ZERO-CYCLES ON DOUBLE EPW SEXTICS

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ABSTRACT. The Chow rings of hyperKähler varieties are conjectured to have a particularly rich structure. In this paper, we focus on the locally complete family of double EPW sextics and establish some properties of their Chow rings. First we prove a Beauville–Voisin type theorem for zero-cycles on double EPW sextics ; precisely, we show that the codimension-4 part of the subring of the Chow ring of a double EPW sextic generated by divisors, the Chern classes and codimension-2 cycles invariant under the anti-symplectic covering involution has rank one. Second, for double EPW sextics birational to the Hilbert square of a K3 surface, we show that the action of the anti-symplectic involution on the Chow group of zero-cycles commutes with the Fourier decomposition of Shen–Vial.

INTRODUCTION

Since the seminal work of Beauville and Voisin on the Chow ring of K3 surfaces [5], it has been observed that the Chow rings (and more generally the Chow motives, considered as algebra objects) of hyperKähler varieties possess a surprisingly rich structure, which seems to parallel that of abelian varieties. Our aim is to study aspects of the Chow ring, which conjecturally should hold for all hyperKähler varieties, in the special case of double EPW sextics. Discovered by O'Grady [34], double EPW sextics form a 20-dimensional locally complete family of hyperKähler fourfolds, deformation equivalent to the Hilbert square of a K3 surface.

For a scheme X of finite type over a field, we denote $\mathrm{CH}^i(X)$ the Chow group of codimension-i cycle classes with rational coefficients (*i.e.* the group of codimension-i algebraic cycles on X with $\mathbb{Q}$-coefficients, modulo rational equivalence).

0.1. The Beauville–Voisin conjecture.

Conjecture 1 (Beauville–Voisin)**.** *Let X be a hyperKähler variety. Consider the $\mathbb{Q}$-subalgebra*

$$\mathrm{R}^*(X) := \langle \mathrm{CH}^1(X), c_j(X) \rangle \quad \subset \ \mathrm{CH}^*(X)$$

generated by divisors and Chern classes. Then the restriction of the cycle class map $\mathrm{R}^i(X) \to H^{2i}(X,\mathbb{Q})$ is injective for all i.

The conjecture was first proven (without being stated as such) in the case of K3 surfaces in the seminal work of Beauville and Voisin [5]. The conjecture was then formulated by Beauville [3]

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without the Chern classes as an explicit workable consequence of a deeper conjecture stipulating the splitting of the conjectural Bloch–Beilinson filtration on the Chow rings of hyperKähler varieties. It was then stated in this form by Voisin [44] who established it in the case of Hilbert schemes of points on K3 surfaces of low dimension and in the case of Fano varieties of lines on smooth cubic fourfolds. The Beauville–Voisin conjecture has by now been established in many cases, including the generic double EPW sextic [13], generalized Kummer varieties [15] and, without the Chern classes, hyperKähler varieties carrying a rational Lagrangian fibration [37] as well as Hilbert schemes of points on K3 surfaces [31]. There are however, so far, only two examples of locally complete families of hyperKähler varieties all of whose members are known to satisfy the Beauville–Voisin conjecture, namely K3 surfaces [5] and Fano varieties of lines on cubic fourfolds [44]. The following result establishes in particular the Beauville–Voisin conjecture for zero-cycles for double EPW sextics, which form a locally complete family of hyperKähler varieties.

Theorem 1. *Let X be a double EPW sextic, and let ι be its anti-symplectic involution. Consider the $\mathbb{Q}$-subalgebra*

$$\mathrm{R}^*(X) := \langle \mathrm{CH}^1(X), \mathrm{CH}^2(X)^+, c_j(X)\rangle \quad \subset \ \mathrm{CH}^*(X)$$

generated by divisors, ι-invariant codimension-2 cycles and Chern classes. The restriction of the cycle class map $\mathrm{R}^i(X) \to H^{2i}(X,\mathbb{Q})$ is injective for $i = 4$.

This builds on and extends the main result of Ferretti [13]. The reason for including $\mathrm{CH}^2(X)^+$ is motivated by Beauville's splitting property conjecture for the Bloch–Beilinson filtration [3], which in fact suggests that all of $\mathrm{R}^*(X)$ should inject in cohomology. Our new input consists in extending a result of Voisin concerning zero-cycles on generic Calabi–Yau hypersurfaces to the case of Calabi–Yau hypersurfaces with quotient singularities (the EPW sextics are such Calabi–Yau hypersurfaces) ; this is Theorem 1.2. Unfortunately, we failed to establish the Beauville–Voisin conjecture for codimension-3 cycles ; we identify in (4) the missing relations.

0.2. **Anti-symplectic involutions and zero-cycles.** In the same way that the action of homomorphisms of abelian varieties preserves the Beauville decomposition [2] of the Chow groups (such a decomposition provides a splitting of the conjectural Bloch–Beilinson filtration), it is conceivable to expect that morphisms (or even rational maps) between hyperKähler varieties preserve the conjectural splitting of the conjectural Bloch–Beilinson filtration. Candidates for such a splitting were constructed for Hilbert schemes of K3 surfaces [38, 40], generalized Kummer varieties [17] and Fano varieties of lines on cubic fourfolds [38]. This expectation was verified for the action of Voisin's rational self-map on the Fano variety of lines on a cubic fourfold in [38, Proposition 21.14], and for the action of finite-order symplectic automorphisms on zero-cycles of generalized Kummer varieties [41, Theorem 5]. We also note that any rational map between K3 surfaces is compatible with the splitting of the Bloch–Beilinson filtration given by $\mathrm{CH}^2(S) = \mathbb{Q}[o] \oplus \mathrm{CH}^2_{hom}(S)$, where o denotes the Beauville–Voisin [5] zero-cycle on S ; indeed o is the class of any point lying on a rational curve of S and hence it is sent to the class of a point lying on a rational curve by the action of any rational map. We provide more evidence for this expectation by determining the action of the anti-symplectic involution attached to a double

EPW sextic birational to the Hilbert square of a K3 surface, and also by determining the action of a birational automorphism of the Hilbert square of a very general K3 surface.

Precisely, we show in Theorem 3.6 that ι^* commutes with the Fourier decomposition of $\mathrm{CH}^4(S^{[2]})$ constructed in [38], which provides an explicit candidate for the splitting of the Bloch–Beilinson conjecture as conjectured by Beauville [3]. As a consequence, we describe explicitly the action of ι on the Chow group of zero-cycles in case X is birational to a Hilbert square $S^{[2]}$ with S a K3 surface (which happens on a dense, countable union of divisors in the moduli space of double EPW sextics) :

Theorem 2. *Let X be a smooth double EPW sextic, and assume that X is birational to a Hilbert square $S^{[2]}$ with S a K3 surface. Let $\iota \in Aut(X)$ be the anti-symplectic involution coming from the double cover $f : X \to Y$, where $Y \subset \mathbb{P}^5$ is an EPW sextic. Then the action of ι on $\mathrm{CH}^4(S^{[2]})$ is given by*

$$\iota^*[x, y] = [x, y] - 2[x, o] - 2[y, o] + 4[o, o],$$

where $o \in S$ denotes any point lying on a rational curve in S and where $[x, y]$ denotes the class in $\mathrm{CH}^4(S^{[2]})$ of any point in $S^{[2]}$ with support $x + y \in S^{(2)} = S^2/\mathfrak{S}_2$ (see §3.3).

Thanks to work of Boissière *et alii* [8] and Debarre–Macrì [10], Theorem 2 implies (and in fact by Remark 3.10 is equivalent to) the following statement :

Theorem 3 (Corollary 3.9)**.** *Let X be a Hilbert scheme $X = S^{[2]}$ where S is a K3 surface with $Pic(S) = \mathbb{Z}$. Let $\iota \in Bir(X)$ be a non-trivial birational automorphism. Then (ι is a non-symplectic birational involution, and) ι acts on $\mathrm{CH}^4(X)$ as in Theorem 2.*

In §3.6, we provide two applications to Theorem 2 : in Corollary 3.11 we extend Theorem 1 to codimension-3 cycles when X is birational to the Hilbert square of a K3 surface, while in Corollary 3.12 we show that the canonical zero-cycle can be characterized as the class of any point lying on a uniruled divisor whose class is ι-invariant.

There are three other explicit families of hyperKähler varieties such that all members have an anti-symplectic involution : the double EPW quartics of [21], the double EPW cubes of [20] and the eightfolds of [29]. It would be interesting to try the argument of the present note for those hyperKählers that are in addition birational to a Hilbert scheme of a K3 surface.

Conventions. In this note, the word *variety* will refer to an integral scheme of finite type over $\mathbb{C}$. By *quotient variety*, we will mean a finite quotient of a smooth variety. For a variety X, $\mathrm{CH}_j(X)$ will denote the Chow group of j-dimensional algebraic cycles on X with $\mathbb{Q}$-coefficients. For X smooth of dimension n the notations $\mathrm{CH}_j(X)$ and $\mathrm{CH}^{n-j}(X)$ will be used interchangeably. We will write $H^j(X)$ to indicate singular cohomology $H^j(X(\mathbb{C}), \mathbb{Q})$. The notation $\mathrm{CH}^j_{hom}(X)$ will be used to indicate the subgroups of homologically trivial cycles. We denote $V = V^+ \oplus V^-$ the eigenspace decomposition of an involution acting on a vector space V.

1. Zero-cycles on Calabi–Yau hypersurfaces with quotient singularities

Our approach to proving Theorem 1 will involve determining the subgroup of $\mathrm{CH}^4(X)$ that is generated by the intersection of ι-invariant cycles on X of positive codimension. Such cycles are pull-backs of the intersection of cycles of positive codimension on the EPW sextic, which is a Calabi–Yau hypersurface that is a quotient variety. We observe that Voisin's [45, Theorem 3.4] can be generalized to the case of Calabi–Yau hypersurfaces that are quotient varieties. First we recall that intersection theory on smooth varieties extends to quotient varieties if one is ready to work with rational coefficients :

Lemma 1.1. *Let M be a* quotient variety, i.e. $M = M'/G$ *where M' is a smooth quasi-projective variety and $G \subset Aut(M')$ is a finite group. Then* $\mathrm{CH}^*(M) := \oplus_i \mathrm{CH}^i(M) := \oplus_i \mathrm{CH}_{\dim M-i}(M)$ *is a commutative graded ring, with the usual functorial properties.*

Proof. According to [19, Example 17.4.10], the natural map

$$\mathrm{CH}^i(M) \ \to \ \mathrm{CH}_{\dim M-i}(M)$$

from operational Chow cohomology (with $\mathbb{Q}$-coefficients) to the usual Chow groups (with $\mathbb{Q}$-coefficients) is an isomorphism. The lemma follows from the good formal properties of operational Chow cohomology. □

Theorem 1.2. *Let $Y \subset \mathbb{P}^{n+1}(\mathbb{C})$ be a hypersurface of degree $n+2$, and assume Y is a quotient variety. Then the image of the intersection product map*

$$\mathrm{CH}^i(Y) \otimes \mathrm{CH}^{n-i}(Y) \ \to \ \mathrm{CH}^n(Y), \quad 0 < i < n,$$

is one-dimensional.

Proof of Theorem 1.2. In the case Y is a *general* hypersurface, this is due to Voisin [45, §3] (this was extended to general Calabi–Yau complete intersections by L. Fu [14]). The genericity assumption is only made in order to ensure that Y is smooth and the Fano variety $F(Y)$ of lines in Y is of the expected dimension (which is $n-3$). Let us check that Voisin's argument extends to all Calabi–Yau hypersurfaces that are quotient varieties.

First we introduce some notation. If $\mathcal{X} \to B$ is a complex morphism to a smooth complex variety B, we denote Z_b the fiber over $b \in B(\mathbb{C})$ of the subscheme $\mathcal{Z} \subseteq \mathcal{X}$, while we denote $\Gamma|_b$ the Gysin fiber [19] in $\mathrm{CH}_*(X_b)$ of the cycle class $\Gamma \in \mathrm{CH}_*(\mathcal{X})$.

Let now $B = \mathbb{P}H^0(\mathbb{P}^{n+1}_{\mathbb{C}}, \mathcal{O}(n+2))$ be the space parameterizing degree $n+2$ hypersurfaces in $\mathbb{P}^{n+1}_{\mathbb{C}}$ and let $\mathcal{Y} \to B$ be the corresponding universal family, *i.e.* $\mathcal{Y} := \{(f, x) : f(x) = 0\} \subseteq B \times \mathbb{P}^{n+1}_{\mathbb{C}}$. Let $B^\circ \subset B$ be the non-empty open subset parameterizing smooth hypersurfaces the Fano varieties of which have dimension $n-3$. Voisin's argument in the proof of [45, Theorem 3.1] provides relative cycle classes $o \in \mathrm{CH}^n(\mathcal{Y}_{B^\circ})$ and $R, \Gamma \in \mathrm{CH}^{2n}(\mathcal{Y} \times_{B^\circ} \mathcal{Y} \times_{B^\circ} \mathcal{Y})$ such that

$$\delta_{Y_b} = p_{12}^*\Delta_{Y_b} \cdot p_3^* o|_b + p_{13}^*\Delta_{Y_b} \cdot p_2^* o|_b + p_{23}^*\Delta_{Y_b} \cdot p_1^* o|_b + R|_b + \Gamma|_b \quad \text{in } \mathrm{CH}^{2n}(Y_b \times Y_b \times Y_b) \quad (1)$$

for all $b \in B^\circ$ with the following properties : $o|_b = \frac{1}{n+2} h^n$ with h the hyperplane class on Y_b, Δ_{Y_b} is the diagonal class in $\mathrm{CH}^n(Y_b \times Y_b)$, $\delta_{Y_b} := p_{12}^*\Delta_{Y_b} \cdot p_{23}^*\Delta_{Y_b}$ is the so-called small diagonal, the restriction $R|_b$ of R at $b \in B^\circ$ is some cycle in the image of the restriction map

$$\mathrm{CH}^{2n}\big(\mathbb{P}^{n+1} \times \mathbb{P}^{n+1} \times \mathbb{P}^{n+1}\big) \to \mathrm{CH}^{2n}\big(Y_b \times Y_b \times Y_b\big),$$

and Γ is a multiple of the cycle class attached to

$$\mathrm{Im}\big(\mathcal{L} \times_{B^\circ} \mathcal{L} \times_{B^\circ} \mathcal{L} \ \to \ \mathcal{Y} \times_{B^\circ} \mathcal{Y} \times_{B^\circ} \mathcal{Y}\big) \quad \subset \ \mathcal{Y} \times_{B^\circ} \mathcal{Y} \times_{B^\circ} \mathcal{Y}\,,$$

where $\mathcal{L} \to \mathcal{Y}$ is the "relative universal line" over B. Here, $p_i : Y_b \times Y_b \times Y_b \to Y_b$ is the projection on the i-th factor and $p_{ij} : Y_b \times Y_b \times Y_b \to Y_b \times Y_b$ is the projection on the product of the i-th and j-th factors. This decomposition implies Theorem 1.2 for Y_b with $b \in B^\circ$ (which is [45, Theorem 3.4]). Indeed, for any $\beta \in \mathrm{CH}^i(Y_b)$ and any $\gamma \in \mathrm{CH}^{n-i}(Y_b)$ with $0 < i < n$ one has

$$\beta \cdot \gamma = (\delta_{Y_b})_*(\beta \times \gamma) = \deg(\beta \cdot \gamma) o|_b + (R|_b + \Gamma|_b)_*(\beta \times \gamma) \quad \text{in } \mathrm{CH}^n(Y_b),$$

where the small diagonal δ_{Y_b} is considered as a correspondence from $Y_b \times Y_b$ to Y_b, and one can check that the right-hand side is proportional to $h^n \in \mathrm{CH}^n(Y_b)$.

Let now $\bar{R}, \bar{\Gamma} \in \mathrm{CH}_*(\mathcal{Y} \times_B \mathcal{Y} \times_B \mathcal{Y})$ be relative cycles over B restricting over B° to R and Γ, respectively. (Here we use the $\mathrm{CH}_*(-)$ notation, since $\mathcal{Y} \to B$ is not a smooth morphism and so the fiber product $\mathcal{Y} \times_B \mathcal{Y} \times_B \mathcal{Y}$ may not be smooth and contain components of various dimensions.) A standard Hilbert scheme argument [48, Proposition 2.4] implies that the decomposition (1) extends to the whole parameter space B, in the sense that

$$\delta_{Y_b} = p_{12}^*\Delta_{Y_b} \cdot p_3^* o|_b + p_{13}^*\Delta_{Y_b} \cdot p_2^* o|_b + p_{23}^*\Delta_{Y_b} \cdot p_1^* o|_b + \bar{R}|_b + \bar{\Gamma}|_b \quad \text{in } \mathrm{CH}_n(Y_b \times Y_b \times Y_b)$$

for any $b \in B$.

Since the formalism of correspondences with $\mathbb{Q}$-coefficients goes through unchanged for quotient varieties, the equality $\beta \cdot \gamma = (\delta_{Y_b})_*(\beta \times \gamma)$ is still valid for quotient varieties. Hence, to prove the theorem we just need to understand the action of the correspondences $\bar{R}|_b$ and $\bar{\Gamma}|_b$ for b parameterizing a hypersurface that is a quotient variety. The first is easy : the action of $\bar{R}|_b$ still factors over $\mathrm{CH}^n(\mathbb{P}^{n+1})$ and so $(\bar{R}|_b)_*(\beta \times \gamma)$ is proportional to h^n. As for the second, we can consider the locus swept out by lines

$$\mathcal{Z} := \mathrm{Im}\big(\mathcal{L} \times_B \mathcal{L} \times_B \mathcal{L} \to \mathcal{Y} \times_B \mathcal{Y} \times_B \mathcal{Y}\big) \quad \subset \mathcal{Y} \times_B \mathcal{Y} \times_B \mathcal{Y}\,.$$

The natural morphism $\mathcal{L} \to B$ has fibers of dimension $n-3 \geq 0$ over B°, but the fiber dimension may jump outside of B°. Because of upper-semicontinuity, any fiber L_b has dimension $\geq n-3$. By construction of the refined Gysin homomorphism [19, §6], we have that the zero-cycle $(\bar{\Gamma}|_b)_*(\beta \times \gamma)$ is supported on the image of L_b under the restriction over b of the natural morphism $\mathcal{L} \to \mathcal{Y}$. It follows that $(\bar{\Gamma}|_b)_*(\beta \times \gamma)$ is supported on a finite union of lines contained in Y_b. We can conclude since any 0-cycle on a line is proportional to h^n in Y_b. ☐

Remark 1.3. The image of the intersection product map

$$\mathrm{CH}^1(Y) \otimes \mathrm{CH}^i(Y) \to \mathrm{CH}^{i+1}(Y)\,, \quad 0 \leq i < n\,,$$

is one-dimensional for every hypersurface Y of dimension > 2 that is a quotient variety. Indeed in that case we have $\mathrm{CH}^1(Y) = \mathbb{Q}[c_1(O_Y(1))]$ and $c_1(O_Y(1)) \cdot \alpha$ is the restriction of a cycle on $\mathbb{P}^{n+1}$ to Y. In particular, for Calabi–Yau hypersurfaces of dimension $n \leq 4$ with quotient singularities, the image of the intersection product map

$$\mathrm{CH}^i(Y) \otimes \mathrm{CH}^j(Y) \to \mathrm{CH}^{i+j}(Y)\,, \quad i, j > 0 \text{ and } i + j \leq n$$

is one-dimensional.

The following lemma is used in Remark 1.3.

Lemma 1.4. *Let $\tau\colon Y\subset\mathbb{P}^{n+1}(\mathbb{C})$ be a hypersurface of degree d that is a quotient variety. Let $h\in\mathrm{CH}^1(Y)$ denote the restriction of $c_1(\mathcal{O}_{\mathbb{P}^{n+1}}(1))$ to Y.*

(i) The composition

$$\mathrm{CH}^i(Y)\xrightarrow{\tau_*}\mathrm{CH}^{i+1}(\mathbb{P}^{n+1})\xrightarrow{\tau^*}\mathrm{CH}^{i+1}(Y)$$

is the same as intersecting with dh.

(ii) Assume $n\geq 3$. Then $\mathrm{CH}^1(Y)=\mathbb{Q}[h]$.

Proof. We recall that by convention, $\mathrm{CH}^i(Y)$ is identified with $\mathrm{CH}_{n-i}(Y)$ (Lemma 1.1). Point (i) is [19, Proposition 2.6]. As for (ii), any (possibly singular) hypersurface Y of dimension ≥ 3 has $\mathrm{Pic}(Y)=\mathbb{Z}[h]$ (Grothendieck–Lefschetz). Because quotient varieties are $\mathbb{Q}$-factorial, this implies that $\mathrm{CH}^1(Y)=\mathrm{CH}_{n-1}(Y)=\mathbb{Q}[h]$. □

Remark 1.5. It is possible to relax the hypotheses on the hypersurface Y in Theorem 1.2. Instead of demanding that Y is a quotient variety, the argument proving Theorem 1.2 goes through as soon as Y is an *Alexander scheme* (in the sense of [42], [24]). In this case, $\mathrm{CH}^*(Y)$ (the operational Chow cohomology of Fulton [19]) is isomorphic to the Chow groups $\mathrm{CH}_{n-*}(Y)$, and so the conclusion of Lemma 1.1 holds for Y. Varieties with quotient singularities are examples of Alexander schemes.

2. The Beauville–Voisin conjecture for zero-cycles on double EPW sextics

2.1. **Double EPW sextics and some of their geometry.**

As the name suggests, double EPW sextics are double covers of certain so-called EPW sextics :

Definition 2.1 (Eisenbud–Popescu–Walter [11])**.** Let $A\subset\wedge^3\mathbb{C}^6$ be a subspace which is Lagrangian with respect to the symplectic form on $\wedge^3\mathbb{C}^6$ given by the wedge product. The *EPW sextic associated to A* is

$$Y_A:=\Big\{[v]\in\mathbb{P}(\mathbb{C}^6)\mid \dim\big(A\cap(v\wedge\wedge^2\mathbb{C}^6)\big)\geq 1\Big\}\ \subset\mathbb{P}(\mathbb{C}^6)\,.$$

An *EPW sextic* is an Y_A for some $A\subset\wedge^3\mathbb{C}^6$ Lagrangian.

Theorem 2.2 (O'Grady)**.** *Let Y be an EPW sextic such that the singular locus $S:=Sing(Y)$ is a smooth irreducible surface. Let $f:X\to Y$ be the double cover branched over S. Then X is a smooth hyperKähler fourfold of $K3^{[2]}$ type (called a* double EPW sextic*), and the class $h:=f^*c_1(O_Y(1))\in\mathrm{CH}^1(X)$ defines a polarization of square 2 for the Beauville–Bogomolov form. Double EPW sextics form a 20-dimensional locally complete family.*

Proof. This is [34, Theorem 1.1(2)]. Let us remark that the hypothesis on $\mathrm{Sing}(Y)$ is satisfied by a generic EPW sextic (more precisely, it suffices that the Lagrangian subspace A be in $\mathrm{LG}(\wedge^3V)^0$, which is a certain open dense subset of $\mathrm{LG}(\wedge^3V)$ defined in [34, Section 2]). Letting A vary in $\mathrm{LG}(\wedge^3V)^0$, one obtains a locally complete family with 20 moduli (as noted in [34, Introduction]). □

Let Z be the invariant locus of the involution $\iota:X\to X$, which is also the pre-image of the singular locus S of Y along $f:X\to Y$. The following proposition summarizes the information that will be needed concerning Z and its class in $\mathrm{CH}^2(X)$.

Proposition 2.3. *The surface Z is smooth projective, regular and Lagrangian. Moreover its class modulo rational equivalence satisfies the following relation*

$$3Z = 15h^2 - c_2(X) \quad \text{in } \mathrm{CH}^2(X)\,. \tag{2}$$

Proof. The surface Z is isomorphic to the singular locus of the EPW sextic Y, which is smooth irreducible by assumption. The fixed locus of an anti-symplectic involution on a hyperKähler variety is (smooth and) Lagrangian [4, Lemma 1]. The relation (2) is due to Ferretti [13, Lemma 4.1].

It remains to show that Z is regular, *i.e.* that $q(Z) := h^1(O_Z) = 0$. This is done in [12, Corollary 3.19]. (Alternatively, the irregularity and the geometric genus of Z can also be computed using the Chow-theoretic results of [43], as explained in [27, Corollary 4.2(iv)].) ☐

2.2. **Proof of Theorem 1.** We start by recalling the main result of [13], which establishes the Beauville–Voisin conjecture for the generic double EPW sextic.

Theorem 2.4 (Ferretti [13])**.** *Let X be a double EPW sextic. Consider the $\mathbb{Q}$-subalgebra*

$$\mathrm{R}^*(X) := \langle h, c_j(X)\rangle \quad \subset \mathrm{CH}^*(X)$$

generated by the polarization $h = f^*c_1(O_Y(1))$ and the Chern classes. The restriction of the cycle class map $\mathrm{R}^i(X) \to H^{2i}(X)$ is injective for all i. Moreover,

$$c_2(X)\cdot h = 5h^3 \quad \text{in } \mathrm{CH}^3(X)\,. \tag{3}$$

Proof. For illustrative purposes, let us briefly show how Ferretti's original argument can be slightly simplified by exploiting Theorem 1.2.

First we observe that the tangent bundle T_X is ι-invariant, so that $\mathrm{R}^*(X)$ consists of ι-invariant cycles. In degree 1, the injectivity is obvious. In degree 2, the injectivity follows from the fact that h^2 and $c_2(X)$ are linearly independent in $H^4(X)$. In degree 3, the generators of $\mathrm{R}^3(X)$ are h^3 and $h\cdot c_2(X)$ and, by Remark 1.3, $h\cdot c_2(X)$ is a multiple of h^3, thereby yielding the injectivity in degree 3. In degree 4, the generators of $\mathrm{R}^4(X)$ are h^4, $h^2\cdot c_2(X)$, $c_2(X)\cdot c_2(X)$ and $c_4(X)$. Since $c_2(X)$ is ι-invariant, Theorem 1.2 yields that the subspace of $\mathrm{R}^*(X)$ spanned by h^4, $h^2\cdot c_2(X)$ and $c_2(X)\cdot c_2(X)$ is one-dimensional. The injectivity in degree 4 now follows from the Chern class computation carried out by Ferretti in [13, Proposition 4.5], where the relation (3) is also established. ☐

Consider now the eigenspace decomposition

$$\mathrm{CH}^1(X) = \mathrm{CH}^1(X)^+ \oplus \mathrm{CH}^1(X)^-$$

for the action of the involution ι. Note that $\mathrm{CH}^1(X)^+ = f^*\,\mathrm{CH}^1(Y)$ is one-dimensional spanned by $h := f^*c_1(O_Y(1))$ and that $\mathrm{CH}^1(X)^-$ consists of primitive divisors. The proof of Theorem 1 is a combination of Theorem 1.2, which describes the intersection of ι-invariant cycles on X of positive and complementary codimension, and of the following lemma and elementary claim.

Lemma 2.5. *Let Z be the smooth Lagrangian surface which is the invariant locus of the involution $\iota : X \to X$. Then $Z\cdot D = 0$ in $\mathrm{CH}^3(X)$ for all $D \in \mathrm{CH}^1(X)^-$.*

Proof. Denote $j : Z \hookrightarrow X$ the embedding. Since Z is Lagrangian and defined for all double EPW sextics and since $H^2_{prim}(X) = H^2_{tr}(X)$ for the very general double EPW sextic, we have that $j^*H^2(X)_{prim} = 0$ and hence that $j^* \mathrm{CH}^1(X)^-$ consist of homologically trivial divisors on Z. Since Z is regular (Proposition 2.3), we conclude that $j^* \mathrm{CH}^1(X)^- = 0$ and hence that $Z \cdot \mathrm{CH}^1(X)^- = 0$.

(Alternative proof: since $Z \in \mathrm{CH}^2(X)^+$, we have $Z \cdot \mathrm{CH}^1(X)^- \subset \mathrm{CH}^3(X)^-$. On the other hand, $Z \cdot \mathrm{CH}^1(X)$ is generated by $j_* \mathrm{CH}^1(Z)$. But any divisor in Z is ι-invariant (as Z is the fixed locus of ι) and so $Z \cdot \mathrm{CH}^1(X) \subset \mathrm{CH}^3(X)^+$. The lemma follows from the fact that $\mathrm{CH}^3(X)^+ \cap \mathrm{CH}^3(X)^- = 0$.) ☐

Claim 2.6. *$h \cdot \alpha$ is a multiple of h^3 for all $\alpha \in \mathrm{CH}^2(X)^+$.*

Proof. By the projection formula, the cycle $h \cdot \alpha$ is the pull-back along $f : X \to Y$ of the intersection of $c_1(O_Y(1))$ with the codimension-2 cycle $f_*\alpha$, and so (Remark 1.3) is the pull-back along f of a multiple of $c_1(O_Y(1))^3$, *i.e.* it is a multiple of h^3. ☐

Proof of Theorem 1. First we observe that $c_2(X)$ is ι-invariant (since T_X is ι-invariant) and that $c_4(X)$ belongs to the image of $\mathrm{CH}^2(X)^+ \otimes \mathrm{CH}^2(X)^+ \to \mathrm{CH}^4(X)$ (this follows from [13, Proposition 4.5]). Next, let $D_k \in \mathrm{CH}^1(X)^-$ be anti-invariant divisors on X. We note from Claim 2.6 that, since $D_1 \cdot D_2$ is ι-invariant, $h \cdot D_1 \cdot D_2$ is a multiple of h^3. Thus $h \cdot D_1 \cdot D_2 \cdot D_3$ is a multiple of $h^3 \cdot D_3$. Intersecting Ferretti's relation (2) with $h \cdot D_3$, Lemma 2.5 yields that

$$0 = Z \cdot D_3 \cdot h = 15h^3 \cdot D_3 - c_2(X) \cdot h \cdot D_3 = 15h^3 \cdot D_3 - 5h^3 \cdot D_3 = 10h^3 \cdot D_3,$$

where the third equality comes from Ferretti's relation (3). It follows that $\mathrm{R}^4(X)$ is spanned by cycles in $\mathrm{im}\big(\mathrm{CH}^i(X)^+ \otimes \mathrm{CH}^{4-i}(X)^+ \to \mathrm{CH}^4(X)\big)$ for $i = 1, 2$. We conclude with Theorem 1.2. ☐

Remark 2.7. The alternative proof of Lemma 2.5 also shows that $Z \cdot \mathrm{CH}^2(X)^- = 0$. Combined with Theorem 1, this implies that

$$Z \cdot \mathrm{CH}^2(X) = \mathbb{Q}[c_4(X)] .$$

This is an indication that Z might perhaps be a constant cycle surface in X.

2.3. **A variant of Theorem 1.** If one is willing to drop primitive divisors, then one can deal with codimension-3 cycles:

Theorem 2.8. *Let X be a double EPW sextic, and let ι be its anti-symplectic involution. Consider the $\mathbb{Q}$-subalgebra*

$$\mathrm{R}^*(X) := \langle h, \mathrm{CH}^2(X)^+, c_j(X) \rangle \quad \subset \mathrm{CH}^*(X)$$

generated by $h = f^*c_1(O_Y(1))$, ι-invariant codimension-2 cycles and Chern classes. The restriction of the cycle class map $\mathrm{R}^i(X) \to H^{2i}(X)$ is injective for $i \geq 3$.

Proof. In view of Theorem 1, we only need to prove the injectivity for $i = 3$. Since $c_2(X)$ is ι-invariant, this follows readily from Claim 2.6. ☐

2.4. **Towards the Beauville–Voisin conjecture for double EPW sextics.** Since [7] the cup-product map $\mathrm{Sym}^2 H^2(X) \to H^4(X)$ is injective for all hyperKähler varieties of dimension larger than 2, the Beauville–Voisin conjecture holds in codimension 2 for any hyperKähler variety X of dimension > 2 for which $[c_2(X)]$ does not belong to the image of the restriction $\mathrm{Sym}^2 \mathrm{NS}(X) \to H^4(X)$ of the above cup-product map to the Néron–Severi group of X. This is in particular the case for hyperKähler varieties of dimension > 2 that are deformation equivalent to Hilbert schemes of K3 surfaces or to generalized Kummer varieties. The following proposition thus shows that the Beauville–Voisin conjecture for double EPW sextics reduces to showing that

$$(\mathrm{CH}^1(X)^-)^{\cdot 3} \subseteq h^2 \cdot \mathrm{CH}^1(X)^-. \tag{4}$$

Proposition 2.9. *Let X be a double EPW sextic. Then $c_2(X) \cdot \mathrm{CH}^1(X) = h^2 \cdot \mathrm{CH}^1(X)$.*

Proof. Let Z be the smooth Lagrangian surface which is the invariant locus of the involution $\iota : X \to X$. By Lemma 2.5, we have $Z \cdot \mathrm{CH}^1(X)^- = 0$. Since $3Z = 15h^2 - c_2(X)$ in $\mathrm{CH}^2(X)$ (see (2)), we have $c_2(X) \cdot \mathrm{CH}^1(X)^- = h^2 \cdot \mathrm{CH}^1(X)^-$. Finally, the relation (3) concludes the proof of the proposition. □

Finally, let us mention that, due to Theorem 1 and precisely to the fact that $(\mathrm{CH}^1(X)^-)^{\cdot 4}$ injects in cohomology, it is likely that the recent result of Voisin [49, Theorem 0.3] applies to double EPW sextics ; this would imply that $(\mathrm{CH}^1(X)^-)^{\cdot 3}$ injects in cohomology. Although this would provide new information concerning the Beauville–Voisin conjecture in codimension 3 for double EPW sextics, it is however not clear how to establish the missing relation (4).

3. Hilbert squares of K3 surfaces with an involution

3.1. **MCK decomposition.** Multiplicative Chow–Künneth decompositions were introduced in [38, §8] as a motivic way to provide an explicit candidate for Beauville's conjectural splitting of the conjectural Bloch–Beilinson filtration on the Chow rings of hyperKähler varieties. First, we recall what a Chow–Künneth decomposition is.

Definition 3.1 (Murre [32])**.** Let X be a smooth projective variety of dimension n. We say that X has a *Chow–Künneth decomposition* (CK decomposition for short) if there exists a decomposition of the diagonal

$$\Delta_X = \Pi_X^0 + \Pi_X^1 + \cdots + \Pi_X^{2n} \quad \text{in } \mathrm{CH}^n(X \times X)\,,$$

such that the Π_X^i are mutually orthogonal idempotents and $(\Pi_X^i)_* H^*(X) = H^i(X)$.

Assuming the Bloch–Beilinson conjectures, Jannsen [23] proved that all smooth projective varieties admit a CK decomposition and moreover the CK projectors Π_X^i induce a splitting of the Bloch–Beilinson filtration on the Chow *groups*. A sufficient condition for the induced splitting to be compatible with the ring structure is given by the following definition.

Definition 3.2 (Shen–Vial [38])**.** Let X be a smooth projective variety of dimension n. Let $\delta_X \in \mathrm{CH}^{2n}(X \times X \times X)$ be the class of the small diagonal

$$\delta_X := \big\{(x,x,x) \mid x \in X\big\} \subset X \times X \times X\,.$$

A *multiplicative Chow–Künneth decomposition* (MCK decomposition for short) is a CK decomposition $\{\Pi^i_X\}$ of X that is *multiplicative*, *i.e.* that satisfies

$$\Pi^k_X \circ \delta_X \circ (\Pi^i_X \times \Pi^j_X) = 0 \quad \text{in } \mathrm{CH}^{2n}(X \times X \times X) \quad \text{for all } i + j \neq k\,.$$

An MCK decomposition is necessarily *self-dual*, *i.e.* it satisfies $\Pi^k_X = {}^t\Pi^{2n-k}_X$ for all k, where the superscript t indicates the transpose correspondence ; see [18, footnote 24].

From the definition, it follows that if X has an MCK decomposition $\{\Pi^i_X\}$, then setting

$$\mathrm{CH}^i_{(j)}(X) := (\Pi^{2i-j}_X)_*\,\mathrm{CH}^i(X)\,,$$

one obtains a bigraded ring structure on the Chow ring : that is, the intersection product sends $\mathrm{CH}^i_{(j)}(X) \otimes \mathrm{CH}^{i'}_{(j')}(X)$ to $\mathrm{CH}^{i+i'}_{(j+j')}(X)$. In other words, an MCK decomposition induces a splitting of the conjectural Bloch–Beilinson filtration on the Chow ring of X.

While the existence of a CK decomposition for any smooth projective variety is expected (either as part of the Bloch–Beilinson conjectures, or as part of Murre's conjectures [32, 23]), the property of having an MCK decomposition is severely restrictive ; for example, a very general curve of genus ≥ 3 does not admit an MCK decomposition (although the conjectural BB filtration on the Chow ring of curves splits). The existence of an MCK decomposition is closely related to Beauville's "weak splitting property" [3], and it is conjectured in [38, Conjecture 4] that hyperKähler varieties should admit an MCK. The seminal work of Beauville–Voisin [5] establishes for K3 surfaces S the existence of a canonical zero-cycle $o \in \mathrm{CH}^2(S)$ of degree 1 that "decomposes" the small diagonal in $S \times S \times S$. By [38, Proposition 8.14], this can be reinterpreted as saying that the Chow–Künneth decomposition defined by $\Pi^0_S := o \times S$, $\Pi^4_S := S \times o$ and $\Pi^2_S = \Delta_S - \Pi^0_S - \Pi^4_S$ is multiplicative. Beyond the case of K3 surfaces, the MCK conjecture for hyperKähler varieties has been established for Hilbert squares of K3 surfaces in [38], more generally for Hilbert schemes of length-n subschemes on K3 surfaces in [40], and for generalized Kummer varieties in [17]. Other examples of varieties admitting an MCK can be found in [39] and [28]. For more ample discussion and examples of varieties with an MCK decomposition, we refer to [38, Section 8] and also [40], [39], [17], [28].

3.2. **The Chow rings of birational hyperKähler varieties.** Consider two birational hyperKähler varieties. We recall a result of Rieß showing the existence of a correspondence inducing an isomorphism between their Chow rings.

Theorem 3.3 (Rieß [36]). *Let $\phi\colon X \dashrightarrow X'$ be a birational map between two hyperKähler varieties of dimension n.*

(i) There exists a correspondence $R_\phi \in \mathrm{CH}^n(X \times X')$ such that $(R_\phi)_*\colon \mathrm{CH}^*(X) \to \mathrm{CH}^*(X')$ is a graded ring isomorphism.

(ii) For any j, there is equality $(R_\phi)_* c_j(X) = c_j(X')$ in $\mathrm{CH}^j(X')$.

(iii) There is equality

$$(\Gamma_R)_* = (\bar{\Gamma}_\phi)_*\colon \quad \mathrm{CH}^j(X) \;\to\; \mathrm{CH}^j(X') \quad \textit{for } j = 0, 1, n-1, n$$

(where $\bar{\Gamma}_\phi$ denotes the closure of the graph of ϕ).

Proof. Item (i) is [36, Theorem 3.2], while item (ii) is [36, Lemma 4.4].

In a nutshell, the construction of the correspondence R_ϕ is as follows : [36, Section 2] provides a diagram

$$\begin{array}{ccc} \mathcal{X} & \stackrel{\Psi}{\dashrightarrow} & \mathcal{X}' \\ & \searrow \quad \swarrow & \\ & C & \end{array}$$

where $\mathcal{X}, \mathcal{X}'$ are algebraic spaces over a quasi-projective curve C such that the fibers $\mathcal{X}_0, \mathcal{X}'_0$ are isomorphic to X resp. X', and where $\Psi\colon \mathcal{X} \dashrightarrow \mathcal{X}'$ is a birational map inducing an isomorphism $\mathcal{X}_{C\setminus 0} \cong \mathcal{X}'_{C\setminus 0}$ and whose restriction $\Psi|_{\mathcal{X}_0}$ coincides with ϕ. The correspondence R_ϕ is then defined as the specialization (in the sense of [19], extended to algebraic spaces) of the graphs of the isomorphisms $\mathcal{X}_c \cong \mathcal{X}'_c$, $c \neq 0$.

Point (iii) is not stated explicitly in [36], and can be seen as follows. Let $U \subset X$ be the locus on which ϕ induces an isomorphism. The complement $T := X \setminus U$ has codimension ≥ 2. Using complete intersections of hypersurfaces, one can find a closed subset $\mathcal{T} \subset \mathcal{X}$ of codimension 2 such that $\mathcal{T}_0$ contains T, *i.e.* Ψ restricts to an isomorphism on $V := \mathcal{X}_0 \setminus \mathcal{T}_0$. Since specialization commutes with pullback, the restriction of R_ϕ to $V \times X'$ is the specialization of the graphs of the morphisms $\mathcal{X}_c \setminus \mathcal{T}_c \to \mathcal{X}'_c$, $c \neq 0$ to $V \times X'$, which is exactly the graph of the morphism $\phi|_V$, *i.e.*

$$R_\phi|_{V\times X'} = \Gamma_{(\phi|_V)} = \bar\Gamma_\phi|_{V\times X'} \ \text{ in } \mathrm{CH}^n(V \times X')\,.$$

It follows that one has

$$R_\phi = \bar\Gamma_\phi + \gamma \ \text{ in } \mathrm{CH}^n(X \times X')\,,$$

where γ is some cycle supported on $\mathcal{T}_0 \times X'$. This proves the "moreover" statement : 0-cycles and 1-cycles on X can be moved to be disjoint of $\mathcal{T}_0$, and so γ_* acts as zero on $\mathrm{CH}^j(X)$ for $j \geq n-1$. Likewise, γ^* acts as zero on $\mathrm{CH}^j(X')$ for $j \leq 1$ for dimension reasons, and so (by inverting the roles of X and X') statement (iii) is proven. ☐

Remark 3.4. As observed in [40, Introduction], the correspondence R_ϕ of Theorem 3.3 actually induces an isomorphism of Chow motives as $\mathbb{Q}$-algebra objects. This implies that the property "having an MCK decomposition" is birationally invariant among hyperKähler varieties.

3.3. **MCK for $K3^{[2]}$.** Let S be a K3 surface. We denote $S^{[2]}$ the Hilbert scheme of length-2 subschemes of S and $Z := \{(\zeta, x) \in S^{[2]} \times S : x \in \mathrm{Supp}(\zeta)\}$ the corresponding universal family. The former is naturally the quotient of the blow-up $\widetilde{S \times S}$ of $S \times S$ along the diagonal under the natural involution switching the factors, while the latter comes equipped with two projection maps :

$$\begin{array}{ccc} Z & \xrightarrow{p} & S^{[2]} \\ {\scriptstyle q}\downarrow & & \\ S & & \end{array}\,.$$

Recall that every divisor class on $S^{[2]}$ is of the form $p_*q^*D_S + a\delta$ for some divisor class D_S on S and some integer a, where δ denotes half of the image of the exceptional divisor under the quotient morphism $\widetilde{S \times S} \to S^{[2]}$.

For a closed point $x \in S$, $S_x := p(q^{-1}(x))$ defines a smooth subvariety of $S^{[2]}$ (which canonically identifies with the blow-up of S at x) and its class in $\mathrm{CH}^2(S^{[2]})$ is $Z_*[x]$. For two distinct points $x, y \in S$, $[x, y]$ denotes the point of $S^{[2]}$ that corresponds to the subscheme $\{x, y\} \subset S$. When $x = y$, $[x, x]$ denotes the element in $\mathrm{CH}^4(S^{[2]})$ represented by any point corresponding to a nonreduced subscheme of length 2 of S supported at x. As cycles, we have $S_x \cdot S_y = [x, y]$.

The following statement summarizes the results concerning the Chow ring of hyperKähler varieties birational to the Hilbert square of a K3 surface that will be needed for the proofs of Theorems 2 and 3.6.

Theorem 3.5 (Shen–Vial [38]). *Let S be a K3 surface, and let X be a hyperKähler fourfold birational to the Hilbert square $S^{[2]}$. Then X admits a self-dual MCK decomposition such that the induced bigraded ring structure $\mathrm{CH}^*_{(*)}(X)$ on $\mathrm{CH}^*(X)$ coincides with the bigrading on $\mathrm{CH}^*(X)$ defined by the "Fourier transform" of [38] and enjoys the following properties :*

(i) $c_j(X) \in \mathrm{CH}^j_{(0)}(X)$ *for all* j *;*
(ii) The multiplication map $\cdot D^2 \colon \mathrm{CH}^2_{(2)}(X) \to \mathrm{CH}^4_{(2)}(X)$ *is an isomorphism for any choice of divisor* $D \in \mathrm{CH}^1(X)$ *with* $\deg(D^4) \neq 0$ *;*
(iii) The intersection product map $\mathrm{CH}^2_{(2)}(X) \otimes \mathrm{CH}^2_{(2)}(X) \to \mathrm{CH}^4_{(4)}(X)$ *is surjective.*

Moreover, with respect to the choice of any birational map $X \overset{\sim}{\dashrightarrow} S^{[2]}$*, the bigraded pieces of the Chow groups have the following explicit descriptions :*

- $\mathrm{CH}^0(X) = \mathrm{CH}^0_{(0)}(X) = \mathbb{Q}[X]$ *;*
- $\mathrm{CH}^1(X) = \mathrm{CH}^1_{(0)}(X)$ *injects in cohomology* via *the cycle class map ;*
- $\mathrm{CH}^2(X) = \mathrm{CH}^2_{(0)}(X) \oplus \mathrm{CH}^2_{(2)}(X)$*, where* $\mathrm{CH}^2_{(2)}(X) = \langle S_x - S_y : x, y \in S \rangle$ *;*
- $\mathrm{CH}^3(X) = \mathrm{CH}^3_{(0)}(X) \oplus \mathrm{CH}^3_{(2)}(X)$*,* $\mathrm{CH}^3_{(2)}(X) = \mathrm{CH}^3_{hom}(X)$ *;*
- $\mathrm{CH}^4(X) = \mathrm{CH}^4_{(0)}(X) \oplus \mathrm{CH}^4_{(2)}(X) \oplus \mathrm{CH}^4_{(4)}(X)$*, where* $\mathrm{CH}^4_{(0)}(X) = \mathbb{Q}[o, o]$*,* $\mathrm{CH}^4_{(2)}(X) = \langle [x, o] - [y, o] : x, y \in S \rangle$ *and* $\mathrm{CH}^4_{(4)}(X) = \langle [x, y] - [x, o] - [y, o] + [o, o] : x, y \in S \rangle$.

Here, o denotes any point lying on a rational curve on S.

Proof. By the result of Rieß (Theorem 3.3), birational hyperKähler varieties have isomorphic Chow motives as algebra objects (and Chern classes are sent to Chern classes). It follows that the proof of the theorem reduces to the case of $X = S^{[2]}$; see *e.g.* [38, Section 6] and [40, Introduction].

We consider the MCK on X constructed in [38, Theorem 13.4] ; its relation with the Fourier transform is [38, Theorem 15.8]. Statement (i) about the Chern classes is [38, Lemma 13.7(iv)], while statement (iii) is [38, Proposition 15.6]. The explicit description of the Chow groups is the combination of [38, Theorem 2], [38, Proposition 15.6] and [38, Proposition 12.9].

It remains to check (ii). Let $D \in \mathrm{CH}^1(S^{[2]})$. Then $D = p_*q^*D_S + a\delta$, where $a \in \mathbb{Q}$ and $D_S \in \mathrm{CH}^1(S)$. As in the proof of [38, Proposition 12.8], one computes $D^2 \cdot S_x = -a^2[x, x] + \deg(D_S^2)[x, o]$. Combined with the fact that $[x, x] = 2[x, o] - [o, o]$ ([38, Proposition 12.6]) and the fact that $q(D) = \deg(D_S^2) - 2a^2$, one finds

$$D^2 \cdot (S_x - S_y) = q(D)([x, o] - [y, o]).$$

Now if D is such that $\deg(D^4) \neq 0$, since $q(D)^2 = \lambda \deg(D^4)$ with λ the Fujiki–Beauville–Bogomolov constant (which is non-zero), we see that intersecting with D^2 induces an isomorphism $\mathrm{CH}^2_{(2)}(X) \xrightarrow{\sim} \mathrm{CH}^4_{(2)}(X)$. □

3.4. **Proof of Theorem 2.** Before proving Theorem 2, we first prove the following statement, which determines the action of ι on the relevant pieces of the Fourier decomposition $\mathrm{CH}^*_{(*)}(X)$.

Theorem 3.6. *Let X be a smooth double EPW sextic, and assume that X is birational to a Hilbert square $S^{[2]}$ with S a K3 surface. Let $\iota \in Aut(X)$ be the anti-symplectic involution coming from the double cover $f\colon X \to Y$, where $Y \subset \mathbb{P}^5$ is an EPW sextic. Then ι^* commutes with the Fourier decomposition,* i.e. *ι^* respects the grading of $\mathrm{CH}^i_{(*)}(X)$ given by Theorem 3.5 for all i. Moreover, we have $\mathrm{CH}^1(X)^+ = \mathbb{Q}[h]$ and $\mathrm{CH}^3_{(0)}(X)^+ = \mathbb{Q}[h^3]$, together with*

$$\begin{aligned} \iota^* = \mathrm{id}\colon \quad & \mathrm{CH}^4_{(0)}(X) \to \mathrm{CH}^4_{(0)}(X)\,; \\ \iota^* = -\mathrm{id}\colon \quad & \mathrm{CH}^4_{(2)}(X) \to \mathrm{CH}^4_{(2)}(X)\,; \\ \iota^* = \mathrm{id}\colon \quad & \mathrm{CH}^4_{(4)}(X) \to \mathrm{CH}^4_{(4)}(X)\,; \\ \iota^* = -\mathrm{id}\colon \quad & \mathrm{CH}^2_{(2)}(X) \to \mathrm{CH}^2_{(2)}(X)\,. \end{aligned}$$

Proof. Recall that $h = f^*c_1(O_Y(1))$; in particular, h is ι-invariant. Since X satisfies the assumptions of Theorem 3.5, we know that $\mathrm{CH}^4(X)$ is generated by $\mathrm{CH}^2_{(2)}(X)$ and by h. Therefore, Theorem 3.6 for $\mathrm{CH}^4(X)$ would follow from knowing that $\mathrm{CH}^2_{(2)}(X)$ is ι-anti-invariant. Considering that $\mathrm{CH}^2_{(2)}(X)$ consists of homologically trivial cycles, the latter would follow from knowing that $\mathrm{CH}^2_{hom}(Y) = 0$, where Y is the EPW sextic. However, although it is conjectured that $\mathrm{CH}^2_{hom}(Y) = 0$ for any hypersurface Y of dimension ≥ 4, this is an intractable problem for Calabi–Yau hypersurfaces and for hypersurfaces of general type – not a single case is known. Instead, our strategy consists in first showing that $\mathrm{CH}^4_{(2)}(X)$ is ι-anti-invariant and then deduce that $\mathrm{CH}^2_{(2)}(X)$ is ι-anti-invariant.

We now proceed to the proof of Theorem 3.6 and argue step-by-step. Since $\mathrm{CH}^i(X) = \mathrm{CH}^i_{(0)}(X)$ for $i = 0, 1$, it is clear that ι preserves the multiplicative grading in these cases. If X is very general in moduli, then $\mathrm{CH}^1(X) = \mathbb{Q}[h]$, and consequently $H^2(X) = h^\perp \oplus \mathbb{Q}[h]$ where $h^\perp$ is an irreducible polarized Hodge structure. Since ι is anti-symplectic, we get that ι acts as $-\mathrm{id}$ on $h^\perp$. This must hold for all smooth double EPW sextics. It follows that $\mathrm{CH}^1(X)^+ = \mathbb{Q}[h]$. Alternatively, $\mathrm{CH}^1(X)^+ = f^*\,\mathrm{CH}^1(Y)$, but $\mathrm{CH}^1(Y) = \mathbb{Q}[c_1(O_Y(1))]$ by the Grothendieck–Lefschetz theorem.

By Theorem 3.5, $\mathrm{CH}^3(X) = \mathrm{CH}^3_{(0)}(X) \oplus \mathrm{CH}^3_{(2)}(X)$, where $\mathrm{CH}^3_{(2)}(X) = \mathrm{CH}^3_{hom}(X)$ is clearly stable under the action of ι. In particular, $\mathrm{CH}^3_{(0)}(X)$ identifies with the image of the cycle class map $\mathrm{CH}^3(X) \to H^6(X)$. Therefore, by the hard Lefschetz theorem together with the multiplicativity of the bigrading on $\mathrm{CH}^*_{(*)}(X)$, we have $\mathrm{CH}^3_{(0)}(X) = h^2 \cdot \mathrm{CH}^1(X)$. It follows readily that $\mathrm{CH}^3_{(0)}(X)^+ = \mathbb{Q}[h^3]$.

We now deal with the remaining cases of codimension-4 and codimension-2 cycles. First, note that $\mathrm{CH}^4_{(0)}(X) = \mathbb{Q}[h^4]$ (apply Theorem 3.5), so that $\mathrm{CH}^4_{(0)}(X)$ is stable under the action of ι, *i.e.* we have $\mathrm{CH}^4_{(0)}(X) = \mathrm{CH}^4_{(0)}(X)^+$. Let now $\alpha \in \mathrm{CH}^2_{(2)}(X)$, and consider the ι-invariant zero-cycle $\alpha \cdot h^2 + \iota^*(\alpha \cdot h^2)$. By the projection formula, we have

$$\alpha \cdot h^2 + \iota^*(\alpha \cdot h^2) = f^*\big((f_*\alpha) \cdot c_1(O_Y(1))^2\big) \ \in \ \mathrm{CH}^4(X).$$

It follows readily from Remark 1.3 that this cycle is a multiple of h^4. Since α is algebraically trivial, the zero-cycles $\alpha \cdot h^2$ and $\iota^*(\alpha \cdot h^2)$ are of degree zero, and hence

$$\alpha \cdot h^2 + \iota^*(\alpha \cdot h^2) = 0 \quad \text{in } \mathrm{CH}^4(X)\,. \tag{5}$$

As any element in $\mathrm{CH}^4_{(2)}(X)$ is of the form $\alpha \cdot h^2$ with $\alpha \in \mathrm{CH}^2_{(2)}(X)$ (Theorem 3.5(ii)), equality (5) proves that $\mathrm{CH}^4_{(2)}(X) = \mathrm{CH}^4_{(2)}(X)^-$.

Thus the action of ι commutes with the covariant action of Π^6_X on $\mathrm{CH}^4(X)$. By Bloch–Srinivas [6], this means that

$$\iota^* \circ \Pi^6_X = \Pi^6_X \circ \iota^* + \Gamma \quad \text{in } \mathrm{CH}^4(X \times X)$$

for some correspondence Γ supported on $D \times X$ for some divisor D in X. Let $\tilde{D} \to D$ be a desingularization of D. Since $\mathrm{CH}^2_{(2)}(X)$ consists of algebraically trivial cycles, and since $H^3(X) = 0$, the contravariant action $\Gamma^* : \mathrm{CH}^2_{(2)}(X) \to \mathrm{CH}^2(X)$ factors through $\ker(AJ : \mathrm{CH}^1(\tilde{D}) \to \mathrm{Pic}^0(\tilde{D}))$, which is zero. Using that Π^2_X is the transpose of Π^6_X, it follows that the action of ι and that of Π^2_X commute on $\mathrm{CH}^2_{(2)}(X)$; in particular, the action of ι on $\mathrm{CH}^2(X)$ preserves $\mathrm{CH}^2_{(2)}(X)$. Equation (5) together with Theorem 3.5(ii) yields that $\alpha = -\iota^*\alpha$, proving that $\mathrm{CH}^2_{(2)}(X) = \mathrm{CH}^2_{(2)}(X)^-$.

It then follows from Theorem 3.5(iii) that the action of ι on $\mathrm{CH}^4(X)$ preserves $\mathrm{CH}^4_{(4)}(X)$ and that $\mathrm{CH}^4_{(4)}(X) = \mathrm{CH}^4_{(4)}(X)^+$.

It remains to check that the action of ι on $\mathrm{CH}^2(X)$ preserves $\mathrm{CH}^2_{(0)}(X)$. Let α be a cycle in $\mathrm{CH}^2_{(0)}(X)$ and let us write $\iota^*\alpha = \beta_0 + \beta_2$, where $\beta_0 \in \mathrm{CH}^2_{(0)}(X)$ and $\beta_2 \in \mathrm{CH}^2_{(2)}(X)$. Since $h^2 \cdot \mathrm{CH}^2_{(2j)}(X) \subseteq \mathrm{CH}^4_{(2j)}(X)$, and since $\mathrm{CH}^4_{(0)}(X) = \mathrm{CH}^4_{(0)}(X)^+$ and $\mathrm{CH}^4_{(2)}(X) = \mathrm{CH}^4_{(2)}(X)^-$, we have

$$\beta_0 \cdot h^2 + \beta_2 \cdot h^2 = \alpha \cdot h^2 = \iota^*(\alpha \cdot h^2) = \iota^*(\beta_0 \cdot h^2) + \iota^*(\beta_2 \cdot h^2) = \beta_0 \cdot h^2 - \beta_2 \cdot h^2.$$

Consequently, $\beta_2 \cdot h^2 = 0$, and hence, thanks to Theorem 3.5(ii), also $\beta_2 = 0$. Thus $\iota^*\alpha$ belongs to $\mathrm{CH}^2_{(0)}(X)$, thereby concluding the proof of the theorem. □

We now prove Theorem 2 stated in the introduction :

Proof of Theorem 2. Let us write $\phi\colon X' \to X$ for the birational map from a Hilbert square X', and $\iota' := \phi^{-1} \circ \iota \circ \phi \in \mathrm{Bir}(X')$ for the birational automorphism induced by $\iota \in \mathrm{Aut}(X)$.

To prove Theorem 2 for X', it will suffice, thanks to the explicit generators for $\mathrm{CH}^4_{(2j)}(X')$ given in Theorem 3.5, to prove that

$$(\iota')_* = (-1)^j \,\mathrm{id}\colon \quad \mathrm{CH}^4_{(2j)}(X') \ \to \ \mathrm{CH}^4(X')\,. \tag{6}$$

Let R_ϕ be Rieß's correspondence from Theorem 3.3 ; it induces, thanks to Remark 3.4, an isomorphism of bigraded rings $(R_\phi)_*\colon \mathrm{CH}^*_{(*)}(X') \xrightarrow{\simeq} \mathrm{CH}^*_{(*)}(X)$. We claim that there is a commutative diagram

$$\begin{array}{ccc} \mathrm{CH}^4(X') & \xrightarrow{(R_\phi)_*} & \mathrm{CH}^4(X) \\ \downarrow{\scriptstyle \iota'_*} & & \downarrow{\scriptstyle \iota_*} \\ \mathrm{CH}^4(X') & \xrightarrow{(R_\phi)_*} & \mathrm{CH}^4(X). \end{array}$$

Clearly this claim, combined with Theorem 3.6, establishes equality (6). To prove the claim, we use Theorem 3.3(iii). We note that any zero-cycle $b \in \mathrm{CH}^4(X')$ can be represented by a cycle β supported on an open $U' \subseteq X'$ such that the birational map ϕ restricts to an isomorphism $\phi|_{U'} \colon U' \xrightarrow{\simeq} U$ and $U \subseteq X$ is ι-stable. Because $(R_\phi)_*$ and $(\bar{\Gamma}_\phi)_*$ coincide on 0-cycles (Theorem 3.3(iii)), the image $(R_\phi)_* b \in \mathrm{CH}^4(Z)$ is supported on $(\phi|_{U'})(\mathrm{Supp}(\beta)) \subset U$, which gives the claimed commutativity

$$\iota_*(R_\phi)_*(b) = (R_\phi)_*(\iota')_*(b) \ \text{ in } \mathrm{CH}^4(X)\ .$$

□

3.5. **A concise reformulation of Theorem 2.** In order to restate Theorem 2 in a concise way, we invoke the following result :

Theorem 3.7 (Debarre–Macrì [10])**.** *Let S be a polarized K3 surface of degree d and Picard number 1, and let $X := S^{[2]}$. Then Bir(X) is trivial except in the following cases :*

- *$d = 2$, or $d > 2$ and d verifies*
$$(*) \quad a^2 d = 2n^2 + 2\ , \quad a, n \in \mathbb{Z}\ ,$$
while the Pell equation
$$\mathcal{P}_{2d}(5)\colon \quad n^2 - 2da^2 = 5$$
has no solution. In this case, Aut(X) = Bir$(X) = \mathbb{Z}/2\mathbb{Z}$.
- *$d = 10$, or d is not divisible by 10 and d verifies $(*)$ and the Pell equation $\mathcal{P}_{2d}(5)$ is solvable. In this case, Aut$(X) = 0$ and Bir$(X) = \mathbb{Z}/2\mathbb{Z}$.*

Moreover, if Bir(X) is non-trivial and $d \notin \{2, 4, 10\}$, X is birational to a double EPW sextic.

Proof. This is [10, Proposition B.3]. This extends and builds on prior work of Boissière *et alii* [8, Theorem 1.1], who had proven the result for Aut(X). □

Remark 3.8. Double EPW sextics isomorphic to the Hilbert square of a K3 surface can be explicitly described [10, Corollary 7.6] and are dense (for the euclidean topology) in the moduli space of double EPW sextics [10, Proposition 7.9]. Double EPW sextics birational to the Hilbert square of a K3 surface are explicitly described by Pertusi [35, Theorem 1.4] (this completes earlier work of Iliev–Madonna [22]).

Corollary 3.9. *Let X be a Hilbert scheme $X = S^{[2]}$ where S is a K3 surface with Pic$(S) = \mathbb{Z}$. Let $\iota \in$ Bir(X) be a non-trivial birational automorphism. Then (ι is a non-symplectic birational involution, and) ι acts on $\mathrm{CH}^4(X)$ as in Theorem 2.*

Proof. Theorem 3.7 implies that ι is the unique non-symplectic birational involution. In case the degree of S is 2, ι must be the automorphism induced by the covering involution $S \to \mathbb{P}^2$, and it is elementary to prove that ι acts on $\mathrm{CH}^4(X)$ as requested. In case the degree of S is 4, ι must be the famous Beauville involution [1], for which Theorem 3.6 (and hence Theorem 2) is known to hold ([16, Corollary 1.8] or [26]). In case the degree of S is 10, ι must be the O'Grady involution [33]. For this case, it was proven in [25] that ι acts as $-\mathrm{id}$ on $\mathrm{CH}^4_{(2)}(X)$. Reasoning as in the proof of Theorem 3.6, this proves that ι acts on $\mathrm{CH}^4(X)$ as in Theorem 2 (a simpler, more geometric argument for the case of the O'Grady involution is given by Lin [30]). Finally, in case the degree of S is ≥ 6 and not equal to 10, X must be birational to a double EPW sextic X' and ι is induced by the covering involution ι' of the double EPW sextic (Theorem 3.7). It follows that ι acts on $\mathrm{CH}^4_{(j)}(X)$ as in Theorem 2. □

Remark 3.10. We observe that Corollary 3.9 in turn implies Theorem 2. Indeed, assume X is a double EPW sextic birational to a Hilbert square $S^{[2]}$. Let d be the degree of S. Since "being birational to a Hilbert square" can be translated, *via* lattice-theory, into a numerical condition on double EPW sextics (*cf.* [35]), all Hilbert squares of degree-d K3 surfaces are birational to a double EPW sextic (and d satisfies the numerical condition of Theorem 3.7). Let $\mathcal{X}_d \to \mathcal{F}_d$ denote the universal family of Hilbert squares of degree-d K3 surfaces. Corollary 3.9 applies to the very general element of this family. But then a standard spread argument ([46, Lemma 3.2]), plus the fact that the graph of the covering involution and the MCK decomposition are universally defined, implies that Theorem 3.6 (and hence Theorem 2) holds for all elements of $\mathcal{X}_d \to \mathcal{F}_d$.

3.6. **Some applications of Theorem 2.** We provide two corollaries to Theorem 2, which conjecturally should hold for all double EPW sextics. First, when X is birational to the Hilbert square of a K3 surface, we can improve Theorem 1 to the case of codimension-3 cycles.

Corollary 3.11. *Let X be a double EPW sextic, and assume either that $\mathrm{CH}^1(X) = \mathbb{Q}[h]$, or that X is birational to the Hilbert square of a K3 surface. Consider the subring*

$$\mathrm{R}^*(X) := \langle \mathrm{CH}^1(X), \mathrm{CH}^2(X)^+, c_j(X) \rangle \quad \subset \mathrm{CH}^*(X) .$$

The cycle class map $\mathrm{R}^i(X) \to H^{2i}(X)$ is injective for $i \geq 3$.

Proof. The case where $\mathrm{CH}^1(X) = \mathbb{Q}[h]$ is the content of Theorem 2.8. In the case X is assumed to be birational to the Hilbert square of a K3 surface, it suffices to show, due to Theorem 3.5, that $\mathrm{CH}^2(X)^+ \subset \mathrm{CH}^2_{(0)}(X)$. Since $\mathrm{CH}^2(X) = \mathrm{CH}^2_{(0)}(X) \oplus \mathrm{CH}^2_{(2)}(X)$, this follows from the facts proven in Theorem 3.6 that ι^* acts as -1 on $\mathrm{CH}^2_{(2)}(X)$ and that $\mathrm{CH}^2_{(0)}(X)$ is stable under the action of ι^*. □

We note that Corollary 3.11 should hold for all i (the problem in proving this is that it is not known whether $\mathrm{CH}^2_{(0)}(X)$ injects into cohomology) and for all double EPW sextics. Second, still when X is birational to the Hilbert square of a K3 surface or when X is generic, we can characterize the canonical zero-cycle on X as being the class of any point lying on a uniruled divisor whose class is ι-invariant.

Corollary 3.12. *Let X be a double EPW sextic, and assume either that $\mathrm{CH}^1(X)=\mathbb{Q}[h]$, or that X is birational to the Hilbert square of a K3 surface. Let $S_1\,\mathrm{CH}^4(X)\subset \mathrm{CH}^4(X)$ denote the subgroup generated by points on uniruled divisors in X. Then*

$$S_1\,\mathrm{CH}^4(X)\cap \mathrm{CH}^4(X)^+ = \mathbb{Q}[c_4(X)]\ .$$

Proof. Let X be a double EPW sextic. From [13], we know that $c_4(X)$ is a multiple of h^4 in $\mathrm{CH}^4(X)$, where $h=f^*c_1(O_Y(1))$ is the ι-invariant polarization on X. From [9, Theorem 1.6 and Remark 1.2] we have

$$S_1\,\mathrm{CH}^4(X)=\mathrm{CH}^1(X)\cdot \mathrm{CH}^3(X). \tag{7}$$

First assume that $\mathrm{CH}^1(X)=\mathbb{Q}[h]$. In that case, we have $S_1\,\mathrm{CH}^4(X)\cap\mathrm{CH}^4(X)^+ = h\cdot\mathrm{CH}^3(X)^+$. The corollary in that case then follows immediately from Remark 1.3.

Second, assume that X is birational to the Hilbert square of a K3 surface. In that case, the subgroup $S_1\,\mathrm{CH}^4(X)$ coincides with $\mathrm{CH}^4_{(0)}(X)\oplus \mathrm{CH}^4_{(2)}(X)$, where $\mathrm{CH}^*_{(*)}(X)$ refers to the Fourier decomposition of Theorem 3.5. This can be seen either as a combination of [47, End of Section 4.1] and the fact that $S_1\,\mathrm{CH}^4(X)$ is a birational invariant due to [36] and the characterization (7), or as a direct consequence of Theorem 3.5. Theorem 3.6 then implies that the ι-invariant part of $S_1\,\mathrm{CH}^4(X)$ is $\mathrm{CH}^4_{(0)}(X)\cong\mathbb{Q}[h^4]$. □

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