Niveau and coniveau filtrations on cohomology groups and Chow groups

Charles Vial

March 19, 2012

Abstract

The Bloch–Beilinson–Murre conjectures predict the existence of a descending filtration on Chow groups of smooth projective varieties which is functorial with respect to the action of correspondences and whose graded parts depend solely on the topology – i.e. the cohomology – of smooth projective varieties. In this paper, given a smooth projective complex variety X, we wish to explore, at the cost of having to assume general conjectures about algebraic cycles, how the coniveau filtration on the cohomology of X has an incidence on the Chow groups of X. However, by keeping such assumptions minimal, we are able to prove some of these conjectures either in low-dimensional cases or when a variety is known to have small Chow groups. For instance, we give a new example of a fourfold of general type with trivial Chow group of zero-cycles and we prove Murre's conjectures for threefolds dominated by a product of curves, for threefolds rationally dominated by the product of three curves, for rationally connected fourfolds and for complete intersections of low degree. The BBM conjectures are closely related to Kimura–O'Sullivan's notion of finite-dimensionality. Assuming the standard conjectures on algebraic cycles the former is known to imply the latter. We show that the missing ingredient for finite-dimensionality to imply the BBM conjectures is the coincidence of a certain niveau filtration with the coniveau filtration on Chow groups.
2010 Mathematics Subject Classification 14C15, 14C25, 14C30

Introduction

Chow groups and cohomology groups constitute important invariants of algebraic varieties. While cohomology groups can usually be computed explicitly, Chow groups are still largely mysterious and computing them is a difficult problem in algebraic geometry. On the arithmetic side it is expected via Beilinson's conjecture that the Chow groups of smooth projective varieties defined over number fields are finite in some sense. Here we wish to focus on the geometric side. Given a smooth projective variety X defined over the complex numbers, Mumford [27] (in the case of surfaces), Roitman [35] and Bloch and Srinivas [5] proved that if the Chow group $\mathrm{CH}_0(X)$ of 0-cycles on X is generated by a point then the cohomology groups of X are supported on a divisor in positive degree. In other words if $\mathrm{CH}_0(X) = \mathbf{Q}$ then $H^i(X) = N^1H^i(X)$ for all $i > 0$. Here N is the so-called coniveau filtration. Bloch has asked if the converse holds true, that is if the coniveau filtration on the cohomology of X controls its Chow groups. More generally as part of the Bloch–Beilinson philosophy it is expected that the coniveau filtration on cohomology should reflect on Chow groups.

Rather than working with the coniveau filtration N on cohomology, we introduce a finer filtration – the niveau filtration $\widetilde{N}$ on homology – whose definition and relation with the usual coniveau filtration N is given in §1.1. The relevance of $\widetilde{N}$ can be read off Proposition 2.2 which shows that if $\mathrm{CH}_0(X) = \mathbf{Q}$ then $H^i(X) = \widetilde{N}^1H^i(X)$ for all $i > 0$. Proposition 1.1 shows, however, that these two filtrations are expected to be the same. This niveau filtration was implicitly used by Schoen in [37] in order to prove his Theorem 0.3 which we recall below as Theorem 2.7.

In this paper, we want to make more precise the link between Chow groups and the support of cohomology groups. A first step consists in giving an algebraic origin to the niveau filtration on homology. This is embodied in Theorem 1. The main general result of this first section is then Theorem 2. In order to state Theorem 1 we have to restrict our attention to a certain class of varieties, those that satisfy the property ($\star$) below. Likewise Theorem 2 only applies to those varieties that in addition to ($\star$) satisfy the property ($\star\star$) below. Let's immediately mention that those conditions are not empty; abelian varieties and Fermat varieties [40] of dimension less than or equal to 5 satisfy those conditions. More importantly, Theorems 1 and 2 make it possible to exhibit some new examples of varieties, the Chow groups of which can be computed, see §2.3. Moreover, the properties ($\star$) and ($\star\star$) are expected to be satisfied by all smooth projective varieties by general conjectures on algebraic cycles.

In the second section, we consider a variety X that satisfies the properties ($\star$) and ($\star\star$) and we derive several propositions relating the support of the cohomology groups of X to the Chow groups of X. Proposition 2.8 gives the most general statement. Proposition 2.3 treats the case of zero-cycles and proves that if there is an integer i such that $H_k(X) = \widetilde{N}^1 H_k(X)$ for all $k > i$, then $\mathrm{CH}_0(X)$ is supported in dimension i. This proposition generalizes previous results : Kimura treated the case when X is a surface [20, Cor. 7.7] and Voisin treated the case $i = 0$ assuming Grothendieck's generalized Hodge conjecture [43, Th. 3]. This section culminates in §2.3 where we exhibit a new example of a fourfold of general type Y whose Chow group of zero-cycles is generated by a point. Precisely, Y is the quotient of the Fermat hypersurface $\{x_0^7 + x_1^7 + \ldots + x_5^7 = 0\} \subset \mathbf{P}^5_{\mathbf{C}}$ by an appropriate free action of the group of seventh roots of unity. Given Proposition 2.8, the key point consists in proving Theorem 2.15 which implies that $H_4(Y) = \widetilde{N}^1 H_4(Y)$, in other words, that the cohomology of Y is generated by that of a surface.

In the spirit of Jannsen's paper [18], we are then interested in relating various conjectures on algebraic cycles. Grothendieck's standard conjectures [21] imply that homological equivalence of algebraic cycles coincides with the coarsest of equivalence relations, namely numerical equivalence. Kimura's finite-dimensionality conjecture [20] (independently stated by O'Sullivan) is an attempt to fill in the gap there exists between the finest equivalence relation, i.e. rational equivalence, and homological equivalence of cycles. However this conjecture is less precise, i.e. weaker (see [1]), than the Bloch–Beilinson–Murre conjectures [17] which describe such a gap in terms of the cohomology of X. In §3, we wish to give a sufficient condition for Kimura's conjecture to imply Murre's conjectures (these are equivalent to Bloch's and Beilinson's by Jannsen [17]). For this purpose, assuming Grothendieck's standard conjectures and Kimura's conjecture, we define thanks to Theorem 2 a filtration on the Chow groups of X that "lifts" the coniveau filtration on the cohomology of X. We prove in Proposition 3.8 that if this filtration on Chow groups coincides with Jannsen's coniveau filtration on Chow groups [18, 5.10(b)] then the BBM conjectures hold.

Finally, in the fourth section, we settle Murre's conjectures (except perhaps for the "independency" conjecture) in some new cases. Examples of varieties for which Murre's conjectures are known to hold true include uniruled 3-folds [8], 3-folds with a nef tangent bundle [14] and elliptic modular 3-folds [11]. Our new examples include 3-folds rationally dominated by the product of three curves, rationally connected 3-folds, Calabi-Yau 3-folds, abelian 3-folds, rationally connected 4-folds, 4-folds admitting a curve as their base for their maximal rationally connected fibration, rationally connected 5-folds with vanishing Hodge number $h^{3,1}$, and some complete intersections of low degree, e.g. cubic 6-folds, quartic 7-folds and the smooth intersection of two quadrics in $\mathbf{P}^{10}$. A known case due to Murre [29] that we recover through our method is given by the product of a curve with a surface.

Notations. Throughout, k denotes a subfield of the field of complex numbers $\mathbf{C}$ and X denotes a smooth projective variety over k of pure dimension d given with an ample line bundle L. We set

$H_i(X) := H_i(X(\mathbf{C}), \mathbf{Q})$; this group is isomorphic to $H^{2d-i}(X(\mathbf{C}), \mathbf{Q})$. The Chow group $\mathrm{CH}_l(X)$ is the group of algebraic cycles of dimension l on X with rational coefficients modulo rational equivalence. Given $\sim$ an adequate equivalence relation [18, 1.3], e.g. algebraic or homological equivalence, $\mathrm{CH}_l(X)_\sim$ denotes those l-cycles that are ~ 0.

Acknowledgements. I am grateful to the referee for his careful reading and for his insightful comments, especially for suggesting Propositions 1.8 and 1.9 and Lemma 4.20. Thanks to François Charles, Burt Totaro and Claire Voisin for their interest, comments and encouragement. This work started while I was a student at the École Normale Supérieure and a PhD student at Trinity College, Cambridge. It is now supported by a Nevile Research Fellowship at Magdalene College, Cambridge and an EPSRC Postdoctoral Fellowship under grant EP/H028870/1. I would like to thank all four institutions for their support.

1 Niveau and Coniveau filtrations on cohomology

Statement of the results

The homology groups of X are endowed with a coniveau filtration that was studied, for example, by Bloch and Ogus [6], but also by Jannsen [17]. There is yet another filtration on $H_i(X)$ which is finer than the coniveau filtration and which is called the niveau filtration. Such a filtration appears implicitly, for instance, in [37]. The niveau and the coniveau filtrations are both of interest and Proposition 1.1 shows that these filtrations are expected to coincide.

The *coniveau filtration* on $H_i(X)$ can be defined as follows :

$$N^jH_i(X) := \sum \mathrm{Im}\left(\Gamma_* : H^{i-2j}(Y) \to H_i(X)\right),$$

where the sum runs through all smooth projective varieties Y and through all correspondences $\Gamma \in \mathrm{CH}_{i-j}(Y \times X)$. The subgroup $N^jH_i(X)$ should be thought of as those classes in $H_i(X)$ that are supported on a closed subscheme of dimension $i - j$.

The *niveau filtration* on $H_i(X)$ can be defined as follows :

$$\widetilde{N}^jH_i(X) := \sum \mathrm{Im}\left(\Gamma_* : H_{i-2j}(Y) \to H_i(X)\right),$$

where the sum runs through all smooth projective varieties Y and through all correspondences $\Gamma \in \mathrm{CH}^{d-j}(Y \times X)$. It can be shown that this sum can be restricted to those varieties Y of dimension $i - 2j$ in which case the correspondence Γ can be thought of as a family of j-cycles parametrized by Y. In particular we immediately see that $\widetilde{N}^jH_i(X) \subseteq N^jH_i(X)$. Because $H_i(X)$ is a finite-dimensional $\mathbf{Q}$-vector space, we can find a smooth projective variety $Z_{i,j}$ of pure dimension $i - 2j$ and a correspondence $\Gamma_{i,j} \in \mathrm{CH}_{i-j}(Z_{i,j} \times X)$ such that $\widetilde{N}^jH_i(X) = (\Gamma_{i,j})_*H_{i-2j}(Z_{i,j})$.

The ample line bundle L on X defines an embedding $X \subseteq \mathbf{P}^N_k$ and hence, for any integer $i \leq d$, a map $L^{d-i} : H_{2d-i}(X) \to H_i(X)$ given by intersecting $d - i$ times with a hyperplane. This map is obviously induced by a correspondence on $X \times X$ and is an isomorphism of Hodge structures (Hard Lefschetz theorem). Given $i \leq d$, we say that X satisfies the property B_i if the isomorphism $L^{i-d} := (L^{d-i})^{-1} : H_i(X) \to H_{2d-i}(X)$ is induced by a correspondence. If X satisfies the property B_i for every i, then we say that X satisfies the property B. The filtration $\widetilde{N}$ on $H_i(X)$ splits canonically into orthogonal pieces as $H_i(X) = \bigoplus_j \mathrm{Gr}^j_{\widetilde{N}} H_i(X)$ with respect to the choice of a polarization. If X satisfies property B then the splitting is also induced by the bilinear form $Q_i := \langle L^{i-d}-, -\rangle$, see Remark 1.5. Condition B is satisfied by abelian varieties [25, 26], curves, surfaces, complete intersections and any products, hypersurface sections or finite quotients thereof. It is a conjecture of Grothendieck (the Lefschetz standard conjecture) that all smooth projective varieties should satisfy property B.

Kahn, Murre and Pedrini [19, §7] gave a decomposition of the numerical motive of varieties satisfying property B with respect to a certain coniveau filtration. Here we wish to refine their result and construct a decomposition modulo homological equivalence. For this purpose we will have to consider varieties satisfying the following property :

$$(\star)\qquad \begin{array}{c} X \text{ satisfies B and, for all } i \leq d \text{ and all } j \geq 1, \\ \text{either there exists } Z_{i,j} \text{ as before satisfying } \mathrm{B}_l \text{ for all } l \leq i-2j-2 \\ \text{or } N^{j+1}H_i(X) = \widetilde{N}^{j+1}H_i(X). \end{array}$$

Because the property B_1 holds for all smooth projective varieties, we see that the property $(\star)$ holds for all smooth projective varieties of dimension at most 5 that satisfy property B. In particular, it holds for curves, surfaces, abelian varieties of dimension ≤ 5, complete intersections of dimension ≤ 5, uniruled 3-folds, rationally connected varieties of dimension ≤ 4, rationally connected 5-folds with $H^3(X, \Omega_X) = 0$ [41]. Property $(\star)$ obviously holds for all varieties if property B holds for all varieties.

Theorem 1. *If X satisfies $(\star)$ then, for all integers i and j, there exists a cycle $\pi_{i,j} \in \mathrm{CH}_d(X \times X)$ inducing the orthogonal projection $H_*(X) \to \mathrm{Gr}^j_{\widetilde{N}} H_i(X) \to H_*(X)$. Moreover, $\pi_{i,j}$ can be chosen to factor through $Z_{i,j}$, i.e. $\pi_{i,j} = f \circ g$ for some $g \in \mathrm{CH}_{d-j}(X \times Z_{i,j})$ and some $f \in \mathrm{CH}_{i-j}(Z_{i,j} \times X)$.*

With the notations of Kahn, Murre and Pedrini [19], Theorem 1 says that the endomorphism $\pi_{i,j}$ of the motive $h(X)$ factors through the motive $h(Z_{i,j})(j)$. Let's mention that the cycles $\pi_{i,j}$ coincide modulo numerical equivalence with the ones constructed in [19, 7.7.3] if X satisfies $(\star)$. This is proved in Proposition 1.8. It follows that modulo homological equivalence the correspondences $\pi_{i,j}$ are central idempotents independent of the choice of L, and we deduce in Proposition 1.9 that the splitting of the niveau filtration on $H_i(X)$ is independent of the choice of a polarization,

Our aim is to show how such a filtration on cohomology reflects on Chow groups. For this purpose it is natural to lift the decomposition of the diagonal $\Delta_X \in \mathrm{CH}_d(X \times X)$ modulo homological equivalence given above to a decomposition into a sum of mutually orthogonal idempotents in $\mathrm{CH}_d(X \times X)$. This is achieved (cf. [17, Lemma 3.1]) by considering varieties X satisfying

$$(\star\star)\qquad \mathrm{Ker}\big(cl : \mathrm{CH}_d(X \times X) \to H_{2d}(X \times X)\big) \text{ is a nilpotent ideal.}$$

Here $\mathrm{CH}_d(X \times X)$ is endowed with a ring structure given by the composition law. For instance, varieties dominated by a product of curves satisfy $(\star\star)$. This includes abelian varieties and Fermat varieties [40]. More generally, any variety that is finite dimensional in the sense of Kimura satisfies $(\star\star)$; see [20, 7.2].

Theorem 2. *Let X be a smooth projective variety that satisfies $(\star)$ and $(\star\star)$. Then there exist mutually orthogonal idempotents $\Pi_{i,j} \in \mathrm{CH}_d(X \times X)$ which are homologically equivalent to the cycles $\pi_{i,j}$ of Theorem 1 and such that $\Delta_X = \sum \Pi_{i,j} \in \mathrm{CH}_d(X \times X)$. For any such choice of idempotents, we have*

1. *$\Pi_{i,j}$ acts as 0 on $\mathrm{CH}_l(X)$ if either $l < j$ or $l > i - j$.*
2. *$\Pi_{i,j}$ acts as 0 on $\mathrm{CH}_l(X)$ if $l = i - j$ and $i < 2l$.*
3. *$\Pi_{i,j}$ acts as 0 on $\mathrm{CH}_l(X)$ if $l + 1 = i - j$ and $i \leq 2l$.*
4. *$\Pi_{i,j} = 0$ if and only if $\mathrm{Gr}^j_{\widetilde{N}} H_i(X) = 0$.*

5. $\mathrm{CH}_i(X)_{\mathrm{hom}} = \mathrm{Ker}\,\big(\Pi_{2i,i} : \mathrm{CH}_i(X) \to \mathrm{CH}_i(X)\big)$.

6. in the case the base field k is algebraically closed, if $AJ_i : \mathrm{CH}_i(X)_{\mathrm{alg}} \to J_i(X) \otimes \mathbf{Q}$ is the Abel–Jacobi map to Griffith's i^{th} intermediate Jacobian tensored with $\mathbf{Q}$ restricted to algebraically trivial cycles, then

$$\mathrm{Ker}\,\big(AJ_i\big) = \mathrm{Ker}\,\big(\Pi_{2i+1,i} : \mathrm{CH}_i(X)_{\mathrm{alg}} \to \mathrm{CH}_i(X)_{\mathrm{alg}}\big).$$

Examples of varieties satisfying ($\star$) and ($\star\star$) include curves, abelian varieties of dimension ≤ 5, Fermat varieties of dimension ≤ 5, as well as their blow-ups along smooth curves. More generally, varieties of dimension ≤ 5 which are dominated by a product of curves satisfy ($\star$) and ($\star\star$). In [41] we show that varieties X for which AJ_i is injective for all $i < d/2$ satisfy ($\star$) and ($\star\star$), this includes Godeaux surfaces [42], Fano threefolds [22] and hypersurfaces of very small degree [31]. Moreover, any variety dominated by one of the varieties above also satisfies ($\star$) and ($\star\star$).

The first three points in the theorem are consequences of theorem 1 together with standard results for varieties satisfying ($\star\star$) (Lemmas 1.10, 1.12, 1.13 and 1.14). The proofs of the two last points are based on the construction of explicit mutually orthogonal projectors $\Pi_{2i,i}$ and $\Pi_{2i+1,i}$ as done in [41], see Propositions 1.16 and 1.17. However, the construction in [41] is valid under less restrictive hypotheses than the ones we are working with. For the sake of completeness, we provide a direct construction of such idempotents under the assumption that X satisfies B, see Theorem 1.15.

1.1 Filtrations on cohomology

Both the niveau and the coniveau filtration define decreasing filtrations on the cohomology ring of a smooth projective variety X over k, each of which satisfies

$$H_i(X) = N^0H_i(X) \supseteq N^1H_i(X) \supseteq \ldots \supseteq N^{\lfloor i/2 \rfloor}H_i(X) \supseteq N^{\lfloor i/2 \rfloor+1}H_i(X) = 0.$$

The aim of this paragraph is to compare such filtrations as well as gather well known and easy properties thereof.

The coniveau filtration N. This filtration is more naturally defined on cohomology. The j^{th} filtered piece of $H^i(X)$ defines a rational sub-Hodge structure of $H^i(X)$:

$$\begin{aligned} N^jH^i(X) &= \sum_{Z \subset X} \mathrm{Ker}\,\big(H^i(X) \to H^i(X-Z)\big) \\ &= \sum_{Z \subset X} \mathrm{Im}\,\big(H^i_Z(X) \to H^i(X)\big) \\ &= \sum_{f : Y \to X} \mathrm{Im}\,\big(f_* : H^{i-2j}(Y) \to H^i(X)\big), \end{aligned}$$

where Z runs through all subschemes of X of codimension $\geq j$, and $f : Y \to X$ runs through all morphisms from a smooth projective variety Y of pure dimension $\leq \dim X - j$ to X. The last equality holds in characteristic zero under resolution of singularities by Hironaka. It holds in positive characteristic (for ℓ-adic cohomology) by de Jong's theorem on alterations; see Jannsen [15, 7.7] and [18, 6.5] for more details. We then transcribe this definition in terms of homology and set

$$N^jH_i(X) := \sum_{\Gamma \in \mathrm{CH}_{i-j'}(Y \times X)} \mathrm{Im}\,\big(\Gamma_* : H^{i-2j'}(Y) \to H_i(X)\big),$$

where the sum runs through all integers $j' \geq j$, all smooth projective varieties Y and all correspondences $\Gamma \in \mathrm{CH}_{i-j'}(Y \times X)$. By considering $\mathbf{P}^{j'-j} \times Y$ instead of Y and $\mathbf{P}^{j'-j} \times \Gamma$ instead of Γ we see that it is superfluous to consider those integers $j' > j$. Therefore

$$N^j H_i(X) = \sum_{\Gamma \in \mathrm{CH}_{i-j}(Y \times X)} \mathrm{Im}\,\big(\Gamma_* : H^{i-2j}(Y) \to H_i(X)\big),$$

where the sum runs through all smooth projective varieties Y and all correspondences $\Gamma \in \mathrm{CH}_{i-j}(Y \times X)$. As before, using resolution of singularities, we see that actually

$$N^j H_i(X) = \sum_{f : Y \to X} \mathrm{Im}\,\big(f_* : H^{i-2j}(Y) \to H_i(X)\big),$$

where the sum runs through all morphisms $f : Y \to X$ from a smooth projective variety Y of pure dimension $\leq i - j$ to X.

The niveau filtration $\widetilde{N}$. This filtration is more naturally defined in homology. By the weak Lefschetz theorem, instead of considering varieties Y of dimension $i - j$, we can consider smooth linear sections Z of Y of dimension $i - 2j$. Such a section induces a surjection $H_{i-2j}(Z) \twoheadrightarrow H_{i-2j}(Y) \xrightarrow{\simeq} H_i(Y)$ which is induced by an algebraic correspondence if property B holds for Y.

We are thus led to consider families $\Gamma \in \mathrm{CH}_{i-j}(Z \times X)$ of j-cycles parametrized by Z and look at the image of $\Gamma_* : H_{i-2j}(Z) \to H_i(X)$. We then define

$$\widetilde{N}^j H_i(X) := \sum_{\Gamma \in \mathrm{CH}^{d-j'}(Z \times X)} \mathrm{Im}\,\big(\Gamma_* : H_{i-2j'}(Z) \to H_i(X)\big),$$

where the union runs through all integers $j' \geq j$, all smooth projective varieties Z and all correspondences $\Gamma \in \mathrm{CH}^{d-j'}(Z \times X)$. Here we let j' and the dimension of Z vary for greater flexibility. However, this is not needed. Indeed, as in the case of the coniveau filtration, if $j' > j$, then replace Z by $\mathbf{P}^{j'-j} \times Z$ and Γ by $\{0\} \times \Gamma \in \mathrm{CH}^{d-j}(\mathbf{P}^{j'-j} \times Z \times X)$. Therefore, we can fix j' to be equal to j in the sum defining the niveau filtration. Let's now show that it is possible to restrict the sum to varieties Z of dimension $i - 2j$. If $\dim Z > i - 2j$, then any smooth linear section $\iota : Z' \hookrightarrow Z$ of dimension $i - 2j$ induces a surjection $H_{i-2j}(Z') \to H_{i-2j}(Z)$. Replace Z by Z' and Γ by $\Gamma \circ \iota$. If $\dim Z < i - 2j$, then replace Z with $\mathbf{P}^a \times Z$ ($a = i - 2j - \dim Z$) and Γ by $\mathbf{P}^a \times \Gamma \in \mathrm{CH}^{d-j}(\mathbf{P}^a \times Z \times X)$.

Therefore, we have

$$\widetilde{N}^j H_i(X) = \sum_{\Gamma \in \mathrm{CH}_{i-j}(Z \times X)} \mathrm{Im}\,\big(\Gamma_* : H_{i-2j}(Z) \to H_i(X)\big),$$

where the union runs through all smooth projective varieties Z of dimension $i - 2j$ and all correspondences $\Gamma \in \mathrm{CH}_{i-j}(Z \times X)$ $(= \mathrm{CH}^{d-j}(Z \times X))$.

Properties. The last given characterization of the niveau filtration shows

$$\widetilde{N} \subseteq N.$$

Because the Lefschetz isomorphism $H_i(Z) \to H^i(Z)$ is induced by a correspondence for $i = 0$ or $i = 1$, we always have $\widetilde{N}^{\lfloor i/2 \rfloor} H_i(X) = N^{\lfloor i/2 \rfloor} H_i(X)$. More generally, it is expected that these two filtrations agree:

Proposition 1.1. *Suppose property B holds for all smooth projective varieties of dimension $< \dim X$. Then, property B holds for X if and only if $N^j H_i(X) = \widetilde{N}^j H_i(X)$ for all i and all j.*

Proof. Assume $f : Y \to X$ is a morphism of smooth projective varieties with $\dim Y = i - j$ and let Z be a smooth linear section of Y of dimension $i - 2j$. The discussion above shows that the composition $H_{i-2j}(Z) \twoheadrightarrow H_{i-2j}(Y) \xrightarrow{\simeq} H_i(Y) \xrightarrow{f_*} H_i(X)$ is induced by a correspondence $\Gamma \in \mathrm{CH}_{i-j}(Z \times X)$; it clearly maps $H_{i-2j}(Z)$ onto the image of f_*, which proves that $N^j H_i(X) = \widetilde{N}^j H_i(X)$.

If N and $\widetilde{N}$ agree on $H_i(X)$ for $i > d$, then in particular $N^{i-d} H_i(X) = \widetilde{N}^{i-d} H_i(X)$. It is easy to see from the definition that $N^{i-d} H_i(X) = H_i(X)$. Therefore, there is a smooth projective variety Z of dimension $2d - i$ and a correspondence $\Gamma \in \mathrm{CH}_d(Z \times X)$ such that $\Gamma_* : H_{2d-i}(Z) \to H_i(X)$ is surjective. Because Z is assumed to satisfy property B, there is a correspondence $s \in \mathrm{CH}_{2d-i}(Z \times Z)$ such that $H_{2d-i}(Z)$ endowed with the pairing $\langle -, s_* - \rangle$ is polarized, see [21]. Hence, thanks to Lemma 1.6 below, the correspondence $\Gamma \circ s \circ {}^t\Gamma$ induces an isomorphism $H_{2d-i}(X) \xrightarrow{\simeq} H_i(X)$. Thus $\alpha := \Gamma \circ s \circ {}^t\Gamma \circ L^{i-d}$ induces an automorphism of $H_i(X)$. By the theorem of Cayley–Hamilton, its inverse is given by $P(\alpha)_*$ for some rational polynomial P. It follows that $P(\alpha) \circ \Gamma \circ s \circ {}^t\Gamma$ induces the inverse to $(L^{i-d})_*$. □

The niveau and the coniveau filtrations behave functorially with respect to the action of correspondences:

Proposition 1.2. *Let* $\alpha \in \mathrm{CH}_{d+l}(X \times Y)$*, then* $\alpha_* N^j H_i(X) \subseteq N^{j+l} H_{i+2l}(Y)$ *and* $\alpha_* \widetilde{N}^j H_i(X) \subseteq \widetilde{N}^{j+l} H_{i+2l}(Y)$*.*

Proof. The proof is straightforward from the definition of the niveau and coniveau filtrations. □

There is a non-degenerate bilinear form Q_i on $H_i(X)$ given by

$$Q_i(\alpha, \beta) = \langle L^{i-d}\alpha, \beta \rangle,$$

where $\langle -, - \rangle : H_i(X) \otimes_{\mathbf{Q}} H_{2d-i}(X) \to \mathbf{Q}$ denotes the cup product. The primitive part of $H_i(X)$ is defined to be $H_i(X)_{\mathrm{prim}} := \mathrm{Ker}\big(L^{i-d+1} : H_i(X) \to H_{2d-i-2}(X)\big)$. We have the Lefschetz decomposition formula

$$H_i(X) = \bigoplus_{0 \leq 2r \leq 2d-i} L^r H_{i+2r}(X)_{\mathrm{prim}}.$$

The Hodge index theorem states that the above decomposition is orthogonal for Q_i and that the sub-Hodge structure $L^r H_{i+2r}(X, \mathbf{Q})_{\mathrm{prim}}$ endowed with the form $(-1)^r Q_i$ is polarized. We let $p_{i,r}$ denote the orthogonal projector

$$H_i(X, \mathbf{Q}) \to L^r H_{i+2r}(X, \mathbf{Q})_{\mathrm{prim}} \to H_i(X, \mathbf{Q}).$$

Polarized Hodge structures have the following well-known particularity.

Lemma 1.3. *Let H be a polarized rational Hodge structure and let K be a sub-Hodge structure. Then the pairing on H remains non-degenerate after restriction to K.*

Proof. This is because the associated hermitian form on $H \otimes \mathbf{C}$ remains non-degenerate when restricted to the pieces $K_{p,q}$ of the Hodge decomposition of $K \otimes \mathbf{C}$. □

Proposition 1.4. *If X satisfies property B, then, for all i and all j, the cup product pairings*

$$\widetilde{N}^j H_i(X) \otimes \widetilde{N}^{d-i+j} H_{2d-i}(X) \to \mathbf{Q} \quad \text{and} \quad N^j H_i(X) \otimes N^{d-i+j} H_{2d-i}(X) \to \mathbf{Q}$$

are non-degenerate.

Proof. The arguments that follow work equally well for the filtration N.

First we have $L_*^{d-i}\widetilde{N}^j H_i(X) = \widetilde{N}^{d-i+j}H_{2d-i}(X)$. Indeed, if $i \leq d$, then, by Proposition 1.2, $L_*^{d-i}\widetilde{N}^j H_i(X) \subseteq \widetilde{N}^{d-i+j}H_{2d-i}(X)$. Because L_*^{d-i} is invertible as a correspondence, the reverse inclusion holds. It is thus enough to show that $(Q_i)|_{\widetilde{N}^j H_i(X)}$ is non-degenerate.

Secondly, Kleiman showed [21, Theorem 4.1.(3)] that if X satisfies B, then the projectors $p_{i,r}$ are induced by algebraic correspondences $P_{i,r} \in \mathrm{CH}_d(X \times X)$. Therefore, $(p_{i,r})_*\widetilde{N}^j H_i(X) = (P_{i,r})_*\widetilde{N}^j H_i(X)$ and, by Proposition 1.2, we have $(p_{i,r})_*\widetilde{N}^j H_i(X) \subseteq \widetilde{N}^j H_i(X)$. Hence

$$\widetilde{N}^j H_i(X) = \bigoplus_{0 \leq 2r \leq 2d-i} (p_{i,r})_*\widetilde{N}^j H_i(X).$$

Therefore, thanks to Lemma 1.3, the form Q_i restricts to a non-degenerate form on $\widetilde{N}^j H_i(X)$. □

Remark 1.5. The proof of the proposition shows in particular that if X satisfies property B, then both filtrations N and $\widetilde{N}$ on $H_i(X)$ split into orthogonal pieces for the form Q_i, e.g. as $H_i(X) = \bigoplus \mathrm{Gr}^j_{\widetilde{N}} H_i(X)$. Moreover we see that if $H_i(X)$ is polarized with respect to the choice of the ample line bundle L, then the splitting of the coniveau (resp. niveau) filtration N (resp. $\widetilde{N}$) on $H_i(X)$ induced by the polarization identifies canonically with the splitting induced by the bilinear form Q_i.

1.2 Proof of Theorem 1

We start with two well-known lemmas.

Lemma 1.6 (Lemma 5 in [43])**.** *Let H and H' be rational Hodge structures endowed, respectively, with a polarization Q and Q'. Let $\gamma : H \to H'$ be a morphism of Hodge structures. Then*

$$\mathrm{Im}\,(\gamma) = \mathrm{Im}\,(\gamma \circ \gamma^\vee),$$

where H is identified with its dual $H^\vee$ via the polarization Q it carries.

Proof. By linear algebra, we have $\dim_{\mathbf{Q}} H = \dim_{\mathbf{Q}} \mathrm{Ker}\,\gamma + \dim_{\mathbf{Q}} \mathrm{Im}\,\gamma^\vee$. Therefore, it is enough to prove that the subspace $\mathrm{Ker}\,\gamma \cap \mathrm{Im}\,\gamma^\vee$ is zero. The subspace $\mathrm{Ker}\,\gamma \cap \mathrm{Im}\,\gamma^\vee$ actually defines a sub-Hodge structure of H and we claim that the pairing Q restricts to zero on this subspace. Indeed, let z and $\gamma^\vee y$ be elements in $\mathrm{Ker}\,\gamma \cap \mathrm{Im}\,\gamma^\vee$. Then we have

$$Q(\gamma^\vee y, z) \ = \ Q'(y, \gamma z) \ = \ Q'(y, 0) \ = 0.$$

By Lemma 1.3, $\mathrm{Ker}\,\gamma \cap \mathrm{Im}\,\gamma^\vee = \{0\}$. □

Lemma 1.7 (Lemma 7 in [7])**.** *Let Z be a smooth projective variety of dimension $\leq k$ endowed with an ample line bundle L_Z. If Z satisfies properties B_l for all $l \leq 2\dim Z - k - 2$, then for all $r \geq 0$ there exists a correspondence whose action on $H_k(Z)$ is the orthogonal projector $H_k(Z) \to L_Z^r H_{k+2r}(Z)_{\mathrm{prim}} \to H_k(Z)$.*

Proof. By induction on r it is enough to prove that there is a correspondence whose action on $H_k(Z)$ is the projector $H_k(Z) \to L_Z H_{k+2}(Z) \to H_k(Z)$. For this purpose, let α be a correspondence in $\mathrm{CH}_{k+2}(Z \times Z)$ inducing the inverse to the Lefschetz isomorphism $L_Z^{k+2-\dim Z} : H_{k+2}(Z) \to H_{2\dim Z-k-2}(Z)$. Then $L_Z \circ \alpha \circ L_Z^{k+1-\dim Z}$ is the required correspondence because $H_k(Z)_{\mathrm{prim}}$ is the kernel of the action of $L_Z^{k+1-\dim Z}$ on $H_k(Z)$. □

We fix i and j with $j \geq 1$. By definition of the niveau filtration $\widetilde{N}$, let Z be a smooth projective variety of pure dimension $i - 2j$ and let $\Gamma \in \mathrm{CH}_{i-j}(Z \times X)$ be a correspondence such that

$$\widetilde{N}^j H_i(X) = \mathrm{Im}\,\big(\Gamma_* : H_{i-2j}(Z) \to H_i(X)\big).$$

Because X satisfies B, there is a cycle $\pi_i \in \mathrm{CH}_d(X \times X)$ whose homology class is the projector $H_*(X) \twoheadrightarrow H_i(X) \hookrightarrow H_*(X)$ (see [21]). Hence, considering $\pi_i \circ \Gamma$ instead of Γ we can assume that

$$\widetilde{N}^j H_i(X) = \mathrm{Im}\big(\Gamma_* : H_*(Z) \to H_*(X)\big).$$

Moreover, for weight reasons, Γ acts as zero on $H_k(Z)$ for $k \neq i - 2j$.

Supposing that we have already exhibited the cycles $\pi_{i,k} \in \mathrm{CH}_d(X \times X)$ for $k > j$, we are going to construct a cycle $\pi_{i,j}$ as in the statement of the theorem. Note that, for $k > i/2$, $\pi_{i,k} = 0$ is suitable. Consider then the cycle $\gamma := \big(\pi_i - \sum_{k>j} \pi_{i,k}\big) \circ \Gamma$. The induced morphism $\gamma_* : H_*(Z) \to H_*(X)$ has $\mathrm{Gr}^j_{\widetilde{N}} H_i(X)$ for image, that is the orthogonal complement of $\widetilde{N}^{j+1} H_i(X)$ inside $\widetilde{N}^j H_i(X)$ for the form Q_i.

Write L for the Lefschetz isomorphism $L : H_i(X) \xrightarrow{\simeq} H_{2d-i}(X)$, which is assumed to be induced by an algebraic correspondence. Because $\gamma_* H_*(Z) = \mathrm{Gr}^j_{\widetilde{N}} H_i(X)$ and by Remark 1.5, we see that the transpose of γ induces a morphism ${}^t\gamma_* \circ L : H_i(X) \to H_{i-2j}(Z)$ which is zero on the orthogonal complement of $\mathrm{Gr}^j_{\widetilde{N}} H_i(X)$ (for the form Q_i or for the polarization induced by L; these split $\widetilde{N}$ the same way). Consider now the Lefschetz decomposition $H_{i-2j}(Z) = \bigoplus_{r \geq 0} L_Z^r H_{i-2j+2r}(Z)_{\mathrm{prim}}$ and write p_r for the orthogonal projector onto $L_Z^r H_{i-2j+2r}(Z)_{\mathrm{prim}}$. There are two cases at hand.

If we are assuming that $N^{j+1} H_i(X) = \widetilde{N}^{j+1} H_i(X)$, then $\gamma \circ p_r = 0$ unless $r = 0$. Indeed, if $x \in L_Z^r H_{i-2j+2r}(Z)_{\mathrm{prim}}$, then $\gamma_*(x) = (\gamma \circ L_Z^r)_*(y)$ for some $y \in H_{i-2j+2r}(Z)_{\mathrm{prim}} = H^{i-2j-2r}(Z)_{\mathrm{prim}}$ so that $\gamma_*(x) \in N^{j+r} H_i(X)$. Thus, if $r > 0$, then $\gamma_*(x) \in N^{j+1} H_i(X) = \widetilde{N}^{j+1} H_i(X)$ and hence $\gamma_*(x) = 0$. Therefore, the action of γ on $H_*(Z)$ factors through the polarized Hodge structure $H_{i-2j}(Z)_{\mathrm{prim}}$. We then let $\varphi := \gamma \circ {}^t\gamma \circ L \in \mathrm{CH}_d(X \times X)$.

If we are assuming that Z satisfies property B_l for all $l \leq i - 2j - 2$, then the morphism $s_Z := \sum (-1)^r p_r$ on $H_{i-2j}(Z)$ is induced by a correspondence thanks to Lemma 1.7 and $\langle -, s_Z(-) \rangle$ defines a polarization on $H_{i-2j}(Z)$. We then let $\varphi := \gamma \circ s_Z \circ {}^t\gamma \circ L \in \mathrm{CH}_d(X \times X)$.

Either way, Lemma 1.6 gives

$$\varphi_* H_*(X) = \gamma_* H_*(Z) = \mathrm{Gr}^j_{\widetilde{N}} H_i(X).$$

Therefore, φ restricts to an isomorphism of the finite dimensional $\mathbf{Q}$-vector space $\mathrm{Gr}^j_{\widetilde{N}} H_i(X)$. By the theorem of Cayley–Hamilton, there exists a polynomial $P \in \mathbf{Q}[X]$ such that $(\varphi|_{\mathrm{Gr}^j_{\widetilde{N}} H_i(X)})^{-1} = P(\varphi|_{\mathrm{Gr}^j_{\widetilde{N}} H_i(X)})$. Then let ψ be the correspondence $P(\varphi)$. The composite $\pi_{i,j} := \psi \circ \varphi$ thus induces the identity on the rational Hodge structure $\mathrm{Gr}^j_{\widetilde{N}} H_i(X)$ and is zero on its orthogonal complement, i.e. it is the required cycle.

Finally, we set $\pi_{i,0} = \pi_i - \sum_{j \geq 1} \pi_{i,j}$ and this completes the proof of Theorem 1. □

We now show that the cycles of Theorem 1 coincide with the ones given by Kahn–Murre–Pedrini [19, §7.7].

Proposition 1.8. *Let X be a smooth projective variety that satisfies $(\star)$. Then the cycles $\pi_{i,j}$ of Theorem 1 coincide modulo homological equivalence with the ones given in [19, §7.7]*

Proof. The construction in [19] is performed under the only assumption that X satisfies B. Thus, to start with, suppose merely that X satisfies B. Since we are working in characteristic zero, it follows [21] that homological and numerical equivalence coincide on X and $X \times X$. It follows from Jannsen's theorem [16] that the homological motive $h^{\mathrm{hom}}(X)$ and its Tate twists are contained in a full semisimple abelian subcategory of motives for homological equivalence. We have as in [19, 7.7.3] a unique decomposition

$$h_i^{\mathrm{hom}}(X) = \bigoplus_j h_{i,j}^{\mathrm{hom}}(X),$$

where $h^{\hom}_{i,j}(X)(-j)$ is effective while $h^{\hom}_{i,j}(X)(-j-1)$ has no non-zero direct summand that is effective. Any direct summand M of $h^{\hom}_i(X)$ for which $M(-j)$ is effective is then contained in $\bigoplus_{j'\geq j} h^{\hom}_{i,j'}(X)$. Applying the homology functor H to the above decomposition gives a similar decomposition of $H_i(X)$ into subspaces $H_{i,j}(X)$. The $H_{i,j}(X)$ are mutually orthogonal, because the polarization on $H_i(X)$ arises from a polarization on $h^{\hom}_i(X)$. Since $h^{\hom}_{i,j}(X)(-j)$ is effective, there is, for some Y, a retraction $h^{\hom}(Y)(j) \to h^{\hom}_{i,j}(X)$. If $\Gamma : h^{\hom}(Z)(j) \to h^{\hom}(X)$ is its composite with the embedding, then $\Gamma_* : H_{i-2j}(Z) \to H_i(X)$ has image $H_{i,j}(X)$. Thus,

$$\bigoplus_{j'\geq j} H_{i,j'}(X) \subseteq \widetilde{N}^j H_i(X). \tag{1}$$

Suppose now that $(\star)$ holds for X and let $\pi_{i,j}$ be a cycle as in Theorem 1 : its action on $H_*(X)$ induces the orthogonal projection on $\mathrm{Gr}^j_{\widetilde{N}} H_i(X)$ and there are a smooth projective variety Z and correspondences $f \in \mathrm{CH}_{i-j}(Z \times X)$ and $g \in \mathrm{CH}_{d-j}(X \times Z)$ such that $\pi_{i,j} = f \circ g$. In particular, $f \circ g$ is an idempotent modulo homological equivalence. The image of $f \circ g$ is a homological motive M which is a direct summand of $h^{\hom}(X)$. The composite of the map $h^{\hom}(Z)(j) \to h^{\hom}(X)$ induced by f with the projection $\mathfrak{h}^{\hom}(X) \to M$ is a retraction, and thus M is a direct summand of $h^{\hom}(Z)(j)$. It follows that M is contained in $\bigoplus_{j'\geq j} h^{\hom}_{i,j'}(X)$. Applying H then shows that the image $\mathrm{Gr}^j_{\widetilde{N}} H_i(X)$ of $(f \circ g)_*$ is contained in the left-hand side of (1). Thus (1) is an equality. Both the $\mathrm{Gr}^j_{\widetilde{N}} H_i(X)$ and the $H_{i,j}(X)$ therefore give orthogonal splittings of the filtration $\widetilde{N}$ of $H_i(X)$, so that

$$\mathrm{Gr}^j_{\widetilde{N}} H_i(X) = H_{i,j}(X)$$

as wanted. □

Since the decomposition of Kahn–Murre–Pedrini is unique modulo homological equivalence, it follows that the idempotents $\pi_{i,j}$ when considered as endomorphisms of the homological motive $h^{\hom}(X)$ are unique. In particular, the cycles $\pi_{i,j}$ are central endomorphisms of $h^{\hom}(X)$.

Proposition 1.9. *Let X be a smooth projective variety that satisfies $(\star)$. Then the splitting of the niveau filtration on $H_i(X)$ is independent of the choice of polarization.*

Proof. Let's actually show that any two splittings of the niveau filtration on $H_i(X)$ that are defined by algebraic cycles coincide. This will prove the proposition because a splitting induced by the choice of a polarization is induced by algebraic cycles by Theorem 1. Let $\pi'_{i,j}$ be cycles in $\mathrm{CH}_d(X \times X)$ (not necessarily satisfying the conditions of Theorem 1) such that the endomorphism $(\pi'_{i,j})_*$ of $H_i(X)$ induced by the cycles $\pi'_{i,j}$ are idempotents which give a splitting (not necessarily defined by a polarization) of the niveau filtration. Then it is easily seen that the action of $\pi'_{i,j}$ on $H_i(X)$ coincides with the action of $\pi_{i,j}$, using descending induction on j and the fact that two idempotent endomorphisms of a vector space coincide when they commute with one another and have the same image. □

1.3 Proof of Theorem 2

A standard lifting lemma, proof of 4. in Theorem 2.

Lemma 1.10 (Particular case of [17], Lemma 3.1)**.** *Let X be a variety satisfying $(\star\star)$. Then the following statements hold.*

1. *Let $p \in \mathrm{CH}_d(X \times X)$ be an idempotent. If $p_* H_*(X) = 0$ then $p = 0$.*
2. *Let $c_1, \ldots, c_n \in \mathrm{CH}_d(X \times X)$ be correspondences such that $cl(c_i) \in H_*(X \times X) = \mathrm{End}(H_*(X))$ define mutually orthogonal idempotents adding to the identity. Then there exist mutually orthogonal idempotents $p_1, \ldots, p_n \in \mathrm{CH}_d(X \times X)$ adding to the identity in*

$\mathrm{CH}_d(X \times X)$ *such that* $cl(p_i) = cl(c_i)$ *for all* $1 \leq i \leq n$. *Moreover, any two such choices* $\{p_1, \ldots, p_n\}$ *and* $\{p'_1, \ldots, p'_n\}$ *are conjugate by an element lying above the identity, i.e. there exists a nilpotent correspondence* $n \in \mathrm{CH}_d(X \times X)$ *such that* $p'_i = (1+n) \circ p_i \circ (1+n)^{-1}$ *for all* $1 \leq i \leq n$.

Therefore if X is a variety satisfying both $(\star)$ and $(\star\star)$ then the correspondences $\pi_{i,j}$ of Theorem 1 can be chosen to be idempotents adding to $\Delta_X \in \mathrm{CH}_d(X \times X)$ and any two such choices are conjugate. Let's fix such a choice and write such idempotents $\Pi_{i,j}$. As an immediate consequence of the above lemma, we settle 4. of Theorem 2 :

Proposition 1.11. *If* X *satisfies* $(\star)$ *and* $(\star\star)$ *and if* $\mathrm{Gr}^j_{\widetilde{N}} H_i(X) = 0$ *for some integers* i *and* j, *then* $\Pi_{i,j} = 0$.

Proof. The identity $\mathrm{Gr}^j_{\widetilde{N}} H_i(X) = 0$ means that the homology class of the idempotent $\Pi_{i,j}$ is zero. Part (1) of Lemma 1.10 implies that $\Pi_{i,j} = 0$. □

Three lemmas.

Lemma 1.12. *Let* X *be a variety that satisfies* $(\star\star)$. *Let* π *be an idempotent and* γ *be a correspondence in* $\mathrm{CH}_d(X \times X)$, *both acting on* $\mathrm{CH}_*(X)$. *Let also* S *be a subset of* $\mathrm{Ker}\,\gamma$ *which is stabilized by all nilpotent correspondences in* $\mathrm{CH}_d(X \times X)$. *Then, if* π *and* γ *have same homology class,* $S \subseteq \mathrm{Ker}\,\pi$.

Proof. By Lemma 1.10, there exists a nilpotent correspondence $n \in \mathrm{CH}_d(X \times X)$ such that $\pi = \gamma + n$. Let N be the nilpotent index of n. Let $x \in S \subseteq \mathrm{Ker}\,\gamma$. Then $\pi(x) = (\gamma + n)(x)$. Therefore, $\pi^{\circ N}(x) = (\gamma + n)^{\circ N}(x)$. By assumption n stabilizes S which is a subset of the kernel of γ. Moreover, $n^{\circ N} = 0$. Hence $\pi(x) = 0$, that is $x \in \mathrm{Ker}\,\pi$. □

Lemma 1.13. *Let* X *be a variety that satisfies* $(\star\star)$. *Let* π *and* π' *be two idempotents in* $\mathrm{CH}_d(X \times X)$, *both acting on* $\mathrm{CH}_*(X)$. *Suppose* $\mathrm{Ker}\,\pi'$ *is stabilized by all nilpotent correspondences in* $\mathrm{CH}_d(X \times X)$. *Then, if* π *and* π' *have same homology class,* $\mathrm{Ker}\,\pi = \mathrm{Ker}\,\pi'$.

Proof. The previous lemma shows that $\mathrm{Ker}\,\pi' \subseteq \mathrm{Ker}\,\pi$. By Lemma 1.10, there exists a nilpotent correspondence $n \in \mathrm{CH}_d(X \times X)$ such that $\pi = (1+n) \circ \pi' \circ (1+n)^{-1}$. Therefore, $\mathrm{Ker}\,\pi = (1+n)\big(\mathrm{Ker}\,\pi'\big)$ and we conclude that $\mathrm{Ker}\,\pi \subseteq \mathrm{Ker}\,\pi'$ since by assumption n stabilizes $\mathrm{Ker}\,\pi'$. □

Lemma 1.14. *Let* X *be a variety that satisfies* $(\star\star)$ *and let* Π *be an idempotent in* $\mathrm{CH}_d(X \times X)$ *which is homologically equivalent to a correspondence* π *that factors through a smooth projective variety* Z : $\pi = f \circ g$ *for* $f \in \mathrm{CH}^{d-j}(Z \times X)$ *and* $g \in \mathrm{CH}_{d-j}(X \times Z)$. *Then there exist* $f_1, \ldots, f_m \in \mathrm{CH}^{d-j}(Z \times X)$ *and* $g_1, \ldots, g_m \in \mathrm{CH}_{d-j}(X \times Z)$ *such that* $\Pi = \sum f_i \circ g_i$.

Proof. There is a nilpotent correspondence $n \in \mathrm{CH}_d(X \times X)$ such that $\Pi = \pi + n$. Let N be large enough so that $n^{\circ N} = 0$. Then $\Pi = \Pi^{\circ N} = (\pi + n)^{\circ N}$. Expanding this last term, we see that each summand factors through π. □

Proof of 5. and 6. in Theorem 2 The base field k is an algebraically closed subfield of $\mathbf{C}$. In [41, Th. 3], we proved the following (Proposition 1.4 shows that the conditions on X in *loc. cit.* are met when X satisfies property B). For the sake of completeness, we provide a proof in the specific situation when X satisfies B.

Theorem 1.15. *Let* X *be a smooth projective variety of dimension* d *over* k *that satisfies B. Then there exists a set of mutually orthogonal idempotents* $\widetilde{\Pi}_{i,\lfloor i/2 \rfloor} \in \mathrm{CH}_d(X \times X)$ $(0 \leq i \leq 2d)$ *that factor through curves and whose homology classes are the idempotents* $H_*(X) \twoheadrightarrow \widetilde{N}^{\lfloor i/2 \rfloor} H_i(X) \hookrightarrow H_*(X)$. *Moreover if* $\widetilde{\Pi}_{i,\lfloor i/2 \rfloor}$ *is any idempotent that factors through a curve and that induces the orthogonal projection* $H_*(X) \twoheadrightarrow \widetilde{N}^{\lfloor i/2 \rfloor} H_i(X) \hookrightarrow H_*(X)$, *then we have*

$$\mathrm{CH}_i(X)_{\mathrm{hom}} = \mathrm{Ker}\left(\widetilde{\Pi}_{2i,i} : \mathrm{CH}_i(X) \to \mathrm{CH}_i(X)\right) \text{ and}$$

$$\mathrm{Ker}\left(AJ_i : \mathrm{CH}_i(X)_{\mathrm{alg}} \to J_i(X) \otimes \mathbf{Q}\right) = \mathrm{Ker}\left(\widetilde{\Pi}_{2i+1,i} : \mathrm{CH}_i(X)_{\mathrm{alg}} \to \mathrm{CH}_i(X)_{\mathrm{alg}}\right).$$

Proof. First we show that there exist mutually orthogonal idempotents $\widetilde{\Pi}_{i,\lfloor i/2\rfloor}$ with the properties that the $\widetilde{\Pi}_{i,\lfloor i/2\rfloor}$ factor through a curve and that $(\widetilde{\Pi}_{i,\lfloor i/2\rfloor})_*H_*(X) = \widetilde{N}^{\lfloor i/2\rfloor}H_i(X)$. Note that for each i there is a Z_i of dimension 1 and a morphism

$$q_i : h^{\mathrm{hom}}(Z_i)(\lfloor i/2\rfloor) \to h^{\mathrm{hom}}(X)$$

such that $(q_i)_*$ has image $\widetilde{N}^{\lfloor i/2\rfloor}H_i(X)$. Since X and the Z_i satisfy B, there is a full subcategory of the category of motives for homological equivalence containing $h^{\mathrm{hom}}(X)$ and the $h^{\mathrm{hom}}(Z_i)$ and their Tate twists which is semisimple abelian by Jannsen's theorem [16], and on which H is therefore exact. Thus q_i has an image $\overline{M}_i$, and $H_*(\overline{M}_i) = \widetilde{N}^{\lfloor i/2\rfloor}H_i(X)$. Since the polarization on $H_*(X)$ arises from a polarization on $h^{\mathrm{hom}}(X)$, the decomposition of $H_*(X)$ as the sum of $\widetilde{N}^{\lfloor i/2\rfloor}H_i(X)$ and its orthogonal complement arises from a similar direct sum decomposition of $h^{\mathrm{hom}}(X)$. Thus, the embedding $\overline{s}_i : \overline{M}_i \to h^{\mathrm{hom}}(X)$ has a left inverse $\overline{r}_i : h^{\mathrm{hom}}(X) \to \overline{M}_i$ such that $(\overline{s}_i \circ \overline{r}_i)_*$ is the orthogonal projection onto $\widetilde{N}^{\lfloor i/2\rfloor}H_i(X)$. By semisimplicity, the image $\overline{M}_i$ of q_i is a direct summand of $h^{\mathrm{hom}}(Z_i)(\lfloor i/2\rfloor)$. Since $h(Z_i)$ is finite-dimensional in the sense of Kimura, there exists a direct summand M_i of $h(Z_i)(\lfloor i/2\rfloor)$ lying above $\overline{M}_i$. The direct sum M of the M_i is then finite dimensional, and lies above the direct sum $\overline{M}$ of the $\overline{M}_i$. Write $\overline{s} : \overline{M} \to h^{\mathrm{hom}}(X)$ and $\overline{r} : h^{\mathrm{hom}}(X) \to \overline{M}$ for the morphisms with respective components the $\overline{s}_i$ and the $\overline{r}_i$. Then $\overline{r}$ is left inverse to $\overline{s}$. If $s : M \to h(X)$ is a lifting of $\overline{s}$, then there is by finite-dimensionality of M a lifting $r : h(X) \to M$ of $\overline{r}$ which is left inverse to s. Write $s_i : M_i \to h(X)$ and $r_i : h(X) \to M_i$ for the respective components of s and r. Then r_i is left inverse to s_i, and we may take as $\widetilde{\Pi}_{i,\lfloor i/2\rfloor}$ the idempotent $s_i \circ r_i$. That the $\widetilde{\Pi}_{i,\lfloor i/2\rfloor}$ are mutually orthogonal is clear, and $\widetilde{\Pi}_{i,\lfloor i/2\rfloor}$ factors through $h(Z_i)(\lfloor i/2\rfloor)$ because it factors through the direct summand M_i of $h(Z_i)(\lfloor i/2\rfloor)$.

The rest of the theorem then follows as in the proof of Lemma 4.16 or 4.17 by functoriality of the cycle class maps and of the Abel–Jacobi maps with respect to the action of correspondences. □

If, moreover, X satisfies $(\star\star)$, then it is possible to prove that the above kernels do not depend on the choice of a lift $\Pi_{i,\lfloor i/2\rfloor}$ for the idempotent $\pi_{i,\lfloor i/2\rfloor}$.

Proposition 1.16. *If X satisfies B and $(\star\star)$ then*

$$\mathrm{CH}_i(X)_{\mathrm{hom}} = \mathrm{Ker}\left(\Pi_{2i,i} : \mathrm{CH}_i(X) \to \mathrm{CH}_i(X)\right)$$

for any choice of idempotent $\Pi_{2i,i}$ such that $(\Pi_{2i,i})_*H_*(X) = \widetilde{N}^iH_{2i}(X)$.

Proof. By Theorem 1.15, $\mathrm{CH}_i(X)_{\mathrm{hom}} = \mathrm{Ker}\left(\widetilde{\Pi}_{2i,i} : \mathrm{CH}_i(X) \to \mathrm{CH}_i(X)\right)$. The group $\mathrm{CH}_i(X)_{\mathrm{hom}}$ is stabilized by the action of correspondences in $\mathrm{CH}_d(X \times X)$. Therefore, by Lemma 1.13, $\mathrm{Ker}\left(\Pi_{2i,i}\right) = \mathrm{Ker}\left(\widetilde{\Pi}_{2i,i}\right)$, where both idempotents act on $\mathrm{CH}_i(X)$. □

Proposition 1.17. *If X satisfies B and $(\star\star)$, then*

$$\mathrm{Ker}\left(AJ_i : \mathrm{CH}_i(X)_{\mathrm{alg}} \to J_i(X) \otimes \mathbf{Q}\right) = \mathrm{Ker}\left(\Pi_{2i+1,i} : \mathrm{CH}_i(X)_{\mathrm{alg}} \to \mathrm{CH}_i(X)_{\mathrm{alg}}\right),$$

where $\Pi_{2i+1,i}$ is any idempotent in $\mathrm{CH}_d(X \times X)$ such that $(\Pi_{2i+1,i})_*H_*(X) = \widetilde{N}^iH_{2i+1}(X)$.

Proof. The idempotents $\widetilde{\Pi}_{2i+1,i}$ and $\Pi_{2i+1,i}$ are homologous. By Lemma 1.13 it is thus enough to see that $\mathrm{Ker}\left(AJ_i : \mathrm{CH}_i(X)_{\mathrm{alg}} \to J_i(X) \otimes \mathbf{Q}\right) = \mathrm{Ker}\left(\widetilde{\Pi}_{2i+1,i}\right)$ is stabilized by the action of correspondences in $\mathrm{CH}_d(X \times X)$. This is indeed the case by functoriality of the Abel–Jacobi map with respect to the action of correspondences. □

proof of 1., 2. and 3. in Theorem 2. The idempotent $\Pi_{i,j}$ is homologically equivalent to a correspondence $\pi_{i,j}$ that factors through a variety $Z_{i,j}$ of dimension $i-2j$. The action of this correspondence $\pi_{i,j}$ on $\mathrm{CH}_l(X)$ factors through $\mathrm{CH}_{l-j}(Z_{i,j})$, and hence $\mathrm{Ker}\,\pi_{i,j} = \mathrm{CH}_l(X)$ when $l < j$ or $l > i-j$ for dimension reasons. Clearly, $\mathrm{CH}_l(X)$ is stabilized by the action of any correspondence in $\mathrm{CH}_d(X \times X)$. Therefore Lemma 1.12 applies, giving $\mathrm{Ker}\,\Pi_{i,j} \supseteq \mathrm{Ker}\,\pi_{i,j} = \mathrm{CH}_l(X)$ for $l < j$ or $l > i-j$. Alternately one could have used Lemma 1.14 directly.

In the case $l = i-j$ and $i < 2l$, the action of $\pi_{i,j}$ on $\mathrm{CH}_l(X)$ factors through $\mathrm{CH}_{i-2j}(Z_{i,j}) = \mathrm{CH}^0(Z_{i,j})$. In particular, $\pi_{i,j}$ acts trivially on $\mathrm{CH}_l(X)_{\mathrm{hom}}$. Because $i < 2l$, $\pi_{i,j}$ acts trivially on $H_{2l}(X)$ and hence on $\mathrm{Im}\,(\mathrm{CH}_l(X) \to H_{2l}(X))$. Therefore, by functoriality of the cycle class map, $\pi_{i,j}$ sends $\mathrm{CH}_l(X)$ to $\mathrm{CH}_l(X)_{\mathrm{hom}}$. Hence, $\pi_{i,j} \circ \pi_{i,j}$ acts as zero on $\mathrm{CH}_l(X)$. Besides, $\pi_{i,j} \circ \pi_{i,j}$ is homologically equivalent to $\Pi_{i,j} \circ \Pi_{i,j} = \Pi_{i,j}$. By Lemma 1.12, we get $\mathrm{Ker}\,\Pi_{i,j} \supseteq \mathrm{Ker}\,\big(\pi_{i,j} \circ \pi_{i,j}\big) = \mathrm{CH}_l(X)$.

Similarly, in the case $l+1 = i-j$ and $i \leq 2l$, the action of $\pi_{i,j}$ on $\mathrm{CH}_l(X)$ factors through $\mathrm{CH}_{i-2j-1}(Z_{i,j}) = \mathrm{CH}^1(Z_{i,j})$. We have an isomorphism $AJ^1 : \mathrm{CH}^1(Z_{i,j})_{\mathrm{hom}} \to J^1(Z_{i,j})$. Therefore, by functoriality of the Abel-Jacobi map on homologically trivial cycles, $\pi_{i,j}$ acts trivially on $\mathrm{Ker}\,(\mathrm{CH}_l(X)_{\mathrm{hom}} \to J_l(X))$. Now, by assumption on i, j and l, $\pi_{i,j}$ acts trivially on $\mathrm{Im}\,(\mathrm{CH}_l(X) \to H_{2l}(X))$ and on $H_{2l+1}(X)$ and hence on $J_l(X)$. As such, $\pi_{i,j}$ sends $\mathrm{CH}_l(X)$ to $\mathrm{Ker}\,(\mathrm{CH}_l(X)_{\mathrm{hom}} \to J_l(X))$. As before, we get $\mathrm{Ker}\,\Pi_{i,j} \supseteq \mathrm{Ker}\,\big(\pi_{i,j} \circ \pi_{i,j}\big) = \mathrm{CH}_l(X)$. □

If X satisfies $(\star)$ and $(\star\star)$, then a star in the diagram below with coordinates (i,j) $(0 \leq i \leq j)$ indicates that $\mathrm{Gr}^i_{\widetilde{N}} H_{i+j}(X)$ does not induce a "motivic" action on $\mathrm{CH}_l(X)$, i.e. that $\Pi_{i+j,i}$ acts as zero on $\mathrm{CH}_l(X)$. A bullet with coordinates (i,j) $(0 \leq i \leq j)$ indicates that $\Pi_{i+j,i}$ should act as zero on $\mathrm{CH}_l(X)$ if one believes in the BBM conjectures, see Proposition 3.4.

	0	1				l							d
0	∗	∗	∗	∗	∗	∗	∗	•	•	•	•		
1	∗	∗	∗	∗	∗	∗	∗	•	•	•			
	∗	∗	∗	∗	∗	∗	∗	•	•				
	∗	∗	∗	∗	∗	∗	∗	•					
	∗	∗	∗	∗	∗	∗	∗						
l	∗	∗	∗	∗	∗								
	∗	∗	∗	∗	∗		∗	∗	∗	∗	∗	∗	∗
	•	•	•	•			∗	∗	∗	∗	∗	∗	∗
	•	•	•				∗	∗	∗	∗	∗	∗	∗
	•	•					∗	∗	∗	∗	∗	∗	∗
	•						∗	∗	∗	∗	∗	∗	∗
							∗	∗	∗	∗	∗	∗	∗
d							∗	∗	∗	∗	∗	∗	∗

We have "symmetrized" the diagram by including points with coordinates (i,j) with $i > j \geq 0$. As such, a point with coordinates (i,j) represents $\mathrm{Gr}^{\min(i,j)}_{\widetilde{N}} H_{i+j}(X)$. This way the diagram looks like a Hodge diamond. The reason for doing so is the following. Grothendieck's generalized Hodge conjecture (GHC for short) predicts that the coniveau filtration coincides with the Hodge coniveau filtration. Since the Lefschetz standard conjecture is a particular instance of the Hodge conjecture, Grothendieck's GHC for all varieties implies actually that the niveau filtration should coincide with the Hodge coniveau filtration; see Proposition 1.1. Therefore, if $H_{i+j}(X)$ is given with a polarization then, under GHC, $\widetilde{N}^{\min(i,j)} H_{i+j}(X)$ is the sub-Hodge structure of $H_{i+j}(X)$ generated by the orthogonal complement inside $H_{i,j}(X) \oplus H_{j,i}(X)$ of the intersection of $H_{i,j}(X) \oplus H_{j,i}(X)$ with the complexification of the sub-Hodge structure of $H_{i+j}(X)$ spanned by $H_{i+j,0}(X) \oplus \ldots \oplus H_{\max(i,j)+1,\min(i,j)}(X) \oplus H_{\min(i,j),\max(i,j)+1}(X) \oplus \ldots \oplus H_{0,i+j}(X)$. Therefore, $\mathrm{Gr}^{\min(i,j)}_{\widetilde{N}} H_{i+j}(X) = 0$ if and only if $H_{i,j}(X) \oplus H_{j,i}(X)$ is included in the complexification of the sub-Hodge structure of $H_i(X)$ spanned by $H_{i+j,0}(X) \oplus \ldots \oplus H_{\max(i,j)+1,\min(i,j)}(X) \oplus$

$H_{\min(i,j),\max(i,j)+1}(X) \oplus \ldots \oplus H_{0,i+j}(X)$. In particular, if the Hodge number $h_{i,j}(X)$ vanishes, then $\mathrm{Gr}_{\widetilde{N}}^{\min(i,j)} H_{i+j}(X) = 0$.

2 Some applications

In §§2.1 and 2.2 we give direct applications of Theorem 2 and show explicitly how the niveau filtration on $H_*(X)$ reflects on the Chow groups $\mathrm{CH}_*(X)$ for those varieties X that satisfy ($\star$) and ($\star\star$). The most general result in this direction is Proposition 2.8. In particular, we relate (in the spirit of Bloch's conjecture for surfaces) the support of the Chow group of 0-cycles of a variety X satisfying ($\star$) and ($\star\star$) to the support of its cohomology; see Proposition 2.3. In §2.2, we give partial answers to questions raised by Schoen and by Esnault and Levine. In §2.3 we are interested in exhibiting a new example of a variety Y for which we can give a description of its Chow groups. This is achieved by considering a finite quotient of a Fermat 4-fold of degree 7; see Theorem 2.18.

2.1 On the support of the Chow groups

Definition 2.1. The niveau number $g_{i,j}(X)$ of X is equal to $\dim_{\mathbf{Q}} \mathrm{Gr}^i_{\widetilde{N}} H_{i+j}(X)$.

In particular if the niveau numbers $g_{0,i}(X), g_{1,i-1}(X), \ldots, g_{k,i-k}(X)$ vanish then $H_i(X) = \widetilde{N}^{k+1} H_i(X)$. Consequently the Hodge numbers $h_{0,i}(X), \ldots, h_{k,i-k}(X)$ vanish. Here $h_{p,q}(X) := \dim_{\mathbf{C}} H^{d-p,d-q}(X)$.

0-cycles. Bloch and Srinivas [5] proved using a decomposition of the diagonal that if $\mathrm{CH}_0(X \times \mathbf{C})$ is supported in dimension i (i.e. if there exists a closed subscheme Z of $X \times \mathbf{C}$ of dimension i such that the map $\mathrm{CH}_0(Z) \to \mathrm{CH}_0(X \times \mathbf{C})$ induced by the inclusion of Z inside $X \times \mathbf{C}$ is surjective), then $H_k(X) = N^1 H_k(X)$ for all $k > i$. This generalized the result of Roitman [35] who considered the case $i = 1$. In this latter case we actually have a more precise result involving the niveau filtration instead of the coniveau filtration.

Proposition 2.2. *Let X be a smooth projective complex variety. If $\mathrm{CH}_0(X)$ is supported on a curve, then $H_k(X) = \widetilde{N}^1 H_k(X)$ for all $k > 1$.*

Proof. By Bloch and Srinivas [5], there exist a divisor D on X and a curve C in X such that the diagonal Δ_X decomposes as $\Delta_X = \Gamma_1 + \Gamma_2 \in \mathrm{CH}_d(X \times X)$ with Γ_1 supported on $D \times X$ and Γ_2 supported on $X \times C$. Let $\widetilde{C}$ and $\widetilde{D}$ be desingularizations of C and D respectively. Then the action of Γ_1 on $H_k(X)$ factors through $H_{k-2}(\widetilde{D})$ and the action of Γ_2 on $H_k(X)$ factors through $H_k(\widetilde{C})$. This makes it possible to conclude that $H_k(X) = \widetilde{N}^1 H_k(X)$ for all $k > 1$. □

We now give a converse statement for those varieties that satisfy ($\star$) and ($\star\star$). This gives a proof of Jannsen's [17, conjecture 3.3] for varieties that satisfy ($\star$) and ($\star\star$), which is a slight improvement since Jannsen prove gave a proof under the validity of the standard conjectures and the BBM conjectures (the standard conjectures imply ($\star$) and Jannsen [17] proved that the BBM conjectures imply that property ($\star\star$) holds for all smooth projective varieties).

Proposition 2.3. *Let X be a smooth projective variety satisfying ($\star$) and ($\star\star$). If $H_k(X) = \widetilde{N}^1 H_k(X)$ for all $k > i$, then $\mathrm{CH}_0(X)$ is supported in dimension i.*

Proof. By Theorem 2, we have

$$(\Delta_X)_* \mathrm{CH}_0(X) = (\Pi_{0,0} + \Pi_{1,0} + \ldots + \Pi_{i,0})_* \mathrm{CH}_0(X).$$

By Theorem 1, the idempotent $\Pi_{0,0} + \Pi_{1,0} + \ldots + \Pi_{i,0}$ is homologically equivalent to a cycle that factors through an i-dimensional variety. The proposition then follows from Lemma 1.14. □

Remark 2.4. A surface X satisfies ($\star$). Moreover, if X has vanishing geometric genus $p_g := h_{2,0}$, then, by the Lefschetz $(1,1)$-theorem, $H_2(X) = \widetilde{N}^1 H_2(X)$. Therefore, we recover the following result due to Kimura [20, Corollary 7.7] : if X is a surface with $p_g = 0$ whose Chow motive is finite-dimensional, then the Bloch conjecture is true for X.

Remark 2.5. The generalized Bloch conjecture states that if the Hodge numbers $h_{i+1,0}, h_{i+2,0}, \ldots, h_{d,0}$ of X vanish, then $\mathrm{CH}_0(X)$ is supported in dimension i. Proposition 2.3 shows that if Grothendieck's generalized Hodge conjecture holds then the generalized Bloch conjecture holds for those X that satisfy ($\star\star$) (the property ($\star$) is automatically satisfied if Grothendieck's generalized Hodge conjecture holds).

In the cases $i = 0$ and $i = 1$, the integral Chow group $\mathrm{CH}_0^{\mathbf{Z}}(X)$ of 0-cycles of X can be computed explicitly.

Proposition 2.6. *Let X be a smooth projective variety over an algebraically closed field $k \subseteq \mathbf{C}$ satisfying ($\star$) and ($\star\star$). If $H_i(X) = \widetilde{N}^1 H_i(X)$ for all $i > 1$, then $\mathrm{CH}_0^{\mathbf{Z}}(X) = \mathbf{Z} \oplus \mathrm{Alb}_X(k)$.*

Proof. This is a refinement of a result of Voisin [43, Theorem 3]. By Theorem 2, we have $\mathrm{CH}_0(X) = (\Pi_{0,0} + \Pi_{1,0})_* \mathrm{CH}_0(X)$. Still by Theorem 2, we have $\mathrm{Ker}\,(\mathrm{alb}_X : \mathrm{CH}_0(X)_{\mathrm{hom}} \to \mathrm{Alb}_X(k) \otimes \mathbf{Q}) = \mathrm{Ker}\,((\Pi_{1,0})_* : \mathrm{CH}_0(X)_{\mathrm{hom}} \to \mathrm{CH}_0(X)_{\mathrm{hom}})$. Thus $\mathrm{Ker}\,(\mathrm{alb}_X) = 0$. By Roitman's theorem [36], the integral albanese map $\mathrm{alb}_X^{\mathbf{Z}} : \mathrm{CH}_0^{\mathbf{Z}}(X)_{\mathrm{hom}} \to \mathrm{Alb}_X(k)$ is an isomorphism on the torsion. It is also surjective, and hence the proposition. □

l-cycles. Lewis [24] and Schoen [37] proved independently the following theorem.

Theorem 2.7 (Lewis, Schoen)**.** *If $\mathrm{Gr}^l_{\widetilde{N}} H_{2l+k}(X) \neq 0$ for some $k \geq 2$ then $\mathrm{CH}_l(X)_{\mathrm{alg}}$ is not representable, i.e. there is no curve C and no correspondence $\Gamma \in \mathrm{CH}_{l+1}(C \times X)$ such that $\Gamma_* \mathrm{CH}_0(C)_{\mathrm{alg}} = \mathrm{CH}_l(X)_{\mathrm{alg}}$.*

The statement below, when considered in the case $p = 1$ and $q = 0$, gives a partial converse for varieties X satisfying ($\star$) and ($\star\star$).

Proposition 2.8 (Generalization of Proposition 2.3)**.** *Let X be a smooth projective variety satisfying ($\star$) and ($\star\star$) and let l be a non-negative integer. If there exist non-negative integers p and q such that $g_{i,j}(X) = 0$ when*

(i) $i + j \leq 2l$ and $j > l + 1$

and when

(ii) $2l + 1 \leq i + j \leq 2l + p$ and $i < l - q$

and when

(iii) $i + j > 2l + p$ and $i \leq l$

then there exist a smooth projective variety Z of dimension $p + 2q$, a correspondence $\Gamma' \in \mathrm{CH}_{d+q-l}(X \times Z)$ and a correspondence $\Gamma \in \mathrm{CH}_{p+q+l}(Z \times X)$ such that $\Gamma \circ \Gamma'$ acts as the identity on $\mathrm{CH}_l(X)$. In particular $\mathrm{CH}_l(X) = \Gamma_* \mathrm{CH}_q(Z)$.

Proof. Let $\Pi := \Pi_{2l,l} + (\Pi_{2l+1,l} + \ldots + \Pi_{2l+p,l}) + (\Pi_{2l+1,l-1} + \ldots + \Pi_{2l+p,l-1}) + \ldots + (\Pi_{2l+1,l-q} + \ldots + \Pi_{2l+p,l-q})$. Under the assumptions on the niveau numbers, Theorem 2 says $\mathrm{CH}_l(X) = \Pi_* \mathrm{CH}_l(X)$. Theorem 1 says that the idempotent $\Pi_{i,j}$ is homologically equivalent to a cycle of the form $f_{i,j} \circ g_{i,j}$ for some smooth projective variety $Z_{i,j}$ of dimension $i - 2j$, some $f_{i,j} \in \mathrm{CH}_{i-j}(Z_{i,j} \times X)$ and some $g_{i,j} \in \mathrm{CH}_{d-j}(X \times Z_{i,j})$. The idempotents $\Pi_{i,j}$ appearing in the above sum all satisfy $i - 2j \leq p + 2q$ and $l - j \leq q$. Therefore, up to replacing each $Z_{i,j}$ by $Z_{i,j} \times \mathbf{P}^{p+2q-i+2j}$, Lemma 1.14 implies the desired result. □

2.2 Injectivity of some cycle class maps

Proposition 2.9. *Let X be a smooth projective variety satisfying $(\star)$ and $(\star\star)$. If the niveau numbers $g_{i,j}(X)$ vanish for*

(i) $j > l+1$ when $i+j \leq 2l$

and for

(ii) $i \leq l$ when $i+j > 2l$,

then $\mathrm{CH}_l(X)_{\mathrm{hom}} = 0$, i.e. the cycle class map $\mathrm{CH}_l(X) \to H_{2l}(X)$ is injective.

Proof. By assumption on the niveau numbers, Theorem 2 shows that the only idempotent $\Pi_{i,j}$ acting non trivially on $\mathrm{CH}_l(X)$ is $\Pi_{2l,l}$. Therefore $\mathrm{CH}_l(X) = (\Pi_{2l,l})_* \mathrm{CH}_l(X)$. Hence $\mathrm{CH}_l(X)_{\mathrm{hom}} = \mathrm{Ker}\,(\Pi_{2l,l}) = 0$. □

Remark 2.10. The 4-fold Y that will be considered in §2.3 satisfies the assumptions of the proposition with $l = 2$.

Corollary 2.11. *Let X be a smooth projective variety satisfying $(\star)$ and $(\star\star)$. If the niveau numbers $g_{p,q}(X)$ vanish for $p \neq q$ and $p \leq k$, then $\mathrm{CH}_l(X)_{\mathrm{hom}} = 0$ for all $l \leq k$ and all $l \geq d-k-2$.*

Proof. Note that the condition (i) of Proposition 2.9 implies $i \leq l-2$. It is thus easy to check that the vanishing condition on the niveau numbers implies, for all $l \leq k$, the vanishing conditions of Proposition 2.9. Therefore $\mathrm{CH}_l(X)_{\mathrm{hom}} = 0$ for all $l \leq k$. A generalized decomposition of the diagonal as done by Laterveer [23] then shows that $\mathrm{CH}_l(X)_{\mathrm{hom}} = 0$ for all $l \geq d-k-2$. □

Proposition 2.12 (Answers partially question 0.6 of [37])**.** *Let X be a smooth projective variety satisfying $(\star)$ and $(\star\star)$. Suppose that there is an integer $l \geq 0$ such that the niveau numbers $g_{i,j}(X)$ vanish if*

(i) $i+j \leq 2l$ and $j > l+1$

and if

(ii) $i+j = 2l+1$ and $|i-j| > 1$

and if

(iii) $i+j \geq 2l+2$ and $i \leq l$.

Then $\mathrm{CH}_l(X)_{\mathrm{hom}} = \mathrm{CH}_l(X)_{\mathrm{alg}}$ and the Abel-Jacobi map $AJ_l : \mathrm{CH}_l(X)_{\mathrm{hom}} \to J_l(X) \otimes \mathbf{Q}$ is injective.

Proof. The assumptions made on the niveau numbers of X and Theorem 2 (points 1, 2, 3 and 4) give $\mathrm{CH}_l(X) = \left(\Pi_{2l,l} + \Pi_{2l+1,l}\right)_* \mathrm{CH}_l(X)$. The proposition then follows from points 5 and 6 of Theorem 2. □

Corollary 2.13 (see question 1 of [9])**.** *Let X be a smooth projective variety satisfying $(\star)$ and $(\star\star)$. Suppose that there is an integer $s \geq 0$ such that the niveau numbers $g_{i,j}(X)$ vanish if*

(i) $i+j \leq 2s+2$ and $|i-j| > 1$

and if

(ii) $i+j > 2s+2$ and $i \leq s$

Then $\mathrm{CH}_l(X)_{\mathrm{hom}} = \mathrm{CH}_l(X)_{\mathrm{alg}}$ and the Abel–Jacobi maps $AJ_l : \mathrm{CH}_l(X)_{\mathrm{hom}} \to J_l(X) \otimes \mathbf{Q}$ are injective for $l = 0, \ldots, s$ and $l = d-s-1, \ldots, d$.

Proof. It is not too difficult to check that the vanishing condition on the niveau numbers implies for all $l \leq s$ the vanishing conditions of Proposition 2.12. Therefore $\mathrm{CH}_l(X)_{\mathrm{hom}} = \mathrm{CH}_l(X)_{\mathrm{alg}}$ and the Abel–Jacobi maps $AJ_l : \mathrm{CH}_l(X)_{\mathrm{hom}} \to J_l(X) \otimes \mathbf{Q}$ are injective for all $l \leq s$. As before, a generalized decomposition of the diagonal as done by Laterveer [23] makes it possible to conclude for $l \geq d-s-2$. □

The diagram below illustrates the assumptions made on the niveau numbers of X in the above corollary. A star with coordinates (i,j) indicates that the niveau number $g_{\min(i,j),\max(i,j)}(X)$ is possibly non-zero. The absence of a star at the point of coordinate (i,j) indicates that $g_{\min(i,j),\max(i,j)}(X) = 0$.

	0	1	2		$s+1$			$d-s-1$			$d-1$	d
0	*	*										
1	*	*	*									
2		*	*	⋱								
			⋱	⋱	*							
$s+1$				*	*	*	*	*				
					*	*	*	*				
					*	*	*	*				
$d-s-1$					*	*	*	*	*			
								*	⋱	⋱		
									⋱	*	*	
$d-1$										*	*	*
d											*	*

Remark 2.14. In the case $s = d$, the condition $(\star)$ is automatically satisfied since it is always true that $N^{\lfloor i/2 \rfloor +1} H_i(X) = \widetilde{N}^{\lfloor i/2 \rfloor +1} H_i(X) = 0$, cf. also [41, Th. 4].

2.3 A 4-fold Y of general type with $\mathrm{CH}_0^{\mathbf{Z}}(Y) = \mathbf{Z}$

Kollár showed [22] that any Fano variety X satisfies $\mathrm{CH}_0(X) = \mathbf{Q}$. A natural question is to ask whether or not there exist varieties X of general type with $\mathrm{CH}_0(X) = \mathbf{Q}$. This question has a positive answer in the case of surfaces (see, for instance, [3], [13], [42]) and in the case of threefolds [33]. Given any two varieties X and Y which satisfy $\mathrm{CH}_0(X) = \mathbf{Q}$ and $\mathrm{CH}_0(Y) = \mathbf{Q}$, it is easy to see that $\mathrm{CH}_0(X \times Y) = \mathbf{Q}$ as well. Consider indeed P (resp. Q) a closed point of X (resp. Y). Every 0-cycle on X is rationally equivalent to a rational multiple of the class $[P]$ of P. It follows that every 0-cycle on $X \times Y$ is rationally equivalent to a 0-cycle of the form $[P] \times \alpha$ for some 0-cycle $\alpha \in \mathrm{CH}_0(Y)$. But then, α is rationally equivalent to a rational multiple of $[Q]$. Every 0-cycle on $X \times Y$ is therefore rationally equivalent to a rational multiple of $[P \times Q]$. If X and Y are of general type, then $X \times Y$ is of general type. It is thus possible to exhibit, for all $d \geq 2$, examples of varieties X of dimension d which are of general type with $\mathrm{CH}_0(X) = \mathbf{Q}$.

More can be said. By Guletskii and Pedrini [12], the Chow motive of a complex surface S with $\mathrm{CH}_0(S) = \mathbf{Q}$ is a sum of Lefschetz motives. Therefore, the motive of the product of such surfaces is also a sum of Lefschetz motives. Hence, the Chow groups of the product of surfaces with $\mathrm{CH}_0(S) = \mathbf{Q}$ are finite-dimensional $\mathbf{Q}$-vector spaces. If the surfaces involved in the product are all surfaces of general type then the product is itself of general type. It is thus possible to exhibit, in any even dimension, an example of a variety X of general type with finite-dimensional Chow groups. Concerning complex threefolds W with $\mathrm{CH}_0(W) = \mathbf{Q}$, by [41, Theorem 4 & Rk 3.8] the motive of W decomposes as a direct sum of Lefschetz motives with the direct summand of a motive of curve tensored with the Lefschetz motive.

Here we exhibit a new example of a complex fourfold Y of general type with $\mathrm{CH}_0(Y) = \mathbf{Q}$. We will also show that $AJ_1 : \mathrm{CH}_1(Y)_{\mathrm{alg}} \to J_1(Y) \otimes \mathbf{Q}$ is not injective. Thus our variety Y is not isomorphic to the product of two surfaces of general type with $\mathrm{CH}_0(S) = \mathbf{Q}$. (Actually, it is simpler to note that the H^4 of a product of two such surfaces is concentrated in Hodge bidegrees $(2,2)$, but $H^{3,1}(Y) \neq 0$.) It is also not isomorphic to the blow-up of the product of two surfaces of general type with $\mathrm{CH}_0(S) = \mathbf{Q}$ along a smooth surface since such a variety has Picard number > 1 whereas Y will be seen to have Picard number 1. Let's mention that there exist examples of varieties not of general type satisfying the above : Schoen [37, Theorem 0.5] showed that hypersurfaces X of dimension $d \geq 4$ and $d/2+1 \leq \deg X < d+2$ have non injective $AJ_1 : \mathrm{CH}_1(X)_{\mathrm{alg}} \to J_1(X) \otimes \mathbf{Q}$.

Let X^n_m be the complex Fermat n-fold of degree m, that is

$$X^n_m = \{x_0^m + x_1^m + \ldots + x_{n+1}^m = 0\} \subset \mathbf{P}^{n+1}_{\mathbf{C}}.$$

Such a variety satisfies property B (this is the case for any hypersurface) and is dominated by a product of curves [40]. When m is a prime number, Ran [34] and Shioda [38] proved that the Hodge conjecture holds for X^n_m.
From now on X denotes the complex Fermat fourfold of degree 7, i.e. $X = X^4_7 \subset \mathbf{P}^5_{\mathbf{C}}$.

Theorem 2.15. *The Fermat fourfold X satisfies Grothendieck's generalized Hodge conjecture. More is true : $\widetilde{N}^1H^4(X)$ is the largest sub-Hodge structure of $H^4(X)$ whose complexification is included in $H^{1,3}(X) \oplus H^{2,2}(X) \oplus H^{3,1}(X)$.*

Proof. We use Ran's and Shioda's technique (which is summed up in [39, §3 (13)]) to prove

$$\widetilde{N}^1H^4(X) \cap H^4_{prim}(X,\mathbf{Q}) = (H^{1,3}(X) \oplus H^{2,2}(X) \oplus H^{3,1}(X)) \cap H^4_{prim}(X,\mathbf{Q}).$$

From [38, Theorem I], we have a decomposition

$$\big((H^{1,3} \oplus H^{2,2} \oplus H^{3,1}) \cap H^4_{prim}(X,\mathbf{Q})\big) \otimes_{\mathbf{Q}} \mathbf{C} = \bigoplus_{\alpha \in \mathfrak{B}^4_7} V(\alpha),$$

where, if $< a >$ denotes the least positive residue of a modulo 7,

$$\mathfrak{B}^4_7 = \{(a_0,a_1,a_2,a_3,a_4,a_5) \mid 1 \leq a_i \leq 6, \ \sum_{i=0}^{5} < ta_i >= 14, \ 21 \text{ or } 28 \text{ for all } t \in (\mathbf{Z}/7)^\times\}$$

and $V(\alpha)$ is a certain subspace of $H^4_{prim}(X,\mathbf{C})$ of dimension 1. An element $\alpha = (a_i) \in \mathfrak{B}^4_7$ is called *decomposable* if $a_i + a_j = 0 \bmod 7$ for some $i \neq j$ and *quasi-decomposable* if $(a_0,a_1,a_2,a_3,< a_4+a_5 >)$ belongs to $\mathfrak{B}^3_7 = \{(b_0,b_1,b_2,b_3,b_4) \mid 1 \leq b_i \leq 6, \ \sum_{i=0}^4 < tb_i >= 14 \text{ or } 21 \text{ for all } t \in (\mathbf{Z}/7)^\times\}$ after a permutation of the digits a_i. By Theorem II of [38], there is an isomorphism induced by a correspondence

$$[H^3(X^3_7) \otimes H^1(X^1_7)]^{\mu_7} \oplus [H^2_{prim}(X^2_7) \otimes H^0(X^0_7)](-1) \xrightarrow{\simeq} H^4_{prim}(X^4_7).$$

If $\alpha \in \mathfrak{B}^4_7$ is decomposable (resp. quasi-decomposable), then $V(\alpha)$ corresponds to some $V(\beta) \otimes V(\gamma) \in H^2_{prim}(X^2_7) \otimes H^0_{prim}(X^0_7)$ (resp. to some $V(\beta) \otimes V(\gamma) \in H^3_{prim}(X^3_7) \otimes H^1_{prim}(X^1_7)$ with $\beta \in \mathfrak{B}^3_7$). Therefore, if α is decomposable, then it comes from a class supported on a surface, i.e. $V(\alpha) \subset \widetilde{N}^1H^4(X,\mathbf{C})$. If $\beta \in \mathfrak{B}^3_7$, then, by Shioda [39, §3 (13)], $V(\beta) \subset H^3(X^3_7)$ is supported in codimension one. Having in mind that $N^1H_3(Z) = \widetilde{N}^1H_3(Z)$ for a 3-fold Z, we get that $V(\beta)$ comes from a curve. Therefore, if α is quasi-decomposable, then $V(\alpha)$ comes from a class supported on a surface. Now, it can be checked that any element of $\mathfrak{B}^4_7$ is either decomposable or quasi-decomposable[1]. Hence, $(H^{1,3}(X) \oplus H^{2,2}(X) \oplus H^{3,1}(X)) \cap H^4_{prim}(X,\mathbf{Q}) \subseteq \widetilde{N}^1H^4(X)$. Since $H^4(X)$ is the orthogonal sum (for the cup product) of $H^4_{prim}(X,\mathbf{Q})$ with the span of a linear section, the result follows. □

[1]A non-decomposable element in $\mathfrak{B}^4_7$ can only involve at most 3 distinct integers $\in \{1,2,3,4,5,6\}$. It can then be checked that, up to permutation and up to multiplication by an element in $(\mathbf{Z}/7)^\times$, the only non-decomposable elements in $\mathfrak{B}^4_7$ are $(1,1,2,2,4,4)$, $(1,1,1,3,3,5)$ and $(1,1,1,1,5,5)$. But these are quasi-decomposable as one can see after adding the fourth and fifth digits. Also, one can check that the multiples of these three elements are either decomposable or quasi-decomposable. Finally, notice for example that $(2,2,2,2,2,4)$ does not belong to $\mathfrak{B}^4_7$ because it has a multiple, namely $(1,1,1,1,1,2)$, whose digits add to 7.

Let $G = \mu_7$ be the group of complex 7-th roots of unity and let ζ be a generator of μ_7. We let G act on $\mathbf{P}^5_{\mathbf{C}}$ in the following way:

$$\zeta \cdot [x_0, x_1, x_2, x_3, x_4, x_5] = [x_0, \zeta x_1, \zeta^2 x_2, \zeta^3 x_3, \zeta^4 x_4, \zeta^5 x_5].$$

Such an action restricts to a free action of G on X. Thus, the quotient $f : X \to Y = X/G$ exists and defines a smooth projective 4-fold Y. Since X is of general type, so is Y.

Lemma 2.16. *The variety Y above satisfies $H^i(Y, O_Y) = 0$ for $1 \leq i \leq 4$.*

Proof. We have $H^i(Y, O_Y) = H^i(X, O_X)^G$. The weak Lefschetz theorem settles the cases $1 \leq i \leq 3$. For the case $i = 4$, by Hodge theory we have $H^4(X, O_X) = \overline{H^0(X, K_X)}$. Let $\Omega = \sum_i (-1)^i x_i dx_0 \wedge \ldots \widehat{dx_i} \ldots \wedge dx_4$ be a generator of $H^0(\mathbf{P}^5, K_{\mathbf{P}^5}(6))$. We have

$$\zeta^* \Omega = \zeta \Omega.$$

A global 4-form $\omega \in H^0(X, K_X)$ on X is the restriction of $P\Omega/F$ to X for some $P \in H^0(\mathbf{P}^5, O_{\mathbf{P}^5}(1))$ where $F = x_0^7 + x_1^7 + x_2^7 + x_3^7 + x_4^7 + x_5^7 \in H^0(\mathbf{P}^5, O_{\mathbf{P}^5}(7))$. We have

$$\zeta^* \omega = \zeta \Omega / F \cdot (\zeta^* P)|_X.$$

Now, ζ^* does not admit the eigenvalue ζ^{-1} on $H^0(\mathbf{P}^5, O_{\mathbf{P}^5}(1))$. Hence, $H^i(X, O_X)^G = 0$. □

Proposition 2.17. *The variety Y above satisfies the Hodge conjecture as well as the Lefschetz standard conjecture and $\widetilde{N}^1 H_4(Y) = H_4(Y)$. Moreover, Y is dominated by a product of curves.*

Proof. Let $p \in \mathrm{CH}_4(X \times X)$ denote the correspondence $\frac{1}{7} {}^t\Gamma_f \circ \Gamma_f$. Because $\Gamma_f \circ {}^t\Gamma_f = 7\Delta_Y \in \mathrm{CH}_4(Y \times Y)$, p defines an idempotent that identifies $H^4(Y)$ with $p_* H^4(X)$. It follows that Y satisfies the Hodge conjecture and that $\widetilde{N}^1 H_4(Y) = H_4(Y)$. The two other properties are straightforward. □

Theorem 2.18. *The variety Y satisfies the following:* $\mathrm{CH}_0^{\mathbf{Z}}(Y) = \mathrm{CH}_3^{\mathbf{Z}}(Y) = \mathbf{Z}$, $\mathrm{CH}_2(Y)$ *is a finite dimensional* $\mathbf{Q}$*-vector space. Moreover, there exist a surface S and a correspondence* $\Gamma \in \mathrm{CH}_3(S \times Y)$ *such that* $\mathrm{CH}_1(Y) = \Gamma_* \mathrm{CH}_0(S)$.

Proof. The variety Y satisfies $(\star)$ and $(\star\star)$. We can therefore apply the results of the previous sections : Proposition 2.6 for $\mathrm{CH}_0^{\mathbf{Z}}(Y)$, Proposition 2.8 for $\mathrm{CH}_1(Y)$ and Proposition 2.9 for $\mathrm{CH}_2(Y)$. The statement for $\mathrm{CH}_3^{\mathbf{Z}}(Y)$ is obvious. □

Remark 2.19. More precisely the Chow motive of Y can be shown to "come from" the Chow motive of a surface. Also, Murre's conjectures hold for Y (see Corollary 4.12).

Remark 2.20. It can be shown that $H^3(Y, \Omega_Y) \neq 0$ and hence that $\mathrm{Gr}^1_{\widetilde{N}} H_4(Y) \neq 0$. By Lewis' and Schoen's theorem 2.7, there is no curve C and no correspondence $\Gamma \in \mathrm{CH}_2(C \times Y)$ such that $\mathrm{CH}_1(Y) = \Gamma_* \mathrm{CH}_0(C)$.

Remark 2.21. Unfortunately Ran's and Shioda's technique for proving the generalized Hodge conjecture for Fermat varieties does not work for high-degree Fermat varieties. For example, it is mentioned in [39, §3 (13)] that the method fails for Fermat 3-folds of degree 11 because there is an element in $\mathfrak{B}^3_{11} = \{(b_0, b_1, b_2, b_3, b_4) \mid 1 \leq b_i \leq 10, \sum_{i=0}^4 < tb_i >= 22 \text{ or } 33 \text{ for all } t \in (\mathbf{Z}/11)^\times\}$ which is neither decomposable nor quasi-decomposable, namely $(1, 1, 5, 7, 8)$.

3 Niveau and Coniveau filtrations on Chow groups

We use the construction of the idempotents $\Pi_{i,j}$ of Theorem 2 to relate some conjectures on algebraic cycles. André and Kahn [1] proved that the conjunction of Grothendieck's standard conjectures with the Bloch–Beilinson–Murre conjectures imply Kimura's finite-dimensionality conjecture. Jannsen [18, Cor. 6.4(c)] proved that the conjunction of such conjectures implies that the niveau and the coniveau filtrations on Chow groups coincide. Here, assuming Grothendieck's standard conjectures and Kimura's conjecture, we introduce a filtration on Chow groups using our idempotents and, in the spirit of Jannsen's paper, prove that if our filtration on Chow groups coincides with the coniveau filtration considered by Jannsen then the BBM conjectures hold.

Murre's conjectures. These were formulated in [29] and were shown to be equivalent to those formulated by Bloch and Beilinson in [17, Th 5.2]. Let X be a smooth projective variety of dimension d over k. Murre conjectured that:

(A) X has a Chow–Künneth decompositio: There exist mutually orthogonal idempotents $\Pi_0, \ldots, \Pi_{2d} \in \mathrm{CH}_d(X \times X)$ adding to the identity such that $(\Pi_i)_* H_*(X) = H_i(X)$ for all i.

(B) $\Pi_0, \ldots, \Pi_{2l-1}, \Pi_{d+l+1}, \ldots, \Pi_{2d}$ act trivially on $\mathrm{CH}_l(X)$ for all l.

(B') $\Pi_0, \Pi_1, \ldots, \Pi_{2l-1}$ act trivially on $\mathrm{CH}_l(X)$ for all l.

(C) $F^i \mathrm{CH}_l(X) := \mathrm{Ker}\,(\Pi_{2l}) \cap \ldots \cap \mathrm{Ker}\,(\Pi_{2l+i-1})$ doesn't depend on the choice of the idempotents Π_j. Here the idempotents Π_j are acting on $\mathrm{CH}_l(X)$.

(D) $F^1 \mathrm{CH}_l(X) = \mathrm{CH}_l(X)_{\mathrm{hom}}$.

Let X be a smooth projective variety that satisfies $(\star\star)$ and whose Künneth projectors $\pi_i : H_*(X) \to H_i(X) \to H_*(X)$ are algebraic. By Lemma 1.10, the projectors π_i lift to idempotents $\Pi_i \in \mathrm{CH}_d(X \times X)$ and hence X satisfies Murre's conjecture (A). Moreover if $(\Pi'_i)_{0 \leq i \leq 2d}$ is another choice of liftings for the Künneth projectors then, still by Lemma 1.10, there exists a nilpotent correspondence $N \in \mathrm{CH}_d(X \times X)$ such that $\mathrm{Ker}\,(\Pi'_{2l}) \cap \ldots \cap \mathrm{Ker}\,(\Pi'_{2l+i-1}) = (1 + N)_*\big(\mathrm{Ker}\,(\Pi_{2l}) \cap \ldots \cap \mathrm{Ker}\,(\Pi_{2l+i-1})\big)$. We wish to show that if X and $X \times X$ satisfy Murre's conjecture (B') then Murre's conjecture (C) holds for X. For this purpose it is enough to show that $F^i \mathrm{CH}_l(X)$ is stabilized by the action of correspondences in $\mathrm{CH}_d(X \times X)$. The transpose idempotents $\widetilde{\Pi}_i := {}^t\Pi_{2d-i}$ define another set of mutually orthogonal idempotents lifting the Künneth projectors. The correspondences $\widetilde{\Pi}_i \times \Pi_j \in \mathrm{CH}_{2d}((X \times X) \times (X \times X))$ then define mutually orthogonal idempotents and $X \times X$ satisfies Murre's conjecture (A) with respect to the idempotents $\Pi_k^{X \times X} := \sum_{i+j=k} \widetilde{\Pi}_i \times \Pi_j$ by the Künneth formula in cohomology. The proof of the following proposition is essentially contained in the proof of Jannsen's [17, Th 5.2 & Prop. 5.8].

Proposition 3.1. *If X satisfies $(\star\star)$ as well as Murre's conjectures (A) and (B'), and if $X \times X$ given with the idempotents $\Pi_k^{X \times X}$ above satisfies Murre's conjecture (B') then X satisfies Murre's conjecture (C).*

Proof. By the above discussion it is enough to prove that $F^i \mathrm{CH}_l(X)$ is stabilized by the action of correspondences $\alpha \in \mathrm{CH}_d(X \times X)$. Because X satisfies Murre's conjecture (B'), we have $F^i \mathrm{CH}_l(X) = \mathrm{Im}\,(\Pi_{2l+i} + \ldots + \Pi_{2d})$. By orthogonality of the idempotents Π_k, it is thus equivalent to prove that $\Pi_j \circ \alpha \circ \Pi_k = 0$ for $2l + i \leq k \leq 2d$ and $2l \leq j \leq 2l + i - 1$. But $\Pi_j \circ \alpha \circ \Pi_k = ({}^t\Pi_k \times \Pi_j)_* \alpha = (\widetilde{\Pi}_{2d-k} \times \Pi_j)_* \alpha$ and hence $\Pi_j \circ \alpha \circ \Pi_k \in \mathrm{Im}\,(\Pi_{2d-k+j}^{X \times X})$. Now $2d - k + j < 2d$ and because $X \times X$ is assumed to satisfy Murre's conjecture (B') we get that $\Pi_{2d-k+j}^{X \times X}$ acts as zero on $\mathrm{CH}_d(X \times X)$. □

Corollary 3.2. *If $(\star\star)$ and Murre's conjectures (A) and (B') hold for all smooth projective varieties then Murre's conjecture (C) holds also for all smooth projective varieties.*

Remark 3.3. Some authors replace Murre's conjecture (C) by the following weaker conjecture which is satisfied when X has a CK decomposition and satisfies $(\star\star)$.

(C') If $\{\Pi_i\}$ and $\{\Pi'_i\}$ are two Chow–Künneth decompositions for X inducing, respectively, filtrations F and F' on the Chow groups of X, then there exists a nilpotent correspondence $N \in \mathrm{CH}_d(X \times X)$ such that $F^i \mathrm{CH}_l(X) = (1 + N)_*(F')^i \mathrm{CH}_l(X)$ for all i and all l.

If S is a smooth projective surface, then Murre [28] proved that S satisfies Murre's conjectures (A), (B) and (D). If, moreover, S satisfies ($\star\star$), then S does actually satisfy Murre's conjecture (C) (and not just (C')). This can be proved using Proposition 3.1 or can be seen to be a direct consequence of Theorem 4.8 to come.

If X satisfies property B, then the correspondences π_i exist by Kleiman [21] and factor through a smooth projective variety of dimension $\min(i, 2d-i)$. Therefore, thanks to Lemma 1.14, if X satisfies property B and ($\star\star$), then the idempotents $\Pi_{d-l+1}, \ldots, \Pi_{2d}$ act trivially on $\mathrm{CH}_l(X)$ for all l. Thus Murre's conjecture (B') is equivalent to Murre's conjecture (B) for those varieties that satisfy B and ($\star\star$).

If X satisfies ($\star$) and ($\star\star$), then we let $\Pi_i := \sum_j \Pi_{i,j}$. Theorem 2 immediately gives the following proposition.

Proposition 3.4. *If X is a smooth projective variety that satisfies ($\star$) and ($\star\star$) then*

- *X satisfies Murre's conjectures (B) if and only if $(\Pi_{i,j})_* \mathrm{CH}_l(X) = 0$ whenever $i < 2l$ and $i - j > l + 1$.*
- *X satisfies Murre's conjectures (D) if and only if $(\Pi_{i,j})_* \mathrm{CH}_l(X) = 0$ whenever $i = 2l$ and $i - j > l + 1$.*

Niveau filtration on Chow groups. Nori [30] introduced an increasing filtration on $\mathrm{CH}_l(X)_{\mathrm{hom}}$:

$$\mathrm{CH}_l(X)_{\mathrm{alg}} = A_0 \mathrm{CH}_l(X)_{\mathrm{hom}} \subseteq A_1 \mathrm{CH}_l(X)_{\mathrm{hom}} \subseteq \ldots \subseteq A_l \mathrm{CH}_l(X)_{\mathrm{hom}} = \mathrm{CH}_l(X)_{\mathrm{hom}}$$

called the niveau filtration and defined as follows :

$$A_r \mathrm{CH}_l(X)_{\mathrm{hom}} := \sum \mathrm{Im}\,(\Gamma_* : \mathrm{CH}_r(Y)_{\mathrm{hom}} \rightarrow \mathrm{CH}_l(X)),$$

where the sum runs through all varieties Y and all correspondences $\Gamma \in \mathrm{CH}^{d-l+r}(Y \times X)$. From the definition it is clear that this filtration is functorial with respect to the action of correspondences, i.e. if $\alpha \in \mathrm{CH}_{d+k}(X \times Z)$ then $\alpha_* A_r \mathrm{CH}_l(X)_{\mathrm{hom}} \subseteq A_r \mathrm{CH}_{l+k}(Z)_{\mathrm{hom}}$.

If X satisfies ($\star$) and ($\star\star$), then we can define a filtration

$$\widetilde{N}_r \mathrm{CH}_l(X)_{\mathrm{hom}} := \big(\sum_i \sum_{j \geq l-r} \Pi_{i,j}\big)_* \mathrm{CH}_l(X)_{\mathrm{hom}}.$$

By Theorem 1 and Lemma 1.14 we have $\widetilde{N}_r \mathrm{CH}_l(X)_{\mathrm{hom}} \subseteq A_r \mathrm{CH}_l(X)_{\mathrm{hom}}$. If one assumes property B as well as ($\star\star$) being valid for all smooth projective varieties then this filtration is defined for all smooth projective varieties. Unfortunately, I cannot prove (even if I assume Murre's conjectures) that this filtration is functorial with respect to the action of correspondences. If this were the case then the filtration would not depend on the choice of the idempotents $\Pi_{i,j}$ as Lemma 1.10 shows.

Proposition 3.5. *Assume that property B and ($\star\star$) hold for all smooth projective varieties. Then the filtration $\widetilde{N}$ is functorial if and only if it coincides with Nori's filtration A.*

Proof. The "if" part is obvious. Let's prove the "only if" part. Let Y be a smooth projective variety and $\Gamma \in \mathrm{CH}^{d-l+r}(Y \times X)$. We have $\mathrm{CH}_r(Y)_{\mathrm{hom}} = \widetilde{N}_r \mathrm{CH}_r(Y)_{\mathrm{hom}}$. Therefore if the filtration $\widetilde{N}$ is functorial, then $\Gamma_* \mathrm{CH}_r(Y)_{\mathrm{hom}} \subseteq \widetilde{N}_r \mathrm{CH}_l(X)_{\mathrm{hom}}$. This proves $A_r \mathrm{CH}_l(X)_{\mathrm{hom}} \subseteq \widetilde{N}_r \mathrm{CH}_l(X)_{\mathrm{hom}}$. □

Coniveau filtration on Chow groups. In [18, 5.10(b)], Jannsen introduces a coniveau filtration on Chow groups (beware that in *loc. cit.* this filtration is denoted by $\widetilde{N}^\bullet$)

$$N^r \operatorname{CH}_l(X)_{\mathrm{hom}} = \sum \operatorname{Im}(\Gamma_* : \operatorname{CH}^r(Y)_{\mathrm{hom}} \to \operatorname{CH}_l(X)),$$

where the sum runs through all smooth projective varieties Y and all correspondences $\Gamma \in \operatorname{CH}_{r+l}(Y \times X)$. By [30, Prop. 5.3], we have $A_r \operatorname{CH}_l(X) \subseteq N^{r+1} \operatorname{CH}_l(X)$. Jannsen proves the following theorem.

Theorem 3.6 (Cor 6.4(c) [18])**.** *Assume that Murre's conjectures as well as property B hold for all smooth projective varieties. Then $A_r \operatorname{CH}_l(X)_{\mathrm{hom}} = N^{r+1} \operatorname{CH}_l(X)_{\mathrm{hom}}$ for all smooth projective varieties X and all integers r and l.*

André and Kahn [1] showed that Murre's conjectures together with Grothendieck's standard conjecture B imply Kimura's finite dimensionality conjecture and thus that property ($\star\star$) holds for all smooth projective varieties. Thus, we prove a weak converse ("weak" because we cannot prove that the filtrations $\widetilde{N}$ and A agree) to Jannsen's theorem.

Proposition 3.7. *Let X be a smooth projective variety that satisfies ($\star$) and ($\star\star$). If $\widetilde{N}_r \operatorname{CH}_l(X)_{\mathrm{hom}} = N^{r+1} \operatorname{CH}_l(X)_{\mathrm{hom}}$ for all all integers r and l, then Murre's conjectures (A), (B) and (D) hold for X.*

Proof. Recall that by part (5) of Theorem 2 we have $(\Pi_{i,j})_* \operatorname{CH}_l(X) \subseteq \operatorname{CH}_l(X)_{\mathrm{hom}}$ unless $i = 2l$ and $j = l$. Given l and r, by Theorem 1 and Lemma 1.14 we have

$$\Big(\sum_{l<i-j\leq l+r} \Pi_{i,j} \Big)_* \operatorname{CH}_l(X) \subseteq N^r \operatorname{CH}_l(X)_{\mathrm{hom}}.$$

Let's now be given l. By Theorem 2 we already know that $\Pi_{i,j}$ acts trivially on $\operatorname{CH}_l(X)$ when $i \leq 2l$ and $i - j = l + 1$. Let's suppose that we have proved that $\Pi_{i,j}$ acts trivially on $\operatorname{CH}_l(X)$ when $i \leq 2l$ and $l < i - j \leq l + r$ for some $r \geq 1$. Because we are assuming $\widetilde{N}_r \operatorname{CH}_l(X)_{\mathrm{hom}} = N^{r+1} \operatorname{CH}_l(X)_{\mathrm{hom}}$ we get that $\Pi_{i,j}$ acts trivially on $\operatorname{CH}_l(X)$ when $i - j = l + r + 1$ and $j < l - r$, that is when $i - j = l + r + 1$ and $i < 2l + 1$. Therefore, by induction on r, this holds for all positive integers r. This proves that $\Pi_{i,j}$ acts trivially on $\operatorname{CH}_l(X)$ whenever $i - j > l$ and $i \leq 2l$. Proposition 3.4 then shows that X satisfies Murre's conjectures (B) and (D). □

Combining Proposition 3.7 and Corollary 3.2 gives the following proposition.

Proposition 3.8. *Assume that properties B and ($\star\star$) hold for all smooth projective varieties. If $\widetilde{N}_r \operatorname{CH}_l(X)_{\mathrm{hom}} = N^{r+1} \operatorname{CH}_l(X)_{\mathrm{hom}}$ for all smooth projective varieties X and all integers r and l, then Murre's conjectures hold for all smooth projective varieties.* □

4 Murre's conjectures in some new cases

A prerequisite for proving Murre's conjectures for a given variety X is to have a Chow–Künneth decomposition for X at our disposal. Either we know that X satisfies properties B and ($\star\star$), in which case the work done in §1 may apply; or X is given with a specific CK decomposition. In the first paragraph, we go through the known cases of varieties satisfying property B and also through the known cases of varieties having a CK decomposition. In the second paragraph, we use Theorem 2 to establish Murre's conjectures for some varieties satisfying ($\star\star$). In the third paragraph, we establish Murre's conjectures for some varieties having a CK decomposition but for which we cannot prove that they satisfy ($\star\star$). Finally, in the fourth paragraph, we illustrate our results with explicit examples.

4.1 Preliminaries

Varieties satisfying property B. Let E_B be the set of smooth projective varieties satisfying property B. It is known that curves, surfaces, complete intersections and abelian varieties belong to E_B. We have the following lemma.

Lemma 4.1. *The set E_B is stable under product and smooth hypersurface section. If $X \in E_B$ and if $f : X \to Y$ is a dominant morphism, then $Y \in E_B$. If $X \in E_B$ and if Z is a smooth subvariety of X that belongs to E_B, then the blow-up of X along Z belongs to E_B.*

Proof. Stability under product and hypersurface section is well known. The rest is contained in [2, Lemma 4.2]. □

Varieties having a Kimura finite-dimensional Chow motive. Let E_K be the set of smooth projective varieties having a Kimura finite-dimensional Chow motive [20]. Curves, abelian varieties and Fermat hypersurfaces [40] belong to E_K.

Lemma 4.2. *The set E_K is stable under product. If $X \in E_K$ and if $f : X \to Y$ is a dominant morphism, then $Y \in E_K$. If $X \in E_K$ and if Z is a smooth subvariety of X that belongs to E_K, then the blow-up of X along Z belongs to E_K. Moreover if $X \in E_K$ then X satisfies ($\star\star$).*

Proof. For the proof, see [20]. □

Varieties admitting a CK decomposition. Let E_{CK} be the set of smooth projective varieties having a CK decomposition. As for E_B, it is known that curves, surfaces, complete intersections and abelian varieties belong to E_{CK}.

Lemma 4.3. *The set E_{CK} is stable under product. If $X \in E_{CK}$ and if Z is a smooth subvariety of X that belongs to E_{CK}, then the blow-up of X along Z belongs to E_{CK}.*

Proof. The first point is straightforward and the second point follows from the blow-up formula for Chow motives. □

Remark 4.4. It is tempting to think that if Y is a variety dominated by $X \in E_{CK}$, then $Y \in E_{CK}$. Indeed, if $\{\Pi_i\}$ is a CK decomposition for X and if $f : X \to Y$ is a dominant (generically finite for convenience) map, then $\{\frac{1}{\deg f}\Gamma_f \circ \Pi_i \circ {}^t\Gamma_f\}$ gives a Künneth decomposition for X (this is because the action of Π_i on homology is central) and it is tempting to think that they should give a CK decomposition for Y. This last point is, however, far from being clear.

4.2 Murre's conjectures using Theorem 2

4.2.1. We start with an immediate consequence of Theorem 2. Recall that a fourfold that satisfies property B satisfies ($\star$) too.

Theorem 4.5. *If X is a fourfold that satisfies B and ($\star\star$), then X satisfies Murre's conjectures (A), (B) and (C').*

Beauville [4] had already proved that result for abelian fourfolds using a different technique. Because abelian fourfolds satisfy both B and ($\star\star$), Theorem 4.5 can be seen as a generalization of Beauville's result.

Note that the only obstruction for X to satisfying Murre's conjecture (D) is due to the fact that we cannot prove that $\Pi_{4,0}$ acts trivially on $\mathrm{CH}_2(X)$ if $H_4(X) \neq \widetilde{N}^1 H_4(X)$.

4.2.2. Let F be the set of Kimura finite-dimensional smooth projective varieties X that satisfy B and whose cohomology satisfies $H_i(X) = \widetilde{N}^{\lfloor i/2 \rfloor -1} H_i(X)$ for all i. In other words, the cohomology of $X \in F$ in even degree is generated, via the action of correspondences, by the degree 2 homology of surfaces and the cohomology of X in odd degrees is generated by the degree 3 homology of 3-folds. In particular, $X \in F$ satisfies property ($\star$) and ($\star\star$). Let's first mention that the set F is not too small.

Proposition 4.6. *The set F contains Kimura finite dimensional 3-folds that satisfy B (e.g. 3-folds dominated by a product of curves, abelian 3-folds, Fermat 3-folds and rationally connected 3-folds). If $X \in F$ and if Z is a smooth subvariety of X of dimension ≤ 3 that satisfies B and ($\star\star$), then the blow-up of X along Z belongs to F. If $X \in F$ and if $f : X \to Y$ is a dominant morphism, then $Y \in F$.*

Proof. This follows from Lemmas 4.1 and 4.2. □

Corollary 4.7. *Let X be a 3-fold belonging to F and let $f : X \dashrightarrow Y$ be a dominant rational map. Then Y belongs to F.*

Proof. By resolution of singularities, there exists a sequence of blow-ups $\widetilde{X}_n \to \ldots \to \widetilde{X}_1 \to X$ along smooth curves and a dominant map $\widetilde{X}_n \to Y$. By Proposition 4.6, $\widetilde{X}_n \in F$ and hence $Y \in F$. □

Once again, as an immediate consequence of Theorem 2, we get the following theorem.

Theorem 4.8. *If $X \in F$, then X satisfies Murre's conjectures (A), (B), (C') and (D).*
If, moreover, $H_{2i+1}(X) = \widetilde{N}^i H_{2i+1}(X)$ for all i (for example if X is a surface that satisfies ($\star\star$)), then X satisfies Murre's conjecture (C).

Proof. Let's only prove the second point. If $X \in F$ with $H_{2i+1}(X) = \widetilde{N}^i H_{2i+1}(X)$ for all i, then, for all l, Proposition 2.8 provides the existence of a smooth projective surface S, of a correspondence $\Gamma \in \mathrm{CH}_{l+2}(S \times X)$ and of a correspondence $\Gamma' \in \mathrm{CH}_{d-l}(X \times S)$ such that $\Gamma \circ \Gamma'$ acts as the identity on $\mathrm{CH}_l(X)$. (This is the case $p = 2$ and $q = 0$ of the proposition). Because algebraic and homological equivalence agree on surfaces, we get that they also agree on X. Therefore, by Theorem 2, we have that $F^1 \mathrm{CH}_l(X) = \mathrm{CH}_l(X)_{\mathrm{hom}}$, $F^2 \mathrm{CH}_l(X) = \mathrm{Ker}\,(AJ_l : \mathrm{CH}_l(X)_{\mathrm{hom}} \to J_l(X))$ and $F^3 \mathrm{CH}_l(X) = 0$ for all l. In particular, the filtration does not depend on the choice of a CK decomposition for X. □

Corollary 4.9. *If X is rationally dominated by a product of three curves, then X satisfies Murre's conjectures (A), (B), (C') and (D).*

Corollary 4.10. *Abelian 3-folds satisfy Murre's conjectures (A), (B), (C') and (D).*

Remark 4.11. Murre's conjectures (B) and (D) were already known to hold for abelian 3-folds by [4].

Corollary 4.12. *The variety Y of §2.3 satisfies Murre's conjectures (A), (B), (C) and (D).*

Proof. The variety Y is dominated by a product of curves and satisfies $H_3(Y) = H_5(Y) = 0$ and $H_4(Y) = \widetilde{N}^1 H_4(Y)$. Therefore Y satisfies the assumptions of Theorem 4.8. □

4.3 Murre's conjectures for some varieties having a CK decomposition

Let G be the set of smooth projective varieties X that have a CK decomposition $\{\Pi_i\}_{0 \leq i \leq 2 \dim X}$ such that for all i there exist subvarieties Y_i and Z_i of respective dimension $i+1$ and $i+2$ such that Π_{2i} has a representative supported on $X \times Y_i$ and Π_{2i+1} has a representative supported on $X \times Z_i$. The correspondence Π_{2i} should be thought of as factoring through a surface and the correspondence Π_{2i+1} should be thought of as factoring through a 3-fold (although beware that this type of condition is more restrictive). Obvious examples of varieties belonging to G are given by curves, surfaces, products of a curve and a surface, and smooth complete intersections of dimension 3. We will give more examples of varieties belonging to G in the next paragraph. For the moment, let's only mention that the set G is not too small:

Proposition 4.13. *If $X \in G$ and if Z is a smooth subvariety of X that belongs to G, then the blow-up of X along Z belongs to G.*

Proof. This follows from the blow-up formula for Chow motives. □

Remark 4.14. If X is a variety that belongs to F, then its CK projectors Π_{2i} (resp. Π_{2i+1}) are homologically equivalent to some cycles supported on $X \times Y_i$ (resp. $X \times Z_i$) for some subvariety Y_i of dimension $i+1$ (resp. Z_i of dimension $i+2$). However, although it is expected to be the case, it is not clear that the projectors themselves are supported on some $X \times Y_i$ or $X \times Z_i$. Thus it is not clear that $F \subseteq G$. The reverse inclusion is even less clear but $F = G$ would follow from general conjectures on algebraic cycles.

The main result of this section is the following theorem.

Theorem 4.15. *If $X \in G$, then X satisfies Murre's conjectures (A), (B') and (D). If, moreover, X satisfies ($\star\star$), then X satisfies Murre's conjecture (C').*

We split its proof into two lemmas. (the statement about conjecture (C') follows from Remark 3.3).

The first lemma settles Murre's conjecture (D) for varieties belonging to G.

Lemma 4.16. *If $X \in G$, then* $\mathrm{Ker}\left(\Pi_{2i} : \mathrm{CH}_i(X) \to \mathrm{CH}_i(X)\right) = \mathrm{CH}_i(X)_{\mathrm{hom}}$ *for all i.*

Proof. Let's fix i. The functoriality of the cycle class map to singular homology with respect to the action of correspondences combined with the fact that Π_{2i} acts as the identity on $H_{2i}(X)$ immediately implies that $\mathrm{Ker}\,(\Pi_{2i}) \subseteq \mathrm{CH}_i(X)_{\mathrm{hom}}$.

Let's now consider $\widetilde{Y}_i$ a desingularization of Y_i. Then the idempotent Π_{2i} factors through $\widetilde{Y}_i$, i.e. there exist $g \in \mathrm{CH}_d(X \times \widetilde{Y}_i)$ and $f \in \mathrm{CH}_{i+1}(\widetilde{Y}_i \times X)$ such that $\Pi_{2i} = f \circ g$. In particular, the action of Π_{2i} on $\mathrm{CH}_i(X)_{\mathrm{hom}}$ factors through $\mathrm{CH}^1(\widetilde{Y}_i)_{\mathrm{hom}}$. By functoriality of the Abel–Jacobi map, we have the following commutative diagram :

$$\begin{array}{ccccc} \mathrm{CH}_i(X)_{\mathrm{hom}} & \xrightarrow{g_*} & \mathrm{CH}^1(\widetilde{Y}_i)_{\mathrm{hom}} & \xrightarrow{f_*} & \mathrm{CH}_i(X)_{\mathrm{hom}} \\ \downarrow{\scriptstyle AJ_i} & & \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle AJ_i} \\ J_i(X) & \longrightarrow & \mathrm{Pic}^0(\widetilde{Y}_i) & \longrightarrow & J_i(X). \end{array}$$

The composite of the two bottom arrows is zero because Π_{2i} acts trivially on $H_{2i+1}(X)$. Therefore, if $\alpha \in \mathrm{CH}_i(X)_{\mathrm{hom}}$, then we have $(g \circ f \circ g)_*\alpha = 0$ and hence $f_* \circ (g \circ f \circ g)_*\alpha = 0$, i.e. $(\Pi_{2i} \circ \Pi_{2i})_*\alpha = 0$, that is, $(\Pi_{2i})_*\alpha = 0$. □

The second lemma settles Murre's conjecture (B') for varieties belonging to G.

Lemma 4.17. *If $X \in G$, then Π_i acts trivially on* $\mathrm{CH}_l(X)$ *for all $i < 2l$.*

Proof. For obvious dimension reasons it is enough to prove that Π_{2i} acts trivially on $\mathrm{CH}_{i+1}(X)$ and that Π_{2i+1} acts trivially on $\mathrm{CH}_{i+1}(X)$ and on $\mathrm{CH}_{i+2}(X)$.

Let's consider $\widetilde{Z}_i$ a desingularization of Z_i. Then the idempotent Π_{2i+1} factors through $\widetilde{Z}_i$, i.e. there exist $g \in \mathrm{CH}_d(X \times \widetilde{Z}_i)$ and $f \in \mathrm{CH}_{i+2}(\widetilde{Z}_i \times X)$ such that $\Pi_{2i} = f \circ g$.

Let's first prove that Π_{2i+1} acts trivially on $\mathrm{CH}_{i+2}(X)$. By functoriality of the cycle class map, we have the commutative diagram

$$\begin{array}{ccccc} \mathrm{CH}_{i+2}(X) & \xrightarrow{g_*} & \mathrm{CH}^0(\widetilde{Z}_i) & \xrightarrow{f_*} & \mathrm{CH}_{i+2}(X) \\ \downarrow{\scriptstyle cl_{i+2}} & & \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle cl_{i+2}} \\ H_{2i+4}(X) & \longrightarrow & H^0(\widetilde{Z}_i) & \longrightarrow & H_{2i+4}(X). \end{array}$$

By definition of a CK decomposition, the idempotent Π_{2i+1} acts trivially on $H_j(X)$ for $j \neq 2i+1$. Therefore if $\alpha \in \mathrm{CH}_{i+2}(X)$, then $(g \circ f \circ g)_*\alpha = 0$ and hence $f_* \circ (g \circ f \circ g)_*\alpha = 0$, i.e. $(\Pi_{2i+1} \circ \Pi_{2i+1})_*\alpha = 0$, that is $(\Pi_{2i+1})_*\alpha = 0$.

The fact that Π_{2i} acts trivially on $\mathrm{CH}_{i+1}(X)$ is similar.

Let's now prove that Π_{2i+1} acts trivially on $\mathrm{CH}_{i+1}(X)$. For this purpose, let's consider the cycle class map to Deligne cohomology. This map is functorial with respect to the action of correspondences and induces an isomorphism $cl^1_{\mathcal{D}} : \mathrm{CH}^1(Y) \xrightarrow{\simeq} H^2_{\mathcal{D}}(Y, \mathbf{Q}(1))$ for any smooth projective variety Y. Once again we have a commutative diagram

$$\begin{array}{ccccc} \mathrm{CH}_{i+1}(X) & \xrightarrow{g_*} & \mathrm{CH}^1(\widetilde{Z}_i) & \xrightarrow{f_*} & \mathrm{CH}_{i+1}(X) \\ \downarrow{\scriptstyle cl_{\mathcal{D}}} & & \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle cl_{\mathcal{D}}} \\ H^{2d-2i-2}_{\mathcal{D}}(X, \mathbf{Q}(d-i-1)) & \longrightarrow & H^2_{\mathcal{D}}(\widetilde{Z}_i, \mathbf{Q}(1)) & \longrightarrow & H^{2d-2i-2}_{\mathcal{D}}(X, \mathbf{Q}(d-i-1)). \end{array}$$

The idempotent Π_{2i+1} acts trivially on $H_{2i+2}(X)$ and it acts trivially also on the intermediate Jacobian $J_{i+1}(X)$ because it acts trivially on $H_{2i+3}(X)$. Therefore, Π_{2i+1} acts trivially on $H^{2d-2i-2}_{\mathcal{D}}(X, \mathbf{Q}(d-i-1))$ because this last group is an extension of the Hodge classes in $H_{2i+2}(X)$ by $J_{i+1}(X)$. Therefore, as before, if $\alpha \in \mathrm{CH}_{i+1}(X)$, then $(g \circ f \circ g)_*\alpha = 0$ and hence $f_* \circ (g \circ f \circ g)_*\alpha = 0$, i.e. $(\Pi_{2i+1} \circ \Pi_{2i+1})_*\alpha = 0$, that is, $(\Pi_{2i+1})_*\alpha = 0$. □

Remark 4.18. Let X be a smooth projective variety that has a CK decomposition $\{\Pi_i\}$. If Π_{2i} has a representative supported on $X \times Y_i$ with $\dim Y_i = i+2$, then, using the same technique as in the two previous lemmas, it can be shown that Π_{2i} acts trivially on $\mathrm{CH}_l(X)$ for $l > i$. Likewise, it can be shown that if $\dim Y_i = i + n$ for some positive integer n, then Π_{2i} acts trivially on $\mathrm{CH}_l(X)$ for $l > i + n - 2$.

4.4 Examples

4.4.1. Varieties belonging to F. In [41], it is proved that if X is a smooth projective variety with representable Chow groups, then X satisfies B, $H_i(X) = \widetilde{N}^{\lfloor i/2 \rfloor} H_i(X)$ for all i and the Chow motive of X is finite dimensional in the sense of Kimura. Examples of varieties with representable Chow groups are discussed in [41] and include curves, surfaces not of general type with vanishing geometric genus, Godeaux surfaces, Barlow surfaces, rationally connected 3-folds and hypersurfaces of very low degree.

We immediately see that if Y' is the product of three varieties with representable Chow groups, then Y' belongs to F. By Proposition 4.6, a variety Y obtained by repeatedly blowing up Y' along smooth curves belongs to F. Still by Proposition 4.6, any variety dominated by Y belongs to F. Therefore Theorem 4.8 gives the following theorem.

Theorem 4.19. *Let X_1, X_2 and X_3 be smooth projective varieties with representable Chow groups and let Z be a variety dominated by a variety obtained by repeatedly blowing up $X_1 \times X_2 \times X_3$ along smooth curves. Then Z satisfies Murre's conjectures (A), (B), (C') and (D).*
If, moreover, $\mathrm{CH}_l(X_3)$ is a finite-dimensional $\mathbf{Q}$-vector space for all l, then Z satisfies Murre's conjecture (C).

Proof. Only the last point deserves treatment. If $\mathrm{CH}_l(X_3)$ is a finite dimensional $\mathbf{Q}$-vector space for all l, then [41, Th. 5] $H_{2i}(X_3) = \widetilde{N}^i H_{2i}(X_3)$ and $H_{2i+1}(X_3) = 0$ for all i, and the Chow motive of X is finite-dimensional in the sense of Kimura. As such, Z belongs to F and satisfies $H_{2i+1}(Z) = \widetilde{N}^i H_{2i+1}(Z)$ for all i. We are thus reduced to the last point of Theorem 4.8. □

4.4.2. Varieties belonging to G. We wish now to give a criterion on Chow groups for a variety to belong to G.

Before we proceed, let's define the following invariance property which is weaker than Murre's conjecture (C). Let X be a smooth projective variety that admits a CK decomposition $\{\Pi_k\}_{0\leq k\leq 2d}$. We say that the CK decomposition $\{\Pi_k\}_{0\leq k\leq 2d}$ is *special* if, for all $i \neq d$, Π_i factors through a 0-dimensional variety if i is even and Π_i factors through a curve if i is odd. By Murre [28], every surface has a special CK decomposition.

Then we say that X satisfies Murre's conjecture (C") if

(C") For any two special CK decompositions of X, the induced filtrations on the Chow groups of X coincide.

Lemma 4.20. *If $\{\Pi_k\}_{0\leq k\leq 2d}$ and $\{\Pi'_k\}_{0\leq k\leq 2d}$ are special CK decompositions for X, then for every i the Chow motives (X, Π_k) and (X, Π'_k) are isomorphic. Here (X, Π_k) denotes the image of the idempotent endomorphism Π_k of the Chow motive $h(X)$.*

Proof. When $k \neq d$ the proof is clear: both (X, Π_k) and (X, Π'_k) are Kimura finite-dimensional, so that the composite of the embedding $(X, \Pi_k) \to h(X)$ with the projection $h(X) \to (X, \Pi'_k)$ is an isomorphism because it is an isomorphism modulo homological equivalence. Similarly, if M and M' are the respective direct sums of the (X, Π_k) and (X, Π'_k) for $k \neq d$, then the composite of the embedding $s : M \to h(X)$ with the projection $r' : h(X) \to M'$ is an isomorphism u. Thus if $r : h(X) \to M$ is the projection, then r and $u^{-1} \circ r'$ are both left inverse to s, so that

$$(X, \Pi_d) \simeq \operatorname{Ker} r \simeq \operatorname{Coker} s \simeq \operatorname{Ker}(u^{-1} \circ r') \simeq \operatorname{Ker} r' \simeq (X, \Pi'_d),$$

as wanted. □

Let $\{\Pi_k\}_{0\leq k\leq 2d}$ be a CK decomposition for X. Call $\{\Pi_k\}_{0\leq k\leq 2d}$ *very special* if it is special and if Π_d has a representative supported on $X \times Y_n$ with Y_n of dimension $n+1$ when $d = 2n$ is even, and on $X \times Z_n$ with Z_n of dimension $n+2$ when $d = 2n+1$ is odd. One of the reasons for introducing this notion is the following straightforward lemma.

Lemma 4.21. *If X has a very special CK decomposition then X belongs to the set G.*

Let's rephrase in terms of motives what it means for $\{\Pi_k\}_{0\leq k\leq 2d}$ to be very special. In the odd-dimensional case for example, the condition on the support is equivalent to requiring that Π_d factor through $h(\widetilde{Z}_n)$ for a resolution of singularities $\widetilde{Z}_n \to Z_n$. Now an idempotent endomorphism e in a pseudo-abelian category factors through an object N if and only if the image of e is a direct summand of N. Thus the condition on Π_d is equivalent to requiring that (X, Π_d) be a direct summand of $h(\widetilde{Z}_n)$.

Note that if $\{\Pi_k\}_{0\leq k\leq 2d}$ is special, then by the above lemma $(X, \Pi_i)^\vee$ (the dual of (X, Π_i)) is isomorphic to $(X, \Pi_{2d-i}, -d)$. Therefore, taking duals gives the further equivalent conditions that (X, Π_d) be a direct summand of $h(\widetilde{Z}_n)(n-1)$ or that Π_d factor through $h(\widetilde{Z}_n)(n-1)$. Using again the above lemma, we also obtain the following.

Lemma 4.22. *If X has a very special CK decomposition then every special CK decomposition of X is very special.*

We now give a criterion on the Chow groups of X for X to have a very special CK decomposition. We start with the even-dimensional case. The following is taken from [41, Th. 4.5].

Theorem 4.23. *Let X be a smooth projective variety of even dimension $d = 2n$. If $\mathrm{CH}_0(X)_{\mathrm{alg}}, \mathrm{CH}_1(X)_{\mathrm{alg}}, \ldots, \mathrm{CH}_{n-2}(X)_{\mathrm{alg}}$ are representable, then X has a very special CK decomposition.* □

Proposition 4.24. *Suppose that $d = 2n$ is even and that X has a very special CK decomposition. Then X satisfies Murre's conjectures (A), (B), (C") and (D). Precisely, algebraic and homological equivalence agree on i-cycles on X for all i, and AJ_i is injective for $i \neq n-1$. If F is the filtration defined by a special CK decomposition for X, then $F^1 \mathrm{CH}_i(X) = \mathrm{CH}_i(X)_{\mathrm{hom}}$ for every i and $F^2 \mathrm{CH}_i(X) = 0$ for $i \neq n-1$, while $F^2 \mathrm{CH}_{n-1}(X) = \mathrm{Ker}\,(AJ_{n-1} : \mathrm{CH}_{n-1}(X)_{\mathrm{hom}} \to J_{n-1}(X))$ and $F^3 \mathrm{CH}_{n-1}(X) = 0$.*

Proof. Let $\{\Pi_k\}_{0 \leq k \leq 2d}$ be a special CK decomposition for X, which by Lemma 4.22 is very special. By Lemma 4.21, X belongs to G and Theorem 4.15 shows that X satisfies Murre's conjectures (A), (B') and (D) with respect to the CK decomposition $\{\Pi_k\}_{0 \leq k \leq 2d}$. Let's now show that X satisfies Murre's conjecture (B). This is clear because when $i \neq d$, Π_i acts non-trivially only on $\mathrm{CH}_{\lfloor i/2 \rfloor}(X)$ because Π_i factors through a curve and acts only in one degree in homology (see also [41, Prop. 2.9]); and when $i = d$, ${}^t\Pi_d = \Pi_d$ acts trivially on $\mathrm{CH}_l(X)$ for $l < n-1$ for dimension reasons (the action of $({}^t\Pi_d)_*$ on $\mathrm{CH}_l(X)$ factors through $\mathrm{CH}_{l+1-n}(Y_n)$ or the motive (X, Π_d) is a direct summand of $h(\tilde{Y}_n)(n-1)$ for a resolution of singularities $\tilde{Y}_n \to Y_n$).

The above shows that that $F^1 \mathrm{CH}_i(X) = \mathrm{CH}_i(X)_{\mathrm{hom}}$ for all i. It also shows that $\mathrm{CH}_i(X) = (\Pi_{2i} + \Pi_{2i+1})_* \mathrm{CH}_i(X)$ for all $i \neq n-1$. Consequently, for all $i \neq n-1$, homological and algebraic equivalence agree on $\mathrm{CH}_i(X)$ and $F^2 \mathrm{CH}_i(X) = 0$. For dimension reasons, the idempotents Π_i act trivially on $\mathrm{CH}_{n-1}(X)$ for $i < d-2$ and for $i > d$. Therefore, $F^3 \mathrm{CH}_{n-1}(X) = 0$. Moreover, the action of Π_i on $\mathrm{CH}_{n-1}(X)$ factors through the Chow group of zero-cycles of a zero-dimensional variety if $i = d-2$, of a curve if $i = d-1$ and of Y_n if $i = d$. Therefore, homological and algebraic equivalence agree also on $\mathrm{CH}_{n-1}(X)$. By Theorem 1.15 (see also [41, Prop. 2.10]), we have that $F^2 \mathrm{CH}_{n-1}(X) = \mathrm{Ker}\,(AJ_{n-1} : \mathrm{CH}_{n-1}(X)_{\mathrm{hom}} \to J_{n-1}(X))$.

We have thus showed that the filtration F on the Chow groups of X induced by the special CK decomposition $\{\Pi_k\}_{0 \leq k \leq 2d}$ does not depend on the choice of $\{\Pi_k\}_{0 \leq k \leq 2d}$. This settles (C") for X. □

Combining the results above, we immediately obtain the following theorem.

Theorem 4.25. *Let X be a smooth projective variety of even dimension $d = 2n$. If $\mathrm{CH}_0(X)_{\mathrm{alg}}, \mathrm{CH}_1(X)_{\mathrm{alg}}, \ldots, \mathrm{CH}_{n-2}(X)_{\mathrm{alg}}$ are representable, then X satisfies Murre's conjectures (A), (B), (C") and (D).* □

Remark 4.26. When $n = 1$, X is a surface and no assumption is made on the Chow groups of X in the theorem above. In this case, conjectures (A), (B) and (D) were settled by Murre [29].

Corollary 4.27. *Let X be a smooth projective fourfold. If either X is rationally connected or if X admits a curve C as a base for its maximal rationally connected fibration, i.e if there exists a rational map $f : X \dashrightarrow C$ with rationally connected general fiber, then X satisfies Murre's conjectures (A), (B), (C") and (D).*

Proof. Indeed if X is rationally connected, then $\mathrm{CH}_0(X) = \mathbf{Q}$. If X admits a curve C as a base for its maximal rationally connected fibration, then $\mathrm{CH}_0(X)_{\mathrm{alg}}$ is representable. □

Examples 4.28. All smooth projective varieties that are birational to a Fano fourfold are rationally connected by Kollár [22], and hence satisfy Murre's conjectures (A), (B), (C") and (D).

Examples 4.29. Let Y be a generic hypersurface of $\mathbf{P}^4 \times \mathbf{P}^n$ of bidegree (a, b) with $a \leq 4$. The projection of Y onto $\mathbf{P}^n$ has rationally connected general fiber of dimension 3. Let C be a generic curve in $\mathbf{P}^n$. If $X := Y \times_{\mathbf{P}_n} C$, then the projection $X \to C$ has rationally connected general fiber. Therefore, X satisfies Murre's conjectures (A), (B), (C") and (D).

Let's now deal with the odd-dimensional case. In [41, Th.4.10] we proved the following.

Theorem 4.30. *Let X be a smooth projective variety of odd dimension $d = 2n+1$ with $H^{n+1}(X, \Omega_X^{n-1}) = 0$. If $\mathrm{CH}_0(X)_{\mathrm{alg}}, \mathrm{CH}_1(X)_{\mathrm{alg}}, \ldots, \mathrm{CH}_{n-2}(X)_{\mathrm{alg}}$ are representable, then X has a very special CK decomposition.* □

Proposition 4.31. *Suppose that $d = 2n+1$ is odd and that X has a very special CK decomposition. Then X satisfies Murre's conjectures (A), (B), (C") and (D). Precisely, algebraic and homological equivalence agree on i-cycles on X for $i \neq n$, and AJ_i is injective for $i \neq n-1$. If F is the filtration defined by a special CK decomposition for X, then $F^1 \mathrm{CH}_i(X) = \mathrm{CH}_i(X)_{\mathrm{hom}}$ for every i and $F^2 \mathrm{CH}_i(X) = 0$ for $i \neq n-1$, while $F^2 \mathrm{CH}_{n-1}(X) = F^3 \mathrm{CH}_{n-1}(X) = \mathrm{Ker}\,(AJ_{n-1} : \mathrm{CH}_{n-1}(X)_{\mathrm{hom}} \to J_{n-1}(X))$ and $F^4 \mathrm{CH}_{n-1}(X) = 0$.*

Proof. Let $\{\Pi_k\}_{0 \leq k \leq 2d}$ be a special CK decomposition for X, which by Lemma 4.22 is very special. By Lemma 4.21, X belongs to G and Theorem 4.15 shows that X satisfies Murre's conjectures (A), (B') and (D) with respect to the CK decomposition $\{\Pi_k\}_{0 \leq k \leq 2d}$. Let's now show that X satisfies Murre's conjecture (B). This is clear because when $i \neq d$, Π_i acts non-trivially only on $\mathrm{CH}_{\lfloor i/2 \rfloor}(X)$ because Π_i factors through a curve and acts only in one degree in homology (see also [41, Prop. 2.9]); and when $i = d$, ${}^t\Pi_d = \Pi_d$ acts trivially on $\mathrm{CH}_l(X)$ for $l < n-1$ for dimension reasons (the action of $({}^t\Pi_d)_*$ on $\mathrm{CH}^l(X)$ factors through $\mathrm{CH}_{l+1-n}(Z_n)$ or the motive (X, Π_d) is a direct summand of $h(\tilde{Z}_n)(n-1)$).

The above shows that that $F^1 \mathrm{CH}_i(X) = \mathrm{CH}_i(X)_{\mathrm{hom}}$ for all i. It also shows that $\mathrm{CH}_i(X) = (\Pi_{2i} + \Pi_{2i+1})_* \mathrm{CH}_i(X)$ for all $i \neq n-1$. Therefore, for all $i \neq n-1$, $F^2 \mathrm{CH}_i(X) = 0$. (Note that homological and algebraic equivalence do not necessarily agree on $\mathrm{CH}_n(X)$ but do agree on $\mathrm{CH}_i(X)$ for $i \neq n-1, n$.) For the same reasons as in the proof of Theorem 4.25, homological and algebraic equivalence agree on $\mathrm{CH}_{n-1}(X)$. By Theorem 1.15 (see also [41, Prop. 2.10]), we have $\mathrm{Ker}\,\big(\Pi_{d-2} : \mathrm{CH}_{n-1}(X)_{\mathrm{hom}} \to \mathrm{CH}_{n-1}(X)_{\mathrm{hom}}\big) = \mathrm{Ker}\,\big(AJ_{n-1} : \mathrm{CH}_{n-1}(X)_{\mathrm{hom}} \to J_{n-1}(X)\big)$. Because Π_{d-1} acts trivially on $\mathrm{CH}_{n-1}(X)$, we have $F^2 \mathrm{CH}_{n-1}(X) = F^3 \mathrm{CH}_{n-1}(X)$ and then because Π_d is the only remaining CK projector acting possibly non trivially on $\mathrm{CH}_{n-1}(X)$, we have $F^4 \mathrm{CH}_{n-1}(X) = 0$. We therefore see that the filtration does not depend on the particular choice of a special CK decomposition for X. This settles (C") for X. □

Combining the results above gives the following.

Theorem 4.32. *Let X be a smooth projective variety of odd dimension $d = 2n+1$ with $H^{n+1}(X, \Omega_X^{n-1}) = 0$. If $\mathrm{CH}_0(X)_{\mathrm{alg}}, \mathrm{CH}_1(X)_{\mathrm{alg}}, \ldots, \mathrm{CH}_{n-2}(X)_{\mathrm{alg}}$ are representable, then X satisfies Murre's conjectures (A), (B), (C") and (D).* □

Examples 4.33. If X is one of the following :

- a 3-fold with $H^2(X, O_X) = 0$, e.g. a Calabi–Yau 3-fold (more can be said for Fano 3-folds since they belong to F) or a complete intersection of dimension 3;
- a 5-fold with $H^3(X, \Omega_X) = 0$ and with representable $\mathrm{CH}_0(X)_{\mathrm{alg}}$. For instance, this could be a rationally connected 5-fold with $H^3(X, \Omega_X) = 0$. Examples of such varieties are given by blowing up a Fano complete intersection of dimension 5 along a smooth subvariety Z satisfying $H^2(Z, O_Z) = 0$. For instance, Z could be a smooth curve contained in X or, more interestingly, a smooth 3-fold obtained by intersecting X with two (high-degree) hypersurfaces.

Then X satisfies Murre's conjectures (A), (B), (C") and (D).

4.4.3. Complete intersections of low degree. Here is a list of examples of complete intersections for which we can apply Theorems 4.25 and 4.32. All such varieties thus satisfy Murre's conjectures (A), (B), (C") and (D).

- Hypersurfaces of $\mathbf{P}^6$ of degree ≤ 6 and more generally Fano complete intersections of dimension 5; cf. examples above.
- Cubic 5- and 6-folds : Paranjape [32] and Kollár [22] showed that they satisfy $\mathrm{CH}_0(X) = \mathrm{CH}_1(X) = \mathbf{Q}$. Moreover, Theorem 4.19 shows that the cubic 5-fold satisfies Murre's conjecture (C).
- The intersection of a quadric and of a cubic of dimension 6 : Esnault, Levine and Viehweg [10] proved that it satisfies $\mathrm{CH}_0(X) = \mathrm{CH}_1(X) = \mathbf{Q}$.
- The intersection of two quadrics of dimension 8 : Esnault, Levine and Viehweg [10] proved that it satisfies $\mathrm{CH}_0(X) = \mathrm{CH}_1(X) = \mathrm{CH}_2(X) = \mathbf{Q}$.
- Quartic 7-folds : Otwinowska [31, Cor. 1] showed that they satisfy $\mathrm{CH}_0(X) = \mathrm{CH}_1(X) = \mathbf{Q}$.
- Cubic hypersurfaces of dimensions 8, 9, 11, 14 and 15 : Otwinowska [31, Cor. 1] showed that they satisfy the assumptions of Theorems 4.25 and 4.32.

More generally, Otwinowska [31] showed that if X is a smooth hyperplane section of a hypersurface in $\mathbf{P}^{n+1}$ covered by l-planes then $\mathrm{CH}_i(X)_{\mathrm{hom}} = 0$ for $i \leq l-1$ (see also Esnault, Levine and Viehweg [10]). Therefore when $l = \lfloor n/2 \rfloor - 1$, Theorems 4.25 and 4.32 give the following theorem.

Theorem 4.34. *Let $l = \lfloor n/2 \rfloor - 1$ and let X be a smooth hyperplane section of a hypersurface in $\mathbf{P}^{n+1}$ covered by l-planes. Then, X satisfies Murre's conjectures (A), (B), (C") and (D).* □

4.4.4. Product of a surface with a variety with representable Chow groups. Consider a surface S and a smooth projective variety X whose Chow groups are all representable (see §4.4.1. for example). Thanks to [41, Th. 4], it is easy to see that $S \times X$ belongs to G. Let Y be the variety obtained by successively blowing up $X \times S$ along smooth surfaces. Then Y also belongs to G by Proposition 4.13. The following is a corollary of Theorem 4.15.

Theorem 4.35. *The variety Y satisfies Murre's conjectures (A), (B) and (D).*

Proof. By Theorem 4.15, there is only Murre's conjecture (B) left to prove. It is even enough to prove that Π_i acts trivially on $\mathrm{CH}_l(Y)$ for $l < \lfloor i/2 \rfloor - 1$. By the Künneth formula for CK decompositions and by the blow-up formula, we see that the idempotent Π_i factors through a surface if i is even and through a 3-fold if i is odd. In particular, the action of Π_i on $\mathrm{CH}_l(Y)$ factors through $(l - \lfloor i/2 \rfloor + 1)$-cycles on the surface or the 3-fold. □

Remark 4.36. If Y is taken to be the product of a curve with a surface, conjectures (A), (B) and (D) for Y were settled by Murre [29].

In the case when S is Kimura finite-dimensional, Theorem 4.8 gives the following theorem.

Theorem 4.37. *If the surface S is Kimura finite-dimensional and if Z is dominated by $X \times S$, then Z satisfies Murre's conjectures (A), (B), (C') and (D).* □

References

[1] Yves André and Bruno Kahn. (with an appendix by P. O'Sullivan) Nilpotence, radicaux et structures monoïdales. *Rend. Sem. Mat. Univ. Padova*, 108:107–291, 2002.

[2] Donu Arapura. Motivation for Hodge cycles. *Adv. Math.*, 207(2):762–781, 2006.

[3] Rebecca Barlow. Rational equivalence of zero cycles for some more surfaces with $p_g = 0$. *Invent. Math.*, 79(2):303–308, 1985.

[4] Arnaud Beauville. Sur l'anneau de Chow d'une variété abélienne. *Math. Ann.*, 273(4):647–651, 1986.

[5] S. Bloch and V. Srinivas. Remarks on correspondences and algebraic cycles. *Amer. J. Math.*, 105(5):1235–1253, 1983.

[6] Spencer Bloch and Arthur Ogus. Gersten's conjecture and the homology of schemes. *Ann. Sci. École Norm. Sup. (4)*, 7:181–201 (1975), 1974.

[7] F. Charles. Remarks on the Lefschetz standard conjecture and hyperkähler varieties. Preprint.

[8] Pedro Luis del Angel and Stefan Müller-Stach. Motives of uniruled 3-folds. *Compositio Math.*, 112(1):1–16, 1998.

[9] Hélène Esnault and Marc Levine. Surjectivity of cycle maps. *Astérisque*, (218):203–226, 1993. Journées de Géométrie Algébrique d'Orsay (Orsay, 1992).

[10] Hélène Esnault, Marc Levine, and Eckart Viehweg. Chow groups of projective varieties of very small degree. *Duke Math. J.*, 87(1):29–58, 1997.

[11] B. B. Gordon and J. P. Murre. Chow motives of elliptic modular threefolds. *J. Reine Angew. Math.*, 514:145–164, 1999.

[12] V. Guletskiĭ and C. Pedrini. Finite-dimensional motives and the conjectures of Beilinson and Murre. *K-Theory*, 30(3):243–263, 2003. Special issue in honor of Hyman Bass on his seventieth birthday. Part III.

[13] H. Inose and M. Mizukami. Rational equivalence of 0-cycles on some surfaces of general type with $p_g = 0$. *Math. Ann.*, 244(3):205–217, 1979.

[14] Jaya N. N. Iyer. Murre's conjectures and explicit Chow-Künneth projections for varieties with a NEF tangent bundle. *Trans. Amer. Math. Soc.*, 361(3):1667–1681, 2009.

[15] Uwe Jannsen. *Mixed motives and algebraic K-theory*, volume 1400 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1990. With appendices by S. Bloch and C. Schoen.

[16] Uwe Jannsen. Motives, numerical equivalence, and semi-simplicity. *Invent. Math.*, 107(3):447–452, 1992.

[17] Uwe Jannsen. Motivic sheaves and filtrations on Chow groups. In *Motives (Seattle, WA, 1991)*, volume 55 of *Proc. Sympos. Pure Math.*, pages 245–302. Amer. Math. Soc., Providence, RI, 1994.

[18] Uwe Jannsen. Equivalence relations on algebraic cycles. In *The arithmetic and geometry of algebraic cycles (Banff, AB, 1998)*, volume 548 of *NATO Sci. Ser. C Math. Phys. Sci.*, pages 225–260. Kluwer Acad. Publ., Dordrecht, 2000.

[19] Bruno Kahn, Jacob P. Murre, and Claudio Pedrini. On the transcendental part of the motive of a surface. In *Algebraic cycles and motives. Vol. 2*, volume 344 of *London Math. Soc. Lecture Note Ser.*, pages 143–202. Cambridge Univ. Press, Cambridge, 2007.

[20] Shun-Ichi Kimura. Chow groups are finite dimensional, in some sense. *Math. Ann.*, 331(1):173–201, 2005.

[21] Steven L. Kleiman. The standard conjectures. In *Motives (Seattle, WA, 1991)*, volume 55 of *Proc. Sympos. Pure Math.*, pages 3–20. Amer. Math. Soc., Providence, RI, 1994.

[22] János Kollár. *Rational curves on algebraic varieties*, volume 32 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics*. Springer-Verlag, Berlin, 1996.

[23] Robert Laterveer. Algebraic varieties with small Chow groups. *J. Math. Kyoto Univ.*, 38(4):673–694, 1998.

[24] James D. Lewis. Towards a generalization of Mumford's theorem. *J. Math. Kyoto Univ.*, 29(2):267–272, 1989.

[25] David I. Lieberman. Higher Picard varieties. *Amer. J. Math.*, 90:1165–1199, 1968.

[26] David I. Lieberman. Numerical and homological equivalence of algebraic cycles on Hodge manifolds. *Amer. J. Math.*, 90:366–374, 1968.

[27] D. Mumford. Rational equivalence of 0-cycles on surfaces. *J. Math. Kyoto Univ.*, 9:195–204, 1968.

[28] J. P. Murre. On the motive of an algebraic surface. *J. Reine Angew. Math.*, 409:190–204, 1990.

[29] J. P. Murre. On a conjectural filtration on the Chow groups of an algebraic variety. I. The general conjectures and some examples. *Indag. Math. (N.S.)*, 4(2):177–188, 1993.

[30] Madhav V. Nori. Algebraic cycles and Hodge-theoretic connectivity. *Invent. Math.*, 111(2):349–373, 1993.

[31] Anna Otwinowska. Remarques sur les groupes de Chow des hypersurfaces de petit degré. *C. R. Acad. Sci. Paris Sér. I Math.*, 329(1):51–56, 1999.

[32] Kapil H. Paranjape. Cohomological and cycle-theoretic connectivity. *Ann. of Math. (2)*, 139(3):641–660, 1994.

[33] Chris Peters. Bloch-type conjectures and an example of a three-fold of general type. *Commun. Contemp. Math.*, 12(4):587–605, 2010.

[34] Ziv Ran. Cycles on Fermat hypersurfaces. *Compositio Math.*, 42(1):121–142, 1980/81.

[35] A. A. Roĭtman. Rational equivalence of zero-dimensional cycles. *Mat. Sb. (N.S.)*, 89(131):569–585, 671, 1972.

[36] A. A. Roĭtman. The torsion of the group of 0-cycles modulo rational equivalence. *Ann. of Math. (2)*, 111(3):553–569, 1980.

[37] Chad Schoen. On Hodge structures and nonrepresentability of Chow groups. *Compositio Math.*, 88(3):285–316, 1993.

[38] Tetsuji Shioda. The Hodge conjecture for Fermat varieties. *Math. Ann.*, 245(2):175–184, 1979.

[39] Tetsuji Shioda. What is known about the Hodge conjecture? In *Algebraic varieties and analytic varieties (Tokyo, 1981)*, volume 1 of *Adv. Stud. Pure Math.*, pages 55–68. North-Holland, Amsterdam, 1983.

[40] Tetsuji Shioda and Toshiyuki Katsura. On Fermat varieties. *Tôhoku Math. J. (2)*, 31(1):97–115, 1979.

[41] Charles Vial. Projectors on the intermediate algebraic Jacobians. Preprint.

[42] Claire Voisin. Sur les zéro-cycles de certaines hypersurfaces munies d'un automorphisme. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 19(4):473–492, 1992.

[43] Claire Voisin. Remarks on filtrations on Chow groups and the Bloch conjecture. *Ann. Mat. Pura Appl. (4)*, 183(3):421–438, 2004.

DPMMS, University of Cambridge, Wilberforce Road, Cambridge, CB3 0WB, UK
e-mail : C.Vial@dpmms.cam.ac.uk