AROUND THE DE RHAM–BETTI CONJECTURE

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Abstract. A *de Rham–Betti class* on a smooth projective variety X over an algebraic extension K of the rational numbers is a rational class in the Betti cohomology of the analytification of X that descends to a class in the algebraic de Rham cohomology of X via the period comparison isomorphism. The period conjecture of Grothendieck implies that de Rham–Betti classes should be algebraic. We prove that any de Rham–Betti class on a product of elliptic curves is algebraic. This is achieved by showing that the Tannakian torsor associated to a de Rham–Betti object is connected, and by exploiting the analytic subgroup theorem of Wüstholz. In the case of products of non-CM elliptic curves, we prove the stronger result that $\overline{\mathbb{Q}}$-de Rham–Betti classes are $\overline{\mathbb{Q}}$-linear combinations of algebraic classes by showing that the period comparison isomorphism generates the torsor of motivic periods. A key step consists in establishing a version of the analytic subgroup theorem with $\overline{\mathbb{Q}}$-coefficients. Finally, building on results of Deligne and André regarding the Kuga–Satake correspondence, we further show that any de Rham–Betti isometry between the second cohomology groups of hyper-Kähler varieties, with second Betti number not 3, is Hodge. As two applications we show that codimension-2 de Rham–Betti classes on hyper-Kähler varieties of known deformation type are Hodge and we obtain a global de Rham–Betti Torelli theorem for K3 surfaces over $\overline{\mathbb{Q}}$.

Introduction

De Rham–Betti classes. Let X be a smooth projective variety defined over a field $K \subseteq \mathbb{C}$. Serre's GAGA and the analytic Poincaré lemma provide a canonical isomorphism

$$c_X^n : \mathrm{H}^n_{\mathrm{dR}}(X/K) \otimes_K \mathbb{C} \xrightarrow{\simeq} \mathrm{H}^n_{\mathrm{B}}(X^{\mathrm{an}}_{\mathbb{C}}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}, \tag{1}$$

called the *period comparison isomorphism*, between the algebraic de Rham cohomology of $X_{\mathbb{C}}$ and the Betti cohomology with $\mathbb{C}$-coefficients of the analytification $X^{\mathrm{an}}_{\mathbb{C}}$ of $X_{\mathbb{C}}$. Writing as is usual $\mathbb{Q}(k) =_{\mathrm{def}} (2\pi i)^k\mathbb{Q} \subset \mathbb{C}$, the *de Rham–Betti cohomology* of X is defined as the triple

$$\mathrm{H}^n_{\mathrm{dRB}}(X, \mathbb{Q}(k)) =_{\mathrm{def}} \big(\mathrm{H}^n_{\mathrm{dR}}(X/K), \mathrm{H}^n_{\mathrm{B}}(X^{\mathrm{an}}_{\mathbb{C}}, \mathbb{Q}(k)), c^n_X\big).$$

If Z denotes an algebraic cycle of codimension k on X, then its classes in de Rham cohomology $\mathrm{H}^{2k}_{\mathrm{dR}}(X/K)$ and in Betti cohomology $\mathrm{H}^{2k}_{\mathrm{B}}(X^{\mathrm{an}}_{\mathbb{C}}, \mathbb{Q}(k))$ are related, up to a sign, by

$$c^{2k}_X\big(\mathrm{cl}_{\mathrm{dR}}(Z) \otimes_K 1_{\mathbb{C}}\big) = \mathrm{cl}_{\mathrm{B}}(Z) \otimes_{\mathbb{Q}} 1_{\mathbb{C}}, \tag{2}$$

see e.g., [BC16, Prop. 1.1]. A *de Rham–Betti class* in $\mathrm{H}^n_{\mathrm{dRB}}(X, \mathbb{Q}(k))$, also called *Grothendieck class* in *loc. cit.*, is a pair $(\alpha_B, \alpha_{\mathrm{dR}})$ with α_B a class in $\mathrm{H}^n_{\mathrm{B}}(X^{\mathrm{an}}_{\mathbb{C}}, \mathbb{Q}(k))$ and α_{dR} a class in $\mathrm{H}^n_{\mathrm{dR}}(X/K)$ whose complexifications correspond to one another under c_X ; see Definitions 4.1 and 4.3. By (2) the Betti class and the de Rham class of an algebraic cycle of codimension k define a de Rham–Betti class in $\mathrm{H}^{2k}_{\mathrm{dRB}}(X, \mathbb{Q}(k))$. By Poincaré duality, a de Rham class $\omega \in \mathrm{H}^n_{\mathrm{dR}}(X/K)$ can be

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extended to a de Rham–Betti class in $\mathrm{H}^n_{\mathrm{dRB}}(X, \mathbb{Q}(k))$ if and only if the *complex periods*

$$\frac{1}{(2\pi i)^k} \int_\gamma \omega$$

lie in K for every rational homology class $\gamma \in \mathrm{H}_n(X^{\mathrm{an}}_{\mathbb{C}}, \mathbb{Q})$.

The Grothendieck Period Conjecture. The *periods* of X are the complex numbers

$$\int_\gamma \omega, \quad \text{where } \gamma \in \mathrm{H}^{\mathrm{B}}_n(X^{\mathrm{an}}_{\mathbb{C}}, \mathbb{Q}),\ \omega \in \mathrm{H}^n_{\mathrm{dR}}(X/K),\ n \in \mathbb{Z}_{\geq 0}.$$

These form a finite-dimensional K-vector subspace of $\mathbb{C}$ spanned by the coefficients of the *period matrix*, that is, of the matrices of the period comparison isomorphisms c^n_X with respect to the choice of a K-basis of $\mathrm{H}^n_{\mathrm{dR}}(X/K)$ and of a $\mathbb{Q}$-basis of $\mathrm{H}^n_{\mathrm{B}}(X^{\mathrm{an}}_{\mathbb{C}}, \mathbb{Q})$. Any algebraic cycle on a power X^m of X induces a homogeneous polynomial relation of degree m among the coefficients of the period matrix of X. In case $K \subseteq \overline{\mathbb{Q}}$, the *Grothendieck period conjecture* for X stipulates that conversely any polynomial relation with K-coefficients among the periods should be in the ideal generated by polynomials induced by algebraic cycles on some powers of X ; see [And04, Conj. 7.5.2.1]. Using the Tannakian formalism, the Grothendieck Period Conjecture predicts that the comparison isomorphism seen as a complex point of the torsor of *motivic periods* is dense, and in particular [And04, Prop. 7.5.2.2] that the degree of transcendence over $\mathbb{Q}$ of the field generated by the periods of X over $\overline{\mathbb{Q}}$ is equal to the dimension of the conjectural *motivic Galois group* of X, which if the Hodge conjecture holds, is equal to the dimension of the *Mumford–Tate group* of $X_{\mathbb{C}}$. We refer to [And04, Ayo14, BC16] and to §§6.2-6.3 for more details.

As an illustration, suppose E is a smooth elliptic curve over $\overline{\mathbb{Q}}$ given by the equation $y^3 = x^3 - ax - b$, and fix a basis (γ_1, γ_2) of $\mathrm{H}^{\mathrm{B}}_1(E^{\mathrm{an}}_{\mathbb{C}}, \mathbb{Q})$. The $\overline{\mathbb{Q}}$-vector space $\mathrm{H}^1_{\mathrm{dR}}(E/\overline{\mathbb{Q}})$ is spanned by the global differential form $\omega = \frac{dx}{y}$ and the differential form of second kind $\eta = \frac{xdx}{y}$. The field of periods of E is then generated by the four classical elliptic integrals

$$\omega_1 := \int_{\gamma_1} \omega, \quad \omega_2 := \int_{\gamma_2} \omega, \quad \eta_1 := \int_{\gamma_1} \eta, \quad \eta_2 := \int_{\gamma_2} \eta.$$

Note that the field of periods contains π as $\omega_1\eta_2 - \omega_2\eta_1$ is a nonzero rational multiple of $2\pi i$. A classical result of Schneider [Sch37] from the 1930s establishes that the periods ω_1 and ω_2 are transcendental and that they are $\overline{\mathbb{Q}}$-linear independent if and only if E does not have complex multiplication. This was complemented almost 40 years later by Masser [Mas75] who showed that the $\overline{\mathbb{Q}}$-span of $1, 2\pi i, \omega_1, \omega_2, \eta_1, \eta_2$ has dimension 6 if E is without CM and has dimension 4 if E has CM. These results were then overhauled 10 years later by Wüstholz' analytic subgroup theorem [Wüs84] ; see e.g. [BW07, §6.2]. The motivic Galois group of an elliptic curve is well-defined and coincides with its Mumford–Tate group ; it has dimension 4 if E is without CM and it has dimension 2 if E has CM. The Grothendieck Period Conjecture therefore predicts that the field of periods of E has transcendence degree over $\mathbb{Q}$ equal to 4 if E is without CM and equal to 2 if E has CM. The best result in that direction is from the late 1970s and is due to Chudnovsky [Chu80] who showed that the field of periods of an elliptic curve E has transcendence degree at least 2 ; in particular the Grothendieck Period Conjecture is true for CM elliptic curves. This is the only non-trivial example for which the Grothendieck period conjecture is known.

The Grothendieck Period Conjecture and the algebraicity of de Rham–Betti classes. If X satisfies the Grothendieck Period Conjecture 6.6 and if X satisfies the standard conjectures, then every de Rham–Betti class in $\mathrm{H}^j_{\mathrm{dRB}}(X, \mathbb{Q}(k))$ is algebraic, that is, is the class of an algebraic cycle ; see [And04, Prop. 7.5.2.2] and [BC16, Prop. 2.13]. In particular, in view of Chudnovsky's aforementioned theorem [Chu80], de Rham–Betti classes on powers of CM elliptic curves are algebraic.

On the other hand, it has been possible to establish the algebraicity of *non-zero* de Rham–Betti classes in some cases without establishing the Grothendieck Period Conjecture. Here is an exhaustive list : for de Rham–Betti classes in $\mathrm{H}^2_{\mathrm{dRB}}(X, \mathbb{Q}(1))$ for X an abelian variety [And04, Bos13, BC16] as an application of Wüstholz' analytic subgroup theorem [Wüs84] and, via the Kuga–Satake correspondence of André [And96a], for X a hyper-Kähler variety [BC16] ; see Theorem 7.4 and Proposition 9.1. Here a *hyper-Kähler variety* over $K \subseteq \mathbb{C}$ means a variety over K whose base-change to $\mathbb{C}$ is projective, irreducible holomorphic symplectic and, deviating from the usual definition, is such that its second Betti number satisfies $b_2 > 3$. (This latter condition, which holds for all known deformation families of hyper-Kähler varieties, ensures that the deformation space of a hyper-Kähler variety is big enough and is crucial in André's work [And96a].)

As a particular instance of the above, any de Rham–Betti class should be a Hodge class. Even this latter expectation is wide open and we are not aware of any examples beyond the ones mentioned above for which this expectation is met.

De Rham–Betti classes on products of elliptic curves. Our first result is the following extension of the consequence of Chudnovsky's theorem concerned with the algebraicity of de Rham–Betti classes on powers of CM elliptic curves.

Theorem 1. *Let X be an abelian variety over an algebraic extension K of $\mathbb{Q}$ inside $\mathbb{C}$ and assume either that $X_{\overline{\mathbb{Q}}}$ is isogenous to a product of elliptic curves over $\overline{\mathbb{Q}}$, or that $X_{\overline{\mathbb{Q}}}$ is isogenous to the power of an abelian surface over $\overline{\mathbb{Q}}$ with non-trivial endomorphism ring. Then :*

(i) any de Rham–Betti class in $\mathrm{H}^{2k}_{\mathrm{dRB}}(X, \mathbb{Q}(k))$ is algebraic ;

(ii) any de Rham–Betti class in $\mathrm{H}^{j}_{\mathrm{dRB}}(X, \mathbb{Q}(k))$ is zero for $j \neq 2k$.

We refer to Theorem 7.12 for the case of products of elliptic curves, and to Theorem 7.15 for the case of powers of abelian surfaces whose base-change to $\overline{\mathbb{Q}}$ have non-trivial endomorphism ring. In [And09, footnote 12], André mentions that Theorem 1 can be proved for powers of a non-CM elliptic curve. Theorem 1 for products of elliptic curves is the analogue in the de Rham–Betti setting of the following results.

(a) The Hodge conjecture holds for products of complex elliptic curves. Its proof essentially goes back to Tate (unpublished) ; see [Gor99, §3] for a proof and further references.
(b) The Tate conjecture holds for products of elliptic curves over a finitely generated extension of $\mathbb{Q}$. This follows from the validity of the Mumford–Tate conjecture for products of elliptic curves. The latter is established in [Lom16, Cor. 1.2] and builds on the validity of the Mumford–Tate conjecture for elliptic curves due to Serre [Ser68].
(c) The Tate conjecture holds for products of elliptic curves over a finite field. This is due to Spieß [Spi99].

Moreover, both the Hodge and the Mumford–Tate conjectures hold for simple abelian varieties of prime dimension [Tan82]. In contrast to the case of Hodge classes and Tate classes, it is *a priori* not known, for lack of a theory of weights in the de Rham–Betti setting, that for a smooth projective variety X over $\overline{\mathbb{Q}}$ the de Rham–Betti object $\mathrm{H}^{j}_{\mathrm{dRB}}(X, \mathbb{Q}(k))$ does not support any non-zero de Rham–Betti class for $j \neq 2k$. Theorem 1(*ii*) confirms that this is indeed the case for products of elliptic curves. Beyond the case of powers of CM elliptic curves covered by Chudnovsky's theorem [Chu80], the only general results that were known so far in that direction are the following :

(a) Any de Rham–Betti class in $\mathrm{H}^{0}_{\mathrm{dRB}}(X, \mathbb{Q}(k))$ is zero for $k \neq 0$; this is equivalent to the transcendence of π.
(b) Any de Rham–Betti class in $\mathrm{H}^{1}_{\mathrm{dRB}}(X, \mathbb{Q}(k))$ is zero for any $k \in \mathbb{Z}$; this follows from Wüstholz' analytic subgroup theorem (see Theorem 7.6, but also [BC16, Thms. 4.1 & 4.2] for the cases $k = 0$ and $k = 1$).

Tannakian formulation of the Grothendieck Period Conjecture. Let M be a smooth projective variety over $\overline{\mathbb{Q}}$ or more generally a André motive over $\overline{\mathbb{Q}}$ as defined in §3. As outlined in §6, there is a chain of inclusions

$$Z_M \subseteq \Omega_M \subseteq \Omega_M^{\mathrm{dRB}} \subseteq \Omega_M^{\mathrm{And}} \subseteq \mathrm{Iso}_{\overline{\mathbb{Q}}}\big(\mathrm{H}_{\mathrm{dR}}(M), \mathrm{H}_{\mathrm{B}}(M) \otimes \overline{\mathbb{Q}}\big), \quad \text{where}$$

(i) Z_M is the Zariski closure of the comparison isomorphism $c_M \colon \mathrm{H}_{\mathrm{dR}}(M) \otimes_{\overline{\mathbb{Q}}} \mathbb{C} \xrightarrow{\sim} \mathrm{H}_{\mathrm{B}}(M) \otimes_{\mathbb{Q}} \mathbb{C}$ seen as a complex-valued point in the $\overline{\mathbb{Q}}$-torsor $\mathrm{Iso}_{\overline{\mathbb{Q}}}\big(\mathrm{H}_{\mathrm{dR}}(M), \mathrm{H}_{\mathrm{B}}(M) \otimes \overline{\mathbb{Q}}\big)$,
(ii) Ω_M is the smallest $\overline{\mathbb{Q}}$-subtorsor of $\mathrm{Iso}_{\overline{\mathbb{Q}}}\big(\mathrm{H}_{\mathrm{dR}}(M), \mathrm{H}_{\mathrm{B}}(M) \otimes \overline{\mathbb{Q}}\big)$ containing c_M,
(iii) Ω_M^{dRB} is the *torsor of periods* of the de Rham–Betti realization of M,
(iv) Ω_M^{And} is the *torsor of motivated periods* of M.

With this notation, the Grothendieck Period Conjecture for a smooth projective variety X over $\overline{\mathbb{Q}}$ states that the inclusion $Z_X \subseteq \Omega_X^{\mathrm{And}}$ is an equality and that motivated classes on powers of X are algebraic.

The $\overline{\mathbb{Q}}$-subschemes Z_M, Ω_M and Ω_M^{dRB} can in fact be defined for any *de Rham–Betti object*, that is, for any triple $(M_{\mathrm{dR}}, M_{\mathrm{B}}, c_M)$ consisting of a finite-dimensional $\overline{\mathbb{Q}}$-vector space M_{dR}, a finite-dimensional $\mathbb{Q}$-vector space M_{B}, and a $\mathbb{C}$-linear isomorphism $c_M : M_{\mathrm{dR}} \otimes_{\overline{\mathbb{Q}}} \mathbb{C} \xrightarrow{\sim} M_{\mathrm{B}} \otimes_{\mathbb{Q}} \mathbb{C}$. A de Rham–Betti object will be said to be *geometric* if it is the de Rham–Betti realization of a André motive over $\overline{\mathbb{Q}}$. In an effort to distinguish formal properties of de Rham–Betti objects with properties of geometric de Rham–Betti objects, we will focus first in §4 and §5 exclusively on general de Rham–Betti objects (with various fields of coefficients, both on the de Rham and the Betti side), and only then in §6 outline which properties are expected, or have been established, in the geometric setting. For instance, in Theorem 4.7, we show that the torsor of periods of a de Rham–Betti object is connected, while in Theorem 7.11 we observe, as a consequence of Wüstholz' analytic subgroup theorem, that Ω_X^{dRB} is a torsor under a reductive algebraic group for X an abelian variety over $\overline{\mathbb{Q}}$. The latter does not hold for a general de Rham–Betti object since not all de Rham–Betti objects are semi-simple, but is in fact expected to hold, by the Grothendieck Period Conjecture, for the torsor of periods of any geometric de Rham–Betti object. This dichotomy between general de Rham–Betti objects and geometric de Rham–Betti objects also takes in this paper the following form : in §4 and §5 we will study the inclusions $Z_M \subseteq \Omega_M \subseteq \Omega_M^{\mathrm{dRB}}$ for a general de Rham–Betti object, while in §6 we will focus our attention on the inclusion $\Omega_M^{\mathrm{dRB}} \subseteq \Omega_M^{\mathrm{And}}$ for a André motive M. That the inclusion $\Omega_M^{\mathrm{dRB}} \subseteq \Omega_M^{\mathrm{And}}$ is an equality is a special case of the Grothendieck Period Conjecture that we will refer to as the *motivated de Rham–Betti conjecture* :

Conjecture 1 (Conjecture 6.10 and Conjecture 6.13)**.** *Let M be a André motive over $K \subseteq \overline{\mathbb{Q}}$.*

(i) We say M satisfies the motivated de Rham–Betti conjecture *if the inclusion*

$$\Omega_M^{\mathrm{dRB}} \subseteq \Omega_M^{\mathrm{And}}$$

is an equality.

(ii) We say M satisfies the motivated $\overline{\mathbb{Q}}$-de Rham–Betti conjecture *if the inclusion*

$$\Omega_M \subseteq \Omega_M^{\mathrm{And}}$$

is an equality, that is, if the comparison isomorphism c_M generates Ω_M^{And} as a $\overline{\mathbb{Q}}$-torsor.

By Proposition 6.12, the inclusion $\Omega_M^{\mathrm{dRB}} \subseteq \Omega_M^{\mathrm{And}}$ is an equality only if de Rham–Betti classes on the tensor spaces $M^{\otimes n} \otimes (M^\vee)^{\otimes m}$ are motivated. This explains why the Grothendieck Period Conjecture implies the algebraicity of de Rham–Betti classes. The reason for calling Conjecture 1(*ii*) the *motivated $\overline{\mathbb{Q}}$-de Rham–Betti conjecture* will become clear in the next paragraph.

The proof of Theorem 1 uses this Tannakian formalism and follows this dichotomy ; it consists in showing the stronger statement that the de Rham–Betti conjecture holds for $X_{\overline{\mathbb{Q}}}$, that is, that the torsor of periods of $X_{\overline{\mathbb{Q}}}$ agrees with its torsor of motivic periods. Crucial ingredients include the connectedness of the torsor of periods (Theorem 4.7) and the following two consequences (Theorem 7.11) of Wüstholz' analytic subgroup theorem : the torsor of periods of an abelian variety X is a torsor under a reductive group $G_{\mathrm{dRB}}(X) \subseteq \mathrm{GL}(\mathrm{H}_{\mathrm{B}}(X))$ and codimension-1 de Rham–Betti classes on abelian varieties are algebraic. A difficulty is that, in contrast to the Hodge setting and the Mumford–Tate group, it is not known that $G_{\mathrm{dRB}}(X)$ contains the scalar matrices.

On $\overline{\mathbb{Q}}$-de Rham–Betti classes. A $\overline{\mathbb{Q}}$-*de Rham–Betti class* on a de Rham–Betti object $M := (M_{\mathrm{dR}}, M_{\mathrm{B}}, c_M)$ consists of a pair $\alpha_{\mathrm{dR}} \in M_{\mathrm{dR}}$ and $\alpha_{\mathrm{B}} \in M_{\mathrm{B}} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ such that $c_M(\alpha_{\mathrm{dR},\mathbb{C}}) = \alpha_{\mathrm{B},\mathbb{C}}$. It is a priori not at all clear that $\overline{\mathbb{Q}}$-de Rham–Betti classes are $\overline{\mathbb{Q}}$-linear combinations of de Rham–Betti classes. This indeed fails for certain non-geometric de Rham–Betti objects, see Example 5.3. However the Grothendieck Period Conjecture predicts that $\overline{\mathbb{Q}}$-de Rham–Betti classes on geometric de Rham–Betti objects are $\overline{\mathbb{Q}}$-linear combinations of algebraic classes (we say $\overline{\mathbb{Q}}$-*algebraic*) and hence are $\overline{\mathbb{Q}}$-linear combinations of de Rham–Betti classes. We note that if $\overline{\mathbb{Q}}$-de Rham–Betti classes on a André motive M are $\overline{\mathbb{Q}}$-linear combinations of motivated classes (we say $\overline{\mathbb{Q}}$-*motivated*), then de Rham–Betti classes on M are motivated ; see Lemma 6.17.

Coming back to the case of a general de Rham–Betti object M, we identify in Proposition 5.1 the torsor Ω_M with the Tannakian torsor associated to the $\overline{\mathbb{Q}}$-*de Rham–Betti object* $M \otimes \overline{\mathbb{Q}} =_{\mathrm{def}} (M_{\mathrm{dR}}, M_{\mathrm{B}} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}, c_M)$ and show in Proposition 5.2 that, provided both M and $M \otimes \overline{\mathbb{Q}}$ are semi-simple, it agrees with the torsor of periods Ω_M^{dRB} if and only if $\overline{\mathbb{Q}}$-de Rham–Betti classes on tensor spaces $M^{\otimes n} \otimes (M^\vee)^{\otimes m}$ are $\overline{\mathbb{Q}}$-linear combinations of de Rham–Betti classes.

It follows from Chudnovsky [Chu80] that we have the stronger statement that any $\overline{\mathbb{Q}}$-de Rham–Betti class on the power of a CM elliptic curve E is $\overline{\mathbb{Q}}$-algebraic. We extend this result by showing

Theorem 2. *Let X be an abelian variety over $K \subseteq \overline{\mathbb{Q}}$ such that $X_{\overline{\mathbb{Q}}}$ is isogenous to a product of non-CM elliptic curves. Then X satisfies the motivated $\overline{\mathbb{Q}}$-de Rham–Betti conjecture : the comparison isomorphism c_X generates the torsor of motivated periods Ω_X^{And}, that is, $\Omega_X = \Omega_X^{\mathrm{And}}$. In particular, for all integers j and k, any $\overline{\mathbb{Q}}$-de Rham–Betti class in $\mathrm{H}^j_{\mathrm{dRB}}(X, \overline{\mathbb{Q}}(k))$ is $\overline{\mathbb{Q}}$-algebraic.*

We refer to Theorem 7.12(*ii*) for a slightly more general statement and to Theorem 7.15(*ii*) for a similar statement for powers of certain abelian surfaces. While the Grothendieck Period Conjecture remains open in these cases, we are able to prove that the complex comparison c_X generates the torsor of motivated periods. These provide the first examples, beyond the case of varieties whose motive belongs to the Tannakian category generated by the motive of a single CM elliptic curve E, for which it can be showed that $\overline{\mathbb{Q}}$-de Rham–Betti classes are $\overline{\mathbb{Q}}$-linear combinations of de Rham–Betti classes. As a main ingredient we use the connectedness of Ω_M for a general de Rham–Betti object (Theorem 4.7 and Proposition 5.1) and an extension to $\overline{\mathbb{Q}}$-coefficients of Wüstholz' analytic subgroup theorem (Proposition 7.8), which implies that, for any abelian variety $A/\overline{\mathbb{Q}}$, Ω_A is a torsor under a reductive group and any codimension-1 $\overline{\mathbb{Q}}$-de Rham–Betti class on A is $\overline{\mathbb{Q}}$-algebraic. The main new difficulty is that tori over $\overline{\mathbb{Q}}$ can only be distinguished by their rank ; in particular, we are not aware of a proof that $\Omega_X = \Omega_X^{\mathrm{And}}$ for a CM elliptic curve X that does not employ Chudnovsky's theorem.

De Rham–Betti isometries between hyper-Kähler varieties. Recall that the second cohomology group of hyper-Kähler varieties comes equipped with a canonical quadratic form called the Beauville–Bogomolov form. In the case of K3 surfaces, this form coincides with the cup-product pairing. If X is a hyper-Kähler variety over K, we denote $\mathrm{T}^2_{\mathrm{dRB}}(X, \mathbb{Q})$ its de Rham–Betti *transcendental cohomology* – it is the orthogonal complement to the subspace spanned by classes of divisors – and $\mathrm{T}^2_{\mathrm{dRB}}(X,\overline{\mathbb{Q}})$ its base-change to $\overline{\mathbb{Q}}$.

Theorem 3 (special instance of Theorem 9.5)**.** *Let X and X' be hyper-Kähler varieties over $\overline{\mathbb{Q}}$. Any $\overline{\mathbb{Q}}$-de Rham–Betti isometry*

$$\mathrm{T}^2_{\mathrm{dRB}}(X,\overline{\mathbb{Q}}) \xrightarrow{\sim} \mathrm{T}^2_{\mathrm{dRB}}(X',\overline{\mathbb{Q}})$$

is $\overline{\mathbb{Q}}$-motivated, in particular a $\overline{\mathbb{Q}}$-linear combination of Hodge classes.

As an application, we obtain :

Theorem 4 (Global de Rham–Betti Torelli theorem for K3 surfaces over $\overline{\mathbb{Q}}$; see Theorem 9.7)**.** *Let S and S' be two K3 surfaces over $\overline{\mathbb{Q}}$. If there is an integral de Rham–Betti class in $\mathrm{H}^4_{\mathrm{dRB}}(S\times S',\mathbb{Z}(2))$ inducing an isometry*

$$\mathrm{H}^2_{\mathrm{dRB}}(S,\mathbb{Z}) \xrightarrow{\sim} \mathrm{H}^2_{\mathrm{dRB}}(S',\mathbb{Z}),$$

then S and S' are isomorphic.

Our proof of Theorem 3 makes essential use of André's theory of *motivated cycles* [And96b] and André's results regarding the Kuga–Satake construction [And96a]. Motivated cycles and the category of André motives are reviewed in §3, while the Kuga–Satake correspondence is reviewed in §8. We indeed show, via André's Theorem 8.3 establishing that the Kuga–Satake correspondence is motivated and defined over $\overline{\mathbb{Q}}$ but also via Theorem 7.10 which is a consequence of the $\overline{\mathbb{Q}}$-version of Wüstholz' analytic subgroup theorem, that $\overline{\mathbb{Q}}$-de Rham–Betti isometries as in Theorem 3 are $\overline{\mathbb{Q}}$-motivated and hence $\overline{\mathbb{Q}}$-linear combinations of (absolute) Hodge classes.

De Rham–Betti classes on hyper-Kähler varieties. Let X be a hyper-Kähler variety over $\overline{\mathbb{Q}}$. The analytic subgroup theorem for abelian motives with $\overline{\mathbb{Q}}$–coefficients (in the form of Theorem 7.10) together with the Kuga–Satake correspondence implies that $\overline{\mathbb{Q}}$-de Rham–Betti classes in $\mathrm{H}^2_{\mathrm{dRB}}(X,\overline{\mathbb{Q}}(1))$ are $\overline{\mathbb{Q}}$-algebraic ; see Proposition 9.1. A hyper-Kähler variety will be said to be *of known deformation type* if its analytification is deformation equivalent to the Hilbert scheme of points on a K3 surface, a generalized Kummer variety, or one of O'Grady's two sporadic examples (all of these do satisfy $b_2 > 3$). Using Theorem 3, we obtain the following.

Theorem 5 (Proposition 9.8 and Theorem 9.9)**.** *Let X be a hyper-Kähler variety over $\overline{\mathbb{Q}}$ and let $n\in\mathbb{Z}_{>0}$. Then :*

(i) any $\overline{\mathbb{Q}}$-de Rham–Betti class in $\mathrm{H}^2_{\mathrm{dRB}}(X,\overline{\mathbb{Q}}(1))\otimes\mathrm{H}^2_{\mathrm{dRB}}(X,\overline{\mathbb{Q}}(1))$ is $\overline{\mathbb{Q}}$-motivated, in particular a $\overline{\mathbb{Q}}$-linear combination of Hodge classes.

If in addition X is of known deformation type, then :

(ii) any $\overline{\mathbb{Q}}$-de Rham–Betti class in $\mathrm{H}^4_{\mathrm{dRB}}(X,\overline{\mathbb{Q}}(2))$ is $\overline{\mathbb{Q}}$-motivated, in particular a $\overline{\mathbb{Q}}$-linear combination of Hodge classes.

As a first consequence of Theorem 5(*i*), we obtain in Corollary 9.12 that, for any $k\in\mathbb{Z}$, any de Rham–Betti class in $\mathrm{T}^2_{\mathrm{dRB}}(X,\mathbb{Q}(k))$ is zero. Furthermore, building on Theorem 1 and its proof, we can derive from Theorem 5 that Rham–Betti classes of any codimension on hyper-Kähler varieties of large Picard rank are Hodge :

Theorem 6. *Let X be a hyper-Kähler variety over $\overline{\mathbb{Q}}$ of known deformation type. Denote by $\rho^c(X)$ the* Picard corank *of X, that is, $\rho^c(X) =_{\mathrm{def}} h^{1,1}(X^{\mathrm{an}}_{\mathbb{C}}) - \rho(X)$ where $\rho(X)$ is the Picard rank of X.*

(i) If $\rho^c(X)\leq 1$, then any $\overline{\mathbb{Q}}$-de Rham–Betti class in $\mathrm{H}^j_{\mathrm{dRB}}(X^n,\overline{\mathbb{Q}}(k))$ is $\overline{\mathbb{Q}}$-motivated, and hence is a $\overline{\mathbb{Q}}$-linear combination of (absolute) Hodge classes.

(ii) *If $\rho^c(X) = 2$, then any de Rham–Betti class in $\mathrm{H}^j_{\mathrm{dRB}}(X^n, \mathbb{Q}(k))$ is motivated, and hence is (absolute) Hodge.*

Theorem 6 follows (due to Propositions 6.12 and 6.16) from Theorem 9.15, where it is shown that if $\rho^c(X) = 2$ (resp. if $\rho^c(X) \leq 1$), then X satisfies the motivated version of the de Rham–Betti conjecture (resp. the motivated version of the $\overline{\mathbb{Q}}$-de Rham–Betti conjecture). Moreover, we also observe in Theorem 9.15 that the motivated version of the Grothendieck period conjecture holds in case X is of maximal Picard rank.

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1. Reductive groups and semi-simple representations

An *algebraic group* over a field K is a group scheme of finite type over K. Recall that a connected algebraic group over a field K is said to be *reductive* if the unipotent radical $R_u(G_{\overline{K}})$ of $G_{\overline{K}} =_{\mathrm{def}} G \times_K \overline{K}$, which is the largest connected normal unipotent algebraic subgroup of $G_{\overline{K}}$, is trivial.

For the convenience of the reader, we recall the following well-known fact and, for lack of a reference in the non-connected case, provide a proof.

Proposition 1.1. *Assume* $\mathrm{char}(K) = 0$ *and let G be an algebraic group over K. The following statements are equivalent.*

(a) *The connected component G° of G is reductive.*
(b) *Every finite-dimensional representation of G° is semi-simple.*
(c) *Every finite-dimensional representation of G is semi-simple.*
(d) *Some faithful finite-dimensional representation of G is semi-simple.*

Proof. In case G is connected, the Proposition is [Mil17, Thm. 22.42]; in particular, *loc. cit.* establishes $(a) \iff (b)$. The equivalence $(a) \iff (c)$ is [Mil17, Cor. 22.43] or [DM82, Rmk. 2.28]. The implication $(c) \Rightarrow (d)$ is obvious since any algebraic group admits a faithful finite-dimensional representation [Mil17, Thm. 4.9]. For lack of a suitable reference, we now prove the implication $(d) \Rightarrow (a)$. Since a semi-simple representation of G remains semi-simple after base-change to a separable extension [Mil17, Prop. 4.19], we may assume that K is algebraically closed. Let V be a faithful semi-simple representation of G and write $V = \bigoplus_{i=1}^n V_i$ with each V_i being a simple representation of G. Let $U =_{\mathrm{def}} R_u(G^\circ) \subset G^\circ$ be the unipotent radical of G°. Then, for each $g \in G(K)$, we see that $gUg^{-1} \subset G^\circ$ is again a connected normal unipotent subgroup of G°. Since U is the largest such subgroup, we get $gUg^{-1} \subseteq U$. This implies that U is a normal subgroup of G. Hence $V_i^U \subset V_i$ is a sub-representation of G. Since U is unipotent, $V_i^U \neq 0$, and by simplicity of V_i we conclude that $V_i^U = V_i$ for each i. As a consequence, U acts trivially on V. By faithfulness, we conclude that U is trivial and hence that G° is reductive. $\square$

In this work, we will exclusively consider algebraic groups in characteristic zero. We will say that an algebraic group G over K with $\mathrm{char}(K) = 0$ is *reductive* if its connected component is, that is, if it satisfies any of the equivalent conditions of Proposition 1.1.

2. Generalities on Tannakian categories

Our main references concerning Tannakian categories are [DM82, SR72], and [And04, §2.3] for a quick overview.

2.1. **Tannakian categories.** Let F be a field. Suppose that $\mathscr{T}$ is a rigid $\otimes$-category, that $\mathscr{T}$ is abelian, and that $\mathrm{End}_{\mathscr{T}}(\mathbb{1}) = F$. A *fiber functor* on $\mathscr{T}$ is an exact and faithful $\otimes$-functor $\omega : \mathscr{T} \to \mathrm{Vec}_K$ to the rigid $\otimes$-category Vec_K of finite-dimensional vector spaces over a finite field extension K of F. If such a fiber functor exists, we say that $\mathscr{T}$ is a *Tannakian category.* Given a Tannakian category $\mathscr{T}$, equipped with a fiber functor ω, one can define its *Tannakian fundamental group* $\mathrm{Aut}^{\otimes}\omega$; it is the affine group scheme over K such that for all field extensions K'/K the group $(\mathrm{Aut}^{\otimes}\omega)(K')$ is the automorphism group of the extended $\otimes$-functor $\omega_{K'} : \mathscr{T} \to \mathrm{Vec}_{K'}$.

2.2. **Tannakian subcategories.** A *Tannakian subcategory* of a Tannakian category $\mathscr{T}$ is a full subcategory $\mathscr{T}'$ of $\mathscr{T}$, stable under $\otimes$ and duals, and such that any subquotient object in $\mathscr{T}$ of an object in $\mathscr{T}'$ is in $\mathscr{T}'$. The restriction of a fiber functor $\omega : \mathscr{T} \to \mathrm{Vec}_K$ to $\mathscr{T}'$ induces by pull-back a faithfully flat homomorphism of K-group schemes $\mathrm{Aut}^{\otimes}(\omega) \to \mathrm{Aut}^{\otimes}(\omega|_{\mathscr{T}'})$. In case X is an object of $\mathscr{T}$, we denote $\langle X\rangle$ the Tannakian subcategory of $\mathscr{T}$ generated by X; its objects are the subquotients of "tensor spaces" $\bigoplus_{\text{finite}} X^{\otimes n_i} \otimes (X^{\vee})^{\otimes m_i}$.

2.3. **Neutral Tannakian categories.** If there exists a fiber functor ω with $F = K$, we say that ω is *neutral* and that $\mathscr{T}$ is a *neutral Tannakian category.* Let $(\mathscr{T}, \omega : \mathscr{T} \to \mathrm{Vec}_F)$ be a neutral Tannakian category with Tannakian fundamental group $G =_{\text{def}} \mathrm{Aut}^{\otimes}\omega$. Then [DM82, Thm. 2.11] the functor $\omega : \mathscr{T} \to \mathrm{Rep}_F\, G$ to the category $\mathrm{Rep}_F\, G$ of finite-dimensional F-representations of G defined by the fiber functor $\omega : \mathscr{T} \to \mathrm{Vec}_F$ is an equivalence of categories. Moreover, by [DM82, Prop. 2.20(b)], G is algebraic if and only if $\mathscr{T}$ is generated by one of its objects.

If $\phi : \mathscr{T}' \to \mathscr{T}$ is an exact $\otimes$-functor between neutral Tannakian categories and if $\omega : \mathscr{T} \to \mathrm{Vec}_F$ is a neutral fiber functor, then $\omega' := \omega \circ \phi$ defines a neutral fiber functor and we have an induced homomorphism $f = \phi^* : G = \mathrm{Aut}^{\otimes}\omega \to G' = \mathrm{Aut}^{\otimes}\omega'$. Conversely, any homomorphism $f : G \to G'$ of affine group schemes over F gives rise to a $\otimes$-functor $\phi = f^* : \mathscr{T}' = \mathrm{Rep}_F\, G' \to \mathscr{T} = \mathrm{Rep}_F\, G$. By [DM82, Prop. 2.21], f is fully faithful (that is, an epimorphism) if and only if ϕ is fully faithful and, for all objects M' of $\mathscr{T}'$, every subobject of $\phi(M')$ is the image of a subobject of M' (this last condition is automatic if $\mathscr{T}$ is semi-simple); and f is a closed immersion (that is, a monomorphism) if and only if every object M in $\mathscr{T}$ is the subquotient of the image of an object N' in $\mathscr{T}'$.

Finally, for X an object of $\mathscr{T}$, the Tannakian fundamental group $G_X =_{\text{def}} \mathrm{Aut}^{\otimes}(\omega|_{\langle X\rangle})$ is a closed K-subgroup of $\mathrm{GL}(\omega(X))$, and [Wat79, §16.1] there exists a line L in a tensor space $\bigoplus_{\text{finite}} \omega(X)^{\otimes n_i} \otimes (\omega(X)^{\vee})^{\otimes m_i}$ such that G_X is the stabilizer of L.

2.4. **Connectedness of the Tannakian fundamental group in characteristic zero.** In characteristic zero, we have the following criterion for the connectedness of the Tannakian fundamental group of a neutral Tannakian category.

Proposition 2.1. *Assume* $\mathrm{char}(F) = 0$. *Let $\mathscr{T}$ be a neutral Tannakian category with neutral fiber functor $\omega : \mathscr{T} \to \mathrm{Vec}_F$. The following statements are equivalent.*

(i) The Tannakian fundamental group $G =_{\text{def}} \mathrm{Aut}^{\otimes}(\omega)$ is connected.
(ii) For any object X of $\mathscr{T}$, if its Tannakian fundamental group G_X is finite, then it is trivial.

Proof. This is essentially the criterion in [DM82, Cor. 2.22]. In details, since $\mathrm{char}(F) = 0$, the group G is connected if and only if there is no non-trivial epimorphism from G to a finite group scheme. Thus, by [DM82, Prop. 2.21(a)] recalled above, G is connected if and only if $\mathscr{T}$ does not have any Tannakian subcategory with non-trivial finite Tannakian fundamental group. By [DM82, Prop. 2.20(b)], such a Tannakian subcategory is generated by an object X. □

2.5. **Semi-simple neutral Tannakian categories in characteristic zero.** Assume $\mathrm{char}(F) = 0$. A neutral Tannakian category $(\mathscr{T}, \omega)$ is semi-simple if and only if its Tannakian fundamental group $G = \mathrm{Aut}^{\otimes}\,\omega$ is pro-reductive, that is, an inverse limit of (not necessarily connected) reductive groups; see e.g., [And04, §2.3.2] and [DM82, Prop. 2.23 & Rmk. 2.28]. For X an object of $\mathscr{T}$, the Tannakian fundamental group $G_X =_{\mathrm{def}} \mathrm{Aut}^{\otimes}(\omega|_{\langle X\rangle})$ is then a reductive closed K-subgroup of $\mathrm{GL}(\omega(X))$, and [And04, §6.3.1] there exists a vector ℓ in a tensor space $\bigoplus_{\text{finite}} \omega(X)^{\otimes n_i} \otimes (\omega(X)^{\vee})^{\otimes m_i}$ such that G_X is the stabilizer of ℓ. In particular, G_X is the closed K-subgroup of $\mathrm{GL}(\omega(X))$ that fixes tensors of the form $\omega(f)$ for $f \in \mathrm{Hom}_{\mathscr{T}}(\mathbb{1}, X^{\otimes n} \otimes (X^{\vee})^{\otimes m})$.

2.6. **The Tannakian fundamental group of semi-simple objects in neutral Tannakian categories in characteristic zero.** Combined with Proposition 1.1, we can summarize the above discussions into the following well-known proposition.

Proposition 2.2. *Let X be an object of a neutral Tannakian category $(\mathscr{T}, \omega : \mathscr{T} \to F)$ with $\mathrm{char}(F) = 0$. The following statements are equivalent.*

(i) The Tannakian fundamental group $G_X =_{\mathrm{def}} \mathrm{Aut}^{\otimes}(\omega|_{\langle X\rangle})$ is reductive.
(ii) The neutral Tannakian category $(\langle X\rangle, \omega|_{\langle X\rangle})$ is semi-simple.
(iii) The F-vector space $\omega(X)$ is a semi-simple representation of G_X.
(iv) The object X is semi-simple in $\mathscr{T}$.

If any of these statements is satisfied, G_X is the closed subgroup of $\mathrm{GL}(\omega(X))$ that fixes elements in $\omega\big(\mathrm{Hom}_{\mathscr{T}}(\mathbb{1}, M)\big)$ inside $\mathrm{Hom}_F(F, \omega(M))$ for all tensor spaces $M = \bigoplus_{\text{finite}} X^{\otimes n_i} \otimes (X^{\vee})^{\otimes m_i}$.

Proof. Recall from §2.3 that $\omega|_{\langle X\rangle} \colon \langle X\rangle \to \mathrm{Rep}_F G_X$ is an equivalence of categories. Note also that an object X is semi-simple in $\mathscr{T}$ if and only if it is semi-simple in $\langle X\rangle$. The equivalence of (*iii*) and (*iv*) (which in fact holds in any characteristic) is then clear. From the equivalence $\omega|_{\langle X\rangle} \colon \langle X\rangle \xrightarrow{\sim} \mathrm{Rep}_F G_X$ it follows that the neutral Tannakian category $\langle X\rangle$ is semi-simple if and only if all finite-dimensional representations of G_X are semi-simple. Since $\omega(X)$ is a faithful representation of G_X, we get from Proposition 1.1 that the statements (*i*), (*ii*) and (*iii*) are equivalent. □

3. Motivated cycles and André motives

3.1. **Motivated cycles and André motives : definitions and properties.** The notion of *motivated cycle* was introduced by Yves André in [And96b]. Let $K \subseteq \mathbb{C}$ be a subfield of the complex numbers and let X be a smooth projective variety over K. A *motivated cycle* on X is an element in $\mathrm{H}^{2r}_{\mathrm{B}}(X^{\mathrm{an}}_{\mathbb{C}}, \mathbb{Q}(r))$ of the form $p_{X,*}(\alpha \cdot *_L \beta)$, where Y is an arbitrary smooth projective variety over K, α and β are algebraic cycles on $X \times_K Y$, and $*_L$ is the (inverse of the) Lefschetz isomorphism attached to any choice of polarizations on X and Y. As detailed in [And96b, §2], the motivated cycles define a graded $\mathbb{Q}$-subalgebra of $\bigoplus_r \mathrm{H}^{2r}_{\mathrm{B}}(X^{\mathrm{an}}_{\mathbb{C}}, \mathbb{Q}(r))$ that contains the classes of algebraic cycles and that is stable under pullbacks and pushforwards along morphisms of smooth projective varieties. As such, one can define *motivated correspondences* and their compositions. By replacing algebraic correspondences with motivated correspondences in the construction of pure motives (as outlined in [And04, §4]), one obtains a pseudo-abelian rigid $\otimes$-category over $\mathbb{Q}$. From the fact that inverses to the Lefschetz isomorphisms are motivated, one obtains that the Künneth projectors are motivated. This provides a grading on the above category and after changing the commutativity constraint along the Koszul rule of signs we obtain the *André category of motives* $\mathsf{M}^{\mathrm{And}}_K$ over K. Its objects have the form $M = p\mathfrak{h}(X)(n) = (X, p, n)$, where X is a smooth projective variety over K of dimension d_X, p is a motivated idempotent correspondence in $\mathrm{H}^{2d_X}_{\mathrm{B}}((X \times_K X)^{\mathrm{an}}_{\mathbb{C}}, \mathbb{Q}(d_X))$ and n is an integer. The unit object is $\mathbb{1} := \mathfrak{h}(\mathrm{Spec} K)$ and with the above formalism the space of motivated cycles on M is $\mathrm{Hom}_{\mathrm{And}}(\mathbb{1}, M)$.

Theorem 3.1 (André [And96b, §4])**.** *The category $\mathsf{M}_K^{\mathrm{And}}$ is graded Tannakian semi-simple over $\mathbb{Q}$, neutralized by the fiber functor given by the Betti cohomology realization functor*

$$\omega_B : \mathsf{M}_K^{\mathrm{And}} \to \mathrm{Vec}_{\mathbb{Q}}, \quad M := (X, p, n) \mapsto \mathrm{H}^*_{\mathrm{B}}(M) := p_* \mathrm{H}^*_{\mathrm{B}}(X^{\mathrm{an}}_{\mathbb{C}}, \mathbb{Q}(n)).$$

The grading of an object $M = \bigoplus_{k \in \mathbb{Z}} M^k$ is such that $\mathrm{H}^*_{\mathrm{B}}(M^k) = \mathrm{H}^k_B(M)$. We write

$$\mathfrak{h}(X) = \bigoplus_{0 \le k \le 2d_X} \mathfrak{h}^k(X)$$

for the grading of the André motive of X. The various realization functors attached to Weil cohomology theories provide other fiber functors. Under the ℓ-adic realization functor $\omega_\ell : \mathsf{M}_K^{\mathrm{And}} \to \mathrm{Vec}_{\mathbb{Q}_\ell}$, motivated cycles are invariant under the action of the Galois group $\mathrm{Gal}(\overline{K}/K)$. Under the de Rham realization functor $\omega_{\mathrm{dR}} : \mathsf{M}_K^{\mathrm{And}} \to \mathrm{Vec}_K$, motivated cycles are mapped to classes lying in F^0 for the Hodge filtration. In addition these are compatible with the canonical comparison isomorphisms, so that motivated cycles are *absolute Hodge* in the sense of Deligne [Del82] (and in particular Hodge) and they are also *de Rham–Betti* (see Definition 4.3 below). Since the Hodge conjecture is known for codimension-1 cycles (Lefschetz $(1,1)$-theorem), we see that any motivated cycle of codimension 1 on X is algebraic.

3.2. **The motivated Galois group and the Mumford–Tate group.** Using the Tannakian formalism, André makes the following

Definition 3.2 (Motivated Galois group)**.** Given a André motive M over K, its *motivated Galois group* $G_{\mathrm{And}}(M)$ is the Tannakian fundamental group

$$G_{\mathrm{And}}(M) =_{\mathrm{def}} \mathrm{Aut}^{\otimes}(\omega_{\mathrm{B}}|_{\langle M \rangle}).$$

Since the neutral Tannakian category $\mathsf{M}_K^{\mathrm{And}}$ is semi-simple, the motivated Galois group $G_{\mathrm{And}}(M)$ is reductive. In addition, $G_{\mathrm{And}}(M)$ is the closed subgroup of $\mathrm{GL}(\omega_B(M))$ that fixes motivated classes inside tensor spaces $\bigoplus_{\text{finite}} M^{\otimes n_i} \otimes (M^\vee)^{\otimes m_i}$.

Recall that a pure rational Hodge structure H consists of a finite-dimensional $\mathbb{Q}$-vector space H together with a decomposition $H \otimes \mathbb{C} = \bigoplus_{i,j} H^{i,j}$ such that $\overline{H^{i,j}} = H^{j,i}$, where complex conjugation on $H \otimes \mathbb{C}$ acts on the second factor. Equivalently, H is a finite-dimensional $\mathbb{Q}$-vector space equipped with a representation $\rho : \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_{m,\mathbb{C}} \to H \otimes \mathbb{R}$ of the Deligne torus; the pieces $H^{i,j}$ then corresponding to the eigenspace for the character $z^{-i}\bar{z}^{-j}$. The *Mumford–Tate group* $\mathrm{MT}(H)$ of H is the smallest $\mathbb{Q}$-subgroup of $\mathrm{GL}(H)$ that contains the image of the representation ρ. The category of pure Hodge structures naturally defines a neutral Tannakian category HS with fiber functor the forgetful functor ω associating to H the underlying $\mathbb{Q}$-vector space. A *Hodge class* in H is an element of H that lies in $H^{0,0}$ after base-change to $\mathbb{C}$; in other words, it is an element of $\mathrm{Hom}_{\mathsf{HS}}(\mathbb{Q}, H)$, where $\mathbb{Q}$ is the trivial Hodge structure with grading concentrated in bidegree $(0,0)$. The Mumford–Tate group of H can then be described as

$$\mathrm{MT}(H) = \mathrm{Aut}^{\otimes}(\omega|_{\langle H \rangle}).$$

In case H is polarizable, $\langle H \rangle$ is semi-simple and hence $\mathrm{MT}(H)$ is reductive and is the closed subgroup of $\mathrm{GL}(H)$ that fixes all Hodge classes in tensor spaces $\bigoplus_{\text{finite}} H^{\otimes n_i} \otimes (H^\vee)^{\otimes m_j}$.

Since motivated cycles are Hodge, the Betti realization functor $\omega_B : \mathsf{M}_K^{\mathrm{And}} \to \mathrm{Vec}_{\mathbb{Q}}$ factors through $\omega : \mathsf{HS} \to \mathrm{Vec}_{\mathbb{Q}}$. Moreover the Hodge structure associated to a André motive is polarizable. It follows that for a André motive M we have an inclusion $\mathrm{MT}(M) \subseteq G_{\mathrm{And}}(M)$ of reductive groups. For future reference, we then have

Proposition 3.3. *The inclusion* $\mathrm{MT}(M) \subseteq G_{\mathrm{And}}(M)$ *is an equality if and only if Hodge classes in tensor spaces* $\bigoplus_{\text{finite}} M^{\otimes n_i} \otimes (M^\vee)^{\otimes m_i}$ *are motivated.*

Proof. This follows at once from the facts that $\mathrm{MT}(M)$ is the closed subgroup of $\mathrm{GL}(\omega_B(M))$ that fixes all Hodge classes in tensor spaces while $G_{\mathrm{And}}(M)$ is the closed subgroup of $\mathrm{GL}(\omega_B(M))$ that fixes all motivated cycles in tensor spaces. □

3.3. **The case of abelian varieties.** Deligne [Del82] famously proved that any Hodge cycle on a complex abelian variety is absolute Hodge. André [And96b] established the following generalization :

Theorem 3.4 (André [And96b])**.** *Let A be a complex abelian variety. Any Hodge cycle in $\mathrm{H}^{2k}(A, \mathbb{Q}(k))$ is motivated and* $\mathrm{MT}(A) =_{\mathrm{def}} \mathrm{MT}(\mathfrak{h}(A)) = G_{\mathrm{And}}(\mathfrak{h}(A)) =: G_{\mathrm{And}}(A)$. □

4. De Rham–Betti objects

In this section, we fix an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ inside $\mathbb{C}$ and we let K be a subfield of $\overline{\mathbb{Q}}$.

4.1. **The Tannakian category of de Rham–Betti objects : definition.** The following definition will be mostly used in the special case where $L = \mathbb{Q}$; but see §5 for remarks on $\overline{\mathbb{Q}}$-de Rham–Betti classes.

Definition 4.1 (De Rham–Betti objects [And04, §7.1.6])**.** Let L be another subfield of $\overline{\mathbb{Q}}$. The category of *L-de Rham–Betti objects over K* is the $(K \cap L)$-linear category $\mathscr{C}_{\mathrm{dRB},K_{\mathrm{dR}},L_{\mathrm{B}}}$ whose objects M are triples of the form

$$M = (M_{\mathrm{dR}}, M_{\mathrm{B}}, c_M),$$

where M_{dR} is a finite-dimensional K-vector space, M_{B} is a finite-dimensional L-vector space and $c_M : M_{\mathrm{dR}} \otimes_K \mathbb{C} \to M_{\mathrm{B}} \otimes_L \mathbb{C}$ is a $\mathbb{C}$-linear isomorphism. A *de Rham–Betti homomorphism* $f \in \mathrm{Hom}_{\mathrm{dRB}}(M, N)$ between L-de Rham–Betti objects over K consists of a K-linear map $f_{\mathrm{dR}} : M_{\mathrm{dR}} \to N_{\mathrm{dR}}$ together with an L-linear map $f_{\mathrm{B}} : M_{\mathrm{B}} \to N_{\mathrm{B}}$ such that the diagram

$$\begin{tikzcd} M_{\mathrm{dR}} \otimes_K \mathbb{C} \arrow[r, "f_{\mathrm{dR}} \otimes_K \mathrm{id}_{\mathbb{C}}"] \arrow[d, "c_M"'] & N_{\mathrm{dR}} \otimes_K \mathbb{C} \arrow[d, "c_N"] \\ M_{\mathrm{B}} \otimes_L \mathbb{C} \arrow[r, "f_{\mathrm{B}} \otimes_L \mathrm{id}_{\mathbb{C}}"] & N_{\mathrm{B}} \otimes_L \mathbb{C} \end{tikzcd}$$

commutes.

For any $k \in \mathbb{Z}$, we denote $\mathbb{1}(k)$ the object of $\mathscr{C}_{\mathrm{dRB},K_{\mathrm{dR}},L_{\mathrm{B}}}$ defined by

$$\mathbb{1}(k)_{\mathrm{dR}} := K, \quad \mathbb{1}(k)_{\mathrm{B}} := (2\pi i)^k L, \quad \text{and} \quad c_{\mathbb{1}(k)} : \mathbb{C} \to \mathbb{C}, z \mapsto z$$

or equivalently by

$$\mathbb{1}(k)_{\mathrm{dR}} := K, \quad \mathbb{1}(k)_{\mathrm{B}} := L, \quad \text{and} \quad c_{\mathbb{1}(k)} : \mathbb{C} \to \mathbb{C}, z \mapsto (2\pi i)^{-k} z.$$

The category $\mathscr{C}_{\mathrm{dRB},K_{\mathrm{dR}},L_{\mathrm{B}}}$ is then naturally an abelian rigid $\otimes$-category, with unit object $\mathbb{1} := \mathbb{1}(0)$. The two natural forgetful functors

$$\omega_{\mathrm{dR}} \colon \mathscr{C}_{\mathrm{dRB},K_{\mathrm{dR}},L_{\mathrm{B}}} \to \mathrm{Vec}_K \qquad \text{and} \qquad \omega_{\mathrm{B}} \colon \mathscr{C}_{\mathrm{dRB},K_{\mathrm{dR}},L_{\mathrm{B}}} \to \mathrm{Vec}_L$$

define fiber functors, thereby endowing $\mathscr{C}_{\mathrm{dRB},K_{\mathrm{dR}},L_{\mathrm{B}}}$ with the structure of a Tannakian category.

We note that the Tannakian category $\mathscr{C}_{\mathrm{dRB},K_{\mathrm{dR}},L_{\mathrm{B}}}$ is not semi-simple, as can be seen from considering the object

$$M := \left(M_{\mathrm{dR}} = K^2, M_{\mathrm{B}} = L^2, c_M\right), \quad \text{with } c_M = \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix}, \quad \mathrm{degtr}_{\mathbb{Q}} \mathbb{Q}(\alpha, \beta, \gamma) = 3.$$

Definition 4.2 (Base-change of de Rham–Betti objects)**.** Given a L-de Rham–Betti object $M = (M_{\mathrm{dR}}, M_{\mathrm{B}}, c_M)$ over K, that is, an object M in $\mathscr{C}_{\mathrm{dRB},K_{\mathrm{dR}},L_{\mathrm{B}}}$, and given field extensions $K \subseteq K' \subseteq \overline{\mathbb{Q}}$ and $L \subseteq L' \subseteq \overline{\mathbb{Q}}$, we further denote

$$M_{K'} \otimes L' =_{\mathrm{def}} (M_{\mathrm{dR}} \otimes_K K', M_{\mathrm{B}} \otimes_L L', c_M)$$

the object in $\mathscr{C}_{\mathrm{dRB},K'_{\mathrm{dR}},L'_{\mathrm{B}}}$ obtained from base-change of M.

We note that the base-change of a semi-simple object may fail to be semi-simple, as can be seen from considering the de Rham–Betti object

$$M := \left(M_{\mathrm{dR}} = \overline{\mathbb{Q}}^2, M_{\mathrm{B}} = \mathbb{Q}^2, c_M\right), \quad \text{with } c_M = \begin{pmatrix} \alpha & \beta \\ i\alpha & \gamma \end{pmatrix}, \quad \mathrm{degtr}_{\mathbb{Q}}\mathbb{Q}(\alpha,\beta,\gamma) = 3.$$

and its base-change $M \otimes \overline{\mathbb{Q}}$. Conversely, an object M whose base-change $M \otimes \overline{\mathbb{Q}}$ is semi-simple may fail to be semi-simple, as can be seen from considering the de Rham–Betti object

$$M := \left(M_{\mathrm{dR}} = \overline{\mathbb{Q}}^2, M_{\mathrm{B}} = \mathbb{Q}^2, c_M\right), \quad \text{with } c_M = \begin{pmatrix} 1 & i\pi \\ 0 & \pi \end{pmatrix}.$$

In this work we will be interested in the following three cases:

(a) $L = \mathbb{Q}$. In this case, the fiber functor ω_{B} is neutral and we write $\mathscr{C}_{\mathrm{dRB},K}$ for the $\mathbb{Q}$-linear neutral Tannakian category $\mathscr{C}_{\mathrm{dRB},K_{\mathrm{dR}},\mathbb{Q}_{\mathrm{B}}}$ whose objects we call *de Rham–Betti objects* (the field K will usually be clear from the context).

(b) $K = \overline{\mathbb{Q}}$ and $L = \mathbb{Q}$. This is a special instance of (a). We write $\mathscr{C}_{\mathrm{dRB}}$ for $\mathscr{C}_{\mathrm{dRB},\overline{\mathbb{Q}}_{\mathrm{dR}},\mathbb{Q}_{\mathrm{B}}}$.

(c) $K = \overline{\mathbb{Q}}$ and $L = \overline{\mathbb{Q}}$. In this case, both fiber functors ω_{B} and ω_{dR} are neutral and we write $\mathscr{C}_{\overline{\mathbb{Q}}-\mathrm{dRB}}$ for the $\overline{\mathbb{Q}}$-linear neutral Tannakian category $\mathscr{C}_{\mathrm{dRB},\overline{\mathbb{Q}}_{\mathrm{dR}},\overline{\mathbb{Q}}_{\mathrm{B}}}$ whose objects we call *$\overline{\mathbb{Q}}$-de Rham–Betti objects*.

We discuss cases (b) and (c) below in a general context, while case (a) will be discussed in §6.2 in the context of de Rham–Betti objects associated to André motives.

4.2. **The de Rham–Betti group and the torsor of periods.** Let us consider the case (b) of de Rham–Betti objects with $K = \overline{\mathbb{Q}}$ and $L = \mathbb{Q}$.

Definition 4.3 (De Rham–Betti classes [And04, §7.5.1])**.** Let M be a de Rham–Betti object. A *de Rham–Betti class* on M is an element of $\mathrm{Hom}_{\mathscr{C}_{\mathrm{dRB},K}}(\mathbb{1}, M)$. Equivalently, if consists of an element $\alpha_B \in M_{\mathrm{B}}$ such that there exists $\alpha_{\mathrm{dR}} \in M_{\mathrm{dR}}$ with $c_M(\alpha_{\mathrm{dR}} \otimes_K 1_{\mathbb{C}}) = \alpha_{\mathrm{B}} \otimes_{\mathbb{Q}} 1_{\mathbb{C}}$.

Definition 4.4 (The de Rham–Betti group and the torsor of periods)**.** The *de Rham–Betti group* G_{dRB} is the Tannakian fundamental group of the neutral Tannakian category $\mathscr{C}_{\mathrm{dRB}}$, that is,

$$G_{\mathrm{dRB}} =_{\mathrm{def}} \mathrm{Aut}^{\otimes}(\omega_B : \mathscr{C}_{\mathrm{dRB}} \to \mathrm{Vec}_{\mathbb{Q}}).$$

The *torsor of periods* is the Tannakian torsor

$$\Omega^{\mathrm{dRB}} =_{\mathrm{def}} \mathrm{Iso}^{\otimes}(\omega_{\mathrm{dR}}, \omega_{\mathrm{B}} \otimes \overline{\mathbb{Q}})\,;$$

it is a torsor under $\mathrm{Aut}^{\otimes}(\omega_{\mathrm{B}} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})$, which coincides with $G_{\mathrm{dRB},\overline{\mathbb{Q}}} =_{\mathrm{def}} G_{\mathrm{dRB}} \times_{\mathbb{Q}} \overline{\mathbb{Q}}$ by [DM82, Rmk. 3.12].

For a de Rham–Betti object $M \in \mathscr{C}_{\mathrm{dRB}}$, we denote

$$G_{\mathrm{dRB}}(M) =_{\mathrm{def}} \mathrm{Aut}^{\otimes}(\omega_B|_{\langle M\rangle}) \quad \text{and} \quad \Omega_M^{\mathrm{dRB}} =_{\mathrm{def}} \mathrm{Iso}^{\otimes}(\omega_{\mathrm{dR}}|_{\langle M\rangle}, \omega_{\mathrm{B}}|_{\langle M\rangle} \otimes \overline{\mathbb{Q}}).$$

The comparison isomorphism $c_M \colon M_{\mathrm{dR}} \otimes_{\overline{\mathbb{Q}}} \mathbb{C} \xrightarrow{\sim} M_{\mathrm{B}} \otimes_{\mathbb{Q}} \mathbb{C}$ defines a canonical complex-valued point

$$c_M \in \Omega_M^{\mathrm{dRB}}(\mathbb{C}).$$

As an example, we have

$$G_{\mathrm{dRB}}(\mathbb{1}(k)) = \begin{cases} \{1\}, \text{if } k = 0 \\ \mathbb{G}_m, \text{if } k \neq 0. \end{cases}$$

Indeed, $G_{\mathrm{dRB}}(\mathbb{1}(k))$ is a $\mathbb{Q}$-subgroup of $\mathbb{G}_m$. Suppose it is finite, say of order n. Then $G_{\mathrm{dRB}}(\mathbb{1}(k))$ acts trivially on $(\mathbb{1}(k)^{\otimes n})_{\mathrm{B}}$ so that $\mathbb{1}(nk) \simeq \mathbb{1}(k)^{\otimes n} \simeq \mathbb{1}$ in the category of de Rham–Betti objects. This implies that $(2\pi i)^{nk}$ lies in $\overline{\mathbb{Q}}$ and hence that $k = 0$ by the transcendence of π. (Alternately, we will show in Theorem 4.7 that G_{dRB} is connected.) Conversely, it is clear that $G_{\mathrm{dRB}}(\mathbb{1}) = \{1\}$.

From the general formalism of neutral Tannakian categories, for any object $M \in \mathscr{C}_{\mathrm{dRB}}$, there is a natural epimorphism $G_{\mathrm{dRB}} \twoheadrightarrow G_{\mathrm{dRB}}(M)$ and for any object $X \in \langle M \rangle$, the action of G_{dRB} on $\omega_{\mathrm{B}}(X)$ factors through $G_{\mathrm{dRB}}(M)$. In particular, a class in $M_{\mathrm{B}}^{\otimes n} \otimes (M_{\mathrm{B}}^{\vee})^{\otimes m}$ extends to a de Rham–Betti class in $M^{\otimes n} \otimes (M^{\vee})^{\otimes m}$ if and only if it is fixed by G_{dRB}.

4.3. **The $\overline{\mathbb{Q}}$-de Rham–Betti group and the $\overline{\mathbb{Q}}$-torsor of periods.** Likewise in the context of $\overline{\mathbb{Q}}$-de Rham–Betti objects as defined in (c), we have:

Definition 4.5 ($\overline{\mathbb{Q}}$-de Rham–Betti classes)**.** Let M be a $\overline{\mathbb{Q}}$-de Rham–Betti object. A $\overline{\mathbb{Q}}$-*de Rham–Betti class* on M is an element of $\mathrm{Hom}_{\mathscr{C}_{\overline{\mathbb{Q}}-\mathrm{dRB}}}(\mathbb{1}, M)$. Equivalently, if consists of an element $\alpha_B \in M_{\mathrm{B}}$ such that there exists $\alpha_{\mathrm{dR}} \in M_{\mathrm{dR}}$ with $c_M(\alpha_{\mathrm{dR}} \otimes_{\overline{\mathbb{Q}}} 1_{\mathbb{C}}) = \alpha_{\mathrm{B}} \otimes_{\overline{\mathbb{Q}}} 1_{\mathbb{C}}$.

Definition 4.6 (The $\overline{\mathbb{Q}}$-de Rham–Betti group and the $\overline{\mathbb{Q}}$-de Rham–Betti torsor of periods)**.** The $\overline{\mathbb{Q}}$-*de Rham–Betti group* $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}$ is the Tannakian fundamental group of the neutral Tannakian category $\mathscr{C}_{\overline{\mathbb{Q}}-\mathrm{dRB}}$, that is,

$$G_{\overline{\mathbb{Q}}-\mathrm{dRB}} =_{\mathrm{def}} \mathrm{Aut}^{\otimes}(\omega_B : \mathscr{C}_{\overline{\mathbb{Q}}-\mathrm{dRB}} \to \mathrm{Vec}_{\overline{\mathbb{Q}}}).$$

The $\overline{\mathbb{Q}}$-*de Rham–Betti torsor of periods* is the Tannakian torsor

$$\Omega^{\overline{\mathbb{Q}}-\mathrm{dRB}} =_{\mathrm{def}} \mathrm{Iso}^{\otimes}(\omega_{\mathrm{dR}}, \omega_{\mathrm{B}})\,;$$

it is a torsor under $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}$.

As above, we then denote for a $\overline{\mathbb{Q}}$-de Rham–Betti object $M \in \mathscr{C}_{\overline{\mathbb{Q}}-\mathrm{dRB}}$,

$$G_{\overline{\mathbb{Q}}-\mathrm{dRB}}(M) =_{\mathrm{def}} \mathrm{Aut}^{\otimes}(\omega_B|_{\langle M \rangle}) \quad \text{and} \quad \Omega_M^{\overline{\mathbb{Q}}-\mathrm{dRB}} =_{\mathrm{def}} \mathrm{Iso}^{\otimes}(\omega_{\mathrm{dR}}|_{\langle M \rangle}, \omega_{\mathrm{B}}|_{\langle M \rangle}).$$

The comparison isomorphism $c_M \colon M_{\mathrm{dR}} \otimes_{\overline{\mathbb{Q}}} \mathbb{C} \xrightarrow{\sim} M_{\mathrm{B}} \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$ defines a canonical complex-valued point

$$c_M \in \Omega_M^{\overline{\mathbb{Q}}-\mathrm{dRB}}(\mathbb{C}).$$

A similar argument as above establishes

$$G_{\overline{\mathbb{Q}}-\mathrm{dRB}}(\mathbb{1}(k)) = \begin{cases} \{1\}, \text{if } k = 0 \\ \mathbb{G}_m, \text{if } k \neq 0. \end{cases}$$

4.4. **The torsor of periods is connected.** The following observation is purely formal, in the sense that it applies to any de Rham–Betti object (resp. to any $\overline{\mathbb{Q}}$-de Rham–Betti object) and not only to those coming from geometry (that is, that are the realization of André motives; see §6.1 below).

Theorem 4.7 (Connectedness of G_{dRB}, Ω^{dRB}, $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}$ and $\Omega^{\overline{\mathbb{Q}}-\mathrm{dRB}}$)**.**
Both the de Rham–Betti group G_{dRB} and the torsor of periods Ω^{dRB} are connected. In particular, for any de Rham–Betti object M in $\mathscr{C}_{\mathrm{dRB}}$, both $G_{\mathrm{dRB}}(M)$ and Ω^{dRB}_M are connected.

Similarly, both the $\overline{\mathbb{Q}}$-de Rham–Betti group $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}$ and the $\overline{\mathbb{Q}}$-de Rham–Betti torsor of periods $\Omega^{\overline{\mathbb{Q}}-\mathrm{dRB}}$ are connected. In particular, for any $\overline{\mathbb{Q}}$-de Rham–Betti object M in $\mathscr{C}_{\overline{\mathbb{Q}}-\mathrm{dRB}}$, both $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}(M)$ and $\Omega^{\overline{\mathbb{Q}}-\mathrm{dRB}}_M$ are connected.

Proof. It suffices to show that G_{dRB} and $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}$ are connected. We show that G_{dRB} is connected – the case of $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}$ is similar. According to Proposition 2.1, it suffices to show that, for any object $M \in \mathscr{C}_{\mathrm{dRB}}$ with finite de Rham–Betti group, $G_{\mathrm{dRB}}(M)$ is trivial. Since the torsor of periods Ω^{dRB}_M of such an object M is then finite, the comparison isomorphism $c_M : M_{\mathrm{dR}} \otimes_{\overline{\mathbb{Q}}} \mathbb{C} \to M_{\mathrm{B}} \otimes_{\mathbb{Q}} \mathbb{C}$ is defined over $\overline{\mathbb{Q}}$. Choosing a $\mathbb{Q}$-basis (e_i) of M_{B} and letting $(c_M^{-1}(e_i \otimes_{\mathbb{Q}} 1_{\overline{\mathbb{Q}}}))$ be the corresponding $\overline{\mathbb{Q}}$-basis of M_{dR}, we see that $M \simeq \mathbb{1}^{\oplus \dim M_{\mathrm{B}}}$ in $\mathscr{C}_{\mathrm{dRB}}$. This implies $G_{\mathrm{dRB}}(M) = G_{\mathrm{dRB}}(\mathbb{1}) = 1$. □

5. De Rham–Betti objects with $\overline{\mathbb{Q}}$-coefficients

Let $M = (M_{\mathrm{dR}}, M_{\mathrm{B}}, c_M)$ be a de Rham–Betti object in $\mathscr{C}_{\mathrm{dRB}} =_{\mathrm{def}} \mathscr{C}_{\mathrm{dRB},\overline{\mathbb{Q}}_{\mathrm{dR}},\mathbb{Q}_{\mathrm{B}}}$. Its base-change $M \otimes \overline{\mathbb{Q}} =_{\mathrm{def}} (M_{\mathrm{dR}}, M_{\mathrm{B}} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}, c_M)$ defines an object in the $\overline{\mathbb{Q}}$-linear Tannakian category $\mathscr{C}_{\overline{\mathbb{Q}}-\mathrm{dRB}} =_{\mathrm{def}} \mathscr{C}_{\mathrm{dRB},\overline{\mathbb{Q}}_{\mathrm{dR}},\overline{\mathbb{Q}}_{\mathrm{B}}}$ consisting of $\overline{\mathbb{Q}}$-de Rham–Betti objects over $\overline{\mathbb{Q}}$, and we have, e.g. from §2.3, closed embeddings $\Omega^{\overline{\mathbb{Q}}-\mathrm{dRB}}_{M\otimes\overline{\mathbb{Q}}} \subseteq \Omega^{\mathrm{dRB}}_M \subseteq \mathrm{Iso}_{\overline{\mathbb{Q}}}(M_{\mathrm{dR}}, M_{\mathrm{B}} \otimes \overline{\mathbb{Q}})$ of $\overline{\mathbb{Q}}$-torsors. The comparison isomorphism c_M defines a canonical $\mathbb{C}$-point in $\Omega^{\overline{\mathbb{Q}}-\mathrm{dRB}}_{M\otimes\overline{\mathbb{Q}}}$; we denote Z_M its Zariski closure and $\Omega_M \subseteq \Omega^{\overline{\mathbb{Q}}-\mathrm{dRB}}_{M\otimes\overline{\mathbb{Q}}}$ the smallest $\overline{\mathbb{Q}}$-torsor containing c_M.

All in all, we have a chain of natural closed inclusions

$$Z_M \subseteq \Omega_M \subseteq \Omega^{\overline{\mathbb{Q}}-\mathrm{dRB}}_{M\otimes\overline{\mathbb{Q}}} \subseteq \Omega^{\mathrm{dRB}}_M. \tag{3}$$

In this section, we show that the middle inclusion in (3) is an equality, while the left-most and right-most inclusions cannot be expected to be equalities in general.

In fact, for a $\overline{\mathbb{Q}}$-de Rham–Betti object $N = (N_{\mathrm{dR}}, N_{\mathrm{B}}, c_N)$, we also have a chain of natural closed inclusions

$$Z_N \subseteq \Omega_N \subseteq \Omega^{\overline{\mathbb{Q}}-\mathrm{dRB}}_N,$$

where $Z_N \subseteq \Omega^{\overline{\mathbb{Q}}-\mathrm{dRB}}_N$ is the Zariski closure of c_N and $\Omega_N \subseteq \Omega^{\overline{\mathbb{Q}}-\mathrm{dRB}}_N$ is the smallest $\overline{\mathbb{Q}}$-subtorsor containing c_N. We note that, for a de Rham–Betti object M, the complex-valued point c_M agrees with the complex-valued point $c_{M\otimes\overline{\mathbb{Q}}}$ so that we have equalities $Z_M = Z_{M\otimes\overline{\mathbb{Q}}}$ of closed $\overline{\mathbb{Q}}$-subschemes and $\Omega_M = \Omega_{M\otimes\overline{\mathbb{Q}}}$ of $\overline{\mathbb{Q}}$-subtorsors of $\mathrm{Iso}_{\overline{\mathbb{Q}}}(M_{\mathrm{dR}}, M_{\mathrm{B}} \otimes \overline{\mathbb{Q}})$.

5.1. On the inclusion $\Omega_M \subseteq \Omega^{\overline{\mathbb{Q}}-\mathrm{dRB}}_{M\otimes\overline{\mathbb{Q}}}$.

The middle inclusion of (3) is an equality :

Proposition 5.1 (The inclusion $\Omega_M \subseteq \Omega^{\overline{\mathbb{Q}}-\mathrm{dRB}}_{M\otimes\overline{\mathbb{Q}}}$ is an equality)**.** *For any $\overline{\mathbb{Q}}$-de Rham–Betti object $N \in \mathscr{C}_{\overline{\mathbb{Q}}-\mathrm{dRB}}$, we have*

$$\Omega_N = \Omega^{\overline{\mathbb{Q}}-\mathrm{dRB}}_N.$$

In particular, for any de Rham–Betti object $M \in \mathscr{C}_{\mathrm{dRB}}$, we have $\Omega_M = \Omega^{\overline{\mathbb{Q}}-\mathrm{dRB}}_{M\otimes\overline{\mathbb{Q}}}$.

Proof. From the inclusion $\Omega_N \subseteq \Omega_N^{\overline{\mathbb{Q}}-\mathrm{dRB}}$, it follows that Ω_N is a torsor under a $\overline{\mathbb{Q}}$-subgroup $G \subseteq G_{\overline{\mathbb{Q}}-\mathrm{dRB}}(N)$. The group $G \subseteq \mathrm{GL}(N_{\mathrm{B}})$ is thus the stabilizer of a $\overline{\mathbb{Q}}$-line L_{B} in some tensor space $\bigoplus_{\text{finite}} N_{\mathrm{B}}^{\otimes n_i} \otimes N_{\mathrm{B}}^{\vee\otimes m_i}$. We will prove that $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}(N)$ stabilizes the line L_{B}. This will show that $G = G_{\overline{\mathbb{Q}}-\mathrm{dRB}}(N)$ and finish the proof that the two torsors are equal. If we choose a point $x \in \Omega_N(\overline{\mathbb{Q}})$, then $c_N \circ x^{-1} \in G(\mathbb{C})$. As $G(\mathbb{C})$ stabilizes $L_{\mathrm{B}} \otimes \mathbb{C}$ inside $\bigoplus_{\text{finite}} N_{\mathrm{B}}^{\otimes n_i} \otimes N_{\mathrm{B}}^{\vee\otimes m_i}$, this means that

$$c_N \circ x^{-1}(L_{\mathrm{B}} \otimes \mathbb{C}) = L_{\mathrm{B}} \otimes \mathbb{C}.$$

Since $\Omega_N \subseteq \mathrm{Iso}_{\overline{\mathbb{Q}}}(N_{\mathrm{dR}}, N_{\mathrm{B}})$, the line $L_{\mathrm{dR}} := x^{-1}(L_B) \subseteq \bigoplus_{\text{finite}} N_{\mathrm{dR}}^{\otimes n_i} \otimes N_{\mathrm{dR}}^{\vee\otimes m_i}$ is defined over $\overline{\mathbb{Q}}$, and $c_N(L_{\mathrm{dR}} \otimes \mathbb{C}) = L_{\mathrm{B}} \otimes \mathbb{C}$. Hence $(L_{\mathrm{dR}}, L_{\mathrm{B}}, c_N|_{L_{\mathrm{dR}}\otimes\mathbb{C}})$ is a subobject of $\bigoplus_{\text{finite}} N^{\otimes n_i} \otimes N^{\vee\otimes m_i}$ in $\mathscr{C}_{\overline{\mathbb{Q}}-\mathrm{dRB}}$. We conclude that $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}(N)$ stabilizes L_{B}. □

5.2. **On the inclusion $\Omega_{M\otimes\overline{\mathbb{Q}}}^{\overline{\mathbb{Q}}-\mathrm{dRB}} \subseteq \Omega_M^{\mathrm{dRB}}$.** The inclusion $\Omega_{M\otimes\overline{\mathbb{Q}}}^{\overline{\mathbb{Q}}-\mathrm{dRB}} \subseteq \Omega_M^{\mathrm{dRB}}$ is an equality if and only if the functor $\langle M\rangle \otimes \overline{\mathbb{Q}} \to \langle M \otimes \overline{\mathbb{Q}}\rangle$ is an equivalence of Tannakian categories. This is related to the question of whether $\overline{\mathbb{Q}}$-de Rham–Betti classes are $\overline{\mathbb{Q}}$-linear combinations of de Rham–Betti classes, but also to the question of whether the $\overline{\mathbb{Q}}$-torsor $\Omega_{M\otimes\overline{\mathbb{Q}}}^{\overline{\mathbb{Q}}-\mathrm{dRB}}$ is a torsor under a $\overline{\mathbb{Q}}$-group defined over $\mathbb{Q}$:

Proposition 5.2. *Let M be a semi-simple de Rham–Betti object. The following statements are equivalent.*

(i) The comparison c_M generates Ω_M^{dRB} as a $\overline{\mathbb{Q}}$-torsor.
(ii) The inclusion $\Omega_{M\otimes\overline{\mathbb{Q}}}^{\overline{\mathbb{Q}}-\mathrm{dRB}} \subseteq \Omega_M^{\mathrm{dRB}}$ is an equality.
(iii) $M \otimes \overline{\mathbb{Q}}$ is a semi-simple $\overline{\mathbb{Q}}$-de Rham–Betti object and, for any $N \in \langle M\rangle$, any $\overline{\mathbb{Q}}$-de Rham–Betti class in $N \otimes \overline{\mathbb{Q}}$ is a $\overline{\mathbb{Q}}$-linear combinations of de Rham–Betti classes in N.
(iv) $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}(M \otimes \overline{\mathbb{Q}})$ is reductive and defines a $\mathbb{Q}$-subgroup of $G_{\mathrm{dRB}}(M)$.

Proof. The equivalence $(i) \iff (ii)$ holds by Proposition 5.1 and by definition of Ω_M, whether or not M is assumed to be semi-simple. If M is semi-simple, the de Rham–Betti group $G_{\mathrm{dRB}}(M)$ is reductive by Proposition 1.1. By definition, the torsor $\Omega_{M\otimes\overline{\mathbb{Q}}}^{\overline{\mathbb{Q}}-\mathrm{dRB}}$ is included in the intersection of the torsors Ω_α whose $\overline{\mathbb{Q}}$-points are given by

$$\Omega_\alpha(\overline{\mathbb{Q}}) = \{f \in \mathrm{Iso}_{\overline{\mathbb{Q}}}(M_{\mathrm{dR}}, M_{\mathrm{B}} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}) \mid f(\alpha_{\mathrm{dR}}) = \alpha_{\mathrm{B}}\},$$

for α running through the $\overline{\mathbb{Q}}$-de Rham–Betti classes in the $\overline{\mathbb{Q}}$-base change of the various tensor spaces $M^{\otimes n} \otimes (M^\vee)^{\otimes m}$, and this inclusion is an equality if $M \otimes \overline{\mathbb{Q}}$ is semi-simple as a $\overline{\mathbb{Q}}$-de Rham–Betti object. On the other hand, by the semi-simplicity of M, the torsor Ω_M^{dRB} is the intersection of the torsors Ω_α for α running through the de Rham–Betti classes in the various tensor spaces $M^{\otimes n} \otimes (M^\vee)^{\otimes m}$. This establishes the equivalence of (ii) and (iii). The implication $(ii) \Rightarrow (iv)$ is clear ; indeed (ii) is equivalent to the inclusion $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}(M \otimes \overline{\mathbb{Q}}) \subseteq G_{\mathrm{dRB}}(M)_{\overline{\mathbb{Q}}}$ being an equality. Finally, assume both groups $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}(M \otimes \overline{\mathbb{Q}})$ and $G_{\mathrm{dRB}}(M)$ are reductive and assume that $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}(M \otimes \overline{\mathbb{Q}}) = G_{\overline{\mathbb{Q}}}$ where $G \subseteq G_{\mathrm{dRB}}(M)$ is a $\mathbb{Q}$-subgroup. Then, for any element α in a tensor space of M such that α is G-invariant, the class $\alpha \otimes \overline{\mathbb{Q}}$, being $G_{\overline{\mathbb{Q}}}$-invariant, defines a $\overline{\mathbb{Q}}$-de Rham–Betti class. In particular α is a de Rham–Betti class and hence is $G_{\mathrm{dRB}}(M)$-invariant. Therefore the reductive groups G and $G_{\mathrm{dRB}}(M)$ share the same invariants and thus coincide. This establishes the implication $(iv) \Rightarrow (ii)$. □

Example 5.3 (The inclusion $\Omega^{\overline{\mathbb{Q}}-\mathrm{dRB}}_{M\otimes\overline{\mathbb{Q}}} \subseteq \Omega^{\mathrm{dRB}}_M$ can be strict)**.** Consider the de Rham–Betti object

$$M := \left(M_{\mathrm{dR}} = \overline{\mathbb{Q}}^2, M_{\mathrm{B}} = \mathbb{Q}^2, c_M\right), \quad \text{with } c_M = \begin{pmatrix} \pi & a \\ b\pi & c \end{pmatrix}, \quad a,b,c \in \overline{\mathbb{Q}},\ c - ab \neq 0.$$

For a general choice of a, b and c in $\overline{\mathbb{Q}}$, we have $M \otimes \overline{\mathbb{Q}} \cong \mathbb{1}(-1) \oplus \mathbb{1}$, while M is simple and in particular does not have any nonzero de Rham–Betti class. In view of Proposition 5.2, the inclusion $\Omega^{\overline{\mathbb{Q}}-\mathrm{dRB}}_{M\otimes\overline{\mathbb{Q}}} \subseteq \Omega^{\mathrm{dRB}}_M$ is then strict. (In relation to [BC16, Rmk. 2.6], one also notes that the Zariski closure Z_M of c_M is a torsor under $G_{\mathrm{dRB}}(M_{\overline{\mathbb{Q}}}) = \mathbb{G}_{m,\overline{\mathbb{Q}}}$ but that the latter does not descend to a subgroup of GL_2 over $\mathbb{Q}$.)

5.3. **On the inclusion $Z_M \subseteq \Omega_M$.** For an object $N \in \mathscr{C}_{\overline{\mathbb{Q}}-\mathrm{dRB}}$, the comparison isomorphism $c_N : N_{\mathrm{dR}} \otimes_{\overline{\mathbb{Q}}} \mathbb{C} \xrightarrow{\simeq} N_{\mathrm{B}} \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$ defines a $\overline{\mathbb{Q}}$-bilinear map, called the *period pairing*,

$$\int : N_{\mathrm{B}}^\vee \otimes_{\overline{\mathbb{Q}}} N_{\mathrm{dR}} \to \mathbb{C},\ \ \gamma \otimes \omega \mapsto \int_\gamma \omega =_{\mathrm{def}} \gamma_{\mathbb{C}}(c_N(\omega_{\mathbb{C}})).$$

Fixing $\gamma \in N_{\mathrm{B}}^\vee$ and $\omega \in N_{\mathrm{dR}}$ we thereby get $\overline{\mathbb{Q}}$-linear maps

$$\int_\gamma : N_{\mathrm{dR}} \to \mathbb{C},\ \ \omega \mapsto \gamma_{\mathbb{C}}(c_N(\omega_{\mathbb{C}})) \quad \text{and} \qquad \int \omega \colon N_{\mathrm{B}}^\vee \to \mathbb{C},\ \ \gamma \mapsto \gamma_{\mathbb{C}}(c_N(\omega_{\mathbb{C}})).$$

Definition 5.4. For $\gamma \in N_{\mathrm{B}}^\vee$ we define the *annihilator* $\mathrm{Ann}(\gamma) =_{\mathrm{def}} \ker \int_\gamma \subseteq N_{\mathrm{dR}}$. Similarly, for $\omega \in N_{\mathrm{dR}}$ we define $\mathrm{Ann}(\omega) =_{\mathrm{def}} \ker \int \omega \subseteq N_{\mathrm{B}}^\vee$.

The following proposition gives criteria for the inclusion $Z_N \subseteq \Omega_N$ to be an equality. (Criteria similar to (*iii*) and (*iv*) appear in [Hör21, HW22]). We leave it to the reader to state and prove a similar statement for the inclusion $Z_M \subseteq \Omega^{\mathrm{dRB}}_M$ to be an equality for a de Rham–Betti object M.

Proposition 5.5. *Let $M \in \mathscr{C}_{\overline{\mathbb{Q}}-\mathrm{dRB}}$. The following statements are equivalent:*

(i) $Z_M \subseteq \mathrm{Iso}_{\overline{\mathbb{Q}}}(M_{\mathrm{dR}}, M_{\mathrm{B}})$ *is a torsor;*
(ii) $Z_M = \Omega_M$*;*
(iii) For every $N \in \langle M\rangle \subset \mathscr{C}_{\overline{\mathbb{Q}}-\mathrm{dRB}}$ and every $\omega \in N_{\mathrm{dR}}$, there exists a short exact sequence

$$0 \to N' \to N \to N'' \to 0$$

in $\mathscr{C}_{\overline{\mathbb{Q}}-\mathrm{dRB}}$ such that $\omega \in N'_{\mathrm{dR}}$ and $\mathrm{Ann}(\omega) = (N''_{\mathrm{B}})^\vee$;
(iv) For every $N \in \langle M\rangle \subset \mathscr{C}_{\overline{\mathbb{Q}}-\mathrm{dRB}}$ and every $\gamma \in N_{\mathrm{B}}^\vee$, there exists a short exact sequence

$$0 \to N' \to N \to N'' \to 0$$

in $\mathscr{C}_{\overline{\mathbb{Q}}-\mathrm{dRB}}$ such that $\gamma \in (N''_{\mathrm{B}})^\vee$ and $\mathrm{Ann}(\gamma) = N'_{\mathrm{dR}}$.

Proof. The equivalence $(i) \iff (ii)$ follows from the definition of Ω_M. We first introduce another condition (ii') which is equivalent to (ii). The direct sum of the period pairings associated to all objects $N \in \langle M\rangle$ provides a pairing

$$p : \bigoplus_{N\in\langle M\rangle} N_{\mathrm{B}}^\vee \otimes N_{\mathrm{dR}} \to \mathbb{C}.$$

We define the *ring of periods* $\mathcal{P}(M) \subseteq \mathbb{C}$ of M to be the image of p. Note that $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}(M)$ naturally acts on all $N_{\mathrm{B}}^\vee$ and hence on $\bigoplus_{N\in\langle M\rangle} N_{\mathrm{B}}^\vee \otimes N_{\mathrm{dR}}$.

Claim. *Statement (ii) is equivalent to the assertion*

(ii') $\ker(p) \subseteq \bigoplus_{N\in\langle M\rangle} N_{\mathrm{B}}^\vee \otimes N_{\mathrm{dR}}$ *is stable under the action of* $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}(M)$.

Proof of Claim. By Proposition 5.1, the torsor $\Omega_M = \Omega_M^{\overline{\mathbb{Q}}-\mathrm{dRB}}$ is the Tannakian torsor for $\langle M\rangle$. It then follows from the construction (cf. [DM82, proof of Thm. 3.2]) that $\Omega_M = \operatorname{Spec} R$ is the spectrum of an algebra R which fits into the diagram

$$\begin{tikzcd} \bigoplus_{N\in\langle M\rangle} N_{\mathrm{B}}^\vee \otimes N_{\mathrm{dR}} \arrow[r, two heads] \arrow[dr, "p"'] & R \arrow[d, two heads] \\ & \mathcal{P}(M) \subset \mathbb{C}. \end{tikzcd}$$

Note that $Z_M = \operatorname{Spec}\mathcal{P}(M)$. Since Ω_M is a torsor under $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}(M)$, the equality $Z_M = \Omega_M$ holds exactly if the action of $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}(M)$ on $\bigoplus_{N\in\langle M\rangle} N_{\mathrm{B}}^\vee \otimes N_{\mathrm{dR}}$ passes down to an action of $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}(M)$ on the quotient $\mathcal{P}(M)$. This is equivalent to saying that the kernel $\ker(p) \subseteq \bigoplus_{N\in\langle M\rangle} N_{\mathrm{B}}^\vee \otimes N_{\mathrm{dR}}$ is stable under $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}(M)$. ☐

We prove the equivalence $(ii') \iff (iii)$. Suppose $(ii)'$ holds true and let $\omega \in N_{\mathrm{dR}}$. Note that $\gamma \in \operatorname{Ann}(\omega)$ if and only if $\gamma\otimes\omega \in \ker(p)$. It follows that $\operatorname{Ann}(\omega) \subseteq N_B^\vee$ is stable under the action of $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}(M)$ and is therefore the realization of a subobject in $\mathscr{C}_{\overline{\mathbb{Q}}-\mathrm{dRB}}$. The dual object gives the desired N''. Conversely, suppose that (iii) is true. An element of $\bigoplus_{N\in\langle M\rangle} N_{\mathrm{B}}^\vee \otimes N_{\mathrm{dR}}$ is given by a collection $(\sum_{i=1}^{m_N} \gamma_{i,N}\otimes\omega_{i,N})_N$, where only finitely many $N_1, \ldots, N_n$ contribute. Consider the object

$$\tilde{N} := \bigoplus_{k=1}^{n} N_k^{\oplus m_{N_k}} \in \mathscr{C}_{\overline{\mathbb{Q}}-\mathrm{dRB}}.$$

Let $\omega := (\omega_{1,N_k}, \ldots, \omega_{m_{N_k},N_k})_k \in \tilde{N}_{\mathrm{dR}}$. Then $(\sum_{i=1}^{m_N} \gamma_{i,N}\otimes\omega_{i,N})_N$ lies in $\ker(p)$ if and only if $\gamma := (\gamma_{1,N_k}, \ldots, \gamma_{m_{N_k},N_k})_k \in \tilde{N}_{\mathrm{B}}^\vee$ lies in $\operatorname{Ann}(\omega)$. Since $\operatorname{Ann}(\omega)$ is stable under $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}(M)$ by assumption, we see that $\ker(p)$ is stable under $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}(M)$.

The situation is symmetric in the fiber functors ω_{dR} and ω_{B}, and therefore a similar proof shows $(ii') \iff (iv)$. ☐

Example 5.6 (The inclusion $Z_M \subseteq \Omega_M$ can be strict)**.** The object

$$M := \left(M_{\mathrm{dR}} = \overline{\mathbb{Q}}^2, M_{\mathrm{B}} = \mathbb{Q}^2, c_M\right), \quad \text{with } c_M = \begin{pmatrix} \alpha & \beta \\ a & \gamma \end{pmatrix}, \quad \operatorname{degtr}_{\mathbb{Q}}\mathbb{Q}(\alpha,\beta,\gamma) = 3, \quad a \in \overline{\mathbb{Q}} \setminus \{0\}$$

defines a simple de Rham–Betti object such that Z_M is not a torsor. Indeed, $\dim Z_M = 3$ and Ω_M is a torsor under a connected and reductive subgroup of $\mathrm{GL}_{2,\overline{\mathbb{Q}}}$ of dimension ≥ 3 and hence must be a torsor under $\mathrm{GL}_{2,\overline{\mathbb{Q}}}$.

6. The Grothendieck period conjecture

In this section, we fix an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ inside $\mathbb{C}$ and we let K be a subfield of $\overline{\mathbb{Q}}$.

6.1. **The de Rham–Betti realization of André motives.** To a smooth projective variety X defined over K, one associates its *de Rham–Betti cohomology groups*

$$\mathrm{H}^n_{\mathrm{dRB}}(X, \mathbb{Q}(k)) =_{\mathrm{def}} \left(\mathrm{H}^n_{\mathrm{dR}}(X/K), \mathrm{H}^n_{\mathrm{B}}(X^{\mathrm{an}}_{\mathbb{C}}, \mathbb{Q}(k)), c_X\right),$$

where $c_X : \mathrm{H}^n_{\mathrm{dR}}(X/K)\otimes_K \mathbb{C} \xrightarrow{\simeq} \mathrm{H}^n_{\mathrm{B}}(X^{\mathrm{an}}_{\mathbb{C}}, \mathbb{Q})\otimes_{\mathbb{Q}} \mathbb{C}$ is Grothendieck's period comparison isomorphism (1). These are objects in $\mathscr{C}_{\mathrm{dRB},K} =_{\mathrm{def}} \mathscr{C}_{\mathrm{dRB},K_{\mathrm{dR}},\mathbb{Q}_{\mathrm{B}}}$.

Recall from §3 that the Betti and de Rham realizations of motivated cycle classes are compatible with the comparison isomorphisms and therefore that motivated cycle classes are de Rham–Betti. As such, there is a well-defined faithful realization functor

$$\rho_{\mathrm{dRB}} : \mathsf{M}_K^{\mathrm{And}} \to \mathscr{C}_{\mathrm{dRB},K}, \quad M = (X,p,n) \mapsto (p_*\mathrm{H}^*_{\mathrm{dR}}(X/K), p_*\mathrm{H}^*_{\mathrm{B}}(X^{\mathrm{an}}_{\mathbb{C}}, \mathbb{Q}(n)), p_*\circ c_X\circ p_*),$$

and we may speak of de Rham–Betti classes on a André motive M. Note that the composition of the fiber functors $\omega_{\mathrm{dR}}\colon \mathscr{C}_{\mathrm{dRB},K} \to \mathrm{Vec}_K$ and $\omega_{\mathrm{B}}\colon \mathscr{C}_{\mathrm{dRB},K} \to \mathrm{Vec}_{\mathbb{Q}}$ with the de Rham–Betti realization functor $\rho_{\mathrm{dRB}} : \mathsf{M}_K^{\mathrm{And}} \to \mathscr{C}_{\mathrm{dRB},K}$ from the category of André motives over K provide the fiber functors abusively also denoted ω_B and ω_{dR} defined in §3.

Conversely, de Rham–Betti classes are conjectured to be algebraic [And04, Conj. 7.5.1.1]. A weaker form of this expectation is the following

Conjecture 6.1. *The realization functor $\rho_{\mathrm{dRB}} : \mathsf{M}_K^{\mathrm{And}} \to \mathscr{C}_{\mathrm{dRB},K}$ is full. In other words, de Rham–Betti classes are motivated.*

We note in particular that the conjecture implies that de Rham–Betti classes should be Hodge classes, and even absolute Hodge classes (in the sense of Deligne [Del82]). The following easy lemma reduces Conjecture 6.1 to the case $K = \overline{\mathbb{Q}}$:

Lemma 6.2. *Let M be a André motive over $K \subseteq \overline{\mathbb{Q}}$ and let $M_{\overline{\mathbb{Q}}}$ be its base-change to $\overline{\mathbb{Q}}$. If every de Rham–Betti class on $M_{\overline{\mathbb{Q}}}$ is motivated, then every de Rham–Betti class on M is motivated.*

Proof. Recall from [And96b, Scolie p.17] that the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$ acts naturally on the space of motivated cycles on $M_{\overline{\mathbb{Q}}}$ and factors through a finite quotient; moreover, the space of motivated cycles on M is exactly the space of $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$-invariant motivated cycles on $M_{\overline{\mathbb{Q}}}$.

By assumption, we have an isomorphism

$$\mathrm{Hom}_{\mathrm{And}}(\mathbb{1}, M_{\overline{\mathbb{Q}}}) \xrightarrow{\simeq} \mathrm{Hom}_{\mathrm{dRB}}(\mathbb{1}, \mathrm{H}^*_{\mathrm{dRB}}(M_{\overline{\mathbb{Q}}})). \tag{4}$$

This isomorphism gives rise to a Galois action on the right-hand side which is induced from that on the left. Explicitly, the Galois action on the de Rham component of $\mathrm{H}_{\mathrm{dRB}}(M_{\overline{\mathbb{Q}}})$ is the usual Galois action on the algebraic de Rham cohomology. Hence the Galois invariants on the right-hand side are exactly the de Rham–Betti classes in $\mathrm{H}^*_{\mathrm{dRB}}(M_{\overline{\mathbb{Q}}})$ defined over K. The result then follows by taking Galois invariants on both sides of the isomorphism (4). ∎

6.2. **The Grothendieck period conjecture : Tannakian formulation.** We give an overview of [And04, §7.5] and [BC16, §2.2.2] regarding the Grothendieck period conjecture. First, let us define the torsor of periods for case (a) of §4.1.

Definition 6.3 (the de Rham–Betti group and the torsor of periods)**.** The *de Rham–Betti group* $G_{\mathrm{dRB}}(M)$ of a de Rham–Betti object $M \in \mathscr{C}_{\mathrm{dRB},K} = \mathscr{C}_{\mathrm{dRB},K_{\mathrm{dR}},\mathbb{Q}_{\mathrm{B}}}$ is the Tannakian fundamental group

$$G_{\mathrm{dRB}}(M) =_{\mathrm{def}} \mathrm{Aut}^{\otimes}(\omega_B|_{\langle M\rangle}).$$

The *torsor of periods* Ω_M^{dRB} of a de Rham–Betti object $M \in \mathscr{C}_{\mathrm{dRB},K} = \mathscr{C}_{\mathrm{dRB},K_{\mathrm{dR}},\mathbb{Q}_{\mathrm{B}}}$ is

$$\Omega_M^{\mathrm{dRB}} =_{\mathrm{def}} \mathrm{Iso}^{\otimes}(\omega_{\mathrm{dR}}|_{\langle M\rangle}, \omega_{\mathrm{B}}|_{\langle M\rangle} \otimes_{\mathbb{Q}} K)\,;$$

it is a K-torsor under $\mathrm{Aut}^{\otimes}(\omega_{\mathrm{B}}|_{\langle M\rangle}\otimes_{\mathbb{Q}}K)$, which coincides with $G_{\mathrm{dRB}}(M)_K$ by [DM82, Rmk. 3.12].

By definition, the torsor of periods Ω_M^{dRB} is included in the intersection of the Ω_α, where Ω_α is the torsor whose $\overline{\mathbb{Q}}$-points are given by

$$\Omega_\alpha(\overline{\mathbb{Q}}) = \{f \in \mathrm{Iso}_{\overline{\mathbb{Q}}}(M_{\mathrm{dR}} \otimes_K \overline{\mathbb{Q}}, M_{\mathrm{B}} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}) \mid f(\alpha_{\mathrm{dR}} \otimes_K 1_{\overline{\mathbb{Q}}}) = \alpha_{\mathrm{B}} \otimes_{\mathbb{Q}} 1_{\overline{\mathbb{Q}}}\},$$

for α running through the de Rham–Betti classes in the various tensor spaces $M^{\otimes n} \otimes (M^\vee)^{\otimes m}$. In case the Tannakian subcategory $\langle M\rangle$ is semi-simple (or, equivalently, if $G_{\mathrm{dRB}}(M)$ is reductive), then the above inclusion is an equality and one recovers in this case the definition of the torsor of periods Ω_M^{dRB} in terms of invariants given in [BC16, Def. 2.4].

Let now M be a André motive defined over K. By definition, its *de Rham–Betti group* $G_{\mathrm{dRB}}(M)$ is the de Rham–Betti group of its de Rham–Betti realization $\rho_{\mathrm{dRB}}(M)$, and its *torsor of periods*

Ω_M^{dRB} is the torsor of periods of its de Rham–Betti realization $\rho_{\mathrm{dRB}}(M)$. On the other hand, we have

Definition 6.4 (Torsor of motivated periods)**.** The *torsor of motivated periods* Ω_M^{And} of a André motive $M \in \mathsf{M}_K^{\mathrm{And}}$ is

$$\Omega_M^{\mathrm{And}} =_{\mathrm{def}} \mathrm{Iso}^{\otimes}(\omega_{\mathrm{dR}}|_{\langle M\rangle}, \omega_{\mathrm{B}}|_{\langle M\rangle} \otimes_{\mathbb{Q}} K)\,;$$

it is a torsor under $G_{\mathrm{And}}(M)_K$.

Since the neutral Tannakian category of André motives is semi-simple, the torsor of motivated periods has the following description in terms of invariants: it is the intersection of the Ω_α as above, where α runs through the motivated classes on tensor spaces $M^{\otimes n} \otimes (M^\vee)^{\otimes m}$. This description coincides with [BC16, Def. 2.9(3)].

Restricting the intersection to those Ω_α with α algebraic classes yields the notion of torsor of motivic periods. A homological motive M over K is an object of the form $M = (X, p, n)$ with X smooth projective over K of dimension d_X, p an idempotent in $\mathrm{im}(\mathrm{CH}^{d_X}(X \times X) \to \mathrm{H}_B((X \times X)^{\mathrm{an}}_{\mathbb{C}}, \mathbb{Q}))$ and n an integer.

Definition 6.5 (Torsor of motivic periods)**.** The *torsor of motivic periods* Ω_M^{mot} of a homological motive M is defined as the intersection of the Ω_α as above, where α runs through the algebraic classes on tensor spaces $M^{\otimes n} \otimes (M^\vee)^{\otimes m}$.

We note that this torsor has a Tannakian description in case X satisfies Grothendieck's standard conjectures; see [And04, §7.5.2].

From the general theory of neutral Tannakian categories exposed in §2.3, or more simply, from the descriptions above, we have closed immersions $\Omega_M^{\mathrm{dRB}} \subseteq \Omega_M^{\mathrm{And}}$ for M a André motive and $\Omega_M^{\mathrm{And}} \subseteq \Omega_M^{\mathrm{mot}}$ for M a homological motive. It is also clear that, for a de Rham–Betti object M, the period comparison isomorphism $c_M : M_{\mathrm{dR}} \otimes_K \mathbb{C} \to M_{\mathrm{B}} \otimes_{\mathbb{Q}} \mathbb{C}$ defines a complex point of Ω_M^{dRB}, so that the Zariski closure Z_M of c_M inside the K-scheme $\mathrm{Iso}(M_{\mathrm{dR}}, M_{\mathrm{B}} \otimes K)$ is contained in Ω_M^{dRB}. We denote Ω_M the smallest K-subtorsor of $\mathrm{Iso}_K(M_{\mathrm{dR}}, M_{\mathrm{B}} \otimes K)$ containing c_M; we say Ω_M is the K-subtorsor generated by c_M.

Conjecture 6.6 (Grothendieck Period Conjecture [Gro66])**.**

(i) Let M be a André motive over K. We say M satisfies the motivated version of Grothendieck's period conjecture *if the inclusions*

$$Z_M \subseteq \Omega_M \subseteq \Omega_M^{\mathrm{dRB}} \subseteq \Omega_M^{\mathrm{And}}$$

are equalities.

(ii) Let M be a homological motive over $\overline{\mathbb{Q}}$. We say M satisfies Grothendieck's period conjecture *if the inclusions*

$$Z_M \subseteq \Omega_M \subseteq \Omega_M^{\mathrm{dRB}} \subseteq \Omega_M^{\mathrm{And}} \subseteq \Omega_M^{\mathrm{mot}}$$

are equalities.

Note that Conjecture 6.6 predicts that Ω_M^{And} is connected for a André motive M over K and that $G_{\mathrm{And}}(M)$ is connected in case $K = \overline{\mathbb{Q}}$; the latter is also predicted by the Hodge conjecture (or more simply from the conjecture that Hodge classes are motivated) since then $G_{\mathrm{And}}(M)$ coincides with the Mumford–Tate group $\mathrm{MT}(M)$, but is unknown in general.

Example 6.7 (Torsor of periods of Artin motives)**.** Let $K \subseteq F \subseteq \overline{\mathbb{Q}}$ be a finite extension of K and consider the *Artin motive* $M := \mathfrak{h}(\mathrm{Spec}\, F)$ over K. In that case all three K-torsors introduced above agree and we have $\Omega_M = \Omega_M^{\mathrm{And}} = \Omega_M^{\mathrm{mot}} = \mathrm{Spec}\, F^g$ as K-torsors under the constant group scheme $\mathrm{Gal}(F^g/K)$, where F^g denotes the Galois closure of F inside $\overline{\mathbb{Q}}$. Moreover $c_M \in \Omega_M(\mathbb{C}) = \mathrm{Hom}_K(F, \mathbb{C})$ is the canonical element. As such, M satisfies the Grothendieck Period Conjecture.

6.3. **The Grothendieck Period Conjecture : Transcendence of periods.** For the convenience of the reader, let us relate the Grothendieck Period Conjecture as is formulated in Conjecture 6.6 to a perhaps more common formulation in terms of degree of transcendence of the field of periods.

Let M be a André motive over K and fix a K-basis of its de Rham cohomology $\mathrm{H}_{\mathrm{dR}}(M)$ and a $\mathbb{Q}$-basis of its Betti cohomology $\mathrm{H}_B(M)$. The matrix of periods of M is the invertible complex matrix representing the comparison isomorphism $c_M\colon \mathrm{H}_{\mathrm{dR}}(M)\otimes_K \mathbb{C} \longrightarrow \mathrm{H}_{\mathrm{B}}(M)\otimes_{\mathbb{Q}}\mathbb{C}$ in the above bases. The field, abusively denoted $K(c_M)$, generated by the entries of the period matrix does not depend on the choice of the above bases and is the residue field of c_M seen as a point of the torsor of periods Ω_M^{dRB} ; in particular $\mathrm{degtr}_{\mathbb{Q}} K(c_M) \leq \dim \Omega_M^{\mathrm{dRB}}$. In case M is the motive of a smooth projective variety X over $\overline{\mathbb{Q}}$, by Poincaré duality, the residue field $K(c_M)$ is the field generated over K by the *periods* of X :

$$\int_\gamma \omega, \quad \gamma \in \mathrm{H}_n^{\mathrm{B}}(X_{\mathbb{C}}^{\mathrm{an}},\mathbb{Q}),\ \omega\in \mathrm{H}^n_{\mathrm{dR}}(X/K),\ n\in\mathbb{Z}_{\geq 0}.$$

It is then clear, see for instance [And04, Prop. 7.5.2.2], that Conjecture 6.6(i) is equivalent to the following.

Conjecture 6.8 (Grothendieck Period Conjecture)**.** *If M is a André motive over K, then Ω_M^{And} is connected and*

$$\mathrm{degtr}_{\mathbb{Q}} K(c_M) = \dim G_{\mathrm{And}}(M).$$

On the other hand, Hodge classes on M are expected to be motivated, so that by Proposition 3.3 the inclusion $\mathrm{MT}(M)\subseteq G_{\mathrm{And}}(M)$ is expected to identify $\mathrm{MT}(M)$ with the connected component of $G_{\mathrm{And}}(M)$. Hence, conjecturally, the degree of transcendence of the field of periods of M only depends on the Hodge structure on $\mathrm{H}_{\mathrm{B}}(M)$. Precisely :

Conjecture 6.9 (Grothendieck)**.** *If M is a André motive over K, then*

$$\mathrm{degtr}_{\mathbb{Q}} K(c_M) = \dim \mathrm{MT}(M).$$

Over $K=\overline{\mathbb{Q}}$, the Grothendieck Period Conjecture 6.6, as well as Conjectures 6.8 and 6.9, have so far only been fully established in the following cases :

(a) $M=\mathbb{1}(k)$: this is trivial for $k=0$ and this amounts to the transcendence of π for $k\neq 0$.
(b) $M=\mathfrak{h}(E)$ for E a CM elliptic curve : In that case, $G_{\mathrm{And}}(E)=\mathrm{MT}(E)$ is a 2-dimensional torus, and the result follows from Chudnovsky's theorem [Chu80] stating that, for any elliptic curve E, the degree of transcendence of the residue field of the comparison c_E in Ω_E^{dRB} is ≥ 2.

6.4. **The de Rham–Betti Conjecture.** In this work, we will address the following special instance of the Grothendieck Period Conjecture 6.6 :

Conjecture 6.10 (De Rham–Betti Conjecture)**.**

(i) Let M be a André motive over K. We say M satisfies the motivated de Rham–Betti conjecture *if the inclusion*

$$\Omega_M^{\mathrm{dRB}} \subseteq \Omega_M^{\mathrm{And}}$$

is an equality.

(ii) Let M be a homological motive over K. We say M satisfies the de Rham–Betti conjecture *if the inclusions*

$$\Omega_M^{\mathrm{dRB}} \subseteq \Omega_M^{\mathrm{And}} \subseteq \Omega_M^{\mathrm{mot}}$$

are equalities.

The following lemma reduces the motivated de Rham–Betti conjecture for M to that for $M_{\overline{\mathbb{Q}}}$ (and a similar statement holds for homological motives and torsors of motivic periods, in place of André motives and torsors of motivated periods, provided the standard conjectures hold for M) :

Lemma 6.11. *Let M be a André motive over K. Then*
$$\Omega^{\mathrm{dRB}}_{M_{\overline{\mathbb{Q}}}} = \Omega^{\mathrm{And}}_{M_{\overline{\mathbb{Q}}}} \implies \Omega^{\mathrm{dRB}}_{M} = \Omega^{\mathrm{And}}_{M}.$$

Proof. In general, we have the chain of closed immersions
$$\begin{tikzcd} \Omega^{\mathrm{dRB}}_{M_{\overline{\mathbb{Q}}}} \arrow[r, hook] \arrow[d, hook] & \Omega^{\mathrm{And}}_{M_{\overline{\mathbb{Q}}}} \arrow[d, hook] \\ (\Omega^{\mathrm{dRB}}_{M})_{\overline{\mathbb{Q}}} \arrow[r, hook] & (\Omega^{\mathrm{And}}_{M})_{\overline{\mathbb{Q}}}. \end{tikzcd}$$
If the top horizontal inclusion is an equality, then we have inclusions $\Omega^{\mathrm{And}}_{M_{\overline{\mathbb{Q}}}} \subseteq (\Omega^{\mathrm{dRB}}_{M})_{\overline{\mathbb{Q}}} \subseteq (\Omega^{\mathrm{And}}_{M})_{\overline{\mathbb{Q}}}$. This gives inclusions $G_{\mathrm{And}}(M_{\overline{\mathbb{Q}}})_{\overline{\mathbb{Q}}} \subseteq G_{\mathrm{dRB}}(M)_{\overline{\mathbb{Q}}} \subseteq G_{\mathrm{And}}(M)_{\overline{\mathbb{Q}}}$. Since the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on tensor spaces of $M_{\overline{\mathbb{Q}}}$ preserves motivated cycles [And96b, Scolie p.17], $G_{\mathrm{And}}(M_{\overline{\mathbb{Q}}})$ has finite index in $G_{\mathrm{And}}(M)$. Indeed, $G_{\mathrm{And}}(M_{\overline{\mathbb{Q}}})$ can be defined as the closed subgroup of $\mathrm{GL}(\omega_B(M_{\overline{\mathbb{Q}}}))$ fixing a finite number of motivated classes in tensors spaces of $M_{\overline{\mathbb{Q}}}$ and the quotient set $G_{\mathrm{And}}(M)_{\overline{\mathbb{Q}}}/G_{\mathrm{And}}(M_{\overline{\mathbb{Q}}})$ can be shown to preserve the Galois orbit of each of these classes and therefore is finite ; see e.g. [And96b, §4.6] and [DM82, Prop. 6.23(a)]. As a consequence, $G_{\mathrm{dRB}}(M)_{\overline{\mathbb{Q}}}$ is a finite extension of the reductive group $G_{\mathrm{And}}(M_{\overline{\mathbb{Q}}})_{\overline{\mathbb{Q}}}$. Hence $G_{\mathrm{dRB}}(M)$ is reductive. From Lemma 6.2 and Proposition 6.12, we conclude that $\Omega^{\mathrm{dRB}}_{M} = \Omega^{\mathrm{And}}_{M}$. ☐

The following proposition shows that Conjecture 6.10(i) for a André motive M over L is a strengthening of Conjecture 6.1 restricted to $\langle M\rangle$; see also [BC16, Prop. 2.14].

Proposition 6.12. *Let M be a André motive over K. The following statements are equivalent :*

(i) M satisfies the motivated de Rham–Betti conjecture, i.e., $\Omega^{\mathrm{dRB}}_{M} = \Omega^{\mathrm{And}}_{M}$;

(ii) The functor $(\rho_{\mathrm{dRB}})|_{\langle M\rangle}$ is full and $G_{\mathrm{dRB}}(M)$ is reductive.

In particular, if M satisfies the motivated de Rham–Betti conjecture, then any de Rham–Betti class on a tensor space $M^{\otimes n} \otimes (M^\vee)^{\otimes m}$ is motivated, and if $M = \mathfrak{h}(X)$, then any de Rham–Betti class in $\mathrm{H}^j_{\mathrm{dRB}}(X^n, \mathbb{Q}(k))$ is motivated and in particular zero if $j \neq 2k$.

Proof. The equivalence of (i) and (ii) is analogue to Proposition 3.3 ; it is a direct consequence of the basic facts concerning Tannakian categories (see Section 2) and the fact that $G_{\mathrm{And}}(M)$ is reductive (see Section 3). Assume now that $(\rho_{\mathrm{dRB}})|_{\langle M\rangle}$ is full. Then any de Rham–Betti classes on a tensor space $M^{\otimes n} \otimes (M^\vee)^{\otimes m}$ is motivated. If now $M = \mathfrak{h}(X)$, we note that $\mathbb{1}(-1)$ is a direct summand of $\mathfrak{h}(X)$, so that $\mathfrak{h}(X^n)(k)$ is a direct summand of $\mathfrak{h}(X)^{\otimes r} \otimes (\mathfrak{h}(X)^\vee)^{\otimes s}$ for some $r, s \geq 0$. Hence any de Rham–Betti class in $\mathrm{H}^j_{\mathrm{dRB}}(X^n, \mathbb{Q}(k))$ is motivated. That de Rham–Betti classes in $\mathrm{H}^j_{\mathrm{dRB}}(X^n, \mathbb{Q}(k))$ are zero for $j \neq 2k$ follows at once from the fact that a André motive with no grade zero component does not support any non-zero motivated class. ☐

6.5. **The $\overline{\mathbb{Q}}$-de Rham–Betti Conjecture.** Let $\mathsf{M}^{\mathrm{And}}_K \otimes \overline{\mathbb{Q}}$ be the $\overline{\mathbb{Q}}$-linear category of André motives over K with $\overline{\mathbb{Q}}$-coefficients ; it is the pseudo-abelian envelope of the base-change to $\overline{\mathbb{Q}}$ of $\mathsf{M}^{\mathrm{And}}_K$. For clarity, we note that, if $K \subsetneq \overline{\mathbb{Q}}$, although one may consider $\mathrm{H}^j_{\mathrm{dRB}}(X, \mathbb{Q}(k)) \otimes \overline{\mathbb{Q}}$ as an object in $\mathscr{C}_{\mathrm{dRB},K_{\mathrm{dR}},\overline{\mathbb{Q}}_{\mathrm{B}}}$ for X smooth projective over K, there is no linear functor $\mathsf{M}^{\mathrm{And}}_K \otimes \overline{\mathbb{Q}} \to \mathrm{Vec}_K$ and hence no de Rham–Betti realization functor $\mathsf{M}^{\mathrm{And}}_K \otimes \overline{\mathbb{Q}} \to \mathscr{C}_{\mathrm{dRB},K_{\mathrm{dR}},\overline{\mathbb{Q}}_{\mathrm{B}}}$. (This is the reason we did not consider the case $K \subsetneq L$ among the cases (a)–(c) in §4.1.) Recall however from Proposition 5.1 that, if M is a André motive over $K = \overline{\mathbb{Q}}$, then $\Omega_M = \Omega^{\overline{\mathbb{Q}}-\mathrm{dRB}}_{M\otimes\overline{\mathbb{Q}}}$. This justifies calling, in analogy to the de Rham–Betti Conjecture 6.10, the following conjecture the $\overline{\mathbb{Q}}$-de Rham–Betti Conjecture.

Conjecture 6.13 ($\overline{\mathbb{Q}}$-de Rham–Betti Conjecture)**.**

(i) Let M be a André motive over K. We say M satisfies the motivated $\overline{\mathbb{Q}}$-de Rham–Betti conjecture *if the inclusions*

$$\Omega_M \subseteq \Omega_M^{\mathrm{dRB}} \subseteq \Omega_M^{\mathrm{And}}$$

are equalities, that is, if the comparison isomorphism c_M generates Ω_M^{And} as a K-torsor.

(ii) Let M be a homological motive over K. We say M satisfies the $\overline{\mathbb{Q}}$-de Rham–Betti conjecture *if the inclusions*

$$\Omega_M \subseteq \Omega_M^{\mathrm{dRB}} \subseteq \Omega_M^{\mathrm{And}} \subseteq \Omega_M^{\mathrm{mot}}$$

are equalities, that is, if the comparison isomorphism c_M generates Ω_M^{mot} as a K-torsor.

Obviously, Conjecture 6.13 for M implies Conjecture 6.10 for M. We have the analogue of Lemma 6.11, which reduces Conjecture 6.13 to motives over $\overline{\mathbb{Q}}$:

Lemma 6.14. *Let M be a André motive over K. Then*

$$\Omega_{M_{\overline{\mathbb{Q}}}} = \Omega^{\mathrm{And}}_{M_{\overline{\mathbb{Q}}}} \implies \Omega_M = \Omega_M^{\mathrm{And}}.$$

Proof. As in the proof of Lemma 6.11, we have the chain of closed immersions

$$\begin{tikzcd} \Omega_{M_{\overline{\mathbb{Q}}}} \arrow[r, hook] \arrow[d, hook] & \Omega^{\mathrm{And}}_{M_{\overline{\mathbb{Q}}}} \arrow[d, hook] \\ (\Omega_M)_{\overline{\mathbb{Q}}} \arrow[r, hook] & (\Omega_M^{\mathrm{And}})_{\overline{\mathbb{Q}}}. \end{tikzcd}$$

If the top horizontal inclusion is an equality, then we have inclusions $\Omega^{\mathrm{And}}_{M_{\overline{\mathbb{Q}}}} \subseteq (\Omega_M)_{\overline{\mathbb{Q}}} \subseteq (\Omega_M^{\mathrm{And}})_{\overline{\mathbb{Q}}}$. By [And96b, §4.6], there is a short exact sequence

$$1 \to G_{\mathrm{And}}(M_{\overline{\mathbb{Q}}}) \to G_{\mathrm{And}}(M) \to \mathrm{Gal}(F/K) \to 1$$

for some finite Galois extension F of K. This implies that

$$(\Omega_M^{\mathrm{And}})_{\overline{\mathbb{Q}}} = \coprod_{\mathrm{Gal}(F/K)} \Omega^{\mathrm{And}}_{M_{\overline{\mathbb{Q}}}}. \tag{5}$$

The inclusion $\Omega_M \subseteq \Omega_M^{\mathrm{And}}$ is defined over K, and therefore the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$ preserves the subvariety $(\Omega_M)_{\overline{\mathbb{Q}}} \subseteq (\Omega_M^{\mathrm{And}})_{\overline{\mathbb{Q}}}$. Since $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$ permutes the components on the right-hand side of (5) and $\Omega^{\mathrm{And}}_{M_{\overline{\mathbb{Q}}}} \subseteq (\Omega_M)_{\overline{\mathbb{Q}}}$, we conclude that $(\Omega_M)_{\overline{\mathbb{Q}}} = (\Omega_M^{\mathrm{And}})_{\overline{\mathbb{Q}}}$. ☐

Remark 6.15. Let $N \in \mathscr{C}_{\mathrm{dRB},K}$ be a de Rham–Betti object. For the object $N\otimes K \in \mathscr{C}_{\mathrm{dRB},K_{\mathrm{dR}},K_{\mathrm{B}}}$ we can define a Tannakian torsor $\Omega^{\mathrm{dRB},K_{\mathrm{dR}},K_{\mathrm{B}}}_{N\otimes K}$ similarly as in Definition 6.3, but using the Tannakian category $\mathscr{C}_{\mathrm{dRB},K_{\mathrm{dR}},K_{\mathrm{B}}}$. If $N = \mathrm{H}_{\mathrm{dRB}}(M)$ is the de Rham–Betti realization of a André motive M over K and $\Omega_{M_{\overline{\mathbb{Q}}}} = \Omega^{\mathrm{And}}_{M_{\overline{\mathbb{Q}}}}$, then Lemma 6.14 implies that $\Omega_N = \Omega^{\mathrm{dRB},K_{\mathrm{dR}},K_{\mathrm{B}}}_{N\otimes K}$. We want to emphasize that this last equality is not true for a general de Rham–Betti object N. For example, let $a \in \overline{\mathbb{Q}}$ be such that $a^n \notin K$ for all $n \geq 1$. Then we can define

$$N := (N_{\mathrm{dR}} = K, N_{\mathrm{B}} = \mathbb{Q}, c_N),$$

where c_N is multiplication by a. It is easy to see that $\Omega_N = \mathrm{Spec}\, F^g$, where F^g denotes the Galois closure of $K(a)$. But $N^{\otimes m}$ does not admit a K-de Rham–Betti class for $m > 0$, and therefore

$\Omega_{N\otimes K}^{\mathrm{dRB},K_{\mathrm{dR}},K_{\mathrm{B}}}$ is a torsor under $\mathbb{G}_{m,K}$. Note that this is in contrast to Proposition 5.1, which holds for arbitrary de Rham–Betti objects.

We also have the analogue of Proposition 6.12 :

Proposition 6.16. *Let M be a André motive over $\overline{\mathbb{Q}}$. The following statements are equivalent :*

(i) M satisfies the motivated $\overline{\mathbb{Q}}$-de Rham–Betti conjecture, i.e., $\Omega_M = \Omega_M^{\mathrm{And}}$;

(ii) The $\overline{\mathbb{Q}}$-de Rham–Betti realization functor $\rho_{\overline{\mathbb{Q}}-\mathrm{dRB}} \colon \mathsf{M}_{\overline{\mathbb{Q}}}^{\mathrm{And}} \otimes \overline{\mathbb{Q}} \to \mathscr{C}_{\overline{\mathbb{Q}}-\mathrm{dRB}}$ restricted to $\langle M \otimes \overline{\mathbb{Q}} \rangle$ is full and $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}(M)$ is reductive.

In particular, if M satisfies the motivated $\overline{\mathbb{Q}}$-de Rham–Betti conjecture, then any $\overline{\mathbb{Q}}$-de Rham–Betti class on a tensor space $M^{\otimes n} \otimes (M^\vee)^{\otimes m}$ is $\overline{\mathbb{Q}}$-motivated (meaning a $\overline{\mathbb{Q}}$-linear combination of motivated cycles) and if $M = \mathfrak{h}(X)$, then any $\overline{\mathbb{Q}}$-de Rham–Betti class in $\mathrm{H}^j_{\mathrm{dRB}}(X^n, \overline{\mathbb{Q}}(k))$ is motivated and in particular zero if $j \neq 2k$.

Proof. From Proposition 5.1, we have $\Omega_M = \Omega_{M\otimes\overline{\mathbb{Q}}}^{\overline{\mathbb{Q}}-\mathrm{dRB}}$ and the proof is then the same as that of Proposition 6.12. □

Finally, we have :

Lemma 6.17. *Let $M \in \mathsf{M}_{\overline{\mathbb{Q}}}^{\mathrm{And}}$ be a André motive over $\overline{\mathbb{Q}}$. If $\overline{\mathbb{Q}}$-de Rham–Betti classes on M are $\overline{\mathbb{Q}}$-motivated, then de Rham–Betti classes on M are motivated.*

Proof. This is clear. □

7. De Rham–Betti classes on abelian varieties

In this section, we fix an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ inside $\mathbb{C}$ and we let K be a subfield of $\overline{\mathbb{Q}}$.

7.1. **Consequences of Wüstholz' analytic subgroup theorem.** Let $M \in \mathscr{C}_{\mathrm{dRB},\overline{\mathbb{Q}}_{\mathrm{dR}},K_{\mathrm{B}}}$ and $\gamma \in M_{\mathrm{B}}^\vee$. The comparison $c_M : M_{\mathrm{dR}} \otimes_{\overline{\mathbb{Q}}} \mathbb{C} \xrightarrow{\simeq} M_{\mathrm{B}} \otimes_K \mathbb{C}$ defines a $\overline{\mathbb{Q}}$-linear map

$$\int_\gamma : M_{\mathrm{dR}} \to \mathbb{C}, \;\; \omega \mapsto \gamma_{\mathbb{C}}(c_M(\omega_{\mathbb{C}})).$$

As in Definition 5.4 we define the *annihilator* of $\gamma \in M_{\mathrm{B}}^\vee$ to be $\mathrm{Ann}(\gamma) =_{\mathrm{def}} \ker \int_\gamma \subseteq M_{\mathrm{dR}}$.

We denote by $\mathcal{AB} \subset \mathsf{M}_{\overline{\mathbb{Q}}}^{\mathrm{And}}$ the full abelian subcategory of the category of André motives generated by the motives $\mathfrak{h}^1(A)$, where A is an abelian variety over $\overline{\mathbb{Q}}$; it is equivalent to the category of abelian varieties over $\overline{\mathbb{Q}}$ up to isogeny.

The following formulation of Wüstholz' analytic subgroup theorem [Wüs84] is derived from the more general formulation for 1-motives taken from [HW22, Thm. 9.7]. It has the advantage that it admits a natural extension to motives with $\overline{\mathbb{Q}}$-coefficients (see §7.2) and will allow for applications regarding $\overline{\mathbb{Q}}$-de Rham–Betti classes.

Proposition 7.1 (Analytic subgroup theorem for abelian motives ; [HW22, Thm. 9.7])**.** *Let $M \in \mathcal{AB}$ and $\gamma \in \mathrm{H}_{\mathrm{B}}(M)^\vee$. There exists a decomposition*

$$M = M' \oplus M''$$

in the category $\mathcal{AB}$ such that $\gamma \in \mathrm{H}_{\mathrm{B}}(M'')^\vee$ and $\mathrm{Ann}(\gamma) = \mathrm{H}_{\mathrm{dR}}(M')$.

Proof. This is [HW22, Thm. 9.7], where a stronger version for 1–motives is proved. More precisely, we obtain a short exact sequence

$$0 \to M' \to M \to M'' \to 0$$

in the category of 1–motives such that $\gamma \in \mathrm{H_B}(M'')^\vee$ and $\mathrm{Ann}(\gamma) = \mathrm{H_{dR}}(M')$. Since the category $\mathcal{AB}$ is a full subcategory of the category of 1–motives which is stable under taking subquotients, we know that $M', M'' \in \mathcal{AB}$. The short exact sequence is split since the category $\mathcal{AB}$ is semi-simple. $\square$

Remark 7.2. Proposition 7.1 can also be derived without reference to 1-motives by applying the more classical version of the analytic subgroup theorem [HW22, Thm. 6.2] to the universal vector extensions of abelian varieties.

Remark 7.3. Let $M \in \mathcal{AB}$. By applying the de Rham–Betti realization, Proposition 7.1 tells us that for every $\gamma \in H_\mathrm{B}(M)^\vee$, there is a decomposition $H_\mathrm{dRB}(M) = N' \oplus N''$ in the category $\mathscr{C}_\mathrm{dRB}$ such that $\gamma \in (N''_\mathrm{B})^\vee$ and $\mathrm{Ann}(\gamma) = N'_\mathrm{dR}$. This is not true for an arbitrary de Rham–Betti object $N \in \mathscr{C}_\mathrm{dRB}$ and therefore forces a strong restriction on the possible de Rham–Betti objects which arise as realizations of motives in $\mathcal{AB}$. As an example, consider the de Rham–Betti object

$$N := \left(N_\mathrm{dR} = \overline{\mathbb{Q}}^2, N_\mathrm{B} = \mathbb{Q}^2, c_N\right), \quad \text{with } c_N = \begin{pmatrix} \alpha & \beta \\ -\beta & \gamma \end{pmatrix}, \quad \mathrm{degtr}_\mathbb{Q}\mathbb{Q}(\alpha,\beta,\gamma) = 3.$$

Denote by γ_1, γ_2 the dual basis of the natural basis e_1, e_2 of N_B. Similarly, denote by ω_1, ω_2 the natural basis of N_dR. If we let $\gamma := (\gamma_1, \gamma_2) \in (N_B^{\oplus 2})^\vee$, then $\mathrm{Ann}(\gamma) \subset N_\mathrm{dR}^{\oplus 2}$ is the one-dimensional subspace generated by (ω_2, ω_1). But one checks that the de Rham–Betti object N is simple, hence there does not exist a subobject $N' \subset N^{\oplus 2}$ such that $\mathrm{Ann}(\gamma) = N'_\mathrm{dR}$.

Theorem 7.4 **(André, Bost, Wüstholz).** *The functor $\mathcal{AB} \to \mathscr{C}_\mathrm{dRB}$ is fully faithful and the image is closed under taking subobjects. In particular, the essential image forms a full abelian subcategory of $\mathscr{C}_\mathrm{dRB}$ which is semi-simple.*

Proof. We first prove that the image is closed under taking subobjects. Let $M \in \mathcal{AB}$ and suppose that $0 \subsetneq N \subset \mathrm{H_{dRB}}(M)$ is a subobject in the category $\mathscr{C}_\mathrm{dRB}$ of de Rham–Betti objects. We may assume that N is not contained in the de Rham–Betti realization of a submotive $M' \subsetneq M$, otherwise we replace M by M'. We aim to show that $N = \mathrm{H_{dRB}}(M)$. To get a contradiction, assume that $N \subsetneq \mathrm{H_{dRB}}(M)$. Then we denote by $N' \in \mathscr{C}_\mathrm{dRB}$ the quotient of $\mathrm{H_{dRB}}(M)$ by N. If we choose $0 \neq \gamma \in {N'_\mathrm{B}}^\vee$, then Proposition 7.1 gives a decomposition $M = M' \oplus M''$ in $\mathcal{AB}$ such that $0 \neq \gamma \in \mathrm{H_B}(M'')^\vee$ and $\mathrm{Ann}(\gamma) = \mathrm{H_{dR}}(M')$. But then $N_\mathrm{dR} \subset \mathrm{Ann}(\gamma) = \mathrm{H_{dR}}(M')$. This contradicts the assumption that N is not contained in the realization of a proper submotive $M' \subsetneq M$.

We now prove the fully faithfulness. Let $M, N \in \mathcal{AB}$ and let $(f_\mathrm{dR}, f_\mathrm{B}) : \mathrm{H_{dRB}}(M) \to \mathrm{H_{dRB}}(N)$ be a morphism in $\mathscr{C}_\mathrm{dRB}$. We denote by $\Gamma_{f_\mathrm{dR}} \subset \mathrm{H_{dR}}(M) \oplus \mathrm{H_{dR}}(N)$ and $\Gamma_{f_\mathrm{B}} \subset \mathrm{H_B}(M) \oplus \mathrm{H_B}(N)$ the graphs of f_dR and f_B, respectively. Then $(\Gamma_{f_\mathrm{dR}}, \Gamma_{f_\mathrm{B}})$ defines a subobject of $\mathrm{H_{dRB}}(M) \oplus \mathrm{H_{dRB}}(N)$ in $\mathscr{C}_\mathrm{dRB}$. By the first part of the proof, this subobject is the realization of a submotive $\Gamma \subset M \oplus N$. The composition

$$r \colon \Gamma \subset M \oplus N \xrightarrow{\mathrm{pr}_1} M$$

is an isomorphism. Then

$$f \colon M \xrightarrow{r^{-1}} \Gamma \subset M \oplus N \xrightarrow{\mathrm{pr}_2} N,$$

defines a morphism in $\mathcal{AB}$ whose de Rham–Betti realization is $(f_\mathrm{dR}, f_\mathrm{B})$. $\square$

Remark 7.5. Theorem 7.4 is stated in [And04, §7.5.3] and established as a consequence of a weaker form of Wüstholz' analytic subgroup theorem. Bost [Bos13, Thm. 5.1 & 5.3] also provides a proof of the full faithfulness based on the older transcendence theorems of Schneider and Lang. However, in the next subsection we will need the full power of the analytic subgroup theorem in the version of Proposition 7.1 to prove an analog of Theorem 7.4 with $\overline{\mathbb{Q}}$–coefficients.

An easy consequence of Theorem 7.4 is the following:

Theorem 7.6. *Let X be a smooth projective variety over $\overline{\mathbb{Q}}$. Then the de Rham–Betti object $\mathrm{H}^1_{\mathrm{dRB}}(X,\mathbb{Q})$ does not have any odd-dimensional subobject. In particular, for any $k \in \mathbb{Z}$, any de Rham–Betti class on $\mathrm{H}^1_{\mathrm{dRB}}(X,\mathbb{Q}(k))$ is zero.*

Proof. The Poincaré bundle induces an isomorphism $\mathfrak{h}^1(X) \simeq \mathfrak{h}^1(\mathrm{Pic}^0_X)$ of André motives. It follows from Theorem 7.4 that any subobject of $\mathrm{H}^1_{\mathrm{dRB}}(X,\mathbb{Q})$ is isomorphic to the de Rham–Betti realization of $\mathfrak{h}^1(A)$ for some abelian variety $A/\overline{\mathbb{Q}}$ and hence is even-dimensional. We have the identification

$$\mathrm{Hom}_{\mathrm{dRB}}(\mathbb{1},\mathrm{H}^1_{\mathrm{dRB}}(X,\mathbb{Q}(k))) = \mathrm{Hom}_{\mathrm{dRB}}(\mathbb{1}(-k),\mathrm{H}^1_{\mathrm{dRB}}(X,\mathbb{Q})),$$

which then shows that any de Rham–Betti class on $\mathrm{H}^1_{\mathrm{dRB}}(X,\mathbb{Q}(k))$ is zero. □

Remark 7.7. That $\mathrm{H}^1_{\mathrm{dRB}}(X,\mathbb{Q}(k))$ does not support any non-zero de Rham–Betti class appears for the cases $k=0$ and $k=1$ in [BC16, Thms. 4.1 & 4.2]. Note that the proof of [BC16, Thm. 4.2] (the case $k=1$) relies on [BC16, Cor. 3.4], which itself relies on [BC16, Thm. 3.3]. Although [BC16, Cor. 3.4] is correct, it appears that [BC16, Thm. 3.3] is wrong as can be seen by considering the case $G=\mathbb{G}_{m,\overline{\mathbb{Q}}}$ therein. Likewise, the functor ω of [And04, §7.5.3] is not full nor is a subobject in the image of ω the image of a subobject, as can be seen with $0=\mathrm{Hom}(\mathbb{G}_a,\mathbb{G}_m)$ but $\mathrm{Hom}(\omega(\mathbb{G}_a),\omega(\mathbb{G}_m)) = \overline{\mathbb{Q}}$.

7.2. **A $\overline{\mathbb{Q}}$-version of Wüstholz' analytic subgroup theorem.** We write $\mathcal{AB}\otimes\overline{\mathbb{Q}} \subset \mathsf{M}^{\mathrm{And}}_{\overline{\mathbb{Q}}}\otimes\overline{\mathbb{Q}}$ for the full abelian subcategory of the category of André motives with coefficients in $\overline{\mathbb{Q}}$ generated by the motives $\mathfrak{h}^1(A)$, where A is an abelian variety over $\overline{\mathbb{Q}}$.

Proposition 7.8 (Analytic subgroup theorem for abelian motives with $\overline{\mathbb{Q}}$–coefficients)**.** *Let $M \in \mathcal{AB}\otimes\overline{\mathbb{Q}}$ and $\gamma \in \mathrm{H}_{\mathrm{B}}(M)^\vee$. There exists a decomposition*

$$M = M' \oplus M''$$

in the category $\mathcal{AB}\otimes\overline{\mathbb{Q}}$ such that $\gamma \in \mathrm{H}_{\mathrm{B}}(M'')^\vee$ and $\mathrm{Ann}(\gamma) = \mathrm{H}_{\mathrm{dR}}(M')$.

Proof. We first handle the case where $M = \mathfrak{h}^1(A)\otimes\overline{\mathbb{Q}}$ is the motive of an abelian variety. Write

$$\gamma = \sum_{i=1}^n \lambda_i\gamma_i \in \mathrm{H}^1_{\mathrm{B}}(A,\mathbb{Q})\otimes\overline{\mathbb{Q}}, \quad \text{with } \lambda_i \in \overline{\mathbb{Q}} \text{ and } \gamma_i \in \mathrm{H}^1_{\mathrm{B}}(A,\mathbb{Q}).$$

Inspired by the proof of [HW22, Thm. 9.10], we apply Proposition 7.1 to $\underline{\gamma} := (\gamma_1,\dots,\gamma_n) \in \mathrm{H}^1_{\mathrm{B}}(A,\mathbb{Q})^{\oplus n}$. This gives a decomposition $\mathfrak{h}^1(A)^{\oplus n} = N' \oplus N''$ in $\mathcal{AB}$ such that $\underline{\gamma} \in \mathrm{H}_{\mathrm{B}}(N'')^\vee$ and $\mathrm{Ann}(\underline{\gamma}) = \mathrm{H}_{\mathrm{dR}}(N')$. The motive N' is the kernel of the second projection $\mathrm{pr}_2 : \mathfrak{h}^1(A)^{\oplus n} \to N''$. Via the inclusion $\overline{\mathbb{Q}} \subset \mathrm{End}(A)\otimes\overline{\mathbb{Q}}$, we can view $\lambda_i \in \overline{\mathbb{Q}}$ as an endomorphism of $\mathfrak{h}^1(A)\otimes\overline{\mathbb{Q}}$. We can thus define the morphism

$$f\colon \mathfrak{h}^1(A)\otimes\overline{\mathbb{Q}} \xrightarrow{(\lambda_1,\dots,\lambda_n)} \mathfrak{h}^1(A)^{\oplus n}\otimes\overline{\mathbb{Q}} \xrightarrow{\mathrm{pr}_2} N''\otimes\overline{\mathbb{Q}}$$

in $\mathcal{AB}\otimes\overline{\mathbb{Q}}$. We let $M' := \ker f \subset M = \mathfrak{h}^1(A)\otimes\overline{\mathbb{Q}}$ and write $M = M'\oplus M''$ for some motive $M'' \in \mathcal{AB}\otimes\overline{\mathbb{Q}}$. We have to show that $\gamma \in \mathrm{H}_{\mathrm{B}}(M'')^\vee$ and $\mathrm{Ann}(\gamma) = \mathrm{H}_{\mathrm{dR}}(M')$. For the former, note that by construction the dual morphism

$$(\lambda_1,\dots,\lambda_n)^\vee\colon (\mathrm{H}^1_{\mathrm{B}}(A,\mathbb{Q})^\vee)^{\oplus n}\otimes\overline{\mathbb{Q}} \longrightarrow \mathrm{H}^1_{\mathrm{B}}(A,\mathbb{Q})^\vee\otimes\overline{\mathbb{Q}}$$

maps $\underline{\gamma}$ to γ and the subspace $\mathrm{H}_{\mathrm{B}}(N'')^\vee \otimes \overline{\mathbb{Q}}$ to $\mathrm{H}_{\mathrm{B}}(M'')^\vee$. For the latter, one computes that $\mathrm{Ann}(\gamma) \subset \mathrm{H}^1_{\mathrm{dR}}(A) \otimes \overline{\mathbb{Q}}$ is precisely the preimage of $\mathrm{Ann}(\underline{\gamma})$ under

$$(\lambda_1, ..., \lambda_n) \colon \mathrm{H}^1_{\mathrm{dR}}(A) \otimes \overline{\mathbb{Q}} \longrightarrow \mathrm{H}^1_{\mathrm{dR}}(A)^{\oplus n} \otimes \overline{\mathbb{Q}}.$$

By construction of N', we have $\mathrm{Ann}(\underline{\gamma}) = \mathrm{H}_{\mathrm{dR}}(N') = \ker \mathrm{pr}_{2,\mathrm{dR}}$. This proves that $\mathrm{Ann}(\gamma) = \ker f_{\mathrm{dR}} = \mathrm{H}_{\mathrm{dR}}(M')$ and concludes the proof for $M = \mathfrak{h}^1(A) \otimes \overline{\mathbb{Q}}$.

In general, $M \in \mathcal{AB} \otimes \overline{\mathbb{Q}}$ will be a direct summand of $\mathfrak{h}^1(A) \otimes \overline{\mathbb{Q}}$ for some abelian variety A. Let $\gamma \in \mathrm{H}_{\mathrm{B}}(M)^\vee$. Via the projection, this gives an element $\tilde{\gamma} \in \mathrm{H}_{\mathrm{B}}(A)^\vee \otimes \overline{\mathbb{Q}}$. Applying the proposition to the motive $\mathfrak{h}^1(A) \otimes \overline{\mathbb{Q}}$, we get a decomposition $\mathfrak{h}^1(A) \otimes \overline{\mathbb{Q}} = N' \oplus N''$ such that $\tilde{\gamma} \in \mathrm{H}_{\mathrm{B}}(N'')^\vee$ and $\mathrm{Ann}(\tilde{\gamma}) = \mathrm{H}_{\mathrm{dR}}(N')$. Setting $M' := M \cap N'$ and $M'' = M/M'$, we get $\gamma \in \mathrm{H}_{\mathrm{B}}(M'')^\vee$ and

$$\mathrm{Ann}(\gamma) = \mathrm{Ann}(\tilde{\gamma}) \cap \mathrm{H}_{\mathrm{dR}}(M) = \mathrm{H}_{\mathrm{dR}}(M'),$$

which concludes the proof of the proposition. □

Remark 7.9 (A $\overline{\mathbb{Q}}$-version of the analytic subgroup theorem for 1-motives)**.** Using the arguments in the proof of Proposition 7.8, one can more generally establish a version with $\overline{\mathbb{Q}}$-coefficients of the analytic subgroup theorem for 1-motives [HW22, Thm. 9.7].

Theorem 7.10. *The functor $\mathcal{AB} \otimes \overline{\mathbb{Q}} \to \mathscr{C}_{\mathrm{dRB},\overline{\mathbb{Q}}_{\mathrm{dR}},\overline{\mathbb{Q}}_{\mathrm{B}}}$ is fully faithful and the image is closed under taking subobjects. In particular, the essential image forms a full abelian subcategory of $\mathscr{C}_{\mathrm{dRB},\overline{\mathbb{Q}}_{\mathrm{dR}},\overline{\mathbb{Q}}_{\mathrm{B}}}$ which is semi-simple.*

Proof. The proof is the same as in Theorem 7.4, with essential input the $\overline{\mathbb{Q}}$–version of the analytic subgroup theorem for abelian motives (Proposition 7.8). □

7.3. **The de Rham–Betti group of an abelian variety.** Let A be an abelian variety over $\overline{\mathbb{Q}}$ and let $G_{\mathrm{dRB}}(A) =_{\mathrm{def}} G_{\mathrm{dRB}}(\mathfrak{h}(A))$ be its de Rham–Betti group. Since $\mathrm{H}^*_{\mathrm{dRB}}(A, \mathbb{Q})$ is the exterior algebra on $\mathrm{H}^1_{\mathrm{dRB}}(A, \mathbb{Q})$, we have $G_{\mathrm{dRB}}(A) = G_{\mathrm{dRB}}(\mathrm{H}^1_{\mathrm{dRB}}(A))$. Likewise, we let $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}(A) =_{\mathrm{def}} G_{\overline{\mathbb{Q}}-\mathrm{dRB}}(\mathfrak{h}(A) \otimes \overline{\mathbb{Q}})$ be its $\overline{\mathbb{Q}}$-de Rham–Betti group and we have $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}(A) = G_{\overline{\mathbb{Q}}-\mathrm{dRB}}(\mathrm{H}^1_{\mathrm{dRB}}(A, \overline{\mathbb{Q}}))$.

Theorem 7.11. *Let $A/\overline{\mathbb{Q}}$ be an abelian variety of positive dimension. Its de Rham–Betti group $G_{\mathrm{dRB}}(A)$ has the following properties.*

(i) $G_{\mathrm{dRB}}(A) \subseteq \mathrm{MT}(A)$ *is a connected reductive group.*
(ii) $\mathrm{End}_{G_{\mathrm{dRB}}(A)}(\mathrm{H}^1_{\mathrm{B}}(A, \mathbb{Q})) = \mathrm{End}(A)_{\mathbb{Q}}$.
(iii) $\det : G_{\mathrm{dRB}}(A) \to \mathbb{G}_m$ *is surjective.*
(iv) A has complex multiplication if and only if $G_{\mathrm{dRB}}(A)$ is a torus.

Likewise, its $\overline{\mathbb{Q}}$-de Rham–Betti group $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}(A)$ has the following properties.

(i') $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}(A) \subseteq \mathrm{MT}(A)$ *is a connected reductive group.*
(ii') $\mathrm{End}_{G_{\overline{\mathbb{Q}}-\mathrm{dRB}}(A)}(\mathrm{H}^1_{\mathrm{B}}(A, \overline{\mathbb{Q}})) = \mathrm{End}(A)_{\mathbb{Q}} \otimes \overline{\mathbb{Q}}$.
(iii') $\det : G_{\overline{\mathbb{Q}}-\mathrm{dRB}}(A) \to \mathbb{G}_m$ *is surjective.*
(iv') A has complex multiplication if and only if $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}(A)$ is a torus.

Proof. That $G_{\mathrm{dRB}}(A)$ lies in $\mathrm{MT}(A)$ is due to André's Theorem 3.4 stating that the inclusion $\mathrm{MT}(A) \subseteq G_{\mathrm{And}}(A)$ is an equality. That $G_{\mathrm{dRB}}(A)$ is connected is Theorem 4.7. As a consequence of Theorem 7.4, the de Rham–Betti cohomology group $\mathrm{H}^1_{\mathrm{dRB}}(A, \mathbb{Q})$ is semi-simple as an object in $\mathscr{C}_{\mathrm{dRB}}$. It follows from Proposition 2.2 that $G_{\mathrm{dRB}}(A)$ acts faithfully and semi-simply on $\mathrm{H}^1_{\mathrm{B}}(A, \mathbb{Q})$ and from Proposition 1.1 that, since we are working in characteristic zero, $G_{\mathrm{dRB}}(A)$ is reductive. Statement *(ii)* is Theorem 7.4. Regarding *(iii)*, the image of det is connected. Assume it is trivial. Then $G_{\mathrm{dRB}}(A)$ acts trivially on $\det(\mathrm{H}^1_{\mathrm{B}}(A, \mathbb{Q})) = \mathrm{H}^{2\dim A}_{\mathrm{B}}(A, \mathbb{Q})$. But $\mathfrak{h}^{2\dim A}(A) \simeq \mathbb{1}(-\dim A)$. This is a contradiction since $2\pi i$ is transcendental. For *(iv)*, suppose

A has complex multiplication. Then $G_{\mathrm{dRB}}(A)$ is a subgroup of the Mumford–Tate group $\mathrm{MT}(A)$, which is a torus. Since $G_{\mathrm{dRB}}(A)$ is reductive and connected by (i), it has to be a torus. Conversely, assume that $G_{\mathrm{dRB}}(A)$ is a torus. We recall the following classical argument (usually used for the Mumford–Tate group): $G_{\mathrm{dRB}}(A)$ is contained in a maximal torus $T \subseteq \mathrm{GL}(\mathrm{H}^1_{\mathrm{B}}(A, \mathbb{Q}))$. Then

$$\mathrm{End}_T(\mathrm{H}^1_{\mathrm{B}}(A, \mathbb{Q})) \subseteq \mathrm{End}_{G_{\mathrm{dRB}}(A)}(\mathrm{H}^1_{\mathrm{B}}(A, \mathbb{Q})) = \mathrm{End}(A)_{\mathbb{Q}}.$$

But $\mathrm{End}_T(\mathrm{H}^1_{\mathrm{B}}(A, \mathbb{Q}))$ is a commutative $\mathbb{Q}$-algebra of dimension $2g$, as can be seen after extending scalars to an algebraically closed field. It follows that A has complex multiplication.

Statements (i'), (ii'), (iii') and (iv') are proven similarly by replacing Theorem 7.4 with its $\overline{\mathbb{Q}}$-analogue Theorem 7.10. □

7.4. **The de Rham–Betti conjecture for products of elliptic curves.** We prove the following stronger (but equivalent, by the motivic analogue of Proposition 6.12) version of Theorem 1 in the case of products of elliptic curves.

Theorem 7.12. *Let $E_1, \cdots, E_s$ be pairwise non-isogenous elliptic curves over $\overline{\mathbb{Q}}$ and let A be an abelian variety over K such that $A_{\overline{\mathbb{Q}}}$ is isogenous to $E_1^{n_1} \times \cdots \times E_s^{n_s}$.*

(i) The de Rham–Betti conjecture (Conjecture 6.10(ii)) holds for $\mathfrak{h}(A)$, i.e.,

$$\Omega_A^{\mathrm{dRB}} = \Omega_A^{\mathrm{mot}}.$$

In particular, by Proposition 6.12, for any $n \geq 0$ and any $k \in \mathbb{Z}$, any de Rham–Betti class on $\mathfrak{h}(A^n)(k)$ is algebraic.

(ii) If at most one of the E_i has CM, then the $\overline{\mathbb{Q}}$-de Rham–Betti conjecture (Conjecture 6.13(ii)) holds for $\mathfrak{h}(A)$, i.e.,

$$\Omega_A = \Omega_A^{\mathrm{mot}}.$$

In particular, by Proposition 6.16, for any $n \geq 0$ and any $k \in \mathbb{Z}$, any $\overline{\mathbb{Q}}$-de Rham–Betti class on $\mathfrak{h}(A^n_{\overline{\mathbb{Q}}})(k)$ is a $\overline{\mathbb{Q}}$-linear combination of algebraic classes.

We start by recalling that the Hodge conjecture is known for products of complex elliptic curves. Recall that the *Hodge group* $\mathrm{Hdg}(H)$ of a rational Hodge structure H is the smallest algebraic $\mathbb{Q}$-subgroup of $\mathrm{GL}(H)$ that contains the image of $\mathrm{U}_{\mathbb{C}/\mathbb{R}} \hookrightarrow \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to \mathrm{GL}(H)_{\mathbb{R}}$, where the right arrow is the morphism defining the Hodge structure on H. The group $\mathrm{Hdg}(H)$ can be characterized as being the largest subgroup of $\mathrm{GL}(H)$ that fixes all Hodge classes in tensor spaces associated to H. In case H is of pure weight 0, then the Hodge group of H agrees with its Mumford–Tate group, while in case H is of pure weight $n \neq 0$, its Mumford–Tate group $\mathrm{MT}(H)$ is the image of the multiplication map $\mathbb{G}_m \times \mathrm{Hdg}(H) \to \mathrm{GL}(H)$. If A is a complex abelian variety, the Hodge group $\mathrm{Hdg}(A)$ (resp. the Mumford–Tate group $\mathrm{MT}(A)$) of A is the Hodge group (resp. the Mumford–Tate group) of the Hodge structure $\mathrm{H}^1_{\mathrm{B}}(A, \mathbb{Q})$.

Proposition 7.13. *Let $E_1, \cdots, E_s$ be pairwise non-isogenous complex elliptic curves and let A be an abelian variety isogenous to $E_1^{n_1} \times \cdots \times E_s^{n_s}$. Denote $V_i := \mathrm{H}^1(E_i, \mathbb{Q})$ and $K_i := \mathrm{End}(E_i) \otimes \mathbb{Q}$. Then the Hodge group of A is*

$$\mathrm{Hdg}(A) = \mathrm{Hdg}(\mathrm{E}_1) \times \cdots \times \mathrm{Hdg}(\mathrm{E}_\mathrm{s}), \quad \textit{where } \mathrm{Hdg}(E_i) = \begin{cases} \mathrm{U}_{K_i} & \textit{if } E_i \textit{ has CM;} \\ \mathrm{SL}(V_i) & \textit{if } E_i \textit{ is without CM,} \end{cases}$$

In particular, the subspace of Hodge classes on A is generated by Hodge classes in $\mathrm{H}^2(A, \mathbb{Q})$ and, consequently, every Hodge class on A is algebraic.

Proof. This is classical and we refer to [Gor99, §3] for a proof. □

Proof of Theorem 7.12(i). By Lemma 6.11, we may and do assume that $K = \overline{\mathbb{Q}}$. The proof goes similarly as for the Mumford–Tate group. By Proposition 7.13, we only need to show that $G_{\mathrm{dRB}}(A) = G_{\mathrm{And}}(A)$. From André's Theorem 3.4, we have $G_{\mathrm{And}}(A) = \mathrm{MT}(A)$. Hence it remains to show that the inclusion $G_{\mathrm{dRB}}(A) \subseteq \mathrm{MT}(A)$ is an equality, which is equivalent to showing that the inclusion $G^1_{\mathrm{dRB}}(A) \subseteq \mathrm{Hdg}(A)$ is an equality, where $G^1_{\mathrm{dRB}}(A)$ denotes the connected component of the kernel of $\det : G_{\mathrm{dRB}}(A) \to \mathbb{G}_m$.

Step 1 : $A = E$ is an elliptic curve. If E has complex multiplication, then $\mathrm{MT}(E) = \mathrm{Res}_{K/\mathbb{Q}}\mathbb{G}_{m,K}$, where $K = \mathrm{End}(E)_{\mathbb{Q}}$. In this case we know from Theorem 7.11 that $G_{\mathrm{dRB}}(E) \subseteq \mathrm{MT}(E)$ is a torus. If $G_{\mathrm{dRB}}(E)$ is one-dimensional, then either $G_{\mathrm{dRB}}(E) = \mathrm{U}_K$ or $G_{\mathrm{dRB}}(E) = \mathbb{G}_m$. Note that the case $G_{\mathrm{dRB}}(E) = \mathrm{U}_K$ violates the surjectivity of the determinant and the case $G_{\mathrm{dRB}}(E) = \mathbb{G}_m$ violates the fact that $\mathrm{End}_{G_{\mathrm{dRB}}(E)}(\mathrm{H}^1_{\mathrm{B}}(E,\mathbb{Q})) = K$. Hence we see that $G_{\mathrm{dRB}}(E)$ is two-dimensional and equal to $\mathrm{MT}(E)$. If E does not have CM, then $\mathrm{MT}(E) = \mathrm{GL}(V)$, where $V = \mathrm{H}^1_{\mathrm{B}}(E,\mathbb{Q})$. In this case, we have $\mathrm{Hdg}(E) = \mathrm{SL}(V)$ and $G^1_{\mathrm{dRB}}(E) \subseteq \mathrm{SL}(V)$ is a connected reductive subgroup. It follows that up to conjugation $G^1_{\mathrm{dRB}}(E) \subseteq \mathrm{SL}(V)$ is either a maximal torus or equal to $\mathrm{SL}(V)$. In the first case, we get a two-dimensional space of invariants in $\mathrm{End}(V)$ which violates the assumption that $\mathrm{End}(E)_{\mathbb{Q}} = \mathbb{Q}$. Hence $G^1_{\mathrm{dRB}}(E) = \mathrm{SL}(V)$ and thus $G_{\mathrm{dRB}}(E) = \mathrm{GL}(V) = \mathrm{MT}(E)$.

Step 2 : $A = E_1^{n_1} \times E_2^{n_2} \times \cdots \times E_r^{n_r}$, for pairwise non-isogenous CM elliptic curves E_i. The fact that $\det(\mathrm{H}^1_{\mathrm{dRB}}(E_i,\mathbb{Q})) \cong \mathbb{Q}(-1)$ for all i implies that

$$G^1_{\mathrm{dRB}}(A) \subseteq G^1_{\mathrm{dRB}}(E_1) \times G^1_{\mathrm{dRB}}(E_2) \times \cdots \times G^1_{\mathrm{dRB}}(E_r) \tag{6}$$

and that $G^1_{\mathrm{dRB}}(A)$ surjects onto each factor $G^1_{\mathrm{dRB}}(E_i) = \mathrm{U}_{K_i}$ with $K_i = \mathrm{End}(E_i)_{\mathbb{Q}}$. By Goursat's lemma [Gor99, Prop. B.71.1] and the fact that the U_{K_i} are pairwise non-isogenous, we conclude that the above inclusion is an identity. It follows from Proposition 7.13 that $G^1_{\mathrm{dRB}}(A) = \mathrm{Hdg}(A)$.

Step 3 : $A = E_1^{n_1} \times E_2^{n_2}$, where E_1 and E_2 are non-isogenous elliptic curves without CM. Due to the lack a theory of weights in the de Rham–Betti setting, we are unable to use the usual argument for the Mumford–Tate group as in [Gor99, §3]. We may assume that $n_1 = n_2 = 1$. Let $V_i := \mathrm{H}^1_{\mathrm{B}}(E_i,\mathbb{Q})$, $i = 1, 2$. Then we know that $\mathrm{MT}(E_i) = \mathrm{GL}(V_i)$, $i = 1, 2$. Consider the inclusion $G_{\mathrm{dRB}}(A) \subseteq \mathrm{MT}(A)$ at the level of Lie algebras. We get

$$\mathfrak{g} = \mathfrak{g}^{ss} \oplus \mathfrak{z}(\mathfrak{g}) \hookrightarrow \mathfrak{mt} = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \oplus \mathfrak{c},$$

where $\mathfrak{g} := \mathrm{Lie}\, G_{\mathrm{dRB}}(A)$ is the Lie algebra of $G_{\mathrm{dRB}}(A)$, $\mathfrak{z}(\mathfrak{g})$ corresponds to the center of $G_{\mathrm{dRB}}(A)$ and $\mathfrak{c}$ is a one-dimensional commutative Lie algebra. The fact that $G_{\mathrm{dRB}}(A)$ surjects onto $G_{\mathrm{dRB}}(E_i)$ implies that $\mathfrak{g}^{ss} \hookrightarrow \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ surjects onto both factors. This implies $\mathfrak{z}(\mathfrak{g}) \hookrightarrow \mathfrak{c}$. Furthermore, by Goursat's lemma [Gor99, Prop. B.71.2], we have either $\mathfrak{g}^{ss} \simeq \mathfrak{sl}_2$, embedded in $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ as the graph of an isomorphism, or $\mathfrak{g}^{ss} = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$. If $\mathfrak{g}^{ss} \simeq \mathfrak{sl}_2$, then $\mathrm{H}^1_{\mathrm{B}}(A,\mathbb{Q}) = V_1 \oplus V_2$ becomes the direct sum of two copies of the standard representation of $\mathfrak{sl}_2$. As a consequence, the space of $G_{\mathrm{dRB}}(A)$-invariants of $\mathrm{End}(\mathrm{H}^1_{\mathrm{B}}(A,\mathbb{Q}))$ is of dimension 4, which contradicts the assumption that E_1 and E_2 are not isogenous. Hence $\mathfrak{g}^{ss} = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$. The fact that $G_{\mathrm{dRB}}(A)$ surjects onto $\mathbb{G}_m$ implies that $\mathfrak{z}(\mathfrak{g}) = \mathfrak{c}$. It follows that $G_{\mathrm{dRB}}(A) = \mathrm{MT}(A)$.

Step 4 : $A = E_1^{n_1} \times E_2^{n_2} \times \cdots \times E_r^{n_r}$ for pairwise non-isogenous non-CM elliptic curves E_i. In this case, by Step 3, the inclusion (6) surjects onto the product of each pair of factors. Then one can apply a form of Goursat's lemma [Gor99, Prop. B.71.3].

Step 5 : $A = B \times C$ where B is a product of non-CM elliptic curves and C is a product of CM elliptic curves. In this case, we apply Goursat's lemma to $G^1_{\mathrm{dRB}}(A) \subset G^1_{\mathrm{dRB}}(B) \times G^1_{\mathrm{dRB}}(C)$ and note that $G^1_{\mathrm{dRB}}(B)$ is semi-simple and $G^1_{\mathrm{dRB}}(C)$ is a torus. □

Proof of Theorem 7.12(ii). By Lemma 6.14, we may and do assume that $K = \overline{\mathbb{Q}}$. By Proposition 7.13, we only need to show that $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}(A) = G_{\mathrm{And}}(A)_{\overline{\mathbb{Q}}}$. This holds for A a non-CM elliptic curve by the exact same argument as in the proof of Theorem 7.12(i). For A a CM elliptic curve, this holds thanks to Chudnovsky's theorem [Chu80]. The proof then follows from the observation that Steps 3 and 4, as well as Step 5 with C the power of a single CM elliptic curve, of the proof of Theorem 7.12(i) carry over to the setting of $\overline{\mathbb{Q}}$-de Rham–Betti classes. □

Remark 7.14. The $\overline{\mathbb{Q}}$-de Rham–Betti conjecture remains open for the product of two non-isogenous CM elliptic curves. In that situation, the arguments in Step 2 of the proof of Theorem 7.12(i) do not carry over as tori over $\overline{\mathbb{Q}}$ are merely classified by their rank.

7.5. **The case of abelian surfaces.** We prove the following equivalent (by the motivic analogue of Proposition 6.12) version of Theorem 1 in the case of powers of abelian surfaces with non-trivial endomorphism ring.

Theorem 7.15. *Let A be an abelian surface over K such that $\mathrm{End}(A_{\overline{\mathbb{Q}}}) \otimes \mathbb{Q} \neq \mathbb{Q}$.*

(i) The de Rham–Betti conjecture (Conjecture 6.10(ii)) holds for $\mathfrak{h}(A)$, i.e.,

$$\Omega_A^{\mathrm{dRB}} = \Omega_A^{\mathrm{mot}}.$$

In particular, by Proposition 6.12, for any $n \geq 0$ and any $k \in \mathbb{Z}$, any de Rham–Betti class on $\mathfrak{h}(A^n)(k)$ is algebraic.

(ii) If in addition A is simple without CM, then the $\overline{\mathbb{Q}}$-de Rham–Betti conjecture (Conjecture 6.13(ii)) holds for $\mathfrak{h}(A)$, i.e.,

$$\Omega_A = \Omega_A^{\mathrm{mot}}.$$

In particular, by Proposition 6.16, for any $n \geq 0$ and any $k \in \mathbb{Z}$, any $\overline{\mathbb{Q}}$-de Rham–Betti class on $\mathfrak{h}(A^n_{\overline{\mathbb{Q}}})(k)$ is a $\overline{\mathbb{Q}}$-linear combination of algebraic classes.

Proof of Theorem 7.15(i). By Lemma 6.11, we may and do assume that $K = \overline{\mathbb{Q}}$. The case where A is isogenous to a product of elliptic curves is handled by Theorem 7.12. We therefore assume that A is a simple abelian surface over $\overline{\mathbb{Q}}$. We recall, e.g. from [MZ99, (2.2)], that there are the following four possibilities for the endomorphism ring $\mathrm{End}(A)_{\mathbb{Q}}$:

(a) $\mathrm{End}(A)_{\mathbb{Q}} = \mathbb{Q}$;
(b) $\mathrm{End}(A)_{\mathbb{Q}}$ is a real quadratic extension of $\mathbb{Q}$;
(c) $\mathrm{End}(A)_{\mathbb{Q}}$ is a quaternion algebra with center $\mathbb{Q}$ which splits over $\mathbb{R}$;
(d) $\mathrm{End}(A)_{\mathbb{Q}}$ is a CM field of degree 4 over $\mathbb{Q}$ which does not contain an imaginary quadratic subfield.

We prove that the inclusion $G_{\mathrm{dRB}}(A) \subseteq \mathrm{MT}(A)$ is an equality when $\mathrm{End}(A)_{\mathbb{Q}} \neq \mathbb{Q}$. We distinguish the different cases for $\mathrm{End}(A)_{\mathbb{Q}}$.

Case (b): Suppose $k := \mathrm{End}(A)_{\mathbb{Q}}$ is a real quadratic field. Then $\mathrm{MT}(A) \subseteq \mathrm{Res}_{k/\mathbb{Q}}\,\mathrm{GL}_{2,k}$ is the subgroup with R-points given by

$$\mathrm{MT}(A)(R) = \{g \in \mathrm{GL}_2(k \otimes_{\mathbb{Q}} R) \,|\, \det g \in R^\times\}.$$

Denote by $\mathfrak{g}$ the Lie algebra of $G_{\mathrm{dRB}}(A)$. Write the reductive Lie algebra $\mathfrak{g} = \mathfrak{g}^{ss} \oplus \mathfrak{z}(\mathfrak{g})$ as the sum of a semi-simple Lie algebra and an abelian Lie algebra. On the other hand, the Lie algebra of $\mathrm{MT}(A)_k$ is $\mathfrak{mt}_k = \mathfrak{sl}_{2,k} \oplus \mathfrak{sl}_{2,k} \oplus \mathfrak{c}$, where $\mathfrak{c}$ is a one-dimensional abelian Lie algebra. Then $\mathfrak{g}_k^{ss} := \mathfrak{g}^{ss} \otimes_{\mathbb{Q}} k \subseteq \mathfrak{sl}_{2,k} \oplus \mathfrak{sl}_{2,k}$ and the semi-simple part is non-trivial since otherwise $G_{\mathrm{dRB}}(A)$ would be a torus, contradicting Theorem 7.11(iv). Since $\mathfrak{g}^{ss}$ is defined over $\mathbb{Q}$, the k-Lie algebra $\mathfrak{g}_k^{ss}$ is invariant under the non-trivial automorphism $\sigma \in \mathrm{Gal}(k/\mathbb{Q})$ which acts on $\mathfrak{sl}_{2,k} \oplus \mathfrak{sl}_{2,k}$ by conjugating and swapping both factors. Consequently, $\mathfrak{g}_k^{ss}$ surjects onto both factors. Thus either

$$\mathfrak{g}_k^{ss} = \mathfrak{sl}_{2,k} \oplus \mathfrak{sl}_{2,k} \quad \text{or} \quad \mathfrak{g}_k^{ss} \cong \mathfrak{sl}_{2,k} \subset \mathfrak{sl}_{2,k} \oplus \mathfrak{sl}_{2,k}$$

is the graph of an automorphism of $\mathfrak{sl}_{2,k}$. We have to exclude the latter. Using Theorem 7.11(iii) it follows that $\mathfrak{z}(\mathfrak{g}) = \mathfrak{c}$ and thus $\mathfrak{g}_k \cong \mathfrak{gl}_{2,k}$, where the action on $\mathrm{H}^1_{\mathrm{B}}(A, \mathbb{Q}) \otimes_{\mathbb{Q}} k$ is isomorphic to the direct sum of two copies of the standard representation. But then the dimension of $\mathrm{End}_{\mathfrak{g}_k}(\mathrm{H}^1_{\mathrm{B}}(A,\mathbb{Q})\otimes_{\mathbb{Q}} k)$ is four, which is different from $\dim_{\mathbb{Q}} \mathrm{End}(A)_{\mathbb{Q}} = 2$. It follows that $\mathfrak{g}_k^{ss} = \mathfrak{sl}_{2,k} \oplus \mathfrak{sl}_{2,k}$. By using Theorem 7.11(*iii*) once more we conclude $\mathfrak{g}_k = \mathfrak{mt}_k$.

Case (c): In this case, $G_{\mathrm{dRB}}(A)_{\mathbb{C}} \subseteq \mathrm{MT}(A)_{\mathbb{C}} \cong \mathrm{GL}_{2,\mathbb{C}}$. Since $G_{\mathrm{dRB}}(A)$ is not a torus by Theorem 7.11(*iv*), the group $G_{\mathrm{dRB}}(A)_{\mathbb{C}}$ is either $\mathrm{SL}_{2,\mathbb{C}}$ or $\mathrm{GL}_{2,\mathbb{C}}$. But $G_{\mathrm{dRB}}(A)$ admits a non-trivial map to $\mathbb{G}_m$ by Theorem 7.11(*iii*), so we have $G_{\mathrm{dRB}}(A)_{\mathbb{C}} = \mathrm{GL}_{2,\mathbb{C}}$.

Case (d): Suppose $F := \mathrm{End}(A)_{\mathbb{Q}}$ is a CM field of degree 4 over $\mathbb{Q}$, which does not contain an imaginary quadratic subfield. In this case, $T := \mathrm{MT}(A) \subseteq \mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_m$ is the torus whose points in a $\mathbb{Q}$-algebra R are given by

$$T(R) = \{g \in (F \otimes_{\mathbb{Q}} R)^{\times} \mid g\bar{g} \in R^{\times}\}.$$

The torus T is determined up to isogeny by the rational vector space of ($\overline{\mathbb{Q}}$-)cocharacters $X_*(T)_{\mathbb{Q}}$ equipped with its $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action. We have the short exact sequence

$$1 \to U_F \to T \to \mathbb{G}_m \to 1,$$

where U_F is the unitary torus given on R-points by

$$U_F(R) = \{g \in (F \otimes_{\mathbb{Q}} R)^{\times} \mid g\bar{g} = 1\}.$$

This gives rise to the short exact sequence of $\mathbb{Q}$-vector spaces with $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action

$$0 \to X_*(U_F)_{\mathbb{Q}} \to X_*(T)_{\mathbb{Q}} \to X_*(\mathbb{G}_m)_{\mathbb{Q}} = \mathbb{Q} \to 0,$$

where the Galois action on the rightmost term is trivial.

Let $T_{\mathrm{dRB}} := G_{\mathrm{dRB}}(A) \subseteq T$, which is a torus by Theorem 7.11(*iv*). It is enough to prove that the inclusion $X_*(T_{\mathrm{dRB}})_{\mathbb{Q}} \subseteq X_*(T)_{\mathbb{Q}}$ is an equality. By Theorem 7.11(*iii*), the group $X_*(T_{\mathrm{dRB}})_{\mathbb{Q}}$ surjects onto $X_*(\mathbb{G}_m)_{\mathbb{Q}}$. The kernel is a Galois-stable subspace of $X_*(U_F)_{\mathbb{Q}}$.

Claim. *There is no non-trivial Galois-stable subspace of $X_*(U_F)_{\mathbb{Q}}$.*

Proof of Claim. Let $\Sigma_F := \mathrm{Hom}(F,\mathbb{C}) = \{\phi_1, \bar{\phi}_1, \phi_2, \bar{\phi}_2\}$ be the set of embeddings of the CM field F into the complex numbers. Then $X^*(\mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_m)_{\mathbb{Q}}$ is naturally identified with the $\mathbb{Q}$-vector space with basis Σ_F with its natural $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action, and we denote by $\{\phi_1^\vee, \bar{\phi}_1^\vee, \phi_2^\vee, \bar{\phi}_2^\vee\}$ the associated dual basis of $X_*(\mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_m)_{\mathbb{Q}}$.

With this notation,

$$X_*(U_F)_{\mathbb{Q}} = \langle \phi_1^\vee - \bar{\phi}_1^\vee, \phi_2^\vee - \bar{\phi}_2^\vee\rangle_{\mathbb{Q}}. \tag{7}$$

Assume there exists a one-dimensional Galois-stable subspace $L \subseteq X_*(U_F)_{\mathbb{Q}}$. As $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts transitively on Σ_F, we know that

$$L \neq \langle \phi_1^\vee - \bar{\phi}_1^\vee\rangle_{\mathbb{Q}} \quad \text{and} \quad L \neq \langle \phi_2^\vee - \bar{\phi}_2^\vee\rangle_{\mathbb{Q}}. \tag{8}$$

Denote by F^g the Galois closure of F. Then $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on L through a character

$$\chi : \mathrm{Gal}(F^g/\mathbb{Q}) \to \mathbb{Z}/2\mathbb{Z}.$$

Let $K := (F^g)^{\ker \chi}$. As F is a CM field, the Galois group $\mathrm{Gal}(F^g/F)$ fixes one pair of complex embeddings $(\phi_i, \bar{\phi}_i)$. Together with (8), we get that $\mathrm{Gal}(F^g/F)$ acts trivially on L, and hence $K \subseteq F$.

Since F does not contain a totally imaginary quadratic subfield, $K \subset F$ is totally real. Consequently, complex conjugation acts trivially on the target of

$$\Sigma_F = \mathrm{Hom}(F,\mathbb{C}) \twoheadrightarrow \mathrm{Hom}(K,\mathbb{C}).$$

It follows that there exist $\sigma_1, \sigma_2 \in \mathrm{Gal}(F^g/K)$ such that $\bar{\phi}_1 = \sigma_1\phi_1$ and $\bar{\phi}_2 = \sigma_2\phi_2$. Applying these relations to the generators in (7), we see that there is no non-zero element of $X_*(U_F)_{\mathbb{Q}}$ on which both σ_1 and σ_2 act trivially. $\square$

The claim shows that either $X_*(T_{\mathrm{dRB}})_{\mathbb{Q}} = X_*(T)_{\mathbb{Q}}$, in which case we are done, or

$$X_*(T_{\mathrm{dRB}})_{\mathbb{Q}} \cong X_*(\mathbb{G}_m)_{\mathbb{Q}}.$$

As $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts transitively on Σ_F, there is a unique one-dimensional subspace of $X_*(T)_{\mathbb{Q}}$ on which $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts trivially. It corresponds to the scalar matrices $\mathbb{G}_m \subseteq T$. Since by Theorem 7.11(ii), T_{dRB} cannot just consist of scalar matrices, we conclude that $X_*(T_{\mathrm{dRB}})_{\mathbb{Q}} = X_*(T)_{\mathbb{Q}}$. This shows $T_{\mathrm{dRB}} = T$, as desired. $\square$

Proof of Theorem 7.15(ii). By Lemma 6.14, we may and do assume that $K = \overline{\mathbb{Q}}$. We have to deal with cases (b) and (c). In case (c) one proves that the inclusion $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}(A) \subseteq \mathrm{MT}(A)_{\overline{\mathbb{Q}}}$ is an equality as in the proof of Theorem 7.15(i) by using the second part of Theorem 7.11 instead of the first. As for case (b), denote by $\mathfrak{g}$ the Lie algebra of $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}(A)$; its semi-simple part $\mathfrak{g}^{ss}$ is a Lie subalgebra of $\mathfrak{mt}^{ss}_{\overline{\mathbb{Q}}} = \mathfrak{sl}_{2,\overline{\mathbb{Q}}} \oplus \mathfrak{sl}_{2,\overline{\mathbb{Q}}}$. First we show that $\mathfrak{g}^{ss}$ surjects onto both factors. Here we may no longer use the Galois argument as in the proof of Theorem 7.15(i). We argue instead as follows. We have the inclusion

$$G_{\overline{\mathbb{Q}}-\mathrm{dRB}}(A) \subseteq \mathrm{MT}(A)_{\overline{\mathbb{Q}}} = \mathbb{G}_{m,\overline{\mathbb{Q}}} \cdot \left(\mathrm{SL}_{2,\overline{\mathbb{Q}}} \times \mathrm{SL}_{2,\overline{\mathbb{Q}}}\right) \subset \mathrm{GL}_{2,\overline{\mathbb{Q}}} \times \mathrm{GL}_{2,\overline{\mathbb{Q}}}\,.$$

Here, the action of $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}(A)$ on $\mathrm{H}^1_{\mathrm{B}}(A, \overline{\mathbb{Q}})$ is via the action of $\mathrm{GL}_{2,\overline{\mathbb{Q}}} \times \mathrm{GL}_{2,\overline{\mathbb{Q}}}$ on two copies of the standard representation. We claim that the projections of $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}(A)$ on both $\mathrm{GL}_{2,\overline{\mathbb{Q}}}$-factors contains $\mathrm{SL}_{2,\overline{\mathbb{Q}}}$. If not, since $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}(A)$ is connected and reductive, then up to switching the factors we would have an inclusion $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}(A) \subseteq \mathrm{GL}_{2,\overline{\mathbb{Q}}} \times \mathrm{T}$ for some maximal torus T inside $\mathrm{GL}_{2,\overline{\mathbb{Q}}}$. But then the space of invariants in $\mathrm{End}(\mathrm{H}^1_{\mathrm{B}}(A, \overline{\mathbb{Q}}))$ under $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}(A)$ would have dimension at least 3, which contradicts Theorem 7.11(ii') and $\mathrm{End}(A) \otimes \overline{\mathbb{Q}} = \overline{\mathbb{Q}} \oplus \overline{\mathbb{Q}}$. We conclude that $\mathfrak{g}^{ss}$ surjects onto both $\mathfrak{sl}_{2,\overline{\mathbb{Q}}}$-factors. We can now argue as in the proof of Theorem 7.15(i) by using the second part of Theorem 7.11 instead of the first and conclude that $\mathfrak{g} = \mathfrak{mt}_{\overline{\mathbb{Q}}}$. $\square$

Remark 7.16. It seems that, in the case of an abelian surface A over $\overline{\mathbb{Q}}$ with $\mathrm{End}(A)_{\mathbb{Q}} = \mathbb{Q}$, neither the properties established in Theorem 7.11 nor Proposition 9.8 below applied to the Kummer surface associated to A are sufficient to determine $G_{\mathrm{dRB}}(A)$ uniquely. In this case, the Mumford–Tate group is $\mathrm{MT}(A) = \mathrm{GSp}_4$. We do not know how to exclude the possibility that the semi-simple part $\mathfrak{g}^{ss} \subseteq \mathfrak{sp}_4$ of the Lie algebra of $G_{\mathrm{dRB}}(A)$ is $\mathfrak{sl}_2$, embedded via the third symmetric power of the standard representation.

8. The Kuga–Satake correspondence

We review the Kuga–Satake construction and André's proof [And96a] that the Kuga–Satake correspondence is motivated and descends. The purpose is to fix notation and to state Theorem 8.3, which will be used in our proof of Theorem 3 ; see Theorem 9.5.

8.1. **The Kuga–Satake construction.** Let R be a commutative ring, *e.g.* $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$. Let V_R be a free R-module of finite rank equipped with a symmetric bilinear form

$$Q : V_R \times V_R \longrightarrow R.$$

The associated *Clifford algebra* $C(V_R)$ is the quotient of the tensor algebra $T(V_R)$ by the two-sided ideal generated by $v \otimes v - Q(v, v)$, for all $v \in V_R$; it admits a natural $\mathbb{Z}/2\mathbb{Z}$-grading and we have

$$C(V_R) = C^+(V_R) \oplus C^-(V_R).$$

The sub-algebra $C^+(V_R)$ is called the *even Clifford algebra*. The *even Clifford group* is

$$\mathrm{CSpin}^+(V_R) =_{\mathrm{def}} \{\gamma \in C^+(V_R)^\times \mid \gamma V \gamma^{-1} = V\}.$$

Each element $\gamma \in \mathrm{CSpin}^+(V_R)$ induces an R-linear homomorphism $\tau(\gamma) : V_R \longrightarrow V_R$, $v \mapsto \gamma v \gamma^{-1}$. One easily checks that the homomorphism $\tau(\gamma)$ preserves the bilinear form Q and hence provides a group homomorphism

$$\tau : \mathrm{CSpin}^+(V_R) \longrightarrow \mathrm{SO}(V_R).$$

The above constructions are compatible with scalar extensions $R \to R'$. As such $\mathrm{CSpin}^+(V_R)$ has the natural structure of an algebraic group over R making τ a surjective homomorphism of R-groups with kernel given by $\mathbb{G}_{m,R}$.

We now consider the special case $R = \mathbb{Z}$. Assume that $V_\mathbb{Z}$ is equipped with a Hodge structure of K3 type, that is, a Hodge structure of type $(-1,1)+(0,0)+(1,-1)$ such that $\dim V^{-1,1} = 1$. Assume that the integral bilinear form $Q : V_\mathbb{Z} \otimes V_\mathbb{Z} \longrightarrow \mathbb{Z}$ is a homomorphism of Hodge structures such that $-Q$ is a polarization. Then the Hodge structure of $V_\mathbb{Z}$ is given by

$$h : \mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{C}^*) \longrightarrow \mathrm{SO}(V_\mathbb{R})$$

and it lifts uniquely to a homomorphism

$$\tilde{h} : \mathbb{S} \longrightarrow \mathrm{CSpin}^+(V_\mathbb{R})$$

such that the restriction of $\tilde{h}$ to $\mathbb{G}_m$ is the natural inclusion of $\mathbb{G}_m$ in $\mathrm{CSpin}^+(V_\mathbb{R})$. The action of $\mathrm{SO}(V_\mathbb{R})$ on the tensor algebra $T(V_\mathbb{R})$ induces an action on the quotient $C(V_\mathbb{R})$ and hence also on $C^+(V_\mathbb{R})$. Thus h induces a Hodge structure of weight 0 on $C^+(V_\mathbb{Z})$. This Hodge structure also has type $(-1,1)+(0,0)+(1,-1)$.

Now let $L_\mathbb{Z}$ be a (left) $C^+(V_\mathbb{Z})$-module that is torsion-free as a $\mathbb{Z}$-module such that

$$L := L_\mathbb{Z} \otimes \mathbb{Q}$$

is a free $C^+(V)$-module of rank r, where $V := V_\mathbb{Z} \otimes \mathbb{Q}$. Then $L_\mathbb{Z}$ is naturally a $\mathrm{CSpin}^+(V_\mathbb{Z})$-module via left multiplication. After base change to $\mathbb{R}$ and composing with $\tilde{h}$, we get a group homomorphism

$$\mathbb{S} \longrightarrow \mathrm{GL}(L_\mathbb{R})$$

that defines a weight one Hodge structure on $L_\mathbb{Z}$.

The assumption that $V_\mathbb{Z}$ is of $K3$-type and that $-Q$ is a polarization implies that Q has signature $(2,N)$. Under these assumptions, the weight 1 Hodge structure on $L_\mathbb{Z}$ is polarizable and hence we get an abelian variety A such that $\mathrm{H}^1(A,\mathbb{Z}) = L_\mathbb{Z}$. Concretely, when $r = 1$, a polarization on L can be given as follows. Pick a generator of L and consider the induced identifications $C^+(V) = L$ of $\mathbb{Q}$-vector spaces. Satake [Sat66, Prop. 3] gave all possible polarizations and one special instance can be described as follows. Pick f_1 and f_2 in V with $Q(f_1,f_1) > 0$, $Q(f_2,f_2) > 0$ and $Q(f_1,f_2) = 0$ and let $a = f_1 f_2 \in C^+(V)$. Then the skew symmetric bilinear form

$$\varphi_a : L \times L \longrightarrow \mathbb{Q}(-1), \quad (x,y) \mapsto \pm \mathrm{tr}_{C(V)}(a x^* y)$$

defines a polarization on L. We note that the formula above also defines a polarization on the free $C^+(V)$-module $L = C(V)$ of rank $r = 2$. In the special case where $L_\mathbb{Z}$ is a free $C^+(V_\mathbb{Z})$-module of rank 1, the associated abelian variety is called the *Kuga–Satake variety* attached to $V_\mathbb{Z}$ and is denoted $\mathrm{KS}(V_\mathbb{Z})$; it has dimension $2^{\mathrm{rk}\, V_\mathbb{Z} - 2}$.

Finally, let C^+ be the opposite ring $(\mathrm{End}_{C^+(V)} L)^{\mathrm{op}}$ of the ring of $C^+(V)$-linear endomorphisms of L. Note that L admits a right-action of C^+ that respects the Hodge structure of L ; in particular, C^+ is endowed with the trivial Hodge structure. More precisely, C^+ is a subring of $\mathrm{End}(A)_\mathbb{Q}$ with action on L given by pull back of cohomology classes. The choice of a basis of L provides an isomorphism $L \simeq C^+(V)^{\oplus r}$ and also an induced isomorphism $C^+ \simeq \mathrm{M}_{r \times r}(C^+(V))$ of algebras (but not of Hodge structures). Note that left-multiplication by $C^+(V)$ provides a canonical algebra isomorphism

$$\psi : C^+(V) \cong \mathrm{End}_{C^+} L$$

which is also an isomorphism of Hodge structures.

8.2. **The Kuga–Satake construction and complex hyper-Kähler varieties.** We compare the Kuga–Satake varieties obtained from the primitive, or transcendental, second cohomology group of a complex hyper-Kähler variety equipped with a polarization form or the Beauville–Bogomolov form.

Example 8.1 (The Kuga–Satake construction for complex hyper-Kähler varieties)**.** Let $X/\mathbb{C}$ be a hyper-Kähler variety of dimension $2n$ with ample divisor class η. The second cohomology group $H := \mathrm{H}^2_{\mathrm{B}}(X, \mathbb{Z}(1))$ is naturally endowed with a Hodge structure of type $(-1,1)+(0,0)+(1,-1)$ such that $\dim H^{-1,1} = 1$ and it comes equipped with two quadratic forms : the one induced by the polarization η, namely $(\alpha,\beta) \mapsto \int_X \alpha \cup \beta \cup \eta^{2n-2}$, and the Beauville–Bogomolov form

$$\mathfrak{B} : \mathrm{H}^2_{\mathrm{B}}(X, \mathbb{Z}(1)) \times \mathrm{H}^2_{\mathrm{B}}(X, \mathbb{Z}(1)) \longrightarrow \mathbb{Z}.$$

We denote $\mathrm{P}^2_{\mathrm{B}}(X, \mathbb{Z}(1))$ the primitive cohomology of X ; by definition it is the orthogonal complement of η in $\mathrm{H}^2_{\mathrm{B}}(X, \mathbb{Z}(1))$ with respect to the polarization form. It is a fact that η and $\mathrm{P}^2_{\mathrm{B}}(X, \mathbb{Z}(1))$ are also orthogonal with respect to the Beauville–Bogomolov form, and that the polarization form and the Beauville–Bogomolov form restricted to $\langle \eta \rangle$ (resp. to $\mathrm{P}^2_{\mathrm{B}}(X, \mathbb{Z}(1))$) differ by a positive rational multiple. Since the restriction of the polarization form on $\mathrm{P}^2_{\mathrm{B}}(X, \mathbb{Z}(1))$ has signature $(2, N)$, the Kuga–Satake construction of §8.1 can be carried out for $V_{\mathbb{Z}} = \mathrm{P}^2_{\mathrm{B}}(X, \mathbb{Z}(1))$ equipped with the Beauville–Bogomolov bilinear form. The corresponding Kuga–Satake variety is denoted $\mathrm{KS}(X)$. The Kuga–Satake variety constructed in [And96a] is the one obtained from $V_{\mathbb{Z}} = \mathrm{P}^2_{\mathrm{B}}(X, \mathbb{Z}(1))$ equipped with the polarization form. This gives however rise to isogenous Kuga–Satake varieties. Indeed, let $Q' = cQ$, $c \in \mathbb{Q}_{>0}$. Then we can define an isomorphism $T^+V \to T^+V$, which is multiplication by c^k on $T^{2k}V$. Then this induces an isomorphism between $C^+(V_{\mathbb{Q}}, Q')$ and $C^+(V_{\mathbb{Q}}, Q)$ and hence an isogeny between the associated Kuga–Satake varieties.

Example 8.2 (The transcendental cohomology of a hyper-Kähler variety)**.** Let $X/\mathbb{C}$ be a hyper-Kähler variety of dimension $2n$ with ample divisor class η. Denote $\mathrm{NS}(X)$ the image of the cycle class map $\mathrm{CH}^1(X) \to \mathrm{H}^2_{\mathrm{B}}(X, \mathbb{Z}(1))$ and denote ρ its rank. The second transcendental cohomology group of X is by definition the orthogonal complement of $\mathrm{NS}(X)$ inside $\mathrm{H}^2_{\mathrm{B}}(X, \mathbb{Z}(1))$, that is,

$$\mathrm{T}^2_{\mathrm{B}}(X, \mathbb{Z}(1)) =_{\mathrm{def}} \mathrm{NS}(X)^{\perp}.$$

Here the orthogonal complement is with respect to either the polarization form or the Beauville–Bogomolov form ; it doesn't matter which, according to Example 8.1. Moreover, the Beauville–Bogomolov form and the polarization form differ by a positive rational multiple when restricted to $\mathrm{T}^2_{\mathrm{B}}(X, \mathbb{Z}(1))$ and have signature $(2, M)$. The Kuga–Satake variety of X is then isogenous to the 2^{ρ}-th power of the Kuga–Satake variety associated to $\mathrm{T}^2_{\mathrm{B}}(X, \mathbb{Z}(1))$.

8.3. **The Kuga–Satake correspondence is motivated.** In this section, we discuss the above Kuga–Satake construction in the setting of André motives. We focus on the special case of hyper-Kähler varieties.

Let X be a hyper-Kähler variety defined over a field $K \subseteq \mathbb{C}$, together with an ample divisor class η. Denote by $\mathfrak{v} := \mathfrak{p}^2(X)(1) \subset \mathfrak{h}^2(X)(1)$ (resp. $\mathfrak{v} := \mathfrak{t}^2(X)(1) \subset \mathfrak{h}^2(X)(1)$) the primitive submotive (resp. transcendental submotive), whose Betti realization is simply $V := \mathrm{P}^2_{\mathrm{B}}(X, \mathbb{Q}(1))$ (resp. $V = \mathrm{T}^2_{\mathrm{B}}(X, \mathbb{Q}(1))$). The Beauville–Bogomolov bilinear form restricted to V is a rational multiple of the one defined by the power of η and hence it is motivated. In other words, we have a morphism

$$Q : \mathfrak{v} \otimes \mathfrak{v} \longrightarrow \mathbb{1}$$

of André motives that induces the Beauville–Bogomolov form $\mathfrak{B}$ under Betti realization.

Now we define the even Clifford algebra motive $C^+(\mathfrak{v}, Q)$. Let

$$\mu = (\mathrm{Id} + \sigma, -2Q) : \mathfrak{v} \otimes \mathfrak{v} \longrightarrow \mathfrak{v} \otimes \mathfrak{v} \oplus \mathbb{1}$$

be the morphism of André motives, where σ is the automorphism of $\mathfrak{v}\otimes\mathfrak{v}$ that switches the two factors. In other words, the Betti realization of μ is given by $(u,v)\mapsto u\otimes v+v\otimes u-2\mathfrak{B}(u,v)$. For each pair of integers $a,b\geq 0$, we define

$$\mu_{a,b}=\mathrm{Id}_{\mathfrak{v}^{\otimes a}}\otimes\mu\otimes\mathrm{Id}_{\mathfrak{v}^{\otimes b}}:\mathfrak{v}^{\otimes(a+b+2)}\longrightarrow\mathfrak{v}^{\otimes(a+b+2)}\oplus\mathfrak{v}^{\otimes(a+b)}.$$

Let $T^+(\mathfrak{v}):=\bigoplus_{n\geq 0}\mathfrak{v}^{\otimes 2n}$ be the even tensor algebra of $\mathfrak{v}$, which is viewed as a formal direct sum. Let $\mathfrak{i}\subset T^+(\mathfrak{v})$ be the image of $\sum\mu_{a,b}$ where a and b run through all non-negative integers such that $a+b$ is even. It turns out that $C^+(\mathfrak{v},Q):=T^+(\mathfrak{v})/\mathfrak{i}$ is a well-defined object in $\mathsf{M}_K^{\mathrm{And}}$. Indeed, let $T_n^+(\mathfrak{v})=\bigoplus_{0\leq i\leq n}\mathfrak{v}^{\otimes 2i}$. Then we have a morphism $T_n^+(\mathfrak{v})\to C^+(\mathfrak{v},Q)$ and let $\mathfrak{i}_n$ be the kernel of this morphism. One checks that $C^+(\mathfrak{v},Q)=T_n^+(\mathfrak{v})/\mathfrak{i}_n$ for all $n\geq\frac{\dim V}{2}$. Let $F_nC^+(\mathfrak{v},Q)$ be the image of $T_n^+(\mathfrak{v})$ in $C^+(\mathfrak{v},Q)$ and this defines an increasing filtration on $C^+(\mathfrak{v},Q)$.

Let A be the Kuga–Satake abelian variety associated to $\mathrm{P}^2_{\mathrm{B}}(X,\mathbb{Z}(1))$ (resp. $\mathrm{T}^2_{\mathrm{B}}(X,\mathbb{Z}(1))$). Then, by [And96a, Lem. 6.5.1], A is defined over some finite extension K' of K; see Theorem 8.3(1) below. Hence L is the Betti realization of $\mathfrak{h}^1(A)\in\mathsf{M}_{K'}^{\mathrm{And}}$.

Note that $C^+\subset\mathrm{End}(A_{\mathbb{C}})$. We may enlarge K' and assume that all endomorphisms of $A_{\mathbb{C}}$ are actually defined over K'. This gives rise to a subalgebra object $\underline{C_{\mathbb{Q}}^+}\subset\underline{\mathrm{End}}(\mathfrak{h}^1(A))$, where $\underline{C_{\mathbb{Q}}^+}$ is the motive associated to the algebra $C_{\mathbb{Q}}^+$ and $\underline{\mathrm{End}}(\mathfrak{h})=\mathfrak{h}^\vee\otimes\mathfrak{h}$ is the internal endomorphism object. Hence we get the subalgebra object

$$\underline{\mathrm{End}}_{C^+}(\mathfrak{h}^1(A))\subset\underline{\mathrm{End}}(\mathfrak{h}^1(A)).$$

Let M and N be two André motives. A homomorphism $\mathrm{H}^*_B(M)\longrightarrow\mathrm{H}^*_B(N)$ is said to be *motivated* if it is induced by a morphism $M\longrightarrow N$ of motives. The following theorem is essentially due to André [And96a].

Theorem 8.3. *Let X be a hyper-Kähler variety defined over a field $K\subseteq\mathbb{C}$ with an ample class η. Let $V_{\mathbb{Z}}:=\mathrm{P}^2_{\mathrm{B}}(X,\mathbb{Z}(1))$ (resp. $V_{\mathbb{Z}}:=\mathrm{T}^2_{\mathrm{B}}(X,\mathbb{Z}(1))$) be the primitive cohomology (resp. the transcendental cohomology). Let $L_{\mathbb{Z}}$ be a torsion-free $C^+(V_{\mathbb{Z}})$-module such that $L=L_{\mathbb{Z}}\otimes\mathbb{Q}$ is a free $C^+(V)$-module of finite rank, where $V=V_{\mathbb{Z}}\otimes\mathbb{Q}$. Let A be the associated abelian variety such that $L_{\mathbb{Z}}=\mathrm{H}^1_{\mathrm{B}}(A,\mathbb{Z})$. Then there exists a finite extension K'/K such that the following statements are true.*

(i) The abelian variety A is defined over K'.

(ii) The canonical algebra isomorphism

$$\psi:C^+(V)\cong\mathrm{End}_{C^+}L$$

is motivated and descends to K'.

(iii) There is a canonical algebra isomorphism

$$\mathrm{End}(L)\cong(\mathrm{End}_{C^+}L)\otimes(C^+)^{\mathrm{op}}$$

which is motivated and descends to K'.

(iv) The natural inclusion $C^+(V)\hookrightarrow\mathrm{End}(L)$, which corresponds to the inclusion

$$\mathrm{End}_{C^+}L\hookrightarrow(\mathrm{End}_{C^+}L)\otimes(C^+)^{\mathrm{op}},\quad\alpha\mapsto\alpha\otimes 1,$$

is motivated and descends to K'.

Proof. Note that A is isogenous to $\mathrm{KS}(V_{\mathbb{Z}})^r$ and hence one can easily reduce to the case $A=\mathrm{KS}(V_{\mathbb{Z}})$. Since $\mathrm{KS}(\mathrm{P}^2_{\mathrm{B}}(X,\mathbb{Z}(1)))$ is isogenous to a power of $\mathrm{KS}(\mathrm{T}^2_{\mathrm{B}}(X,\mathbb{Z}(1)))$, it is enough to prove the theorem for $V_{\mathbb{Z}}=\mathrm{P}^2_{\mathrm{B}}(X,\mathbb{Z}(1))$.

In that setting, statements (i) and (ii) are explicit in [And96a] while statements (iii) and (iv) are easy consequences as we will see. Precisely, statement (i) is [And96a, Lem. 6.5.1]. Further enlarging K' so that all endomorphisms of $A_{\mathbb{C}}$ are defined over K', we get statement (ii) from [And96a, Prop. 6.2.1 and Lem. 6.5.1]. For the remaining part of the proof, all motives are considered as defined over K'. It follows from (ii) that we have an algebra isomorphism $\tilde{\psi} : C^+(\mathfrak{v}, Q) \longrightarrow \underline{\mathrm{End}}_{C^+}(\mathfrak{h}^1(A))$ which induces ψ. Note that both $\underline{C}^+$ and $\underline{\mathrm{End}}_{C^+}(\mathfrak{h}^1(A))$ are subalgebra objects of $\underline{\mathrm{End}}(\mathfrak{h}^1(A))$. Hence we have a canonical algebra homomorphism

$$(\underline{\mathrm{End}}_{C^+}(\mathfrak{h}^1(A))) \otimes (\underline{C}^+)^{\mathrm{op}} \longrightarrow \underline{\mathrm{End}}(\mathfrak{h}^1(A))$$

which induces an isomorphism of the Betti realizations. As a consequence, this is an isomorphism of André motives. Here we use the general fact that $\mathrm{End}_{\mathbb{Q}}(R)$ is canonically isomorphic to $R \otimes_{\mathbb{Q}} R^{\mathrm{op}}$ for any finite dimensional $\mathbb{Q}$-algebra R. This proves statement (iii). The element $1 \in C^+$ induces a morphism $\mathbb{1} \to (\underline{C}^+)^{\mathrm{op}}$ yielding a morphism

$$\underline{\mathrm{End}}_{C^+}(\mathfrak{h}^1(A)) \longrightarrow \underline{\mathrm{End}}_{C^+}(\mathfrak{h}^1(A)) \otimes \mathbb{1} \longrightarrow \underline{\mathrm{End}}_{C^+}(\mathfrak{h}^1(A)) \otimes (\underline{C}^+)^{\mathrm{op}}$$

whose Betti realization is given by $\alpha \mapsto \alpha \otimes 1$. This shows statement (iv). □

From Theorem 8.3, André deduces :

Theorem 8.4 (André [And96a, Prop. 6.4.3 & Cor. 1.5.3])**.**

(i) Let X be a hyper-Kähler variety defined over a field $K \subseteq \mathbb{C}$. Then there is a finite field extension K'/K and an abelian variety A over K' such that the André motive $\mathfrak{h}^2(X_{K'})$ is a direct summand of $\mathfrak{h}(A)$;

(ii) Any Hodge class in a direct summand M of $(\otimes \mathfrak{h}^2(X_i)) \otimes (\otimes \mathfrak{h}(A_j))$, where the X_i are complex hyper-Kähler varieties and the A_j are complex abelian varieties, is motivated. In particular, the Mumford–Tate group $\mathrm{MT}(M)$ of M coincides with its motivated Galois group $G_{\mathrm{And}}(M)$.

Proof. For the sake of completeness, let us outline the proof. Since $\mathfrak{h}^2(X)(1) = \mathbb{1} \oplus \mathfrak{v}$, it is enough to prove (i) for $\mathfrak{v} := \mathfrak{p}^2(X)(1) \subset \mathfrak{h}^2(X)(1)$ the primitive submotive. Denote by $V := \mathrm{P}^2_{\mathrm{B}}(X, \mathbb{Q}(1))$ its Betti realization.

If $\dim V = 2n+1$ is odd, then the primitive motive $\mathfrak{v}$ satisfies

$$\mathfrak{v} \cong \mathfrak{v}^\vee \cong \mathfrak{v}^\vee \otimes \det \mathfrak{v} \cong \wedge^{2n} \mathfrak{v} \cong \mathrm{Gr}_n C^+(\mathfrak{v}, Q)$$

where the last term is the graded piece F_n/F_{n-1}. It follows that $\mathfrak{v}$ is a subquotient of $C^+(\mathfrak{v}, Q)$ and hence a submotive of $C^+(\mathfrak{v}, Q)$ by semi-simplicity. Theorem 8.3(ii) thus provides a split inclusion

$$\mathfrak{v} \hookrightarrow C^+(\mathfrak{v}, Q) \cong \underline{\mathrm{End}}_{C^+}(\mathfrak{h}^1(A)) \hookrightarrow \underline{\mathrm{End}}(\mathfrak{h}^1(A)) = \mathfrak{h}^1(A) \otimes \mathfrak{h}^1(A)^\vee \hookrightarrow \mathfrak{h}(A \times A^\vee)(1).$$

If $\dim V = 2n$ is even, then one reduces to the previous case by considering $\mathfrak{v}^\# := \mathfrak{v} \oplus \mathbb{1}$. Its Betti realization $V^\# := V \oplus \mathbb{Z}(0)$ is equipped with the quadratic form $Q^\# := Q \oplus \langle -1 \rangle$ and one may run the Kuga–Satake construction to get an abelian variety $A^\#$ over K' with split inclusions

$$\mathfrak{v} \hookrightarrow \mathfrak{v}^\# \hookrightarrow \mathfrak{h}(A^\# \times (A^\#)^\vee)(1).$$

Precisely, the analogue of Theorem 8.3(ii) for $V^\#$ is [And96a, Cor. 6.4.4].

Finally, (ii) follows from (i) and the fact (Theorem 3.4) proved by André [And96b] that Hodge cycles on complex abelian varieties are motivated (the statement $\mathrm{MT}(M) = G_{\mathrm{And}}(M)$ is Proposition 3.3). □

Finally, as in [Cha13, MP15], we consider the special case $L_{\mathbb{Z}} = C(V_{\mathbb{Z}})$, viewed as a $C^+(V_{\mathbb{Z}})$-module via left multiplication. The Kuga–Satake construction provides a complex abelian variety A such that $L = \mathrm{H}^1_{\mathrm{B}}(A, \mathbb{Q})$ together with a canonical inclusion

$$V \hookrightarrow \mathrm{End}(L), \quad v \mapsto l_v \tag{9}$$

where $l_v(u) = vu$, for all $u \in L = C(V)$. (Note that A is isogenous to the square of the Kuga–Satake variety of $V_{\mathbb{Z}}$.) This inclusion is a homomorphism of Hodge structures and we call it the *Kuga–Satake correspondence*. Combining Theorem 8.4 and Theorem 3.4, we get the following

Proposition 8.5. *We take up the assumptions of Theorem 8.3 and assume in addition that $L_{\mathbb{Z}} = C(V_{\mathbb{Z}})$. Then the Kuga–Satake correspondence (9) is motivated and defined over $\overline{K}$.* □

9. De Rham–Betti classes on hyper-Kähler varieties

We fix an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ inside $\mathbb{C}$ and we let K be a subfield of $\overline{\mathbb{Q}}$. In this section, we prove Theorems 3, 4, 5 and 6 of the introduction.

9.1. **First observations.** Via André's Theorem 8.4, Theorem 7.11 admits the following two consequences :

Proposition 9.1. *Let X be a hyper-Kähler variety over $K \subseteq \overline{\mathbb{Q}}$.*

(i) Any de Rham–Betti class in $\mathrm{H}^2_{\mathrm{dRB}}(X, \mathbb{Q}(1))$ is algebraic.
(ii) Any $\overline{\mathbb{Q}}$-de Rham–Betti class in $\mathrm{H}^2_{\mathrm{dRB}}(X_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}(1))$ is $\overline{\mathbb{Q}}$-algebraic.

Proof. By Theorem 8.4, there exists an abelian variety A over $\overline{\mathbb{Q}}$ such that the André motive $\mathfrak{h}^2(X_{\overline{\mathbb{Q}}})$ is a direct summand of $\mathfrak{h}^2(A)$. Statement (i) in case $K = \overline{\mathbb{Q}}$, which is [BC16, Thm. 5.6], as well as statement (ii) are then a direct consequence of Theorem 7.11. The case $K \subseteq \overline{\mathbb{Q}}$ in (i) follows from Lemma 6.2. □

Proposition 9.2. *Let X be a hyper-Kähler variety over $\overline{\mathbb{Q}}$.*

(i) The de Rham–Betti group $G_{\mathrm{dRB}}(\mathfrak{h}^2(X))$ is reductive.
(ii) The $\overline{\mathbb{Q}}$-de Rham–Betti group $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}(\mathfrak{h}^2(X) \otimes \overline{\mathbb{Q}})$ is reductive.

Proof. By Theorem 8.4, there exists an abelian variety A over $\overline{\mathbb{Q}}$ such that $\mathfrak{h}^2(X)$ is a direct summand of $\mathfrak{h}(A)$. Hence $G_{\mathrm{dRB}}(\mathfrak{h}^2(X))$ is a quotient of $G_{\mathrm{dRB}}(A)$ and $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}(\mathfrak{h}^2(X) \otimes \overline{\mathbb{Q}})$ is a quotient of $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}(A)$. It follows from Theorem 7.11(i) that both groups are reductive. □

9.2. **De Rham–Betti isometries between hyper-Kähler varieties are motivated.** For a polarized hyper-Kähler variety X over K, we denote $\mathfrak{p}^2(X)$ its degree-2 primitive André motive and $\mathfrak{t}^2(X)$ its degree-2 transcendental motive. Their de Rham–Betti realizations are respectively given by $\mathrm{P}^2_{\mathrm{dRB}}(X, \mathbb{Q})$ and $\mathrm{T}^2_{\mathrm{dRB}}(X, \mathbb{Q})$. The former is the orthogonal complement of the polarization in $\mathrm{H}^2_{\mathrm{dRB}}(X, \mathbb{Q})$, while the latter is the orthogonal complement of the subspace of $\mathrm{H}^2_{\mathrm{dRB}}(X, \mathbb{Q})$ spanned by the classes of divisors.

Theorem 9.3. *Let X and X' be hyper-Kähler varieties over $\overline{\mathbb{Q}}$. Then*

(i) any $\overline{\mathbb{Q}}$–de Rham–Betti isometry $\mathrm{P}^2_{\mathrm{dRB}}(X, \overline{\mathbb{Q}}) \xrightarrow{\sim} \mathrm{P}^2_{\mathrm{dRB}}(X', \overline{\mathbb{Q}})$ is $\overline{\mathbb{Q}}$-motivated.
(ii) any $\overline{\mathbb{Q}}$–de Rham–Betti isometry $\mathrm{T}^2_{\mathrm{dRB}}(X, \overline{\mathbb{Q}}) \xrightarrow{\sim} \mathrm{T}^2_{\mathrm{dRB}}(X', \overline{\mathbb{Q}})$ is $\overline{\mathbb{Q}}$-motivated.
(iii) any $\overline{\mathbb{Q}}$–de Rham–Betti isometry $\mathrm{H}^2_{\mathrm{dRB}}(X, \overline{\mathbb{Q}}) \xrightarrow{\sim} \mathrm{H}^2_{\mathrm{dRB}}(X', \overline{\mathbb{Q}})$ is $\overline{\mathbb{Q}}$-motivated.

Here, the isometries are with respect to either the form induced by η or the Beauville–Bogomolov form (see Example 8.1).

Proof. By Proposition 9.1, if X is a hyper-Kähler variety, then $\mathrm{T}^2_{\mathrm{B}}(X, \mathbb{Q}(1))$ does not support any non-zero $\overline{\mathbb{Q}}$-de Rham–Betti class. Hence $(ii) \implies (iii)$ and it thus suffices to show items (i) and (ii). Let X and X' be two hyper-Kähler varieties over $\overline{\mathbb{Q}}$ with respective polarizations η and η'. Let $V := \mathrm{P}^2_{\mathrm{B}}(X, \mathbb{Q}(1))$ (resp. $V := \mathrm{T}^2_{\mathrm{B}}(X, \mathbb{Q}(1))$) and $V' := \mathrm{P}^2_{\mathrm{B}}(X', \mathbb{Q}(1))$ (resp. $V' := \mathrm{T}^2_{\mathrm{B}}(X', \mathbb{Q}(1))$) be the primitive cohomology (resp. transcendental cohomology). Assume that we have a $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}$-invariant isometry

$$i : V \otimes \overline{\mathbb{Q}} \xrightarrow{\sim} V' \otimes \overline{\mathbb{Q}}.$$

We will show that i is motivated.

Consider $L = C(V)$ and $L' = C(V')$ viewed as $C^+(V)$- and $C^+(V')$-modules respectively, with their natural integral structures. Let A and A' be the corresponding abelian varieties over $\overline{\mathbb{Q}}$ provided by Theorem 8.3; hence we have identifications $L = \mathrm{H}^1_{\mathrm{B}}(A, \mathbb{Q})$ and $L' = \mathrm{H}^1_{\mathrm{B}}(A', \mathbb{Q})$. We fix a $C^+(V \otimes \overline{\mathbb{Q}})$-basis $\{1, \alpha\}$ of $L \otimes \overline{\mathbb{Q}}$ with $\alpha \in V \otimes \overline{\mathbb{Q}}$, as well as the corresponding basis $\{1, \alpha'\}$ of $L' \otimes \overline{\mathbb{Q}}$, where $\alpha' = i(\alpha) \in V' \otimes \overline{\mathbb{Q}}$. This induces identifications of $\overline{\mathbb{Q}}$-algebras

$$C^+ \otimes \overline{\mathbb{Q}} := (\mathrm{End}_{C^+(V)} L)^{\mathrm{op}} \otimes \overline{\mathbb{Q}} = \mathrm{M}_2(C^+(V \otimes \overline{\mathbb{Q}})) \quad \text{and}$$
$$C'^+ \otimes \overline{\mathbb{Q}} := (\mathrm{End}_{C^+(V')} L')^{\mathrm{op}} \otimes \overline{\mathbb{Q}} = \mathrm{M}_2(C^+(V' \otimes \overline{\mathbb{Q}})).$$

The isometry $i : V \otimes \overline{\mathbb{Q}} \longrightarrow V' \otimes \overline{\mathbb{Q}}$ induces $\overline{\mathbb{Q}}$-algebra isomorphisms

$$\begin{tikzcd} C^+ \otimes \overline{\mathbb{Q}} \arrow[r, "j_0"] \arrow[d, equal] & C'^+ \otimes \overline{\mathbb{Q}} \arrow[d, equal] \\ \mathrm{M}_2(C^+(V \otimes \overline{\mathbb{Q}})) \arrow[r, "\mathrm{M}_2(C^+(i))"] & \mathrm{M}_2(C^+(V' \otimes \overline{\mathbb{Q}}))\,. \end{tikzcd}$$

On the other hand, the $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}$-invariant isometry $i : V \otimes \overline{\mathbb{Q}} \longrightarrow V' \otimes \overline{\mathbb{Q}}$ induces a $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}$-invariant algebra isomorphism $C^+(i) : C^+(V \otimes \overline{\mathbb{Q}}) \longrightarrow C^+(V' \otimes \overline{\mathbb{Q}})$. By Theorem 8.3(*ii*) we obtain a $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}$-invariant algebra isomorphism

$$\begin{tikzcd} (\mathrm{End}_{C^+} L) \otimes \overline{\mathbb{Q}} \arrow[r, "j"] \arrow[d, phantom, "\wr\|"] & (\mathrm{End}_{C'^+} L') \otimes \overline{\mathbb{Q}} \arrow[d, phantom, "\wr\|"] \\ C^+(V \otimes \overline{\mathbb{Q}}) \arrow[r, "C^+(i)"] & C^+(V' \otimes \overline{\mathbb{Q}})\,. \end{tikzcd}$$

By Theorem 8.3(*iii*), we then get a $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}$-invariant algebra isomorphism

$$\begin{tikzcd} \mathrm{End}(L \otimes \overline{\mathbb{Q}}) \arrow[r, "J"] \arrow[d, phantom, "\wr\|"] & \mathrm{End}(L' \otimes \overline{\mathbb{Q}}) \arrow[d, phantom, "\wr\|"] \\ ((\mathrm{End}_{C^+} L) \otimes \overline{\mathbb{Q}}) \otimes (C^+ \otimes \overline{\mathbb{Q}})^{\mathrm{op}} \arrow[r, "j \otimes j_0"] & ((\mathrm{End}_{C'^+} L') \otimes \overline{\mathbb{Q}}) \otimes (C'^+ \otimes \overline{\mathbb{Q}})^{\mathrm{op}}\,. \end{tikzcd}$$

Here $C^+ \otimes \overline{\mathbb{Q}}$ and $C'^+ \otimes \overline{\mathbb{Q}}$ are viewed as representations of $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}$ with trivial action.

Our aim is to show that J is motivated. By Lemma 9.4 below, there exists isomorphisms $\nu : L \otimes \overline{\mathbb{Q}} \to L' \otimes \overline{\mathbb{Q}}$ such that

$$J(f) = \nu \circ f \circ \nu^{-1}$$

for all $f \in \mathrm{End}_{\overline{\mathbb{Q}}}(L \otimes \overline{\mathbb{Q}})$. The isomorphism ν is unique up to scaling by a scalar in $\overline{\mathbb{Q}}^*$.

Lemma 9.4. *Let W and W' be two vector spaces over a field k and let $\Phi : \mathrm{End}_k(W) \to \mathrm{End}_k(W')$ be an isomorphism of k-algebras. Then there exists a k-linear isomorphism $\phi : W \to W'$ such that $\Phi(f) = \phi \circ f \circ \phi^{-1}$ for all $f \in \mathrm{End}_k(W)$. The isomorphism ϕ is unique up to a scalar in k^*.*

Proof. Pick a basis $\{w_1, w_2, \ldots, w_n\}$ of W. Let $pr_1 \in \mathrm{End}_k(W)$ be the projector onto the subspace generated by w_1. Let $pr'_1 := \Phi(pr_1) \in \mathrm{End}_k(W')$, which is nonzero. Pick some $w' \in W'$ such that $w'_1 := pr'_1(w')$ is nonzero. Let $f_i \in \mathrm{End}_k(W)$, $i = 2, 3, \ldots, n$, be defined by $f_i(w_1) = w_i$ and $f_i(w_j) = 0$ for all $j > 1$. Let $f'_i := \Phi(f_i)$ and set $w'_i := f'_i(w'_1)$. In this way, we get a basis $\{w'_1, w'_2, \ldots, w'_n\}$ of W' and the isomorphism $\phi : W \to W'$, $\phi(w_i) = w'_i$, satisfies the required condition. If ϕ' is another such isomorphism, then $\phi^{-1} \circ \phi'$ sits in the center of $\mathrm{End}_k(W)$ and hence $\phi' = c\phi$ for some $c \in k^*$. □

The condition that J is a $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}$-invariant isomorphism of algebras means

$$\nu \circ g \circ f \circ g^{-1} \circ \nu^{-1} = g \circ \nu \circ f \circ \nu^{-1} \circ g^{-1}, \quad \forall g \in G_{\overline{\mathbb{Q}}-\mathrm{dRB}},\ f \in \mathrm{End}(L \otimes \overline{\mathbb{Q}}).$$

As a consequence $\lambda(g) := g^{-1} \circ \nu^{-1} \circ g \circ \nu$ commutes with all $f \in \mathrm{End}(L \otimes \overline{\mathbb{Q}})$ and hence $\lambda(g) \in \overline{\mathbb{Q}}^*$. After choosing a $\overline{\mathbb{Q}}$-basis for $L \otimes \overline{\mathbb{Q}}$ and $L' \otimes \overline{\mathbb{Q}}$, we can write $\lambda(g) = YZY^{-1}Z^{-1}$ with $Y, Z \in \mathrm{M}_{r\times r}(\overline{\mathbb{Q}})$. By taking determinant, we see that $\lambda(g) \in \mu_r$. Since by Theorem 4.7 the $\overline{\mathbb{Q}}$-de Rham–Betti group $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}$ is connected, we conclude that $\lambda(g) = 1$ for all $g \in G_{\overline{\mathbb{Q}}-\mathrm{dRB}}$. As a consequence ν is $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}$-invariant. It follows from Theorem 7.10 that ν is a $\overline{\mathbb{Q}}$-linear combination of classes of algebraic cycles. As a consequence, J is $\overline{\mathbb{Q}}$-motivated. Note that the Kuga–Satake correspondence (9), by the definition of J, fits into the following commutative diagram

$$\begin{tikzcd} V \otimes \overline{\mathbb{Q}} \arrow[r, "i"] \arrow[d] & V' \otimes \overline{\mathbb{Q}} \arrow[d] \\ \mathrm{End}(L \otimes \overline{\mathbb{Q}}) \arrow[r, "J"] & \mathrm{End}(L' \otimes \overline{\mathbb{Q}}) \end{tikzcd}$$

We conclude from Proposition 8.5 and the semi-simplicity that $i : V \otimes \overline{\mathbb{Q}} \to V' \otimes \overline{\mathbb{Q}}$ is $\overline{\mathbb{Q}}$-motivated. □

We record the following direct consequence of Theorem 9.3.

Theorem 9.5. *Let X and X' be hyper-Kähler varieties over K. Then*

(i) any de Rham–Betti isometry $\mathrm{P}^2_{\mathrm{dRB}}(X, \mathbb{Q}) \xrightarrow{\sim} \mathrm{P}^2_{\mathrm{dRB}}(X', \mathbb{Q})$ is motivated.
(ii) any de Rham–Betti isometry $\mathrm{T}^2_{\mathrm{dRB}}(X, \mathbb{Q}) \xrightarrow{\sim} \mathrm{T}^2_{\mathrm{dRB}}(X', \mathbb{Q})$ is motivated.
(iii) any de Rham–Betti isometry $\mathrm{H}^2_{\mathrm{dRB}}(X, \mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^2_{\mathrm{dRB}}(X', \mathbb{Q})$ is motivated.

Here, the isometries are with respect to either the form induced by η or the Beauville–Bogomolov form (see Example 8.1).

Proof. In view of Lemma 6.2 and Lemma 6.17, this is a direct consequence of Theorem 9.3 □

Corollary 9.6. *If X and X' are hyper-Kähler varieties over K of K3[n]-deformation type, then any de Rham–Betti isometry $\mathrm{H}^2_{\mathrm{dRB}}(X, \mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^2_{\mathrm{dRB}}(X', \mathbb{Q})$ is algebraic.*

Proof. By Markman [Mar22], any Hodge isometry $\mathrm{H}^2_{\mathrm{B}}(X^{\mathrm{an}}_{\mathbb{C}}, \mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^2_{\mathrm{B}}((X')^{\mathrm{an}}_{\mathbb{C}}, \mathbb{Q})$ is algebraic. (The case where X and X' are K3 surfaces is due to Buskin [Bus19]). The corollary then follows directly from Theorem 9.5(*iii*). □

9.3. **A global de Rham–Betti Torelli theorem for K3 surfaces over $\overline{\mathbb{Q}}$.** We derive from Theorem 9.5 the following result of independent interest :

Theorem 9.7 (A global de Rham–Betti Torelli theorem for K3 surfaces over $\overline{\mathbb{Q}}$)**.** *Let S and S' be two K3 surfaces over $\overline{\mathbb{Q}}$. If there is an integral de Rham–Betti isometry*

$$i : \mathrm{H}^2_{\mathrm{dRB}}(S, \mathbb{Z}) \xrightarrow{\sim} \mathrm{H}^2_{\mathrm{dRB}}(S', \mathbb{Z}),$$

i.e., *an isometry $i : \mathrm{H}^2_{\mathrm{B}}(S^{\mathrm{an}}_{\mathbb{C}}, \mathbb{Z}) \xrightarrow{\sim} \mathrm{H}^2_{\mathrm{B}}(S'^{\mathrm{an}}_{\mathbb{C}}, \mathbb{Z})$ that becomes de Rham–Betti after base-change to $\mathbb{Q}$, then S and S' are isomorphic.*

Proof. Let η be an ample divisor class on S. Then $\eta' = i(\eta)$ is a de Rham–Betti class and hence algebraic by [BC16, Thm. 5.6] ; see also Proposition 9.1. Thus either η' or $-\eta'$ is a positive class on S'. Without loss of generality, we assume that η' is a positive class. Then η' can be moved to an ample class by a series of reflections along (-2)-classes ; see [Huy16, Ch. 8, Cor. 2.9]. Such reflections are all algebraic isometries of $\mathrm{H}^2_{\mathrm{dRB}}(S', \mathbb{Z})$ since each (-2)-class is represented by a rational curve on S'. By composing i with these reflections, we may assume that η' is an ample class. Thus i restricts to a de Rham–Betti isometry

$$i' : \mathrm{P}^2_{\mathrm{dRB}}(S,\mathbb{Z}) \to \mathrm{P}^2_{\mathrm{dRB}}(S',\mathbb{Z}),$$

where $\mathrm{P}^2_{\mathrm{dRB}}(S,\mathbb{Z})$ (resp. $\mathrm{P}^2_{\mathrm{dRB}}(S',\mathbb{Z})$) is the orthogonal complement of η (resp. η'). It follows from Theorem 9.5 that $i' \otimes 1_{\mathbb{Q}}$ is motivated and hence respects Hodge structures. As a consequence, i is a Hodge isometry and the usual Torelli theorem for K3 surfaces provides an isomorphism $S_{\mathbb{C}} \simeq S'_{\mathbb{C}}$, from which we obtain an isomorphism $S \simeq S'$. $\square$

9.4. **Codimension-2 de Rham–Betti classes on hyper-Kähler varieties.** Recall from Section 3 that $G_{\mathrm{And}}(M)$ is reductive and that we have $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}(M\otimes\overline{\mathbb{Q}}) \subseteq G_{\mathrm{dRB}}(M)_{\overline{\mathbb{Q}}} \subseteq G_{\mathrm{And}}(M)_{\overline{\mathbb{Q}}}$ for all André motives M over $\overline{\mathbb{Q}}$. Recall also from Proposition 9.2 that $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}(\mathfrak{h}^2(X))$ is reductive if X is a hyper-Kähler variety over $\overline{\mathbb{Q}}$. The following proposition is analogous to [And96a, Lem. 7.4.1].

Proposition 9.8. *Let X be a hyper-Kähler variety over $\overline{\mathbb{Q}}$. Then the inclusion*

$$\mathrm{End}_{G_{\mathrm{And}}(\mathfrak{h}^2(X))}(\mathrm{H}^2_{\mathrm{B}}(X^{\mathrm{an}}_{\mathbb{C}},\overline{\mathbb{Q}})) \subseteq \mathrm{End}_{G_{\overline{\mathbb{Q}}-\mathrm{dRB}}(\mathfrak{h}^2(X))}(\mathrm{H}^2_{\mathrm{B}}(X^{\mathrm{an}}_{\mathbb{C}},\overline{\mathbb{Q}}))$$

is an equality. Equivalently, any $\overline{\mathbb{Q}}$-de Rham–Betti class in $\mathrm{H}^2_{\mathrm{dRB}}(X,\overline{\mathbb{Q}}(1)) \otimes \mathrm{H}^2_{\mathrm{dRB}}(X,\overline{\mathbb{Q}}(1))$ is $\overline{\mathbb{Q}}$-motivated.

Proof. The two statements in the proposition are equivalent since any choice of polarization on X induces an isomorphism of André motives $\mathfrak{h}^2(X)(1)^\vee \simeq \mathfrak{h}^2(X)(1)$.

Since $G_{\mathrm{And}}(\mathfrak{h}^2(X))$ and $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}(\mathfrak{h}^2(X))$ are reductive, it is enough to show that any simple $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}$-submodule T of $\mathrm{H}^2_{\mathrm{B}}(X^{\mathrm{an}}_{\mathbb{C}},\overline{\mathbb{Q}})$ is a $G_{\mathrm{And},\overline{\mathbb{Q}}}$-submodule. Since the intersection form on $\mathrm{H}^2_{\mathrm{B}}(X^{\mathrm{an}}_{\mathbb{C}},\mathbb{Q})$ is motivated, $T\cap T^\perp$ is a $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}$-submodule of T and it is therefore either $\{0\}$ or T.

If $T\cap T^\perp = \{0\}$, then $\mathrm{H}^2_{\mathrm{B}}(X^{\mathrm{an}}_{\mathbb{C}},\overline{\mathbb{Q}}) = T\oplus^\perp T^\perp$ and $(-\mathrm{id}_T, \mathrm{id}_{T^\perp})$ defines a $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}$-equivariant isometry of $\mathrm{H}^2_{\mathrm{B}}(X^{\mathrm{an}}_{\mathbb{C}},\overline{\mathbb{Q}})$. By Theorem 9.3 it is motivated and hence T is a $G_{\mathrm{And},\overline{\mathbb{Q}}}$-submodule.

If $T\cap T^\perp = T$, i.e. if T is totally isotropic, then choose a $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}$-submodule T' of $\mathrm{H}^2_{\mathrm{B}}(X^{\mathrm{an}}_{\mathbb{C}},\overline{\mathbb{Q}})$ such that $\mathrm{H}^2_{\mathrm{B}}(X^{\mathrm{an}}_{\mathbb{C}},\overline{\mathbb{Q}}) = T^\perp\oplus T'$. The restriction of the quadratic form to the $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}$-submodule $T\oplus T'$ is then nondegenerate. Let $\psi : T\oplus T' \to T^\vee \oplus T'^\vee$ be the induced $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}$-equivariant map and let $\psi_T : T' \to T^\vee$ and $\psi_{T'} : T' \to T'^\vee$ be the induced $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}$-equivariant maps. Since T is totally isotropic, the map ψ_T is an isomorphism. Define the $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}$-submodule $T'' := \mathrm{im}\big(-\frac{1}{2}(\psi_T^{-1})^\vee \circ \psi_{T'} \oplus \mathrm{id}_{T'} : T' \to T\oplus T'\big)$. Then $T\oplus T' = T\oplus T''$ is a decomposition of the nondegenerate quadratic space $T\oplus T'$ into $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}$-submodules such that both T and T'' are totally isotropic. It follows that $(2\mathrm{id}_T, \frac{1}{2}\mathrm{id}_{T''}, \mathrm{id}_{(T\oplus T'')^\perp})$ defines a $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}$-equivariant isometry of $\mathrm{H}^2_{\mathrm{B}}(X^{\mathrm{an}}_{\mathbb{C}},\overline{\mathbb{Q}})$. By Theorem 9.3 it is $\overline{\mathbb{Q}}$-motivated and hence T is a $G_{\mathrm{And},\overline{\mathbb{Q}}}$-submodule. $\square$

Theorem 9.9. *Let X be a hyper-Kähler variety over K of known deformation type and let n be a positive integer.*

(i) Any de Rham–Betti class in $\mathrm{H}^4_{\mathrm{dRB}}(X^n,\mathbb{Q}(2))$ is motivated.
(ii) Any $\overline{\mathbb{Q}}$-de Rham–Betti class in $\mathrm{H}^4_{\mathrm{dRB}}(X^n_{\overline{\mathbb{Q}}},\overline{\mathbb{Q}}(2))$ is $\overline{\mathbb{Q}}$-motivated.

Proof. By Lemma 6.2 and Lemma 6.17, it suffices to show (ii) and we can assume $K = \overline{\mathbb{Q}}$. By [GKLR21, Prop. 2.35(ii) & Prop. 2.38(i)], for any hyper-Kähler variety X, $\mathrm{H}^4_{\mathrm{B}}(X^{\mathrm{an}}_{\mathbb{C}},\mathbb{Q})$ is a sub-Hodge structure of $\big(\mathrm{H}^2_{\mathrm{B}}(X^{\mathrm{an}}_{\mathbb{C}},\mathbb{Q})\otimes \mathrm{H}^2_{\mathrm{B}}(X^{\mathrm{an}}_{\mathbb{C}},\mathbb{Q}) \oplus \mathrm{H}^2_{\mathrm{B}}(X^{\mathrm{an}}_{\mathbb{C}},\mathbb{Q}(-1)) \oplus \mathbb{Q}(-2)\big)^{\oplus N}$ for some N. Since $\mathrm{H}^1_{\mathrm{B}}(X^{\mathrm{an}}_{\mathbb{C}},\mathbb{Q}) = 0$ and since $\mathbb{Q}(-1)$ is a direct summand of $\mathrm{H}^2_{\mathrm{B}}(X^{\mathrm{an}}_{\mathbb{C}},\mathbb{Q})$, we see that for all $n\geq 1$, $\mathrm{H}^4_{\mathrm{B}}((X^{\mathrm{an}}_{\mathbb{C}})^n,\mathbb{Q})$ is a sub-Hodge structure of $\big(\mathrm{H}^2_{\mathrm{B}}(X^{\mathrm{an}}_{\mathbb{C}},\mathbb{Q})\otimes \mathrm{H}^2_{\mathrm{B}}(X^{\mathrm{an}}_{\mathbb{C}},\mathbb{Q})\big)^{\oplus N}$ for some N.

From [FFZ21, Cor. 1.17] and [Sol21, Cor. 1.2], if X is of known deformation type, any Hodge class on a power of $X_{\mathbb{C}}$ is motivated; it follows that the André motive $\mathfrak{h}^4(X^n)$ is a direct summand of $(\mathfrak{h}^2(X)\otimes\mathfrak{h}^2(X))^{\oplus N}$. Now, having proved in Proposition 9.8 that any $\overline{\mathbb{Q}}$-de Rham–Betti class in $\mathfrak{h}^2(X)(1)\otimes\mathfrak{h}^2(X)(1)$ is $\overline{\mathbb{Q}}$-motivated, we deduce that any $\overline{\mathbb{Q}}$-de Rham–Betti class in $\mathfrak{h}^4(X^n)(2)$ is $\overline{\mathbb{Q}}$-motivated. $\square$

Corollary 9.10. *Let X be a hyper-Kähler fourfold over K of known deformation type and let k be a non-negative integer.*

(i) Any de Rham–Betti class in $\mathrm{H}^{2k}_{\mathrm{dRB}}(X,\mathbb{Q}(k))$ is motivated.
(ii) Any $\overline{\mathbb{Q}}$-de Rham–Betti class in $\mathrm{H}^{2k}_{\mathrm{dRB}}(X_{\overline{\mathbb{Q}}},\overline{\mathbb{Q}}(k))$ is $\overline{\mathbb{Q}}$-motivated.

Proof. From Proposition 9.1 and Theorem 9.9, it remains to see that any $\overline{\mathbb{Q}}$-de Rham–Betti class in $\mathrm{H}^6_{\mathrm{dRB}}(X_{\overline{\mathbb{Q}}},\overline{\mathbb{Q}}(3))$ is $\overline{\mathbb{Q}}$-motivated. This follows from Proposition 9.1 together with the Lefschetz isomorphism of André motives $\mathfrak{h}^2(X)(1)\simeq\mathfrak{h}^6(X)(3)$. $\square$

9.5. **The de Rham–Betti group of a hyper-Kähler variety.** Let X be a hyper-Kähler variety over $\overline{\mathbb{Q}}$ and denote by $\mathfrak{t}^2(X)$ its transcendental motive in degree 2. By Zarhin [Zar83], the endomorphism algebra $E:=\mathrm{End}(\mathfrak{t}^2(X))$ is a field of degree dividing $\dim_{\mathbb{Q}}\mathrm{T}^2_{\mathrm{B}}(X,\mathbb{Q})$. We say that X has complex multiplication if $[E:\mathbb{Q}]=\dim_{\mathbb{Q}}\mathrm{T}^2_{\mathrm{B}}(X,\mathbb{Q})$ (this implies E is a CM field). With what we have established so far, we have the following analogue of Theorem 7.11.

Theorem 9.11. *Let $X/\overline{\mathbb{Q}}$ be a hyper-Kähler variety of positive dimension. The de Rham–Betti group $G_{\mathrm{dRB}}(\mathfrak{h}^2(X))$ has the following properties.*

(i) $G_{\mathrm{dRB}}(\mathfrak{h}^2(X))\subseteq\mathrm{MT}(\mathfrak{h}^2(X))$ is a connected reductive group.
(ii) $\mathrm{End}_{G_{\mathrm{dRB}}(\mathfrak{h}^2(X))}(\mathrm{H}^2_{\mathrm{B}}(X,\mathbb{Q}))=\mathrm{End}_{\mathrm{MT}(\mathfrak{h}^2(X))}(\mathrm{H}^2_{\mathrm{B}}(X,\mathbb{Q}))$.
(iii) $\det\colon G_{\mathrm{dRB}}(\mathfrak{h}^2(X))\to\mathbb{G}_m$ is surjective.
(iv) X has complex multiplication if and only if $G_{\mathrm{dRB}}(\mathfrak{h}^2(X))$ is a torus.

Likewise, the $\overline{\mathbb{Q}}$-de Rham–Betti group $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}(\mathfrak{h}^2(X))$ has the following properties.

(i') $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}(\mathfrak{h}^2(X))\subseteq\mathrm{MT}(\mathfrak{h}^2(X))$ is a connected reductive group.
(ii') $\mathrm{End}_{G_{\overline{\mathbb{Q}}-\mathrm{dRB}}(\mathfrak{h}^2(X))}(\mathrm{H}^2_{\mathrm{B}}(X,\overline{\mathbb{Q}}))=\mathrm{End}_{\mathrm{MT}(\mathfrak{h}^2(X))}(\mathrm{H}^2_{\mathrm{B}}(X,\mathbb{Q}))\otimes\overline{\mathbb{Q}}$.
(iii') $\det\colon G_{\overline{\mathbb{Q}}-\mathrm{dRB}}(\mathfrak{h}^2(X))\to\mathbb{G}_m$ is surjective.
(iv') X has complex multiplication if and only if $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}(\mathfrak{h}^2(X))$ is a torus.

Proof. That $G_{\mathrm{dRB}}(\mathfrak{h}^2(X))$ lies in $\mathrm{MT}(\mathfrak{h}^2(X))$ is due to André's Theorem 8.4 stating that the inclusion $\mathrm{MT}(\mathfrak{h}^2(X))\subseteq G_{\mathrm{And}}(\mathfrak{h}^2(X))$ is an equality. That $G_{\mathrm{dRB}}(A)$ is connected is Theorem 4.7 and that it is reductive is Proposition 9.2. Statement (ii) is Proposition 9.8. Regarding (iii), the image of det is connected. Assume it is trivial. Then $G_{\mathrm{dRB}}(\mathfrak{h}^2(X))$ acts trivially on $\det(\mathrm{H}^2_{\mathrm{B}}(X,\mathbb{Q}))$. But, by André's Theorem 8.4, $\det\mathfrak{h}^2(X)\simeq\mathbb{1}(-\dim\mathrm{H}^2_{\mathrm{B}}(X,\mathbb{Q}))$ as André motives. This is a contradiction since $2\pi i$ is transcendental. For (iv), suppose that the transcendental motive $\mathfrak{t}^2(X)$ has complex multiplication. Then $G_{\mathrm{dRB}}(\mathfrak{h}^2(X))$ is a subgroup of the Mumford–Tate group $\mathrm{MT}(\mathfrak{h}^2(X))$, which is a torus. Since $G_{\mathrm{dRB}}(\mathfrak{h}^2(X)$ is reductive and connected by (i), it has to be a torus. Conversely, assume that $G_{\mathrm{dRB}}(\mathfrak{h}^2(X))$ is a torus. Similarly to the case of abelian varieties, if $G_{\mathrm{dRB}}(\mathfrak{h}^2(X))$ is contained in a maximal torus $T\subseteq\mathrm{GL}(\mathrm{T}^2_{\mathrm{B}}(X,\mathbb{Q}))$, then

$$\mathrm{End}_T(\mathrm{T}^2_{\mathrm{B}}(X,\mathbb{Q}))\subseteq\mathrm{End}_{G_{\mathrm{dRB}}(\mathfrak{h}^2(X))}(\mathrm{T}^2_{\mathrm{B}}(X,\mathbb{Q}))=\mathrm{End}_{\mathrm{MT}(\mathfrak{h}^2(X))}(\mathrm{T}^2_{\mathrm{B}}(X,\mathbb{Q})).$$

But $\mathrm{End}_T(\mathrm{T}^2_{\mathrm{B}}(X,\mathbb{Q}))$ is a commutative $\mathbb{Q}$-algebra of dimension $\dim\mathrm{T}^2_{\mathrm{B}}(X,\mathbb{Q})$, as can be seen after extending scalars to an algebraically closed field. It follows that X has complex multiplication.

Statements (i'), (ii'), (iii') and (iv') are proven with the exact same arguments. $\square$

Corollary 9.12. *Let X be a hyper-Kähler variety over $\overline{\mathbb{Q}}$ and let $k\in\mathbb{Z}$. Every de Rham–Betti class in $\mathrm{T}^2_{\mathrm{dRB}}(X,\mathbb{Q}(k))$ is zero.*

Proof. It follows from Theorem 9.11(*ii*) that

$$\mathrm{End}_{G_{\mathrm{dRB}}(\mathfrak{h}^2(X))}(\mathrm{T}^2_{\mathrm{B}}(X,\mathbb{Q})) = \mathrm{End}_{\mathrm{MT}(\mathfrak{h}^2(X))}(\mathrm{T}^2_{\mathrm{B}}(X,\mathbb{Q})).$$

In particular, since the Hodge structure $\mathrm{T}^2_{\mathrm{B}}(X,\mathbb{Q})$ is irreducible, the de Rham–Betti object $\mathrm{T}^2_{\mathrm{dRB}}(X,\mathbb{Q})$ is simple. Since $\dim \mathrm{T}^2_{\mathrm{B}}(X,\mathbb{Q}) \geq 2$, we get

$$\mathrm{Hom}_{\mathrm{dRB}}(\mathbb{1}(-k), \mathrm{T}^2_{\mathrm{dRB}}(X,\mathbb{Q})) = 0,$$

for all $k \in \mathbb{Z}$. □

9.6. **The de Rham–Betti conjecture and hyper-Kähler varieties of large Picard rank.** For convenience we make the following definition.

Definition 9.13. Let X be a smooth projective variety over a subfield of $\mathbb{C}$.

- The *Picard rank* of X is $\rho(X) =_{\mathrm{def}} \mathrm{rk}(\mathrm{CH}^1(X_{\mathbb{C}}) \to \mathrm{H}^2_{\mathrm{B}}(X^{\mathrm{an}}_{\mathbb{C}}, \mathbb{Q}(1)))$.
- The *Picard corank* of X is $\rho^c(X) =_{\mathrm{def}} \mathfrak{h}^{1,1}(X^{\mathrm{an}}_{\mathbb{C}}) - \rho$, where $\mathfrak{h}^{1,1}(X^{\mathrm{an}}_{\mathbb{C}}) = \dim_{\mathbb{C}} \mathrm{H}^1(X^{\mathrm{an}}_{\mathbb{C}}, \Omega^1_{X^{\mathrm{an}}_{\mathbb{C}}})$.

We address the ($\overline{\mathbb{Q}}$-)de Rham–Betti conjecture (Conjectures 6.10 and 6.13) for the second cohomology group of hyper-Kähler varieties of Picard corank ≤ 2 and prove Theorem 6. First we observe that the de Rham–Betti conjectures are stable under direct summand.

Proposition 9.14. *Let N be a André motive over K and let M be an object in $\langle N \rangle$. If the motivated de Rham–Betti conjecture holds for N, then it holds for M. In other words,*

$$\Omega^{\mathrm{dRB}}_N = \Omega^{\mathrm{And}}_N \implies \Omega^{\mathrm{dRB}}_M = \Omega^{\mathrm{And}}_M.$$

If $K = \overline{\mathbb{Q}}$, we also have

$$\Omega_N = \Omega^{\mathrm{And}}_N \implies \Omega_M = \Omega^{\mathrm{And}}_M.$$

Proof. By Proposition 6.12, $\Omega^{\mathrm{dRB}}_N = \Omega^{\mathrm{And}}_N$ if and only if $G_{\mathrm{dRB}}(N)$ is reductive and every de Rham–Betti class on tensor spaces $N^{\otimes n} \otimes (N^{\vee})^{\otimes m}$ is motivated. Now if M belongs to $\langle N \rangle$, then $G_{\mathrm{dRB}}(N)$ is a quotient of $G_{\mathrm{dRB}}(M)$, hence is reductive, and every de Rham–Betti class on tensor spaces $M^{\otimes n} \otimes (M^{\vee})^{\otimes m}$ is motivated. By Proposition 6.12 again, we conclude that $\Omega^{\mathrm{dRB}}_M = \Omega^{\mathrm{And}}_M$. The implication $\Omega_N = \Omega^{\mathrm{And}}_N \Rightarrow \Omega_M = \Omega^{\mathrm{And}}_M$ is proved similarly by using Proposition 6.16 in place of Proposition 6.12. □

Theorem 9.15. *Let X be a hyper-Kähler variety over K.*

(i) If $\rho^c(X) = 0$, then $Z_{\mathfrak{h}^2(X)} = \Omega^{\mathrm{And}}_{\mathfrak{h}^2(X)}$.
(ii) If $\rho^c(X) = 1$, then $\Omega_{\mathfrak{h}^2(X)} = \Omega^{\mathrm{And}}_{\mathfrak{h}^2(X)}$.
(iii) If $\rho^c(X) = 2$, then $\Omega^{\mathrm{dRB}}_{\mathfrak{h}^2(X)} = \Omega^{\mathrm{And}}_{\mathfrak{h}^2(X)}$.

Assume further that X is of known deformation type.

(i') If $\rho^c(X) = 0$, then $Z_X = \Omega^{\mathrm{And}}_X$.
(ii') If $\rho^c(X) = 1$, then $\Omega_X = \Omega^{\mathrm{And}}_X$.
(iii') If $\rho^c(X) = 2$, then $\Omega^{\mathrm{dRB}}_X = \Omega^{\mathrm{And}}_X$.

Proof. Regarding (*i*): By André's Theorem 8.4, $G_{\mathrm{And}}(\mathfrak{h}^2(X_{\overline{\mathbb{Q}}}))$ agrees with the Mumford–Tate group and hence is connected. By a Galois argument we deduce that $\Omega^{\mathrm{And}}(\mathfrak{h}^2(X))$ is connected. Therefore it suffices to show statement (*i*) in the case $K = \overline{\mathbb{Q}}$. Note that the Kuga–Satake variety associated to X of maximal Picard rank is a CM elliptic curve, so that we have an isomorphism of André motives

$$\mathfrak{h}^2(X) \simeq \big(\mathfrak{h}^1(E) \otimes \mathfrak{h}^1(E)\big) \oplus \mathbb{1}(-1)^{\oplus b_2 - 4}.$$

To conclude it suffices to see that $\dim Z_{\mathfrak{h}^2(X)} \geq 2$. This follows at once from Chudnovsky's theorem [Chu80] that the transcendence degree of periods of elliptic curves is at least 2.

Regarding (ii) : By Lemma 6.14, we may and do assume that $K = \overline{\mathbb{Q}}$. Note that if the Picard corank of X is at most 1, then the Kuga–Satake variety A associated to the transcendental motive $\mathfrak{t}^2(X)$ is either a CM elliptic curve or an abelian surface of Picard rank 3. Since $\mathfrak{t}^2(X)$ is a direct summand of $\mathfrak{h}(A \times A)$, we can conclude from Proposition 9.14 together with Theorems 7.12 and 7.15.

Regarding (iii) : By Lemma 6.11, we may and do assume that $K = \overline{\mathbb{Q}}$. Let $E = \mathrm{End}(\mathfrak{t}^2(X))$. By Zarhin [Zar83], E is a field and there are three possibilities for the Mumford–Tate group of $\mathfrak{h}^2(X)(1)$: it is either SO_4, or $\mathrm{U}_2(E)$ for the CM quadratic extension $E/\mathbb{Q}$, or $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{U}_1(F)$ for the CM quartic extension $E/F/\mathbb{Q}$, where F is a real quadratic extension. (Note that the case $\mathrm{Res}_{E/\mathbb{Q}}\mathrm{SO}_2$ for a real quadratic extension $E/\mathbb{Q}$ does not occur since in that case the Hodge group of $\mathfrak{h}^2(X)$ is commutative and hence $\mathfrak{h}^2(X)$ would be CM.) In what follows, we use the shorthand $\mathfrak{h}^2$ for $\mathfrak{h}^2(X)$.

First we assume that $\mathrm{MT}(\mathfrak{h}^2(1)) = \mathrm{SO}_{4,\mathbb{Q}}$. In that case, we actually show the stronger statement that the inclusion $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}(\mathfrak{h}^2(1)) \subseteq G_{\mathrm{And}}(\mathfrak{h}^2(1))_{\overline{\mathbb{Q}}} = \mathrm{SO}_{4,\overline{\mathbb{Q}}}$ is an equality. Let $\mathfrak{g}$ be the Lie algebra of $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}(\mathfrak{h}^2(1))$. We then have an inclusion

$$\mathfrak{g} \subseteq \mathfrak{so}_{4,\overline{\mathbb{Q}}} = \mathfrak{sl}_{2,\overline{\mathbb{Q}}} \times \mathfrak{sl}_{2,\overline{\mathbb{Q}}},$$

where the $\mathfrak{g}$-module structure on $\mathrm{T}^2_{\mathrm{B}}(X, \overline{\mathbb{Q}}(1))$ is induced from the $\mathfrak{sl}_{2,\overline{\mathbb{Q}}} \times \mathfrak{sl}_{2,\overline{\mathbb{Q}}}$-module decomposition $\mathrm{T}^2_{\mathrm{B}}(X, \overline{\mathbb{Q}}(1)) = V_1 \otimes V_2$, where V_i is the standard representation of the i-th factor of $\mathfrak{sl}_{2,\overline{\mathbb{Q}}} \times \mathfrak{sl}_{2,\overline{\mathbb{Q}}}$. Now, by using Theorem 9.11 instead of Theorem 7.11, we may argue exactly as in the proof of Theorem 7.15(ii) in the case of simple abelian surfaces with endomorphism algebra given by a real quadratic extension of $\mathbb{Q}$ to show that the inclusion $\mathfrak{g} \subseteq \mathfrak{so}_{4,\overline{\mathbb{Q}}} = \mathfrak{sl}_{2,\overline{\mathbb{Q}}} \times \mathfrak{sl}_{2,\overline{\mathbb{Q}}}$ is an equality.

Suppose $\mathrm{MT}(\mathfrak{h}^2(1)) = \mathrm{U}_2(E)$ for a CM quadratic extension E of $\mathbb{Q}$, and let $\mathfrak{mt}$ denote the Lie algebra of $\mathrm{MT}(\mathfrak{h}^2(1))$. Then $\mathfrak{mt}_{\overline{\mathbb{Q}}} = \mathfrak{gl}_{2,\overline{\mathbb{Q}}} = \mathfrak{sl}_{2,\overline{\mathbb{Q}}} \oplus \mathfrak{c}$ and the embedding

$$\mathfrak{mt}_{\overline{\mathbb{Q}}} \subset \mathfrak{sl}_{2,\overline{\mathbb{Q}}} \oplus \mathfrak{sl}_{2,\overline{\mathbb{Q}}} \tag{10}$$

corresponding to the embedding $\mathrm{MT}(\mathfrak{h}^2(1)) \subset \mathrm{SO}_4$ is given by mapping $\mathfrak{sl}_{2,\overline{\mathbb{Q}}}$ to one of the factors and $\mathfrak{c}$ to the Lie algebra of a maximal torus in $\mathrm{SL}_{2,\overline{\mathbb{Q}}}$ in the other factor. Here the action of $\mathfrak{sl}_{2,\overline{\mathbb{Q}}} \oplus \mathfrak{sl}_{2,\overline{\mathbb{Q}}}$ on $\mathrm{T}^2_{\mathrm{B}}(1) \otimes \overline{\mathbb{Q}}$ is via the tensor product of two copies of the standard representation. Let $\mathfrak{g} \subseteq \mathfrak{sl}_{2,\overline{\mathbb{Q}}} \oplus \mathfrak{c}$ denote the Lie algebra of $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}(\mathfrak{h}^2(1))$. If $\mathfrak{g}$ was abelian, then $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}(\mathfrak{h}^2(1))$ would be a torus, contradicting by Theorem 9.11 the fact that $\mathrm{End}(\mathfrak{h}^2(1)) = E$. Hence $\mathfrak{g}$ has to contain $\mathfrak{sl}_{2,\overline{\mathbb{Q}}}$. We have to exclude the case $\mathfrak{g} = \mathfrak{sl}_{2,\overline{\mathbb{Q}}}$. In this case the embedding (10) shows that $\mathrm{End}_{G_{\overline{\mathbb{Q}}-\mathrm{dRB}}(\mathfrak{h}^2(1))}(\mathrm{T}^2_{\mathrm{B}}(X, \overline{\mathbb{Q}}(1))$ has dimension 4, which contradicts Theorem 9.11(ii').

Suppose $\mathrm{MT}(\mathfrak{h}^2(1)) = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{U}_1(F)$ for a CM quartic extension $E/\mathbb{Q}$, where $F \subset E$ denotes the real quadratic subfield. We distinguish two cases : if E does not contain an imaginary quadratic subfield, then the proof of Theorem 7.15(i) in case (d) shows that $\mathrm{MT}(\mathfrak{h}^2(1))$ does not contain any non-trivial subtori defined over $\mathbb{Q}$. Since $G_{\mathrm{dRB}}(\mathfrak{h}^2(1))$ cannot be trivial by Proposition 9.8, we conclude that $G_{\mathrm{dRB}}(\mathfrak{h}^2(1)) = \mathrm{MT}(\mathfrak{h}^2(1))$. Suppose now that E contains an imaginary quadratic subfield. Then E is a biquadratic field, and hence $\mathrm{Gal}(E/\mathbb{Q}) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. As we have seen in the proof of Theorem 7.15(i) in case (d), the cocharacter lattice of $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{U}_1(F)$ is given by $X_*(\mathrm{Res}_{F/\mathbb{Q}}\mathrm{U}_1(F))_{\mathbb{Q}} = \langle \phi_1^\vee - \bar{\phi}_1^\vee, \phi_2^\vee - \bar{\phi}_2^\vee \rangle_{\mathbb{Q}}$. Here $\Sigma_E := \mathrm{Hom}(E, \mathbb{C}) = \{\phi_1, \bar{\phi}_1, \phi_2, \bar{\phi}_2\}$ is the set of embeddings of the CM field E into the complex numbers. One computes shows that $X_*(\mathrm{Res}_{F/\mathbb{Q}}\mathrm{U}_1(F))_{\mathbb{Q}}$ has two one-dimensional Galois-stable subspaces, generated by $\phi_1^\vee + \phi_2^\vee - (\bar{\phi}_1^\vee + \bar{\phi}_2^\vee)$ and $\phi_1^\vee - \phi_2^\vee - (\bar{\phi}_1^\vee - \bar{\phi}_2^\vee)$, respectively. The corresponding $\mathbb{Q}$-subtori of rank 1 are precisely the tori of the form $\mathrm{U}_1(K)$ for $K \subset E$ an imaginary quadratic subfield. We have to exclude the possibility that $G_{\mathrm{dRB}}(\mathfrak{h}^2(1)) = \mathrm{U}_1(K)$. If this is the case, then $\mathrm{End}_K(\mathrm{T}^2_{\mathrm{B}}(X, \mathbb{Q})) \subset$

$\mathrm{End}_{G_{\mathrm{dRB}}(\mathfrak{h}^2(1))}(\mathrm{H}^2_{\mathrm{B}}(X,\mathbb{Q}))$. Note that $\mathrm{End}_K(\mathrm{T}^2_{\mathrm{B}}(X,\mathbb{Q}))$ is of dimension 4 over K, and hence of dimension 8 over $\mathbb{Q}$. This contradicts Theorem 9.11(*ii*).

Now that we have established in all cases that $G_{\mathrm{dRB}}(\mathfrak{h}^2(1)) = \mathrm{MT}(\mathfrak{h}^2(1))$, let us prove $G_{\mathrm{dRB}}(\mathfrak{h}^2) = \mathrm{MT}(\mathfrak{h}^2)$. We first show that $\mathrm{MT}(\mathfrak{h}^2(1)) \subset G_{\mathrm{dRB}}(\mathfrak{h}^2)$. Namely, since we know that both groups are reductive, we can argue as follows: Suppose $v \in (\mathrm{H}^2_{\mathrm{B}})^{\otimes a} \otimes (\mathrm{H}^2_{\mathrm{B}})^{\vee\otimes b}$ is fixed by $G_{\mathrm{dRB}}(\mathfrak{h}^2)$. Since

$$(\mathrm{H}^2_{\mathrm{B}}(1))^{\otimes a} \otimes (\mathrm{H}^2_{\mathrm{B}}(1))^{\vee\otimes b} = (\mathrm{H}^2_{\mathrm{B}})^{\otimes a} \otimes (\mathrm{H}^2_{\mathrm{B}})^{\vee\otimes b} \otimes \mathbb{Q}(a-b),$$

we see that the one-dimensional subspace spanned by v is preserved by $G_{\mathrm{dRB}}(\mathfrak{h}^2(1))$. The equality $G_{\mathrm{dRB}}(\mathfrak{h}^2(1)) = \mathrm{MT}(\mathfrak{h}^2(1))$ then shows that this subspace is a one-dimensional sub-Hodge structure of $(\mathrm{H}^2_{\mathrm{B}}(1))^{\otimes a} \otimes (\mathrm{H}^2_{\mathrm{B}}(1))^{\vee\otimes b}$. It therefore has to be of weight zero, and consequently v is fixed by $\mathrm{MT}(\mathfrak{h}^2(1))$. Since $\mathrm{MT}(\mathfrak{h}^2)$ is generated by the scalars $\mathbb{G}_m$ and $\mathrm{MT}(\mathfrak{h}^2(1))$, and we know from Theorem 9.11 that $G_{\mathrm{dRB}}(\mathfrak{h}^2)$ surjects onto the determinant, we conclude that $G_{\mathrm{dRB}}(\mathfrak{h}^2) = \mathrm{MT}(\mathfrak{h}^2)$.

Finally, we show how to derive (i'), (ii') and (iii') from (i), (ii) and (iii), respectively. As before, we may and do assume that $K = \overline{\mathbb{Q}}$. Since $\mathfrak{h}^2(X)$ is a direct summand of $\mathfrak{h}(X)$ we have a commutative diagram with surjective horizontal arrows

$$\begin{tikzcd} G_{\mathrm{And}}(X) \arrow[r, two heads] & G_{\mathrm{And}}(\mathfrak{h}^2(X)) \\ \mathrm{MT}(X) \arrow[r, two heads] \arrow[u, hook] & \mathrm{MT}(\mathfrak{h}^2(X)) \arrow[u, hook] \end{tikzcd}$$

By Theorem 8.4(*ii*) the right inclusion is an equality, while the assumption that X is of known deformation type provides by [FFZ21, Thm. 1.11 & Cor. 1.16] that the left inclusion is an equality. On the other hand, by [FFZ21, Prop. 6.4] the bottom horizontal arrow has finite kernel.

In particular, $G_{\mathrm{And}}(X)$ is connected and $\dim G_{\mathrm{And}}(X) = \dim G_{\mathrm{And}}(\mathfrak{h}^2(X))$. Since clearly $\dim Z_X \geq \dim Z_{\mathfrak{h}^2(X)}$, we get the implication $(i) \Rightarrow (i')$.

On the other hand, we have commutative diagrams with surjective horizontal arrows

$$\begin{tikzcd} G_{\mathrm{dRB}}(X) \arrow[r, two heads] \arrow[d, hook] & G_{\mathrm{dRB}}(\mathfrak{h}^2(X)) \arrow[d, hook] \\ G_{\mathrm{And}}(X) \arrow[r, two heads] & G_{\mathrm{And}}(\mathfrak{h}^2(X)) \end{tikzcd} \qquad \begin{tikzcd} G_{\overline{\mathbb{Q}}-\mathrm{dRB}}(X) \arrow[r, two heads] \arrow[d, hook] & G_{\overline{\mathbb{Q}}-\mathrm{dRB}}(\mathfrak{h}^2(X)) \arrow[d, hook] \\ G_{\mathrm{And}}(X)_{\overline{\mathbb{Q}}} \arrow[r, two heads] & G_{\mathrm{And}}(\mathfrak{h}^2(X))_{\overline{\mathbb{Q}}} \end{tikzcd}$$

and, as explained above, the surjection $G_{\mathrm{And}}(X) \twoheadrightarrow G_{\mathrm{And}}(\mathfrak{h}^2(X))$ is an isogeny of connected algebraic groups. Therefore, if the inclusion $G_{\mathrm{dRB}}(\mathfrak{h}^2(X)) \hookrightarrow G_{\mathrm{And}}(\mathfrak{h}^2(X))$ (resp. the inclusion $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}(\mathfrak{h}^2(X)) \hookrightarrow G_{\mathrm{And}}(\mathfrak{h}^2(X))_{\overline{\mathbb{Q}}}$) is an equality, then the inclusion $G_{\mathrm{dRB}}(X) \subseteq G_{\mathrm{And}}(X)$ (resp. the inclusion $G_{\overline{\mathbb{Q}}-\mathrm{dRB}}(X) \subseteq G_{\mathrm{And}}(X)_{\overline{\mathbb{Q}}}$) is an equality. This establishes the implication $(iii) \Rightarrow (iii')$ (resp. the implication $(ii) \Rightarrow (ii')$). □

Remark 9.16. Let X be a hyper-Kähler variety over K. Assume that X does not have CM and that $\rho^c(X) = 2$. The arguments of the proof of Theorem 9.15 actually show that $\Omega_{\mathfrak{h}^2(X)} = \Omega^{\mathrm{And}}_{\mathfrak{h}^2(X)}$. Moreover, if X is of known deformation type, then $\Omega_X = \Omega^{\mathrm{And}}_X$.

Using the Shioda–Inose structure on K3 surface of Picard corank ≤ 1 and the validity of the Hodge conjecture for powers of abelian surfaces, we obtain:

Corollary 9.17. *Let S be a K3 surface over K.*

(i) If $\rho^c(S) = 0$, then $Z_S = \Omega^{\mathrm{mot}}_S$.
(ii) If $\rho^c(S) = 1$, then $\Omega_S = \Omega^{\mathrm{mot}}_S$.

In particular, in both cases, for any $n \geq 0$ and any $k \in \mathbb{Z}$, any $\overline{\mathbb{Q}}$-de Rham–Betti class on $\mathfrak{h}(S^n)(k)$ is a $\overline{\mathbb{Q}}$-linear combination of algebraic classes.

Proof. On the one hand, Theorem 9.15 provides $Z_S = \Omega_S^{\mathrm{And}}$ and $\Omega_S = \Omega_S^{\mathrm{And}}$ if $\rho^c(S) = 0$ and $\rho^c(S) = 1$, respectively. We are thus left to show that $\Omega_S^{\mathrm{And}} = \Omega_S^{\mathrm{mot}}$ if $\rho^c(S) \leq 1$. Since the standard conjectures hold for surfaces in characteristic zero, we have to show that motivated classes on powers of S are algebraic. For that purpose, we may and do assume that $K = \overline{\mathbb{Q}}$. If S has Picard corank ≤ 1, then it has a Shioda–Inose structure and its transcendental (homological) motive is thus isomorphic to the transcendental motive of an abelian surface over $\overline{\mathbb{Q}}$. Since Hodge classes on powers of complex abelian surfaces are algebraic [Tan82], it follows that motivated classes on powers of S are algebraic. □

Remark 9.18. Corollary 9.17(i) is also established in [Kaw23] in the case of Kummer surfaces associated to squares of CM elliptic curves.

References

[And96a] Yves André, *On the Shafarevich and Tate conjectures for hyper-Kähler varieties*, Math. Ann. **305** (1996), no. 2, 205–248.

[And96b] ———, *Pour une théorie inconditionnelle des motifs*, Inst. Hautes Études Sci. Publ. Math. (1996), no. 83, 5–49.

[And04] ———, *Une introduction aux motifs (motifs purs, motifs mixtes, périodes)*, Panoramas et Synthèses [Panoramas and Syntheses], vol. 17, Société Mathématique de France, Paris, 2004.

[And09] ———, *Galois theory, motives and transcendental numbers*, Renormalization and Galois theories, IRMA Lect. Math. Theor. Phys., vol. 15, Eur. Math. Soc., Zürich, 2009, pp. 165–177.

[Ayo14] Joseph Ayoub, *Periods and the conjectures of Grothendieck and Kontsevich-Zagier*, Eur. Math. Soc. Newsl. (2014), no. 91, 12–18.

[BC16] Jean-Benoît Bost and François Charles, *Some remarks concerning the Grothendieck period conjecture*, J. Reine Angew. Math. **714** (2016), 175–208.

[Bos13] Jean-Benoît Bost, *Algebraization, transcendence, and D-group schemes*, Notre Dame J. Form. Log. **54** (2013), no. 3-4, 377–434.

[Bus19] Nikolay Buskin, *Every rational Hodge isometry between two $K3$ surfaces is algebraic*, J. Reine Angew. Math. **755** (2019), 127–150.

[BW07] A. Baker and G. Wüstholz, *Logarithmic forms and Diophantine geometry*, New Mathematical Monographs, vol. 9, Cambridge University Press, Cambridge, 2007.

[Cha13] François Charles, *The Tate conjecture for $K3$ surfaces over finite fields*, Invent. Math. **194** (2013), no. 1, 119–145.

[Chu80] G. V. Chudnovsky, *Algebraic independence of values of exponential and elliptic functions*, Proceedings of the International Congress of Mathematicians (Helsinki, 1978), Acad. Sci. Fennica, Helsinki, 1980, pp. 339–350.

[Del82] Pierre Deligne, *Hodge cycles on abelian varieties, in* Hodge cycles, motives, and Shimura varieties, Lecture Notes in Mathematics, vol. 900, Springer-Verlag, Berlin-New York, 1982.

[DM82] Pierre Deligne and James Milne, *Tannakian categories, in* Hodge cycles, motives, and Shimura varieties, Lecture Notes in Mathematics, vol. 900, Springer-Verlag, Berlin-New York, 1982.

[FFZ21] Salvatore Floccari, Lie Fu, and Ziyu Zhang, *On the motive of O'Grady's ten-dimensional hyper-Kähler varieties*, Commun. Contemp. Math. **23** (2021), no. 4, Paper No. 2050034, 50.

[GKLR21] Mark Green, Yoon-Joo Kim, Radu Laza, and Colleen Robles, *The LLV decomposition of hyper-Kähler cohomology (the known cases and the general conjectural behavior)*, Mathematische Annalen (2021).

[Gor99] B. Brent Gordon, *Appendix b to "a survey of the Hodge conjecture"*, second ed., CRM Monograph Series, vol. 10, American Mathematical Society, Providence, RI, 1999.

[Gro66] A. Grothendieck, *On the de Rham cohomology of algebraic varieties*, Inst. Hautes Études Sci. Publ. Math. (1966), no. 29, 95–103.

[Hör21] Fritz Hörmann, *A note on formal periods*, 2021.

[Huy16] D. Huybrechts, *Lectures on K3 surfaces*, Cambridge studies in advanced mathematics, vol. 158, Cambridge University Press, 2016.

[HW22] Annette Huber and Gisbert Wüstholz, *Transcendence and linear relations of 1-periods*, Cambridge Tracts in Mathematics, Cambridge University Press, 2022.

[Kaw23] Daiki Kawabe, *Grothendieck's period conjecture for Kummer surfaces of self-product CM type*, 2023.

[Lom16] Davide Lombardo, *On the ℓ-adic Galois representations attached to nonsimple abelian varieties*, Ann. Inst. Fourier (Grenoble) **66** (2016), no. 3, 1217–1245.

[Mar22] Eyal Markman, *Rational Hodge isometries of hyper-Kahler varieties of K3[n]-type are algebraic*, arXiv e-prints (2022), arXiv:2204.00516.

[Mas75] David Masser, *Elliptic functions and transcendence*, Lecture Notes in Mathematics, Vol. 437, Springer-Verlag, Berlin-New York, 1975.

[Mil17] J. S. Milne, *Algebraic groups*, Cambridge Studies in Advanced Mathematics, vol. 170, Cambridge University Press, Cambridge, 2017, The theory of group schemes of finite type over a field.

[MP15] Keerthi Madapusi Pera, *The Tate conjecture for K3 surfaces in odd characteristic*, Invent. Math. **201** (2015), no. 2, 625–668.

[MZ99] B. J. J. Moonen and Yu. G. Zarhin, *Hodge classes on abelian varieties of low dimension*, Math. Ann. **315** (1999), no. 4, 711–733.

[Sat66] I. Satake, *Clifford algebras and families of abelian varieties*, Nagoya J. Math. **27** (1966), 435–446.

[Sch37] Theodor Schneider, *Arithmetische Untersuchungen elliptischer Integrale*, Math. Ann. **113** (1937), no. 1, 1–13.

[Ser68] Jean-Pierre Serre, *Abelian l-adic representations and elliptic curves*, W. A. Benjamin, Inc., New York-Amsterdam, 1968, McGill University lecture notes written with the collaboration of Willem Kuyk and John Labute.

[Sol21] Andrey Soldatenkov, *Deformation Principle and André motives of Projective Hyperkähler Manifolds*, International Mathematics Research Notices (2021), rnab108.

[Spi99] Michael Spieß, *Proof of the Tate conjecture for products of elliptic curves over finite fields*, Math. Ann. **314** (1999), no. 2, 285–290.

[SR72] Neantro Saavedra Rivano, *Catégories Tannakiennes*, Lecture Notes in Mathematics, Vol. 265, Springer-Verlag, Berlin-New York, 1972.

[Tan82] S. G. Tankeev, *Cycles on simple abelian varieties of prime dimension*, Izv. Akad. Nauk SSSR Ser. Mat. **46** (1982), no. 1, 155–170, 192.

[Wat79] William C. Waterhouse, *Introduction to affine group schemes*, Graduate Texts in Mathematics, vol. 66, Springer-Verlag, New York-Berlin, 1979.

[Wüs84] G. Wüstholz, *Zum Periodenproblem*, Invent. Math. **78** (1984), no. 3, 381–391.

[Zar83] Yu. G. Zarhin, *Hodge groups of $K3$ surfaces*, J. Reine Angew. Math. **341** (1983), 193–220.

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