A-1}(A/\mathbb{K})(d_A) \longrightarrow H^{2n-1}(X/\mathbb{K})(n) .$$

Proof. By Proposition 1.1, there is a smooth projective variety W over K of dimension $d_X - n + 1$ and a K -morphism $f : W \rightarrow X$ inducing a surjection

$$f_* : H^1(W_{\bar{K}}, \mathbb{Q}_{\ell_0}(1)) \rightarrow H^{2n-1}(X_{\bar{K}}, \mathbb{Q}_{\ell_0}(n)).$$

Let $Z_{W_{\bar{K}}} \in \text{CH}^1((\text{Pic}_{W_{\bar{K}}}^0)_{\text{red}} \times_{\bar{K}} W_{\bar{K}})$ be the universal divisor on $W_{\bar{K}}$; it induces an isomorphism of \mathbb{Z}_{ℓ_0} -modules $T_{\ell_0} \text{Pic}_{W_{\bar{K}}}^0 \rightarrow H^1(W_{\bar{K}}, \mathbb{Z}_{\ell_0}(1))$. Let L/K be a finite field extension over which $Z_{W_{\bar{K}}}$ is defined. By pushing forward, we obtain a cycle $Z_W \in \text{CH}^1((\text{Pic}_W^0)_{\text{red}} \times_K W_K)$ inducing an isomorphism $V_{\ell_0} \text{Pic}_W^0 \rightarrow H^1(W_{\bar{K}}, \mathbb{Q}_{\ell_0}(1))$ of $\text{Gal}(K)$ -modules. Let us set $B := (\text{Pic}_W^0)_{\text{red}}$. Composing f_* with Z_W we obtain a correspondence $\gamma \in \text{CH}^n(B \times_K X)$ inducing a surjection

$$\gamma_* : V_{\ell_0} B \twoheadrightarrow H^{2n-1}(X_{\bar{K}}, \mathbb{Q}_{\ell_0}(n))$$

of $\text{Gal}(K)$ -modules. Consider now the cycle $\Delta_*(c_1(\mathcal{O}_X(1))^{d-2n+1}) \in \text{CH}^{2d-2n+1}(X \times_K X)$, where $\Delta : X \hookrightarrow X \times_K X$ is the diagonal embedding. By the Hard Lefschetz Theorem, this cycle induces an isomorphism $L : H^{2n-1}(X_{\bar{K}}, \mathbb{Q}_{\ell_0}(n)) \rightarrow H^{2d-2n+1}(X_{\bar{K}}, \mathbb{Q}_{\ell_0}(d-n+1))$ of $\text{Gal}(K)$ -modules and we obtain a homomorphism

$$V_{\ell_0} B \xrightarrow{\gamma_*} H^{2n-1}(X_{\bar{K}}, \mathbb{Q}_{\ell_0}(n)) \xrightarrow[\simeq]{L} H^{2d-2n+1}(X_{\bar{K}}, \mathbb{Q}_{\ell_0}(d-n+1)) \xleftarrow{\gamma^*} (V_{\ell_0} B)^\vee$$

induced by the correspondence $\gamma^* \circ L \circ \gamma_* \in \text{CH}^1(B \times_K B)$. In particular, the above homomorphism, which is $\text{Gal}(K)$ -equivariant, is induced by a K -homomorphism $\varphi : B \rightarrow B^\vee$. It is clear that $\ker \gamma_* = \ker \varphi_*$. By Poincaré reducibility, there exist an abelian variety A and $\psi \in \text{Hom}(A, B) \otimes \mathbb{Q}$ such that $\gamma_* \circ \psi_* : V_{\ell_0} A \rightarrow H^{2n-1}(X_{\bar{K}}, \mathbb{Q}_{\ell_0}(n))$ is an isomorphism. In addition, there exists an idempotent $\theta \in \text{Hom}(B^\vee, B^\vee) \otimes \mathbb{Q}$ with image A such that $\theta \circ \varphi = \varphi$. Setting $\Gamma = \gamma \circ \psi$, it follows

from the independence of ℓ of the ℓ -adic Betti numbers that $\Gamma_* : V_\ell A \rightarrow H^{2n-1}(X_{\bar{K}}, \mathbb{Q}_\ell(n))$ is an isomorphism for all primes $\ell \neq \text{char}(K)$.

Now suppose $\text{char}(K) = p > 0$. Since $\Gamma^* \circ \Gamma_* : V_\ell A \rightarrow V_\ell A$ is an isomorphism, this same cycle induces an automorphism of the F -isocrystal $H^1(A/\mathbb{K})$ ([KM74], after a spread and specialization argument to reduce to K finite). Because crystalline and ℓ -adic Betti numbers coincide,

$$\Gamma_* : H^{2d_A-1}(A/\mathbb{K})(d_A) \longrightarrow H^{2n-1}(X/\mathbb{K})(n)$$

is an isomorphism of crystals. Taking F -invariants shows that (6.1) holds for $l = p$, too. \square

Remark 6.2 ($2n - 1 > d_X$). We note that using the hard Lefschetz theorem, with the notation and assumptions of Proposition 6.1, there also exists a cycle class $\Gamma' \in \text{CH}^{d_X-n}(A \times_K X)$ such that for all primes l

$$\Gamma'_* : V_l A \longrightarrow H^{2d_X-2n+1}(X_{\bar{K}}, \mathbb{Q}_l(d_X - n + 1))$$

is an isomorphism of $\text{Gal}(K)$ -modules.

In case $n = 2$ and under the assumption that $V_l \phi_{X_{\bar{K}}}^2 : V_l A^2(X_{\bar{K}}) \rightarrow V_l \text{Ab}_{X/K}^2$ is an isomorphism for all primes l , one can make Question 2 more precise and ask whether there exists a correspondence $\Gamma \in \text{CH}^2(\text{Ab}_{X/K}^2 \times_K X) \otimes \mathbb{Q}$ inducing for all primes l the canonical identifications (0.5). We provide a positive answer for geometrically uniruled threefolds:

Proposition 6.3. *Let X be a smooth projective variety over a perfect field K and assume $\text{CH}_0(X_{\bar{K}}) \otimes \mathbb{Q}$ is universally supported in dimension 2, e.g. X is a geometrically uniruled threefold. (In particular, due to Proposition 3.8, $V_l \phi_{X_{\bar{K}}}^2 : V_l A^2(X_{\bar{K}}) \rightarrow V_l \text{Ab}_{X/K}^2$ is an isomorphism for all primes l .)*

Then there exists a correspondence $\Gamma \in \text{CH}^2(\text{Ab}_{X/K}^2 \times_K X) \otimes \mathbb{Q}$ inducing for all primes l the canonical identifications (0.5)

$$V_l \text{Ab}_{X/K}^2 \xrightarrow[\simeq]{(V_l \phi_{X_{\bar{K}/\bar{K}}}^2)^{-1}} V_l A^2(X_{\bar{K}}) \xrightarrow{\subset} V_l \text{CH}^2(X_{\bar{K}}) \xrightarrow[\simeq]{V_l \lambda^2} H^3(X_{\bar{K}}, \mathbb{Q}_l(2)).$$

Γ_*

Proof. First we note that the assumption that $\text{CH}_0(X_{\bar{K}}) \otimes \mathbb{Q}$ is universally supported in dimension 2 implies that $\text{im}(V_l \lambda^2) = \mathbb{N}^1 H^3(X_{\bar{K}}, \mathbb{Q}_l(2))$ for all primes l . Together with Proposition 2.1, this implies that $V_l \lambda^2$ is an isomorphism for all l , so that (0.5) is an isomorphism for all l . Second, let $Z \in \text{CH}^2(\text{Ab}_{X/K}^2 \times_K X)$ be a miniversal cycle and let r denote its degree. Then, by [ACMVa, Cor. 11.8] with \mathbb{Q}_l -coefficients, we have that $Z_* = r \cdot V_l \lambda^2 \circ (V_l \phi_{X_{\bar{K}/\bar{K}}}^2)^{-1}$ for all primes l , and it follows that $\Gamma = \frac{1}{r} Z$ induces (0.5) for all primes l . \square

6.2. Modeling \mathbb{Z}_ℓ -cohomology via correspondences: Theorem 15. Combining Proposition 3.8 with Proposition 5.2, we see that, under the assumption that $\Delta_{X_{\bar{K}}} \in \text{CH}^{d_X}(X_{\bar{K}} \times_{\bar{K}} X_{\bar{K}})$ admits a decomposition of type (W_1, W_2) with $\dim W_2 \leq 2$, we obtain for all primes $\ell \neq \text{char}(K)$ canonical isomorphisms

$$T_\ell \text{Ab}_{X/K}^2 \xrightarrow[\simeq]{(T_\ell \phi_{X_{\bar{K}}}^2)^{-1}} T_\ell A^2(X_{\bar{K}}) \xrightarrow[\simeq]{} T_\ell \text{CH}^2(X_{\bar{K}}) \xrightarrow[\simeq]{T_\ell \lambda^2} H^3(X_{\bar{K}}, \mathbb{Z}_\ell(2))_\tau. \quad (6.2)$$

Under the further assumption that $\dim W_2 \leq 1$, $H^3(X_{\bar{K}}, \mathbb{Z}_\ell(2))$ is torsion-free by Proposition 5.2 and the main result below establishes that the isomorphisms (6.2) are induced by a correspondence defined over K . In view of Proposition 3.2 and Remark 3.4, the theorem below establishes Theorem 15.

Theorem 6.4 (Theorem 15). *Let X be a smooth projective variety over a field K of characteristic exponent p , which is assumed to be either finite or algebraically closed. Assume that there is a natural number N such that $N\Delta_X \in \text{CH}^{d_X}(X \times_K X)$ admits a decomposition of type (W_1, W_2) with $\dim W_2 \leq 1$. Then there exists a correspondence $\Gamma \in \text{CH}^2(\text{Ab}_{X/K}^2 \times_K X)$ inducing for all primes l not dividing Np isomorphisms*

$$\Gamma_* : T_l \text{Ab}_{X/K}^2 \xrightarrow{\cong} H^3(X_{\bar{K}}, \mathbb{Z}_l(2)) \quad (6.3)$$

of $\text{Gal}(K)$ -modules. If $p \geq 2$, if $p \nmid N$, and if resolution of singularities holds in dimensions $< d_X$, then (6.3) holds with $l = p$.

Finally, if $\text{char}(K) = 0$, the correspondence Γ induces an isomorphism

$$\Gamma_* : H_1((\text{Ab}_{X/K}^2)_{\mathbb{C}}, \mathbb{Z}) \otimes \mathbb{Z}[\frac{1}{N}] \xrightarrow{\cong} H^3(X_{\mathbb{C}}, \mathbb{Z}(2)) \otimes \mathbb{Z}[\frac{1}{N}]. \quad (6.4)$$

Proof. First we focus on (6.3). With Notation 3.5, we have the diagram

$$\begin{array}{ccccc} \text{A}^2(X_{\bar{K}}) & \xrightarrow{r_1^* \oplus r_2^*} & \text{A}^1(\tilde{W}_1) \oplus \text{A}^2(\tilde{W}_2) & \xrightarrow{s_1^* + s_2^*} & \text{A}^2(X_{\bar{K}}) \\ \downarrow \phi_X^2 & & \downarrow \phi_{\tilde{W}_1}^1 \oplus \phi_{\tilde{W}_2}^2 & & \downarrow \phi_X^2 \\ \text{Ab}_{X/K}^2(\bar{K}) & \longrightarrow & \text{Pic}_{\tilde{W}_1}^0(\bar{K}) \oplus \text{Alb}_{\tilde{W}_2}(\bar{K}) & \longrightarrow & \text{Ab}_{X/K}^2(\bar{K}). \end{array} \quad (6.5)$$

Assuming $\dim W_2 \leq 1$, the diagram (6.5) takes the simpler form

$$\begin{array}{ccccc} \text{A}^2(X_{\bar{K}}) & \xrightarrow{r_1^*} & \text{A}^1(\tilde{W}_1) & \xrightarrow{s_1^*} & \text{A}^2(X_{\bar{K}}) \\ \downarrow \phi_X^2 & & \simeq \downarrow \phi_{\tilde{W}_1}^1 & & \downarrow \phi_X^2 \\ \text{Ab}_{X/K}^2(\bar{K}) & \xrightarrow{g} & \text{Pic}_{\tilde{W}_1}^0(\bar{K}) & \xrightarrow{f} & \text{Ab}_{X/K}^2(\bar{K}) \end{array}$$

where the composition of the horizontal arrows is multiplication by Np^ℓ . Note that the homomorphisms f and g are in fact induced by K -homomorphisms $\text{Ab}_{X/K}^2 \rightarrow (\text{Pic}_{\tilde{W}_1}^0)_{\text{red}}$ and $(\text{Pic}_{\tilde{W}_1}^0)_{\text{red}} \rightarrow \text{Ab}_{X/K}^2$ by [Mur85] in the case K algebraically closed and by the main result of [ACMV17] in the case K perfect. By Proposition 5.2 with Proposition 3.8, we get for all ℓ not dividing Np a commutative diagram

$$\begin{array}{ccccc} H^3(X_{\bar{K}}, \mathbb{Z}_\ell(2)) & \xrightarrow{r_1^*} & H^1(\tilde{W}_1, \mathbb{Z}_\ell(1)) & \xrightarrow{s_1^*} & H^3(X_{\bar{K}}, \mathbb{Z}_\ell(2)) \\ \simeq \uparrow T_\ell \lambda^2 & & \simeq \uparrow T_\ell \lambda^1 & & \simeq \uparrow T_\ell \lambda^2 \\ T_\ell \text{A}^2(X_{\bar{K}}) & \xrightarrow{r_1^*} & T_\ell \text{A}^1(\tilde{W}_1) & \xrightarrow{s_1^*} & T_\ell \text{A}^2(X_{\bar{K}}) \\ \simeq \uparrow (T_\ell \phi_X^2)^{-1} & & \simeq \uparrow (T_\ell \phi_{\tilde{W}_1}^1)^{-1} & & \simeq \uparrow (T_\ell \phi_X^2)^{-1} \\ T_\ell \text{Ab}_{X/K}^2 & \xrightarrow{g^*} & T_\ell \text{Pic}_{\tilde{W}_1}^0 & \xrightarrow{f^*} & T_\ell \text{Ab}_{X/K}^2 \end{array}$$

where the vertical arrows are isomorphisms and the composition of the horizontal arrows is multiplication by Np^ℓ .

Now, the condition that K be either finite or algebraically closed ensures that \tilde{W}_1 admits a universal divisor $Z_{\tilde{W}_1} \in \text{CH}^1(\tilde{W}_1 \times_K (\text{Pic}_{\tilde{W}_1}^0)_{\text{red}})$, meaning that the K -homomorphism $(\text{Pic}_{\tilde{W}_1}^0)_{\text{red}} \rightarrow (\text{Pic}_{\tilde{W}_1}^0)_{\text{red}}$ induced by $Z_{\tilde{W}_1}$ is the identity. In addition, the homomorphism

$$T_\ell \lambda^1 \circ (T_\ell \phi_{\tilde{W}_1}^1)^{-1} : T_\ell \text{Pic}_{\tilde{W}_1}^0 \rightarrow H^1(\tilde{W}_1, \mathbb{Z}_\ell(1))$$

coincides with the action of $Z_{\widetilde{W}_1}$; this follows from Kummer theory, see e.g. [ACMVa, §11]. We then define a codimension-2 cycle $Z \in \text{CH}^2(X \times_K \text{Ab}_{X/K}^2)$ as the composition ${}^t s_1 \circ {}^t Z_{\widetilde{W}_1} \circ g$. By a simple diagram chase, its induced action

$$Z_* = s_1^* \circ Z_{\widetilde{W}_1}^* \circ g_* : T_\ell \text{Ab}_{X/K}^2 \rightarrow H^3(X_{\overline{K}}, \mathbb{Z}_\ell(2))$$

is equal to $Np^e T_\ell \lambda^2 \circ (T_\ell \phi_X^2)^{-1}$, which is an isomorphism.

If K has positive characteristic $p > 0$ and if resolution of singularities holds in dimension $< d_X$, then we may take $e = 0$. Consequently, under this hypothesis, the same argument shows that if $p \nmid N$, then (6.3) is an isomorphism for $l = p$, too.

Finally for the case $K \subseteq \mathbb{C}$ and (6.4), one uses essentially the same argument, but with the canonical identification of the cohomology modulo torsion of a smooth complex projective variety with the first homology of the intermediate Jacobian. \square

7. THE IMAGE OF THE ℓ -ADIC BLOCH MAP IN CHARACTERISTIC 0

In this section we show that we can model the third integral cohomology with an abelian variety for any smooth projective variety liftable to a smooth projective rationally chain connected variety in characteristic 0. We start with the following proposition, which essentially shows that the strategy of the proof of [ACMV17, Thm. B] works with \mathbb{Z}_ℓ -coefficients for rationally chain connected varieties.

Proposition 7.1. *Let X be a smooth projective variety over a field $K \subseteq \mathbb{C}$ such that $N^1 H^3(X_{\mathbb{C}}, \mathbb{Q}) = H^3(X_{\mathbb{C}}, \mathbb{Q})$ (e.g., X is geometrically rationally chain connected, or $\dim X = 3$ and X is geometrically uniruled). Then for any prime ℓ the morphisms*

$$T_\ell \phi_{X_{\overline{K}/\overline{K}}}^2 : T_\ell A^2(X_{\overline{K}}) \rightarrow T_\ell \text{Ab}_{X/K}^2 \quad \text{and} \quad T_\ell \lambda^2 : T_\ell A^2(X_{\overline{K}}) \rightarrow H^3(X_{\overline{K}}, \mathbb{Z}_\ell(2))_\tau$$

are isomorphisms, so that the composition

$$T_\ell \text{Ab}_{X_{\overline{K}/\overline{K}}}^2 \xrightarrow{(T_\ell \phi_{X_{\overline{K}/\overline{K}}}^2)^{-1}} T_\ell A^2(X_{\overline{K}}) \xrightarrow{T_\ell \lambda^2} H^3(X_{\overline{K}}, \mathbb{Z}_\ell(2))_\tau \quad (7.1)$$

is an isomorphism.

Proof. The fact due to Murre that $\phi_{X_{\overline{K}/\overline{K}}}^2[\ell^\infty] : A^2(X_{\overline{K}})[\ell^\infty] \rightarrow \text{Ab}_{X/K}^2[\ell^\infty]$ is an isomorphism was explained in Proposition 3.8(1); now one simply applies Tate modules to get that $T_\ell \phi_{X_{\overline{K}/\overline{K}}}^2$ is an isomorphism.

We now consider the Bloch map. The first observation is that $\phi_{X_{\mathbb{C}}/\mathbb{C}}^2$ agrees with the Abel–Jacobi map $AJ : A^2(X_{\mathbb{C}}) \rightarrow J_a^3(X_{\mathbb{C}})$. Thus, applying Tate modules to the Abel–Jacobi map, we have that $T_\ell \phi_{X_{\mathbb{C}}/\mathbb{C}}^2 = T_\ell AJ$, so that, by the above, $T_\ell AJ$ is an isomorphism. At the same time, the assumption $N^1 H^3(X_{\mathbb{C}}, \mathbb{Q}) = H^3(X_{\mathbb{C}}, \mathbb{Q})$ implies that the Abel–Jacobi map $AJ : A^2(X_{\mathbb{C}}) \rightarrow J^3(X_{\mathbb{C}})$ to the full intermediate Jacobian is surjective (e.g., [Voi07, Thm. 12.22]); i.e., $J_a^3(X_{\mathbb{C}}) = J^3(X_{\mathbb{C}}) = H^{1,2}(X_{\mathbb{C}})/H^3(X_{\mathbb{C}}, \mathbb{Z})_\tau$. We conclude using the fact that $T_\ell AJ = T_\ell \lambda^2$ (see Remark A.19). \square

Remark 7.2. Recall that for a smooth projective variety X the condition $N^1 H^3(X_{\mathbb{C}}, \mathbb{Q}) = H^3(X_{\mathbb{C}}, \mathbb{Q})$ is implied by the diagonal $\Delta_{X_{\mathbb{C}}} \in \text{CH}^{d_X}(X_{\mathbb{C}} \times_{\mathbb{C}} X_{\mathbb{C}})_{\mathbb{Q}}$ admitting a decomposition of type (W_1, W_2) with $\dim W_2 \leq 2$. The diagonal of $X_{\mathbb{C}}$ has such a decomposition if X is geometrically rationally chain connected, or $\dim X = 3$ and X is geometrically uniruled, since in these cases $\text{CH}_0(X_{\overline{K}})_{\mathbb{Q}}$ is universally supported on a surface (see Proposition 3.2 and Remark 3.4).

Remark 7.3. We note here that Proposition 7.1 strengthens Theorem 3 in the case $n = 2$ in that it allows for \mathbb{Z}_ℓ -coefficients; on the other hand, Proposition 7.1 is weaker than Theorem 3 in the sense that it does not provide a correspondence giving the isomorphism (7.1). Similarly, Proposition 7.1 strengthens Theorem 6.4 in the case $\text{char}(K) = 0$ in the sense that it gives isomorphisms for all primes without assuming a decomposition of the diagonal with \mathbb{Z} -coefficients, and it allows for a weaker form of the decomposition (see the previous remark). For example, in characteristic 0, Theorem 6.4 implies (7.1) is an isomorphism for all primes if X is geometrically stably rational, whereas Proposition 7.1 implies the same if X is just assumed to be geometrically rationally chain connected (or even a geometrically uniruled threefold). On the other hand, Proposition 7.1 is weaker than Theorem 6.4 in the sense that it does not provide a correspondence giving the isomorphism (7.1).

We now use Proposition 7.1 to prove the following result on varieties liftable to characteristic 0:

Corollary 7.4. *Let X_\circ be a smooth projective variety over a field κ , and suppose that X_\circ lifts to a smooth projective variety X_η over a field K of characteristic 0; i.e., there is a DVR with spectrum S , generic point $\eta = \text{Spec } K$ with $\text{char}(K) = 0$, closed point $\circ = \text{Spec } \kappa$, and a smooth projective scheme X/S , with special fiber over \circ equal to X_\circ , and generic fiber over η equal to X_η .*

If $\mathbb{N}^1 H^3((X_\eta)_\mathbb{C}, \mathbb{Q}) = H^3((X_\eta)_\mathbb{C}, \mathbb{Q})$, e.g., if X_η is a geometrically rationally chain connected variety, or a smooth geometrically uniruled threefold, then for any prime $\ell \neq \text{char}(\kappa)$ the morphism

$$T_\ell \lambda^2 : T_\ell A^2(X_{\bar{\kappa}}) \rightarrow H^3(X_{\bar{\kappa}}, \mathbb{Z}_\ell(2))_\tau$$

is an isomorphism.

Proof. We consider the diagram

$$\begin{array}{ccc} T_\ell A^2(X_{\bar{\kappa}}) & \xrightarrow{T_\ell \lambda^2} & H^3(X_{\bar{\kappa}}, \mathbb{Z}_\ell)_\tau \\ \downarrow & & \downarrow \simeq \\ T_\ell A^2(X_{\bar{\kappa}}) & \xrightarrow{T_\ell \lambda^2} & H^3(X_{\bar{\kappa}}, \mathbb{Z}_\ell)_\tau \end{array}$$

The vertical arrows are specialization maps. The result holds by commutativity and by Proposition 7.1. \square

APPENDIX A. A REVIEW OF THE l -ADIC BLOCH MAP

The aim of this appendix is to review the original construction [Blo79] of the Bloch map on l -primary torsion and show it in fact yields a direct construction of the l -adic Bloch map. We also review Suwa's construction [Suw88] of the l -adic Bloch map and show it coincides with our direct construction. For future referencing purposes we list in §A.4 the properties of the l -adic Bloch map that can be directly derived from the corresponding properties of the original Bloch map via Suwa's construction. In addition, the construction of Gros–Suwa [GS88] of the p -adic Bloch map is briefly reviewed. Finally, we study the restriction of the Bloch map to the subgroup of algebraically trivial cycles.

A.1. Conventions for l -adic and p -adic cohomology.

A.1.1. *l -adic cohomology.* We fix a variety X over a field K , and consider the étale cohomology groups of $X_{\bar{K}} := X \times_K \bar{K}$ with values in prime-to- $\text{char}(K)$ torsion sheaves. For each n , r , and ν , we use the convention

$$H^n(X_{\bar{K}}, \mathbb{Z}/\ell^\nu \mathbb{Z}(r)) := H_{\text{ét}}^n(X_{\bar{K}}, \mathbf{\mu}_{\ell^\nu}^{\otimes r}),$$

the étale cohomology of the étale sheaf μ_{ℓ^v} of ℓ^v -roots of unity. Note that since étale cohomology is invariant under purely inseparable extensions, we can replace \bar{K} with \bar{K}^a throughout this subsection.

There are maps

$$H^n(X_{\bar{K}}, \mathbb{Z}/\ell^v \mathbb{Z}(r)) \rightarrow H^n(X_{\bar{K}}, \mathbb{Z}/\ell^{v+1} \mathbb{Z}(r))$$

induced from the natural map $\mathbb{Z}/\ell^v \mathbb{Z} \hookrightarrow \mathbb{Z}/\ell^{v+1} \mathbb{Z}$, $[x] \mapsto [\ell x]$, or more precisely, from the natural map $\mu_{\ell^v} \hookrightarrow \mu_{\ell^{v+1}}$, $\zeta \mapsto \zeta^\ell$, as well as maps

$$H^n(X_{\bar{K}}, \mathbb{Z}/\ell^{v+1} \mathbb{Z}(r)) \rightarrow H^n(X_{\bar{K}}, \mathbb{Z}/\ell^v \mathbb{Z}(r))$$

induced from the natural quotient map $\mathbb{Z}/\ell^{v+1} \mathbb{Z} \twoheadrightarrow \mathbb{Z}/\ell^v \mathbb{Z}$, or more precisely the natural map $\mu_{\ell^{v+1}} \twoheadrightarrow \mu_{\ell^{v+1}}/\mu_\ell \simeq \mu_{\ell^v}$. The ℓ -adic cohomology groups of X are defined as follows:

$$H^n(X_{\bar{K}}, \mathbb{Z}_\ell(r)) := \varprojlim_v H^n(X_{\bar{K}}, \mathbb{Z}/\ell^v \mathbb{Z}(r))$$

$$H^n(X_{\bar{K}}, \mathbb{Q}_\ell(r)) := H^n(X_{\bar{K}}, \mathbb{Z}_\ell(r)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

The cohomology groups of X with ℓ -torsion coefficients are defined as follows:

$$H^n(X_{\bar{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(r)) := \varinjlim_v H^n(X_{\bar{K}}, \mathbb{Z}/\ell^v \mathbb{Z}(r)).$$

We denote by

$$\mathbb{Z}_\ell(r) := \varprojlim \mu_{\ell^v}^{\otimes r}$$

$$\mathbb{Q}_\ell(r) := \mathbb{Z}_\ell(r) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

$$\mathbb{Q}_\ell/\mathbb{Z}_\ell(r) := \varinjlim \mu_{\ell^v}^{\otimes r}$$

and we can obtain the various twists in cohomology as:

$$H^n(X_{\bar{K}}, \mathbb{Z}_\ell(r)) = H^n(X_{\bar{K}}, \mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell} \mathbb{Z}_\ell(r)$$

$$H^n(X_{\bar{K}}, \mathbb{Q}_\ell(r)) = H^n(X_{\bar{K}}, \mathbb{Q}_\ell) \otimes_{\mathbb{Q}_\ell} \mathbb{Q}_\ell(r)$$

$$H^n(X_{\bar{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(r)) = H^n(X_{\bar{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \otimes_{\mathbb{Q}_\ell/\mathbb{Z}_\ell} \mathbb{Q}_\ell/\mathbb{Z}_\ell(r)$$

For the sake of completeness, we recall the following basic fact and its proof (see also *e.g.* [Mil80, Ch. III, Rmk. 3.6(d)]):

Proposition A.1. *Viewing $\mathbb{Q}_\ell/\mathbb{Z}_\ell$ as a torsion étale sheaf on X , there is a natural isomorphism*

$$H_{\text{ét}}^n(X_{\bar{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \otimes_{\mathbb{Q}_\ell/\mathbb{Z}_\ell} \mathbb{Q}_\ell/\mathbb{Z}_\ell(r) = H^n(X_{\bar{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(r)) \quad (:= \varinjlim_v H^n(X_{\bar{K}}, \mathbb{Z}/\ell^v \mathbb{Z}(r))).$$

Proof. The Snake Lemma applied to

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z}_\ell & \xrightarrow{\cdot \ell^v} & \mathbb{Z}_\ell & \longrightarrow & \mathbb{Z}/\ell^v \mathbb{Z} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbb{Q}_\ell & \xrightarrow{\cdot \ell^v} & \mathbb{Q}_\ell & \longrightarrow & 0 \longrightarrow 0 \end{array}$$

gives a short exact sequence of étale sheaves

$$0 \longrightarrow \mathbb{Z}/\ell^v \mathbb{Z} \longrightarrow \mathbb{Q}_\ell/\mathbb{Z}_\ell \xrightarrow{\cdot \ell^v} \mathbb{Q}_\ell/\mathbb{Z}_\ell \longrightarrow 0.$$

The map on the left can be written explicitly as $[x] \mapsto [x/\ell^v]$, where we are viewing $\mathbb{Z} \subseteq \mathbb{Z}_\ell$ in the natural way. This gives a long exact sequence in cohomology

$$\cdots \longrightarrow H^n(X_{\bar{K}}, \mathbb{Z}_\ell/\ell^v \mathbb{Z}_\ell) \longrightarrow H^n(X_{\bar{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \xrightarrow{\cdot \ell^v} H^n(X_{\bar{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \longrightarrow \cdots \quad (\text{A.1})$$

In fact we have a diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathbb{Z}/\ell^v\mathbb{Z} & \longrightarrow & \mathbb{Q}_\ell/\mathbb{Z}_\ell & \xrightarrow{\cdot\ell^v} & \mathbb{Q}_\ell/\mathbb{Z}_\ell \longrightarrow 0 \\
& & \downarrow & & \parallel & & \downarrow \cdot\ell \\
0 & \longrightarrow & \mathbb{Z}/\ell^{v+1}\mathbb{Z} & \longrightarrow & \mathbb{Q}_\ell/\mathbb{Z}_\ell & \xrightarrow{\cdot\ell^{v+1}} & \mathbb{Q}_\ell/\mathbb{Z}_\ell \longrightarrow 0
\end{array}$$

which is commutative since given $[x] \in \mathbb{Z}/\ell^v\mathbb{Z}$ we have $[x/\ell^v] = [\ell x/\ell^{v+1}]$. Taking direct limits is exact, and so we obtain an exact sequence

$$\cdots \longrightarrow 0 \longrightarrow \varinjlim_v H^n(X_{\bar{K}}, \mathbb{Z}_\ell/\ell^v\mathbb{Z}_\ell) \longrightarrow H^n(X_{\bar{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \longrightarrow 0 \longrightarrow \cdots,$$

thereby settling the proposition. \square

A.1.2. p -adic cohomology. For K perfect of characteristic $p > 0$, let $\mathbb{W}(K)$ be the ring of Witt vectors of K , with field of fractions $\mathbb{K}(K)$. For X/K , we adopt the standard notation [GS88, §I.3.1], [Mil86, §1]

$$\begin{aligned}
H^n(X, \mathbb{Z}/p^v\mathbb{Z}(r)) &:= H_{\text{ét}}^{n-r}(X, \mathbb{W}_v\Omega_{X,\log}^r) \\
H^n(X_{\bar{K}}, \mathbb{Z}_p(r)) &:= \varprojlim_v H^n(X_{\bar{K}}, \mathbb{Z}/p^v\mathbb{Z}(r)) \\
H^n(X_{\bar{K}}, \mathbb{Q}_p(r)) &:= H^n(X_{\bar{K}}, \mathbb{Z}_p(r)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \\
H^n(X_{\bar{K}}, \mathbb{Q}_p/\mathbb{Z}_p(r)) &:= \varinjlim_v H^n(X_{\bar{K}}, \mathbb{Z}/p^v\mathbb{Z}(r))
\end{aligned}$$

With these conventions, $H_{\text{ét}}^n(X_{\bar{K}}, \mathbb{Q}_p(r)) := \varprojlim_v H_{\text{ét}}^n(X_{\bar{K}}, \mu_{p^v}^{\otimes r}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and $H^n(X, \mathbb{Q}_p(r))$ coincide [Ill79, (5.2.1)], [Mil86, Prop. 1.15], but if $n > 1$ then the corresponding statement for integral coefficients need not hold.

For X/K smooth and projective, we let $H^n(X, \mathbb{W}(K)(r)) := H_{\text{cris}}^n(X/\mathbb{W}(K))(r)$ denote the crystalline cohomology group, and let $H^n(X, \mathbb{K}(K)(r)) := H^n(X, \mathbb{W}(K)(r)) \otimes_{\mathbb{W}(K)} \mathbb{K}(K)$. This group may also be computed as the rigid cohomology group $H_{\text{rig}}^n(X/\mathbb{K}(K))(r)$. Note that if we set $\mathbb{W}_v(K) := \mathbb{W}(K)/p^v\mathbb{W}(K)$, then $H^n(X, \mathbb{W}_v(K)(r)) = H_{\text{cris}}^n(X/\mathbb{W}_v(K)(r))$.

A.2. The ℓ -adic Bloch map. In [Blo79], Bloch constructed a map

$$\lambda^n : \text{CH}^n(X_{\bar{K}^a})[\ell^\infty] \rightarrow H^{2n-1}(X_{\bar{K}^a}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))$$

for smooth projective varieties over a field K . In this section, we review his construction, showing how it also defines a map $T_\ell\lambda^n : T_\ell\text{CH}^n(X_{\bar{K}^a}) \rightarrow H^{2n-1}(X_{\bar{K}^a}, \mathbb{Z}_\ell(n))_\tau$ on Tate modules.

A.2.1. The Abel–Jacobi map on torsion. We start by recalling the definition of the Abel–Jacobi map on torsion. This is rather elementary from the definition of the Abel–Jacobi map, but gives some motivation for Bloch’s approach to his algebraic construction of the map, as well as some motivation for our interest in what we call the ℓ^v -Bloch maps (Definition A.2). We also explain in Remark A.19 that the ℓ -adic Abel–Jacobi map (A.3) agrees with the ℓ -adic Bloch map on homologically trivial cycle classes.

For a complex projective variety X one has the Abel–Jacobi map

$$\text{CH}^n(X)_{\text{hom}} \xrightarrow{AJ} J^{2n-1}(X) = F^n \backslash H^{2n-1}(X, \mathbb{C}) / H^{2n-1}(X, \mathbb{Z})_\tau$$

from the group of homologically trivial cycle classes of codimension- n to the $(2n-1)$ -st intermediate Jacobian $J^{2n-1}(X)$. Here, F^\bullet refers to the Hodge filtration on $H^{2n-1}(X, \mathbb{C})$; in the situation above and in terms of the Hodge decomposition we have $F^n H^{2n-1}(X, \mathbb{C}) = H^{2n-1,0}(X) \oplus$

$H^{2n-2,1}(X) \oplus \dots \oplus H^{n,n-1}(X)$. We can identify the torsion $J^{2n-1}(X)[\ell^\nu]$ as follows. For any complex torus $A = V/\Lambda$, we have $A[\ell^\nu] = \frac{1}{\ell^\nu}\Lambda/\Lambda$. If we consider the commutative diagram of short exact sequences,

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Lambda & \longrightarrow & \Lambda_{\mathbb{Q}} & \longrightarrow & \Lambda_{\mathbb{Q}}/\Lambda \longrightarrow 0 \\ & & \downarrow \cdot \ell^\nu & & \simeq \downarrow \cdot \ell^\nu & & \downarrow \cdot \ell^\nu \\ 0 & \longrightarrow & \Lambda & \longrightarrow & \Lambda_{\mathbb{Q}} & \longrightarrow & \Lambda_{\mathbb{Q}}/\Lambda \longrightarrow \dots \end{array}$$

the snake lemma gives an identification $A[\ell^\nu] = \Lambda/\ell^\nu\Lambda$. In our situation with $A = J^{2n-1}(X)$, we have $\Lambda = H^{2n-1}(X, \mathbb{Z})_\tau$. In other words, $J^{2n-1}(X)[\ell^\nu] = H^{2n-1}(X, \mathbb{Z})_\tau/\ell^\nu H^{2n-1}(X, \mathbb{Z})_\tau$. We then consider the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^{2n-1}(X, \mathbb{Z})_{\text{tors}} & \longrightarrow & H^{2n-1}(X, \mathbb{Z}) & \longrightarrow & H^{2n-1}(X, \mathbb{Z})_\tau \longrightarrow 0 \\ & & \downarrow \cdot \ell^\nu & & \downarrow \cdot \ell^\nu & & \downarrow \cdot \ell^\nu \\ 0 & \longrightarrow & H^{2n-1}(X, \mathbb{Z})_{\text{tors}} & \longrightarrow & H^{2n-1}(X, \mathbb{Z}) & \longrightarrow & H^{2n-1}(X, \mathbb{Z})_\tau \longrightarrow 0 \end{array}$$

For brevity, we denote $\delta_{\ell^\nu}^{an} = \frac{H^{2n-1}(X, \mathbb{Z})_{\text{tors}}}{\ell^\nu H^{2n-1}(X, \mathbb{Z})_{\text{tors}}}$ the cokernel of the vertical map on the left. The snake lemma, and the long exact sequence in cohomology associated to $0 \rightarrow \mathbb{Z} \xrightarrow{\cdot \ell^\nu} \mathbb{Z} \rightarrow \mathbb{Z}/\ell^\nu\mathbb{Z} \rightarrow 0$, together give a diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \delta_{\ell^\nu}^{an} & \longrightarrow & \frac{H^{2n-1}(X, \mathbb{Z})}{\ell^\nu H^{2n-1}(X, \mathbb{Z})} & \longrightarrow & \frac{H^{2n-1}(X, \mathbb{Z})_\tau}{\ell^\nu H^{2n-1}(X, \mathbb{Z})_\tau} \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \delta_{\ell^\nu}^{an} & \longrightarrow & H^{2n-1}(X, \mathbb{Z}/\ell^\nu\mathbb{Z}) & \longrightarrow & H^{2n-1}(X, \mathbb{Z}/\ell^\nu\mathbb{Z})/\delta_{\ell^\nu}^{an} \longrightarrow 0 \end{array}$$

Thus we obtain maps

$$\text{CH}^n(X)_{\text{hom}}[\ell^\nu] \xrightarrow{AJ[\ell^\nu]} J^{2n-1}(X)[\ell^\nu] \longrightarrow H^{2n-1}(X, \mathbb{Z}/\ell^\nu\mathbb{Z})/\delta_{\ell^\nu}^{an}. \quad (\text{A.2})$$

It is easy to see that for sufficiently large ν , we have $\delta_{\ell^\nu}^{an} = H^{2n-1}(X, \mathbb{Z})_{\ell\text{-tors}}$, and via the isomorphism $H^{2n-1}(X, \mathbb{Z}_\ell) = H^{2n-1}(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$, we have that $H^{2n-1}(X, \mathbb{Z})_{\ell\text{-tors}} = H^{2n-1}(X, \mathbb{Z}_\ell)_{\text{tors}}$. It follows that $\varprojlim \delta_{\ell^\nu}^{an} = H^{2n-1}(X, \mathbb{Z}_\ell)_{\text{tors}}$.

We claim now that $\varprojlim H^{2n-1}(X, \mathbb{Z}/\ell^\nu\mathbb{Z})/\delta_{\ell^\nu}^{an} = H^{2n-1}(X, \mathbb{Z}_\ell)_\tau$. For this we consider the short exact sequence $0 \rightarrow \delta_{\ell^\nu}^{an} \rightarrow H^{2n-1}(X, \mathbb{Z}/\ell^\nu\mathbb{Z}) \rightarrow H^{2n-1}(X, \mathbb{Z}/\ell^\nu\mathbb{Z})/\delta_{\ell^\nu}^{an} \rightarrow 0$, and use the fact that since the $\delta_{\ell^\nu}^{an}$ are finite, we have $\varprojlim^1 \delta_{\ell^\nu}^{an} = 0$.

Taking the inverse limit of the maps (A.2), we therefore obtain a map

$$T_\ell \text{CH}^n(X)_{\text{hom}} \xrightarrow{T_\ell AJ} T_\ell J^{2n-1}(X) \longrightarrow H^{2n-1}(X, \mathbb{Z}_\ell)_\tau \quad (\text{A.3})$$

and then tensoring with $- \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$, we obtain a map

$$V_\ell \text{CH}^n(X)_{\text{hom}} \xrightarrow{V_\ell AJ} V_\ell J^{2n-1}(X) \longrightarrow H^{2n-1}(X, \mathbb{Q}_\ell). \quad (\text{A.4})$$

The next claim is that $\varprojlim \delta_{\ell^\nu}^{an} = 0$. This also follows from the fact that for sufficiently large ν , we have $\delta_{\ell^\nu}^{an} = H^{2n-1}(X, \mathbb{Z})_{\ell\text{-tors}}$, since the latter group is finite, and is therefore killed by multiplication by ℓ^N for some sufficiently large N . As a consequence, taking the direct limit in (A.2) we obtain a map

$$\mathrm{CH}^n(X)_{\mathrm{hom}}[\ell^\infty] \xrightarrow{AJ[\ell^\infty]} J^{2n-1}(X)[\ell^\infty] \longrightarrow H^{2n-1}(X, \mathbf{Q}_\ell/\mathbf{Z}_\ell). \quad (\text{A.5})$$

A.2.2. *Bloch's preliminaries.* The set-up in [Blo79] is from the paper [BO74]. It is described in [Blo79] in the following way. One sets $\mathbf{H}^q(\boldsymbol{\mu}_{\ell^v}^{\otimes n})$ to be the Zariski sheaf on $X_{\overline{K}^a}$ associated to the pre-sheaf $U \mapsto H_{\acute{\text{e}}\text{t}}^q(U, \boldsymbol{\mu}_{\ell^v}^{\otimes n})$. In other words, this is the derived push-forward of the sheaf $\boldsymbol{\mu}_{\ell^v}^{\otimes n}$ on $(X_{\overline{K}^a})_{\acute{\text{e}}\text{t}}$ via the morphism of sites $\pi : (X_{\overline{K}^a})_{\acute{\text{e}}\text{t}} \rightarrow (X_{\overline{K}^a})_{\text{zar}}$, from the étale site to the Zariski site :

$$\mathbf{H}^q(\boldsymbol{\mu}_{\ell^v}^{\otimes n}) := R^q \pi_* \boldsymbol{\mu}_{\ell^v}^{\otimes n}.$$

Consider the composition of morphisms of sites $(X_{\overline{K}^a})_{\acute{\text{e}}\text{t}} \rightarrow (X_{\overline{K}^a})_{\text{zar}} \rightarrow \mathrm{Spec} \overline{K}^a$. The Leray spectral sequence is

$$E_2^{p,q} = H_{\text{zar}}^p(X_{\overline{K}^a}, \mathbf{H}^q(\boldsymbol{\mu}_{\ell^v}^{\otimes n})) \implies H_{\acute{\text{e}}\text{t}}^{p+q}(X_{\overline{K}^a}, \boldsymbol{\mu}_{\ell^v}^{\otimes n}).$$

The main tool is the existence of a particular flasque resolution of $\mathbf{H}^q(\boldsymbol{\mu}_{\ell^v}^{\otimes n})$ [Blo79, (1.3)],

$$0 \rightarrow \mathbf{H}^q(\boldsymbol{\mu}_{\ell^v}^{\otimes n}) \rightarrow F^0 \rightarrow F^1 \rightarrow \dots,$$

which of course computes $H_{\text{zar}}^p(X_{\overline{K}^a}, \mathbf{H}^q(\boldsymbol{\mu}_{\ell^v}^{\otimes n}))$ in the p -th place. This resolution has two nice properties. First, it turns out to be easy to read off from the resolution that

$$H_{\text{zar}}^p(X_{\overline{K}^a}, \mathbf{H}^q(\boldsymbol{\mu}_{\ell^v}^{\otimes n})) = 0$$

for $p > q$, and consequently, from the shape of the spectral sequence, one obtains so-called boundary maps for the spectral sequence [Blo79, Cor. 1.4]

$$H_{\text{zar}}^{n-1}(X_{\overline{K}^a}, \mathbf{H}^n(\boldsymbol{\mu}_{\ell^v}^{\otimes n})) \rightarrow H_{\acute{\text{e}}\text{t}}^{2n-1}(X_{\overline{K}^a}, \boldsymbol{\mu}_{\ell^v}^{\otimes n}). \quad (\text{A.6})$$

Second, the precise description of the flasque resolution in [Blo79, (1.3)] shows that the group $H_{\text{zar}}^{n-1}(X_{\overline{K}^a}, \mathbf{H}^n(\boldsymbol{\mu}_{\ell^v}^{\otimes n}))$ on the left in (A.6) is the cohomology of the complex [Blo79, Cor. 1.5]

$$\bigoplus_{W^{n-2} \subseteq X_{\overline{K}^a}} H_{\text{Gal}}^2(Q(W), \boldsymbol{\mu}_{\ell^v}^{\otimes 2}) \rightarrow \bigoplus_{V^{n-1} \subseteq X_{\overline{K}^a}} Q(V)^*/Q(V)^{\otimes \ell^v} \xrightarrow{\partial_{\ell^v}} \bigoplus_{T^n \subseteq X_{\overline{K}^a}} \mathbf{Z}/\ell^v \mathbf{Z},$$

where the sums are taken over irreducible subvarieties of the indicated *codimensions*, and $Q(-)$ indicates the function field. The map ∂_{ℓ^v} is obtained from the standard exact sequence below after reduction modulo ℓ^v :

$$\bigoplus_{V^{n-1} \subseteq X_{\overline{K}^a}} Q(V)^* \xrightarrow{\partial} \bigoplus_{T^n \subseteq X_{\overline{K}^a}} \mathbf{Z} \rightarrow \mathrm{CH}^n(X_{\overline{K}^a}) \rightarrow 0, \quad (\text{A.7})$$

where ∂ sends a rational function and to its divisor of zeros and poles on V , and then one pushes forward via the inclusion $V \subseteq X_{\overline{K}^a}$ to cycles on $X_{\overline{K}^a}$.

In particular, we have surjections

$$\ker \partial_{\ell^v} \twoheadrightarrow H_{\text{zar}}^{n-1}(X_{\overline{K}^a}, \mathbf{H}^n(\boldsymbol{\mu}_{\ell^v}^{\otimes n})). \quad (\text{A.8})$$

A.2.3. *The ℓ^v -Bloch map.* To get the construction started, we simply consider the commutative diagram of short exact sequences of groups [Blo79, (2.1)] :

$$\begin{array}{ccccccc} 0 \rightarrow & \bigoplus_{V^{n-1} \subseteq X_{\overline{K}^a}} Q(V)^*/\overline{K}^{a*} & \xrightarrow{(-)^{\ell^v}} & \bigoplus_{V^{n-1} \subseteq X_{\overline{K}^a}} Q(V)^*/\overline{K}^{a*} & \rightarrow & \bigoplus_{V^{n-1} \subseteq X_{\overline{K}^a}} Q(V)^*/Q(V)^{\otimes \ell^v} & \rightarrow 0 \\ & \downarrow \partial & & \downarrow \partial & & \downarrow \partial_{\ell^v} & \\ 0 \longrightarrow & \bigoplus_{T^n \subseteq X_{\overline{K}^a}} \mathbf{Z} & \xrightarrow{\ell^v} & \bigoplus_{T^n \subseteq X_{\overline{K}^a}} \mathbf{Z} & \longrightarrow & \bigoplus_{T^n \subseteq X_{\overline{K}^a}} \mathbf{Z}/\ell^v \mathbf{Z} & \longrightarrow 0 \end{array} \quad (\text{A.9})$$

The snake lemma yields *via* (A.7) the long exact sequence

$$0 \rightarrow \ker \partial \xrightarrow{\ell^v} \ker \partial \rightarrow \ker \partial_{\ell^v} \rightarrow \mathrm{CH}^n(X_{\overline{K}^a}) \xrightarrow{\ell^v} \mathrm{CH}^n(X_{\overline{K}^a}) \rightarrow \mathrm{CH}^n(X_{\overline{K}^a}) / \ell^v \mathrm{CH}^n(X_{\overline{K}^a}) \rightarrow 0.$$

We obtain a diagram where the top row is a short exact sequence [Blo79, (2.2)] :

$$\begin{array}{ccccccc}
0 & \longrightarrow & \left(\begin{array}{c} \ker \partial \\ \ell^v \ker \partial \end{array} \right) & \longrightarrow & \ker \partial_{\ell^v} & \longrightarrow & \mathrm{CH}^n(X_{\overline{K}^a})[\ell^v] \longrightarrow 0 \\
& & \searrow \rho_{\ell^v} & & \downarrow \text{(A.8)} & & \\
& & & & H_{\mathrm{zar}}^{n-1}(X_{\overline{K}^a}, \mathbf{H}^n(\boldsymbol{\mu}_{\ell^v}^{\otimes n})) & & \\
& & & & \downarrow \text{(A.6)} & & \\
& & & & H_{\mathrm{ét}}^{2n-1}(X_{\overline{K}^a}, \boldsymbol{\mu}_{\ell^v}^{\otimes n}) & &
\end{array} \tag{A.10}$$

where ρ_{ℓ^v} is defined as the indicated composition in the diagram. In fact, we find it convenient to define δ_{ℓ^v} to be the image of ρ_{ℓ^v} , to obtain the diagram :

$$\begin{array}{ccccccc}
0 & \longrightarrow & \left(\begin{array}{c} \ker \partial \\ \ell^v \ker \partial \end{array} \right) & \longrightarrow & \ker \partial_{\ell^v} & \longrightarrow & \mathrm{CH}^n(X_{\overline{K}^a})[\ell^v] \longrightarrow 0 \\
& & \searrow \rho_{\ell^v} & & \downarrow \text{(A.8)} & & \downarrow \\
& & & & H_{\mathrm{zar}}^{n-1}(X_{\overline{K}^a}, \mathbf{H}^n(\boldsymbol{\mu}_{\ell^v}^{\otimes n})) & & \\
& & & & \downarrow \text{(A.6)} & & \\
0 & \longrightarrow & \delta_{\ell^v} & \longrightarrow & H_{\mathrm{ét}}^{2n-1}(X_{\overline{K}^a}, \boldsymbol{\mu}_{\ell^v}^{\otimes n}) & \longrightarrow & H_{\mathrm{ét}}^{2n-1}(X_{\overline{K}^a}, \boldsymbol{\mu}_{\ell^v}^{\otimes n}) / \delta_{\ell^v} \longrightarrow 0
\end{array} \tag{A.11}$$

We now give a name to the vertical arrow on the right; this map is used tacitly by Bloch in many places, and it will be convenient for us to give this a name :

Definition A.2 (The ℓ^v -Bloch map). The map

$$\mathrm{CH}^n(X_{\overline{K}^a})[\ell^v] \xrightarrow{\lambda^n[\ell^v]} H_{\mathrm{ét}}^{2n-1}(X_{\overline{K}^a}, \boldsymbol{\mu}_{\ell^v}^{\otimes n}) / \delta_{\ell^v}$$

which is the negative of the map defined from (A.11) is the ℓ^v -Bloch map in codimension n .

Remark A.3. As explained on [Blo79, p.112], the choice of the negative sign is for compatibility, in the case $n = 1$, with the natural map coming from the Kummer sequence, as in Proposition A.25.

A.2.4. *Bloch's Key Lemma.* Consider as in [Blo79, (2.3)] the map

$$\rho : \ker \partial \longrightarrow H^{2n-1}(X_{\overline{K}^a}, \mathbb{Z}_{\ell}(n)) \tag{A.12}$$

defined as the composition

$$\rho : \ker \partial \longrightarrow \varprojlim (\ker \partial / \ell^v \ker \partial) \xrightarrow{\varprojlim \rho_{\ell^v}} H^{2n-1}(X_{\overline{K}^a}, \mathbb{Z}_{\ell}(n)).$$

The following lemma, whose proof uses the Weil conjectures (*via* specialization to finite fields), is key to constructing the Bloch map on $\mathrm{CH}^n(X_{\overline{K}^a})[\ell^{\infty}]$ and the ℓ -adic Bloch map on $T_{\ell} \mathrm{CH}^n(X_{\overline{K}^a})$.

Lemma A.4 (Bloch's Key Lemma [Blo79, Lem. 2.4]). *The image of ρ is torsion.* \square

What is left tacit by Bloch, but is used in his construction of the Bloch map, is that Lemma A.4 implies :

Lemma A.5. *The image of the map*

$$\varprojlim \rho_{\ell^v} : \varprojlim (\ker \partial / \ell^v \ker \partial) \rightarrow H^{2n-1}(X_{\overline{K}^a}, \mathbb{Z}_\ell(n))$$

is torsion.

Proof. Since $\ker \partial \subseteq \varprojlim (\ker \partial / \ell^v \ker \partial)$ is a dense subset, and the map $\varprojlim \rho_{\ell^v}$ on completions is continuous, the image of $\varprojlim (\ker \partial / \ell^v \ker \partial)$ (i.e., the image of $\varprojlim \rho_{\ell^v}$) is contained in the closure of the image of $\ker \partial$ (i.e., the image of ρ); the image of the closure of a set is contained in the closure of the image. By Bloch's Key Lemma A.4, the image of ρ is contained in $\text{Tors } H_{\text{ét}}^{2n-1}(X_{\overline{K}^a}, \mathbb{Z}_\ell(n))$. Now use that $\text{Tors } H_{\text{ét}}^{2n-1}(X_{\overline{K}^a}, \mathbb{Z}_\ell(n)) \subseteq H_{\text{ét}}^{2n-1}(X_{\overline{K}^a}, \mathbb{Z}_\ell(n))$ is closed. Indeed, find $N = \ell^r$ which kills the torsion; multiplication by N is continuous, and the torsion is the inverse image of 0 under this continuous map. \square

A.2.5. *The ℓ -adic Bloch map.* The ℓ -adic Bloch map is defined using the inverse limit of (A.11). We obtain a diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \varprojlim \left(\frac{\ker \partial}{\ell^v \ker \partial} \right) & \longrightarrow & \varprojlim \ker \partial_{\ell^v} & \longrightarrow & T_\ell \text{CH}^n(X_{\overline{K}^a}) \longrightarrow 0 \\
& & \downarrow & \searrow & \downarrow \text{(A.8)} & & \downarrow \\
& & & & \varprojlim H_{\text{Zar}}^{n-1}(X_{\overline{K}^a}, \mathbf{H}^n(\mu_{\ell^v}^{\otimes n})) & & \\
& & & \swarrow \varprojlim \rho_{\ell^v} & \downarrow \text{(A.6)} & & \\
0 & \longrightarrow & \varprojlim \delta_{\ell^v} & \longrightarrow & H^{2n-1}(X_{\overline{K}^a}, \mathbb{Z}_\ell(n)) & \longrightarrow & H^{2n-1}(X_{\overline{K}^a}, \mathbb{Z}_\ell(n)) / \varprojlim \delta_{\ell^v} \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & (\varprojlim \delta_{\ell^v})_\tau & \longrightarrow & H^{2n-1}(X_{\overline{K}^a}, \mathbb{Z}_\ell(n))_\tau & \longrightarrow & H^{2n-1}(X_{\overline{K}^a}, \mathbb{Z}_\ell(n))_\tau / (\varprojlim \delta_{\ell^v})_\tau \longrightarrow 0
\end{array} \tag{A.13}$$

The top row remains short exact after taking the inverse limit, since $\varprojlim^1 (\ker \partial / \ell^v \ker \partial) = 0$. Indeed, the system $(\ker \partial / \ell^v \ker \partial)$ is clearly surjective, by virtue of the fact that the terms are defined by quotients of an increasing chain of subgroups. The δ_{ℓ^v} , being contained in $H_{\text{ét}}^{2n-1}(X_{\overline{K}^a}, \mu_{\ell^v}^{\otimes n})$, are finite, and thus the middle row, which is obtained as the inverse limit of the bottom row of (A.11), remains a short exact sequence, as well. The bottom row is obtained from the middle row by taking the quotient by torsion subgroups in the left and middle entries.

Lemma A.5 and the commutativity of the diagram (A.13) yield

$$(\varprojlim \delta_{\ell^v})_\tau = 0, \tag{A.14}$$

allowing us to define the ℓ -adic Bloch map:

Definition A.6 (ℓ -adic Bloch map). The map

$$T_\ell \text{CH}^n(X_{\overline{K}^a}) \xrightarrow{T_\ell \lambda^n} H^{2n-1}(X_{\overline{K}^a}, \mathbb{Z}_\ell(n))_\tau,$$

which is the negative of the map defined from (A.13) and Lemma A.5, is defined as the ℓ -adic Bloch map in codimension n .

Remark A.7 (ℓ -adic Bloch map over the separable closure). Using the fact that étale cohomology is invariant under purely inseparable extensions and the fact that the prime-to-char(K) torsion of Chow groups is also invariant under purely inseparable extensions (see e.g., [ACMVb, Lem. 4.10]), the ℓ -adic Bloch map over the algebraic closure induces a well defined map $T_\ell \lambda^n : T_\ell \text{CH}^n(X_{\overline{K}}) \rightarrow H^{2n-1}(X_{\overline{K}}, \mathbb{Z}_\ell(n))_\tau$.

A.2.6. *The Bloch map.* Bloch defines his map by considering the direct limit of (A.11):

$$\begin{array}{ccccccc}
0 & \rightarrow & \varinjlim \left(\frac{\ker \partial}{\ell^v \ker \partial} \right) & \longrightarrow & \varinjlim \ker \partial_{\ell^v} & \longrightarrow & \mathrm{CH}^n(X_{\overline{K}^a})[\ell^\infty] \longrightarrow 0 \quad (\text{A.15}) \\
& & \downarrow & & \downarrow (\text{A.8}) & & \downarrow \\
& & \varinjlim \rho_{\ell^v} & \searrow & \varinjlim H_{\mathrm{zar}}^{n-1}(X_{\overline{K}^a}, \mathbf{H}^n(\boldsymbol{\mu}_{\ell^v}^{\otimes n})) & & \\
& & \downarrow & & \downarrow (\text{A.6}) & & \downarrow \\
0 & \longrightarrow & \varinjlim \delta_{\ell^v} & \longrightarrow & H^{2n-1}(X_{\overline{K}^a}, \mathbf{Q}_\ell / \mathbf{Z}_\ell(n)) & \longrightarrow & H^{2n-1}(X_{\overline{K}^a}, \mathbf{Q}_\ell / \mathbf{Z}_\ell(n)) / \varinjlim \delta_{\ell^v} \longrightarrow 0
\end{array}$$

Bloch's observation is that [Blo79, Lem. 2.4] implies the following:

Lemma A.8 ([Blo79, p.112]). *The map*

$$\varinjlim \rho_{\ell^v} : \varinjlim (\ker \partial / \ell^v \ker \partial) \rightarrow H^{2n-1}(X_{\overline{K}^a}, \mathbf{Q}_\ell / \mathbf{Z}_\ell(n))$$

is the zero map.

Proof. To quote Bloch verbatim, the assertion follows from Lemma A.4 using the fact that the image in $H_{\mathrm{ét}}^{2n-1}(X_{\overline{K}^a}, \boldsymbol{\mu}_{\ell^v}^{\otimes n})$ of the torsion in $H^{2n-1}(X_{\overline{K}^a}, \mathbf{Z}_\ell(n))$ is zero in $H^{2n-1}(X_{\overline{K}^a}, \mathbf{Q}_\ell / \mathbf{Z}_\ell(n))$.

In a little more detail, we consider the commutative diagram

$$\begin{array}{ccccc}
\varprojlim \left(\frac{\ker \partial}{\ell^v \ker \partial} \right) & \longrightarrow & \frac{\ker \partial}{\ell^v \ker \partial} & \longrightarrow & \varinjlim \left(\frac{\ker \partial}{\ell^v \ker \partial} \right) \\
\downarrow \varprojlim \rho_{\ell^v} & & \downarrow \rho_{\ell^v} & & \downarrow \varinjlim \rho_{\ell^v} \\
H_{\mathrm{ét}}^{2n-1}(X_{\overline{K}}, \mathbf{Z}_\ell(n)) = \varprojlim H_{\mathrm{ét}}^{2n-1}(X_{\overline{K}}, \boldsymbol{\mu}_{\ell^v}^{\otimes n}) & \longrightarrow & H_{\mathrm{ét}}^{2n-1}(X_{\overline{K}}, \boldsymbol{\mu}_{\ell^v}^{\otimes n}) & \longrightarrow & \varinjlim H_{\mathrm{ét}}^{2n-1}(X_{\overline{K}}, \boldsymbol{\mu}_{\ell^v}^{\otimes n})
\end{array}$$

where the horizontal maps are the natural maps. To show the direct limit $\varinjlim \rho_{\ell^v}$ is the zero map, it suffices to show that for all v , the image of $\frac{\ker \partial}{\ell^v \ker \partial}$ in $\varinjlim H_{\mathrm{ét}}^{2n-1}(X_{\overline{K}}, \boldsymbol{\mu}_{\ell^v}^{\otimes n})$ is zero. To this end, let $\alpha \in \frac{\ker \partial}{\ell^v \ker \partial}$. As we observed before, $\frac{\ker \partial}{\ell^v \ker \partial}$ forms a surjective system, so we may lift α to $\beta \in \varprojlim \frac{\ker \partial}{\ell^v \ker \partial}$, and then send β to $\gamma \in H_{\mathrm{ét}}^{2n-1}(X_{\overline{K}}, \mathbf{Z}_\ell(n))$. By Lemma A.5, γ is torsion. Now we use that the image of torsion, under the composition in the bottom row, is zero.

Since this last assertion is not immediately obvious, we sketch a proof here. Let $(\alpha_1, \alpha_2, \dots)$ be a torsion element of $H_{\mathrm{ét}}^{2n-1}(X_{\overline{K}}, \mathbf{Z}_\ell)$, say of order ℓ^r . One can show that in order for $(\alpha_1, \alpha_2, \dots)$ to be a consistent system, and also be ℓ^r torsion, one must have a $\mathbf{Z} / \ell^r \mathbf{Z}$ summand of the group $H_{\mathrm{ét}}^{2n-1}(X_{\overline{K}}, \mathbf{Z} / \ell^v \mathbf{Z})$ for all sufficiently large v , with $\alpha_v \in \mathbf{Z} / \ell^r \mathbf{Z} \subseteq H_{\mathrm{ét}}^{2n-1}(X_{\overline{K}}, \mathbf{Z} / \ell^v \mathbf{Z})$. Now since $\ell^r \alpha_v = 0$, by definition, this means that in the directed system for the direct limit the image of α_v in $H_{\mathrm{ét}}^{2n-1}(X_{\overline{K}}, \mathbf{Z} / \ell^{v+r} \mathbf{Z})$ is zero (at each step, we multiply by ℓ). Thus the image of α_v in $\varinjlim H_{\mathrm{ét}}^{2n-1}(X_{\overline{K}}, \mathbf{Z} / \ell^v \mathbf{Z})$ is zero. \square

As a consequence of Lemma A.8, we can define the Bloch map; we also refer to [CT93] where a construction of the Bloch map is discussed and to the recent [Sch20, §10] where a construction of the Bloch map that avoids Bloch–Ogus theory [BO74] is given.

Definition A.9 (Bloch map [Blo79, (2.7)]). The map

$$\mathrm{CH}^n(X_{\overline{K}^a})[\ell^\infty] \xrightarrow{\lambda^n} H^{2n-1}(X_{\overline{K}^a}, \mathbf{Q}_\ell / \mathbf{Z}_\ell(n)),$$

which is the negative of the map defined from (A.15) and Lemma A.8, is the *Bloch map in codimension n* . In some cases we will write $\lambda^n[\ell^\infty]$ for clarity.

Remark A.10 (Bloch map over the separable closure). As in Remark A.7, the ℓ -adic Bloch map induces a well defined map $T_\ell \lambda^n : T_\ell \text{CH}^n(X_{\bar{K}}) \rightarrow H^{2n-1}(X_{\bar{K}}, \mathbb{Z}_\ell(n))$.

Remark A.11 (Abel–Jacobi map on torsion). Bloch shows in [Blo79, Prop. 3.7] that for a complex projective manifold X , the Bloch map (Definition A.9) agrees with the Abel–Jacobi map on homologically trivial ℓ -primary torsion (A.5). We explain below, in Remark A.19, that this implies that the ℓ -adic Bloch map (Definition A.6) agrees with the Abel–Jacobi map on homologically trivial cycle classes (A.3).

A.3. Suwa’s construction of the ℓ -adic Bloch map. In [Suw88], Suwa has given a construction of the ℓ -adic Bloch map by simply taking the Tate module associated to the standard Bloch map. We review the construction here, and show it agrees with the construction given in §A.2. This construction is quite convenient in many cases. Later Gros–Suwa [GS88] constructed an extension of the Bloch map to p -torsion; taking the Tate module gives a p -adic Bloch map. We review this in §A.3.5.

A.3.1. Structure of abelian l -primary torsion groups. Let M be an abelian l -primary torsion group for some prime number l . The set of divisible elements M_{div} forms a divisible abelian subgroup, and since divisible abelian groups are injective, we see that M splits as a direct sum $M = M_{\text{div}} \oplus (M/M_{\text{div}})$. It is a basic fact (e.g., [Fuc70, Ch. IV]) that every divisible abelian l -primary torsion group is a direct sum of factors of the form $\mathbb{Q}_l/\mathbb{Z}_l$, so that $M = (\bigoplus \mathbb{Q}_l/\mathbb{Z}_l) \oplus (M/M_{\text{div}})$. We say that M has *finite corank* if there is an injective homomorphism $M \hookrightarrow (\mathbb{Q}_l/\mathbb{Z}_l)^r$ for some integer $r \geq 0$. It is elementary to show that the following are equivalent:

- M has finite corank.
- $M[l]$ is finite.
- $M \simeq (\mathbb{Q}_l/\mathbb{Z}_l)^r \oplus A$ for some integer $r \geq 0$ and some finite l -primary torsion group A .

A.3.2. ℓ -adic cohomology from cohomology with torsion coefficients. In this subsection, we recall a crucial point used in Suwa’s construction of the ℓ -adic Bloch map (see Proposition A.13). We start with a general statement about cohomology.

Proposition A.12. *There is a natural long exact sequence*

$$\cdots \longrightarrow H^{n-1}(X_{\bar{K}}, \mathbb{Z}_\ell) \longrightarrow H^{n-1}(X_{\bar{K}}, \mathbb{Q}_\ell) \longrightarrow H^{n-1}(X_{\bar{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \longrightarrow H^n(X_{\bar{K}}, \mathbb{Z}_\ell) \longrightarrow \cdots \quad (\text{A.16})$$

In particular, if X is proper,

$$H^n(X_{\bar{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \simeq (\mathbb{Q}_\ell/\mathbb{Z}_\ell)^r \oplus A \quad (\text{A.17})$$

for some integer r and some finite ℓ -primary torsion abelian group A .

Proof. Taking the inverse limit over μ of the long exact sequence associated to the short exact sequence of étale sheaves

$$0 \longrightarrow \mathbb{Z}/\ell^\mu \mathbb{Z} \xrightarrow{\ell^\nu} \mathbb{Z}/\ell^{\mu+\nu} \mathbb{Z} \longrightarrow \mathbb{Z}/\ell^\nu \mathbb{Z} \longrightarrow 0 \quad (\text{A.18})$$

gives a long exact sequence

$$\cdots \longrightarrow H^n(X_{\bar{K}}, \mathbb{Z}_\ell) \xrightarrow{\ell^\nu} H^n(X_{\bar{K}}, \mathbb{Z}_\ell) \longrightarrow H^n(X_{\bar{K}}, \mathbb{Z}/\ell^\nu \mathbb{Z}) \longrightarrow H^{n+1}(X_{\bar{K}}, \mathbb{Z}_\ell) \longrightarrow \cdots$$

Taking the direct limit over ν of the system of long exact sequences

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H^n(X_{\bar{K}}, \mathbb{Z}_\ell) & \xrightarrow{\ell^\nu} & H^n(X_{\bar{K}}, \mathbb{Z}_\ell) & \longrightarrow & H^n(X_{\bar{K}}, \mathbb{Z}/\ell^\nu \mathbb{Z}) \longrightarrow H^{n+1}(X_{\bar{K}}, \mathbb{Z}_\ell) \longrightarrow \cdots \\ & & \parallel & & \downarrow \ell & & \downarrow \ell & & \parallel \\ \cdots & \longrightarrow & H^n(X_{\bar{K}}, \mathbb{Z}_\ell) & \xrightarrow{\ell^{\nu+1}} & H^n(X_{\bar{K}}, \mathbb{Z}_\ell) & \longrightarrow & H^n(X_{\bar{K}}, \mathbb{Z}/\ell^{\nu+1} \mathbb{Z}) \longrightarrow H^{n+1}(X_{\bar{K}}, \mathbb{Z}_\ell) \longrightarrow \cdots \end{array}$$

together with Proposition A.1 provides the desired long exact sequence (A.16). Alternately, this can be obtained from the short exact sequence of sheaves $0 \rightarrow \mathbb{Z}_\ell \rightarrow \mathbb{Q}_\ell \rightarrow \mathbb{Q}_\ell/\mathbb{Z}_\ell \rightarrow 0$ in the pro-étale topology ; see [BS15].

The finiteness property (A.17) follows immediately from the finiteness property of $H^n(X_{\bar{K}}, \mathbb{Z}_\ell)$ when X is proper. More elementarily, the finiteness property (A.17) is a consequence of the finiteness of $H_{\text{ét}}^n(X_{\bar{K}}, \mathbb{Z}/\ell\mathbb{Z})$ and the fact (§A.3.1) that any ℓ -primary torsion abelian group M such that $M[\ell]$ is finite is isomorphic to $(\mathbb{Q}_\ell/\mathbb{Z}_\ell)^r \oplus A$ for some integer r and some finite abelian group A . \square

From the long exact sequence (A.1), we obtain a short exact sequence

$$0 \longrightarrow H^{n-1}(X_{\bar{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)/\ell^v \longrightarrow H^n(X_{\bar{K}}, \mathbb{Z}_\ell/\ell^v\mathbb{Z}_\ell) \longrightarrow H^n(X_{\bar{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)[\ell^v] \longrightarrow 0. \quad (\text{A.19})$$

Now with the finiteness property (A.17) and the notation therein, $(\mathbb{Q}_\ell/\mathbb{Z}_\ell)^r$ is the maximal divisible subgroup of $H^n(X_{\bar{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$, and $A = H^n(X_{\bar{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)_{\text{cotors}}$. As $\mathbb{Q}_\ell/\mathbb{Z}_\ell$ is divisible, the ℓ^v -torsion in $\mathbb{Q}_\ell/\mathbb{Z}_\ell$ forms a surjective system. Thus, since A is finite, the associated \varprojlim^1 for the direct sum vanishes, giving [Suu88, (2.6.2)]:

$$0 \longrightarrow \varprojlim_v \left(H^{n-1}(X_{\bar{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)/\ell^v \right) \longrightarrow H^n(X_{\bar{K}}, \mathbb{Z}_\ell) \longrightarrow T_\ell H^n(X_{\bar{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \longrightarrow 0. \quad (\text{A.20})$$

Proposition A.13. *Assume that X is a proper variety over a field K . Then the morphism $H^n(X_{\bar{K}}, \mathbb{Z}_\ell) \rightarrow T_\ell H^n(X_{\bar{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ from (A.20), obtained as the inverse limit of the morphisms $H^n(X_{\bar{K}}, \mathbb{Z}_\ell/\ell^v\mathbb{Z}_\ell) \rightarrow H^n(X_{\bar{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)[\ell^v]$ from (A.19), factors through the quotient $H^n(X_{\bar{K}}, \mathbb{Z}_\ell) \rightarrow H^n(X_{\bar{K}}, \mathbb{Z}_\ell)_\tau$, giving a natural isomorphism :*

$$\begin{array}{ccc} H^n(X_{\bar{K}}, \mathbb{Z}_\ell) & \longrightarrow & T_\ell H^n(X_{\bar{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \\ & \searrow & \uparrow \simeq \\ & & H^n(X_{\bar{K}}, \mathbb{Z}_\ell)_\tau \end{array} \quad (\text{A.21})$$

Proof. It is a basic fact (e.g., [Mil06, Prop. 0.19]) that $T_\ell H^n(X_{\bar{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ is torsion-free. Thus, considering (A.20), we clearly have $\varprojlim_v \left(H^{n-1}(X_{\bar{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)/\ell^v \right) \supseteq \text{Tors } H^n(X_{\bar{K}}, \mathbb{Z}_\ell)$.

As asserted in [Suu88, (2.6.3)], we claim we have equality, completing the proof. Specifically :

$$\varprojlim_v \left(H^{n-1}(X_{\bar{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)/\ell^v \right) = H^{n-1}(X_{\bar{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)_{\text{cotors}} \quad (\text{A.22})$$

$$= \text{Tors } H^n(X_{\bar{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell). \quad (\text{A.23})$$

Here is an explanation of the claim from Michael Spieß. To prove (A.22), we start with (A.17), that $H^{n-1}(X_{\bar{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) = (\mathbb{Q}_\ell/\mathbb{Z}_\ell)^r \oplus A$, with A a finite ℓ -primary torsion abelian group A . Then we consider the short exact sequence

$$0 \rightarrow \ell^v H^{n-1}(X_{\bar{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \rightarrow H^{n-1}(X_{\bar{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \rightarrow H^{n-1}(X_{\bar{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)/\ell^v \rightarrow 0.$$

Its limit is

$$0 \rightarrow \varprojlim_v \ell^v H^{n-1}(X_{\bar{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \rightarrow H^{n-1}(X_{\bar{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \rightarrow \varprojlim_v H^{n-1}(X_{\bar{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)/\ell^v \rightarrow 0.$$

To see that this stays short exact in the limit, we argue as above. More precisely, as $\mathbb{Q}_\ell/\mathbb{Z}_\ell$ is divisible, $\ell^v(\mathbb{Q}_\ell/\mathbb{Z}_\ell)$ forms a surjective system. We also have that $\ell^v A = 0$ for v sufficiently large. Thus the associated \varprojlim^1 for the direct sum vanishes. Now, the image of the inclusion is

exactly $(\mathbb{Q}_\ell/\mathbb{Z}_\ell)^r$, so that $\varprojlim H^{n-1}(X_{\bar{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)/\ell^v$ identifies with A , i.e., with the cotorsion. This completes the proof of (A.22).

For (A.23), the long exact sequence

$$\cdots \rightarrow H^{n-1}(X_{\bar{K}}, \mathbb{Z}_\ell) \rightarrow H^{n-1}(X_{\bar{K}}, \mathbb{Q}_\ell) \xrightarrow{i} H^{n-1}(X_{\bar{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \rightarrow H^n(X_{\bar{K}}, \mathbb{Z}_\ell) \rightarrow H^n(X_{\bar{K}}, \mathbb{Q}_\ell) \rightarrow \cdots$$

of Proposition A.12 provides an identification

$$\text{Tors } H^n(X_{\bar{K}}, \mathbb{Z}_\ell) = H^{n-1}(X_{\bar{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) / \text{Im}(i).$$

The image of i , being the image of a divisible group, is divisible in $H^{n-1}(X_{\bar{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$, and since $\text{Tors } H^n(X_{\bar{K}}, \mathbb{Z}_\ell)$ is finite, $\text{Im}(i)$ must be the maximal divisible subgroup. This means that we have $H^{n-1}(X_{\bar{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) / \text{Im}(i) = H^{n-1}(X_{\bar{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)_{\text{cotors}}$. \square

Lemma A.14 (Functoriality of (A.21)). *Let $f : X \rightarrow Y$ be a morphism of smooth proper varieties over K . Then there are commutative diagrams :*

$$\begin{array}{ccc} H^n(X_{\bar{K}}, \mathbb{Z}_\ell(d_Y - d_X))_\tau \xrightarrow{\cong} T_\ell H^n(X_{\bar{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(d_X - d_Y)) & & H^n(X_{\bar{K}}, \mathbb{Z}_\ell)_\tau \xrightarrow{\cong} T_\ell H^n(X_{\bar{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \\ f_* \downarrow & & f^* \uparrow \\ H^{n+2(d_X - d_Y)}(Y_{\bar{K}}, \mathbb{Z}_\ell)_\tau \xrightarrow{\cong} T_\ell H^{n+2(d_Y - d_X)}(Y_{\bar{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) & & H^n(Y_{\bar{K}}, \mathbb{Z}_\ell)_\tau \xrightarrow{\cong} T_\ell H^n(Y_{\bar{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \\ & & f^* \uparrow \end{array}$$

where the horizontal arrows are the isomorphisms from (A.21). More generally, given a correspondence $\Gamma : X \dashv Y$, we obtain the corresponding commutative diagrams for Γ_* and Γ^* .

Proof. The commutativity of the diagrams follows from the definitions. \square

Example A.15. We note that while there are natural inclusions

$$f_* T_\ell H^n(X_{\bar{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \subseteq T_\ell f_* H^n(X_{\bar{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell), \quad (\text{A.24})$$

$$f^* T_\ell H^n(Y_{\bar{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \subseteq T_\ell f^* H^n(Y_{\bar{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell), \quad (\text{A.25})$$

these need not be equalities. For instance, consider the case where $X = Y = E$ is an elliptic curve over K , and $f : X \rightarrow Y$ is the multiplication by ℓ map. Then the containment $f_* T_\ell H^1(X_{\bar{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \subseteq T_\ell f_* H^1(X_{\bar{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ is the containment $(\ell\mathbb{Z}_\ell)^2 \subsetneq (\mathbb{Z}_\ell)^2$.

In accordance with the example above, the inclusions (A.24) and (A.25) have torsion co-kernels in the case where $n = 1$:

Lemma A.16. *Let $f : X \rightarrow Y$ be a morphism of smooth proper varieties over K . Then we have isomorphisms*

$$(f_* T_\ell H^1(X_{\bar{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \xrightarrow{\cong} T_\ell f_* H^1(X_{\bar{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell,$$

$$f^* T_\ell H^1(Y_{\bar{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \xrightarrow{\cong} T_\ell f^* H^1(Y_{\bar{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

More generally, given a correspondence $\Gamma : X \dashv Y$, we obtain the corresponding commutation relations for Γ_* and Γ^* .

Proof. We will establish the first isomorphism regarding the push-forward. The second is similar, as are the cases of correspondences. From (A.24), we only need to show that the morphism is surjective. We consider the morphism $f_* : H^1(X_{\bar{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \rightarrow f_* H^1(X_{\bar{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \subseteq H^{1+2(d_Y - d_X)}(Y_{\bar{K}}, \mathbb{Z}_\ell/\mathbb{Q}_\ell)$. Then, using that $H^1(X_{\bar{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ is divisible (it is the cohomology of the abelian variety $\text{Pic}_{X/K}^0$), and that the image of a divisible group is divisible, we apply [Suw88, Lem. 1.4] that given a surjective homomorphism $\phi : M \rightarrow N$ of divisible abelian ℓ -primary torsion groups, with N of finite corank, the associated map $T_\ell \phi \otimes 1 : T_\ell M \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \rightarrow T_\ell N \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ is surjective. \square

A.3.3. Suwa's ℓ -adic Bloch map.

Definition A.17 (Suwa's ℓ -adic Bloch map [Suw88, (2.6.5)]). Assume X is a smooth projective variety over K and ℓ is a prime not equal to $\text{char } K$. The ℓ -adic Bloch map for codimension- n cycles is the map

$$T_\ell \text{CH}^n(X_{\bar{K}}) \xrightarrow{T_\ell \lambda^n} H^{2n-1}(X_{\bar{K}}, \mathbb{Z}_\ell(n))_\tau$$

obtained by applying T_ℓ to the Bloch map $\lambda^n : \text{CH}^n(X_{\bar{K}})[\ell^\infty] \rightarrow H^{2n-1}(X_{\bar{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(n))$ and making the identification $T_\ell H^n(X_{\bar{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) = H^n(X_{\bar{K}}, \mathbb{Z}_\ell)_\tau$ from Proposition A.13.

A.3.4. *The ℓ -adic Bloch map and Suwa's construction.* Here we show that Suwa's construction of the ℓ -adic Bloch map agrees with the direct construction.

Proposition A.18. *Let X be a smooth projective variety over K . Then the ℓ -adic Bloch maps of Definition A.17 and Definition A.6 coincide.*

Proof. We first observe that we have the following commutative diagram

$$\begin{array}{ccc} \text{CH}^n(X_{\bar{K}})[\ell^\nu] & \xrightarrow{\lambda^n[\ell^\nu]} & H^{2n-1}(X_{\bar{K}}, \mathbb{Z}/\ell^\nu\mathbb{Z})/\delta_{\ell^\nu} \\ \downarrow & & \downarrow \\ (\text{CH}^n(X_{\bar{K}})[\ell^\infty])[\ell^\nu] & \xrightarrow{(\lambda^n[\ell^\infty])[\ell^\nu]} & H^{2n-1}(X_{\bar{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)[\ell^\nu] \\ \downarrow & & \downarrow \\ \text{CH}^n(X_{\bar{K}})[\ell^\infty] & \xrightarrow{\lambda^n[\ell^\infty]} & H^{2n-1}(X_{\bar{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \end{array}$$

The bottom arrow, $\lambda^n[\ell^\infty] = \lambda^n$ is the Bloch map (Definition A.9), and the bottom square describes the map induced on the ℓ^ν -torsion for the Bloch map. In any case, the bottom square in the diagram is commutative by definition. The top map in the diagram is the ℓ^ν -Bloch map (Definition A.2). The vertical arrows from the top row to the bottom row are the canonical morphisms from the definition of a direct limit. Thus the outer square is commutative. It is only left to describe the vertical arrows from the top to the middle row. These are defined by the fact that $\text{CH}^n(X_{\bar{K}})[\ell^\nu]$ and $H^i(X_{\bar{K}}, \mathbb{Z}/\ell^\nu\mathbb{Z})/\delta_{\ell^\nu}$ are ℓ^ν -torsion groups, so that the images of the vertical arrows from the top row to the bottom row are contained in the ℓ^ν -torsion.

Taking the inverse limit in the top square, we obtain a commutative diagram

$$\begin{array}{ccc} T_\ell \text{CH}^n(X_{\bar{K}}) & \xrightarrow{T_\ell(\lambda^n[\ell^\nu])} & H^i(X_{\bar{K}}, \mathbb{Z}_\ell)_\tau \\ \parallel & & \downarrow \\ T_\ell \text{CH}^n(X_{\bar{K}}) & \xrightarrow{T_\ell(\lambda^n[\ell^\infty])} & T_\ell H^i(X_{\bar{K}}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \end{array}$$

where the top row is the definition of ℓ -adic Bloch map from Definition A.6, while the bottom row is Suwa's definition of the ℓ -adic Bloch map (Definition A.17). The only thing to check is that the vertical arrow in the diagram is the canonical isomorphism from Proposition A.13. But this is clear from the construction of this isomorphism in Proposition A.13 *via* the limit of the maps on the finite levels. \square

Remark A.19 (ℓ -adic Abel–Jacobi map). The same argument as in the proof of Proposition A.18 shows that for a complex projective manifold X , the ℓ -adic Abel–Jacobi map $T_\ell AJ$ (A.3) is equal to the Tate module of the Abel–Jacobi map on torsion $AJ[\ell^\infty]$ (A.5); *i.e.*, $T_\ell AJ = T_\ell(AJ[\ell^\infty])$. Now using the fact that the Bloch map (Definition A.9) agrees with the Abel–Jacobi map on homologically trivial ℓ -primary torsion (A.5) ([Blo79, Prop. 3.7], Remark A.11), it follows that the ℓ -adic Bloch map $T_\ell \lambda^2$ (Definition A.6) agrees with the ℓ -adic Abel–Jacobi map $T_\ell AJ$ on homologically trivial cycle classes (A.3).

A.3.5. *Gross–Suwa’s p -adic Bloch map.* Now suppose that K is a perfect field of characteristic $p > 0$, and recall the notation concerning p -adic cohomology groups. With these conventions, Gros and Suwa have secured p -adic versions of the results reviewed above. So, Proposition A.1 holds by definition; the proof of Proposition A.12 is valid at p , provided one replaces (A.18) with the exact sequence of étale sheaves

$$0 \longrightarrow \mathbb{W}_\mu \Omega_{X,\log}^r \longrightarrow \mathbb{W}_{\mu+\nu} \Omega_{X,\log}^r \longrightarrow \mathbb{W}_\nu \Omega_{X,\log}^r \longrightarrow 0$$

(see also [GS88, Prop. I.4.18] for (A.17)). Gros and Suwa construct a group homomorphism $\lambda^n = \lambda_p^n : \mathrm{CH}^n(X)[p^\infty] \rightarrow H^{2n-1}(X, \mathbb{Q}_p/\mathbb{Z}_p(n))$ [GS88, Def. III.1.25], and, as in Definition A.17, by applying the Tate module obtain a map

$$T_p \mathrm{CH}^n(X_{\bar{K}}) \xrightarrow{T_p \lambda^n} H^{2n-1}(X_{\bar{K}}, \mathbb{Z}_p(n))_\tau.$$

Remark A.20. Moreover, Lemma A.14 holds for p -adic coefficients, as well. Finally, Lemma A.16 holds with $\mathbb{Q}_p/\mathbb{Z}_p$ -coefficients since, as we have seen, $H^1(X_{\bar{K}}, \mathbb{Q}_p/\mathbb{Z}_p)$ is p -divisible of finite corank.

A.4. Properties of the Bloch maps. In this section we fix a field K . The aim of this section consists simply, for future reference, in restating known results due to Bloch [Blo79] concerning the usual Bloch map $\lambda^n : \mathrm{CH}^n(X_{\bar{K}})[l^\infty] \rightarrow H^{2n-1}(X_{\bar{K}}, \mathbb{Q}_l/\mathbb{Z}_l(n))$ in the setting of the l -adic Bloch map $T_l \lambda^n : T_l \mathrm{CH}^n(X_{\bar{K}}) \rightarrow H^{2n-1}(X_{\bar{K}}, \mathbb{Z}_l(n))_\tau$, as well as the finite level Bloch maps. All statements for the l -adic Bloch map are direct consequences of the fact that $T_l \lambda^n$ is simply obtained from λ^n by applying the Tate module functor. Alternatively for $\ell \neq \mathrm{char}(K)$, using the direct definition of the ℓ -adic Bloch map via the ℓ^v -Bloch maps, the proofs in [Blo79] carry over directly. In fact, the proofs in [Blo79] regarding the Bloch map go directly through the corresponding assertions about the ℓ^v -Bloch maps.

Proposition A.21 (Flat pull back and proper push forward). *The Bloch map, the l -adic Bloch, and for all primes $\ell \neq \mathrm{char}(K)$, the ℓ^v -Bloch maps, are functorial for flat pull back and proper push forward.*

Proof. The case of the Bloch map is [Blo79, Prop. 3.3] for $\ell \neq \mathrm{char}(K)$ and [GS88, III. Prop. 2.3] in the p -adic case. The proposition for the l -adic Bloch map can be obtained simply obtained by applying T_l to the case of the Bloch map. For the ℓ^v -Bloch maps, this follows directly from the proof of [Blo79, Prop. 3.3]. \square

Proposition A.22 (Bloch maps and Galois-equivariance). *The Bloch map, the l -adic Bloch map, and for $\ell \neq \mathrm{char}(K)$, the ℓ^v -Bloch maps, are $\mathrm{Aut}(\bar{K}/K)$ -equivariant.*

Proof. Fix $\ell \neq \mathrm{char}(K)$. Let L/K be a finite Galois extension, and let X/K be a smooth projective variety. Then for each $\sigma \in \mathrm{Gal}(L/K)$, applying the previous proposition to the morphisms $\sigma : X_L \rightarrow X_L$ given by the Galois descent data, shows that each of the various Bloch maps is $\mathrm{Gal}(L/K)$ -equivariant. The general case follows by passing to the limit over all finite Galois extensions. For the p -adic case, this is [GS88, III. Prop. 2.1]. \square

Proposition A.23 (Bloch maps and correspondences). *The Bloch map, l -adic Bloch map, and for all primes $\ell \neq \mathrm{char}(K)$, the ℓ^v -Bloch maps, are compatible with the action of correspondences. Precisely, in the case of the l -adic Bloch map, let X and Y be smooth projective varieties over K and let Γ be a cycle on*

$X \times_K Y$ of codimension $\dim Y + n - m$. Then the following natural diagram

$$\begin{array}{ccc} T_l \mathrm{CH}^m(Y_{\bar{K}}) & \xrightarrow{\Gamma_*} & T_l \mathrm{CH}^n(X_{\bar{K}}) \\ T_l \lambda^m \downarrow & & T_l \lambda^n \downarrow \\ H^{2m-1}(Y_{\bar{K}}, \mathbb{Z}_l(m))_{\tau} & \xrightarrow{\Gamma_*} & H^{2n-1}(X_{\bar{K}}, \mathbb{Z}_l(n))_{\tau} \end{array}$$

is commutative.

Proof. The case of the Bloch map is [Blo79, Prop. 3.5] for $\ell \neq \mathrm{char}(K)$ and [GS88, III. Prop. 2.9] for the p -adic case. The proposition for the l -adic Bloch map is simply obtained by applying T_l to the case of the Bloch map. For the ℓ^v -Bloch maps, this follows directly from the proof of [Blo79, Prop. 3.5]. \square

Proposition A.24 (Bloch maps and specialization). *The Bloch map, ℓ -adic Bloch map, and ℓ^v -Bloch maps are compatible with specialization. Precisely, in the case of the ℓ -adic Bloch map, given a local ring R with fraction field K and residue field K_0 , and a smooth projective morphism $\mathcal{X} \rightarrow \mathrm{Spec} R$ with generic fiber X and special fiber X_0 , the following diagram*

$$\begin{array}{ccc} T_{\ell} \mathrm{CH}^n(\bar{X}) & \longrightarrow & T_{\ell} \mathrm{CH}^n(\bar{X}_0) \\ T_{\ell} \lambda^n \downarrow & & T_{\ell} \lambda^n \downarrow \\ H^{2n-1}(\bar{X}, \mathbb{Z}_{\ell}(n))_{\tau} & \xrightarrow{\simeq} & H^{2n-1}(\bar{X}_0, \mathbb{Z}_{\ell}(n))_{\tau} \end{array}$$

is commutative for ℓ prime to $\mathrm{char}(X_0)$. Here, \bar{X} and \bar{X}_0 denote the base-changes of X and X_0 to \bar{K} and \bar{K}_0 , respectively. The top horizontal arrow is obtained by applying T_{ℓ} to the specialization map (e.g., [Ful98, Ex. 20.3.5]), while the bottom horizontal arrow is obtained from smooth proper base-change.

Proof. The case of the Bloch map is [Blo79, Prop. 3.8]. The proposition in the case of the ℓ -adic Bloch map is simply obtained by applying T_{ℓ} to the case of the Bloch map. For the ℓ^v -Bloch maps, this follows directly from the proof of [Blo79, Prop. 3.8]. \square

Proposition A.25 (Bloch maps and Kummer sequence). *Let X be a smooth projective variety over K . The l -adic Bloch map*

$$T_l \lambda^1 : T_l \mathrm{CH}^1(X_{\bar{K}}) \rightarrow H^1(X_{\bar{K}}, \mathbb{Z}_l(1))$$

is the natural isomorphism arising from the Kummer sequence

$$0 \longrightarrow \mathbb{Z}/l^v \mathbb{Z}(1) \longrightarrow \mathbb{G}_m \xrightarrow{l^v} \mathbb{G}_m \longrightarrow 0$$

and the identification $\mathrm{CH}^1(X_{\bar{K}}) = H^1(X_{\bar{K}}, \mathbb{G}_m)$.

Proof. The case of the Bloch map is [Blo79, Prop. 3.6] in the case $\ell \neq \mathrm{char}(K)$ and [GS88, III. Prop. 3.1, Cor. 3.2] in the p -adic case. The proposition is simply obtained by applying T_l to the case of the Bloch map, and noting that $H^1(X_{\bar{K}}, \mathbb{Z}_l(1))$ is torsion-free. \square

Proposition A.26 (Bloch maps and Albanese morphism, Roitman's theorem). *Let X be a smooth projective variety of dimension d over K ; if $l = \mathrm{char}(K)$, assume K is perfect. Then the following diagram*

$$\begin{array}{ccc} T_l \mathrm{CH}^d(X_{\bar{K}}) & \xrightarrow{T_l \lambda^d} & H^{2d-1}(X_{\bar{K}}, \mathbb{Z}_l(d))_{\tau} \\ & \searrow \mathrm{alb} & \parallel \\ & & T_l \mathrm{Alb}_X \end{array}$$

is commutative, where alb is obtained by applying T_l to the map $\text{CH}^d(X_{\bar{K}}) \rightarrow \text{Alb}_X(\bar{K})$ mapping a zero cycle on $X_{\bar{K}}$ to the sum of the corresponding points on the Albanese. Moreover, $T_l \lambda^d$ is an isomorphism.

Proof. The case of the Bloch map is [Blo79, Prop. 3.9] for $\ell \neq \text{char}(K)$ and [GS88, III. Prop. 3.14, Cor. 3.17] in the p -adic case. The commutativity of the diagram is simply obtained by applying T_l to the case of the Bloch map. Finally, that $\text{alb} : T_l \text{CH}^d(X_{\bar{K}}) \rightarrow T_l \text{Alb}_X$ is an isomorphism is due to Roitman [Blo79, Thm. 4.1]. Note that in *loc. cit.* it is stated that $\text{alb} : \text{CH}^d(X_{\bar{K}}) \rightarrow \text{Alb}_X(\bar{K})$ is an isomorphism on torsion for \bar{K} algebraically closed; this implies the needed fact that $\text{CH}^d(X_{\bar{K}}) \rightarrow \text{Alb}_X(\bar{K})$ is an isomorphism on prime-to- $\text{char}(K)$ torsion for \bar{K} separably closed. Indeed, this follows from the general fact that the prime-to- $\text{char}(K)$ torsion in the Chow group of a scheme of finite type over K is invariant under purely inseparable extensions (see e.g., [ACMVb, Lem. 4.10]), and the fact that the prime-to- $\text{char}(K)$ torsion of an abelian variety is invariant under extension of separably closed fields. \square

Finally, we have the following l -adic analogue of a result of due to Merkurjev–Suslin [MS82].

Proposition A.27 (Injectivity of the second Bloch map). *Let X be a smooth projective variety over K ; if $l = \text{char}(K)$, assume K is perfect. The second l -adic Bloch map*

$$T_l \lambda^2 : T_l \text{CH}^2(X_{\bar{K}}) \longrightarrow H^3(X_{\bar{K}}, \mathbb{Z}_l(2))_{\tau}$$

is injective, as are the second ℓ^v -Bloch maps $\lambda^2[\ell^v] : \text{CH}^2(X_{\bar{K}})[\ell^v] \rightarrow H^3(X_{\bar{K}}, \boldsymbol{\mu}_{\ell^v}^{\otimes 2}) / \delta_{\ell^v}$ for $\ell \neq \text{char}(K)$.

Proof. The injectivity of the Bloch map λ^2 in the case $\ell \neq \text{char}(K)$ is due to Merkurjev–Suslin [MS82]; see also [Mur85, Prop. 9.2] and [CTR85, Prop. 3.1 and Rmk. 3.2]. For the p -adic case this is [GS88, III. Prop. 3.4]. The injectivity of the second l -adic Bloch map follows *via* applying T_l to the Bloch map.

For $\ell \neq \text{char}(K)$, the fact that the second ℓ^v -Bloch maps are injective is [Mur85, Prop. 6.1]. Indeed, we consider diagram (A.11). Using K -theoretic techniques, Murre showed that (A.11) can be factored as follows [Mur85, Rem. 6.3]:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \left(\frac{\ker \partial}{\ell^v \ker \partial} \right) & \longrightarrow & \ker \partial_{\ell^v} & \longrightarrow & \text{CH}^n(X_{\bar{K}^a})[\ell^v] \longrightarrow 0 & \quad (\text{A.26}) \\
& & \downarrow & & \downarrow & & \parallel & \\
0 & \longrightarrow & \ker \alpha_v & \longrightarrow & H_{\text{zar}}^{n-1}(X_{\bar{K}^a}, \mathcal{K}_n / \ell^v) & \xrightarrow{\alpha_v} & \text{CH}^n(X_{\bar{K}^a})[\ell^v] \longrightarrow 0 & \\
& & \downarrow & & \cong \downarrow \beta_v & & \downarrow & \\
& & & & H_{\text{zar}}^{n-1}(X_{\bar{K}^a}, \mathbf{H}^n(\boldsymbol{\mu}_{\ell^v}^{\otimes n})) & & \downarrow -\lambda^n[\ell^v] & \\
& & \downarrow & & \downarrow \gamma_v & & \downarrow & \\
0 & \longrightarrow & \delta_{\ell^v} & \longrightarrow & H_{\text{ét}}^{2n-1}(X_{\bar{K}^a}, \boldsymbol{\mu}_{\ell^v}^{\otimes n}) & \longrightarrow & H_{\text{ét}}^{2n-1}(X_{\bar{K}^a}, \boldsymbol{\mu}_{\ell^v}^{\otimes n}) / \delta_{\ell^v} \longrightarrow 0 &
\end{array}$$

Here \mathcal{K}_n is the sheaf on $X_{\bar{K}^a}$ associated to the pre-sheaf that assigns to each Zariski open subset $U \subseteq X_{\bar{K}^a}$ the algebraic K-group $\mathbf{K}_n(\mathcal{O}_{X_{\bar{K}^a}}(U))$ (see e.g., [Mur85, §4.1]). The maps α_v and β_v are defined in [Mur85, Rem. 6.3], and the map γ_v is Murre’s notation for the map defined in (A.6). The fact that β_v is an isomorphism is explained in [Mur85, Rem. 6.3] using [MS82], and this isomorphism then defines the top vertical map in the center of (A.26). The fact that the top right square of the diagram is commutative comes from the construction of α_v in [Mur85, Rem. 6.3]; in fact, the

commutativity of (A.26) is tacitly asserted in [Mur85], so that the construction of the Bloch map given there agrees with that given in [Blo79]. The rest of (A.26) can be established via a diagram chase.

The key point in the case $n = 2$ is that Murre shows in [Mur85, Prop. 6.1, Cor. 5.4(c)] that γ_ν is an inclusion. Applying the snake lemma to the bottom half of (A.26) shows that the second ℓ^ν -Bloch maps are injective. Note that taking the direct limit of the injective ℓ^ν -Bloch maps gives Murre's proof that λ^2 is injective, while taking the inverse limit gives another proof that $T_\ell \lambda^2$ is injective. \square

A.5. Restriction of the Bloch map to algebraically trivial cycle classes. Let X be a smooth projective variety over a field K . From (A.16), we have a diagram with exact row (see e.g., [GS88, (3.33)] for the p -adic case)

$$\begin{array}{c} \text{CH}^n(X_{\bar{K}})[l^\infty] \hspace{15em} \text{(A.27)} \\ \downarrow \lambda^n \quad \dashrightarrow \\ 0 \longrightarrow H^{2n-1}(X_{\bar{K}}, \mathbb{Z}_l(n)) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l/\mathbb{Z}_l \longrightarrow H^{2n-1}(X_{\bar{K}}, \mathbb{Q}_l/\mathbb{Z}_l(n)) \longrightarrow H^{2n}(X_{\bar{K}}, \mathbb{Z}_l(n)) \end{array}$$

where the dashed arrow is, up to sign, the cycle class map ([CTSS83, Cor. 4], [GS88, Prop. III.1.16 and Prop. III.1.21]). Since algebraically trivial cycles are homologically trivial, it follows that the image of $A^n(X_{\bar{K}})[l^\infty]$ under λ^n is contained in $H^{2n-1}(X_{\bar{K}}, \mathbb{Z}_l(n)) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l/\mathbb{Z}_l \subseteq H^{2n-1}(X_{\bar{K}}, \mathbb{Q}_l/\mathbb{Z}_l(n))$. In other words, when we restrict the Bloch map to algebraically trivial cycle classes we obtain a map

$$\lambda^n : A^n(X_{\bar{K}})[l^\infty] \longrightarrow H^{2n-1}(X_{\bar{K}}, \mathbb{Z}_l(n)) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l/\mathbb{Z}_l \subseteq H^{2n-1}(X_{\bar{K}}, \mathbb{Q}_l/\mathbb{Z}_l(n)). \quad \text{(A.28)}$$

Proposition A.28 (Codimension-1). *Let X be a smooth projective variety over K ; if $l = \text{char}(K)$, assume K is perfect. The Bloch map (A.28)*

$$\lambda^1 : A^1(X_{\bar{K}})[l^\infty] \longrightarrow H^1(X_{\bar{K}}, \mathbb{Z}_l(1)) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l/\mathbb{Z}_l$$

is an isomorphism, and taking Tate modules yields an isomorphism

$$T_l \lambda^1 : T_l A^1(X_{\bar{K}}) \longrightarrow H^1(X_{\bar{K}}, \mathbb{Z}_l(1)).$$

Proof. This follows from Proposition A.25. Indeed, we start with the fact that $\text{CH}^1(X_{\bar{K}})/A^1(X_{\bar{K}}) \simeq \text{Pic}_{X/K}(\bar{K})/\text{Pic}_{X/K}^0(\bar{K}) = \text{NS}(X_{\bar{K}})$ is a finitely generated \mathbb{Z} -module. From this we can conclude that $T_l A^1(X_{\bar{K}}) = T_l \text{CH}^1(X_{\bar{K}})$. This gives the result for $T_l \lambda^1$. Then from the identification $A^1(X_{\bar{K}}) = \text{Pic}_{X/K}^0(\bar{K})$, the torsion and Tate modules are free of the same rank, and so we have $A^1(X_{\bar{K}})[l^\infty] = T_l A^1(X_{\bar{K}}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l/\mathbb{Z}_l$. This gives the result for the Bloch map. \square

Proposition A.29 (Bloch maps and Albanese morphisms, Roitman's theorem). *Let X be a smooth projective variety of dimension d over K . Then the following diagram*

$$\begin{array}{ccc} A^d(X_{\bar{K}})[l^\infty] & \xrightarrow{\lambda^d} & H^{2d-1}(X_{\bar{K}}, \mathbb{Z}_l(d)) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l/\mathbb{Z}_l \hookrightarrow H^{2d-1}(X_{\bar{K}}, \mathbb{Q}_l/\mathbb{Z}_l(d)) \\ & \searrow \text{alb} & \parallel \\ & & \text{Alb}_X[l^\infty] \end{array} \quad \text{(A.29)}$$

is commutative, where alb is obtained by restricting to $A^d(X_{\bar{K}})$ the map $\text{CH}^d(X_{\bar{K}}) \rightarrow \text{Alb}_X(\bar{K})$ mapping a zero cycle on $X_{\bar{K}}$ to the sum of the corresponding points on the Albanese. Moreover, λ^d , as well as the inclusion $H^{2d-1}(X_{\bar{K}}, \mathbb{Z}_l(d)) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l/\mathbb{Z}_l \hookrightarrow H^{2d-1}(X_{\bar{K}}, \mathbb{Q}_l/\mathbb{Z}_l(d))$, are isomorphisms.

Taking Tate modules, we obtain a commutative diagram

$$\begin{array}{ccc} T_l A^d(X_{\bar{K}}) & \xrightarrow{T_l \lambda^d} & H^{2d-1}(X_{\bar{K}}, \mathbb{Z}_l(d))_{\tau} \\ & \searrow \text{alb} & \parallel \\ & & T_l \text{Alb}_X \end{array}$$

Moreover, $T_l \lambda^d$ is an isomorphism.

Proof. This just follows from Proposition A.26 and the fact that $A^d(X_{\bar{K}})[l^{\infty}] = \text{CH}^d(X_{\bar{K}})[l^{\infty}]$. Indeed, the commutativity of (A.29) follows from this and the discussion above. Then Proposition A.26 and the fact that $A^d(X_{\bar{K}})[l^{\infty}] = \text{CH}^d(X_{\bar{K}})[l^{\infty}]$ implies that the composition of the top row is an isomorphism. This forces λ^d to be an isomorphism. The result for Tate modules follows immediately. \square

Finally, we have the following l -adic analogue of a result due to Merkurjev–Suslin [MS82].

Proposition A.30 (Injectivity of the second Bloch map). *Let X be a smooth projective variety over K . The second Bloch map*

$$\lambda^2 : A^2(X_{\bar{K}})[l^{\infty}] \longrightarrow H^3(X_{\bar{K}}, \mathbb{Z}_l(2)) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l / \mathbb{Z}_l,$$

the second l -adic Bloch map

$$T_l \lambda^2 : T_l A^2(X_{\bar{K}}) \longrightarrow H^3(X_{\bar{K}}, \mathbb{Z}_l(2))_{\tau},$$

and for all primes $\ell \neq \text{char}(K)$, the second ℓ^v -Bloch maps $\lambda^2[\ell^v] : A^2(X_{\bar{K}})[\ell^v] \rightarrow H^3(X_{\bar{K}}, \mu_{\ell^v}^{\otimes 2}) / \delta_{\ell^v}$, are injective.

Proof. This just follows from Proposition A.27 and from the inclusion of $A^2(X_{\bar{K}}) \subseteq \text{CH}^2(X_{\bar{K}})$. \square

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