Representations of Quivers with Applications to Collections of Matrices with Fixed Similarity Types and Sum Zero

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Abstract

Given a collection of matrix similarity classes $C_1, \dots, C_k$, the additive matrix problem asks under what conditions do there exist matrices $A_i \in C_i$ for $i = 1, \dots, k$ such that $A_1 + \cdots + A_k = 0$. This and similar problems have been examined under various guises in the literature. The results of Crawley-Boevey use the representation theory of quivers to link the additive matrix problem to the root systems of quivers. We relate the results of Crawley-Boevey to another partial solution offered by Silva et al. and develop some tools to interpret the solutions of Silva et al. in terms of root systems.

The results of Crawley-Boevey require us to know the precise Jordan form of the similarity classes; we address the problem of invoking Crawley-Boevey's results when only the invariant polynomials are known and we are not permitted to use polynomial factorization.

We use the machinery of symmetric quivers and symmetric representations to study the problem of finding symmetric matrix solutions to the additive matrix problem. We show the reflection functors, defined for representations of deformed preprojective algebras, can be defined for symmetric representations. We show every rigid irreducible solution to the additive matrix problem can be realized by symmetric matrices and we use algebraic geometry to show that in some circumstances there are solutions which cannot be realized by symmetric matrices. We show there exist symmetric representations of deformed preprojective algebras of root dimension vectors when the underlying quiver is Dynkin or extended Dynkin of type $\tilde{A}_n$ or $\tilde{D}_n$.

Contents

Chapter 1

Introduction and Preliminary Material

Representation theory builds connections between different types of objects in mathematics that may have no obvious connection, such as groups, algebras and quivers. This is achieved by defining representations on the objects which “represent”, in some suitable sense, the objects with structures from linear algebra. The collection of representations, along with suitably defined morphisms, forms a category. Representing complicated mathematical objects with appropriate linear algebra objects allows us to use the power of linear algebra and category theory to study them under a common, well-understood framework.

Let $C_1, \dots, C_k \subseteq M_n(K)$ be a collection of matrix similarity classes. The additive matrix problem asks under what necessary and sufficient conditions on the $C_1, \dots, C_k$ do there exist matrices $A_1, \dots, A_k \in M_n(K)$ such that $A_i \in C_i$ for $i = 1, \dots, k$ and $A_1 + \cdots + A_k = 0$. The existence of such matrix tuples is linked to the existence of certain representations of a particular deformed preprojective algebra, which is linked to certain representations of a quiver. Crawley-Boevey shows that the resolution of several variants of the additive matrix problem is linked, via representation theory, to the root systems of star-shaped quivers.

The results of this thesis can be divided into three areas.

1. We compare the results obtained by Crawley-Boevey to the results obtained by Silva et al. To do this we present some new techniques for studying the dimension

vectors of star-shaped quivers, in particular we present a set of conditions for a dimension vector to be a root. The results of Silva et al. do not explicitly involve quiver representations or root systems, but the results of Crawley-Boevey suggest it is useful to consider these results in such terms. We use the new techniques to express some of the results of Silva et al. in terms of root systems, this allows us to compare them to Crawley-Boevey's results.

2. The results of Crawley-Boevey assume precise knowledge of the Jordan structure of the similarity classes. We consider these results when only the invariant polynomials are known and it is not possible to perform any polynomial factorization. We present a solution to this problem which is applicable when the similarity classes are closed. In this case the Jordan blocks of the classes all have order one, but the precise eigenvalues cannot be directly computed.

3. We consider the symmetric variant of the additive matrix problem, that is in which we require the matrices to be symmetric. We show this is linked to the existence of symmetric representations in a way analogous to the general additive matrix problem. We explore symmetric representations and define symmetric representations of deformed preprojective algebras. We show that reflection functors, which are important tools for understanding representations of deformed preprojective algebras, are well-defined for symmetric representations also. We present several results on the structure of the category of symmetric representations of deformed preprojective algebras, some showing the structure is analogous to the general case, some showing it is not.

In Chapter 2 we recall some basic facts about matrices and present some theorems that are needed throughout. We develop some new polynomial operations in Chapter 3 which allow us to manipulate the roots of polynomials without factorizing them. We look at the results of Crawley-Boevey and Silva et al. in Chapter 4 and develop some machinery to compare them. We also use the machinery developed in Chapter 3 to present a different approach to the results of Crawley-Boevey when only the invariant polynomials are known. We define symmetric representations in Chapter 5, prove the reflection functors have a symmetric analogue and explore the symmetric additive matrix problem. In Chapter 6 we use algebraic geometry to show there exist certain solutions to the additive matrix problem which are not conjugate to solutions to the symmetric additive matrix problem. We also show there always exist symmetric representations of a given dimension vector of the deformed preprojective algebras constructed from Dynkin or extended Dynkin (in most cases) quivers where the given dimension vector is a root.

1.1 Quivers, Roots and Representations

Here we recall the notions of quivers, quiver representations and root systems. These elementary definitions and results are found in many references such as [ASS06], [ARS95] and [Kac90].

1.1.1 Quivers

Definition 1.1.1. A quiver Q is a quadruple (Q_0, Q_1, h, t) where Q_0 and Q_1 are sets and $h, t\colon Q_1 \to Q_0$ are functions from Q_1 to Q_0. We call Q_0 the set of *vertices*, Q_1 the set of *arrows*, h the *head* function and t the *tail* function. Given an arrow $a \in Q_1$ we call $h(a)$ and $t(a)$ the *head* and *tail of a* respectively.

From hereon all quivers are assumed to be *finite*, that is Q_0 and Q_1 are finite sets.

1.1.2 Roots and Dimension Vectors

Let Q be a quiver. A *dimension vector* α of Q is a vector with integer values indexed by the vertices of Q, that is $\alpha \in \mathbb{Z}^{Q_0}$. A dimension vector is *positive* if it is nonzero and all of its entries are non-negative.

Let α, β be dimension vectors. The *Ringel form* $\langle \cdot, \cdot \rangle$ is defined by

$$\langle \alpha, \beta \rangle = \sum_{i \in Q_0} \alpha_i \beta_i - \sum_{a \in Q_1} \alpha_{t(a)} \beta_{h(a)}.$$

The associated *symmetric bilinear form* $(\cdot, \cdot)$ is defined by $(\alpha, \beta) = \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle$.

A *simple root* of Q, also called a *coordinate vector*, is a dimension vector with a 1 at a single loop-free vertex and 0 elsewhere. We denote the simple root with a 1 at $i \in Q_0$ (where i is loop-free) by ϵ_i. Let $i \in Q_0$ the *reflection at i* is the function $s_i\colon \mathbb{Z}^{Q_0} \to \mathbb{Z}^{Q_0}$ defined by $s_i(\alpha) = \alpha - (\alpha, \epsilon_i)\epsilon_i$. This is equivalent to $s_i(\alpha)_j = \alpha_j$ for all $j \neq i$ and

$$s_i(\alpha)_i = \sum_{j \text{ adjacent to } i} \alpha_j - \alpha_i.$$

We say a dimension vector α is in the *fundamental region* if it is positive, has connected support (that is the full subquiver formed by the nonzero vertices is a connected quiver) and for every loop-free vertex $i \in Q_0$ we have $s_i(\alpha)_i \geq \alpha_i$.

Definition 1.1.2. Let $\alpha \in \mathbb{Z}^{Q_0}$. We say α is a *real root* if there is a finite sequence of reflections which take α to a simple root. We say α is an *imaginary root* if there is a finite sequence of reflections which take either α or $-\alpha$ to a vector in the fundamental region. We call a dimension vector a *root* if it is either a real root or an imaginary root.

If α is a root, then either α is a positive dimension vector, or $-\alpha$ is a positive dimension vector.

1.1.3 Representations of Quivers

Here we recall the definition of a quiver representation and of a homomorphism between two quivers representations.

Let Q be a quiver and K an algebraically closed field of characteristic zero. Unless otherwise stated all fields are algebraically closed and of characteristic zero. A *K-representation V of Q* (or simply a *representation* when the field is clear from the context) is an assignment of a K-vector space V_i to each vertex $i \in Q_0$ and of a K-linear map $V_a \colon V_{t(a)} \to V_{h(a)}$ to each arrow $a \in Q_1$. Let V and W be K-representations. A *homomorphism ϕ from V to W* is an assignment of a linear map $\phi_i \colon V_i \to W_i$ to each vertex $i \in Q_0$ such that for each arrow $a \in Q_1$ the *intertwining relation* $\phi_{h(a)} V_a = W_a \phi_{t(a)}$ holds.

Unless otherwise stated all representations are finite dimensional, that is all their vector spaces have finite dimension. If V is finite dimensional, then there is an associated dimension vector $\mathbf{dim}(V) \in \mathbb{Z}^{Q_0}$ defined by $\mathbf{dim}(V)_i = \dim(V_i)$ for all $i \in Q_0$. We denote by $\mathrm{Rep}_K(Q)$ the category whose objects are K-representations of Q and whose morphisms are the homomorphisms of K-representations. Given a representation X the *identity morphism*, that is the homomorphism assigning the identity map 1_{X_i} to each vertex $i \in Q_0$, is denoted 1_X.

The *path algebra KQ* of Q over K is the K-algebra whose underlying vector space has basis: the set of all paths of Q (See [ARS95, Sec. III.1]). A *path p* is either an ordered sequence of arrows $p = a_r \dots a_1$ of Q such that $h(a_i) = t(a_{i+1})$ for $i = 1, \dots, r-1$ called a *nontrivial path*, or the symbol e_i for $i \in Q_0$ called the *trivial path at i*. If p is a nontrivial path we define the *head of p* as $h(p) = h(a_r)$ and the *tail of p* as $t(p) = t(a_1)$. We define the head and tail of a trivial path e_i by $h(e_i) = i$ and $t(e_i) = i$ respectively. Multiplication of basis elements is defined by path concatenation, that is if $p, q \in KQ$ are paths (either trivial or nontrivial) then

$$pq = \begin{cases} \text{path formed by attaching } p \text{ to the end of } q & \text{if } p, q \text{ nontrivial and } t(p) = h(q), \\ p & \text{if } q = e_{t(p)}, \\ q & \text{if } p = e_{h(q)}, \\ 0 & \text{otherwise.} \end{cases}$$

It is well known that the category of left KQ-modules with finite K-dimension is equivalent to the category of finite dimensional K-representations of Q. See [ARS95, Sec. III.1, Thm. 1.5].

1.2 Deformed Preprojective Algebras

Let Q be a quiver and K an algebraically closed field of characteristic zero. Let $\lambda \in K^{Q_0}$, that is λ is a K-vector with entries indexed by vertices of Q.

1.2.1 Definition

Definition 1.2.1. We denote by $\overline{Q}$ the *doubled quiver* of Q, that is $\overline{Q}_0 = Q_0$ and $\overline{Q}_1 = Q_1 \cup \{a^* \colon h(a) \to t(a) \forall a \in Q_1\}$. Informally we form $\overline{Q}_1$ from Q_1 by adjoining an extra arrow a^* for each $a \in Q_1$ going in the reverse direction.

Example 1.2.2. Suppose we have a quiver Q

$$\begin{tikzcd} 1 & 2 \arrow[dr] & 3 \arrow[d] \\ 4 \arrow[u] \arrow[ur] & 5 \arrow[l] & 6 \arrow[l] \end{tikzcd}$$

the doubled quiver is $\overline{Q}$

$$\begin{tikzcd} 1 \arrow[d, bend right] & 2 \arrow[dr] \arrow[dl, bend right] & 3 \arrow[d] \\ 4 \arrow[u] \arrow[ur] \arrow[r, bend right] & 5 \arrow[l] \arrow[r, bend right] & 6 \arrow[l] \arrow[u, bend right] \arrow[ul, bend right] \end{tikzcd}$$

The following definition comes from [CBH98, Sec. 2].

Definition 1.2.3. Let I be the ideal of $K\overline{Q}$ given by

$$I = \left\langle \sum_{a \in Q_1} (aa^* - a^*a) - \sum_{i \in Q_0} \lambda_i e_i \right\rangle.$$

Let $\Pi^\lambda(Q) = K\overline{Q}/I$, we call $\Pi^\lambda(Q)$ the *deformed preprojective algebra* of Q. When $\lambda = 0$ we call $\Pi^0(Q)$ the *preprojective algebra* of Q and denote it $\Pi(Q)$.

The category of left $\Pi^\lambda(Q)$ modules with finite K-dimension is equivalent to the full subcategory of representations of $\overline{Q}$ whose objects are representations V of $\overline{Q}$ which, for each $i \in Q_0$, satisfy:

$$\sum_{a \in Q_1 \colon h(a)=i} V_a V_{a^*} - \sum_{a \in Q_1 \colon t(a)=i} V_{a^*} V_a = \lambda_i 1_{V_i}.$$

We denote this category by $\mathrm{Rep}_K(\Pi^\lambda(Q))$.

1.2.2 Reflection Functors

Recall the definition of a reflection $s_i \colon \mathbb{Z}^{Q_0} \to \mathbb{Z}^{Q_0}$, where $i \in Q_0$, of dimension vectors. The definitions in the section come from [CBH98, Sec. 5] and [CB01, Sec. 2].

Definition 1.2.4. Let $i \in Q_0$ be a loop-free vertex. Define $r_i \colon K^{Q_0} \to K^{Q_0}$ to be

$$r_i(\lambda)_j = \lambda_j - (\epsilon_i, \epsilon_j)\lambda_i.$$

Note that r_i is dual to s_i in the sense that $r_i(\lambda) \cdot \alpha = \lambda \cdot s_i(\alpha)$ for all $\lambda \in K^{Q_0}$ and $\alpha \in \mathbb{Z}^{Q_0}$. We say the reflection at loop-free $i \in Q_0$ is *admissible for the pair* (λ, α) if $\lambda_i \neq 0$. If the reflection at i is admissible for (λ, α), then there is an equivalence of categories between $\mathrm{Rep}_K(\Pi^\lambda(Q))$ and $\mathrm{Rep}_K(\Pi^{r_i(\lambda)}(Q))$ which acts as s_i on dimension vectors (see [CBH98, Thm. 5.1]). Such equivalences are known as *reflection functors*. We discuss reflection functors in more detail in Section 5.2.

Definition 1.2.5. Let $\lambda, \lambda' \in K^{Q_0}$ and $\alpha, \alpha' \in \mathbb{Z}^{Q_0}$. We say the pairs (λ, α) and (λ', α') are equivalent if there is a finite sequence $i_1, \dots, i_l \in Q_0$ of loop-free vertices such that

- $r_{i_l} \dots r_{i_1}(\lambda) = \lambda'$,
- $s_{i_l} \dots s_{i_1}(\alpha) = \alpha'$, and
- the reflection at i_j is admissible for the pair $(r_{i_{j-1}} \dots r_{i_1}(\lambda), s_{i_{j-1}} \dots s_{i_1}(\alpha))$ for all $j = 1, \dots, l$.

So when (λ, α) and (λ', α') are equivalent, there exists an equivalence from $\mathrm{Rep}_K(\Pi^\lambda(Q))$ to representations of $\mathrm{Rep}_K(\Pi^{\lambda'}(Q))$ which acts as $s_{i_l} \dots s_{i_1}$ on dimension vectors where $\alpha' = s_{i_l} \dots s_{i_1}(\alpha)$.

1.2.3 Lifting of Representations

Definition 1.2.6. Let V be a representation of $\overline{Q}$ (this could also be a representation of $\Pi^\lambda(Q)$ for some $\lambda \in K^{Q_0}$). Let $\pi_1(V)$ be the representation of Q obtained by discarding the linear maps V_{a^*} for $a \in Q_1$ and let $\pi_2(V)$ be the representation of Q^{op} obtained by discarding the linear maps V_a for $a \in Q_1$ (Q^{op} is the *opposite quiver* of Q, obtained from Q by switching the orientation of each arrow).

Let $\lambda \in K^{Q_0}$ and let V be a representation of Q, we say V *lifts* to a representation of $\Pi^\lambda(Q)$ if there exists a representation X of $\Pi^\lambda(Q)$ such that $\pi_1(X) = V$.

Given a positive dimension vector α we say a collection of positive dimension vectors $(\beta_1, \dots, \beta_r)$ is a *decomposition of* α if $\alpha = \beta_1 + \cdots + \beta_r$. We say a decomposition $(\beta_1, \dots, \beta_r)$ of α is a *root decomposition* if each $\beta_1, \dots, \beta_r$ is a root. We say a decomposition $(\beta_1, \dots, \beta_r)$ of α is *compatible with* λ if $\lambda \cdot \beta_i = 0$ for each $i = 1, \dots, r$.

Let α be a dimension vector. Recall the Ringel form $\langle \cdot, \cdot \rangle$, the corresponding quadratic form is *the Tits form*, $q(\alpha) = \langle \alpha, \alpha \rangle = \frac{1}{2}(\alpha, \alpha)$, that is

$$q(\alpha) = \sum_{i \in Q_0} \alpha_i^2 - \sum_{a \in Q_1} \alpha_{h(\alpha)} \alpha_{t(\alpha)},$$

(see [CB01, Sec. 2]). Let $p(\alpha) = 1 - q(\alpha)$.

Definition 1.2.7. Let R_λ^+ be the set of positive roots α with the property that $\lambda \cdot \alpha = 0$. Let Σ_λ be the set of $\alpha \in R_\lambda^+$ such that $p(\alpha) > p(\beta_1) + \cdots + p(\beta_r)$ for all root decompositions $(\beta_1, \dots, \beta_r)$ of α compatible with λ with $r \geq 2$.

Lemma 1.2.8. Suppose $\alpha \in \Sigma_\lambda$. There exists $\alpha' \in \mathbb{Z}^{Q_0}$ and $\lambda' \in K^{Q_0}$ such that (α, λ) and (α', λ') are equivalent, where either α' is a simple root at a loop-free vertex or α' is in the fundamental region.

Proof. [CB01, Thm. 5.8]. □

Theorem 1.2.9. There exists a simple representation of $\Pi^\lambda(Q)$ of dimension vector α if and only if $\alpha \in \Sigma_\lambda$. If α is a real root, then this simple representation is unique up to isomorphism and if α is imaginary, then there are infinitely many nonisomorphic simple representations of dimension vector α.

Proof. The first part is proved in [CB01, Thm. 1.2] and the second part in the subsequent remarks. □

Chapter 2

Matrices

Matrices are one of the central objects of study in both linear algebra and representation theory. Certain problems involving matrices can be expressed in terms of the representation theory of quivers. The additive matrix problem, which is the central focus this thesis, is formally introduced in Chapter 4. We devote this chapter to material concerning matrices (and endomorphisms of a vector space). In Section 2.1 we establish notation and recall many definitions and results concerning matrices and matrix similarity, most of which are standard. In Section 2.2 we give a construction for vector space endomorphisms of a particular conjugacy class or conjugacy class closure, which we use in Chapter 4. We also introduce a new construction for self-adjoint endomorphisms which we use in Chapter 5. Throughout let K be an algebraically closed field of characteristic zero.

2.1 Matrices and Matrix Similarity

The set $M_n(K)$ is the K-algebra of square n by n matrices (with entries in K). Recall the definition of a Jordan block, a Jordan matrix and matrix similarity. The definitions and results in this section are standard concepts and results in elementary linear algebra and can be found in references such as [Hal58] and [Gan59].

Let $A \in M_n(K)$. The *spectrum* $\Psi(A)$ of A is the set of eigenvalues of A. The *characteristic polynomial* $\mathrm{char}_A \in K[x]$ of A is defined by $\mathrm{char}_A(x) = \det(xI_n - A)$. The *minimal polynomial* $\min_A \in K[x]$ of A is the unique nonzero monic polynomial of minimal degree such that $\min_A(A) = 0$. Let $\lambda \in K$. We have two notions of the multiplicity of λ in A. The *algebraic multiplicity* of λ in A, denoted $\mathrm{algr}_A(\lambda)$, is the number of factors of $(x - \lambda)$ that appear in

$\mathrm{char}_A(x)$, and the *geometric multiplicity* of $\lambda \in A$, denoted $\mathrm{geom}_A(\lambda)$, is the dimension of the eigenspace of λ, that is $\dim\{x \in K^n \colon (A - \lambda I_n)x = 0\}$. Alternatively, $\mathrm{geom}_A(\lambda)$ is the number of linearly independent eigenvectors associated to λ. We say the *index* of λ in A, denoted $\mathrm{idx}_A(\lambda)$, is the size of the largest Jordan block with eigenvalue λ appearing in a Jordan decomposition of A. Note that $\mathrm{geom}_A(\lambda)$, $\mathrm{algr}_A(\lambda)$ and $\mathrm{idx}_A(\lambda)$ are nonzero if and only if $\lambda \in \Psi(A)$.

Remark 2.1.1. We make use of the following well-known properties of matrices. Two similar matrices A and B have the same spectrum, characteristic and minimal polynomial, and for each $\lambda \in K$ we have $\mathrm{algr}_A(\lambda) = \mathrm{algr}_B(\lambda)$, $\mathrm{geom}_A(\lambda) = \mathrm{geom}_B(\lambda)$ and $\mathrm{idx}_A(\lambda) = \mathrm{idx}_B(\lambda)$. Suppose $p \in K[x]$ is a polynomial, we have $p(X^{-1}AX) = X^{-1}p(A)X$ for each nonsingular $X \in M_n(K)$.

Example 2.1.2. Both the characteristic and minimal polynomials are invariant under similarity, however there exist matrices which have the same characteristic and minimal polynomials but which are not similar. For instance consider the matrices

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Both of these have characteristic polynomial $(x-1)^4$ and minimal polynomial $(x-1)^2$ but they are not similar as they have different Jordan normal forms (up to permutation of blocks). However, there does exist a collection of polynomials which fully classify the similarity of matrices, these are known as the *invariant polynomials* of a matrix.

Classifying matrices up to similarity is one of the central problems in linear algebra. There are a variety of ways to determine whether two matrices are similar, most involve computing some invariant of the similarity transformation. A collection of invariants which completely classifies matrix similarity is called *a complete set of invariants*. The most well-known of these is the collection of Jordan blocks which make up the Jordan normal form of a matrix, another important complete set of invariants is the collection of invariant polynomials. We discuss both of these below and show how to compute one from the other.

2.1.1 Invariant Polynomials

Definition 2.1.3. Let R be a ring and let $A \in R^{m\times n}$ be an m by n matrix (i.e. A has m rows and n columns) with entries in R, and suppose $X \subseteq \{1,\ldots,m\}$ and $Y \subseteq \{1,\ldots,n\}$ such that $|X| = |Y| = j$ for some $j \in \{0,\ldots,\min(m,n)\}$. We denote by $A_{X,Y} \in M_j(K)$ the j by j *submatrix* of A consisting of only those rows indexed by X and only those columns indexed by Y. The quantity $\det(A_{X,Y})$ is called a *jth minor of A*.

A submatrix of the form $A_{X,X}$ is called a *principal submatrix*, and $\det(A_{X,X})$ is called a *principal jth minor of A*. By convention we take $\det(A_{\emptyset,\emptyset}) = 1$, that is the determinant of the empty matrix (i.e. the matrix of size zero) is one.

Example 2.1.4. Let $A = \begin{pmatrix} 2 & 2 & 0 \\ 0 & 4 & -2 \\ 1 & -4 & -6 \end{pmatrix}$. The principal submatrices of A of order two are

$$A_{\{1,2\},\{1,2\}} = \begin{pmatrix} 2 & 2 \\ 0 & 4 \end{pmatrix}, \qquad A_{\{1,3\},\{1,3\}} = \begin{pmatrix} 2 & 0 \\ 1 & -6 \end{pmatrix}, \qquad A_{\{2,3\},\{2,3\}} = \begin{pmatrix} 4 & -2 \\ -4 & -6 \end{pmatrix}$$

The submatrix $A_{\{1,2,3\},\{1,2,3\}} = A$ and $A_{\emptyset,\emptyset} = []$. The principal second minors are $\det(A_{\{1,2\},\{1,2\}}) = 8$, $\det(A_{\{1,3\},\{1,3\}}) = -12$ and $\det(A_{\{2,3\},\{2,3\}}) = -32$.

Each square matrix of order n has a collection of n polynomials associated to it, these are known as *invariant polynomials* and denoted $\iota_{A,1},\ldots,\iota_{A,n}$. In many cases not all of these polynomials are nontrivial. Given two nonzero polynomials $p,q \in K[x]$ the *highest common factor* $\mathrm{hcf}(p,q)$ is the unique monic polynomial of highest degree which divides both p and q. The following definition comes from [Gan59, Vol. I, Chap. VI, Sec. 3].[1]

Definition 2.1.5. Let $A \in M_n(K)$ and let

$$P_{A,j}(x) = \mathrm{hcf}\{\det((I_n x - A)_{X,Y})\colon X,Y \subseteq \{1,\ldots,n\}, |X| = |Y| = j\},$$

for all $j = 0,\ldots,n$, that is $P_{A,j}$ is the highest common factor of all the jth-minors of $I_n x - A$. Note that the $P_{A,j}$ are *not* the invariant polynomials, which are defined below.

The *invariant polynomial* $\iota_{A,p}(x)$ of A is defined by

$$\iota_{A,p}(x) = \frac{P_{A,p}(x)}{P_{A,p-1}(x)}, \text{ for } p = 1,\ldots,n.$$

[1] N.B. Gantmacher indexes the invariant polynomials in the reverse order, so Gantmacher defines $\iota_{A,p}(x) = \frac{P_{A,n-p+1}(x)}{P_{A,n-p}(x)}$ for $p = 1,\ldots,n$. This is analogous to the definition we use, however under Gantmacher's definition we would have $\iota_{A,n} \mid \cdots \mid \iota_{A,1}$ rather than $\iota_{A,1} \mid \cdots \mid \iota_{A,n}$.

The invariant polynomials satisfy the following properties.

- $\iota_{A,1}(x) \mid \iota_{A,2}(x) \mid \cdots \mid \iota_{A,n}(x) = \min_A(x)$, where $p \mid q$ means p divides q,
- $\iota_{A,1}(x)\iota_{A,2}(x)\ldots\iota_{A,n}(x) = \mathrm{char}_A(x) = \det(xI_n - A)$,
- λ is a root of $\iota_{A,p}$ if and only if λ is an eigenvalue of A with geometric multiplicity no less than $n - p + 1$,
- $A \sim B$ if and only if they have the same set of invariant polynomials.

See [Gan59] for the proof of the above properties.

Example 2.1.6. We show here how to calculate the invariant polynomials of the matrix

$$A = \begin{pmatrix} 2 & -2 & 0 \\ 0 & 2 & 0 \\ 0 & -2 & 2 \end{pmatrix}.$$

We first compute the $P_{A,j}(x)$.

$$\begin{aligned} P_{A,3}(x) &= \mathrm{hcf}\{\det(I_n x - A)\} \\ &= (x-2)^3 \\ &= x^3 - 6x^2 + 12x - 8 \\ P_{A,2}(x) &= \mathrm{hcf}\{(x-2)^2, 2(x-2), 0\} \\ &= x - 2 \\ P_{A,1}(x) &= \mathrm{hcf}\{x-2, 2, 0\} \\ &= 1 \\ P_{A,0}(x) &= 1. \end{aligned}$$

Now we calculate the invariant polynomials using $\iota_{A,p}(x) = P_{A,p}(x)/P_{A,p-1}(x)$:

$$\begin{aligned} \iota_{A,3}(x) &= (x^3 - 6x^2 + 12x - 8)/(x-2) \\ &= (x-2)^2 \\ &= x^2 - 4x + 4 \\ \iota_{A,2}(x) &= (x-2)/1 \\ &= x - 2 \\ \iota_{A,1}(x) &= 1 \end{aligned}$$

Note that when we calculate the Jordan normal form of A it has two blocks of eigenvalue 2, of sizes 2 and 1.

As both the invariant polynomials and the Jordan normal form (up to the order of the blocks) form a complete set of invariants under similarity we show how to go back and forth from a given Jordan normal form to a set of invariant polynomials. First we develop some terminology to express the Jordan normal form.

2.1.2 Jordan Normal Form

We say a matrix is a *Jordan normal form* (or is *in Jordan normal form*) if it is block diagonal with Jordan blocks along the diagonal. Let A be a square matrix of order n, let $\mathcal{J}(A)$ be the set of all Jordan normal forms similar to A. We denote by $\mathrm{J}_n(\lambda)$ the Jordan block of size n with eigenvalue λ.

Lemma 2.1.7. Let $N \in \mathcal{J}(A)$ and let $\lambda \in \Psi(A)$. The algebraic multiplicity $\mathrm{algr}_A(\lambda)$ is equal to the number of times λ appears on the diagonal of N and the geometric multiplicity $\mathrm{geom}_A(\lambda)$ is equal to the number of Jordan blocks associated to λ appearing in N.

Proof. The first claim follows because the determinant of an upper-triangular matrix (in this case $xI_n - N$) is the product of its diagonal elements. The second follows because we can write the kernel of $N - \lambda I_n$, as

$$\ker(N - \lambda I_n) \cong \ker(J_1 - \lambda I_{n_1}) \oplus \cdots \oplus \ker(J_p - \lambda I_{n_p})$$

where $\ker(J_i - \lambda I_{n_i})$ are the kernels of the Jordan blocks $J_1, \ldots, J_p$ of N and $n_1, \ldots, n_p$ are their respective sizes. So

$$\dim(\ker(N - \lambda I_n)) = \sum_{i=1}^{p} \dim(\ker(J_i - \lambda I_{n_i}))$$

Now as J_i is a Jordan block, we have $\dim(\ker(J_i - \lambda I_{n_i})) = 1$ if J_i has eigenvalue λ (as the kernal of a nilpotent Jordan block is the space of vectors with zeros in all but the first entry) and 0 otherwise. So $\mathrm{geom}_N(\lambda) = \dim(\ker(N - \lambda I_n))$ is the number of Jordan blocks associated to λ. □

Let J be a Jordan block, we say J is *a Jordan block of A* if J appears as a block in a Jordan normal form of A.

An *integer partition* $\mu = (\mu_1, \ldots, \mu_n)$ is a weakly descending sequence of non-negative integers, that is $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n \geq 0$. We regard two partitions which differ by a string of zeroes at the end as equivalent. Let $\mathcal{P}$ be the *set of all integer partitions*.

Definition 2.1.8. Let $\mu_A \colon \Psi(A) \to \mathcal{P}$ be defined by $\mu_A(\lambda) = (m_1, \ldots, m_{\mathrm{geom}_A(\lambda)})$ where $m_1, \ldots, m_{\mathrm{geom}_A(\lambda)}$ are the sizes of the Jordan blocks associated to λ arranged in descending order of size.

If an integer partition μ is given by $\mu = (\mu_1, \ldots, \mu_n)$, then μ_{n+i} for $i > 0$ is taken to be zero. This is compatible with our convention of considering partitions which only differ by a string of zeros at the end as equivalent.

Example 2.1.9. Let $A, B \in M_{13}(K)$ such that

$$A \sim \mathrm{diag}(\mathrm{J}_2(4), \mathrm{J}_1(-3), \mathrm{J}_4(4), \mathrm{J}_1(-3), \mathrm{J}_3(4), \mathrm{J}_2(4)) \text{ and } B \sim \mathrm{diag}(\mathrm{J}_4(-1), \mathrm{J}_4(-1), \mathrm{J}_1(0), \mathrm{J}_4(0)).$$

We have $\mu_A(4) = (4, 3, 2, 2)$ and $\mu_A(-3) = (1, 1)$, and we have $\mu_B(-1) = (4, 4)$ and $\mu_B(0) = (4, 1)$.

Given the spectrum of A, the function μ_A completely describes the associated Jordan normal forms (up-to reordering).

2.1.3 The Correspondence between Jordan Normal Form and Invariant Polynomials

We introduce the concept of a derogatory matrix and define the companion matrix of a polynomial. The companion matrix can be computed directly from the coefficients of the polynomial, we use the fact that companion matrices are always nonderogatory and the resulting properties to describe the correspondence between the Jordan normal form and the invariant polynomials of a matrix.

Definition 2.1.10. Let $A \in M_n(K)$. The matrix A is called *derogatory* if $\min_A(x) \neq \mathrm{char}_A(x)$.

Theorem 2.1.11. Let $A \in M_n(K)$. We have $\min_A(x) = \prod_{\lambda \in \Psi(A)} (x - \lambda)^{\mathrm{idx}_A(\lambda)}$.

Proof. See [Fin78, Thm. 7.17], in which the collection of sizes of Jordan blocks is called the Segre characteristic (written with sizes of Jordan blocks of like eigenvalues bracketed together). □

Theorem 2.1.12. Let $A \in M_n(K)$. The following are equivalent.

1. A is derogatory,

2. There exists some $\lambda \in \Psi(A)$ such that $\mathrm{idx}_A(\lambda) \neq \mathrm{algr}_A(\lambda)$,

3. There exists some $\lambda \in \Psi(A)$ such that $\mathrm{geom}_A(\lambda) > 1$.

Proof. (1) $\Rightarrow$ (2): By definition $\mathrm{char}_A(x) = \prod_{\lambda \in \Psi(A)} (x - \lambda)^{\mathrm{algr}_A(\lambda)}$. If $\mathrm{char}_A(x) \neq \min_A(x)$, then by Theorem 2.1.11 there must be some $\lambda \in \Psi(A)$ whose index differs from its algebraic multiplicity.

(2) $\Rightarrow$ (3): If the index of $\lambda \in \Psi(A)$ is smaller (by Lemma 2.1.7 it cannot be greater) than the algebraic multiplicity, then the largest Jordan block associated to λ must have size smaller than the number of times λ appears as a root of char_A, so there must be more than one Jordan block associated to λ, by Lemma 2.1.7 this implies a geometric multiplicity greater than one.

(3) $\Rightarrow$ (1): Let $\lambda \in K$ be such that $\mathrm{geom}_A(\lambda) > 1$. By Lemma 2.1.7 there exists more than one Jordan block associated to λ. As the sum of their sizes must add up to $\mathrm{algr}_A(\lambda)$ the size of the largest (that is the $\mathrm{idx}_A(\lambda)$) must be strictly smaller than $\mathrm{algr}_A(\lambda)$. By Theorem 2.1.11 $\min_A(x) \neq \prod_{\lambda' \in \Psi(A)} (x - \lambda')^{\mathrm{algr}_A(\lambda')} = \mathrm{char}_A(x)$. □

We now introduce the companion matrix.

Definition 2.1.13. [Gan59, Vol. I, Chap. VI, Sec. 6] Let $f \in K[x]$ be a monic polynomial, we define the *companion matrix* C_f of f as follows. Suppose $f(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} + x^n$, the companion matrix of f is:

$$C_f = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{pmatrix}$$

The next lemma implies that a companion matrix is nonderogatory.

Lemma 2.1.14. Given a monic polynomial $f \in K[x]$ we have $\mathrm{char}_{C_f}(x) = \min_{C_f}(x) = f(x)$.

Proof. By [Gan59, Vol. I, Chap. VI, Sec. 6] and the fact that $\iota_{C_f,n} = \min_{C_f}(x)$. □

Theorem 2.1.15. Let $A \in M_n(K)$, with invariant polynomials $\iota_{A,1} \mid \cdots \mid \iota_{A,n}$, we have that A is similar to $\mathrm{diag}(C_{\iota_{A,1}}, \ldots, C_{\iota_{A,n}})$.

Proof. [Gan59, Vol. I, Chap. VI, Sec. 6]. □

Definition 2.1.16. Let f be a polynomial and let ξ be a root of f, we denote the *multiplicity* of ξ in f by $\mathrm{mul}_f(\xi)$.

As companion matrices are nonderogatory we can see, by Theorems 2.1.12 and 2.1.15 and Lemma 2.1.7, that for each distinct root ξ of $\iota_{A,p}$ the block $C_{\iota_{A,p}}$ contributes precisely one instance of $\mathrm{J}_{\mathrm{mul}_{\iota_{A,p}}(\xi)}(\xi)$ to the Jordan structure A, that is a Jordan matrix with eigenvalue ξ of size $\mathrm{mul}_{\iota_{A,p}}(\xi)$.

The following theorem allows us to go between invariant polynomials and the Jordan structure, as long as the spectrum is known.

Theorem 2.1.17. Let A be a square matrix with Jordan structure μ_A. We have, for $i = 1, \dots, n$,

$$\iota_{A,i}(x) = \prod_{\lambda \in \Psi(A)} (x - \lambda)^{\mu_A(\lambda)_{n-i}}.$$

Let B be a square matrix with invariant polynomials $\iota_{B,1} \mid \cdots \mid \iota_{B,n}$, B has the following Jordan structure μ_B. Let $\lambda \in \Psi(B)$,

$$\mu_B(\lambda) = \left(\mathrm{mul}_{\iota_{B,n}}(\lambda), \mathrm{mul}_{\iota_{B,n-1}}(\lambda), \dots, \mathrm{mul}_{\iota_{B,2}}(\lambda), \mathrm{mul}_{\iota_{B,1}}(\lambda)\right).$$

Proof. The results follow quite easily from Theorem 2.1.15.

Let $\lambda \in \Psi(A)$ suppose $\mu_A(\lambda) = (p_1, \dots, p_r)$, that is there are r Jordan blocks of A with eigenvalue λ. Theorem 2.1.15 implies λ must appear as a root in r invariant polynomials, namely $\iota_{A,n}, \dots, \iota_{A,n-r}$. Furthermore the multiplicity of λ in $\iota_{A,i}$ must be less than or equal to the multiplicity of λ in $\iota_{A,i+1}$ as $\iota_{A,i} \mid \iota_{A,i+1}$. So, as $p_1, \dots, p_r$ appear in descending order, we conclude λ has multiplicity p_i in $\iota_{A,n-i}$ for $i = 1, \dots, r$. By convention $\mu_A(\lambda)_i = 0$ for $i > r$, which proves the correctness of the first formula.

Let $\lambda \in \Psi(B)$. By Theorem 2.1.15 it is easy to see the sizes of the Jordan blocks associated to λ are given by the multiplicities of λ in the invariant polynomials. Recalling that $\iota_{B,i} \mid \iota_{B,i+1}$ shows $\mathrm{mul}_{\iota_{B,n}}(\lambda), \mathrm{mul}_{\iota_{B,n-1}}(\lambda), \dots, \mathrm{mul}_{\iota_{B,1}}(\lambda)$ is a nonincreasing sequence, that is: a partition. □

So given either a Jordan normal form of a matrix or the invariant polynomials of a matrix we can calculate one from the other, as long as we know the spectrum of the matrix. If we have the Jordan structure then the spectrum must already be known, however if we begin with the invariant polynomials we need to be able to factorize them to obtain the spectrum. The following definition and theorem provides another way to classify matrix similarity.

Definition 2.1.18. Let $A \in M_n(K)$ and let $\xi_1, \ldots, \xi_d$ be a list of all roots (possibly repeating) of the minimal polynomial of A. Define $n_0, n_1, \ldots, n_d$ by $n_i = \operatorname{rank}(\prod_{j=1}^{i}(A - \xi_j I_n))$, we call $(n_0, n_1, \ldots, n_d)$ a *dimension vector* of A.

Note that a dimension vector of a matrix depends on how we order the list of roots of the minimal polynomial, also note that in all cases $n_0 = n$ and $n_d = 0$.

Theorem 2.1.19. Let $A, B \in M_n(K)$ and let $\xi_1, \ldots, \xi_d$ be the roots of the minimal polynomial of A and let $(n_0, \ldots, n_d)$ be the dimension vector of A (with respect to the chosen ordering). We have $A \sim B$ if and only if B has the same minimal polynomial and dimension vector (with respect to the same ordering), i.e. $\operatorname{rank}(\prod_{j=1}^{i}(B - \xi_j I_n)) = n_i$ for $i = 0, 1, \ldots, d$.

Proof. As noted previously similar matrices must have the same minimal polynomial. Let $\lambda \in \Psi(A)$, suppose $\{a_1, \ldots, a_{\operatorname{idx}_A(\lambda)}\}$ is the biggest subset of $\{1, \ldots, d\}$ such that $\xi_{a_1} = \cdots = \xi_{a_{\operatorname{idx}_A(\lambda)}} = \lambda$. As $\prod_{j=1}^{i}(A - \xi_j I_n)$ (for some $i = 0, \ldots, d$) is a polynomial in A and rank is invariant under similarity we can assume A is in Jordan normal form. It is clear, for $i = 1, \ldots, \operatorname{idx}_A(\lambda)$, that the difference in rank

$$\operatorname{rank}\left(\prod_{j=1}^{a_i - 1}(A - \xi_j I_n)\right) - \operatorname{rank}\left(\prod_{j=1}^{a_i}(A - \xi_j I_n)\right) = n_{a_i - 1} - n_{a_i}$$

is precisely the number of Jordan blocks with eigenvalue λ of size i or greater. This information is uniquely defined by the $n_0, \ldots, n_d$ and the $n_0, \ldots, n_d$ are themselves uniquely defined by this information (as n_d is always zero). Using this information we can precisely construct the Jordan structure $\mu_A \colon \Psi(A) \to \mathcal{P}$. So as $A \sim B$ if and only if they have the same Jordan structure, they are similar if and only if $\min_A(x) = \min_B(x)$ and $\operatorname{rank}(\prod_{j=1}^{i}(B - \xi_j I_n)) = n_i$ for $i = 0, 1, \ldots, d$. □

2.1.4 Matrix Similarity Classes

Definition 2.1.20. A *similarity class* (or *matrix similarity class*) C is an orbit of $M_n(K)$ under the action of similarity. Similarity classes are irreducible quasi-projective varieties (see [Hum75, Sec. 8.3]).

Given a similarity class C we extend the various invariants defined for individual matrices to the class C. In particular we define $\Psi(C)$, algr_C, geom_C, idx_C, char_C, $\min_C$ and $\iota_{C,1}, \ldots, \iota_{C,n}$ to be $\Psi(A)$, algr_A, geom_A, idx_A, char_A, $\min_A$ and $\iota_{A,1}, \ldots, \iota_{A,n}$ respectively, for some $A \in C$. As these objects are invariants under similarity these definitions are well-defined.

Definition 2.1.21. Let C be a a similarity class.

- The *closure* $\overline{C}$ of C is the topological closure in the Zariski sense.
- We say a C is *closed* if it is Zariski-closed, that is if $C = \overline{C}$.

It is well known that a closed similarity class consists only of diagonalizable matrices.

Theorem 2.1.22. The Gerstenhaber-Hesselink theorem. If $A, B \in M_n(K)$, then B is in the closure of the similarity class of A if and only if $\mathrm{rank}((B - \lambda I_n)^m) \leq \mathrm{rank}((A - \lambda I_n)^m)$ for all $\lambda \in K$ and $m \geq 0$.

Proof. [Ger59, Thm. 1.7]. □

The following corollary follows from the Gerstenhaber-Hesselink theorem and is also a consequence of Theorem 2.1.19.

Corollary 2.1.23. If $A, B \in M_n(K)$, then A is in the similarity class of B if and only if $\mathrm{rank}((B - \lambda I_n)^m) = \mathrm{rank}((A - \lambda I_n)^m)$ for all $\lambda \in K$ and $m \geq 0$.

Proof. Suppose A is in the similarity class of B, A is in the closure of the similarity class of B and B is in the closure of the similarity class of A. Applying the Gersetenhaber-Hesselink theorem to these facts derives the result.

Now suppose $\mathrm{rank}((B-\lambda I_n)^m) = \mathrm{rank}((A-\lambda I_n)^m)$ the Gerstenhaber-Hesselink theorem shows A and B are in the closures of the similarity classes of one another, hence are similar. □

2.1.5 The Class of Matrices with Given Eigenvalues

Now we wish to discuss the set of all matrices in $M_n(K)$ with given eigenvalues, say $\xi_1, \ldots, \xi_n \in K$. If the eigenvalues are all pairwise distinct then this is precisely the similarity class C^{diag} of matrices similar to the diagonal matrix $D = \mathrm{diag}(\xi_1, \ldots, \xi_n)$. If the eigenvalues are not all pairwise distinct then C^{diag} does not contain all matrices with these eigenvalues. For instance if $\xi_1 = \xi_2$, then $E = \operatorname{diag}\left(\left(\begin{smallmatrix}\xi_1 & 1\\ 0 & \xi_2\end{smallmatrix}\right), \xi_3, \ldots, \xi_n\right)$ has the required eigenvalues but $E \notin C^{\mathrm{diag}}$. Theorem 2.1.25 gives a description of the set of all matrices with eigenvalues $\xi_1, \ldots, \xi_n$.

Lemma 2.1.24. Let A be a nonderogatory matrix and $\lambda \in K$, we have

$$\operatorname{rank}((A-\lambda I_n)^m) = \begin{cases} n-m & \text{if} \quad m = 0,1,\ldots,\operatorname{idx}_A(\lambda), \\ n-\operatorname{idx}_A(\lambda) & \text{if} \quad m > \operatorname{idx}_A(\lambda). \end{cases}$$

Proof. By Lemma 2.1.12 the nonderogatory matrix A has one Jordan block of size $\operatorname{idx}_A(\lambda)$, for each distinct eigenvalue $\lambda \in \Psi(A)$, so we have $\operatorname{rank}((A-\lambda I_n)^m) = n-m$ for $m = 0,1,\ldots,\operatorname{idx}_A(\lambda)$ with the sequence stabilizing after that as required. If $\lambda \notin \Psi(A)$, then $\operatorname{idx}_A(\lambda) = 0$ and $\operatorname{rank}((A-\lambda I_n)^m) = n$ for all $m \geq 0$, so $\operatorname{rank}((A-\lambda I_n)^m) = n-0 = n-\operatorname{idx}_A(\lambda)$ as required. □

Theorem 2.1.25. Let $\xi_1, \ldots, \xi_n \in K$. The set of all matrices in $M_n(K)$ with eigenvalues $\xi_1, \ldots, \xi_n$ is precisely $\overline{C}$ where C is the similarity class of all nonderogatory matrices with eigenvalues $\xi_1, \ldots, \xi_n \in K$.

Proof. Suppose $B \in \overline{C}$, by Theorem 2.1.22 (the Gerstenhaber-Hesselink theorem) we have $\operatorname{rank}((B-\lambda I_n)^m) \leq \operatorname{rank}((A-\lambda I_n)^m)$ (for any $A \in C$). So, as $A \in C$ is nonderogatory, we have by Lemma 2.1.24

$$\operatorname{rank}((B-\lambda I_n)^m) \leq \begin{cases} n-m & \text{if} \quad m = 0,1,\ldots,\operatorname{idx}_C(\lambda), \\ n-\operatorname{idx}_C(\lambda) & \text{if} \quad m > \operatorname{idx}_C(\lambda). \end{cases} \tag{2.1}$$

Suppose $\lambda \in \Psi(C)$, (2.1) implies

$$\operatorname{rank}((B-\lambda I_n)^{\operatorname{idx}_C(\lambda)}) \leq n-\operatorname{idx}_C(\lambda),$$

which, by Theorem 2.1.12, implies

$$\operatorname{rank}((B-\lambda I_n)^{\operatorname{algr}_C(\lambda)}) \leq n-\operatorname{algr}_C(\lambda).$$

So B has at least $\operatorname{algr}_C(\lambda)$ eigenvalues equal to λ. That is $\operatorname{algr}_B(\lambda) \geq \operatorname{algr}_C(\lambda)$. However as $\sum_{\lambda\in\Psi(C)} \operatorname{algr}_C(\lambda) = n$ this implies $\operatorname{algr}_B(\lambda) = \operatorname{algr}_C(\lambda)$ for $\lambda \in \Psi(C)$ and $\operatorname{algr}_B(\lambda) = 0$ for all $\lambda \notin \Psi(C)$. So B has precisely the eigenvalues $\xi_1, \ldots, \xi_n$.

For the converse, suppose B has precisely the eigenvalues $\xi_1, \ldots, \xi_n$. Let us consider $\lambda = \xi_i$ for some $i = 1, \ldots, n$ and the sequence $r_m = \operatorname{rank}((B-\lambda I_n)^m)$ for $m \geq 0$. It is clear that

$$n = r_0 > r_1 > \cdots > r_{\mathrm{idx}_B(\lambda)} = r_{\mathrm{idx}_B(\lambda)+1} = \cdots = r_{\mathrm{algr}_B(\lambda)} = r_{\mathrm{algr}_B(\lambda)+1} = \cdots$$

From r_0 to $r_{\mathrm{idx}_B(\lambda)}$ the inequalities are strict and the value of $r_{\mathrm{idx}_B(\lambda)}$, at which the sequence stabilizes, must be equal to $n - \mathrm{algr}_B(\lambda)$. These two facts imply $r_i \leq n - i$ for $i = 0, 1, \ldots, \mathrm{algr}_B(\lambda)$, in terms of rank this is

$$\mathrm{rank}((B - \lambda I_n)^m) \leq n - m = \mathrm{rank}((A - \lambda I_n)^m),$$

for some $A \in C$ and $m = 0, 1, \ldots, \mathrm{algr}_B(\lambda) = \mathrm{idx}_C(\lambda)$ (by Lemma 2.1.24). For $m > \mathrm{algr}_B(\lambda)$ we have

$$\mathrm{rank}((B - \lambda I_n)^m) = n - \mathrm{algr}_B(\lambda) = n - \mathrm{idx}_C(\lambda) = \mathrm{rank}((A - \lambda I_n)^m).$$

If $\lambda \notin \Psi(B)$, then $\mathrm{rank}((B - \lambda I_n)^m) = n = \mathrm{rank}((A - \lambda I_n)^m)$, for some $A \in C$ and $m \geq 0$, as $\lambda \notin \Psi(C)$. So by the Gerstenhaber-Hesselink theorem $B \in \overline{C}$. □

2.2 Constructions of Endomorphisms in a Given Conjugacy Class or Class Closure

We now work with linear maps from a vector space to itself, that is endomorphisms of a vector space, rather than matrices. We say two endomorphisms are *conjugate* if the matrices they define, under some choice of basis, are similar. The classification of vector space endomorphism by conjugacy works in exactly the same way as the classification of matrices by similarity. The purpose of this section is twofold, first we give a construction of all endomorphisms which lie in a given conjugacy class (or closure of a conjugacy class), these constructions are found in the literature ([CB04] and [CB03]). The second purpose is to give a construction of the self-adjoint endomorphisms in a given conjugacy class (or its closure). The constructions of self-adjoint endomorphisms are new results. Let V be a finite dimensional vector space over K, we denote the set of endomorphisms of V by $\mathrm{End}(V)$. All vector spaces in this section are over K.

2.2.1 Endomorphisms of a Vector Space

Lemma 2.2.1. Let V_0, V_1 be vector spaces and let $\phi\colon V_1 \to V_0$ and $\phi^*\colon V_0 \to V_1$ be linear maps, we have $\dim(V_0) - \mathrm{rank}((\phi\phi^* + \lambda 1_{V_0})^m) = \dim(V_1) - \mathrm{rank}((\phi^*\phi + \lambda 1_{V_1})^m)$ for any

$0 \neq \lambda \in K$ and $m \geq 0$.

Proof. [CB04, Lem. 2.2]. □

The next theorem gives a construction of all endomorphisms in the closure of a given conjugacy class.

Theorem 2.2.2. Let $A, B \in \mathrm{End}(V)$. Let $\xi_1, \ldots, \xi_d \in K$ be a list of all roots of the minimal polynomial of A, let $(n_0, n_1, \ldots, n_d)$ be the dimension vector with respect to this list. The following statements are equivalent.

1. B is in the closure of the conjugacy class of A.

2. There exists a flag of subspaces

$$V = V_0 \supseteq V_1 \supseteq \cdots \supseteq V_d = 0$$

such that $\dim(V_i) = n_i$ and $(B - \xi_j 1_V)(V_{j-1}) \subseteq V_j$ for all $1 \leq j \leq d$.

3. There exist vector spaces $V = V_0, V_1, \ldots, V_d = 0$, where $\dim(V_i) = n_i$, and linear maps

$$\begin{tikzcd} V_0 \arrow[r, shift left, "\phi_1"] & V_1 \arrow[l, shift left, "\phi_1^*"] \arrow[r, shift left, "\phi_2"] & V_2 \arrow[l, shift left, "\phi_2^*"] \arrow[r, shift left, "\phi_3"] & \cdots \arrow[l, shift left, "\phi_3^*"] \arrow[r, shift left, "\phi_{d-2}"] & V_{d-2} \arrow[l, shift left, "\phi_{d-2}^*"] \arrow[r, shift left, "\phi_{d-1}"] & V_{d-1} \arrow[l, shift left, "\phi_{d-1}^*"] \arrow[r, shift left, "\phi_d"] & V_d \arrow[l, shift left, "\phi_d^*"] \end{tikzcd}$$

such that

(a) $B = \phi_1^* \phi_1 + \xi_1 1_V$,

(b) $\phi_i \phi_i^* - \phi_{i+1}^* \phi_{i+1} = (\xi_{i+1} - \xi_i) 1_{V_i}$, for $i = 1, \ldots, d-1$.

Proof. [CB04, Thm. 2.1]. □

Theorem 2.2.4 gives a construction of all endomorphisms which lie precisely in a given conjugacy class.

Lemma 2.2.3. Suppose we have the vector spaces and linear maps

$$V \xrightarrow{h} U \xrightarrow{g} W \xrightarrow{f} X .$$

If f is injective, then $\mathrm{rank}(fg) = \mathrm{rank}(g)$ and if h is surjective, then $\mathrm{rank}(gh) = \mathrm{rank}(g)$.

Proof. A basis of $\mathrm{im}(g)$ is mapped by f to a linearly independent set of equal cardinality as f is injective, so $\mathrm{rank}(g) = \dim(\mathrm{im}(g)) = \dim(\mathrm{im}(fg)) = \mathrm{rank}(fg)$.

We have $\mathrm{rank}(gh) = \dim(\mathrm{im}(gh))$, but as $\mathrm{im}(h) = U$ we have $\mathrm{im}(gh) = \mathrm{im}(g)$ so $\mathrm{rank}(gh) = \mathrm{rank}(g)$. □

The following theorem is essentially the same as the one found in [CB03, Sec. 3] but written in a manner analogous to Theorem 2.2.2.

Theorem 2.2.4. Let $A, B \in \mathrm{End}(V)$. Let $\xi_1, \dots, \xi_d \in K$ be a list of the zeros of the minimal polynomial of A, let $(n_0, n_1, \dots, n_d)$ be the associated dimension vector. The following statements are equivalent.

1. The map B is in the conjugacy class of A.

2. There exists a flag of subspaces

$$V = V_0 \supseteq V_1 \supseteq \cdots \supseteq V_d = 0$$

where $\dim(V_i) = n_i$, such that $(B - \xi_j 1_V)(V_{j-1}) = V_j$ for all $1 \leq j \leq d$.

3. There exist vector spaces $V = V_0, V_1, \dots, V_d = 0$, where $\dim(V_i) = n_i$, and linear maps

$$\begin{tikzcd} V_0 \arrow[r, shift left, "\phi_1"] & V_1 \arrow[l, shift left, "\phi_1^*"] \arrow[r, shift left, "\phi_2"] & V_2 \arrow[l, shift left, "\phi_2^*"] \arrow[r, shift left, "\phi_3"] & \cdots \arrow[l, shift left, "\phi_3^*"] \arrow[r, shift left, "\phi_{d-2}"] & V_{d-2} \arrow[l, shift left, "\phi_{d-2}^*"] \arrow[r, shift left, "\phi_{d-1}"] & V_{d-1} \arrow[l, shift left, "\phi_{d-1}^*"] \arrow[r, shift left, "\phi_d"] & V_d \arrow[l, shift left, "\phi_d^*"] \end{tikzcd}$$

such that:

(a) $B = \phi_1^* \phi_1 + \xi_1 1_V$,

(b) $\phi_i \phi_i^* - \phi_{i+1}^* \phi_{i+1} = (\xi_{i+1} - \xi_i) 1_{V_i}$, for $i = 1, \dots, d-1$,

(c) the maps ϕ_i are surjective and the maps ϕ_i^* are injective for $i = 1, \dots, d$.

Proof. (1) ⇒ (2): Let $V_i = \mathrm{im}(\prod_{j=1}^{i}(B - \xi_j 1_V))$. Clearly $(B - \xi_i 1_V)(V_{i-1}) = \mathrm{im}(\prod_{j=1}^{i}(B - \xi_j 1_V)) = V_i$ and as B is conjugate to A it has the same dimension vector, so $\dim(V_i) = n_i$.

(2) ⇒ (3): Let $\phi_i = (B - \xi_i 1_V)|_{V_{i-1}}$ and let ϕ_i^* be the inclusion. Now for $i = 1, \dots, d-1$ we have $\phi_i \phi_i^* = (B - \xi_i 1_V)|_{V_{i-1}} \phi_i^* = (B - \xi_i 1_V)|_{V_i}$ which is an endomorphism of V_i. Now $\phi_{i+1}^* \phi_{i+1} = \phi_{i+1}^* (B - \xi_{i+1} 1_V)|_{V_i} = (B - \xi_{i+1} 1_V)|_{V_i}$ when thought of as an endomorphism of

V_i. So we have

$$\begin{aligned}\phi_i\phi_i^* - \phi_{i+1}^*\phi_{i+1} &= (B - \xi_i 1_V)\big|_{V_i} - (B - \xi_{i+1}1_V)\big|_{V_i} \\ &= (\xi_{i+1} - \xi_i)1_V\big|_{V_i} \\ &= (\xi_{i+1} - \xi_i)1_{V_i},\end{aligned}$$

so (3b) holds. Similarly $\phi_1^*\phi_1 + \xi_1 1_V = B - \xi_1 1_V + \xi_1 1_V = B$, so (3a) holds. Clearly ϕ_i^* is injective and by the definition of the flag $V_0, \dots, V_d$ the map ϕ_i is surjective for $i = 1, \dots, d$, so (3c) holds.

(3) $\Rightarrow$ (1): We prove by induction on d. If $d = 1$, then there exists $V_0 = V, V_1 = 0$ and maps $\phi_1 = 0$, $\phi_1^* = 0$ and $B = \phi_1^*\phi_1 + \xi_1 1_V = \xi_1 1_V$, that is B is scalar. From the dimension vector and minimal polynomial A must also be equal to $\xi_1 1_V$ so B is trivially conjugate to A.

Suppose $d > 1$. Let us identify V_1 with $\mathrm{im}(A - \xi_1 1_V)$ and let $A_1 = A|_{V_1}$. Clearly A_1 has minimal polynomial roots $\xi_2, \dots, \xi_d$ and associated dimension vector $(n_1, \dots, n_d)$. Let $B_1 = \phi_1\phi_1^* + \xi_1 1_{V_1}$, hence we have

$$\begin{aligned}B_1 &= \phi_2^*\phi_2 + \xi_2 1_{V_1} \\ \phi_i\phi_i^* - \phi_{i+1}^*\phi_{i+1} &= (\xi_{i+1} - \xi_i)1_{V_i}, \text{ for } i = 2, \dots, d-1.\end{aligned}$$

So by the inductive hypothesis B_1 is contained in the conjugacy class of A_1. By Corollary 2.1.23 we have

$$\mathrm{rank}((B_1 - \lambda 1_{V_1})^m) = \mathrm{rank}((A_1 - \lambda 1_V)^m), \text{ for } \lambda \in K, m \geq 1$$

Now A_1 is the restriction of A to $V_1 = \mathrm{im}(A - \xi_1 1_V)$, so

$$\mathrm{rank}((A_1 - \lambda 1_V)^m) = \mathrm{rank}((A - \xi_1 1_V)(A - \lambda 1_V)^m). \tag{2.2}$$

If $\lambda = \xi_1$ and $m = 0$, then $\mathrm{rank}(B - \lambda 1_V)^m = \mathrm{rank}(A - \lambda 1_V)^m$ trivially. Suppose $m \geq 1$, we have

$$(B - \lambda 1_V)^m = (B - \xi_1 1_V)^m = (\phi_1^*\phi_1)^m = \phi_1^*(\phi_1\phi_1^*)^{m-1}\phi_1 = \phi_1^*(B_1 - \lambda 1_{V_1})^{m-1}\phi_1.$$

So, using (2.2), we have

$$\mathrm{rank}((B - \lambda 1_V)^m) = \mathrm{rank}(\phi_1^*(B_1 - \lambda 1_{V_1})^{m-1}\phi_1) = \mathrm{rank}((B_1 - \lambda 1_{V_1})^{m-1}) = \mathrm{rank}((A - \lambda 1_V)^m)$$

We use the fact that ϕ_1^* is injective and ϕ_1 surjective and Lemma 2.2.3 in the above rank formula. If $\lambda \neq \xi_1$, then by Lemma 2.2.1 we have

$$\operatorname{rank}((B-\lambda 1_V)^m) = n_0 - n_1 + \operatorname{rank}((B_1 - \lambda 1_{V_1})^m).$$

Using the fact that

$$\ker((A-\xi_1 1_V)(A-\lambda 1_V)^m) = \ker(A-\xi_1 1_V) \oplus \ker((A-\lambda 1_V)^m)$$

we have

$$\operatorname{rank}((A-\lambda 1_V)^m) = n_0 - n_1 + \operatorname{rank}((A_1 - \lambda 1_V)^m). \tag{2.3}$$

So we conclude $\operatorname{rank}((B-\lambda 1_V)^m) = \operatorname{rank}((A-\lambda 1_V)^m)$ for all $\lambda \in K$ and $m \geq 0$. Hence, by Corollary 2.1.23, B is in the conjugacy class of A. $\square$

2.2.2 Self-Adjoint Endomorphisms of a Vector Space

We introduce symmetric bilinear forms in this section. The basic definitions and well-known results can be found in references such as [Hal58], [Kap03] and [Lam05].

Let $(\cdot,\cdot)$ be a bilinear form on V. We say $(\cdot,\cdot)$ is *symmetric* if $(x,y) = (y,x)$ for all $x,y \in V$. Let $x \in V$, we say $(\cdot,\cdot)$ is *nondegenerate* if $(x,y) = 0$, for all $y \in V$, implies $x = 0$.

Given a linear map $g\colon V \to W$ between two vector spaces V, W endowed with nondegenerate symmetric bilinear forms $(\cdot,\cdot)_V, (\cdot,\cdot)_W$ respectively, the *adjoint of g with respect to the bilinear forms* is the unique linear map $g^*\colon W \to V$ such that $(v, g^*(w))_V = (g(v), w)_W$ for all $v \in V$ and $w \in W$.[2]

If $(\cdot,\cdot)_V, (\cdot,\cdot)_W$ are symmetric bilinear forms (but not necessarily nondegenerate), then we say two map $g\colon V \to W$ and $g^*\colon W \to V$ are *adjoint to each another with respect to the bilinear forms* if $(v, g^*(w))_V = (g(v), w)_W$ for all $v \in V$ and $w \in W$. If we are given only g, then without the nondegeneracy of the bilinear forms the existence and uniqueness of the adjoint is not guaranteed.

A linear map $f \in \operatorname{End}(V)$ where V is endowed with a symmetric bilinear form $(\cdot,\cdot)$ is *self-adjoint with respect to* $(\cdot,\cdot)$ if $(f(x), y) = (x, f(y))$ for all $x, y \in V$.

[2]Note that we sometimes use the superscripted star g^* notation simply to denote a function distinct from g which is not necessarily the adjoint of g. If we intend g^* to be the adjoint of g, then this must be explicitly stated when defining g^*, otherwise it need not be the adjoint.

Unless otherwise stated $(\cdot,\cdot)$ is a nondegenerate symmetric bilinear form defined on V.

Definition 2.2.5. Let $W \subseteq V$ be a subspace of V. The *orthogonal complement* $W^{\perp} \subseteq V$ is a vector subspace of V defined by $W^{\perp} = \{v \in V : (v, w) = 0, \forall w \in W\}$.

Remark 2.2.6. Let $W \subseteq V$ be a subspace of V and let $(\cdot,\cdot)$ be a nondegenerate symmetric bilinear form, we have $\dim(W^{\perp}) = \dim(V) - \dim(W)$ (see [Lam05, Chap. I, Prop. 1.3]), this implies $\dim(W) = \dim(V/W^{\perp})$. There is a natural isomorphism from W to $V/W^{\perp}$ given by $w \mapsto w + W^{\perp}$ (see [Hal58, Sec. 48]).

Note: we use the notation $v + W$ to write elements of V/W where $v \in V$.

Lemma 2.2.7. Suppose V is a vector space endowed with a symmetric bilinear form $(\cdot,\cdot)_V$. Let W be a vector space and let $\phi\colon V \to W$ and $\psi\colon W \to V$ be linear maps such that ϕ is surjective and $\psi\phi\colon V \to V$ is self-adjoint with respect to $(\cdot,\cdot)_V$. Let $(\cdot,\cdot)_W$ be the bilinear form defined on W given by $(w, w')_W = (v, \psi(w'))_V$ where $v \in \phi^{-1}(w)$, (where $\phi^{-1}(w)$ is the preimage of w). We have that $(\cdot,\cdot)_W$ is well-defined and symmetric, ϕ and ψ are adjoint with respect to $(\cdot,\cdot)_V$ and $(\cdot,\cdot)_W$ and $\phi\psi$ is self-adjoint with respect to $(\cdot,\cdot)_W$.

Furthermore if ψ is injective and $(\cdot,\cdot)_V$ is nondegenerate, then $(\cdot,\cdot)_W$ is also nondegenerate.

Proof. We first show $(\cdot,\cdot)_W$ is well-defined, let $w, w' \in W$ and $v, v' \in \phi^{-1}(w)$, we have

$$\begin{aligned}
(v, \psi(w'))_V - (v', \psi(w'))_V &= (v - v', \psi(w'))_V \\
&= (v - v', \psi\phi(v''))_V, && \text{(where } v'' \in \phi^{-1}(w'),\text{ as } \phi \text{ is onto)} \\
&= (\psi\phi(v - v'), v'')_V, && \text{(by self-adjointness)} \\
&= (\psi(w - w), v'')_V \\
&= (0, v'')_V = 0
\end{aligned}$$

So $(v, \psi(w'))_V = (v', \psi(w'))_V$. We now show symmetry. Let $w, w' \in W$, we have

$$\begin{aligned}
(w, w')_W &= (v, \psi(w'))_V, && \text{(for some } v \in \phi^{-1}(w)) \\
&= (v, \psi\phi(v'))_V, && \text{(for some } v' \in \phi^{-1}(w')) \\
&= (\psi(w), v')_V, && \text{(by self-adjointness of } \psi\phi \text{ and } \phi(v) = w) \\
&= (v', \psi(w))_V \\
&= (w', w)_W.
\end{aligned}$$

By definition we have $(\phi(v), w)_W = (v, \psi(w))_V$ for all $v \in V$ and $w \in W$, as $v \in \phi^{-1}(\phi(v))$, so ϕ and ψ are adjoints. We now show $\phi\psi$ is self-adjoint with respect to $(\cdot,\cdot)_W$. Let $w, w' \in W$, we have

$$(\phi\psi w, w')_W = (\psi w, \psi w')_V = (\psi w', \psi w)_V = (\phi\psi w', w)_W = (w, \phi\psi w')_W.$$

Now suppose ψ is injective and $(\cdot,\cdot)_V$ is nondegenerate. Let $w' \in W$ and suppose $(w, w')_W = 0$ for all $w \in W$, as $\mathrm{im}(\phi) = W$ we have $(v, \psi(w'))_V = 0$ for all $v \in V$. By the nondegeneracy of $(\cdot,\cdot)_V$ we have $\psi(w') = 0$ which implies $w' = 0$ therefore $(\cdot,\cdot)_W$ is nondegenerate. □

The next theorem is a new result which gives a construction of all self-adjoint endomorphisms which lie in the closure of a given conjugacy class.

Theorem 2.2.8. Let V be a vector space endowed with a nondegenerate symmetric bilinear form $(\cdot,\cdot)$. Let $A, B \in \mathrm{End}(V)$ such that A is self-adjoint with respect to $(\cdot,\cdot)$. Let $\xi_1, \ldots, \xi_d \in K$ be a list of the zeros of the minimal polynomial of A, let $(n_0, n_1, \ldots, n_d)$ be the associated dimension vector. The following statements are equivalent.

1. The map B is in the closure of the conjugacy class of A and B is self-adjoint with respect to $(\cdot,\cdot)$.

2. There exists vector spaces $V = V_0, V_1, \ldots, V_d = 0$ endowed with symmetric bilinear forms $(\cdot,\cdot)_i$ (not necessarily nondegenerate), where $\dim(V_i) = n_i$, $(\cdot,\cdot)_0 = (\cdot,\cdot)$ and there exist linear maps

$$V_0 \underset{\phi_1^*}{\overset{\phi_1}{\rightleftarrows}} V_1 \underset{\phi_2^*}{\overset{\phi_2}{\rightleftarrows}} V_2 \underset{\phi_3^*}{\overset{\phi_3}{\rightleftarrows}} \cdots \underset{\phi_{d-2}^*}{\overset{\phi_{d-2}}{\rightleftarrows}} V_{d-2} \underset{\phi_{d-1}^*}{\overset{\phi_{d-1}}{\rightleftarrows}} V_{d-1} \underset{\phi_d^*}{\overset{\phi_d}{\rightleftarrows}} V_d$$

such that

(a) $B = \phi_1^*\phi_1 + \xi_1 1_V$,

(b) $\phi_i\phi_i^* - \phi_{i+1}^*\phi_{i+1} = (\xi_{i+1} - \xi_i)1_{V_i}$, for $i = 1, \ldots, d-1$,

(c) ϕ_i^* is the adjoint of ϕ_i with respect to the appropriate bilinear forms for $i = 1, \ldots, d$.

Proof. (1) ⇒ (2): By Theorem 2.2.2 there exists a flag of subspaces

$$V = W_0 \supseteq W_1 \supseteq \cdots \supseteq W_{d-1} \supseteq W_d = 0$$

such that $\dim(W_j) = n_j$ for $j = 0, 1, \ldots, d$ and $(B - \xi_j 1_V)(W_{j-1}) \subseteq W_j$ for $j = 1, \ldots, d$. Let $V_i = V/W_i^\perp$. By Remarks 2.2.6 $\dim(V_i) = n_i$ for $i = 0, 1, \ldots, d$. Let $\phi_i \colon V_{i-1} \to V_i$ be the surjection defined by

$$\phi_i(v + W_{i-1}^\perp) = v + W_i^\perp$$

and let $\phi_i^* \colon V_i \to V_{i-1}$ be defined by

$$\phi_i^*(v + W_i^\perp) = (B - \xi_i 1_V)v + W_{i-1}^\perp.$$

To show ϕ_i^* is well-defined we show $(B - \xi_i 1_V)$ maps $W_i^\perp$ into $W_{i-1}^\perp$. Let $v \in W_i^\perp$, so $(v, v') = 0$ for all $v' \in W_i$. Now let $v'' \in W_{i-1}$ we have $((B - \xi_i 1_V)v, v'') = (v, (B - \xi_i 1_V)v'')$ by self-adjointness. Now $(B - \xi_i 1_V)v'' \in W_i$ so $(v, (B - \xi_i 1_V)v'') = 0$ as $v \in W_i^\perp$. Therefore $((B - \xi_i 1_V)v, v'') = 0$ for all $v'' \in W_{i-1}$, so $(B - \xi_i 1_V)v \in W_{i-1}^\perp$.

We prove the linear maps satisfy the deformed preprojective relations, we make use of the natural isomorphisms between W_i and $V/W_i^\perp$ for $i = 0, 1, \ldots, d$ referred to in Remark 2.2.6. We have, for all $v \in V$,

$$\begin{aligned}
(\phi_1^*\phi_1 + \xi_1 1_V)v &= \phi_1^*\phi_1(v + W_0^\perp) + \xi_1 v \\
&= (B - \xi_1 1_V)(v + W_1^\perp) + \xi_1 v \\
&= ((B - \xi_1 1_V)v + W_0^\perp) + \xi_1 v \\
&= (B - \xi_1 1_V + \xi_1 1_V)v \\
&= Bv.
\end{aligned}$$

For all $i = 1, \ldots, d-1$ and $v \in V_i$ we have

$$\begin{aligned}
(\phi_i\phi_i^* - \phi_{i+1}^*\phi_{i+1})v &= \phi_i(B - \xi_i 1_V)(v + W_i^\perp) - (B - \xi_{i+1} 1_V)\phi_{i+1}(v + W_i^\perp) \\
&= \phi_i((B - \xi_i 1_V)v + W_{i-1}^\perp) - (B - \xi_{i+1} 1_V)(v + W_{i+1}^\perp) \\
&= ((B - \xi_i 1_V)v + W_i^\perp) - ((B - \xi_{i+1} 1_V)v + W_i^\perp) \\
&= (B - \xi_i 1_V)v - (B - \xi_{i+1} 1_V)v \\
&= (\xi_{i+1} - \xi_i)v
\end{aligned}$$

Let $(\cdot,\cdot)_0 = (\cdot,\cdot)$. By induction on $i = 1, \ldots, d$ we endow each V_i with a symmetric bilinear form such that ϕ_i is the adjoint of ϕ_i^*. Assume $(\cdot,\cdot)_{i-1}$ is a symmetric bilinear form on V_{i-1}, using the fact that ϕ_i is surjective let $(\cdot,\cdot)_i$ be the symmetric bilinear form on V_i

constructed in Lemma 2.2.7 using

$$\begin{tikzcd} V_{i-1} \arrow[r, shift left, "\phi_i"] & V_i \arrow[l, shift left, "\phi_i^*"] \end{tikzcd}.$$

The lemma ensures ϕ_i and ϕ_i^* are adjoints.

(2) $\Rightarrow$ (1): By Theorem 2.2.2 we have that B is in the closure of the conjugacy class of A, all we need to show is that B is self-adjoint with respect to $(\cdot,\cdot) = (\cdot,\cdot)_0$. Now $\phi_1^*\phi_1 = B - \xi_1 1_V$ is clearly self-adjoint with respect to $(\cdot,\cdot)_0$, as is a multiple of the identity map, so as self-adjointness is closed under addition B is self-adjoint. □

The next theorem is a new result which gives a construction of all self-adjoint endomorphisms which lie precisely in a given conjugacy class.

Theorem 2.2.9. Let V be a vector space endowed with a nondegenerate symmetric bilinear form $(\cdot,\cdot)$. Let $A, B \in \mathrm{End}(V)$ such that A is self-adjoint with respect to $(\cdot,\cdot)$. Let $\xi_1, \dots, \xi_d \in K$ be a list of the zeros of the minimal polynomial of A, let $(n_0, n_1, \dots, n_d)$ be the associated dimension vector. The following statements are equivalent.

1. The map B is in the conjugacy class of A and B is self-adjoint with respect to $(\cdot,\cdot)$.

2. There exist vector spaces $V = V_0, V_1, \dots, V_d = 0$ endowed with nondegenerate symmetric bilinear forms $(\cdot,\cdot)_i$, where $\dim(V_i) = n_i$, $(\cdot,\cdot)_0 = (\cdot,\cdot)$ and there exists linear maps

$$\begin{tikzcd} V_0 \arrow[r, shift left, "\phi_1"] & V_1 \arrow[l, shift left, "\phi_1^*"] \arrow[r, shift left, "\phi_2"] & V_2 \arrow[l, shift left, "\phi_2^*"] \arrow[r, shift left, "\phi_3"] & \cdots \arrow[l, shift left, "\phi_3^*"] \arrow[r, shift left, "\phi_{d-2}"] & V_{d-2} \arrow[l, shift left, "\phi_{d-2}^*"] \arrow[r, shift left, "\phi_{d-1}"] & V_{d-1} \arrow[l, shift left, "\phi_{d-1}^*"] \arrow[r, shift left, "\phi_d"] & V_d \arrow[l, shift left, "\phi_d^*"] \end{tikzcd}$$

   such that

   (a) $B = \phi_1^*\phi_1 + \xi_1 1_V$,

   (b) $\phi_i\phi_i^* - \phi_{i+1}^*\phi_{i+1} = (\xi_{i+1} - \xi_i)1_{V_i}$, for $i = 1, \dots, d-1$,

   (c) the maps ϕ_i are surjective and ϕ_i^* are injective,

   (d) ϕ_i^* is the adjoint of ϕ_i with respect to the appropriate bilinear forms for $i = 1, \dots, d$.

Proof. (1) $\Rightarrow$ (2): By Theorem 2.2.4 there exist vector spaces $V = V_0, V_1, \dots, V_{d-1}, V_d = 0$ with $\dim(V_i) = n_i$ and linear maps ϕ_i and ϕ_i^* satisfying all the conditions except for adjointness of ϕ_i and ϕ_i^*. We show there exist nondegenerate symmetric bilinear forms on the vector spaces which ensure adjointness.

Let $(\cdot,\cdot)_0 = (\cdot,\cdot)$ (note this is nondegenerate). By induction on $i = 1,\ldots,d$ we endow each V_i with a symmetric bilinear form such that ϕ_i is the adjoint of ϕ_i^*. Assume $(\cdot,\cdot)_{i-1}$ is a nondegenerate symmetric bilinear form on V_{i-1}, using the fact that ϕ_i is surjective let $(\cdot,\cdot)_i$ be the symmetric bilinear form on V_i constructed in Lemma 2.2.7 using

$$\begin{tikzcd} V_{i-1} \arrow[r, bend left, "\phi_i"] & V_i \arrow[l, bend left, "\phi_i^*"] \end{tikzcd} .$$

The lemma ensures ϕ_i and ϕ_i^* are adjoints and futhermore, as $(\cdot,\cdot)_{i-1}$ is nondegenerate and ϕ_i^* is injective, $(\cdot,\cdot)_i$ is nondegenerate.

(2) $\Rightarrow$ (1): By Theorem 2.2.4 B is in the conjuugacy class of A and by the same argument of Theorem 2.2.8 B is self-adjoint with respect to $(\cdot,\cdot)$. □

Given a vector space V endowed with a symmetric nondegenerate bilinear form $(\cdot,\cdot)$, the orthogonal group $\mathrm{O}(V) \subseteq \mathrm{End}(V)$ is the group of automorphisms which preserve $(\cdot,\cdot)$, that is $\phi \in \mathrm{O}(V)$ if and only if $(\phi(x),\phi(y)) = (x,y)$ for all $x,y \in V$. The conditions of Theorem 2.2.9 are equivalent to A and B being orthogonally conjugate, that is conjugate via an orthogonal transformation. This is proven in [Kap03, Thm. 70] as well as in Theorem 6.1.7. We attempted to show that the conditions of Theorem 2.2.8 are equivalent to B being in the closure of the conjugacy class of A under the orthogonal group (i.e. B is in the closure of $\mathrm{O}(V)A = \{U^{-1}AU \colon U \in \mathrm{O}(V)\}$) but were unable to do so. This question remains an open problem.

Chapter 3

The Kronecker and Root Sum

We introduce some new machinary in this chapter which we use in Section 4.4. Here we define two polynomial operations: the Kronecker sum of two polynomials and the root sum operation on one polynomial. Let R be an integral domain. The Kronecker sum takes two polynomials in $R[x]$ and returns another polynomial in $R[x]$. The root sum takes a polynomial in $R[x]$ of degree n and returns a polynomial in $R[x, y_1, \ldots, y_n]$. The usefulness of the Kronecker sum and root sum become apparent in Theorems 3.1.3 and 3.2.11 respectively, where we show how they can be used to manipulate the roots of their operands. In Section 4.4 we make use of the fact that it is possible to compute both operations without having to factor the given polynomials (i.e. the operations can be computed from the coefficients in a finite number of steps).

Suppose polynomials f and g have degrees n and m and can be written as a product of distinct linear factors, suppose f has roots $\xi_1, \ldots, \xi_n$ and g has roots $\zeta_1, \ldots, \zeta_m$. Their Kronecker sum has degree nm and has roots: all possible sums of the roots of f and g, that is $\xi_i + \zeta_j$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$. The root sum of f has degree $n!$ in x and has roots $\xi_{\sigma(1)} y_1 + \cdots \xi_{\sigma(n)} y_n$ where σ is a permutation of $\{1, \ldots, n\}$. It may be helpful to think of the root sum as having roots which are linear combinations of $y_1, \ldots, y_n$ where the coefficients are the roots of f.

3.1 The Kronecker Sum of Two Polynomials

Remark 3.1.1. Let $f, g \in R[x]$ where $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ and $g(x) = b_0 + b_1 x + \cdots + b_m x^m$. Recall the resultant $\mathrm{Res}(f, g, x)$ of f and g in x is the determinant of the Sylvester matrix.

The Sylvester matrix is a square matrix of size $n + m$ given by

$$\begin{pmatrix} a_n & a_{n-1} & \cdots & a_1 & a_0 & 0 & \cdots & 0 \\ 0 & a_n & a_{n-1} & \cdots & a_1 & a_0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_n & a_{n-1} & \cdots & a_1 & a_0 \\ b_m & b_{m-1} & \cdots & b_1 & b_0 & 0 & \cdots & 0 \\ 0 & b_m & b_{m-1} & \cdots & b_1 & b_0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & b_m & b_{m-1} & \cdots & b_1 & b_0 \end{pmatrix}$$

That is the first m rows are built by progressively shifting the coefficients $a_n, \dots, a_0$ and the final n rows are built by shifting $b_m, \dots, b_0$. For example, suppose $f(x) = 3x^2 + 6x - 3$ and $g(x) = 8x - 4$, the Sylvester matrix is given by

$$\begin{pmatrix} 3 & 6 & -3 \\ 8 & -4 & 0 \\ 0 & 8 & -4 \end{pmatrix},$$

so the resultant of f and g in x is 48.

Suppose we can write f and g in factored form $f(x) = a_n(x - \xi_1)\dots(x - \xi_n)$ and $g(x) = b_m(x - \zeta_1)\dots(x - \zeta_m)$ where $\xi_i, \zeta_j \in R$, we can write the resultant in terms of the linear factors of f and g. By [Lan02, Prop. 8.3] we have

$$\operatorname{Res}(f(x), g(x), x) = a_n^m b_m^n \prod_{i=1}^{n} \prod_{j=1}^{m} (\xi_i - \zeta_j).$$

Definition 3.1.2. Let $f, g \in R[x]$ where $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ and $g(x) = b_0 + b_1 x + \cdots + b_m x^m$. Let y be an indeterminate, we define the *Kronecker sum* of f and g with respect to x by

$$f \oplus^K g = (-1)^{nm} \operatorname{Res}(f(x - y), g(y), y).$$

If the polynomials are multivariate polynomials, then it must be made clear from the context which variable the operation is in terms of, for example suppose $f, g \in R[x, y, z]$ we must say which of x, y, z the Kronecker sum is in terms of.

The following theorem is applicable whenever we can write the operands of the Kronecker sum as products of linear factors.

Theorem 3.1.3. Suppose $f, g \in R[x]$ such that we can write $f(x) = a_n(x-\xi_1)\dots(x-\xi_n)$ and $g(x) = b_m(x-\zeta_1)\dots(x-\zeta_m)$ where $a_n, \xi_1, \dots, \xi_n, b_m, \zeta_1, \dots, \zeta_m \in R$, we have

$$f \oplus^K g = a_n^m b_m^n \prod_{i=1}^{n} \prod_{j=1}^{m} (x - \xi_i - \zeta_j).$$

Proof. The roots of $f(x)$ are $\xi_1, \dots, \xi_n$ and the roots of $g(x)$ are $\zeta_1, \dots, \zeta_m$. The resultant of f and g in x can be written as $\mathrm{Res}(f(x), g(x), x) = a_n^m b_m^n \prod_{i=1}^n \prod_{j=1}^m (\xi_i - \zeta_j)$. Now consider $f(x-y)$ and $g(y)$ as polynomials in y. The roots of $f(x-y)$ are $x-\xi_1, \dots, x-\xi_n$ as $f(x-y) = (-1)^n a_n(y-x+\xi_1)\dots(y-x+\xi_n)$. So $\mathrm{Res}(f(x-y), g(y), y) = (-1)^{nm} a_n^m b_m^n \prod_{i=1}^n \prod_{j=1}^m (x-\xi_i-\zeta_j)$ as required. □

Theorem 3.1.4. The Kronecker sum is commutative and associative.

Proof. Let $f, g \in R[x]$, it is always possible to find an extension field L such that $f, g \in L[x]$ and f and g factor into linear factors with roots in L (see [Art91, Chap. 13, Prop. 5.3]). We can then write $f \oplus^K g$ as in Theorem 3.1.3. As the roots of the Kronecker sum are: all the possible sums of the roots of the operands, both commutativity and associativity follow from the commutativity and associativity of addition. □

Example 3.1.5. Suppose $f(x) = x^2-10x+24 = (x-4)(x-6)$, $g(x) = x^3+x^2-6x = (x-2)(x+3)x$, then by Theorem 3.1.3

$$\begin{aligned} f \oplus^K g &= (x-4-2)(x-6-2)(x-4+3)(x-6+3)(x-4)(x-6) \\ &= (x-8)(x-1)(x-3)(x-4)(x-6)^2 \\ &= x^6 - 28x^5 + 311x^4 - 1736x^3 + 5052x^2 - 7056x + 3456. \end{aligned}$$

We now calculate $f \oplus^K g$ from the definition to show this yields the same answer. We have $f(x-y) = y^2 + (10-2x)y + (x^2 - 10x + 24)$. The Kronecker sum is therefore:

$$\begin{aligned} f \oplus^K g &= (-1)^6 \mathrm{Res}(f(x-y), f(y), y) \\ &= \det \begin{pmatrix} 1 & 10-2x & x^2-10x+24 & 0 & 0 \\ 0 & 1 & 10-2x & x^2-10x+24 & 0 \\ 0 & 0 & 1 & 10-2x & x^2-10x+24 \\ 1 & 1 & -6 & 0 & 0 \\ 0 & 1 & 1 & -6 & 0 \end{pmatrix} \\ &= x^6 - 28x^5 + 311x^4 - 1736x^3 + 5052x^2 - 7056x + 3456. \end{aligned}$$

3.2 The Root Sum of a Polynomial

The root sum, $\mathcal{R}_g$ is a unary operation on a polynomial g in $R[x]$ of degree n, with respect to x. The resulting polynomial has n extra indeterminates so is in $R[x, y_1, \dots, y_n]$. The definition is given in terms of the coefficients of g but in Theorem 3.2.11 we show it is possible to write the root sum in terms of the roots of g (whenever g can be written as a product of linear factors), the theorem therefore justifies the name root sum. The definition and Theorem 3.2.11 give two methods for computing the root sum, in Section 3.2.3 we give a third method. In Section 3.2.4 we give an example of computing the root sum using all three methods and show they agree.

3.2.1 Integer Partitions, Integer Compositions and Symmetric Polynomials

We establish some notation before giving the definition of the root sum. For a non-negative integer n we denote by Σ_n the *group of permutations* of $\{1, \dots, n\}$. Recall the definition of an integer partition introduced in Section 2.1.2, we introduce the related concept of an integer composition.

Definition 3.2.1. An *integer composition* is a finite sequence of non-negative integers $\mathbf{r} = (r_1, \dots, r_n)$ for some n such that $r_1, \dots, r_n \geq 0$, the integers $r_1, \dots, r_n$ are called the parts of $\mathbf{r}$. Let $\mathbf{r} = (r_1, \dots, r_n)$ be an integer composition. Let $\nu \in \Sigma_n$, define $\nu \cdot \mathbf{r}$ to be the permuted composition given by $(r_{\nu^{-1}(1)}, \dots, r_{\nu^{-1}(n)})$.

Let the group of permutations in Σ_n which fix $\mathbf{r}$ be denoted $\Sigma_n^{\mathbf{r}}$, that is $\Sigma_n^{\mathbf{r}} = \{\sigma \in \Sigma_n : r_{\sigma^{-1}(i)} = r_i, i = 1, \dots, n\}$ (this defines a left group action of Σ_n on the set of integer compositions with n parts). We denote the set of left-cosets of $\Sigma_n^{\mathbf{r}}$ in Σ_n by $\Sigma_n/\Sigma_n^{\mathbf{r}}$. This is not generally a group as $\Sigma_n^{\mathbf{r}}$ is not generally a normal subgroup of Σ_n. We identify $\Sigma_n^{\mathbf{r}}$ with a set of representatives of the cosets.

Example 3.2.2. We have $\Sigma_3 = \{\mathrm{id}, (1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)\}$. Let $\mathbf{r} = (1, 1, 2)$, so we have $\Sigma_3^{\mathbf{r}} = \{\mathrm{id}, (1\ 2)\}$ and $\Sigma_n/\Sigma_3^{\mathbf{r}} = \{\mathrm{id}, (2\ 3), (1\ 3)\}$ is a set of representatives of the cosets. Let $\mathbf{r} = (2, 1, 3)$, so we have $\Sigma_3^{\mathbf{r}} = \{\mathrm{id}\}$ and so $\Sigma_n/\Sigma_3^{\mathbf{r}} = \Sigma_n$.

Recall integer partitions from Section 2.1.2, we can think of an integer partition $\mu = (\mu_1, \dots, \mu_n)$ as an integer composition such that the parts are arranged in descending order (recall that we don't distinguish between partitions which differ only by a string a zeros at the end, we think of a partition consiting only of zeros as being equivalent to (0)). We denote the set of all integer partitions by $\mathcal{P}$. If $\mu \neq (0)$, then let $\mathcal{L}(\mu)$ be the number of nonzero parts of μ, if $\mu = (0)$, then let $\mathcal{L}(\mu) = 1$. If $i > \mathcal{L}(\mu)$, then we define $\mu_i = 0$.

Let $t \geq 0$ and let $\mathcal{P}(t)$ be the set of integer partitions which sum to t. Let $n \geq 0$ and let $\mathcal{P}_n(t)$ be the set of integer partitions which sum to t and have at most n parts. let $\mathcal{P}^n(t)$ be the set of integer partitions which sum to t and have no part exceeding n.

For each composition $\mathbf{r}$ we can uniquely write $\mathbf{r} = \nu \cdot \mu$ where $\nu \in \Sigma_n/\Sigma_n^{\mathbf{r}}$ and $\mu \in \mathcal{P}$. We introduce two families of symmetric polynomials: the monomial symmetric polynomials and the elementary symmetric polynomials, these definitions can be found in [Mac95] and [Art91, Chap. 14].

Definition 3.2.3. Let $\mu \in \mathcal{P}_n(t)$ be a integer partition, summing to $t \geq 0$. The *monomial symmetric polynomials* $m_\mu(y_1, \ldots, y_n)$ in n variables is defined by

$$m_\mu(y_1, \ldots, y_n) = \sum_{\nu \in \Sigma_n/\Sigma_n^\mu} y_1^{(\nu\cdot\mu)_1} \ldots y_n^{(\nu\cdot\mu)_n}.$$

Definition 3.2.4. Let $0 \leq i \leq n$. The ith *elementary symmetric polynomial* in n variables is denoted $e_i(y_1, \ldots, y_n)$ and defined by $e_0(y_1, \ldots, y_n) = 1$ and, for $i > 0$, by

$$e_i(y_1, \ldots, y_n) = \sum_{1 \leq j_1 < \cdots < j_i \leq n} y_{j_1} \ldots y_{j_i}$$

Let $\mu \in \mathcal{P}^n(t)$ be an integer partition for some $t \geq 0$ with no part greater than n. We define

$$e_\mu(y_1, \ldots, y_n) = \prod_{i=1}^{\mathcal{L}(\mu)} e_{\mu_i}(y_1, \ldots, y_n).$$

Let T be a finite set, we denote the power-set of T as $\mathbb{P}(T)$, that is the set of all subsets of T, and denote by $\mathbb{P}^i(T)$ the set of all subsets of T of cardinality i, that is $\mathbb{P}^i(T) = \{S \in \mathbb{P}(T) \colon |S| = i\}$. Using this notation we can write the ith elementary symmetric polynomial as

$$e_i(y_1, \ldots, y_n) = \sum_{S \in \mathbb{P}^i(\{1,\ldots,n\})} \prod_{s \in S} y_s.$$

Lemma 3.2.5. If $p \in R[x]$ is a polynomial with coefficients in a ring R such that we can write $p(x) = \prod_{i=1}^n (x - \xi_i)$ for some roots $\xi_1, \ldots, \xi_n \in R$, then the expanded form of p is written

$$p(x) = \sum_{i=0}^n x^{n-i} e_i(-\xi_1, \ldots, -\xi_n).$$

Proof. [Art91, Chap 14, 3.3] □

Lemma 3.2.6. Let $t \geq 0$ and let $\mu \in \mathcal{P}_n(t)$ be an integer partition, we can write m_μ in terms of elementary symmetric polynomials by

$$m_\mu(y_1, \ldots, y_n) = \sum_{\eta \in \mathcal{P}^n(t)} \mathcal{E}_{\mu,\eta} e_\eta(y_1, \ldots, y_n),$$

where $\mathcal{E}_{\mu,\eta}$ are integer coefficients defined combinatorially in [BRW96, Sec. 2] (where it is denoted $M(e,m)_{\eta\mu}$), which depend only on μ and η.

Proof. The monomial symmetric and elementary symmetric polynomials both form bases over the vector space of symmetric polynomials of degree t. See [BRW96, Sec. 2]. □

The values of $\mathcal{E}_{\mu,\eta}$ can be computed from the formula above or from combinatorial descriptions given in [BRW96, Sec. 2].

Definition 3.2.7. Let $n \geq 1$ be fixed. Given a set S and an integer composition $\mathbf{r} = (r_1, \ldots, r_n)$ let $F(S, \mathbf{r})$ be the set of maps from S to $\{1, \ldots, n\}$ with fibres of size $r_1, \ldots, r_n$, that is

$$F(S, \mathbf{r}) = \{p \colon S \to \{1, \ldots, n\} \colon |p^{-1}(j)| = r_j, j = 1, \ldots, n\}.$$

Given an integer composition $\mathbf{r} = (r_1, \ldots, r_n)$ and a set S we can think of $F(S, \mathbf{r})$ as the set of functions which distribute the elements of S between n ordered boxes such that r_i elements go into box i, for $i = 1, \ldots, n$. Suppose $S = \{a, b, c\}$ and $\mathbf{r} = (0, 1, 2)$ then $F(S, \mathbf{r})$ contains three functions, each function places one of a, b, c into box 1, places the other two in box 2 and places no elements in box 0.

3.2.2 Root Sum Definition and Properties

We now give the definition of the root sum, although the purpose of this definition may not be immediately clear, the important thing to note is that it can be constructed using the coefficients of the polynomial $g(x)$ without any need to factorize. Theorem 3.2.11 proves the key property which justifies the name "root sum" and explains its purpose.

Definition 3.2.8. Let $g \in R[x]$ be $g(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ where $a_0, \ldots, a_n \in R$. Let $t \geq 0$ and let $\eta \in \mathcal{P}^n(t)$, we define $\gamma_\eta \in R[y_1, \ldots, y_n]$ by

$$\gamma_\eta(y_1, \ldots, y_n) = \sum_{\mu \in \mathcal{P}_n(t)} \mathcal{E}_{\mu,\eta} \sum_{S \in \mathbb{P}^t(\Sigma_n)} \sum_{p \in F(S,\mu)} \prod_{\sigma \in S} y_{\sigma(p(\sigma))}. \tag{3.1}$$

The *root sum* $\mathcal{R}_g \in R[x, y_1, \ldots, y_n]$ of g is defined by

$$\mathcal{R}_g(x, y_1, \ldots, y_n) = \sum_{t=0}^{n!} x^{n!-t} \sum_{\eta \in \mathcal{P}^n(t)} \left(\prod_{i=1}^{\mathcal{L}(\eta)} a_{n-\eta_i} \right) \gamma_\eta(y_1, \ldots, y_n). \tag{3.2}$$

The following lemmas and theorem justify the name "root sum", by showing how we can write it in terms of the roots of $g(x)$.

Lemma 3.2.9. Let $t \geq 0$ and let $\mu \in \mathcal{P}_n(t)$. Suppose we are given constants $\beta_\nu \in R$ where $\nu \in \Sigma_n/\Sigma_n^\mu$ and indeterminates $y_1, \dots, y_n$, we have

$$\sum_{\nu \in \Sigma_n/\Sigma_n^\mu} \beta_\nu \sum_{S \in \mathbb{P}^t(\Sigma_n)} \sum_{p \in F(S, \nu \cdot \mu)} \prod_{\sigma \in S} y_{\sigma(p(\sigma))} = \left(\sum_{\nu \in \Sigma_n/\Sigma_n^\mu} \beta_\nu \right) \left(\sum_{S \in \mathbb{P}^t(\Sigma_n)} \sum_{p \in F(S, \mu)} \prod_{\sigma \in S} y_{\sigma(p(\sigma))} \right).$$

Proof. There is a bijection from $F(S, \nu \cdot \mu)$ to $F(S, \mu)$ given by $p \mapsto \nu \cdot p$ (where $(\nu \cdot p)(\sigma) = \nu(p(\sigma))$ for $\sigma \in S$). To see this note that $p \in F(S, \nu \cdot \mu)$ if and only if $|p^{-1}(j)| = (\nu \cdot \mu)_j = \mu_{\nu^{-1}(j)}$ for all $j = 1, \dots, n$ which occurs if and only if $|p^{-1}(\nu^{-1}(j))| = |(\nu \cdot p)^{-1}(j)| = \mu_j$ for all $j = 1, \dots, n$. So we can write

$$\sum_{\nu \in \Sigma_n/\Sigma_n^\mu} \beta_\nu \sum_{S \in \mathbb{P}^t(\Sigma_n)} \sum_{p \in F(S, \nu \cdot \mu)} \prod_{\sigma \in S} y_{\sigma(p(\sigma))} = \sum_{\nu \in \Sigma_n/\Sigma_n^\mu} \beta_\nu \sum_{S \in \mathbb{P}^t(\Sigma_n)} \sum_{p \in F(S, \mu)} \prod_{\sigma \in S} y_{\sigma(\nu(p(\sigma)))}$$

There is a bijection from $\mathbb{P}^t(\Sigma_n)$ to $\mathbb{P}^t(\Sigma_n)$ given by $S \mapsto S'$ where $S' = \{\sigma\nu \colon \sigma \in S\}$, and a bijection from $F(S, \mu)$ to $F(S', \mu)$ given by $p \mapsto p'$ where $p'(\sigma) = p(\sigma\nu^{-1})$ for $\sigma \in S'$. So we can write

$$\begin{aligned}
\sum_{\nu \in \Sigma_n/\Sigma_n^\mu} \beta_\nu \sum_{S \in \mathbb{P}^t(\Sigma_n)} \sum_{p \in F(S, \mu)} \prod_{\sigma \in S} y_{\sigma(\nu(p(\sigma)))} &= \sum_{\nu \in \Sigma_n/\Sigma_n^\mu} \beta_\nu \sum_{S \in \mathbb{P}^t(\Sigma_n)} \sum_{p \in F(S, \mu)} \prod_{\sigma \in S} y_{(\sigma\nu)(p(\sigma\nu\nu^{-1}))} \\
&= \sum_{\nu \in \Sigma_n/\Sigma_n^\mu} \beta_\nu \sum_{S \in \mathbb{P}^t(\Sigma_n)} \sum_{p \in F(S, \mu)} \prod_{\sigma \in S} y_{(\sigma\nu)(p'(\sigma\nu))} \\
&= \sum_{\nu \in \Sigma_n/\Sigma_n^\mu} \beta_\nu \sum_{S \in \mathbb{P}^t(\Sigma_n)} \sum_{p \in F(S, \mu)} \prod_{\sigma \in S'} y_{\sigma(p'(\sigma))}
\end{aligned}$$

As the summations run through every $S \in \mathbb{P}^t(\Sigma_n)$ and $p \in F(S, \mu)$ exactly once, the bijections $S \mapsto S'$ and $p \mapsto p'$ have no effect on the sum so can be reversed. As β_ν is the only part of the expression depending on ν, we can place the summation over Σ_n/Σ_n^μ within parentheses as required. $\square$

Lemma 3.2.10. Suppose $g \in R[x]$ and we are able to write $g(x) = (x - \xi_1) \dots (x - \xi_n)$. The product $\prod_{\sigma \in \Sigma_n}(x - \sum_{i=1}^n y_{\sigma_i}\xi_i)$ is equal to

$$\sum_{t=0}^{n!} x^{n!-t} \sum_{\mu \in \mathcal{P}_n(t)} m_\mu(-\xi_1, \dots, -\xi_n) \sum_{S \in \mathbb{P}^t(\Sigma_n)} \sum_{p \in F(S, \mu)} \prod_{\sigma \in S} y_{\sigma(p(\sigma))}. \tag{3.3}$$

Proof. Let $\chi = \prod_{\sigma \in \Sigma_n}(x - \sum_{i=1}^n y_{\sigma_i}\xi_i)$. By Lemma 3.2.5 we have

$$\chi = \sum_{t=0}^{n!} x^{n!-t} e_t(\underbrace{\dots, -\sum_{i=1}^n y_{\sigma_i}\xi_i, \dots}_{\sigma \in \Sigma_n})$$

In the above e_t has one variable for each element of Σ_n, as e_t is a symmetric polynomial the order does not matter. Below we write e_t as a sum over subsets of Σ_n.

$$= \sum_{t=0}^{n!} x^{n!-t} \sum_{S\in\mathbb{P}^t(\Sigma_n)} \prod_{\sigma\in S} (-\sum_{i=1}^{n} y_{\sigma_i}\xi_i)$$

We expand the product of sums into a sum of products.

$$= \sum_{t=0}^{n!} x^{n!-t} \sum_{S\in\mathbb{P}^t(\Sigma_n)} \sum_{p\colon S\to\{1,\dots,n\}} \prod_{\sigma\in S} y_{\sigma(p(\sigma))}(-\xi_{p(\sigma)})$$

$$= \sum_{t=0}^{n!} x^{n!-t} \sum_{S\in\mathbb{P}^t(\Sigma_n)} \sum_{p\colon S\to\{1,\dots,n\}} \prod_{\sigma\in S} y_{\sigma(p(\sigma))} \prod_{\sigma\in S} (-\xi_{p(\sigma)})$$

Now $\prod_{\sigma\in S}(-\xi_{p(\sigma)})$ is equal to $\prod_{i=1}^{n}(-\xi_i)^{r_i}$ where $r_i = |p^{-1}(i)|$ for $i = 1,\dots,n$, that is $p \in F(S, \mathbf{r})$ where $\mathbf{r} = (r_1,\dots,r_n)$. We can write $\mathbf{r} = \nu\cdot\mu$ for some unique $\mu \in \mathcal{P}_n(t)$ and $\nu \in \Sigma_n/\Sigma_n^{\mu}$. We introduce summations over $\mathcal{P}_n(t)$ and Σ_n/Σ_n^{μ} and restrict the summation over $p\colon S \to \{1,\dots,n\}$ to $p \in F(S, \nu\cdot\mu)$.

$$= \sum_{t=0}^{n!} x^{n!-t} \sum_{\mu\in\mathcal{P}_n(t)} \sum_{\nu\in\Sigma_n/\Sigma_n^{\mu}} \sum_{S\in\mathbb{P}^t(\Sigma_n)} \sum_{p\in F(S,\nu\cdot\mu)} \prod_{i=1}^{n}(-\xi_i)^{(\nu\cdot\mu)_i} \prod_{\sigma\in S} y_{\sigma(p(\sigma))}$$

$$= \sum_{t=0}^{n!} x^{n!-t} \sum_{\mu\in\mathcal{P}_n(t)} \underbrace{\sum_{\nu\in\Sigma_n/\Sigma_n^{\mu}} \left(\prod_{i=1}^{n}(-\xi_i)^{(\nu\cdot\mu)_i}\right) \sum_{S\in\mathbb{P}^t(\Sigma_n)} \sum_{p\in F(S,\nu\cdot\mu)} \prod_{\sigma\in S} y_{\sigma(p(\sigma))}}_{\text{This part is of the form found in Lemma 3.2.9}}$$

By Lemma 3.2.9 we can replace $p \in F(S, \nu\cdot\mu)$ with $p \in F(S, \mu)$ and place the $\sum_{\nu\in\Sigma_n/\Sigma_n^{\mu}}$ within the parentheses, where it defines a monomial symmetric polynomial.

$$= \sum_{t=0}^{n!} x^{n!-t} \sum_{\mu\in\mathcal{P}_n(t)} m_\mu(-\xi_1,\dots,-\xi_n) \sum_{S\in\mathbb{P}^t(\Sigma_n)} \sum_{p\in F(S,\mu)} \prod_{\sigma\in S} y_{\sigma(p(\sigma))},$$

which is equal to (3.3). □

Theorem 3.2.11. Let $g \in R[x]$ be $g(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ where $a_0,\dots,a_n \in R$ such that we are able to write $g(x) = (x-\xi_1)\dots(x-\xi_n)$ for $\xi_1,\dots,\xi_n \in R$, note that this requirement forces g to be monic so $a_n = 1$. We have

$$\mathcal{R}_g(x, y_1,\dots,y_n) = \prod_{\sigma\in\Sigma_n} (x - \sum_{i=1}^{n} y_{\sigma_i}\xi_i).$$

Proof. By Lemma 3.2.10 the right hand side of the above expression is equal to (3.3), i.e.

$$\sum_{t=0}^{n!} x^{n!-t} \sum_{\mu\in\mathcal{P}_n(t)} m_\mu(-\xi_1,\dots,-\xi_n) \sum_{S\in\mathbb{P}^t(\Sigma_n)} \sum_{p\in F(S,\mu)} \prod_{\sigma\in S} y_{\sigma(p(\sigma))},$$

by Lemma 3.2.6 we can write $m_\mu(-\xi_1, \ldots, -\xi_n)$ as $\sum_{\eta \in \mathcal{P}^n(t)} \mathcal{E}_{\mu,\eta} e_\eta(-\xi_1, \ldots, -\xi_n)$ which by Lemma 3.2.5 is equal to $\sum_{\eta \in \mathcal{P}^n(t)} \mathcal{E}_{\mu,\eta} \prod_{i=1}^{\mathcal{L}(\eta)} a_{n-\eta_i}$. Substituting this into (3.3) and rearranging $\sum_{\mu \in \mathcal{P}_n(t)}$ and $\mathcal{E}_{\mu,\eta}$ gives us an expression equal to $\mathcal{R}_g(x, y_1, \ldots, y_n)$. □

It would be preferable if Theorem 3.2.11 could be formulated such that g is not required to be monic. It is likely that such a formulation could be found however time restrictions prevent us from determining this. The requirement that g is monic is however suitable for our purposes.

3.2.3 An Recursive Method for Computing the Root Sum

Suppose $g \in R[x]$ is monic. We have two methods for computing the root sum $\mathcal{R}_g$: directly from the definition or from the roots of g if the roots are known. Theorem 3.2.12 gives a recursive method for computing the root sum which does not depend on knowing the roots of g. An example of each method is given in Section 3.2.4.

Theorem 3.2.12. Suppose $g \in R[x]$ is monic. Let

$$F_{g,1}(x, y_1) = y_1^n g(x/y_1),$$

$$F_{g,m}(x, y_1, \ldots, y_m) = \frac{F_{g,m-1}(x, y_1, \ldots, y_{m-1}) \oplus^K y_m^n g(x/y_m)}{\prod_{i=1}^{m-1} F_{g,m-1}(x, y_1, \ldots, \underbrace{y_i - y_m}_{i\text{th position}}, \ldots, y_{m-1})},$$

for $m = 2, 3, \ldots$. We have $\mathcal{R}_g(x, y_1, \ldots, y_n) = F_{g,n}(x, y_1, \ldots, y_n)$.

Proof. By [Art91, Chap. 13, Prop. 5.3] it is always possible to find a field containing R such that g can be written as a product of linear factors with each root in the field. Let L be such a field. Let $g(x) = (x - \xi_1) \ldots (x - \xi_n)$ where $\xi_1, \ldots, \xi_n \in L$. Let $P_m = \{\eta \colon \{1, \ldots, m\} \hookrightarrow \{1, \ldots, n\}\}$, that is the set of injective maps sending $\{1, \ldots, m\}$ to $\{1, \ldots, n\}$. We prove by induction that [1]

$$F_{g,m}(x, y_1, \ldots, y_m) = \prod_{\eta \in P_m} (x - \sum_{i=1}^{m} y_i \xi_{\eta(i)}).$$

[1] N.B. we are taking the product over P_m on the right-hand side.

Suppose $m = 1$, we have

$$\begin{aligned} F_{g,1}(x, y_1) &= y_1^n g(x/y_1) \\ &= y_1^n (x/y_1 - \xi_1) \dots (x/y_1 - \xi_n) \\ &= \prod_{i=1}^{n} (x - \xi_i y_1). \end{aligned}$$

In this case $\eta \in P_1$ maps to a single value in $\{1, \dots, n\}$.

Suppose the induction hypothesis is true for $m - 1$, we prove it is true for m.

$$F_{g,m}(x, y_1, \dots, y_m) = \frac{F_{g,m-1}(x, y_1, \dots, y_{m-1}) \oplus^K y_m^n g(x/y_m)}{\prod_{i=1}^{m-1} F_{g,m-1}(x, y_1, \dots, \underbrace{y_i - y_m}_{i\text{th position}}, \dots, y_{m-1})}$$

We expand the numerator using Theorem 3.1.3:

$$\begin{aligned} \prod_{\eta \in P_{m-1}} (x - \sum_{i=1}^{m-1} y_i \xi_{\eta(i)}) \oplus^K \prod_{i=1}^{n} (x - \xi_i y_m) &= \prod_{i=1}^{n} \prod_{\eta \in P_{m-1}} (x - \sum_{j=1}^{m-1} y_j \xi_{\eta(j)} - \xi_i y_m) \\ &= \prod_{\eta \in Q_m} (x - \sum_{j=1}^{m} y_j \xi_{\eta(j)}) \end{aligned}$$

where $Q_m = \{\eta \colon \{1, \dots, m\} \to \{1, \dots, n\} \colon \eta\mid_{\{1,\dots,m-1\}}$ is injective $\}$, that is: all maps η from $\{1, \dots, m\}$ to $\{1, \dots, n\}$ such that when restricted to the first $m - 1$ domain values, η is injective, though $\eta(m)$ can take any value in $\{1, \dots, n\}$.

Now the denominator:

$$\begin{aligned} \prod_{i=1}^{m-1} \prod_{\eta \in P_{m-1}} (x - \sum_{j=1:\ j \neq i}^{m-1} y_j \xi_{\eta(j)} - (y_i - y_m) \xi_{\eta(i)}) &= \prod_{i=1}^{m-1} \prod_{\eta \in P_{m-1}} (x - \sum_{j=1}^{m-1} y_j \xi_{\eta(j)} + \xi_{\eta(i)} y_m) \\ &= \prod_{\eta \in R_m} (x - \sum_{j=1}^{m} y_j \xi_{\eta(j)}) \end{aligned}$$

where $R_m = \{\eta \colon \{1, \dots, m\} \to \{1, \dots, n\} \colon \eta\mid_{\{1,\dots,m-1\}}$ is injective and $\exists i \in \{1, \dots, m-1\}$ such that $\eta(m) = \eta(i)\}$. It is clear that R_m is a subset of Q_m, therefore

$$F_{g,m}(x, y_1, \dots, y_m) = \prod_{\eta \in Q_m \setminus R_m} (x - \sum_{j=1}^{m} y_j \xi_{\eta(j)}).$$

We now show $P_m = Q_m \setminus R_m$.

If $\eta \in P_m$, then P_m restricted to $\{1, \dots, m-1\}$ is injective, so $\eta \in Q_m$. By injectivity $\eta(m) \neq \eta(i)$ for any $i = 1, \dots, m-1$, so $\eta \notin R_m$.

Suppose $\eta \in Q_m \setminus R_m$, and suppose $\eta(i) = \eta(j)$ for some $i, j = 1, \dots, m$. If $i, j < m$, then (as η restricted to $\{1, \dots, m-1\}$ is injective) we have $i = j$. Suppose $i < m$ and $j = m$, this would imply $\eta \in R_m$ contradicting the definition of η. Similarly for $j < m$ and $i = m$. As the only other possibility is $i = j = m$ this proves η is injective.

So $F_{g,m}(x, y_1, \ldots, y_m) = \prod_{\eta \in P_m}(x - \sum_{j=1}^{m} y_j \xi_{\eta(j)})$. Let us consider $m = n$. In this case $P_n = \{\eta \colon \{1, \ldots, n\} \hookrightarrow \{1, \ldots, n\}\}$, which implies $\eta \in P_n$ is surjective. So $P_n = \Sigma_n$. By Theorem 3.2.11 $F_{g,m}(x, y_1, \ldots, y_m)$ is equal to $\mathcal{R}_g(x, y_1, \ldots, y_n)$. □

3.2.4 Worked Example

We compute the root sum of an example polynomial below using all three method to show they agree.

Example 3.2.13. Let $f(x) = x^2 - x - 6 = (x-3)(x+2)$. The root sum $\mathcal{R}_f(x, y_1, y_2)$ is of degree 2 in x. We compute $\mathcal{R}_f$ from the definition. Recall that $\mathcal{R}_f$ is built from polynomials $\gamma_\eta(y_1, y_2)$ where $\eta \in \mathcal{P}^2(t)$ and $t = 0, 1, 2$, which are defined by

$$\gamma_\eta(y_1, y_2) = \sum_{\mu \in \mathcal{P}_2(t)} \mathcal{E}_{\mu,\eta} \sum_{S \in \mathbb{P}^t(\Sigma_2)} \sum_{p \in F(S,\mu)} \prod_{\sigma \in S} y_{\sigma(p(\sigma))}.$$

The partition sets which appear in the first summation (and those which contain the arguments $\eta \in \mathcal{P}^2(t)$ for $t = 0, 1, 2$) are:

$$\mathcal{P}_2(0) = \mathcal{P}^2(0) = \{(0)\}, \qquad \mathcal{P}_2(1) = \mathcal{P}^2(1) = \{(1)\}, \qquad \mathcal{P}_2(2) = \mathcal{P}^2(2) = \{(2), (1,1)\}.$$

The powersets of $\Sigma_2 = \{\mathrm{id}, (1\ 2)\}$ in the second summation are:

$$\mathbb{P}^0(\Sigma_2) = \{\emptyset\}, \qquad \mathbb{P}^1(\Sigma_2) = \{\{\mathrm{id}\}, \{(1\ 2)\}\}, \qquad \mathbb{P}^2(\Sigma_2) = \{\Sigma_2\}.$$

The sets of maps in the third summation are: (the notation $[\sigma, \ldots] \mapsto [l_1, \ldots]$ means the permutation σ is mapped to the integer $l_1, \ldots$)

$$\begin{aligned}
F(\emptyset, (0)) &= \{0\}, \text{ i.e. the empty map } 0 \colon \emptyset \to \{1,2\} \\
F(\{\mathrm{id}\}, (1)) &= \Big\{[\mathrm{id}] \mapsto [1]\Big\}, \\
F(\{(1\ 2)\}, (1)) &= \Big\{[(1\ 2)] \mapsto [1]\Big\}, \\
F(\Sigma_2, (2)) &= \Big\{[\mathrm{id}, (1\ 2)] \mapsto [1,1]\Big\}, \\
F(\Sigma_2, (1,1)) &= \Big\{[\mathrm{id}, (1\ 2)] \mapsto [1,2], [\mathrm{id}, (1\ 2)] \mapsto [2,1]\Big\}.
\end{aligned}$$

Using these we compute the γ_η by substituting them into (3.1):

$$\begin{aligned}
\gamma_{(0)}(y_1, y_2) &= \mathcal{E}_{(0),(0)} \\
\gamma_{(1)}(y_1, y_2) &= \mathcal{E}_{(1),(1)}(y_1 + y_2) \\
\gamma_{(2)}(y_1, y_2) &= \mathcal{E}_{(2),(2)}(y_1 y_2) + \mathcal{E}_{(1,1),(2)}(y_1^2 + y_2^2) \\
\gamma_{(1,1)}(y_1, y_2) &= \mathcal{E}_{(2),(1,1)}(y_1 y_2) + \mathcal{E}_{(1,1),(1,1)}(y_1^2 + y_2^2)
\end{aligned}$$

To compute $\gamma_\eta(y_1, y_2)$ we need the coefficients $\mathcal{E}_{\mu,\eta}$ (see Lemma 3.2.6). We compute these directly from their definition:

$$\begin{aligned}
m_{(0)}(y_1, y_2) &= 1 \\
&= e_{(0)}(y_1, y_2) \\
m_{(1)}(y_1, y_2) &= y_1 + y_2 \\
&= e_{(1)}(y_1, y_2) \\
m_{(1,1)}(y_1, y_2) &= y_1 y_2 \\
&= e_{(2)}(y_1, y_2) \\
m_{(2)}(y_1, y_2) &= y_1^2 + y_2^2 \\
&= (y_1 + y_2)^2 - 2y_1 y_2 \\
&= e_{(1,1)}(y_1, y_2) - 2e_{(2)}(y_1, y_2)
\end{aligned}$$

From this we read off the coefficients: $\mathcal{E}_{(0),(0)} = 1$, $\mathcal{E}_{(1),(1)} = 1$, $\mathcal{E}_{(2),(2)} = -2$, $\mathcal{E}_{(2),(1,1)} = 1$, $\mathcal{E}_{(1,1),(1,1)} = 0$ and $\mathcal{E}_{(1,1),(2)} = 1$. We substitute these values into the γ_η polynomials.

$$\begin{aligned}
\gamma_{(0)}(y_1, y_2) &= 1 \\
\gamma_{(1)}(y_1, y_2) &= y_1 + y_2 \\
\gamma_{(2)}(y_1, y_2) &= y_1^2 + y_2^2 - 2y_1 y_2 \\
\gamma_{(1,1)}(y_1, y_2) &= y_1 y_2
\end{aligned}$$

The products of coefficients of g that appear in the root sum are: $a_0 = -6$, $a_1 = -1$, $a_2 = 1$

and $a_1 a_1 = 1$. So the root sum (3.2) $\mathcal{R}_f(x, y_1, y_2)$ is:

$$
\begin{aligned}
x^2 \sum_{\eta \in \mathcal{P}^2(0)} \left(\prod_{i=1}^{\mathcal{L}(\eta)} a_{2-\eta_i} \right) \gamma_\eta(y_1, y_2) + x \sum_{\eta \in \mathcal{P}^2(1)} \left(\prod_{i=1}^{\mathcal{L}(\eta)} a_{2-\eta_i} \right) \gamma_\eta(y_1, y_2) + \sum_{\eta \in \mathcal{P}^2(2)} \left(\prod_{i=1}^{\mathcal{L}(\eta)} a_{2-\eta_i} \right) \gamma_\eta(y_1, y_2) \\
= x^2 a_2 \gamma_{(0)}(y_1, y_2) + x a_1 \gamma_{(1)}(y_1, y_2) + a_0 \gamma_{(2)}(y_1, y_2) + a_1^2 \gamma_{(1,1)}(y_1, y_2) \\
= x^2 a_2 + x a_1 (y_1 + y_2) + a_0 (y_1^2 + y_2^2 - 2 y_1 y_2) + a_1^2 y_1 y_2 \\
= x^2 - x(y_1 + y_2) - 6(y_1^2 + y_2^2 - 2 y_1 y_2) + y_1 y_2 \\
= x^2 - (y_1 + y_2)x - 6y_1^2 + 13 y_1 y_2 - 6 y_2^2.
\end{aligned}
$$

We now use Theorem 3.2.11 to compute the root sum polynomial from the roots of $g(x)$ to ensure it agrees with the definition.

$$
\begin{aligned}
\mathcal{R}_f(x, y_1, y_2) &= (x - 3y_1 + 2y_2)(x - 3y_2 + 2y_1) \\
&= x^2 - (y_1 + y_2)x - 6y_1^2 + 13 y_1 y_2 - 6 y_2^2,
\end{aligned}
$$

which is equal to the result obtained from the definition.

We now use Theorem 3.2.12 to compute the root sum using the third method. We have

$$
\begin{aligned}
F_{f,1}(x, y_1) &= y_1^2 f(x/y_1) \\
&= y_1^2((x/y_1)^2 - x/y_1 - 6) \\
&= x^2 - x y_1 - 6 y_1^2.
\end{aligned}
$$

$$
F_{f,2}(x, y_1, y_2) = \frac{F_{f,1}(x, y_1) \oplus^K y_2^2 f(x/y_2)}{F_{f,1}(x, y_1 + y_2)}
$$

We use the resultant formula in Definition 3.1.2 to compute $F_{f,1}(x, y_1) \oplus^K y_2^2 f(x/y_2)$. We have

$$
\begin{aligned}
F_{f,1}(x - z, y_1) &= (x - z)^2 - (x - z) y_1 - 6 y_1^2 \\
&= z^2 + z(y_1 - 2x) + x^2 - x y_1 - 6 y_1^2,
\end{aligned}
$$

$$
\begin{aligned}
y_2^2 f(z/y_2) &= y_2^2((z/y_2)^2 - z/y_2 - 6) \\
&= z^2 - z y_2 - 6 y_2^2,
\end{aligned}
$$

so we have

$$
\begin{aligned}
F_{f,1}(x,y_1) \oplus^K y_2^2 f(x/y_2) &= (-1)^4 \mathrm{Res}(F_{f,1}(x-z,y_1), y_2^2 f(z/y_2), z) \\
&= \det \begin{pmatrix} 1 & y_1 - 2x & x^2 - xy_1 - 6y_1^2 & 0 \\ 0 & 1 & y_1 - 2x & x^2 - xy_1 - 6y_1^2 \\ 1 & -y_2 & -6y_2^2 & 0 \\ 0 & 1 & -y_2 & -6y_2^2 \end{pmatrix} \\
&= x^4 - (2y_1 + 2y_2)x^3 - (11y_2^2 - 3y_2y_1 + 11y_1^2)x^2 + (12y_2^3 + 11y_1^2y_2 + 11y_2^2y_1 + 12y_1^3)x \\
&\qquad + 36y_2^4 - 6y_2^3y_1 - 84y_1^2y_2^2 - 6y_1^3y_2 + 36y_1^4.
\end{aligned}
$$

The denominator is: $F_{f,1}(x, y_1 + y_2) = x^2 - x(y_1 + y_2) - 6y_1^2 - 12y_1y_2 - 6y_2^2$.

We determine the coefficients of $F_{f,2}(x, y_1, y_2)$ by expanding

$$F_{f,1}(x, y_1 + y_2)F_{f,2}(x, y_1, y_2) = F_{f,1}(x, y_1) \oplus^K y_2^2 f(x/y_2)$$

and then building a linear system by comparing coefficients of the monomials in x, y_1 and y_2. The linear system has a unique solution as $F_{f,1}(x, y_1+y_2)$ divides $F_{f,1}(x, y_1) \oplus^K y_2^2 f(x/y_2)$. This method yields

$$\mathcal{R}_f(x, y_1, y_2) = F_{f,2}(x, y_1, y_2) = x^2 - (y_2 + y_1)x - 6y_1^2 + 13y_1y_2 - 6y_2^2.$$

So all three methods yields the same result for $\mathcal{R}_f(x, y_1, y_2)$.

Chapter 4

The Additive Matrix Problem

Given a k-tuple of matrix similarity classes $C_1, \dots, C_k$ of order n in an algebraically closed field K of characteristic zero, the *additive matrix problem* asks under what necessary and sufficient conditions on $C_1, \dots, C_k$ does there exist matrices $A_1, \dots, A_k$ such that $A_i \in C_i$ for $i = 1, \dots, k$ and $A_1 + \dots + A_k = 0$. The problem is currenly open in its most general form but solutions exist to several related problems, for instance when the $C_1, \dots, C_k$ are closed, or where we require the $A_1, \dots, A_k$ to be irreducible (irreducible is defined in Section 4.1.3). This chapter looks at the solutions obtained in the literature by Crawley-Boevey and Silva et al. and introduces a new approach of the author to the Crawley-Boevey results. The solutions of Crawley-Boevey and Silva et al. each solve a different variant of the general additive matrix problem, but we compare the two solutions in the case where the two problems intersect.

The solutions of Crawley-Boevey in Section 4.1 relate the additive matrix problem to the existence of particular representations of a deformed preprojective algebra constructed from the similarity classes. The existence of these representations are related to the existence of certain roots of the quiver used to construct the deformed preprojective algebra. In Section 4.2 we derive some results about the positive roots and dimension vectors of star-shaped quivers (which are defined in Section 4.1). The solutions of Silva et al. in Section 4.3 do not explicitly involve quivers and representations so we use the results derived in Section 4.2 to write them in terms comparable to the results of Crawley-Boevey. We introduce a new approach devised by the author to solving the additive matrix problem in Section 4.4 which enables us to obtain necessary and sufficient conditions for the existence of solutions when the classes are closed and the eigenvalues are not known apart from the invariant polynomials.

Throughout this chapter K is an algebraically closed field of characteristic zero.

4.1 Crawley-Boevey's Solutions

The main results of this section are Theorems 4.1.14 and 4.1.16. Both are partial answers to the additive matrix problem and both rely on being able to compute the Jordan normal forms of the similarity classes.

Section 4.1.1 defines some crucial terminology that shall be used often. Section 4.1.2 describes the categorical relationship between solutions to the additive matrix problem and representations of deformed preprojective algebras. Section 4.1.3 contains the theorems described above.

4.1.1 Tuples of Similarity Classes

Definition 4.1.1. A *star-shaped quiver* Q is a connected quiver which consists of a central vertex and a number of *arms*, that is linear quivers which are joined to each other only at the central vertex. A linear quiver is a quiver of the form $\circ \longrightarrow \circ \longrightarrow \cdots \longrightarrow \circ$.

Given a star-shaped quiver Q with k arms, we assume an ordering on the arms so they are indexed $i = 1, \ldots, k$. The arm vertices are denoted $[i, j]$ (or sometimes i, j when it does not result in confusion), that is $[i, j]$ is the jth vertex from the centre on the ith arm. The central vertex is denoted 0 and by convention $[i, 0]$ means 0 for every $i = 1, \ldots, k$. The arrow between $[i, j]$ and $[i, j-1]$ is denoted $a_{i,j}$.

We say an arm is of length $d \geq 1$ if it has d arrows. The number of vertices, excluding the central vertex, is also d but we define the length by the number of arrows rather than vertices to avoid ambiguity. It is sometimes useful to think of an arm as including the central vertex and sometimes useful to exclude the central vertex. When referring to the arms of a star-shaped quiver it is crucial to state explicitly whether or not the central vertex is included.

Example 4.1.2. The quiver below is a star-shaped quiver.

$$\begin{tikzcd}
& {[1,1]} \arrow[dl] & {[1,2]} \arrow[l] & {[1,3]} \arrow[l] \\
Q = 0 & {[2,1]} \arrow[l] & {[2,2]} \arrow[l] \\
& {[3,1]} \arrow[ul] & {[3,2]} \arrow[l]
\end{tikzcd}$$

To fully describe a star-shaped quiver we need only specify the number of arms, the length of each arm and the orientation of the arrows. Let V be a K-vector space with $\dim(V) = n$. Let $k \geq 2$ and $C_1, \ldots, C_k \subseteq \operatorname{End}(V)$ be a k-tuple of nonscalar conjugacy classes. The restriction to nonscalar conjugacy classes entails no loss of generality. To see this suppose $i, j \in \{1, \ldots, k\}$ such that $i \neq j$ and C_i is scalar. The class C_i contains only one endomorphism $A_i = c1_V$ where $c \in K$. Let $C_j' = \{A + c1_V \colon A \in C_j\}$, it is easy to see this is also a conjugacy class. The additive matrix problem with $C_1, \ldots, C_k$ is equivalent to the problem with C_j replaced with C_j' and C_i removed. This process continues until either all scalar classes are removed, or we get down to two conjugacy classes. Suppose there are two classes C_1, C_2, if precisely one of these is scalar, then there is no solution to the additive matrix problem, if both are scalar then a solution exists if and only if $\operatorname{trace}(C_1) = -\operatorname{trace}(C_2)$.

Definition 4.1.3. For $i = 1, \ldots, k$ let $d_i = \sum_{\lambda \in \Psi(C_i)} \operatorname{idx}_{C_i}(\lambda)$ (which is $\deg(\min_{C_i}(x))$ by Theorem 2.1.11). We define the *quiver Q associated to* $C_1, \ldots, C_k$ to be the star-shaped quiver with k arms, where arm i is of length $d_i - 1$ and all arrows are orientated towards the central vertex.

For each $i = 1, \ldots, k$ let $\xi_{i,1}, \ldots, \xi_{i,d_i}$ be a list of the roots of the minimal polynomial $\min_{C_i}(x)$ of C_i. The following definitions are in respect to these lists, that is they depend on the orders chosen.

Definition 4.1.4. Let Q be the star-shaped quiver associated to $C_1, \ldots, C_k$.

- Let $\alpha \in \mathbb{Z}^{Q_0}$ be the dimension vector of Q defined by $\alpha_0 = n$ and $\alpha_{i,j} = \operatorname{rank}(\prod_{l=1}^{j}(A - \xi_{i,l} 1_V))$ where $A \in C_i$ for $i = 1, \ldots, k$ and $j = 1, \ldots, d_i - 1$. This does not depend on the choice of $A \in C_i$.

- Let $\lambda \in K^{Q_0}$ be the K-vector of Q defined by $\lambda_0 = -\sum_{i=1}^{k} \xi_{i,1}$ and for each $i = 1, \ldots, k$ and $j = 1, \ldots, d_i - 1$ let $\lambda_{i,j} = \xi_{i,j} - \xi_{i,j+1}$.

Note that the ith arm (with the central vertex) of α, with a zero appended to the end, is precisely the dimension vector of C_i in the sense of Definition 2.1.18 (with respect to the list $\xi_{i,1}, \dots, \xi_{i,d_i}$).

Remark 4.1.5. The quiver Q, dimension vector $\alpha \in \mathbb{Z}^{Q_0}$ and K-vector $\lambda \in K^{Q_0}$ associated to $C_1, \dots, C_k$ are fixed by the conjugacy classes. It is not true, however, that Q, α and λ entirely fix the conjugacy classes. Fixing Q fixes the number of conjugacy classes and the degrees of their minimal polynomials, fixing α fixes the dimension vectors of $C_1, \dots, C_k$ and fixing λ fixes the differences between roots of the minimal polynomial of each $C_1, \dots, C_k$ and fixes $\sum_{i=1}^{k} \operatorname{trace}(C_i)$.

To see how far Q, α and λ fix the conjugacy classes let $C_1', \dots, C_k' \subseteq \operatorname{End}(V)$ be a collection of conjugacy classes. The quiver, dimension vector and K-vector associated to $C_1', \dots, C_k'$ is Q, α and λ respectively if and only if $C_i' = \{A_i + b_i 1_V : A_i \in C_i\}$ for some $b_1, \dots, b_k \in K$ such that $b_1 + \dots + b_k = 0$.

4.1.2 Functors from Representations to Tuples of Endomorphisms

We define a functor from the category of representations of $\Pi^\lambda(Q)$ of a particular dimension vector to a category consisting of certain endomorphism tuples. We are given matrix similarity classes in this part of the section from which we define conjugacy classes for the endomorphisms of a given vector space. We say an endomorphism θ of an n-dimensional vector space V *corresponds* to a matrix $A \in M_n(K)$ if V can be given a basis such that $\theta(v) = Av$ for all $v \in K^n$.

Let $C_1, \dots, C_k \subseteq M_n(K)$ be matrix similarity classes. Let Q, α and λ be defined as in Section 4.1.1. Given a vector space V of dimension n let $C_i(V)$ and $\overline{C}_i(V)$ be defined by

$$C_i(V) = \left\{\theta \in \operatorname{End}(V) : \theta \text{ corresponds to a matrix in } C_i\right\},$$
$$\overline{C}_i(V) = \left\{\theta \in \operatorname{End}(V) : \theta \text{ corresponds to a matrix in } \overline{C}_i\right\}.$$

Definition 4.1.6. Let X be a representation of $\overline{Q}$, where Q is a star-shaped quiver, we call the representation of the linear quiver consisting of the central vertex and the ith arm, obtained by selecting the corresponding maps and vector spaces, the *ith arm component of* X.

Definition 4.1.7. We say a representation X of $\Pi^\lambda(Q)$ is *strict* if for each $a \in Q_1$ we have X_a injective and X_{a^*} surjective.

Lemma 4.1.8. Given a vector space V of dimension n and a tuple $(A_1, \dots, A_k)$ of linear maps such that $A_i \in \overline{C}_i(V)$ (resp. $A_i \in C_i(V)$) for $i = 1, \dots, k$ and $\sum_{i=1}^k A_i = 0$ there exists a representation (resp. strict representation) X of $\Pi^\lambda(Q)$ of dimension vector α such that $A_i = X_{a_{i,1}} X_{a^*_{i,1}} + \xi_{i,1} 1_V$ for $i = 1, \dots, k$.

Proof. For each arm $i = 1, \dots, k$ the ith arm component of X is obtained from A_i using Theorem 2.2.2 (resp. Theorem 2.2.4), this is well-defined as the central vector space of each component is the same, i.e. V. The theorem ensures that the deformed preprojective relations on the arms are satisfied and that $A_i = X_{a_{i,1}} X_{a^*_{i,1}} + \xi_{i,1} 1_V$. The central deformed preprojective relation is also satisfied as $0 = \sum_{i=1}^k A_i = \sum_{i=1}^k (X_{a_{i,1}} X_{a^*_{i,1}} + \xi_{i,1} 1_V)$ which implies $\sum_{i=1}^k X_{a_{i,1}} X_{a^*_{i,1}} = -\sum_{i=1}^k \xi_{i,1} 1_V = \lambda_0 1_V$. □

Representations and Closures of Similarity Classes

Recall that the category of representations of $\Pi^\lambda(Q)$ is denoted $\mathrm{Rep}_K(\Pi^\lambda(Q))$. Let $\mathcal{R}(\alpha)$ be the full subcategory of $\mathrm{Rep}_K(\Pi^\lambda(Q))$ consisting of all representations of dimension vector α.

Let $\overline{C}$ be the category with objects given by

$$\mathrm{ob}(\overline{C}) = \Big\{(V, A_1, \dots, A_k)\colon V \text{ is an } n\text{-dimensional } K\text{-vector space},$$
$$A_i \in \overline{C}_i(V) \text{ for } i = 1, \dots, k \text{ and } \sum_{i=1}^k A_i = 0\Big\}.$$

and morphisms between $\mathbf{A}, \mathbf{A}' \in \mathrm{ob}(\overline{C})$, where $\mathbf{A} = (V, A_1, \dots, A_k)$ and $\mathbf{A}' = (V', A'_1, \dots, A'_k)$, given by

$$\hom_{\overline{C}}(\mathbf{A}, \mathbf{A}') = \{\phi\colon V \to V'\colon A'_i\phi = \phi A_i \text{ for } i = 1, \dots, k\}.$$

We define a functor $\mathcal{F}$ from $\mathcal{R}(\alpha)$ to $\overline{C}$. Given a representation X of $\mathcal{R}(\alpha)$ we define the target object $\mathcal{F}(X) = (X_0, A_1, \dots, A_k)$ by $A_i = X_{a_{i,1}} X_{a^*_{i,1}} + \xi_{i,1} 1_{X_0}$ for $i = 1, \dots, k$. Given a morphism $\phi\colon X \to Y$ of representations X, Y of $\mathcal{R}(\alpha)$ we define the target morphism $\mathcal{F}(\phi)\colon \mathcal{F}(X) \to \mathcal{F}(Y)$ by $\mathcal{F}(\phi) = \phi_0$.

Theorem 4.1.9. $\mathcal{F}$ is a functor.

Proof. Let X be a representation of $\mathcal{R}(\alpha)$. We first show $\mathcal{F}(X) = (X_0, A_1, \dots, A_k)$ is an object in $\overline{C}$. By Theorem 2.2.2 we have $A_i \in \overline{C}_i(X_0)$ for $i = 1, \dots, k$ and by the deformed preprojective relation at the central vertex we have $\sum_{i=1}^{k} A_i = \sum_{i=1}^{k} X_{a_{i,1}} X_{a_{i,1}^*} + \sum_{i=1}^{k} \xi_{i,1} 1_{X_0} = 0$.

Let $\phi \colon X \to Y$ be a morphism of representations where X, Y are objects of $\mathcal{R}(\alpha)$, we show $\mathcal{F}(\phi)$ is a target morphism. Let $\mathcal{F}(X) = (X_0, A_1, \dots, A_k)$ and $\mathcal{F}(Y) = (Y_0, A_1', \dots, A_k')$. Now $\mathcal{F}(\phi) = \phi_0$ and for each $i = 1, \dots, k$ we have

$$\begin{aligned}
\phi_0 A_i &= \phi_0 X_{a_{i,1}} X_{a_{i,1}^*} + \xi_{i,1} \phi_0 \\
&= Y_{a_{i,1}} \phi_{i,1} X_{a_{i,1}^*} + \xi_{i,1} \phi_0 \\
&= Y_{a_{i,1}} Y_{a_{i,1}^*} \phi_0 + \xi_{i,1} \phi_0 \\
&= A_i' \phi_0.
\end{aligned}$$

So for $\mathcal{F}(\phi) \colon \mathcal{F}(X) \to \mathcal{F}(Y)$ we have $\mathcal{F}(\phi) \in \hom_{\overline{C}}(\mathcal{F}(X), \mathcal{F}(Y))$. Let X be a representation of $\mathcal{R}(\alpha)$, we have $\mathcal{F}(1_X) = 1_{X_0}$ which is clearly the identity morphism of $\mathcal{F}(X)$. Let X, Y, Z be representations of $\mathcal{R}(\alpha)$ and $\phi \colon X \to Y, \psi \colon Y \to Z$ be homomorphisms, we have $\mathcal{F}(\psi\phi) = (\psi\phi)_0 = \psi_0 \phi_0 = \mathcal{F}(\psi)\mathcal{F}(\phi)$. So $\mathcal{F}$ is a functor from $\mathcal{R}(\alpha)$ to $\overline{C}$. □

A functor $\mathcal{H}$ from a category $\mathcal{L}$ to a category $\mathcal{M}$ is *dense* (or *essentially surjective*) if for each $B \in \mathcal{M}$ there is an object $A \in \mathcal{L}$ such that $\mathcal{H}(A)$ is isomorphic to B. We say $\mathcal{H}$ is *surjective* if for each $B \in \mathcal{M}$ there is an object $A \in \mathcal{L}$ such that $\mathcal{H}(A) = B$. A surjective functor is also dense by definition.

Lemma 4.1.10. The functor $\mathcal{F}$ is surjective.

Proof. Let $(V, A_1, \dots, A_k)$ be an object of $\overline{C}$, and let X be the representation of $\Pi^\lambda(Q)$ obtained from this given by Lemma 4.1.8, note that $X_0 = V$. Let $(X_0, A_1', \dots, A_k') = \mathcal{F}(X)$ so for $i = 1, \dots, k$ we have $A_i' = X_{a_{i,1}} X_{a_{i,1}^*} + \xi_{i,1} 1_{X_0}$ which, by Lemma 4.1.8, is equal to A_i so $\mathcal{F}(X) = (X_0, A_1', \dots, A_k') = (V, A_1, \dots, A_k)$. □

Strict Representations and Similarity Classes

Let $\widetilde{\operatorname{Rep}}_K(\Pi^\lambda(Q))$ be the full subcategory of strict representations of $\Pi^\lambda(Q)$. Let $\tilde{\mathcal{R}}(\alpha)$ be the full subcategory $\widetilde{\operatorname{Rep}}_K(\Pi^\lambda(Q))$ consisting of strict representations of dimension vector α. Let C be the full subcategory of $\overline{C}$ consisting of objects $(V, A_1, \dots, A_k)$ such that $A_i \in C_i(V)$ for $i = 1, \dots, k$. Let $\mathcal{G}$ be the functor $\mathcal{F}$ restricted to $\tilde{\mathcal{R}}(\alpha)$.

Let X be an object of $\tilde{\mathcal{R}}(\alpha)$ and write $\mathcal{G}(X) = (V, A_1, \dots, A_k)$. By Theorem 2.2.4 we have $A_i \in C_i(V)$ for each $i = 1, \dots, k$ so $\mathcal{G}(X)$ is a functor from $\tilde{\mathcal{R}}(\alpha)$ to C.

Lemma 4.1.11. The functor $\mathcal{G}$ is surjective.

Proof. The proof is essentially the same as in Lemma 4.1.10. □

Lemma 4.1.12. The functor $\mathcal{G}$ is fully faithful.

Proof. Let X, Y be strict representations, the map $\hom_{\tilde{\mathcal{R}}(\alpha)}(X,Y) \to \hom_{\mathcal{C}}(\mathcal{G}(X), \mathcal{G}(Y))$ is given by $\phi \mapsto \phi_0$, for each $\phi\colon X \to Y$.

We show $\mathcal{G}$ is faithful, i.e. the map between hom-spaces is injective. Let $\phi, \psi \in \hom_{\tilde{\mathcal{R}}(\alpha)}(X,Y)$ and suppose $\mathcal{G}(\phi) = \mathcal{G}(\psi)$, we show $\phi = \psi$. Now by definition $\mathcal{G}(\phi) = \mathcal{G}(\psi)$ implies $\phi_0 = \psi_0$, we prove the remaining maps are equal by induction. Let $i = 1, \ldots, k$ and $j = 1, \ldots, d_i - 1$, we show that $\phi_{i,j-1} = \psi_{i,j-1}$ implies $\phi_{i,j} = \psi_{i,j}$. Recall that the intertwining relations for ϕ at the arrows $a_{i,j}$ and $a^*_{i,j}$ are the respective commutative squares.

$$\begin{tikzcd} X_{i,j-1} \arrow[d, "\phi_{i,j-1}"'] & X_{i,j} \arrow[l, hook', "X_{a_{i,j}}"'] \arrow[d, "\phi_{i,j}"] \\ Y_{i,j-1} & Y_{i,j} \arrow[l, hook', "Y_{a_{i,j}}"] \end{tikzcd} \tag{4.1}$$

$$\begin{tikzcd} X_{i,j-1} \arrow[r, two heads, "X_{a^*_{i,j}}"] \arrow[d, "\phi_{i,j-1}"'] & X_{i,j} \arrow[d, "\phi_{i,j}"] \\ Y_{i,j-1} \arrow[r, two heads, "Y_{a^*_{i,j}}"'] & Y_{i,j} \end{tikzcd} \tag{4.2}$$

The intertwining relation (4.1) is $\phi_{i,j-1}X_{a_{i,j}} = Y_{a_{i,j}}\phi_{i,j}$. For ψ the same relation is $\psi_{i,j-1}X_{a_{i,j}} = Y_{a_{i,j}}\psi_{i,j}$. Suppose $\phi_{i,j-1} = \psi_{i,j-1}$, the intertwining relations imply $Y_{a_{i,j}}\phi_{i,j} = Y_{a_{i,j}}\psi_{i,j}$ which, by the injectivity of $Y_{a_{i,j}}$, implies $\phi_{i,j} = \psi_{i,j}$.

We show $\mathcal{G}$ is full, i.e. the map between hom-spaces is surjective. Suppose $\theta \in \hom_{\mathcal{C}}(\mathcal{G}(X), \mathcal{G}(Y))$. We construct a map ϕ from X to Y such that $\mathcal{G}(\phi) = \theta$. Let $\phi_0 = \theta$. Suppose $\mathcal{G}(X) = (X_0, A_1, \ldots, A_k)$ and $\mathcal{G}(Y) = (Y_0, A'_1, \ldots, A'_k)$, the property $A'_i\theta = \theta A_i$ gives the relation $Y_{a_{i,1}}Y_{a^*_{i,1}}\phi_0 = \phi_0 X_{a_{i,1}}X_{a^*_{i,1}}$ for each $i = 1, \ldots, k$. We construct the remaining maps of ϕ by induction. Let $i = 1, \ldots, k$ and $j = 1, \ldots, d_i - 1$. Assume $\phi_{i,j-1}$ exists and the relation

$$Y_{a_{i,j}}Y_{a^*_{i,j}}\phi_{i,j-1} = \phi_{i,j-1}X_{a_{i,j}}X_{a^*_{i,j}} \tag{4.3}$$

is satisfied. We construct $\phi_{i,j}\colon X_{i,j} \to Y_{i,j}$ such that the intertwining relations (4.1) and (4.2) for $a_{i,j}$ and $a^*_{i,j}$ are satisfied. For each $x \in X_{i,j}$ let $\phi_{i,j}(x) = Y_{a^*_{i,j}}\phi_{i,j-1}(x')$ where $x' \in X_{i,j-1}$ is chosen such that $x = X_{a^*_{i,j}}(x')$. Such an x' exists because $X_{a^*_{i,j}}$ is surjective. We show this is well-defined. Suppose $x', x'' \in X_{i,j-1}$ have $x = X_{a^*_{i,j}}(x') = X_{a^*_{i,j}}(x'')$, by (4.3) we have $Y_{a_{i,j}}Y_{a^*_{i,j}}\phi_{i,j-1}(x' - x'') = \phi_{i,j-1}X_{a_{i,j}}X_{a^*_{i,j}}(x' - x'')$. Now $X_{a^*_{i,j}}(x' - x'') = x - x = 0$ so $Y_{a_{i,j}}Y_{a^*_{i,j}}\phi_{i,j-1}(x' - x'') = 0$. Since $Y_{a_{i,j}}$ is injective we have $Y_{a^*_{i,j}}\phi_{i,j-1}(x' - x'') = 0$. So $Y_{a^*_{i,j}}\phi_{i,j-1}(x')$ and $Y_{a^*_{i,j}}\phi_{i,j-1}(x'')$ give the same definition of $\phi_{i,j}(x)$.

Now we show the intertwining relation (4.1) holds for $\phi_{i,j-1}$ and $\phi_{i,j}$. Let $x \in X_{i,j}$, we have $Y_{a_{i,j}}\phi_{i,j}(x) = Y_{a_{i,j}}Y_{a^*_{i,j}}\phi_{i,j-1}(x')$ where $x' \in X_{a_{i,j-1}}$ is such that $x = X_{a^*_{i,j}}(x')$. By (4.3) we have $Y_{a_{i,j}}Y_{a^*_{i,j}}\phi_{i,j-1}(x') = \phi_{i,j-1}X_{a_{i,j}}X_{a^*_{i,j}}(x') = \phi_{i,j-1}X_{a_{i,j}}(x)$, so $Y_{a_{i,j}}\phi_{i,j}(x) = \phi_{i,j-1}X_{a_{i,j}}(x)$.

Now we show the intertwining relation (4.2) holds for $\phi_{i,j-1}$ and $\phi_{i,j}$. Let $x' \in X_{i,j-1}$ and let $x = X_{a^*_{i,j}}(x')$. We have $\phi_{i,j}X_{a^*_{i,j}}(x') = \phi_{i,j}(x) = Y_{a^*_{i,j}}\phi_{i,j-1}(x'')$ where $x'' \in X_{a_{i,j-1}}$ is such that $x = X_{a^*_{i,j}}(x'')$. As $x = X_{a^*_{i,j}}(x')$, and the definition of $\phi_{i,j}$ is well-defined, it is permissible to take $x'' = x'$, so we have $\phi_{i,j}X_{a^*_{i,j}}(x') = Y_{a^*_{i,j}}\phi_{i,j-1}(x')$ as required.

To complete the induction we show that, when $j < d_i - 1$, we have $Y_{a_{i,j+1}}Y_{a^*_{i,j+1}}\phi_{i,j} = \phi_{i,j}X_{a_{i,j+1}}X_{a^*_{i,j+1}}$. The deformed preprojective relations for X and Y at i, j are

$$\begin{aligned} \lambda_{i,j}1_{X_{i,j}} &= X_{a^*_{i,j}}X_{a_{i,j}} - X_{a_{i,j+1}}X_{a^*_{i,j+1}}, \\ \lambda_{i,j}1_{Y_{i,j}} &= Y_{a^*_{i,j}}Y_{a_{i,j}} - Y_{a_{i,j+1}}Y_{a^*_{i,j+1}}. \end{aligned} \tag{4.4}$$

By the intertwining relations (4.2) and (4.1) we have $\phi_{i,j}X_{a^*_{i,j}}X_{a_{i,j}} = Y_{a^*_{i,j}}Y_{a_{i,j}}\phi_{i,j}$, which when substituted into (4.4) yields $\lambda_{i,j}\phi_{i,j} + \phi_{i,j}X_{a_{i,j+1}}X_{a^*_{i,j+1}} = \lambda_{i,j}\phi_{i,j} + Y_{a_{i,j+1}}Y_{a^*_{i,j+1}}\phi_{i,j}$, which completes the induction. □

A functor is an equivalence if it is both fully-faithful and dense. Both $\mathcal{F}$ and $\mathcal{G}$ are surjective, and therefore dense, but only $\mathcal{G}$ is necessarily fully-faithful. The diagram below shows the relations between the categories and functors.

$$\begin{tikzcd} \mathcal{R}(\alpha) \arrow[d, "\mathcal{F}"', "\text{dense}"] & \tilde{\mathcal{R}}(\alpha) \arrow[l, hook', "\text{full subcategory}"'] \arrow[d, "\mathcal{G}"', "\cong"] \\ \overline{\mathcal{C}} & \mathcal{C} \arrow[l, hook', "\text{full subcategory}"'] \end{tikzcd} \tag{4.5}$$

4.1.3 The Theorems

Most of the following definitions and theorems come from [CB03]. Let V be a vector space of dimension n.

Definition 4.1.13. Let $k \geq 1$. A tuple of linear maps $A_1, \ldots, A_k \in \operatorname{End}(V)$ is said to be *irreducible* if there is no common nontrivial invariant subspace of the maps, that is for any subspace $U \subseteq V$ such that $A_iU \subseteq U$ for all $i = 1, \ldots, k$ we have either $U = 0$ or $U = V$.

We say two k-tuples of endomorphisms are *conjugate* if they are simultaneously conjugate. Let $C_1, \ldots, C_k \subseteq \mathrm{End}(V)$ be a tuple of conjugacy classes and let Q, α and λ be the associated star-shaped quiver, dimension vector and K-vector as defined in Section 4.1.1. Recall the definition of Σ_λ given in Section 1.2.

Theorem 4.1.14. There exist endomorphisms $A_i \in C_i$ for each $i = 1, \ldots, k$, such that $\sum_{i=1}^k A_i = 0$ and $A_1, \ldots, A_k$ is an irreducible k-tuple of linear maps if and only if $\alpha \in \Sigma_\lambda$. Furthermore if α is a real root then $A_1, \ldots, A_k$ is the only such irreducible solution (up to conjugacy), and if α is imaginary then there are infinitely many nonconjugate irreducible solutions.

Proof. [CB03, Thm. 1] □

In the case where an irreducible solution is the only irreducible solution to the additive matrix problem, up to conjugacy, we say it is a *rigid* solution.

Theorem 4.1.16 gives a necessary and sufficient condition for the existence of matrices in conjugacy class closures. This condition is weaker than the one in Theorem 4.1.14, in particular note the absence of the irreducibility condition.

Lemma 4.1.15. There exist endomorphisms $A_i \in \overline{C}_i$ for each $i = 1, \ldots, k$, such that $\sum_{i=1}^k A_i = 0$ if and only if there exists a representation in $\mathrm{Rep}(\Pi^\lambda(Q))$ of dimension vector α.

Proof. Recall the functor $\mathcal{F}$ from the previous section. As $\mathcal{F}$ is dense the existence of a representation in $\mathcal{R}(\alpha)$ is equivalent to the existence of the required tuple. □

Theorem 4.1.16. There exist endomorphisms $A_i \in \overline{C}_i$ for each $i = 1, \ldots, k$, such that $\sum_{i=1}^k A_i = 0$ if and only if there exists a root decomposition $(\beta_1, \cdots, \beta_r)$ of α such that $\beta_1, \ldots, \beta_r \in R_\lambda^+$.

Proof. Use Lemma 4.1.15 and either [CB06, Thm. 2] or [CB01, Thm. 3.3]. □

4.2 Star-Shaped Quivers

Section 4.2 presents some results that are needed in Sections 4.3 and 4.4. Section 4.3 examines the solutions obtained to certain additive matrix problems by Silva et al. and shows that in certain circumstances the results of Silva et al. and Crawley-Boevey agree.

Section 4.4 introduces a new method for solving a particular subset of the additive matrix problem.

This section specifically looks at star-shaped quivers, defined in Section 4.1.1, and their dimension vectors. The main result of this section is a new condition, for a certain class of dimension vectors, which determines whether the dimension vectors are roots. The result is given by Theorems 4.2.25 and 4.2.28. Let Q be a star-shaped quiver with k arms. The length of the ith arm is written $d_i - 1$, the arm-lengths are described by $d_i - 1$ rather than d_i to be consistent with our use of d_i in other sections.

Definition 4.2.1. Let α be a dimension vector of Q. We say α is *nonincreasing* if for each arm $i = 1, \dots, k$ we have $\alpha_{i,j-1} \geq \alpha_{i,j}$ for $j = 1, \dots, d_i - 1$, that is the values on the arms do not get larger as one goes out along them.

Definition 4.2.2. Let α be a positive nonincreasing dimension vector. The *nonzero i-arm length of α* is the number of nonzero elements on the i-arm of α and is denoted $z_i(\alpha)$.

Example 4.2.3. The following dimension vector α is positive, nonincreasing with nonzero arm lengths $z_1(\alpha) = 1$, $z_2(\alpha) = 2$ and $z_3(\alpha) = 3$.

$$\begin{tikzcd} & 5 \arrow[r, dash] & 0 \arrow[r, dash] & 0 \\ \alpha = 16 \arrow[ur, dash] \arrow[r, dash] \arrow[dr, dash] & 9 \arrow[r, dash] & 4 \\ & 6 \arrow[r, dash] & 6 \arrow[r, dash] & 2 \arrow[r, dash] & 0 \end{tikzcd}$$

Definition 4.2.4. Let α be a positive nonincreasing dimension vector of Q. We say α is *arm-fundamental* if any reflection on the arm vertices increases the value at that vertex, that is $s_{i,j}(\alpha)_{i,j} \geq \alpha_{i,j}$ where $i = 1, \dots, k$ and $j = 1, \dots, d_i - 1$.

Definition 4.2.5. Let Q be a star-shaped quiver and let α be a dimension vector of Q. Let $i = 1, \dots, k$ and $j = 1, \dots, d_i$. We define the *gradient of α at i, j* to be

$$\nabla_{i,j}(\alpha) = \begin{cases} \alpha_{i,d_i-1} & \text{if } j = d_i, \\ \alpha_{i,j-1} - \alpha_{i,j} & \text{otherwise.} \end{cases}$$

Lemma 4.2.6. Let α be a dimension vector of Q. Let $i = 1, \dots, k$ and $j = 1, \dots, d_i - 1$ and let $\tilde{\alpha}$ be the dimension vector obtained by reflecting α at the $[i, j]$ vertex. The effect of reflecting at an arm vertex is to permute the gradients on either side of the vertex, that is $\nabla_{i,j}(\tilde{\alpha}) = \nabla_{i,j+1}(\alpha)$, $\nabla_{i,j+1}(\tilde{\alpha}) = \nabla_{i,j}(\alpha)$ and $\nabla_{i',j'}(\tilde{\alpha}) = \nabla_{i',j'}(\alpha)$ for all $i' = 1, \dots, k$ and $j' = 1, \dots, d_i$ such that if $i' = i$, then $j' \notin \{j, j+1\}$.

Proof. A reflection at i, j only changes the $[i, j]$ vertex which in turn only affects the $[i, j]$ and $[i, j+1]$ gradient. So

$$\nabla_{i,j}(\tilde{\alpha}) = \tilde{\alpha}_{i,j-1} - \tilde{\alpha}_{i,j} = \alpha_{i,j-1} - (\alpha_{i,j-1} + \alpha_{i,j+1} - \alpha_{i,j}) = \alpha_{i,j} - \alpha_{i,j+1} = \nabla_{i,j+1}(\alpha)$$
$$\nabla_{i,j+1}(\tilde{\alpha}) = \tilde{\alpha}_{i,j} - \tilde{\alpha}_{i,j+1} = (\alpha_{i,j-1} + \alpha_{i,j+1} - \alpha_{i,j}) - \alpha_{i,j+1} = \alpha_{i,j-1} - \alpha_{i,j} = \nabla_{i,j}(\alpha)$$

The formula for $\tilde{\alpha}_{i,j}$ is derived from the definition of reflection in Section 1.1.2. □

Lemma 4.2.7. A positive dimension vector α of Q is arm-fundamental if and only if for each $i = 1, \ldots, k$ we have $\nabla_{i,j}(\alpha) \geq \nabla_{i,j+1}(\alpha)$ for all $j = 1, \ldots, d_i - 1$.

Proof. α is arm-fundamental if $s_{i,j}(\alpha)_{i,j} \geq \alpha_{i,j}$ for all $i = 1, \ldots, k$ and $j = 1, \ldots, d_i - 1$, that is if $\alpha_{i,j-1} + \alpha_{i,j+1} - \alpha_{i,j} \geq \alpha_{i,j}$. Rearranging this gives $\alpha_{i,j-1} - \alpha_{i,j} \geq \alpha_{i,j} - \alpha_{i,j+1}$, which is $\nabla_{i,j}(\alpha) \geq \nabla_{i,j+1}(\alpha)$. □

The next theorem says that if a positive dimension vector with positive central vertex is a root, then it must be nonincreasing.

Theorem 4.2.8. Let β be a positive dimension vector of Q such that $\beta_0 > 0$. If $\beta_{i,j-1} < \beta_{i,j}$ for any $i = 1, \ldots, k$ and $j = 1, \ldots, d_i - 1$, then β is not a root.

Proof. Suppose for some i, j we have $\beta_{i,j-1} < \beta_{i,j}$, this implies $\nabla_{i,j} = \beta_{i,j-1} - \beta_{i,j} < 0$. By Lemma 4.2.6 a sequence of reflections along the i-arm (excluding the central vertex) can permute $\nabla_{i,j}(\beta)$ and $\nabla_{i,d_i}(\beta)$. Let $\tilde{\beta}$ be the dimension vector resulting from this sequence of reflection. We have $\nabla_{i,d_i}(\tilde{\beta}) = \tilde{\beta}_{i,d_i-1} < 0$, yet as $\beta_0 > 0$ we must have $\tilde{\beta}_0 > 0$ as we have not reflected on the central vertex. So $\tilde{\beta}$ (and therefore β) cannot be a root as the nonzero entries must be either all positive or all negative. □

Definition 4.2.9. Let α be arm-fundamental, we say an arm i of α is *steadily sloping* if $\nabla_{i,1}(\alpha) = \nabla_{i,2}(\alpha)$.

Definition 4.2.10. Let α be a dimension vector. Let $P(\alpha)$ be the *set of all dimension vectors of Q obtained from α by a sequence of reflections* of α on the arm vertices.

Lemma 4.2.11. If α is nonincreasing and positive then all $P(\alpha)$ are nonincreasing and positive furthermore there is a unique arm-fundamental dimension vector in $P(\alpha)$.

Proof. If α is nonincreasing and positive, then this says precisely that each $\nabla_{i,j}$ is non-negative. Reflections on the arms of α permute the sequence $\nabla_{i,1}(\alpha), \ldots, \nabla_{i,d_i}(\alpha)$ so each dimension vector of $P(\alpha)$ is nonincreasing and positive. By Lemma 4.2.7, permuting each sequence of gradients so they are in descending order gives an arm-fundamental dimension vector, clearly this is unique. □

Definition 4.2.12. Let α (with $\alpha_0 \geq 2$) be an arm-fundamental dimension vector. Let $S(\alpha)$ be the dimension vector defined by $S(\alpha)_{i,j} = \alpha_{i,j}$ for $j = 1, \ldots, d_i - 1$ and $i = 1, \ldots, k$ and $S(\alpha)_0 = \alpha_0 - 1$. Let $T(\alpha)$ be the unique arm-fundamental dimension vector in $P(S(\alpha))$.

$T(\alpha)$ is well-defined as $S(\alpha)$ has $S(\alpha)_0 \geq 1$ so is positive, and as $\nabla_{i,1}(\alpha) \geq 1$ for each $i = 1, \ldots, k$ (due to α being arm-fundamental) we have $\nabla_{i,1}(S(\alpha)) \geq 0$ so $S(\alpha)$ is nonincreasing.

Theorem 4.2.13. Let α be an arm-fundamental dimension vector. Let I be the index set of all steadily sloping arms. Let $\beta = S(\alpha)$ and $\gamma = T(\alpha)$. We have $\gamma_{i,1} = \alpha_{i,1} - 1$ for all $i \in I$ and $\gamma_{i',1} = \alpha_{i',1}$ for all $i' \notin I$.

Proof. For each steadily sloping arm i of α we have $\nabla_{i,1}(\alpha) = \nabla_{i,2}(\alpha)$. Therefore $\nabla_{i,1}(\beta) = \nabla_{i,1}(\alpha) - 1$ and $\nabla_{i,2}(\beta) = \nabla_{i,2}(\alpha)$. Clearly the unique arm-fundamental element γ of $P(\beta)$ has $\nabla_{i,1}(\gamma) = \nabla_{i,2}(\beta) = \nabla_{i,2}(\alpha) = \nabla_{i,1}(\alpha)$. So $\nabla_{i,1}(\gamma) = \alpha_0 - 1 - \gamma_{i,1} = \alpha_0 - \alpha_{i,1}$ hence $\gamma_{i,1} = \alpha_{i,1} - 1$.

For each non-steadily sloping arm i' we have $\nabla_{i',1}(\alpha) \geq \nabla_{i',2}(\alpha) + 1$. Therefore $\nabla_{i',1}(\beta) = \nabla_{i',1}(\alpha) - 1 \geq \nabla_{i',2}(\alpha)$ and $\nabla_{i,2}(\beta) = \nabla_{i,2}(\alpha)$. So the unique arm-fundamental $\gamma \in P(\beta)$ has $\nabla_{i',1}(\gamma) = \nabla_{i',1}(\alpha) - 1$ which gives $\alpha_0 - 1 - \gamma_{i',1} = \alpha_0 - \alpha_{i',1} - 1$ and hence $\gamma_{i',1} = \alpha_{i',1}$. □

Lemma 4.2.14. If α is an arm-fundamental dimension vector (with $\alpha_0 \geq 3$) such that each arm i has $z_i(\alpha) \geq 1$, then each arm i of $T(\alpha)$ has $z_i(T(\alpha)) \geq 1$.

Proof. Let $i = 1, \ldots, k$ be an arm of α. If $\nabla_{i,1}(\alpha) = 1$, then as α is arm-fundamental the ith arm (with the central vertex) must look like

$$\alpha_0 \xrightarrow{\nabla_{i,1}(\alpha)=1} \alpha_0 - 1 \xrightarrow{\nabla_{i,2}(\alpha)=1} \alpha_0 - 2 \xrightarrow{\nabla_{i,3}(\alpha)=1} \cdots \text{———} 1 \text{———} 0 \text{———} \cdots \text{———} 0$$

and as $\alpha_0 \geq 3$ we must have $z_i(\alpha) \geq 2$. Now the ith arm of $S(\alpha)$ looks like

$$\alpha_0 - 1 \xrightarrow{\nabla_{i,1}(S(\alpha))=0} \alpha_0 - 1 \xrightarrow{\nabla_{i,2}(S(\alpha))=1} \alpha_0 - 2 \xrightarrow{\nabla_{i,3}(S(\alpha))=1} \cdots \text{———} 1 \text{———} 0 \text{———} \cdots \text{———} 0$$

To reflect this to $T(\alpha)$ we permute the gradients until they are in descending order. As there are at least two nonzero gradients on the ith arm in $S(\alpha)$ we have $\nabla_{i,1}(T(\alpha)) = 1$ so $z_i(T(\alpha)) \geq 1$.

Suppose $\nabla_{i,1}(\alpha) \geq 2$, as $z_i(\alpha) \geq 1$ we must have $\nabla_{i,2}(\alpha) \neq 0$. Now we have $\nabla_{i,1}(S(\alpha)) \geq 1$ and $\nabla_{i,2}(S(\alpha)) \neq 0$. As the ith arm gradients of $T(\alpha)$ are the (possibly permuted) ith arm gradients of $S(\alpha)$, $T(\alpha)$ has at least two nonzero i-arm gradients. Therefore $z_i(T(\alpha)) \geq 1$. □

Theorem 4.2.15. Let α be an arm-fundamental dimension vector such that $z_i(\alpha) \geq 1$ for $i = 1, \ldots, k$. Let $I \subseteq \{1, \ldots, k\}$ be a subset of the arms of Q such that $|I| \geq 3$. If $\sum_{i \in I} \alpha_{i,1} = \alpha_0$, then there can be at most one steadily sloping arm indexed by I.

Proof. Suppose $i, i' \in I$ such that $i \neq i'$ are steadily sloping. So $\nabla_{i,1}(\alpha) = \nabla_{i,2}(\alpha)$ which implies $\alpha_0 - \alpha_{i,1} = \alpha_{i,1} - \alpha_{i,2}$, hence $\alpha_{i,1} \geq \alpha_0/2$ and similarly $\alpha_{i',1} \geq \alpha_0/2$. Let $J = I \setminus \{i, i'\}$, we have

$$\alpha_0 = \sum_{r \in I} \alpha_{r,1} = \alpha_{i,1} + \alpha_{i',1} + \sum_{r \in J} \alpha_{r,1} \geq \alpha_0 + \sum_{r \in J} \alpha_{r,1}.$$

We have $|J| \geq 1$ (as $|I| \geq 3$) and $\alpha_{i,1} \geq 1$ (as $z_i(\alpha) \geq 1$) for each $i = 1, \ldots, k$ therefore we have $\sum_{r \in J} \alpha_{r,1} > 0$ so $\sum_{r \in I} \alpha_{r,1} > \alpha_0$ which is a contradiction. □

4.2.1 Arm-Fundamental Dimension Vectors with Shallowest Slope

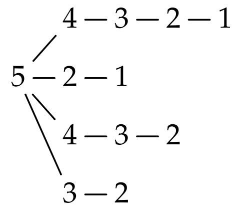
Let $k \geq 2$ and let Q be a quiver with $k+1$ arms. We now consider positive dimension vectors such that the value on the $(k+1)$-arm drops by one each vertex as one moves away from the centre. Such dimension vectors appear in Section 4.3.2 when we consider the conjugacy class of nonderogatory matrices with prescribed eigenvalues. The main results of this section are Theorems 4.2.25 and 4.2.28 which give necessary and sufficient conditions for such a dimension vector (under additional restrictions) to be a root of Q.

Definition 4.2.16. Let α be a positive dimension vector. We say the ith arm of α has *shallowest slope* if the ith arm (with the central vertex) is of the form

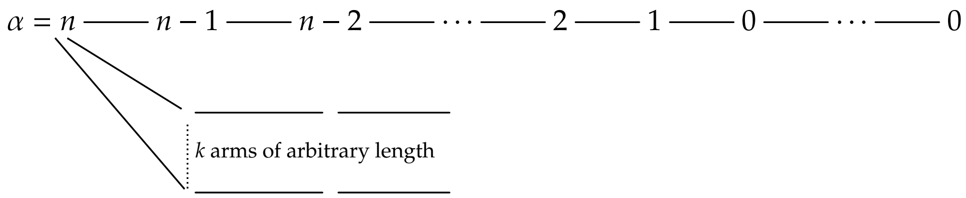
$$\alpha_0 \text{ ——— } \alpha_0 - 1 \text{ ——— } \alpha_0 - 2 \text{ ——— } \cdots \text{ ——— } 2 \text{ ——— } 1 \text{ ——— } 0 \text{ ——— } \cdots \text{ ——— } 0\,,$$

that is: the ith arm is of length greater or equal to $\alpha_0 - 1$, $\alpha_{i,j} = \alpha_0 - j$ for $j = 1, \ldots, \alpha_0 - 1$ and $\alpha_{i,j} = 0$ for $j \geq \alpha_0$. Equivalently if $\nabla_{i,j}(\alpha) = 1$ for $j = 1, \ldots, \alpha_0 - 1$ and zero otherwise.

Example 4.2.17. Suppose $k = 3$, the following dimension vector of a star-shaped quiver, with $k+1$ arms, has shallowest slope on its $(k+1)$th arm.



We are interested in positive nonincreasing dimension vectors of Q whose $k+1$ arm has shallowest slope, that is dimension vectors of the form given below for some $n \geq 1$ (the $(k+1)$th arm is at the top).



Definition 4.2.18. Let H be the set of dimension vectors α of Q such that α is arm-fundamental, has $\alpha_0 \geq 2$, has $z_i(\alpha) \geq 1$ for $i = 1, \ldots, k$ and has shallowest $k+1$ slope. Let H_n be the set of dimension vectors α of H such that $\alpha_0 = n$, clearly $H = \bigcup_{n \geq 2} H_n$.

We have $H_n = \emptyset$ whenever $n-1$ is greater than the length of the $k+1$ arm (as it cannot have shallowest slope), therefore the set H is finite. The finiteness of H, however, is not important for this section and, if one prefers, one can think of the arms of Q as having arbitrarily large length.

Theorem 4.2.19. If $\alpha \in H$ is a root, then $\sum_{i=1}^{k} \alpha_{i,1} \geq \alpha_0$.

Proof. Suppose $\sum_{i=1}^{k} \alpha_{i,1} < \alpha_0$. Let $\beta = s_0(\alpha)$, that is the dimension vector obtained by reflecting α at the central vertex. So

$$\beta_0 = \sum_{i=1}^{k+1} \alpha_{i,1} - \alpha_0 = \sum_{i=1}^{k} \alpha_{i,1} + (\alpha_0 - 1) - \alpha_0 = \sum_{i=1}^{k} \alpha_{i,1} - 1 < \alpha_0 - 1 = \alpha_{k+1,1} = \beta_{k+1,1},$$

so $\beta_0 < \beta_{k+1,1}$, by Theorem 4.2.8 β is not a root. □

Theorem 4.2.20. Let $\alpha \in H$. If $\sum_{i=1}^{k} \alpha_{i,1} > \alpha_0$, then α is an imaginary root.

Proof. Let $\beta = s_0(\alpha)$, that is the dimension vector obtained by reflecting α at the central vertex.

$$\beta_0 = \sum_{i=1}^{k} \alpha_{i,1} + (\alpha_0 - 1) - \alpha_0 > \alpha_0 - 1,$$

which implies $\beta_0 \geq \alpha_0$ so α is in the fundamental region and hence an imaginary root. □

Recall from the previous section $S(\alpha)$ is the same as α except for $S(\alpha)_0 = \alpha - 1$ and $T(\alpha)$ is the unique arm-fundamental vector in $P(S(\alpha))$.

Lemma 4.2.21. Let $\alpha \in H$. If $\sum_{i=1}^{k} \alpha_{i,1} = \alpha_0$, then $s_0(\alpha) = S(\alpha)$ and $T(\alpha)$ is the unique arm-fundamental dimension vector of $P(s_0(\alpha))$.

Proof. $s_0(\alpha)_0 = \sum_{i=1}^{k+1} \alpha_{i,1} - \alpha_0 = \sum_{i=1}^{k} \alpha_{i,1} + (\alpha_0 - 1) - \alpha_0 = \alpha_0 - 1 = S(\alpha)_0$, so $s_0(\alpha) = S(\alpha)$. The second statement follows by definition. □

Lemma 4.2.22. Let $n \geq 3$, if $\alpha \in H_n$ such that $\sum_{i=1}^{k} \alpha_{i,1} = \alpha_0$, then $T(\alpha) \in H_{n-1}$.

Proof. By definition we have $T(\alpha)_0 = n - 1$ and clearly $T(\alpha)_0$ has shallowest $k + 1$ slope (all gradients on the $(k + 1)$th arm are either one or zero). By Lemma 4.2.14 we have $z_i(T(\alpha)) \geq 1$ for each $i = 1, \ldots, k$, so we have $T(\alpha) \in H_{n-1}$ □

N.B. let $\alpha \in H$, although the $(k + 1)$th arm of α is steadily sloping, by convention we exclude this arm from consideration when discussing the steadily sloping arms of α, for instance if we say α has no steadily sloping arms, then we mean that none of the arms $1, \ldots, k$ is steadily sloping.

Theorem 4.2.23. If $\alpha \in H$ such that $\alpha_0 \geq 3$, $\sum_{i=1}^{k} \alpha_{i,1} = \alpha_0$ and α has no steadily sloping arms, then α is an imaginary root.

Proof. By Lemma 4.2.21 the dimension vector obtained by reflecting α on the central vertex is $S(\alpha)$ so let $\gamma = T(\alpha)$. Let $n = \alpha_0$ so that $\alpha \in H_n$. By Lemma 4.2.22 we have $\gamma \in H_{n-1}$. Clearly $\gamma_0 = \alpha_0 - 1$ and by Theorem 4.2.13 $\gamma_{i,1} = \alpha_{i,1}$ for all $i = 1, \ldots, k$, as α has no steadily sloping arms. So we have

$$\sum_{i=1}^{k} \gamma_{i,1} = \sum_{i=1}^{k} \alpha_{i,1} = \alpha_0 = \gamma_0 + 1,$$

therefore $\sum_{i=1}^{k} \gamma_{i,1} > \gamma_0$. As $\gamma \in H$, Theorem 4.2.20 implies γ and α are imaginary roots. □

Lemma 4.2.24. Let $\alpha \in H$ such that $\alpha_0 \geq 4$. If $\sum_{i=1}^{k} \alpha_{i,1} = \alpha_0$, then for each $i = 1, \ldots, k$ $z_i(\alpha) \geq 2$ implies $z_i(T(\alpha)) \geq 2$.

Proof. The condition $\sum_{i=1}^{k} \alpha_{i,1} = \alpha_0$ implies $S(\alpha)_0 = \alpha_0 - 1$. So $\nabla_{i,1}(S(\alpha)) = \nabla_{i,1}(\alpha) - 1$. As we obtain $T(\alpha)$ from $S(\alpha)$ by permuting the gradients, the only circumstance in which $z_i(T(\alpha)) < z_i(\alpha)$, for some $i \in \{1, \ldots, k\}$ is if $\nabla_{i,1}(\alpha) = 1$. Suppose $\nabla_{i,1}(\alpha) = 1$, as α is arm-fundamental we have $z_i(\alpha) = \alpha_0 - 1 \geq 3$. Now $\nabla_{i,1}(S(\alpha)) = 0$ and by performing the reflection to obtain $T(\alpha)$ this gradient is permuted to the end of the arm, thus reducing the nonzero length by one. So $z_i(T(\alpha)) = z_i(\alpha) - 1 \geq 2$. □

The following theorems give necessary and sufficient conditions for a dimension vector in H to be a root. We consider two separate cases depending on the number of arms, we first consider the case where $k \geq 3$. Remember however that Q has $k+1$ arms.

Theorem 4.2.25. Let $k \geq 3$, $\alpha \in H$ is a root if and only if $\sum_{i=1}^{k} \alpha_{i,1} \geq \alpha_0$.

Proof. If α is a root, then $\sum_{i=1}^{k} \alpha_{i,1} \geq \alpha_0$ by Theorem 4.2.19.

We use induction to show that if $\sum_{i=1}^{k} \alpha_{i,1} \geq \alpha_0$, then α is a root. We first prove this for $\alpha \in H_2$. As α is arm-fundamental and $z_i(\alpha) \geq 1$ for each $i = 1, \ldots, k$ we must have $\alpha_{i,1} = 1$ for $i = 1, \ldots, k$ (clearly we must also have $\alpha_{k+1,1} = 1$). As $k \geq 3$, this implies that α satisfies $\sum_{i=1}^{k} \alpha_{i,1} \geq \alpha_0$, we show this property implies α is a root. Let $\beta = s_0(\alpha)$, so

$$\beta_0 = \sum_{i=1}^{k} \alpha_{i,1} + \alpha_{k+1,1} - \alpha_0 = k + 1 - 2 = k - 1 \geq 2.$$

So reflecting α at the central vertex does not decrease the value at the central vertex, therefore α is in the fundamental region, and hence an (imaginary) root.

Now we show that if the induction hypothesis is true for all dimension vectors in H_{n-1}, where $n \geq 3$, then it is true for all dimension vectors in H_n.

Let $\alpha \in H_n$ and let $\beta = s_0(\alpha)$. If $\sum_{i=1}^{k} \alpha_{i,1} > \alpha_0$, then by Theorem 4.2.20 α is an imaginary root. Suppose instead that $\sum_{i=1}^{k} \alpha_{i,1} = \alpha_0$, by Lemma 4.2.21 this implies $\beta = S(\alpha)$. By Theorem 4.2.15 α can have no more than one steadily sloping arm. If α has no steadily sloping arms then by Theorem 4.2.23 α is a root. Suppose that α has one steadily sloping arm, say $i' \in \{1, \ldots, k\}$. Let $\gamma = T(\alpha)$. By Theorem 4.2.13 we have

$$\sum_{i=1}^{k} \gamma_{i,1} = \sum_{\substack{i=1: \\ i \neq i'}}^{k} \alpha_{i,1} + (\alpha_{i',1} - 1) = \sum_{i=1}^{k} \alpha_{i,1} - 1 = \alpha_0 - 1 = \gamma_0.$$

So we have $\sum_{i=1}^{k} \gamma_{i,1} \geq \gamma_0$. By Lemma 4.2.22 we have $\gamma \in H_{n-1}$ which, by the induction hypothesis, implies γ and α are roots. □

We now look at the case where $k = 2$.

Lemma 4.2.26. Let $k = 2$. If $\alpha \in H$ such that $\sum_{i=1}^{k} \alpha_{i,1} = \alpha_0$, then α has two steadily sloping arms if and only if $z_1(\alpha) = z_2(\alpha) = 1$.

Proof. Suppose $z_1(\alpha) = z_2(\alpha) = 1$, as α is arm-fundamental we have, for $i = 1, 2$, $\nabla_{i,1}(\alpha) = \alpha_0 - \alpha_{i,1} \geq \alpha_{i,1}$ so $\alpha_{i,1} \leq \alpha_0/2$. Let $i \in \{1, 2\}$, if we have $\alpha_{i,1} < \alpha_0/2$, then $\alpha_{1,1} + \alpha_{2,1} = \alpha_0$ implies $\alpha_{3-i,1} > \alpha_0/2$ so $\alpha_{1,1} = \alpha_{2,1} = \alpha_0/2$ (note that this case arrises only if α_0 is even). So for $i = 1, 2$ we have $\nabla_{i,1}(\alpha) = \alpha_0 - \alpha_0/2 = \alpha_0/2$ and $\nabla_{i,2}(\alpha) = \alpha_0/2$. So both arms are steadily sloping.

Suppose α has two steadily sloping arms, we have $\nabla_{i,1}(\alpha) = \nabla_{i,2}(\alpha)$ for $i = 1, 2$, this implies $\alpha_0 - \alpha_{i,1} = \alpha_{i,1} - \alpha_{i,2}$, which gives $\alpha_{i,1} \geq \alpha_0/2$. Using a similar argument to the one above we conclude from $\alpha_{1,1} + \alpha_{2,1} = \alpha_0$ that $\alpha_{1,1} = \alpha_{2,1} = \alpha_0/2$. As both arms are steadily sloping we conclude $\alpha_{1,2} = \alpha_{2,2} = 0$ so $z_1(\alpha) = z_2(\alpha) = 1$. □

Lemma 4.2.27. Let $k = 2$ and $\alpha \in H$. If α is a root and $z_1(\alpha) = z_2(\alpha) = 1$, then $\alpha_0 = 2$.

Proof. If α is a root and $z_1(\alpha) = z_2(\alpha) = 1$, then we can think of α as a dimension vector of a quiver of underlying type D_{n+2}, so by the representation theory of quivers of underlying type D_{n+2} we must have $\alpha_0 = 2$. □

Theorem 4.2.28. Suppose $k = 2$, $\alpha \in H$ is a root if and only if $\sum_{i=1}^{k} \alpha_{i,1} \geq \alpha_0$ and either $\alpha_0 = 2$ or at least one of $z_1(\alpha), z_2(\alpha)$ is not equal to 1.[1]

Proof. If $\alpha \in H$ is a root, then by Theorem 4.2.19 $\sum_{i=1}^{k} \alpha_{i,1} \geq \alpha_0$ and by Lemma 4.2.27 we have either $\alpha_0 = 2$ or at least one of $z_1(\alpha), z_2(\alpha)$ is not equal to 1.

We use induction to show that if $\alpha \in H$ satisfies $\sum_{i=1}^{k} \alpha_{i,1} \geq \alpha_0$ and either $\alpha_0 = 2$ or at least one of $z_1(\alpha), z_2(\alpha)$ is not equal to 1, then α is a root. We first prove this for $\alpha \in H_2$, in this case we have $\alpha_0 = 2, \alpha_{1,1} = \alpha_{2,1} = \alpha_{3,1} = 1$ with all other entries zero. This is both a (real) root and satisfies $\sum_{i=1}^{k} \alpha_{i,1} \geq \alpha_0$.

Now we show that if the hypothesis is true for all dimension vectors in H_{n-1}, where $n \geq 3$, then it is true for all dimension vectors in H_n. Let $\alpha \in H_n$. Suppose at least one of $z_1(\alpha), z_2(\alpha)$ is not equal to 1.

If $\sum_{i=1}^{k} \alpha_{i,1} > \alpha_0$, then by Theorem 4.2.20 α is an (imaginary) root. Suppose that $\sum_{i=1}^{k} \alpha_{i,1} = \alpha_0$ and let $\beta = s_0(\alpha)$. We have

$$\beta_0 = \sum_{i=1}^{k} \alpha_{i,1} + (\alpha_0 - 1) - \alpha_0 = n - 1,$$

[1] The condition "either $\alpha_0 = 2$ or at least one of $z_1(\alpha), z_2(\alpha)$ is not equal to 1" is equivalent to "$z_1(\alpha) = z_2(\alpha) = 1$ implies $\alpha_0 = 2$".

that is $\beta = S(\alpha)$. Let $\gamma = T(\alpha)$, by Lemma 4.2.22 we have $\gamma \in H_{n-1}$. If $n \geq 4$, then by Lemma 4.2.24 for any $i = 1, 2$ such that $z_i(\alpha) \geq 2$ we have $z_i(\gamma) \geq 2$ so at least one of $z_1(\gamma), z_2(\gamma)$ is not equal to 1. By the induction hypothesis this proves γ is a root, which proves α is also. Suppose finally that $n = 3$, in this case $\gamma_0 = 2$. By Lemma 4.2.26 α can have no more than one steadily sloping arm. If α has no steadily sloping arms then α is a root by Theorem 4.2.23. Suppose that α has one steadily sloping arm, say $i' \in \{1, 2\}$, we have by Theorem 4.2.13

$$\sum_{i=1}^{k} \gamma_{i,1} = \sum_{\substack{i=1:\\ i \neq i'}}^{k} \alpha_{i,1} + (\alpha_{i',1} - 1) = \sum_{i=1}^{k} \alpha_{i,1} - 1 = \alpha_0 - 1 = \gamma_0.$$

So we have $\sum_{i=1}^{k} \gamma_{i,1} \geq \gamma_0$ and $\gamma_0 = 2$. By the induction hypothesis, as $\gamma \in H_{n-1}$, we have that γ is a root, therefore so is α. $\square$

4.3 Silva's Solutions

Let K be an algebraically closed field of characteristic zero, let V be a K-vector space of dimension $n \geq 2$. Let $k \geq 2$ and let $C_1, \ldots, C_k \subset \mathrm{End}(V)$ be nonscalar conjugacy classes. As seen in Section 4.1.1 the nonscalar restriction entails no loss of generality. Let $c_1, \ldots, c_n \in K$. The problem addressed by Silva et al. in [NS99], [Sil90] and several other papers is called the *recognition problem* and is of the form: given the conjugacy classes $C_1, \ldots, C_k$ and eigenvalues $c_1, \ldots, c_n$ under what conditions does there exist $A_i \in C_i$ for $i = 1, \ldots, k$ such that $A_1 + \cdots + A_k$ has eigenvalues $c_1, \ldots, c_n$. This is close to the problems addressed in Section 4.1 but differs in that we do not specify the exact Jordan structure of $A_1 + \cdots + A_k$ but only the eigenvalues.

We give the solution presented by Silva et al. in Section 4.3.1 and in Section 4.3.2 we compare this solution to the one given by Crawley-Boevey.

4.3.1 The Main Theorem

For each $i = 1, \ldots, k$ let η_i be an eigenvalue of C_i of maximal geometric multiplicity, let $r_i = \mathrm{rank}(A_i - \eta_i 1_V)$ where $A \in C_i$ (this is equal to $\min_{\xi \in \Psi(C_i)}\{\mathrm{rank}(A_i - \xi I_n)\}$), let $t_i = \sum_{l=1}^{k} r_l - r_i$, and let s_i be the number of nonconstant invariant polynomials of C_i. Recall that we denote the invariant polynomials of C_i by $\iota_{C_i,1}, \ldots, \iota_{C_i,n}$.

Theorem 4.3.1. There exists $A_i \in C_i$ such that $A_1 + \cdots + A_k$ has eigenvalues $c_1, \ldots, c_k$ if and only if

1. $\sum_{i=1}^{k} \operatorname{trace}(C_i) = c_1 + \cdots + c_k$,
2. For each $i = 1, \ldots, k$ we have
$$\prod_{j=1}^{n-t_i} \iota_{C_i,j}(x + \eta_i - \sum_{l=1}^{k} \eta_l) \Big| \prod_{p=1}^{n} (x - c_p). \tag{4.6}$$
3. If $k = 2$ and $\deg(\iota_{C_1,n}) = \deg(\iota_{C_2,n}) = 2$ (in this case let $\iota_{C_i,n}(x) = (x - \eta_i)(x - \upsilon_i)$ for $i = 1, 2$), then there exists a permutation $\pi\colon \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that:
$$\begin{aligned} c_{\pi(2i-1)} + c_{\pi(2i)} &= \eta_1 + \eta_2 + \upsilon_1 + \upsilon_2 && \text{for } 1 \le i \le n - s_2, \\ c_{\pi(i)} &= \eta_1 + \eta_2 && \text{for } 2(n - s_2) < i \le n + s_1 - s_2, \\ c_{\pi(i)} &= \upsilon_1 + \eta_2 && \text{for } n + s_1 - s_2 < i \le n. \end{aligned}$$

Proof. [Sil90, Thm. 7] and [NS99, Thm. 2]. [2] □

Although the problem of Silva et al. is not to find endomorphisms which sum to zero, we can easily rephrase the problem so that it does. Finding k endomorphisms in prescribed conjugacy classes which *sum to an endomorphism with prescribed eigenvalues* (i.e. $c_1, \ldots, c_n$) is equivalent to finding $k + 1$ endomorphisms, which *sum to zero*, where the first k are in prescribed conjugacy classes and the last one has prescribed eigenvalues (i.e. $-c_1, \ldots, -c_n$).

To say an endomorphism has prescribed eigenvalues is equivalent to saying it has prescribed characteristic polynomial (in this case $(x + c_1) \ldots (x + c_n)$). By Theorem 2.1.25 the set of endomorphisms with prescribed characteristic polynomial is the closure of the conjugacy class containing nonderogatory endomorphisms with the prescribed characteristic polynomial. This is not, in general, the same problem considered by Crawley-Boevey, however under a certain condition, given below, the two problems coincide.

Remark 4.3.2. If $C_1, \ldots, C_k$ are closed and C_{k+1} consists of nonderogatory endomophisms, then Theorems 4.3.1 and 4.1.16 coincide (with appropriate modifications to the $c_1, \ldots, c_n$, i.e. negation).

[2] Silva et al. considers the $k = 2$ and $k \ge 3$ cases in separate papers, but we amalgamate the theorems into one here.

We use the machinary developed in Section 4.2 to interpret the conditions of Theorem 4.3.1 in terms of dimension vectors and relate it to Theorem 4.1.16 in the case where the two conditions coincide.

4.3.2 Comparing the Results of Crawley-Boevey and Silva et al.

Let $C_1, \dots, C_k$ be closed and let C_{k+1} be the conjugacy class consisting of nonderogatory endomorphisms with characteristic polynomial $(x + c_1) \dots (x + c_n)$.

Both Theorem 4.3.1 and 4.1.16 are of the form: "there exists $A_i \in C_i$ such that $A_1 + \cdots + A_k$ has eigenvalues $c_1, \dots, c_n$ if and only if some conditions on the $C_1, \dots, C_k$ and the $c_1, \dots, c_n$ are satisifed". The purpose of this section to show the "conditions on the $C_1, \dots, C_k$ and the $c_1, \dots, c_n$" in both Theorem 4.3.1 and 4.1.16 are equivalent. Thought time restrictions prevent us from showing this completely we make some progress towards it.

Let $\xi_{i,1}, \dots, \xi_{i,d_i}$ for $i = 1, \dots, k+1$, Q, $\alpha \in \mathbb{Z}^{Q_0}$ and $\lambda \in K^{Q_0}$ be defined as in Section 4.1.1. Note $\alpha_0 = n$, $\xi_{k+1,j} = -c_j$ for $j = 1, \dots, n$. We assume α is arm-fundamental. By the following lemma this does not result in any loss of generality.

Lemma 4.3.3. If we choose an ordering of the $\xi_{i,1}, \dots, \xi_{i,d_i}$ such that $\mathrm{algr}_{C_i}(\xi_{i,j}) > \mathrm{algr}_{C_i}(\xi_{i,j'})$ implies $j < j'$ for $j, j' \in \{1, \dots, d_i\}$ for each $i = 1, \dots, k$, then α is arm-fundamental.

Proof. Suppose we choose such an ordering, this is equivalent to saying: for each $i = 1, \dots, k$ the $\xi_{i,1}, \dots, \xi_{i,d_i}$ are arranged in descending order of algebraic multiplicity. As the $C_1, \dots, C_k$ are closed the $\xi_{i,1}, \dots, \xi_{i,d_i}$ are distinct, therefore we have $\alpha_{i,j} = \mathrm{rank}(\prod_{l=1}^{j}(A - \xi_{i,l}1_V)) = n - \sum_{l=1}^{j} \mathrm{algr}_{C_i}(\xi_{i,l})$ (for $A \in C_i$). So $\nabla_{i,j}(\alpha) = n - \sum_{l=1}^{j-1} \mathrm{algr}_{C_i}(\xi_{i,l}) - (n - \sum_{l=1}^{j} \mathrm{algr}_{C_i}(\xi_{i,l})) = \mathrm{algr}_{C_i}(\xi_{i,j})$. So α is arm-fundamental on the ith arm for $i = 1, \dots, k$. The C_{k+1} class is nonderogatory (but not necessarily closed), this implies the $(k+1)$th arm has shallowest slope, hence is trivially arm-fundamental (recall Theorem 2.1.24). □

By the above lemma we can safely assume the eigenvalue η_i of C_i, chosen to have maximal geometric (and algebraic, as C_i is closed) multiplicity, is $\xi_{i,1}$ for $i = 1, \dots, k$. We consider the case where the underlying graph of Q is not D_{n+2}, by the following lemma this assumption implies Condition 3 of Theorem 4.3.1 is satisfied trivially.

Lemma 4.3.4. The associated quiver Q has underlying graph D_{n+2} if and only if $k = 2$ and $\deg(\iota_{C_1,n}) = \deg(\iota_{C_2,n}) = 2$.

Proof. Q has underlying graph D_{n+2} if and only if $k = 2$ and $d_1 = d_2 = 2$. As $\min_{C_i}(x) = \iota_{C_i,n}(x)$ for $i = 1, 2$, it follows that $d_1 = d_2 = 2$ is equivalent to $\deg(\iota_{C_1,n}) = \deg(\iota_{C_2,n}) = 2$. □

Remark 4.3.5. It can be shown that $\lambda \cdot \alpha = 0$ is equivalent to $\sum_{i=1}^{k+1} \text{trace}(C_i) = 0$. By Theorem 2.2.4, for any $A_i \in C_i$, there exists vector spaces and maps as given in the theorem. Using this result we take the trace of each A_i and use the fact that $\text{trace}(AB) = \text{trace}(BA)$ for approprately sized endomorphisms A and B to derive the result.

It follows easily from Theorem 4.1.16 that $\lambda \cdot \alpha = 0$ is a necessary condition for a solution to the additive matrix problem to exist. By Remark 4.3.5 this is equivalent to $\sum_{i=1}^{k+1} \text{trace}(C_i) = 0$ which is equivalent to Condition 1 in Theorem 4.3.1.

The following lemma gives the necessary and sufficient condition for Condition 2 of Theorem 4.3.1 to be satisfied trivially, that is for the left-hand sides of (4.6) to be equal to 1.

Lemma 4.3.6. Let Q have underlying type different from D_{n+2}. Condition 2 of Theorem 4.3.1 is satisfied trivially if and only if α is a root.

Proof. As C_{k+1} is a conjugacy class consisting of nonderogatory matrices, Lemma 2.1.24 implies the $k+1$ arm of α has shallowest slope. As the $C_1, \ldots, C_k$ are nonscalar, the nonzero arm-lengths $z_1(\alpha), \ldots, z_k(\alpha)$ are all nonzero. By Lemma 4.3.3 we assume α is arm-fundamental so that $\alpha \in H$, and that $\sum_{i=1}^{k} r_i \geq n$ is equivalent to $\sum_{i=1}^{k} \alpha_{i,1} \geq \alpha_0$. By Theorem 4.2.25 or Theorem 4.2.28 (depending on where $k \geq 3$ or $k = 2$ respectively) we have $\sum_{i=1}^{k} \alpha_{i,1} \geq \alpha_0$ if and only if α is a root.

We show $\sum_{i=1}^{k} r_i \geq n$ if and only if the divisibility condition (4.6) is trivial. Let $i \in \{1, \ldots, k\}$. The nonconstant invariant polynomials of C_i are $\iota_{C_i,n-s_i+1}(x), \ldots, \iota_{C_i,n}(x)$. As the left-hand side of (4.6) is a product of $\iota_{C_i,1}(x), \ldots, \iota_{C_i,n-t_i}(x)$, it is equal to 1 if and only if $n - t_i < n - s_i + 1$, which (as $n - s_i = r_i$ by their definitions) is equivalent to $n - \sum_{l=1}^{k} r_l + r_i < r_i + 1$, which is equivalent to $n - \sum_{l=1}^{k} r_l \leq 0$, which is equivalent to $\sum_{l=1}^{k} r_l \geq n$. □

So if α is a root and Q does not have underlying type D_{n+2}, then both Theorem 4.3.1 and Theorem 4.1.16 imply there exists a solution to the additive matrix problem if and only if $\lambda \cdot \alpha = 0$.

Due to time limitations it was not possible to address the case where α is not a root or where Q has underlying type D_{n+2}. It seems reasonable to conjecture that if α is not a root (and Q does not have underlying type D_{n+2}), then the nontrivial factors of the left-hand side of (4.6) can be used to construct a root decomposition $(\beta_1, \dots, \beta_r)$ of α compatible with λ. Each linear factor of the left-hand side of (4.6), for some $i \in \{1, \dots, k\}$, implies there exists some $p \in \{1, \dots, n\}$ and $j \in \{1, \dots, d_i\}$ such that $c_p = \xi_{i,1} - \sum_{l=1}^{k} \xi_{l,1} - \xi_{i,j}$. This can be expressed as $\lambda \cdot \beta = 0$ where β is a particular positive real root consisting only of ones and zeros. Let $\beta_1, \dots, \beta_r$ be a list of such positive roots (the possible sizes of such lists is not immediately clear) and let $\gamma = \alpha - (\beta_1 + \cdots + \beta_r)$, we have $\alpha = \beta_1 + \cdots + \beta_r + \gamma$ and $\lambda \cdot \beta_1 = \cdots = \lambda \cdot \beta_r = \lambda \cdot \gamma = 0$. If it can be shown that each summand is a positive root, then this result could be the beginning of a more general theorem comparing the results of Silva et al. and Crawley-Boevey.

4.4 A New Approach to the Additive Matrix Problem with Closed Conjugacy Classes using Invariant Polynomials

We introduce a new approach to solving the additive matrix problem when all conjugacy classes are closed. This approach builds on Theorem 4.1.16 of Crawley-Boevey. Let V be a K-vector space. Let $k \geq 2$ and let $C_1, \dots, C_k \subseteq \mathrm{End}(V)$ be a k-tuple of closed conjugacy classes, so $C_i = \overline{C}_i$ for $i = 1, \dots, k$. Let Q be the star-shaped quiver associated to $C_1, \dots, C_k$.

Let $\xi_{i,1}, \dots, \xi_{i,d_i}$ be a list of the roots of the minimal polynomial of C_i for $i = 1, \dots, k$. Recall the definitions of $\alpha \in \mathbb{Z}^{Q_0}$ and $\lambda \in K^{Q_0}$ in Section 4.1.1 and recall that these depend on the orderings of the $\xi_{i,1}, \dots, \xi_{i,d_i}$. Let us consider those orderings which makes α arm-fundamental. There may be many such orderings and these may give different values for λ. Let Λ be the set of all such λ over these orderings.

In this setting Theorem 4.1.16 is equivalent to: there exists $A_i \in C_i$ for $i = 1, \dots, k$ such that $\sum_{i=1}^{k} A_i = 0$ if and only if there exists some root decomposition $\underline{\beta} = (\beta_1, \dots, \beta_l)$ of α and some $\lambda \in \Lambda$ such that $(\beta_1, \dots, \beta_l)$ is compatible with λ (recall the definitions from Section 1.2.3). It is easy to see this occurs if and only if

$$M_{\underline{\beta}}(z_1, \dots, z_l) = \prod_{\lambda \in \Lambda} (\lambda \cdot (\beta_1 z_1 + \cdots + \beta_l z_l)) = 0, \tag{4.7}$$

where $z_1, \dots, z_l$ is a collection of indeterminates.

As there is a finite number of root decompositions of α, determining whether the conditions of Theorem 4.1.16 are satisfied is equivalent to checking whether for any of the finite number of root decompositions $(\beta_1, \dots, \beta_l)$ of α we have $M_{\underline{\beta}}(z_1, \dots, z_l) = 0$.

We show that (4.7) can be computed using the coefficients of the invariant polynomials of $C_1, \dots, C_k$ without calculating the eigenvalues directly. Note that given $\lambda \in \Lambda$ and $\beta \in \mathbb{Z}^{Q_0}$ the scalar product $\lambda \cdot \beta$ can be expressed as an linear combination of the $\xi_{i,1}, \dots, \xi_{i,d_i}$ for $i = 1, \dots, k$ with integral coefficients. The different elements of Λ are produced by different orderings of the $\xi_{i,1}, \dots, \xi_{i,d_i}$. In Section 4.4.1 we establish notation to make the orderings explicit. We show, in Remark 4.4.9, that it is possible to calculate the unique arm-fundamental dimension vector α without factoring the invariant polynomials.

In Section 4.4.1 we introduce explicit notation for the different orderings, in Section 4.4.2 we introduce, for each $i = 1, \dots, k$, an n-tuple of polynomials $g_{i,1}, \dots, g_{i,n}$ from which (4.7) can be constructed. In Section 4.4.3 we show how to write (4.7) in terms of the roots of $g_{i,1}, \dots, g_{i,n}$. Recall the two operations introduced in Chapter 3: the *Root sum* and *Kronecker sum* of a polynomial. In Section 4.4.4 we define the matrix sum polynomial $\mathcal{S}_{\underline{\beta}}(z_1, \dots, z_l)$ using the Kronecker sum and the root sum and show this is precisely equal to (4.7) as defined above.

4.4.1 Orderings of Eigenvalues

Definition 4.4.1. Let R be an integral domain, let $\xi_1, \dots, \xi_n \in R$ be distinct and let $g(x) = (x - \xi_1) \dots (x - \xi_n)$. An *ordering* of the roots of $g(x)$ is a bijective map ω from the set $\{1, \dots, n\}$ to the set of roots of g, that is a bijection $\omega \colon \{1, \dots, n\} \to \{\xi_1, \dots, \xi_n\}$. Such a map gives the set of roots an ordering, that is $\omega(i)$ is the ith root of g under the ordering ω. Let Σ_g be the set of all orderings of the roots of g.

As the roots are distinct it is clear that Σ_g is isomorphic to Σ_n, the group of permutations of $\{1, \dots, n\}$.

Definition 4.4.2. Let $C \subseteq \mathrm{End}(V)$ be a closed conjugacy class. An *ordering of the distinct eigenvalues* of C is an ordering of the roots of the minimal polynomial of C. Let the set of such orderings be denoted Σ_C.

Let $\underline{\mathbf{C}} = (C_1, \dots, C_k)$ and let $\Sigma_{\underline{\mathbf{C}}} = \prod_{i=1}^{k} \Sigma_{C_i}$. Recall the definitions of α and λ given in Section 4.1.1, and recall that these depend on the orderings chosen for the roots of the minimal polynomials $\min_{C_i}$. We introduce terminology for these objects which makes the dependence on the orderings explicit.

Definition 4.4.3. Let $\underline{\omega} \in \Sigma_{\underline{\mathbf{C}}}$. Let $\alpha^{\underline{\omega}} \in \mathbb{Z}^{Q_0}$ be the dimension vector of Q given by $\alpha_0^{\underline{\omega}} = n$ and

$$\alpha_{i,j}^{\underline{\omega}} = \operatorname{rank}\Big(\prod_{l=1}^{j}(A_i - \omega_i(l)1_V)\Big), \text{ for } A_i \in C_i,$$

let $\lambda^{\underline{\omega}} \in K^{Q_0}$ be the K-vector given by $\lambda_0^{\underline{\omega}} = -\sum_{i=1}^{k}\omega_i(1)$ and $\lambda_{i,j}^{\underline{\omega}} = \omega_i(j) - \omega_i(j+1)$ for each $i = 1, \dots, k$ and $j = 1, \dots, d_i - 1$. The definition of $\alpha^{\underline{\omega}}$ is well-defined as the rank does not depend on our choices of $A_i \in C_i$ for $i = 1, \dots, k$.

Recall the definition of the gradient $\nabla_{i,j}$ of a dimension vector at vertex $[i, j]$, given in Section 4.2. Recall from Section 2.1 that the algebraic multiplicity of an eigenvalue ξ of C (that is the multiplicity of ξ in the characteristic polynomial) is denoted $\operatorname{algr}_C(\xi)$. As C is closed, the geometric and algebraic multiplicities of C are the same, however we continue to use the term algebraic multiplicity to distinguish from the concept of the multiplicity of a root of a polynomial.

Lemma 4.4.4. Let $\underline{\omega} \in \Sigma_{\underline{\mathbf{C}}}$. We have $\nabla_{i,j}(\alpha^{\underline{\omega}}) = \operatorname{algr}_{C_i}(\omega_i(j))$.

Proof. Recall from the proof of Lemma 4.3.3 that $\operatorname{rank}(\prod_{l=1}^{j-1}(A_i - \xi_{i,l}1_V)) = n - \sum_{l=1}^{j-1}\operatorname{algr}_{C_i}(\xi_{i,l})$ for $A_i \in C_i$ as C_i is closed.

$$\begin{aligned}\alpha_{i,j-1}^{\underline{\omega}} - \alpha_{i,j}^{\underline{\omega}} &= \operatorname{rank}\Big(\prod_{l=1}^{j-1}(A_i - \omega_i(l)1_V)\Big) - \operatorname{rank}\Big(\prod_{l=1}^{j}(A_i - \omega_i(l)1_V)\Big) \\ &= (n - \sum_{l=1}^{j-1}\operatorname{algr}_{C_i}(\omega_i(l))) - (n - \sum_{l=1}^{j}\operatorname{algr}_{C_i}(\omega_i(l))) = \operatorname{algr}_{C_i}(\omega_i(j)),\end{aligned}$$

where $A_i \in C_i$ for $i = 1, \dots, k$. □

4.4.2 Multiplicity Preserving Orderings and the Multiplicity Factorization

Definition 4.4.5. Let C be a closed conjugacy class. We say an ordering $\omega \in \Sigma_C$ of the distinct eigenvalues of C *preserves multiplicity* if $\omega^{-1}(\psi) < \omega^{-1}(\psi')$ implies $\operatorname{algr}_C(\psi) \geq \operatorname{algr}_C(\psi')$ where $\psi, \psi' \in \Psi(C)$ (Recall $\Psi(C)$ is the set of eigenvalues of C). Let the set of multiplicity preserving orderings be denoted Ξ_C.

Effectively, a multiplicity preserving ordering ensures eigenvalues with higher algebraic multiplicities appear before eigenvalues with lower algebraic multiplicities. Let $\underline{\mathbf{C}} = (C_1, \dots, C_k)$ be a set of closed conjugacy classes and let $\Xi_{\underline{\mathbf{C}}} = \Xi_{C_1} \times \cdots \times \Xi_{C_k}$.

Lemma 4.4.6. $\underline{\omega} \in \Xi_{\underline{\mathbf{C}}}$ if and only if $\alpha^{\underline{\omega}}$ is arm-fundamental.

Proof. Use Lemmas 4.2.6 and 4.4.4. □

Recall the definition of the invariant polynomials $\iota_{C,1} \mid \cdots \mid \iota_{C,n}$ for a conjugacy class C.

Definition 4.4.7. Let $C \subseteq \mathrm{End}(V)$ be a closed conjugacy class. For each $i = 1, \ldots, k$ define the *multiplicity factorization of* C by

$$g_j(x) = \frac{\iota_{C,n-j+1}(x)}{\iota_{C,n-j}(x)}$$

for $j = 1, \ldots, n-1$ and let $g_n(x) = \iota_{C,1}(x)$. As $\iota_{C,p}$ are monic so are the g_j. Note that many of the g_j may be trivial (i.e. $g_j(x) = 1$). As $\iota_{C,1} \mid \cdots \mid \iota_{C,n}$ we have $g_1, \ldots, g_n \in K[x]$.

The next lemma explains the name multiplicity factorization.

Lemma 4.4.8. Let $C \subseteq \mathrm{End}(V)$ be a closed conjugacy class with multiplicity factorization $g_1, \ldots, g_n$. The roots of g_j are precisely the eigenvalues of C which have multiplicity j.

Proof. By definition the roots of g_j are the roots of $\iota_{C,n-j+1}(x)$ which are not also roots of $\iota_{C,n-j}(x)$. From the properties of invariant polynomials given in Section 2.1.1 we see that g_j has for its roots the eigenvalues of C which have geometric (and algebraic) multiplicity less than $j+1$ but not less than $n - n + j = j$. As each invariant polynomial has distinct roots the result follows. □

For each $i = 1, \ldots, k$, let $g_{i,1}(x), \ldots, g_{i,n}(x)$ be the multiplicity factorization of C_i. and let $d_{i,j} = \deg(g_{i,j})$.

Remark 4.4.9. Given the invariant polynomials of a conjugacy class C_i the multiplicity factorization $g_{i,1}, \ldots, g_{i,n}$ can be computed without polynomial factorization. We have $g_{i,n}(x) = \iota_{C_i,1}(x)$ by definition and for $j = 1, \ldots, n-1$ we expand the equation $\iota_{C_i,n-j}(x) g_{i,j}(x) = \iota_{C_i,n-j+1}(x)$ and obtain the coefficients of $g_{i,j}(x)$ by solving a linear system. The fact that $\iota_{C_i,n-j}(x)$ divides $\iota_{C_i,n-j+1}(x)$ ensures the system has a unique solution.

Let $\underline{\omega} \in \Sigma_{\underline{\mathbf{C}}}$, by Lemma 4.4.8, the degree $d_{i,j}$ of $g_{i,j}$ is the number of eigenvalues of C_i which have algebraic multiplicity j. By Lemma 4.4.4 the $d_{i,j}$'s give us the number of gradients $\nabla_{i,j}(\alpha^{\underline{\omega}})$ equal to j on the ith arm of $\alpha^{\underline{\omega}}$. To ensure $\alpha^{\underline{\omega}}$ is arm-fundamental (and so $\underline{\omega} \in \Xi_{\underline{\mathbf{C}}}$) we construct each arm of $\alpha^{\underline{\omega}}$ such that it has the appropriate number of gradients of each size and such that the gradients are arranged in descending order. Although we do not give an algorithm to construct arm-fundamental $\alpha^{\underline{\omega}}$ here, it is clear that such an algorithm exists and relies only on the degrees of the multiplicity factorizations of $C_1, \ldots, C_k$.

4.4.3 The Scalar Product $\lambda^{\underline{\omega}} \cdot \beta$ where $\underline{\omega}$ is Multiplicity Preserving

Let β be a dimension vector. As previously noted we can write the scalar product $\lambda^{\underline{\omega}} \cdot \beta$ as an integral combination of the roots of the minimal polynomials. Let $z_1, \dots, z_l$ be indeterminates and let $(\beta_1, \dots, \beta_l)$ be a decomposition of α. It is easy to see that we can also write $\lambda^{\underline{\omega}} \cdot (\beta_1 z_1 + \cdots \beta_l z_l)$ as an integral combination of the $z_1, \dots, z_l$ and the roots of the minimal polynomials and that these integer coeficients depend only on the $\beta_1, \dots, \beta_l$.

The following lemma shows, for general orderings $\underline{\omega}$ of the distinct eigenvalues, how to write $\lambda^{\underline{\omega}} \cdot \beta$ in terms the values of $\underline{\omega}$ and the gradients of β.

Lemma 4.4.10. Let β be a dimension vector of Q and let $\underline{\omega} \in \Sigma_{\underline{C}}$, we have

$$\lambda^{\underline{\omega}} \cdot \beta = -\sum_{i=1}^{k} \sum_{j=1}^{d_i} \omega_i(j) \nabla_{i,j}(\beta). \tag{4.8}$$

Proof.

$$\begin{aligned}
\lambda^{\underline{\omega}} \cdot \beta &= \lambda_0^{\underline{\omega}} \beta_0 + \sum_{i=1}^{k} \sum_{j=1}^{d_i - 1} \lambda_{i,j}^{\underline{\omega}} \beta_{i,j} \\
&= -\sum_{i=1}^{k} \Big(\beta_0 \omega_i(1) - \sum_{j=1}^{d_i-1} \beta_{i,j} \omega_i(j) + \sum_{j=2}^{d_i} \beta_{i,j-1} \omega_i(j) \Big) \\
&= -\sum_{i=1}^{k} \Big(\beta_0 \omega_i(1) - \sum_{j=2}^{d_i-1} \beta_{i,j} \omega_i(j) + \sum_{j=2}^{d_i-1} \beta_{i,j-1} \omega_i(j) - \beta_{i,1}\omega_i(1) + \beta_{i,d_i-1}\omega_i(d_i) \Big) \\
&= -\sum_{i=1}^{k} \Big(\omega_i(1)(\beta_0 - \beta_{i,1}) + \sum_{j=2}^{d_i-1} \omega_i(j)(\beta_{i,j-1} - \beta_{i,j}) + \beta_{i,d_i-1}\omega_i(d_i) \Big) \\
&= -\sum_{i=1}^{k} \Big(\sum_{j=1}^{d_i-1} \omega_i(j)(\beta_{i,j-1} - \beta_{i,j}) + \omega_i(d_i)\beta_{i,d_i-1} \Big) = -\sum_{i=1}^{k} \sum_{j=1}^{d_i} \omega_i(j) \nabla_{i,j}(\beta).
\end{aligned}$$

□

Remark 4.4.11. Multiplicity preserving orderings ensure eigenvalues with the same multiplicity are arranged in contiguous blocks. Let C be a closed conjugacy class with multiplicity factorization $g_1, \dots, g_n$. Let $\xi_{j,1}, \dots, \xi_{j,d_j}$ be the roots of g_j for $j = 1, \dots, n$ (collectively these are also the roots of $\min_C$). A multiplicity preserving ordering $\omega \in \Xi_C$ arranges the roots of $\min_C$ in blocks as below

$$\overbrace{\xi_{n,1}, \dots, \xi_{n,d_n}}^{\text{roots of } g_n}, \dots, \overbrace{\xi_{j,1}, \dots, \xi_{j,d_j}}^{\text{roots of } g_j}, \overbrace{\xi_{j-1,1}, \dots, \xi_{j-1,d_{j-1}}}^{\text{roots of } g_{j-1}}, \dots, \overbrace{\xi_{2,1}, \dots, \xi_{2,d_2}}^{\text{roots of } g_2}, \overbrace{\xi_{1,1}, \dots, \xi_{1,d_1}}^{\text{roots of } g_1} \tag{4.9}$$

Different multiplicity preserving orderings permute the roots within the contiguous blocks.

We can identify $\omega \in \Xi_C$ with a unique n-tuple of orderings $\underline{\rho} = (\rho_1, \ldots, \rho_n)$ where ρ_j is an ordering of the roots of g_j, that is there is a bijection from $\prod_{j=1}^n \Sigma_{g_j}$ to Ξ_C. Although we do not give an explicit construction of the bijection here, it is straight-forward to see how it can be constructed using the degrees of the multiplicity factorizations.

Remark 4.4.12. For $i = 1, \ldots, k$ let F_i be the bijection from $\prod_{j=1}^n \Sigma_{g_{i,j}}$ to Ξ_{C_i}. Let $H = \prod_{i=1}^k \prod_{j=1}^n \Sigma_{g_{i,j}}$. There is a bijection $\mathcal{F}$ from H to $\Xi_{\underline{C}}$ given by $\mathcal{F}(\underline{\rho}) = (F_1(\underline{\rho}^{(1)}), \ldots, F_k(\underline{\rho}^{(k)}))$, where $\underline{\rho} = (\underline{\rho}^{(1)}, \ldots, \underline{\rho}^{(k)}) \in H$, that is $\underline{\rho}^{(i)} = (\rho_{i,1}, \ldots, \rho_{i,n})$ where $\rho_{i,j} \in \Sigma_{g_{i,j}}$ for $i = 1, \ldots, k$ and $j = 1, \ldots, n$.

The following lemma shows how to write $\lambda^{\underline{\omega}} \cdot \beta$ in terms of the roots of the multiplicity factorizations when the ordering $\underline{\omega}$ is multiplicity preserving. Let $R_i(j) = \sum_{l=j+1}^n d_{i,l}$, note that $R_i(j)+1$ is the position of the first root of multiplicity j under a multiplicity preserving ordering and $R_i(j) + d_{i,j}$ is the position of the last.

Lemma 4.4.13. Let β be a dimension vector of Q. For some $\underline{\omega} \in \Xi_{\underline{C}}$ we have

$$\lambda^{\underline{\omega}} \cdot \beta = -\sum_{i=1}^{k} \sum_{j=1}^{n} \sum_{p=1}^{d_{i,j}} \nabla_{i,R_i(j)+p}(\beta) \mathcal{F}^{-1}(\underline{\omega})_{i,j}(p),$$

Proof. By Lemma 4.4.10 we have $\lambda^{\underline{\omega}} \cdot \beta = -\sum_{i=1}^k \sum_{j=1}^{d_i} \omega_i(j) \nabla_{i,j}(\beta)$ and by Remarks 4.4.12 $\underline{\omega}$ is uniquely identified with a tuple $\mathcal{F}^{-1}(\underline{\omega}) \in H$ and $\underline{\omega}$ assigns the indices $R_i(j)+1, \ldots, R_i(j)+d_{i,j}$ to the roots of $g_{i,j}$, therefore

$$\lambda^{\underline{\omega}} \cdot \beta = -\sum_{i=1}^{k} \sum_{j=1}^{n} \sum_{p=1}^{d_{i,j}} \underline{\omega}_i(R_i(j) + p) \nabla_{i,R_i(j)+p}(\beta)$$

We use the fact that $\mathcal{F}\colon H \to \Xi_{\underline{C}}$ is a bijection to get

$$\lambda^{\underline{\omega}} \cdot \beta = -\sum_{i=1}^{k} \sum_{j=1}^{n} \sum_{p=1}^{d_{i,j}} \mathcal{F}^{-1}(\underline{\omega})_{i,j}(p) \nabla_{i,R_i(j)+p}(\beta).$$

□

Remark 4.4.14. In Lemma 4.4.13 we are given $\underline{\omega} \in \Xi_{\underline{C}}$, if instead we are given $\underline{\rho} \in H$ we can rewrite the statement of the lemma as

$$\lambda^{\mathcal{F}(\underline{\rho})} \cdot \beta = -\sum_{i=1}^{k} \sum_{j=1}^{n} \sum_{p=1}^{d_{i,j}} \nabla_{i,R_i(j)+p}(\beta) \rho_{i,j}(p).$$

4.4.4 Main Theorem

Recall the Kronecker sum and root sum operations on polynomials introduced in Chapter 3. Let $\underline{\beta} = (\beta_1, \ldots, \beta_l)$ be a root decomposition of $\alpha^{\underline{\omega}}$ where $\underline{\omega} \in \Xi_{\underline{\mathbf{C}}}$ (recall that $\alpha^{\underline{\omega}}$ is arm-fundamental and can be computed as in Remark 4.4.9). Let $\underline{\mathbf{z}} = (z_1, \ldots, z_l)$ be an l-tuple of indeterminates. We show how the operations defined in Chapter 3, applied to the multiplicity factorization, are used to construct a polynomial $\mathcal{S}_{\underline{\beta}}(z_1, \ldots, z_l)$ which is precisely equal to (4.7). For $i = 1, \ldots, k$, $j = 1, \ldots, n$ and $p = 1, \ldots, d_{i,j}$, let

$$D_{i,j,p}(\underline{\mathbf{z}}) = \sum_{q=1}^{l} \nabla_{i,R_i(j)+p}(\beta_q) z_q.$$

Definition 4.4.15. We define the *matrix sum polynomial* by

$$\mathcal{S}_{\underline{\beta}}(z_1, \ldots, z_n) = \bigoplus_{i,j=1}^{k,n}{}^{\mathcal{K}} \mathcal{R}_{g_{i,j}}(x, D_{i,j,1}(\underline{\mathbf{z}}), \ldots, D_{i,j,d_{i,j}}(\underline{\mathbf{z}}))\Big|_{x=0}.$$

Note that the above is constructed by a finite number of applications of the root sum and Kronecker sum operations (all with respect to x) and variable substitution. The following theorem along with Remark 4.4.14 shows the matrix sum polynomial is equivalent to (4.7). Recall that $H = \prod_{i=1}^{k} \prod_{j=1}^{n} \Sigma_{g_{i,j}}$.

Theorem 4.4.16. We have

$$\bigoplus_{i,j=1}^{k,n}{}^{\mathcal{K}} \mathcal{R}_{g_{i,j}}(x, D_{i,j,1}(\underline{\mathbf{z}}), \ldots, D_{i,j,d_{i,j}}(\underline{\mathbf{z}})) = \prod_{\underline{\rho} \in H} (x - \sum_{i=1}^{k} \sum_{j=1}^{n} \sum_{p=1}^{d_{i,j}} \rho_{i,j}(p) D_{i,j,p}(\underline{\mathbf{z}}))$$

Proof. For each $i = 1, \ldots, k$ and $j = 1, \ldots, n$ let $(y_{\mathrm{i,j},1}, \ldots, y_{\mathrm{i,j},d_{i,j}})$ be a family of variables, where $d_{i,j} = \deg(g_{i,j})$. For each polynomial of the multiplicity factorization $g_{i,1}, \ldots, g_{i,n}$ of C_i we compute the root sum, by Theorem 3.2.11 the root sum is:

$$\mathcal{R}_{g_{i,j}}(x, y_{\mathrm{i,j},1}, \ldots, y_{\mathrm{i,j},d_{i,j}}) = \prod_{\rho \in \Sigma_{g_{i,j}}} (x - \sum_{p=1}^{d_{i,j}} y_{i,j,p} \rho(p))$$

By repeated use of the Kronecker sum (with respect to x) we "glue together" each of the root sums, by Theorem 3.1.3 this expression is:

$$\bigoplus_{i,j=1}^{k,n}{}^{\mathcal{K}} \mathcal{R}_{g_{i,j}}(x, y_{\mathrm{i,j},1}, \ldots, y_{\mathrm{i,j},d_{i,j}}) = \prod_{\underline{\rho} \in H} (x - \sum_{i=1}^{k} \sum_{j=1}^{n} \sum_{p=1}^{d_{i,j}} y_{i,j,p} \rho_{i,j}(p)).$$

Note that the order we do the "gluing" does not matter as the Kronecker sum is commutative and associative. For each i, j, p we substitute $D_{i,j,p}(\underline{\mathbf{z}})$ in to $y_{i,j,p}$. $\square$

Recall the bijection $\mathcal{F}\colon H \to \Xi_{\underline{\mathbf{C}}}$ from Section 4.4.3.

Corollary 4.4.17. Recall (4.7), we have $M_{\underline{\beta}}(z_1, \ldots, z_n) = 0$ if and only if $\mathcal{S}_{\underline{\beta}}(z_1, \ldots, z_n) = 0$.

Proof. By Remark 4.4.14 we have, for $\underline{\rho} \in H$,

$$\sum_{i=1}^{k}\sum_{j=1}^{n}\sum_{p=1}^{d_{i,j}} D_{i,j,p}(\underline{\mathbf{z}})\rho_{i,j}(p) = \sum_{i=1}^{k}\sum_{j=1}^{n}\sum_{p=1}^{d_{i,j}}\sum_{q=1}^{l} \nabla_{i,R_i(j)+p}(\beta_q)\rho_{i,j}(p)z_q = \sum_{q=1}^{l} \lambda^{\mathcal{F}(\underline{\rho})} \cdot \beta_q z_q,$$

so by Theorem 4.4.16 $\mathcal{S}_{\underline{\beta}}(z_1, \ldots, z_n) = 0$ if and only if $\sum_{q=1}^{l} \lambda^{\mathcal{F}(\underline{\rho})} \cdot \beta_q z_q = 0$ for some $\underline{\rho} \in H$. As $\mathcal{F}$ is a bijection this is equivalent to $\sum_{q=1}^{l} \lambda^{\underline{\omega}} \cdot \beta_q z_q = 0$ for some $\underline{\omega} \in \Xi_{\underline{\mathbf{C}}}$, which, by (4.7), occurs if and only if $M_{\underline{\beta}}(z_1, \ldots, z_n) = 0$. □

Theorem 4.4.18. There exists $A_i \in C_i$ for $i = 1, \ldots, k$ such that $A_1 + \cdots + A_k = 0$ if and only there exists some root decomposition $\underline{\beta} = (\beta_1, \ldots, \beta_l)$ of α such that $\mathcal{S}_{\underline{\beta}}(z_1, \ldots, z_l) = 0$.

Proof. By Corollary 4.4.17 and the remarks in the introduction. □

The results in this section show that it is possible to solve the additive matrix problem where the conjugacy classes are closed in such a way that does not require knowing the eigenvalues exactly, as long as we know the invariant polynomials. These results only apply in the case where the conjugacy classes are closed, if we work with more general conjugacy classes then we cannot use multiplicity preserving orderings in order to make the dimension vector arm-fundamental. Due to time restrictions it was not possible to study this case.

Chapter 5

Symmetric Quivers and Symmetric Representations

In Section 5.1 we define the concepts of a symmetric quiver and of a symmetric representation of a symmetric quiver. These definitions appear in the literature (see for example [Shm06]). We extend these concepts and define the symmetric representations of a deformed preprojective algebra. In Section 5.2 we extend the concept of the reflection functor, defined in Section 1.2.2 for representations of deformed preprojective algebras, to symmetric representations. In Section 5.3 we describe the relationships between various categories of symmetric representations and categories of symmetric solutions to the additive matrix problem. This is analogous to the result in Section 4.1.2. We use the results of this chapter to show that every rigid irreducible solution to the additive matrix problem is conjugate to a symmetric solution.

Let K be an algebraically closed field of characteristic zero.

5.1 Symmetric Representations of Deformed Preprojective Algebras

The concept of a symmetric quiver has been studied by several authors in the literature, see for instance [DW02] or [Shm06]. We use the definitions given in [Shm06], though we use them in a far less general setting. Symmetric quivers are quivers along with an involution, defined on the vertices and arrows, which defines the symmetry. Symmetric representations are isomorphic to their "dual representations" via particular isomorphisms called symmetrizations (they are called signed-forms in [Shm06]). We show in Section 5.1.2 that for certain symmetric quivers, having such an isomorphism on a representation is equivalent to defining a nondegenerate symmetric bilinear form on each vector space such that the involution on arrows pairs each linear map with its adjoint.

5.1.1 Symmetric Quivers and Symmetric Representations

Definition 5.1.1. Let Q be a quiver. Let $\iota\colon Q_0 \cup Q_1 \to Q_0 \cup Q_1$ be a bijective map such that $\iota(Q_0) = Q_0$ and $\iota(Q_1) = Q_1$, $\iota^2 = \mathrm{id}$ and $t(\iota(a)) = \iota(h(a))$ and $h(\iota(a)) = \iota(t(a))$ for all $a \in Q_1$. We say (Q, ι) is a *symmetric quiver*.

Definition 5.1.2. Let (Q, ι) be a symmetric quiver. A dimension vector $\alpha \in \mathbb{Z}^{Q_0}$ is *symmetric* if $\alpha_i = \alpha_{\iota(i)}$.

From hereon every α is assumed to be symmetric unless otherwise specified. We are almost ready to introduce the definition of a representation of a symmetric quiver and the dual representation, but first we recall some facts from linear algebra.

Remark 5.1.3. Let V be a K-vector space, we recall the defintion of the *dual space* V^* of V. The dual space is the K-vector space defined by $V^* = \{f\colon V \to K$ where f is a linear functional$\}$. Let V, W be finite dimensional vector spaces and let $\theta\colon V \to W$ be a linear map. Every such map gives rise to a linear map $\theta^*\colon W^* \to V^*$ called the *dual of* θ given by $\theta^*(f)(v) = f(\theta(v))$ for all $f \in W^*$, $v \in V$.[1] Taking the dual of linear maps gives an isomorphism: $*\colon \hom(V, W) \to \hom(W^*, V^*)$. The map is injective as if $\theta^*(f)(v) = 0$ for all $v \in V$ and $f \in W^*$, then $f(\theta(v)) = 0$ for all $v \in V$ and $f \in W^*$. This implies $\theta(v) = 0$ for all $v \in V$ so $\theta = 0$, and as $\dim(\hom(W^*, V^*)) = \dim(\hom(V, W))$ the map is surjective.

Now we take $W = V^*$ so that $\theta^*\colon V^{**} \to V^*$. The double dual V^{**} is $\{g\colon V^* \to K\colon g$ is a linear functional$\}$. If V is finite dimensional, then V^{**} is naturally isomorphic to V. The isomorphism $\tau_V\colon V \to V^{**}$ is given by $\tau_V(v)(f) = f(v)$ for all $f \in V^*$ and $v \in V$.

We show this is a natural isomorphism: τ_V is injective as if $\tau_V(v) = 0$, then $\tau_V(v)(f) = f(v) = 0$ for all $f \in V^*$, which implies $v = 0$. We have $\dim(V) = \dim(V^*) = \dim(V^{**})$ which implies τ_V is surjective also. We show τ_V is natural, let $g \in \hom(V, U)$ for some finite

[1] Throughout Section 5.1 we use the superscript $*$ notation to denote either the dual space or the dual map. That we have previously used this notation for adjoints of linear maps is intentional as nondegenerate symmetric bilinear forms allow us to identify adjoint maps with dual maps.

dimensional vector space U. Taking the dual gives an isomorphism from $\hom(V,U)$ to $\hom(V^{**},U^{**})$ via $\theta \mapsto \theta^{**}$.

$$\begin{tikzcd} V^{**} \arrow[r, "\theta^{**}"] & U^{**} \\ V \arrow[u, "\tau_V"] \arrow[r, "\theta"] & U \arrow[u, "\tau_U"'] \end{tikzcd}$$

We want to show $\tau_U(\theta(v)) = \theta^{**}(\tau_V(v))$ for all $v \in V$. We have $\tau_U(\theta(v)) \in U^{**}$ so for all $f \in U^*$ and $v \in V$ we have $\tau_U(\theta(v))(f) = f(\theta(v)) = \theta^*(f)(v) = \tau_V(v)(\theta^*(f)) = \theta^{**}(\tau_V(v))(f)$.

So as V is naturally isomorphic to V^{**} we can define equality between the maps $\theta\colon V \to V^*$ and $\theta^*\colon V^{**} \to V^*$, from hereon we write $\theta = \theta^*$ to mean $\theta(v) = \theta^*(\tau_V(v))$ for all $v \in V$.

Definition 5.1.4. Let (Q,ι) be a symmetric quiver. A *representation* V of (Q,ι) is just a representation of Q. We define the *dual* representation V^* of V to be the representation of (Q,ι) given by: $(V^*)_i = (V_{\iota(i)})^*$ for each $i \in Q_0$ and $(V^*)_a = (V_{\iota(a)})^*$ for each $a \in Q_1$.

Definition 5.1.5. Let (Q,ι) be a symmetric quiver. Let V be a representation of (Q,ι). A *symmetrization* on V is a collection of isomorphisms $(J_i\colon V_i \to V_i^*)_{i\in Q_0}$ such that $J_{\iota(i)} = J_i^*$.

Definition 5.1.6. Let (Q,ι) be a symmetric quiver. Given a representation V of (Q,ι) and a symmetrization $(J_i)_{i\in Q_0}$ on V we say V is *symmetric with respect to J* if J is an isomorphism between V and V^*.[2]

Definition 5.1.7. Let (Q,ι) be a symmetric quiver. Let V be a representation of (Q,ι). We say V is *symmetrizable* if there exists a symmetrization J on V such that that V is symmetric with respect to J.

Definition 5.1.8. Let (Q,ι) be a symmetric quiver. Let V,W be representations of (Q,ι), and J^V, J^W be symmetrizations such that V and W are symmetric with respect to J^V and J^W respectively. We define a morphism of symmetric representations $f\colon V \to W$ to be a morphism of representations such that $J_i^V(v)(v') = J_i^W(f_i(v))(f_{\iota(i)}(v'))$ for all $v \in V_i, v' \in V_{\iota(i)}$ and $i \in Q_0$.

Let (Q,ι) be a symmetric quiver. Let $\mathrm{Rep}_K^\Sigma(Q,\iota)$ be the *category of symmetric representations* of (Q,ι), that is the category whose objects are pairs (V,J) where V is a representation of Q and J is a symmetrization on V such that V is symmetric with respect to V. The morphisms are the morphisms of symmetric representations.

[2]That is J an isomorphism of representations rather than just a collection of isomorphisms of vector spaces.

5.1.2 The Symmetric Double of a Quiver

Let Q be a quiver without loops, we form a symmetric quiver $(\overline{Q}, \iota)$ as follows: $\overline{Q}$ is the doubled quiver (defined in Section 1.2), let $\iota(i) = i$ for all $i \in Q_0$ and let $\iota(a) = a^*$ and $\iota(a^*) = a$ for all $a \in Q_1$. We call $(\overline{Q}, \iota)$ the *symmetric double* of Q.

We write $\mathrm{Rep}_K^{\Sigma}(\Pi^{\lambda}(Q))$ for the full subcategory of $\mathrm{Rep}_K^{\Sigma}(\overline{Q}, \iota)$, consisting of symmetric representations of $(\overline{Q}, \iota)$ which satisfy

$$\sum_{a \in Q_1 \colon h(a)=i} V_a V_{a^*} - \sum_{a \in Q_1 \colon t(a)=i} V_{a^*} V_a = \lambda_i 1_{V_i}$$

for each $i \in Q_0$.

Given an isomorphism from a vector space V to its dual we can identify V with V^*. Let Q be a quiver without loops and $(\overline{Q}, \iota)$ its symmetric double. Let V be a representation of $\overline{Q}$. A symmetrization J on V identifies each vector space V_i with its dual, this is equivalent to assigning a nondegenerate symmetric bilinear form on each vector space given by $(x, y)_i = J_i(x)(y)$. We show that the defining properties of a symmetrization are equivalent to the bilinear form being nondegenerate and symmetric and that V being symmetric with respect to J is equivalent to V_a and V_{a^*} being adjoints of one another for each $a \in Q_1$, with respect to the appropriate bilinear forms.

Theorem 5.1.9. Let V be a representation of $(\overline{Q}, \iota)$. Let J be an assignment of linear maps $J_i \colon V_i \to V_i^*$ to each $i \in \overline{Q}_0$ and let $(x, y)_i = J_i(x)(y)$ for $x, y \in V_i$.

- For $i \in \overline{Q}_0$: J_i is an isomorphisms if and only if $(\cdot, \cdot)_i$ is nondegenerate and $J_i = J_i^*$ if and only if $(\cdot, \cdot)_i$ is symmetric.

- If the above two properties hold (i.e. J is a symmetrization), then V is symmetric with respect to J if and only if for each $a \in \overline{Q}_1$ we have $(V_a(x), y)_{h(a)} = (x, V_{\iota(a)}(y))_{t(a)}$ for all $x \in V_{t(a)}$, $y \in V_{h(a)}$.

Proof. Let $i \in \overline{Q}_0$. Let $x \in V_i$. Suppose $J_i \colon V_i \to V_i^*$ is an isomorphism we show $(\cdot, \cdot)_i$ is nondegenerate. Suppose $(x, y)_i = 0$ for all $y \in V_i$, that is $J_i(x)(y) = 0$, this implies $J_i(x) = 0$. As J_i is an isomorphism we have $x = 0$ so $(\cdot, \cdot)_i$ is nondegenerate. Now suppose $(\cdot, \cdot)_i$ is nondegenerate we show J_i is an isomorphism. If $J_i(x) = 0$ for some $x \in V_i$, then $(x, y)_i = 0$ for all $y \in V_i$. By nondegeneracy this implies $x = 0$ so J_i is injective, and as $\dim(V) = \dim(V^*)$ we have that J_i is an isomorphism.

Suppose $J_i = J_i^*$ we show $(\cdot,\cdot)_i$ is symmetric. Recall the definition of the natural isomorphism $\tau_{V_i}\colon V_i \to V_i^{**}$, given by $\tau_{V_i}(v)(f) = f(v)$, in Remark 5.1.3. Let $x, y \in V_i$, we have

$$(x,y)_i = J_i(x)(y) = J_i^*(\tau_{V_i}(x))(y) = \tau_{V_i}(x)(J_i(y)) = J_i(y)(x) = (y,x)_i.$$

Now suppose $(\cdot,\cdot)_i$ is symmetric we show $J_i = J_i^*$. Let $x, y \in V_i$, we have

$$J_i(x)(y) = (x,y)_i = (y,x)_i = J_i(y)(x) = \tau_{V_i}(x)(J_i(y)) = J_i^*(\tau_{V_i}(x))(y)$$

So $J_i = J_i^*$ in the sense of Remark 5.1.3, as required.

Now suppose J is a symmetrization on V. Let $a \in \overline{Q}_1$. Let V be symmetric with respect to J. We show V_a is the adjoint of $V_{\iota(a)}$. As V is symmetric we have $J_{h(a)}V_a = V^*_{\iota(a)}J_{t(a)}$ for each $a \in \overline{Q}_1$. Let $x \in V_{t(a)}$ and $y \in V_{h(a)}$ we have

$$\begin{aligned}
(V_a(x), y)_{h(a)} &= J_{h(a)}(V_a(x))(y) \\
&= (J_{h(a)}V_a)(x)(y) \\
&= (V^*_{\iota(a)}J_{t(a)})(x)(y) \\
&= V^*_{\iota(a)}(J_{t(a)}(x))(y) \\
&= J_{t(a)}(x)(V_{\iota(a)}(y)) \\
&= (x, V_{\iota(a)}(y))_{t(a)}.
\end{aligned}$$

Now suppose for each $a \in \overline{Q}_1$ we have $(V_a(x), y)_{h(a)} = (x, V_{\iota(a)}(y))_{t(a)}$ for all $x \in V_{t(a)}$, $y \in V_{h(a)}$ we show this implies $(J_i)_{i\in Q_0}$ is an isomorphism from V to V^*. Let $x \in V_{t(a)}$ and $y \in V_{h(a)}$, we have

$$\begin{aligned}
(J_{h(a)}V_a)(x)(y) &= (V_a(x), y)_{h(a)} \\
&= (x, V_{\iota(a)}(y))_{t(a)} \\
&= J_{t(a)}(x)(V_{\iota(a)}(y)) \\
&= V^*_{\iota(a)}(J_{t(a)}(x))(y) \\
&= (V^*_{\iota(a)}J_{t(a)})(x)(y).
\end{aligned}$$

So $V^*_{\iota(a)}J_{t(a)} = J_{h(a)}V_a$ for each $a \in Q_1$, which shows the J satisfies the intertwining relations. So J is an isomorphism from V to V^*. Therefore V is symmetric with respect to J. □

So given a pair (V, J) of the category $\mathrm{Rep}_K^{\Sigma}(\overline{Q}, \iota)$, we can define a nondegenerate symmetric bilinear form on each V_i such that the V_a and V_{a^*} are adjoints of each other with respect to the appropriate bilinear forms. Given such an assignment of bilinear forms it is clear by Theorem 5.1.9 that these define a symmetrization on V such that V is symmetric. Furthermore given pairs (V, J^V) and (W, J^W) of $\mathrm{Rep}_K^{\Sigma}(\overline{Q}, \iota)$, it is clear that a morphism of representations $f\colon V \to W$ is a morphism of symmetric represenations if and only if $(f(x), f(y))_i^W = (x, y)_i^V$ for all $x, y \in V_i$ where $(\cdot,\cdot)_i^V = J_i^V(x)(y)$ and $(\cdot,\cdot)_i^W = J_i^W(x)(y)$ for all $i \in \overline{Q}_0$.

5.2 Symmetric Reflection Functors

We show here that the reflection functor introduced in Section 1.2, which establishes equivalences between certain categories of representations of deformed preprojective algebras, can be extended to establish equivalences between certain categories of symmetric representations.

Let Q be a quiver and $(\overline{Q}, \iota)$ its symmetric double, without loss of generality we can assume there exists a vertex $i \in Q_0$ such that no arrow in Q_1 has its tail at i. Let $\lambda \in K^{Q_0}$ and let $\alpha \in \mathbb{Z}^{Q_0}$ be a positive dimension vector. Let V be a symmetric representation of $\Pi^\lambda(Q)$ of dimension vector α with respect to a symmetrization, recall from Theorem 5.1.9 that fixing the symmetrization is equivalent to fixing a nondegenerate symmetric bilinear form at each vertex. After giving some preliminary results in Section 5.2.1 we give the target objects of the reflection functor in Section 5.2.2 and the target morphisms in Section 5.2.3. We prove the reflection functor is an equivalence of categories in Section 5.2.4. The methodology in this section is adapted from the results on reflection functors of categories of representations found in [CBH98, Sec. 5] and [CB01, Sec. 2].

5.2.1 Preliminary Results and Notation

Let us fix $i \in Q_0$ such that no arrow $a \in Q_1$ has $t(a) = i$. Let us suppose that $\lambda_i \neq 0$, that is we assume the reflection at i is admissible for the pair (λ, α) (note that our assumption that no arrow has a tail at i implies i is loop-free). Let $H_i = \{a_1, \dots, a_k\}$ be the set of arrows which have i as their head, and let us write $t_j = t(a_j)$. Let $V_\oplus = \bigoplus_{j=1}^k V_{t_j}$. For each $j = 1, \dots, k$ let $\mu_j\colon V_{t_j} \hookrightarrow V_\oplus$ and $\pi_j\colon V_\oplus \twoheadrightarrow V_{t_j}$ be the standard inclusions and projections respectively. So we have the following vector spaces and maps

$$
\begin{tikzcd}
 & V_{t_1} \arrow[dl, bend left=10, "V_{a_1}"] \arrow[dr, hook, bend left=10, "\mu_1"] \arrow[d, dotted, no head] & \\
V_i \arrow[ur, bend left=20, "V_{a_1^*}"] \arrow[dr, bend right=10, "V_{a_k^*}"'] & \, & V_\oplus \arrow[ul, two heads, bend left=10, "\pi_1"] \arrow[dl, two heads, bend left=20, "\pi_k"] \\
 & V_{t_k} \arrow[ul, bend left=20, "V_{a_k}"] \arrow[ur, hook, bend left=10, "\mu_k"] \arrow[u, dotted, no head] &
\end{tikzcd}
$$

The space $V_\oplus$ is equipt with the symmetric nondegenerate bilinear form

$$(x, y)_\oplus = \sum_{j=1}^{k} (\pi_j(x), \pi_j(y))_{t_j}.$$

Definition 5.2.1. Let $\mu\colon V_i \to V_\oplus$ be defined by $\mu = \sum_{j=1}^{k} \mu_j V_{a_j^*}$ and $\pi\colon V_\oplus \to V_i$ by $\pi = \frac{1}{\lambda_i} \sum_{j=1}^{k} V_{a_j} \pi_j$.

Lemma 5.2.2. We have that π and $\frac{1}{\lambda_i}\mu$ are adjoints.

Proof. We compute the adjoint of μ. Let $x \in V_i, y \in V_\oplus$, we have

$$
\begin{aligned}
(\mu x, y)_\oplus &= \sum_{j=1}^{k} (\mu_j V_{a_j^*} x, y)_\oplus = \sum_{j=1}^{k} \sum_{r=1}^{k} (\pi_r \mu_j V_{a_j^*} x, \pi_r y)_{t_r} \\
&= \sum_{j=1}^{k} (V_{a_j^*} x, \pi_j y)_{t_j} \quad \text{(as } \pi_r \mu_j = 1 \text{ if } r = j \text{ and zero otherwise)} \\
&= \sum_{j=1}^{k} (x, V_{a_j} \pi_j y)_i = (x, \sum_{j=1}^{k} V_{a_j} \pi_j y)_i
\end{aligned}
$$

So the adjoint of μ is $\sum_{j=1}^{k} V_{a_j} \pi_j$ hence π and $\frac{1}{\lambda_i}\mu$ are adjoints. □

Lemma 5.2.3. The composition $\pi\mu\colon V_i \to V_i$ is $\pi\mu = 1_{V_i}$.

Proof. Follows from the deformed preprojective relations. □

So $\mu\pi\colon V_\oplus \to V_\oplus$ is an idempotent endomorphism of $V_\oplus$, as $(\mu\pi)(\mu\pi) = \mu(\pi\mu)\pi = \mu\pi$.

Lemma 5.2.4. The map $(1_{V_\oplus} - \mu\pi)\colon V_\oplus \to V_\oplus$ has $\operatorname{im}(1_{V_\oplus} - \mu\pi) \subseteq \ker(\pi)$.

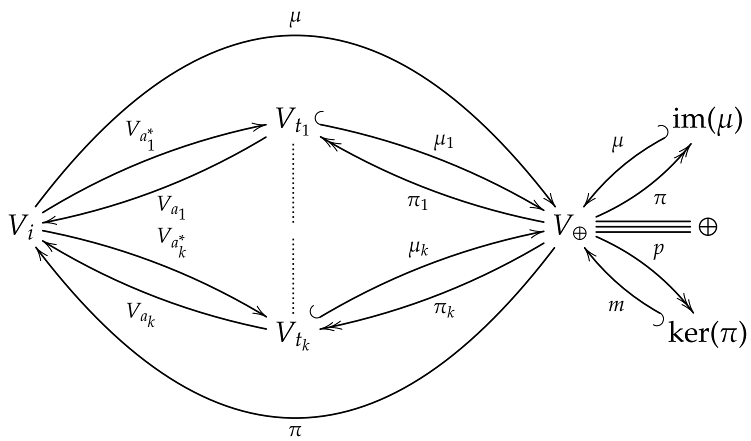
Proof. Apply π to it. $\pi(1_{V_\oplus} - \mu\pi) = \pi - \pi\mu\pi = \pi - 1_{V_\oplus}\pi = 0$ by Lemma 5.2.3. □

Lemma 5.2.5. We can write $V_\oplus$ as $\operatorname{im}(\mu) \oplus \ker(\pi)$.

Proof. We first show $\operatorname{im}(\mu) \cap \ker(\pi) = 0$. If $x \in \operatorname{im}(\mu) \cap \ker(\pi)$, then $\pi x = 0$. There exists some $y \in V_i$ such that $\mu y = x$, so $\pi\mu y = 0$ but $\pi\mu = 1_{V_i}$ so $y = 0$, hence $x = 0$.

Now we show $V_\oplus = \mathrm{im}(\mu) + \ker(\pi)$. Suppose $x \in V_\oplus$, let $y = \mu\pi x$, which is clearly in $\mathrm{im}(\mu)$, and let $z = (1_{V_\oplus} - \mu\pi)x$, by Lemma 5.2.4 we have $z \in \ker(\pi)$. We have $x = \mu\pi x + (1_{V_\oplus} - \mu\pi)x = y + z$, which shows $x \in \mathrm{im}(\mu) + \ker(\pi)$. $\square$

Let $m\colon \ker(\pi) \to V_\oplus$ and $p\colon V_\oplus \to \ker(\pi)$ be the canonical inclusion and projection maps. We have the following vector spaces and maps. As μ is injective we can identify V_i with $\mathrm{im}(\mu)$, under this identification μ is the canonical inclusion and π the canonical projection.



Lemma 5.2.6. Let $j \in \{1, \ldots, k\}$. If $u \in V_{t_j}$ and $w \in \ker(\pi)$, then we have $(\mu\pi\mu_j u, w)_\oplus = 0$.

Proof. By Lemma 5.2.2 μ^* is given by $\lambda_i\pi$ so $(\mu\pi\mu_j u, w)_\oplus = \lambda_i(\pi\mu_j u, \pi w)_\oplus = 0$. $\square$

Lemma 5.2.7. The symmetric bilinear form $(\cdot,\cdot)_\oplus$ restricted to $\ker(\pi)$ is nondegenerate.

Proof. We have $V_\oplus = \mathrm{im}(\mu) \oplus \ker(\pi)$. Let $u \in \mathrm{im}(\mu)$ so $u = \mu x$ for some $x \in V_i$ and $w \in \ker(\pi)$ so $\pi w = 0$. So we have $(u, w)_\oplus = (\mu x, w)_\oplus$ and by Lemma 5.2.2 this is equal to $(x, \lambda_i \pi w)_i = 0$. So $V_\oplus = \mathrm{im}(\mu) \perp \ker(\pi)$, i.e. $\mathrm{im}(\mu)$ and $\ker(\pi)$ are orthogonal complements in $V_\oplus$ under $(\cdot,\cdot)_\oplus$. So the symmetric bilinear form $(\cdot,\cdot)_\oplus$ restricted to $\ker(\pi)$ inherits the nondegeneracy of $(\cdot,\cdot)_\oplus$. $\square$

5.2.2 Construction of the Reflected Representation

We construct a new representation V' of $(\overline{Q}, \iota)$. This definition is adapted from the one for reflection functors of representations in [CBH98, Sec. 5] and [CB01, Sec. 2].

Definition 5.2.8. The representation V' is constructed as follows:

- To the i vertex we assign $V'_i = \ker(\pi)$ and to each $j \neq i$ we assign $V'_j = V_j$.
- For each arrow $a_j \in H_i$:

  - to a_j we assign $V'_{a_j}\colon V_{t_j} \to V'_i$ defined by $V'_{a_j} = -\lambda_i(1_{V_\oplus} - \mu\pi)\mu_j$ (this definition is valid by Lemma 5.2.4),
  - to a_j^* we assign $V'_{a_j^*}\colon V'_i \to V_{t_j}$ defined by π_j.

- For $a \notin H_i$: to a we assign $V'_a = V_a$ and to a^* we assign $V'_{a^*} = V_{a^*}$.
- We equip V'_i with the nondegenerate symmetric bilinear form $(x, y)'_i = (-\frac{1}{\lambda_i})(x, y)_\oplus$ and equip V'_j for $j \neq i$ with $(x, y)_j$.

Theorem 5.2.9. If V is a symmetric representation of $\Pi^\lambda(Q)$ of dimension vector α, then V' is a representation of $\Pi^{r_i(\lambda)}(Q)$ of dimension vector $s_i(\alpha)$.

Proof. The reflected representation in Section 1.2.2 has the same definition as the reflected symmetric representation here except for the nondegenerate symmetric bilinear form. So the theorem follows from [CBH98, Thm. 5.1]. □

Theorem 5.2.10. If V is a symmetric representation, then so is V'.

Proof. We show for $a \in Q_1$ that V'_a and V'_{a^*} are adjoints with respect to the relevant bilinear forms. This follows by definition for $a \notin H_i$ so we consider $a_j \in H_i$. Let $x \in V'_{t_j} = V_{t_j}$ and $y \in V'_i = \ker(\pi)$, so $V'_{a_j}x \in V'_i = \ker(\pi)$. We show $(V'_{a_j}x, y)'_i = (x, V'_{a_j^*}y)_{t_j}$ holds

$$\begin{aligned}
(V'_{a_j}x, y)'_i &= (-\lambda_i(1_{V_\oplus} - \mu\pi)\mu_j x, y)'_i \\
&= -\lambda_i(-\frac{1}{\lambda_i})((1_{V_\oplus} - \mu\pi)\mu_j x, y)_\oplus \\
&= ((1_{V_\oplus} - \mu\pi)\mu_j x, y)_\oplus \\
&= (\mu_j x, y)_\oplus - (\mu\pi\mu_j x, y)_\oplus = (\mu_j x, y)_\oplus \text{ (by Lemma 5.2.6)} \\
&= \sum_{r=1}^{k} (\pi_r\mu_j x, \pi_r y)_{t_r} = (x, \pi_j y)_{t_j} \\
&= (x, V'_{a_j^*}y)_{t_j}
\end{aligned}$$

The last step follows because $y \in \ker(\pi)$ so π_j on this domain is the definition of $V'_{a_j^*}$. □

5.2.3 Construction of the Reflected Morphisms

Let V, W be symmetric representations of $(\overline{Q}, \iota)$, let V', W' be the respective representations obtained as defined in the previous section. For $U \in \{V, W\}$ let $U_\oplus$, $\mu_j^U\colon U_{t_j} \to U_\oplus$,

$\pi_j^U \colon U_\oplus \to U_{t_j}$ for $j = 1, \dots, k$, $\mu^U \colon U_i \to U_\oplus$ and $\pi^U \colon U_\oplus \to U_i$ be defined as in the previous section.

Let $f \colon V \to W$ be a morphism of symmetric representations. Recall that $f = (f_{i'} \colon V_{i'} \to W_{i'})_{i' \in Q_0}$ must, for each $a \in Q_1$, make the following two squares commute

$$\begin{tikzcd} V_{t(a)} \arrow[r, "V_a"] \arrow[d, "f_{t(a)}"'] & V_{h(a)} \arrow[d, "f_{h(a)}"] \\ W_{t(a)} \arrow[r, "W_a"] & W_{h(a)} \end{tikzcd} \qquad \begin{tikzcd} V_{t(a)} \arrow[d, "f_{t(a)}"'] & V_{h(a)} \arrow[l, "V_{a^*}"'] \arrow[d, "f_{h(a)}"] \\ W_{t(a)} & W_{h(a)} \arrow[l, "W_{a^*}"'] \end{tikzcd}$$

That is $f_{h(a)} V_a = W_a f_{t(a)}$ and $f_{t(a)} V_{a^*} = W_{a^*} f_{h(a)}$. When $a_j \in H_i$ we have $h(a_j) = i$ and $t(a_j) = t_j$ so the conditions become: $f_i V_{a_j} = W_{a_j} f_{t_j}$ and $f_{t_j} V_{a_j^*} = W_{a_j^*} f_i$.

Lemma 5.2.11. The map $\sum_{j=1}^k \mu_j^W f_{t_j} \pi_j^V \colon \ker(\pi^V) \to W_\oplus$ has image contained in $W_i' = \ker(\pi^W)$.

Proof. Let $x \in V_i' = \ker(\pi^V)$. The defining relation for $x \in \ker(\pi^V)$ is $\pi^V(x) = 0$, that is $\frac{1}{\lambda_i} \sum_{j=1}^k V_{a_j} \pi_j^V(x) = 0$. Let $y = \sum_{r=1}^k \mu_r^W f_{t_r} \pi_r^V x$, we show this is in $W_i' = \ker(\pi^W)$.

$$\begin{aligned} \pi^W(y) &= \frac{1}{\lambda_i} \sum_{j=1}^k W_{a_j} \pi_j^W y &&= \frac{1}{\lambda_i} \sum_{j=1}^k W_{a_j} \pi_j^W \sum_{r=1} \mu_r^W f_{t_r} \pi_r^V x \\ &= \frac{1}{\lambda_i} \sum_{j=1}^k W_{a_j} f_{t_j} \pi_j^V x &&= \frac{1}{\lambda_i} \sum_{j=1}^k f_i V_{a_j} \pi_j^V x \\ &= f_i \frac{1}{\lambda_i} \sum_{j=1}^k V_{a_j} \pi_j^V x &&= f_i \pi^V(x) &&= 0, \end{aligned}$$

as $\pi^V(x) = 0$. □

We now define the target morphisms of the reflection functor. The reflection functor takes the morphism $f \colon V \to W$ to a collection of maps $(f_j' \colon V_j' \to W_j')_{j \in Q_0}$ given by $f_j' = f_j$ for $j \neq i$ and $f_i' = \sum_{j=1}^k \mu_j^W f_{t_j} \pi_j^V$ (the definition of f_i' is well-defined because of Lemma 5.2.11).

Theorem 5.2.12. The collection of maps $(f_j' \colon V_j' \to W_j')_{j \in Q_0}$ satisfies the intertwining relations $f_{h(a)}' V_a' = W_a' f_{t(a)}'$ and $f_{t(a)}' V_{a^*}' = W_{a^*}' f_{h(a)}'$ for each $a \in Q_1$.

Proof. For $a \notin H_i$ the intertwining relations for f' follow trivially from the intertwining relations for f, so let us consider a_j for $j = 1, \dots, k$. In this case the relations are $f_i' V_{a_j}' =$

$W'_{a_j}f'_{t_j}$ and $f'_{t_j}V'_{a^*_j} = W'_{a^*_j}f'_i$. To show $f'_iV'_{a_j} = W'_{a_j}f'_{t_j}$ we first expand $f'_iV'_{a_j}$ and $W'_{a_j}f'_{t_j}$.

$$\begin{aligned}
f'_iV'_{a_j} &= \sum_{r=1}^{k} \mu^W_r f_{t_r}\pi^V_r(-\lambda_i(1_{V_\oplus} - \mu^V\pi^V)\mu^V_j) \\
&= \lambda_i \sum_{r=1}^{k} \mu^W_r f_{t_r}\pi^V_r(\mu^V\pi^V)\mu^V_j - \lambda_i \sum_{r=1}^{k} \mu^W_r f_{t_r}\pi^V_r\mu^V_j \\
&= \lambda_i \sum_{r=1}^{k} \mu^W_r f_{t_r}\pi^V_r\Big(\big(\sum_{s=1}^{k}\mu^V_s V_{a^*_s}\big)\frac{1}{\lambda_i}\big(\sum_{u=1}^{k} V_{a_u}\pi^V_u\big)\Big)\mu^V_j - \lambda_i\mu^W_j f_{t_j} \\
&= \sum_{r=1}^{k} \mu^W_r f_{t_r} V_{a^*_r} V_{a_j} - \lambda_i \mu^W_j f_{t_j}
\end{aligned}$$

We show this is equal to $W'_{a_j}f'_{t_j}$.

$$\begin{aligned}
W'_{a_j}f'_{t_j} &= (-\lambda_i(1_{W_\oplus} - \mu^W\pi^W)\mu^W_j)f_{t_j} \\
&= \lambda_i(\mu^W\pi^W)\mu^W_j f_{t_j} - \lambda_i\mu^W_j f_{t_j} \\
&= \lambda_i\Big(\big(\sum_{r=1}^{k}\mu^W_r W_{a^*_r}\big)\frac{1}{\lambda_i}\big(\sum_{s=1}^{k} W_{a_s}\pi^W_s\big)\Big)\mu^W_j f_{t_j} - \lambda_i\mu^W_j f_{t_j} \\
&= \sum_{r=1}^{k} \mu^W_r W_{a^*_r} W_{a_j} f_{t_j} - \lambda_i\mu^W_j f_{t_j}
\end{aligned}$$

The equality $f'_iV'_{a_j} = W'_{a_j}f'_{t_j}$ follows because the intertwining relations of f and V at a_j imply $W_{a^*_r}W_{a_j}f_{t_j} = W_{a^*_r}f_iV_{a_j} = f_{t_r}V_{a^*_r}V_{a_j}$. We now show $f'_{t_j}V'_{a^*_j} = W'_{a^*_j}f'_i$. We use $f_{t_j} = \sum_{r=1}^{k}\pi^W_j\mu^W_r f_{t_j}$ to get

$$f'_{t_j}V'_{a^*_j} = f_{t_j}\pi^V_j = \sum_{r=1}^{k}(\pi^W_j\mu^W_r)f_{t_r}\pi^V_r = \pi^W_j\sum_{r=1}^{k}\mu^W_r f_{t_r}\pi^V_r = W'_{a^*_j}f'_i.$$

So f' is a morphism of representations from V to W. □

The next theorem shows that if f respects the symmetric bilinear forms of V then so does f'.

Theorem 5.2.13. If $f\colon V \to W$ is a morphism of symmetric representations, then so is $f'\colon V' \to W'$.

Proof. For all $j \in Q_0$ we have $(x,y)^V_j = (f_j(x), f_j(y))^W_j$ for all $x, y \in V_j$, we want to show $(x,y)^{V'}_j = (f'_j(x), f'_j(y))^{W'}_j$ for all $x, y \in V'_j$. This follows trivially for $j \neq i$. Suppose $j = i$ and let $x, y \in V_i'$.

$$\begin{aligned}
(f_i'(x), f_i'(y))_i^{W'} &= \frac{1}{\lambda_i}(f_i'(x), f_i'(y))_{\oplus}^{W} \\
&= \frac{1}{\lambda_i}\sum_{j=1}^{k}(\pi_j^W f_i'(x), \pi_j^W f_i'(y))_{t_j}^{W} \\
&= \frac{1}{\lambda_i}\sum_{j=1}^{k}\sum_{p=1}^{k}\sum_{q=1}^{k}(\pi_j^W \mu_p^W f_{t_p}\pi_p^V(x), \pi_j^W \mu_q^W f_{t_q}\pi_q^V(y))_{t_j}^{W} \\
&= \frac{1}{\lambda_i}\sum_{j=1}^{k}(f_{t_j}\pi_j^V(x), f_{t_j}\pi_j^V(y))_{t_j}^{W} \\
&= \frac{1}{\lambda_i}\sum_{j=1}^{k}(\pi_j^V(x), \pi_j^V(y))_{t_j}^{V} \\
&= \frac{1}{\lambda_i}(x, y)_{\oplus}^{V},
\end{aligned}$$

which is equal to $(x, y)_i^{V'}$ so $(f_i'(x), f_i'(y))_i^{W'} = (x, y)_i^{V'}$ for all $x, y \in V_i$. □

5.2.4 Equivalence of Categories

Recall the definitions and results in Section 1.2.2. The nonsymmetric reflection functor at $i \in \overline{Q}_0$ gives an equivalence from $\mathrm{Rep}_K(\Pi^\lambda(Q))$ and $\mathrm{Rep}_K(\Pi^{r_i(\lambda)}(Q))$ which acts as s_i on dimension vectors. That the symmetric reflection functor at i acts as s_i on dimension vectors is shown by Theorem 5.2.9. We show in this section that the symmetric reflection functor at i establishes an equivalence between $\mathrm{Rep}_K^\Sigma(\Pi^\lambda(Q))$ and $\mathrm{Rep}_K^\Sigma(\Pi^{r_i(\lambda)}(Q))$.

Suppose $\mathcal{C}$ and $\mathcal{D}$ are categories, to establish an equivalence between them we need to show there exist functors $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ and $\mathcal{G}\colon \mathcal{D} \to \mathcal{C}$ such that there is a pair of natural isomorphisms $\eta^{\mathcal{C}}\colon \mathcal{GF} \to \mathrm{id}_{\mathcal{C}}$ and $\eta^{\mathcal{D}}\colon \mathcal{FG} \to \mathrm{id}_{\mathcal{D}}$ where $\mathrm{id}_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}$ and $\mathrm{id}_{\mathcal{D}}\colon \mathcal{D} \to \mathcal{D}$ are the identity functors. To say $\eta^{\mathcal{C}}\colon \mathcal{GF} \to \mathrm{id}_{\mathcal{C}}$ is a natural isomorphism means that, for each pair of objects X, Y in $\mathcal{C}$, $\eta^{\mathcal{C}}$ assigns isomorphisms $\eta^{\mathcal{C}}_X\colon \mathcal{GF}(X) \to X$ and $\eta^{\mathcal{C}}_Y\colon \mathcal{GF}(Y) \to Y$ such that for each $f \in \hom_{\mathcal{C}}(X, Y)$ we have $\eta^{\mathcal{C}}_Y \circ \mathcal{GF}(f) = f \circ \eta^{\mathcal{C}}_X$. The definition of $\eta^{\mathcal{D}}$ is the same with the appropriate changes. To establish the equivalence between $\mathrm{Rep}_K^\Sigma(\Pi^\lambda(Q))$ and $\mathrm{Rep}_K^\Sigma(\Pi^{r_i(\lambda)}(Q))$ we let $\mathcal{F}$ be the reflection functor at i from $\mathrm{Rep}_K^\Sigma(\Pi^\lambda(Q))$ to $\mathrm{Rep}_K^\Sigma(\Pi^{r_i(\lambda)}(Q))$ and $\mathcal{G}$ be the reflection functor at i from $\mathrm{Rep}_K^\Sigma(\Pi^{r_i(\lambda)}(Q))$ to $\mathrm{Rep}_K^\Sigma(\Pi^\lambda(Q))$. The composition is equivalent to applying the reflection functor twice and as $\lambda \in K^{Q_0}$ is arbitrary (subject to $\lambda_i \neq 0$) we need only show the composition is naturally isomorphic to the identity in one direction.

We define the representation V'' then prove it is obtained by applying the reflection functor twice. It then follows from Theorem 5.2.9 that V'' is a representation of $\Pi^{\lambda}(Q)$ of dimension vector α. Note that we do not need any new definition for μ_j, π_j when working with V' as $V'_{t_j} = V_{t_j}$ for $j = 1, \ldots, k$ and $V'_{\oplus} = \bigoplus_{j=1}^{k} V'_{t_j} = V_{\oplus}$.

In Section 5.2.3, as we were working with morphisms from one representation to another, we found it necessary to specify the representation in the superscript of the functions μ_j^V, π_j^V, μ^V and π^V. As we only work with morphisms of representations at the end of this section it is convenient to suppress the superscripts, as in Section 5.2.2, until they are needed.

Definition 5.2.14. We define V'' as follows.

- To the i vertex we assign vector space $V''_i = \ker(\pi') = \operatorname{im}(\mu)$ and to each $j \neq i$ we assign $V''_j = V_j$.
- For each arrow $a_j \in H_i$:
  - to a_j we assign $V''_{a_j}\colon V_{t_j} \to V''_i$ defined by $V''_{a_j} = \mu V_{a_j}$,
  - to a_j^* we assign $V''_{a_j^*}\colon V''_i \to V_{t_j}$ defined by $V''_{a_j^*} = \pi_{t_j}$.
- For $a \notin H_i$: to a we assign $V''_a = V_a$ and to a^* we assign $V''_{a^*} = V_{a^*}$.
- We equip V''_i with the nondegenerate symmetric bilinear form $(x,y)''_i = \frac{1}{\lambda_i}(x,y)_{\oplus}$ and equip $V''_j = V_j$ for $j \neq i$ with $(x,y)''_j = (x,y)_j$.

Lemma 5.2.15. If $\lambda' = r_i(\lambda)$, then $\lambda'_i = -\lambda_i$.

Proof. From the definition in Section 1.2.2 we have $\lambda'_i = \lambda_i - (\epsilon_i, \epsilon_i)\lambda_i$, where $(\cdot,\cdot)$ is the symmetric bilinear form associated to the Ringel form defined Section 1.1.2. From the definition $(\epsilon_i, \epsilon_i) = 2$. □

Let λ' be defined as in Lemma 5.2.15. Recall the definitions of μ, π and μ_j, π_j for $j = 1, \ldots, k$ for V. We define μ', π' for V'. Let $\mu'\colon V'_i \to V_{\oplus}$ be defined by

$$\mu' = \sum_{j=1}^{k} \mu_j V'_{a_j^*} = \sum_{j=1}^{k} \mu_j \pi_j.$$

Now as $\sum_{j=1}^{k} \mu_j \pi_j = 1_{V_\oplus}$ it is clear that μ' is the injection of V'_i into $V_\oplus$ (previously denoted by m). Let $\pi' \colon V_\oplus \to V'_i$ be defined by

$$\pi' = \frac{1}{\lambda'_i} \sum_{j=1}^{k} V'_{a_j} \pi_j = -\frac{1}{\lambda_i} \sum_{j=1}^{k} [-\lambda_i (1_{V_\oplus} - \mu\pi)\mu_j]\pi_j = (1_{V_\oplus} - \mu\pi) \sum_{j=1}^{k} \mu_j \pi_j = (1_{V_\oplus} - \mu\pi).$$

Lemma 5.2.16. We have $\ker(\pi') = \operatorname{im}(\mu)$.

Proof. If $x \in \ker(\pi')$, then $(1_{V_\oplus} - \mu\pi)x = 0$, so $x = \mu\pi x$, which implies $x \in \operatorname{im}(\mu)$. If $x \in \operatorname{im}(\mu)$, then we have $x = \mu y$ for some $y \in V_i$, so $(1_{V_\oplus} - \mu\pi)x = (1_{V_\oplus} - \mu\pi)\mu y = \mu y - \mu\pi\mu y = \mu y - \mu y = 0$ (by Lemma 5.2.3), so $x \in \ker(\pi')$. □

Lemma 5.2.17. We have $\mu'\pi' = 1_{V_\oplus} - \mu\pi$.

Proof. We have $\mu'\pi' = (\sum_{r=1}^{k} \mu_r \pi_r)(1_{V_\oplus} - \mu\pi) = \sum_{r=1}^{k} \mu_r \pi_r - \sum_{r=1}^{k} \mu_r \pi_r \mu\pi = 1_{V_\oplus} - \mu\pi$. □

Theorem 5.2.18. V'' is obtained by applying the reflection functor twice at i to V.

Proof. The assignment of vector spaces is clear from the definition, Lemma 5.2.16 and the identification of V_i and $\operatorname{im}(\mu)$. To see that $V''_{a_j} = \mu V_{a_j}$ we substitute λ'_i, μ', π' into the the respective terms in the construction of V' in Definition 5.2.8. We have, by Lemmas 5.2.17 and 5.2.15

$$V''_{a_j} = -\lambda'_i(1 - \mu'\pi')\mu_j = -(-\lambda_i)(\mu\pi)\mu_j = \lambda_i(\mu \frac{1}{\lambda_i} \sum_{r=1}^{k} V_{a_r}\pi_r)\mu_j = \mu V_{a_j},$$

as required. The assignment $V''_{a^*_j} = \pi_j$ is obvious. To see that $(x, y)''_i = \frac{1}{\lambda_i}(x, y)_i$ we note that $\lambda'_i = -\lambda_i$ by Lemma 5.2.15, so $(x, y)''_i = -\frac{1}{\lambda'_i}(x, y)'_\oplus = \frac{1}{\lambda_i}(x, y)_\oplus$. □

Let $\phi \colon V \to V''$ be given by $\phi_j = 1_{V_j}$ for $j \neq i$ (recall that $V''_j = V_j$ for $j \neq i$) and $\phi_i = \mu$. Though μ goes from V_i to $V_\oplus$, ϕ_i goes from V_i to $\operatorname{im}(\mu)$ so is surjective.

Theorem 5.2.19. We have that $\phi \colon V \to V''$ is an isomorphism of symmetric representations.

Proof. As μ is injective $\phi_i \colon V_i \to \operatorname{im}(\mu)$ is an isomorphism, so ϕ is a collection of isomorphisms of vector spaces. We show ϕ intertwines V and V''. This is trivial for $a \notin H_i$ and for each $j = 1, \ldots, k$ we have to show $\mu V_{a_j} = V''_{a_j} 1_{V_{t_j}}$ and $V''_{a^*_j}\mu = 1_{V_j} V_{a^*_j}$. Now $V''_{a_j} 1_{V_{t_j}} = V''_{a_j} = \mu V_{a_j}$ and $V''_{a^*_j}\mu = \pi_j \sum_{r=1}^{k} \mu_r V_{a^*_r} = V_{a^*_j} = 1_{V_j} V_{a^*_j}$. So ϕ is an isomorphism of representations.

Finally we show ϕ respects the symmetric structure, this is trivial for $j \neq i$ so we prove it for i. Recall from Lemma 5.2.2 that the adjoint of μ is $\lambda_i \pi$. For $x, y \in V_i$ we have

$$(\phi_i x, \phi_i y)''_i = (\mu x, \mu y)''_i = \frac{1}{\lambda_i}(\mu x, \mu y)_\oplus = \frac{\lambda_i}{\lambda_i}(x, \pi\mu y)_i = (x, y)_i$$

So ϕ is an isomorphism of symmetric representations. □

We now consider morphisms between representations of $\Pi^\lambda(Q)$ so it is necessary to switch back to specifying the representation in the superscript of the functions π_j, μ_j, π and μ. We consider the collection of isomorphisms $\phi^V : V \to V''$, where V is a representation of $\Pi^\lambda(Q)$ and $\phi_i^V = \mu^V$ and $\phi_j^V = 1_{V_j}$ for $j \neq i$.

Theorem 5.2.20. The collection of isomorphisms ϕ is a natural isomorphism from the reflection functor, applied twice, to the identity functor.

Proof. Given symmetric representations V and W of $\mathrm{Rep}_K^\Sigma(\Pi^\lambda(Q))$, we need to show that, for each $f : V \to W$, the square

$$\begin{tikzcd} V_j \arrow[r, "f_j"] \arrow[d, "\phi^V"'] & W_j \arrow[d, "\phi^W"] \\ V''_j \arrow[r, "f''_j"] & W''_j \end{tikzcd}$$

commutes for each $j \in Q_0$, that is $\phi^W f_j = f''_j \phi^V$. Now $f''_j = f_j$ for $j \neq i$ and $f''_i = \sum_{j=1}^k \mu_j^W f_{t_j} \pi_j^V$. For $j \neq i$ the square commutes trivially as $\phi_j^V = 1_{V_j}$ and $\phi_j^W = 1_{W_j}$, so we prove the relation holds for i.

$$\begin{aligned} f''_i \phi_i^V = \sum_{j=1}^k \mu_j^W f_{t_j} \pi_j^V \mu^V &= \sum_{j=1}^k \mu_j^W f_{t_j} \pi_j^V \sum_{r=1}^k \mu_r^V V_{a_r^*} \\ = \sum_{j=1}^k \mu_j^W f_{t_j} \sum_{r=1}^k \pi_j^V \mu_r^V V_{a_r^*} &= \sum_{j=1}^k \mu_j^W f_{t_j} V_{a_j^*} \\ = \sum_{j=1}^k \mu_j^W W_{a_j^*} f_i &= \phi_i^W f_i. \end{aligned}$$

So ϕ is a natural isomorphism. □

So the reflection functor at i from $\mathrm{Rep}_K^\Sigma(\Pi^\lambda(Q))$ to $\mathrm{Rep}_K^\Sigma(\Pi^{\lambda'}(Q))$ has an inverse, namely the reflection functor at i from $\mathrm{Rep}_K^\Sigma(\Pi^{\lambda'}(Q))$ to $\mathrm{Rep}_K^\Sigma(\Pi^\lambda(Q))$. The composition of these two functors is naturally isomorphic to the identity functor so the reflection functor establishes an equivalence of categories.

5.3 The Symmetric Additive Matrix Problem

We saw in Section 4.1.2 how solutions to the additive matrix problem correspond to representations of certain deformed preprojective algebras of certain dimension vectors. We see in this section how a similar correspondence exists for symmetric solutions and symmetric representations.

Let K be an algebraically closed field of characteristic zero. As in Section 4.1.2 let $C_1, \ldots, C_k \subseteq M_n(K)$ be a tuple of matrix similarity classes. Let Q, α and λ be the star-shaped quiver, dimension vector and K-vector associated to the similarity classes (as defined in Section 4.1.1).

Definition 5.3.1. Let V be a K-vector space. We say a tuple of linear maps $(A_1, \ldots, A_k)$ of V is *symmetrizable* if there exists a nondegenerate symmetric bilinear form $(\cdot,\cdot)$ on V such that $A_1, \ldots, A_k$ are self-adjoint with respect to this bilinear form, that is $(A_i x, y) = (x, A_i y)$ for all $x, y \in V$ and $i = 1, \ldots, k$.

A *symmetric* tuple $((A_1, \ldots, A_k), (\cdot,\cdot))$ is a pair consisting of a symmetrizable tuple and a nondegenerate symmetric bilinear form on V such that $A_1, \ldots, A_k$ are self-adjoint.

5.3.1 Functors from Symmetric Representations to Tuples of Symmetric Matrices

Recall the notation $C_i(V)$ and $\overline{C}_i(V)$ in Section 4.1.2 where V is a vector space.

Lemma 5.3.2. Given a vector space V and a symmetric tuple $((A_1, \ldots, A_k), (\cdot,\cdot))$ of linear maps of V such that $A_i \in \overline{C}_i(V)$ (resp. $A_i \in C_i(V)$) for $i = 1, \ldots, k$ and $\sum_{i=1}^k A_i = 0$ there exists a symmetric representation (resp. strict symmetric representation) X of $\Pi^\lambda(Q)$ such that $A_i = X_{a_{i,1}} X_{a^*_{i,1}} + \xi_{i,1} 1_V$.

Proof. For each arm $i = 1, \ldots, k$ the ith arm component of X is obtained from A_i using Theorem 2.2.8 (resp. Theorem 2.2.9), this is well-defined as the central vector spaces and symmetric nondegenerate bilinear form of each component are the same, i.e. V and $(\cdot,\cdot)$. The theorem ensures that the deformed preprojective relations on the arms are satisfied, that $X_{a_{i,j}}$ and $X_{a^*_{i,j}}$ are adjoint for $j = 1, \ldots, d_i - 1$ and $A_i = X_{a_{i,1}} X_{a^*_{i,1}} + \xi_{i,1} 1_V$. The central deformed preprojective relation is also satisfied as $0 = \sum_{i=1}^k A_i = \sum_{i=1}^k (X_{a_{i,1}} X_{a^*_{i,1}} + \xi_{i,1} 1_V)$ which implies $\sum_{i=1}^k X_{a_{i,1}} X_{a^*_{i,1}} = -\sum_{i=1}^k \xi_{i,1} 1_V = \lambda_0 1_V$. $\square$

Representations and Closures of Similarity Classes

The category of symmetric representations of $\Pi^\lambda(Q)$ is denoted $\mathrm{Rep}^\Sigma_K(\Pi^\lambda(Q))$. Let $\mathcal{R}^\Sigma(\alpha)$ be the full subcategory of $\mathrm{Rep}^\Sigma_K(\Pi^\lambda(Q))$ consisting of all symmetric representations of dimension vector α. Let $\overline{C}^\Sigma$ be a category with objects given by

$$\mathrm{ob}(\overline{C}^\Sigma) = \Big\{(V, A_1, \ldots, A_k, (\cdot,\cdot)) \colon V \text{ is an } n\text{-dimensional } K\text{-vector space,}$$
$$(A_1, \ldots, A_k, (\cdot,\cdot)) \text{ is a symmetric tuple, } A_i \in \overline{C}_i(V) \text{ for } i = 1, \ldots, k \text{ and } \sum_{i=1}^k A_i = 0\Big\},$$

and morphisms between objects $\mathbf{A}, \mathbf{A}' \in \mathrm{ob}(\overline{C}^\Sigma)$, where $\mathbf{A} = (V, A_1, \ldots, A_k, (\cdot,\cdot))$ and $\mathbf{A}' = (V', A'_1, \ldots, A'_k, (\cdot,\cdot)')$, given by

$$\hom_{\overline{C}^\Sigma}(\mathbf{A}, \mathbf{A}') = \{\phi \colon V \to V' \colon (\phi x, \phi y)' = (x, y) \text{ for all } x, y \in V, A'_i\phi = \phi A_i \text{ for } i = 1, \ldots, k\}.$$

We define a functor $\mathcal{F}^\Sigma$ from $\mathcal{R}^\Sigma(\alpha)$ to $\overline{C}^\Sigma$. The definition of $\mathcal{F}^\Sigma$ in this section is similar to the definition of $\mathcal{F}$ in Section 4.1.1 though it operates on different objects and morphisms. Given a representation $X \in \mathcal{R}^\Sigma(\alpha)$ we define a tuple $\mathcal{F}^\Sigma(X) = (X_0, A_1, \ldots, A_k, (\cdot,\cdot)_0)$ where $A_i = X_{a_{i,1}} X^*_{a_{i,1}} + \xi_{i,1} 1_{X_0}$ for $i = 1, \ldots, k$. Given a morphism $\phi \colon X \to Y$ of representations $X, Y \in \mathcal{R}^\Sigma(\alpha)$ we define a morphism $\mathcal{F}^\Sigma(\phi) \colon \mathcal{F}^\Sigma(X) \to \mathcal{F}^\Sigma(Y)$ given by $\mathcal{F}^\Sigma(\phi) = \phi_0$.

Theorem 5.3.3. $\mathcal{F}^\Sigma$ is a functor.

Proof. Let X be a representation in $\mathcal{R}^\Sigma(\alpha)$. We first show $\mathcal{F}^\Sigma(X) = (X_0, A_1, \ldots, A_k, (\cdot,\cdot)_0)$ is an object in $\overline{C}^\Sigma$. By Theorem 2.2.8 we have $A_i \in \overline{C}_i(X_0)$, A_i is self-adjoint with respect to $(\cdot,\cdot)_0$ for $i = 1, \ldots, k$, and $\sum_{i=1}^k A_i = \sum_{i=1}^k X_{a_{i,1}} X_{a^*_{i,1}} + \sum_{i=1}^k \xi_{i,1} 1_{X_0} = 0$ by the deformed preprojective relation at the central vertex.

Let $\phi \colon X \to Y$ be a morphism of symmetric representations where X, Y are objects of $\mathcal{R}^\Sigma(\alpha)$, we show $\mathcal{F}^\Sigma(\phi)$ is a morphism from $\mathcal{F}^\Sigma(X)$ to $\mathcal{F}^\Sigma(Y)$. Let $\mathcal{F}^\Sigma(X) = (X_0, A_1, \ldots, A_k, (\cdot,\cdot)^X_0)$ and $\mathcal{F}^\Sigma(Y) = (Y_0, A'_1, \ldots, A'_k, (\cdot,\cdot)^Y_0)$. Now $\mathcal{F}^\Sigma(\phi) = \phi_0$ and by the proof of Theorem 4.1.9 we have that $A'_i\phi_0 = \phi_0 A_i$ for each $i = 1, \ldots, k$. By definition we have $(\phi_0 x, \phi_0 x')^Y_0 = (x, x')^X_0$ for all $x, x' \in X_0$. These properties show $\mathcal{F}^\Sigma(\phi) \in \hom_{\overline{C}^\Sigma}(\mathcal{F}^\Sigma(X), \mathcal{F}^\Sigma(Y))$. The proof that that $\mathcal{F}^\Sigma$ respects identity morphisms and respects morphism composition is virtually the same as in Theorem 4.1.9. So $\mathcal{F}^\Sigma$ is a functor from $\mathcal{R}^\Sigma(\alpha)$ to $\overline{C}^\Sigma$. □

Lemma 5.3.4. The functor $\mathcal{F}^\Sigma$ is surjective.

Proof. Let $(V, A_1, \ldots, A_k, (\cdot,\cdot)) \in \overline{C}^{\Sigma}$, and let X be the symmetric representation of $\Pi^{\lambda}(Q)$ obtained from this given by Lemma 5.3.2, note that $X_0 = V$ and $(\cdot,\cdot)_0^X = (\cdot,\cdot)$. Let $(X_0, A_1', \ldots, A_k', (\cdot,\cdot)_0^X) = \mathcal{F}^{\Sigma}(X)$ so for $i = 1, \ldots, k$ we have $A_i' = X_{a_{i,1}} X_{a_{i,1}^*} + \xi_{i,1} 1_{X_0}$ which, by Lemma 5.3.2, is equal to A_i so $\mathcal{F}^{\Sigma}(X) = (X_0, A_1, \ldots, A_k, (\cdot,\cdot)_0^X) = (V, A_1, \ldots, A_k, (\cdot,\cdot))$. □

Strict Representations and Similarity Classes

Let $\widetilde{\mathrm{Rep}}_K^{\Sigma}(\Pi^{\lambda}(Q))$ be the full sub-category of strict symmetric representations of $\Pi^{\lambda}(Q)$. Recall from Definition 4.1.7 that a strict representation of $\mathrm{Rep}_K(\Pi^{\lambda}(Q))$, where Q is star-shaped, is a representation V such that each map V_a is injective and each map V_{a^*} is surjective, for all $a \in Q_1$. Let $\tilde{\mathcal{R}}^{\Sigma}(\alpha)$ be the full subcategory $\widetilde{\mathrm{Rep}}_K^{\Sigma}(\Pi^{\lambda}(Q))$ consisting of strict symmetric representations of dimension vector α. Let C^{Σ} be the full sub-category of $\overline{C}^{\Sigma}$ consisting of objects $(V, A_1, \ldots, A_k, (\cdot,\cdot))$ such that $A_i \in C_i(V)$ for $i = 1, \ldots, k$. Let $\mathcal{G}^{\Sigma}$ be the functor $\mathcal{F}^{\Sigma}$ restricted to $\tilde{\mathcal{R}}^{\Sigma}(\alpha)$.

Let X be an object of $\tilde{\mathcal{R}}^{\Sigma}(\alpha)$ and write $\mathcal{G}^{\Sigma}(X) = (X_0, A_1, \ldots, A_k, (\cdot,\cdot)_0^X)$. By Theorem 2.2.9 we have $A_i \in C_i(V)$ and A_i is self-adjoint with respect to $(\cdot,\cdot)_0^X$ for each $i = 1, \ldots, k$ so $\mathcal{G}^{\Sigma}$ is a functor from $\tilde{\mathcal{R}}^{\Sigma}(\alpha)$ to C^{Σ}.

Lemma 5.3.5. The functor $\mathcal{G}^{\Sigma}$ is surjective.

Proof. The proof is essentially the same as in Lemma 5.3.4. □

Lemma 5.3.6. The functor $\mathcal{G}^{\Sigma}$ is fully faithful.

Proof. The proof that $\mathcal{G}^{\Sigma}$ is faithful is essentially the same as in Lemma 4.1.12. The proof that $\mathcal{G}^{\Sigma}$ is full is an extension of the argument in Lemma 4.1.12. Suppose $\psi \in \hom_{C^{\Sigma}}(\mathcal{G}^{\Sigma}(X), \mathcal{G}^{\Sigma}(Y))$. We construct a morphism of symmetric representations ϕ from X to Y such that $\mathcal{G}^{\Sigma}(\phi) = \psi$. The morphism ϕ is given by $\phi_0 = \psi$ and $\phi_{i,j}(x) = Y_{a_{i,j}^*}\phi_{i,j-1}(y)$ for all $x \in X_{i,j}$ where $y \in X_{i,j-1}$ such that $x = X_{a_{i,j}^*}(y)$ for $j = 1, \ldots, d_i - 1$ and $i = 1, \ldots, k$. That this is well-defined and a morphism of representations is proven in Lemma 4.1.12. What remains to show is that it is a morphism of symmetric representations. We want to show $(\phi_{i,j}x, \phi_{i,j}x')_{i,j}^Y = (x, x')_{i,j}^X$ for all $x, x' \in X_{i,j}$, for $i = 1, \ldots, k$ and $j = 1, \ldots, d_i - 1$. Let $i \in \{1, \ldots, k\}$. It is clear that this holds for $j = 0$. We prove for $j \in \{1, \ldots, d_i - 1\}$ by induction. Assume the hypothesis holds for $j - 1$ and let $z, z' \in X_{i,j-1}$ be such that $x = X_{a_{i,j}^*}(z)$ and

$x' = X_{a^*_{i,j}}(z')$ respectively. Then

$$\begin{aligned}
(\phi_{i,j}x, \phi_{i,j}x')^Y_{i,j} &= (\phi_{i,j}X_{a^*_{i,j}}z, \phi_{i,j}X_{a^*_{i,j}}z')^Y_{i,j} &=& \quad (Y_{a^*_{i,j}}\phi_{i,j-1}z, \phi_{i,j}X_{a^*_{i,j}}z')^Y_{i,j} \\
&= (\phi_{i,j-1}z, Y_{a_{i,j}}\phi_{i,j}X_{a^*_{i,j}}z')^Y_{i,j-1} &=& \quad (\phi_{i,j-1}z, \phi_{i,j-1}X_{a_{i,j}}X_{a^*_{i,j}}z')^Y_{i,j-1} \\
&= (z, X_{a_{i,j}}X_{a^*_{i,j}}z')^X_{i,j-1} &=& \quad (X_{a^*_{i,j}}z, X_{a^*_{i,j}}z')^X_{i,j} \\
&= (x, x')^X_{i,j}
\end{aligned}$$

So $\mathcal{G}^\Sigma(\phi)$ is a morphism of symmetric representations. □

The diagram below is the symmetric analog of (4.5). Both $\mathcal{F}^\Sigma$ and $\mathcal{G}^\Sigma$ are surjective, and therefore dense, but only $\mathcal{G}^\Sigma$ is necessarily fully-faithful.

$$\begin{tikzcd}
\mathcal{R}^\Sigma(\alpha) \arrow[d, "\mathcal{F}^\Sigma"', "\text{dense}"] & \tilde{\mathcal{R}}^\Sigma(\alpha) \arrow[l, hook', "\text{full subcategory}"] \arrow[d, "\mathcal{G}^\Sigma"', "\cong"] \\
\overline{\mathcal{C}}^\Sigma & \mathcal{C}^\Sigma \arrow[l, hook', "\text{full subcategory}"]
\end{tikzcd} \tag{5.1}$$

The functors show that symmetric tuples (and hence symmetrizable tuples) of solutions correspond to symmetric representations of the associated deformed preprojective algebra. This says that for a given vector space V, there exists $A_i \in \overline{C}_i(V)$ for each $i = 1, \ldots, k$ such that $\sum_{i=1}^k A_i = 0$ and $(A_1, \ldots, A_k)$ is symmetrizable if and only if there exists a symmetric representation of $\Pi^\lambda(Q)$ of dimension vector α. Similarly there exists $A_i \in C_i(V)$ for each $i = 1, \ldots, k$ such that $\sum_{i=1}^k A_i = 0$ and $(A_1, \ldots, A_k)$ is irreducible and symmetrizable if and only if there exists a strict symmetric representation of $\Pi^\lambda(Q)$ of dimension vector α.

5.3.2 An Application of the Reflection Functors

Let $C_1, \ldots, C_k \subseteq M_n(K)$ be similarity classes. Let Q, α and λ be the associated star-shaped quiver, dimension vector and K-vector respectively. Recall the definition of a rigid solution to the additive matrix problem from Section 4.1.3. By Theorem 4.1.14 an irreducible solution is rigid if and only if α is a real root.

Theorem 5.3.7. Every rigid irreducible solution to the additive matrix problem is symmetrizable.

Proof. Let V be a vector space and $(A_1, \ldots, A_k)$ a rigid irreducible solution to the additive matrix problem. By the surjectivity of the functor $\mathcal{G}$ in Section 4.1.2 there exists a strict representation X of $\Pi^\lambda(Q)$ of dimension vector α such that $\mathcal{G}(X) = (V, A_1, \ldots, A_k)$.

By Theorem 4.1.14 $\alpha \in \Sigma_\lambda$ and α is a real root, so by Lemma 1.2.8 there is a sequence of admissible reflections $s_{j_1}, \ldots, s_{j_l}$ such that $\epsilon_i = s_{j_1} \ldots s_{j_l}(\alpha)$ for some $i \in Q_0$ (where ϵ_i is the simple root at i). Let $\lambda' = r_{j_1} \ldots r_{j_l}(\lambda)$. As the reflections are admissible this implies there is a categorical equivalence between $\mathrm{Rep}(\Pi^\lambda(Q))$ and $\mathrm{Rep}(\Pi^{\lambda'}(Q))$ which acts on dimension vectors by $s_{j_1} \ldots s_{j_l}$. This means the isomorphism classes of representations of $\Pi^{\lambda'}(Q)$ of dimension vector ϵ_i are in one-one correspondence with the isomorphism classes of representations of $\Pi^\lambda(Q)$ of dimension vector α. It is easy to see that all representations of dimension vector ϵ_i are trivially symmetrizable (as all maps are zero) so there is an equivalence between the category of representations of $\Pi^{\lambda'}(Q)$ of dimension vector ϵ_i and the category of symmetric representations of $\Pi^{\lambda'}(Q)$ of dimension vector ϵ_i. Using reflection functors of the category of symmetric representations (at the reverse sequence of admissible reflections $s_{j_l}, \ldots, s_{j_1}$) we see there is an equivalence between $\mathcal{R}(\alpha)$ and $\mathcal{R}^\Sigma(\alpha)$. This says there is a one-one correspondence between the isomorphism classes of $\mathcal{R}(\alpha)$ and the isomorphism classes of $\mathcal{R}^\Sigma(\alpha)$.

Let Y be an object of $\mathcal{R}^\Sigma(\alpha)$ in the isomorphism class which corresponds to the isomorphism class of X in $\mathcal{R}(\alpha)$. The equivalence implies X is isomorphic to Y as (nonsymmetric) representations of $\Pi^\lambda(Q)$. Let $\phi\colon X \to Y$ be such an isomorphism. Let $(V', A'_1, \ldots, A'_k, (\cdot,\cdot)')$ be such that $\mathcal{G}^\Sigma(Y) = (V', A'_1, \ldots, A'_k, (\cdot,\cdot)')$ (where $\mathcal{G}^\Sigma$ is the symmetric functor of Section 5.3.1). Let $(x, y) = (\mathcal{G}(\phi)x, \mathcal{G}(\phi)y)'$ for $x, y \in V$ (where $\mathcal{G}$ is the functor of Section 4.1.2), so $(\cdot,\cdot)$ is a nondegenerate symmetric bilinear form on V such that $(A_i x, y) = (x, A_i y)$ for $i = 1, \ldots, k$ as

$$
\begin{aligned}
(A_i x, y) = (\mathcal{G}(\phi)A_i x, \mathcal{G}(\phi)y)' \quad &= \quad (A'_i\mathcal{G}(\phi)x, \mathcal{G}(\phi)y)' \\
= (\mathcal{G}(\phi)x, A'_i\mathcal{G}(\phi)y)' \quad &= \quad (\mathcal{G}(\phi)x, \mathcal{G}(\phi)A_i y)' \quad = \quad (x, A_i y).
\end{aligned}
$$

So $(V, A_1, \ldots, A_k)$ is symmetrizable. $\square$

Chapter 6

Some Results on the Existence of Symmetric Representations

We present two results about the existence of symmetric representations of certain quivers. In Section 6.1 we show there exist irreducible solutions to the additive matrix problem which are not symmetrizable. This implies, under certain circumstances, there exist strict representations of $\Pi^\lambda(Q)$ of dimension vector α (where Q, λ and α are as in Section 4.1.1) which are not symmetrizable. This contrasts with Theorem 5.3.7 which states that every rigid irreducible solution to the additive matrix problem is symmetrizable. In Section 6.2 we prove there always exists a symmetric representation of $\Pi^\lambda(Q)$ of dimension vector α where Q is a Dynkin or extended Dynkin quiver of type $\tilde{A}_n$ or $\tilde{D}_n$, α is a positive root and $\lambda \cdot \alpha = 0$, and we conjecture that the result holds in the $\tilde{E}_n$ case also. Due to time restrictions we have not been able prove the result in the $\tilde{E}_n$ case but we describe a promising method.

Let K be an algebraically closed field of characteristic zero.

6.1 The Number of Parameters of Symmetrizable Representations

We prove in this section that there exist irreducible solutions to certain additive matrix problems which are not symmetrizable, we show that in certain circumstances the number of parameters of general irreducible solutions must be strictly greater than the number of parameters of symmetrizable irreducible solutions. We work with square matrices in $M_n(K)$ in this section rather than endomorphisms of a K-vector space. If we have a vector space V endowed with a nondegenerate symmetric bilinear form $(\cdot,\cdot)$, then we can choose an orthonormal basis for V. Under such a basis $(\cdot,\cdot)$ corresponds to the scalar product and taking the adjoint of an endomorphism correponds to taking the transpose of the corresponding matrix. So the results of this section can easily be framed in terms of the endomorphisms of a vector space if desired.

In Section 6.1.1 we recall some important definitions and results from algebraic geometry. In Sections 6.1.2 and 6.1.3 we derive inequalities which relate the numbers of parameters of the varieties of tuples and symmetric tuples of irreducible solutions to the dimensions of the similarity classes and symmetric similarity classes respectively. In Sections 6.1.4 and 6.1.5 we compute the dimensions of a similarity and symmetric similarity class respectively. Finally in Section 6.1.6 we use the inequalities to show there are circumstances in which solutions to the additive matrix problem exist which are not symmetrizable. Throughout this section let $C_1,\ldots,C_k \subseteq M_n(K)$ be similarity classes such that $\sum_{i=1}^{k} \operatorname{trace}(C_i) = 0$.

6.1.1 Preliminary Material

We recall some definitions and results from algebraic geometry. We use these to prove the set of symmetrizable irreducible solutions is, in some cases, a strict subset of the set of irreducible solutions. Recall the definition of a variety, an algebraic group, an irreducible component of a variety and the dimension of a variety. These definition and results can be found in [Mum88] and [CB93].

Definition 6.1.1. Let V be an algebraic variety and G an algebraic group, *the number of parameters* $\nu_G(V)$ of V over G is defined by $\nu_G(V) = \max_t(\dim(V_{(t)}) - t)$ where $V_{(t)}$ is the union of all G-orbits of V with dimension t, or equivalently $V_{(t)} = \{v \in V \colon \dim(Gv) = t\}$.

Definition 6.1.2. Let X, Y be varieties and $f\colon X \to Y$ a morphism. We say f is *dominant* if $\overline{f(X)} = Y$, i.e. its image is dense in Y.

Lemma 6.1.3. Let $f\colon X \to Y$ be a dominant morphism of irreducible varieties. Any irreducible component of the fibre $f^{-1}(y)$ for $y \in Y$ has dimension at least $\dim(X)-\dim(Y)$.

Proof. [Mum88, Chap. I, Sec. 8, Thm. 2]. □

6.1.2 Number of Parameters of General Tuples

Let $\mathbf{C} = C_1 \times \cdots \times C_k$ be the product of the similarity classes, that is the set of tuples of matrices $(A_1, \ldots, A_k)$ with $A_i \in C_i$ for $i = 1, \ldots, k$. As $\mathbf{C}$ is the product of similarity classes it is an irreducible variety, futhermore $\mathrm{GL}_n(K)$ acts on $\mathbf{C}$ by simultaneous similarity. Let $\mathbf{C}_{\mathrm{Irr}} \subseteq \mathbf{C}$ be the subset of irreducible tuples of $\mathbf{C}$ (recall the definition of irreducible from Section 4.1.3). The set $\mathbf{C}_{\mathrm{Irr}}$ is an open subset of $\mathbf{C}$ so, as $\mathbf{C}$ is an irreducible variety, $\mathbf{C}_{\mathrm{Irr}}$ is dense in $\mathbf{C}$.

The *special linear Lie algebra* $\mathfrak{sl}_n(K)$ is the Lie algebra consisting of traceless square n by n matrices (with entries in K), that is $\mathfrak{sl}_n(K) = \{A \in M_n(K) \colon \mathrm{trace}(A) = 0\}$. We define the morphisms:

$$\sigma' \colon \mathbf{C} \to \mathfrak{sl}_n, \quad \text{and} \quad \tilde{\sigma}' \colon \mathbf{C}_{\mathrm{Irr}} \to \mathfrak{sl}_n,$$

by summation, i.e. $(A_1, \ldots, A_k) \mapsto \sum_{i=1}^{k} A_i$. From these we define the dominant morphisms:

$$\sigma \colon \mathbf{C} \to \overline{\mathrm{im}(\sigma')}, \quad \text{and} \quad \tilde{\sigma} \colon \mathbf{C}_{\mathrm{Irr}} \to \overline{\mathrm{im}(\tilde{\sigma}')}.$$

Lemma 6.1.4. If $\tilde{\sigma}^{-1}(0)$ is nonempty, then $\dim(\tilde{\sigma}^{-1}(0)) \geq \dim(\mathbf{C}_{\mathrm{Irr}}) - \dim(\mathrm{im}(\tilde{\sigma}'))$.

Proof. Apply Lemma 6.1.3 to $\tilde{\sigma}^{-1}(0)$. □

If $\mathbf{C}_{\mathrm{Irr}}$ is nonempty, then $\dim(\mathbf{C}_{\mathrm{Irr}}) = \dim(\mathbf{C})$ as $\mathbf{C}_{\mathrm{Irr}}$ is dense in $\mathbf{C}$. We have $\dim(\mathrm{im}(\tilde{\sigma}')) \leq \dim(\mathfrak{sl}_n(K)) = n^2 - 1$ as $\mathrm{im}(\tilde{\sigma}') \subseteq \overline{\mathrm{im}(\tilde{\sigma}')} \subseteq \mathfrak{sl}_n(K)$. We apply these results to the number of parameters of $\tilde{\sigma}^{-1}(0)$ over $\mathrm{GL}_n(K)$ in the following corollary.

Corollary 6.1.5. If $\tilde{\sigma}^{-1}(0)$ is nonempty, then $\nu_{\mathrm{GL}_n(K)}(\tilde{\sigma}^{-1}(0)) \geq \dim(\mathbf{C}) - 2(n^2 - 1)$.

Proof. Each irreducible tuple $(A_1, \ldots, A_k) \in \mathbf{C}_{\mathrm{Irr}}$ has stabilizer of dimension one (the stabilizer is precisely the set of nonzero scalar matrices, this follows from the irreducibility of $(A_1, \ldots, A_k)$), so

$$\dim(\mathrm{Orb}_{\mathrm{GL}_n(K)}((A_1, \ldots, A_k))) = \dim(\mathrm{GL}_n(K)) - \dim(\mathrm{Stab}_{\mathrm{GL}_n(K)}((A_1, \ldots, A_k))) = n^2 - 1.$$

So each $\mathrm{GL}_n(K)$-orbit of $\mathbf{C}_{\mathrm{Irr}}$ has the same dimension, and similarly for the $\mathrm{GL}_n(K)$-orbits of $\tilde{\sigma}^{-1}(0)$. So we have $\nu_{\mathrm{GL}_n(K)}(\tilde{\sigma}^{-1}(0)) = \dim(\tilde{\sigma}^{-1}(0)) - (n^2 - 1)$. We have $\nu_{\mathrm{GL}_n(K)}(\tilde{\sigma}^{-1}(0)) \geq \dim(\mathbf{C}_{\mathrm{Irr}}) - \dim(\mathrm{im}(\tilde{\sigma}')) - (n^2 - 1)$, by Lemma 6.1.4, which gives us $\nu_{\mathrm{GL}_n(K)}(\tilde{\sigma}^{-1}(0)) \geq \dim(\mathbf{C}) - (n^2 - 1) - (n^2 - 1)$ by the remarks before the corollary. □

6.1.3 Number of Parameters of Symmetrizable Tuples

For each $i = 1, \dots, k$ let $C_i^\Sigma \subseteq C_i$ be the subset consisting of symmetric matrices in the similarity class C_i. The *set of symmetric tuples* is given by $\mathbf{C}^\Sigma = C_1^\Sigma \times \cdots \times C_k^\Sigma \subseteq \mathbf{C}$. Let the set of *symmetrizable tuples* $\mathbf{C}^T$ be defined by $\{(A_1, \dots, A_k) \in \mathbf{C} \colon \exists P \in \mathrm{GL}_n(K) \text{ with } P^{-1}(A_1, \dots, A_k)P \in \mathbf{C}^\Sigma\}$, that is the subset consisting of tuples which are in the same $\mathrm{GL}_n(K)$-orbit as a symmetric tuple. We define the morphisms:

$$\sigma'_T \colon \mathbf{C}^T \to \mathfrak{sl}_n, \quad \text{and} \quad \tilde{\sigma}'_T \colon \mathbf{C}^T_{\mathrm{Irr}} \to \mathfrak{sl}_n$$

by summation. From these we define the dominant morphisms:

$$\sigma_T \colon \mathbf{C}^T \to \overline{\mathrm{im}(\sigma'_T)}, \quad \text{and} \quad \tilde{\sigma}_T \colon \mathbf{C}^T_{\mathrm{Irr}} \to \overline{\mathrm{im}(\tilde{\sigma}'_T)}.$$

Lemma 6.1.6. Given a nonsingular matrix $Y \in M_n(K)$ there exists a nonsingular matrix H such that $H^2 = Y$ and H is a polynomial in Y.

Proof. [Kap03, Thm. 68]. □

Recall the definition of the orthogonal group of matrices $\mathrm{O}_n(K) = \{A \in M_n(K) \colon AA^T = I_n\}$. The next theorem is a generalization of [Kap03, Thm. 70] for symmetric matrices.

Theorem 6.1.7. Let $(A_1, \dots, A_k)$ and $(B_1, \dots, B_k)$ be k-tuples of symmetric matrices in $M_n(K)$. If there exists some $X \in \mathrm{GL}_n(K)$ such that for each $i = 1, \dots, k$ we have $X^{-1}A_iX = B_i$, then there exists some $U \in \mathrm{O}_n(K)$ such that $U^TA_iU = B_i$ for all $i = 1, \dots, k$.

Proof. Let $i = 1, \dots, k$. By the symmetry of A_i and B_i, $X^{-1}A_iX = B_i$ implies $X^{-1}A_i^TX = B_i^T$. So we apply the transpose to $X^{-1}A_i^TX = B_i^T$ to get $X^TA_iX^{-T} = B_i$, and we substitute this into $X^{-1}A_iX = B_i$ to get $X^{-1}A_iX = X^TA_iX^{-T}$, so $A_iXX^T = XX^TA_i$, that is XX^T commutes with each A_i for $i = 1, \dots, k$.

By Lemma 6.1.6 there exists a matrix H such that $H^2 = XX^T$ and H is a polynomial in XX^T, therefore H is symmetric, nonsingular and commutes with each A_i for $i = 1, \dots, k$. Now let $U = H^{-1}X$. Now $UU^T = H^{-1}XX^TH^{-T} = H^{-1}H^2H^{-1} = I_n$ so U is orthogonal. For each $i = 1, \dots, k$ we have $U^{-1}A_iU = (H^{-1}X)^{-1}A_iH^{-1}X = X^{-1}HA_iH^{-1}X = X^{-1}A_iX = B_i$. □

Corollary 6.1.8. The number of parameters of the set of symmetrizable tuples over the general linear group is equal to the number of parameters of the set of symmetric tuples over the orthogonal group. That is $\nu_{\mathrm{GL}_n(K)}(\mathbf{C}^T) = \nu_{\mathrm{O}_n(K)}(\mathbf{C}^\Sigma)$, and similarly we have $\nu_{\mathrm{GL}_n(K)}(\mathbf{C}^T_{\mathrm{Irr}}) = \nu_{\mathrm{O}_n(K)}(\mathbf{C}^\Sigma_{\mathrm{Irr}})$.

Proof. Use Theorem 6.1.7. □

Lemma 6.1.9. We have $\nu_{\mathrm{GL}_n(K)}(\tilde{\sigma}_T^{-1}(0)) \leq \nu_{\mathrm{GL}_n(K)}(\mathbf{C}^T_{\mathrm{Irr}})$.

Proof. This is because $\tilde{\sigma}_T^{-1}(0) \subseteq \mathbf{C}^T_{\mathrm{Irr}}$. □

Lemma 6.1.10. $\nu_{\mathrm{GL}_n(K)}(\mathbf{C}^T_{\mathrm{Irr}}) = \sum_{i=1}^k \dim(C_i^\Sigma) - n(n-1)/2$

Proof. By Corollary 6.1.8 we know $\nu_{\mathrm{GL}_n(K)}(\mathbf{C}^T_{\mathrm{Irr}}) = \nu_{\mathrm{O}_n(K)}(\mathbf{C}^\Sigma_{\mathrm{Irr}})$. So $\nu_{\mathrm{GL}_n(K)}(\mathbf{C}^T_{\mathrm{Irr}}) = \dim(\mathbf{C}^\Sigma_{\mathrm{Irr}}) - (\dim(\mathrm{O}_n(K)) - \dim(\mathrm{Stab}_{\mathrm{O}_n(K)}((A_1,\dots,A_k))))$ for some $(A_1,\dots,A_k) \in \mathbf{C}^\Sigma_{\mathrm{Irr}}$ (this is true and well-defined as every $\mathrm{O}_n(K)$-orbit in $\mathbf{C}^\Sigma_{\mathrm{Irr}}$ has the same dimension). As $\mathbf{C}^\Sigma_{\mathrm{Irr}}$ is dense in $\mathbf{C}^\Sigma$ we have that $\dim(\mathbf{C}^\Sigma_{\mathrm{Irr}}) = \dim(\mathbf{C}^\Sigma)$ and that $\dim(\mathrm{Stab}_{\mathrm{O}_n(K)}((A_1,\dots,A_k))) = 0$ for $(A_1,\dots,A_k) \in \mathbf{C}^\Sigma_{\mathrm{Irr}}$ (as there are only finitely many diagonal orthogonal matrices), so $\nu_{\mathrm{GL}_n(K)}(\mathbf{C}^T_{\mathrm{Irr}}) = \sum_{i=1}^k \dim(C_i^\Sigma) - n(n-1)/2$ (as $\dim(\mathrm{O}_n(K)) = n(n-1)/2$). □

The results of Sections 6.1.2 and 6.1.3 give us the following inequalities:

$$\nu_{\mathrm{GL}_n(K)}(\tilde{\sigma}^{-1}(0)) \geq \sum_{i=1}^k \dim(C_i) - 2(n^2-1), \tag{6.1}$$

$$\nu_{\mathrm{GL}_n(K)}(\tilde{\sigma}_T^{-1}(0)) \leq \sum_{i=1}^k \dim(C_i^\Sigma) - n(n-1)/2. \tag{6.2}$$

The next two sections compute $\dim(C_i)$ and $\dim(C_i^\Sigma)$.

6.1.4 The Dimension of a Similarity Class

A similarity class is an orbit of $M_n(K)$ under the action of $\mathrm{GL}_n(K)$, therefore the dimension of a similarity class C is $\dim(\mathrm{GL}_n(K)) - \dim(\mathrm{Stab}_{\mathrm{GL}_n(K)}(A))$ (for some $A \in C$). The dimension of $\mathrm{GL}_n(K)$ is $n^2 - 1$ so we spend most of this section computing the dimension of the stabilizer.

Recall from Section 2.1.2 the function $\mu_C \colon \Psi(C) \to \mathcal{P}$ describes the Jordan normal form of the similarity class C (or of a matrix). If $\xi \in \Psi(C)$, then $\mu_C(\xi)$ is the integer partition giving the sizes of the Jordan blocks of C associated to ξ (in decreasing order of size).

Lemma 6.1.11. Given $A \in M_n(K)$, $B \in M_m(K)$ and $X \in K^{m\times n}$ (X has m rows and n columns) such that A and B have no eigenvalues in common, if $XA - BX = 0$, then $X = 0$.

Proof. [Gan59, Vol. I, Chap. VIII, Sec. 1]. □

Lemma 6.1.12. Let B be a block-diagonal matrix such that each block has a single eigenvalue and all blocks have pairwise distinct eigenvalues. Any matrix commuting with B must have the same block structure.

Proof. To see this let $B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} \in M_n(K)$ where B_1 and B_2 are blocks with no eigenvalue in common. Let $X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} \in M_n(K)$ be partitioned such that the blocks are conformal with the blocks of B, we have

$$\begin{aligned} XB - BX &= \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}\begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} - \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}\begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} \\ &= \begin{pmatrix} X_1B_1 & X_2B_2 \\ X_3B_1 & X_4B_2 \end{pmatrix} - \begin{pmatrix} B_1X_1 & B_1X_2 \\ B_2X_3 & B_2X_4 \end{pmatrix} \\ &= 0. \end{aligned}$$

We get relations $X_1B_1 - B_1X_1 = 0$, $X_4B_2 - B_2X_4 = 0$, $X_2B_2 - B_1X_2 = 0$ and $X_3B_1 - B_2X_3 = 0$. By Lemma 6.1.11 $X_2 = X_3 = 0$. □

Definition 6.1.13. We say a matrix $A \in K^{m\times n}$ *consists of diagonal bands* $a_1, \ldots, a_{m+n-1} \in K$ if $A_{i,j} = A_{i+1,j+1}$ for all $i = 1, \ldots, n-1$ and $j = 1, \ldots, m-1$, and if $A_{1,j} = a_{n-j+1}$ for $j = 1, \ldots, n$ and $A_{i,1} = a_{n+i-1}$ for $i = 1, \ldots, m$. We call $a_1, \ldots, a_{m+n-1}$ the *diagonals of* A.

To illustrate the above definition we consider the example where $m = 5$ and $n = 7$. The following matrix consists of diagonal bands $a_1, \ldots, a_{11} \in K$.

$$\begin{pmatrix} a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 \\ a_8 & a_7 & a_6 & a_5 & a_4 & a_3 & a_2 \\ a_9 & a_8 & a_7 & a_6 & a_5 & a_4 & a_3 \\ a_{10} & a_9 & a_8 & a_7 & a_6 & a_5 & a_4 \\ a_{11} & a_{10} & a_9 & a_8 & a_7 & a_6 & a_5 \end{pmatrix}$$

Let $A \in M_n(K), B \in M_m(K)$ be square matrices. Let $V(A, B)$ be the vector space of rectangular matrices $X \in K^{m\times n}$ such that $XA = BX$.

Lemma 6.1.14. All matrices in $V(\mathrm{J}_n(0), \mathrm{J}_m(0))$ have entries consisting of diagonal bands $a_1, \ldots, a_{m+n-1} \in K$ such that the diagonals $a_{\min(m,n)+1} = \cdots = a_{m+n} = 0$. We illustrate this with the $m = 5$ and $n = 7$ example:

$$\begin{pmatrix} 0 & 0 & a_5 & a_4 & a_3 & a_2 & a_1 \\ 0 & 0 & 0 & a_5 & a_4 & a_3 & a_2 \\ 0 & 0 & 0 & 0 & a_5 & a_4 & a_3 \\ 0 & 0 & 0 & 0 & 0 & a_5 & a_4 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_5 \end{pmatrix} \in V(\mathrm{J}_7(0), \mathrm{J}_5(0)).$$

Proof. The matrix $X\mathrm{J}_n(0)$ is the same as X but with all entries shifted one place right (with zeros in the left-most column) and $\mathrm{J}_m(0)X$ is the matrix X with all entries shifted one place up (with zeros in the bottom row).

$$\begin{pmatrix} 0 & X_{1,1} & \cdots & X_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & X_{m-1,1} & \cdots & X_{m-1,n-1} \\ 0 & X_{m,1} & \cdots & X_{m,n-1} \end{pmatrix} = \begin{pmatrix} X_{2,1} & X_{2,2} & \cdots & X_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ X_{m,1} & X_{m,2} & \cdots & X_{m,n} \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

So the entries $X_{i,j}$ must satisfy $X_{i,j} = X_{i+1,j+1}$ for all $i = 1, \ldots, m-1$, $j = 1, \ldots, n-1$. This implies X consists of diagonal bands. Comparing the two sides it is clear that some of the bands are zero, in particular the bands containing $X_{2,1}, \ldots, X_{m,1}, \ldots X_{m,n-1}$ are zero. If $m = n$ then every band under the main diagonal is zero. If $m > n$ then the $m-1$ lower-most bands are zero. If $m < n$ then the $n-1$ left-most bands are zero. $\square$

Remark 6.1.15. In light of Lemma 6.1.14 it is clear that $\dim(V(\mathrm{J}_n(0), \mathrm{J}_m(0))) = \min(n, m)$.

We calculate, in the following theorem, the number of parameters necessary to describe the stabilizer of a matrix. Recall that given an integer partition μ the length $\mathcal{L}(\mu)$ is the number of nonzero parts of μ if $\mu \neq (0)$ and one if $\mu = (0)$.

Theorem 6.1.16. Let $A \in M_n(K)$. We have

$$\dim(\mathrm{Stab}_{\mathrm{GL}_n}(A)) = \sum_{\xi \in \Psi(A)} \sum_{j_1=1}^{\mathcal{L}(\mu_A(\xi))} \sum_{j_2=1}^{\mathcal{L}(\mu_A(\xi))} \min(\mu_A(\xi)_{j_1}, \mu_A(\xi)_{j_2}).$$

Proof. Without loss of generality suppose A is a Jordan matrix. As the set of nonsingular matrices is dense the proof reduces to finding the dimension of the variety of matrices which commute with A.

By Lemma 6.1.12 we can concentrate on the case where A has only one eigenvalue. Let $\xi \in K$ be the eigenvalue of A. We can write $A = \xi I_n + N$ where N is nilpotent. Let

X be an arbitrary matrix commuting with A, so $XA = AX = X\xi + XN = \xi X + NX$ so we have $XN = NX$, in other words we need to think about matrices which commute with the nilpotent part of A.

Say $N = N_1 \oplus \cdots \oplus N_p$ where N_i is a nilpotent Jordan block of size $\mu_N(0)_i$. Let X be partitioned into blocks $X_{i,j}$, for $i, j = 1, \ldots, p$ such that the blocks are conformal with the blocks of N, that is $X_{i,j}$ has size $\mu_N(0)_i \times \mu_N(0)_j$. Then $XN = NX$ gives p^2 matrix relations of the form $X_{i,j}N_j = N_iX_{i,j}$. In each case we have $X_{i,j} \in V(N_j, N_i)$. By Lemma 6.1.14 $X_{i,j}$ consists precisely of $\min(\mu_N(0)_j, \mu_N(0)_i)$ arbitrary bands. So the number of parameters needed to describe $V(N_j, N_i)$ is $\min(\mu_N(0)_i, \mu_N(0)_j)$. Each $X_{i,j}$ is independent of every other block so the total number of parameters for X is

$$\sum_{j_1=1}^{\mathcal{L}(\mu_N(0))} \sum_{j_2=1}^{\mathcal{L}(\mu_N(0))} \min(\mu_N(0)_{j_1}, \mu_N(0)_{j_2}).$$

Clearly we have $\min(\mu_N(0)_{j_1}, \mu_N(0)_{j_2}) = \min(\mu_A(\xi)_{j_1}, \mu_A(\xi)_{j_2})$. When we generalize back to the case of multiple different eigenvalues we simply sum over each distinct eigenvalue to get the required formula. □

So we have, by Theorem 6.1.16, for $A \in M_n(K)$

$$\dim(\mathrm{Orb}_{\mathrm{GL}_n(K)}(A)) = n^2 - \sum_{\xi \in \Psi(A)} \sum_{j_1=1}^{\mathcal{L}(\mu_A(\xi))} \sum_{j_2=1}^{\mathcal{L}(\mu_A(\xi))} \min(\mu_A(\xi)_{j_1}, \mu_A(\xi)_{j_2}). \tag{6.3}$$

Theorem 6.1.17. If $\tilde{\sigma}^{-1}(0)$ is nonempty then

$$\nu_{\mathrm{GL}_n(K)}(\tilde{\sigma}^{-1}(0)) \geq (k-2)n^2 + 2 - \sum_{i=1}^{k} \sum_{\xi \in \Psi(C_i)} \sum_{r_1=1}^{\mathcal{L}(\mu_{C_i}(\xi))} \sum_{r_2=1}^{\mathcal{L}(\mu_{C_i}(\xi))} \min(\mu_{C_i}(\xi)_{r_1}, \mu_{C_i}(\xi)_{r_2}).$$

Proof. Substitute (6.3) into (6.1). □

6.1.5 The Dimension of a Symmetric Similarity Class

The symmetric similarity class C^Σ consisting of all symmetric matrices from a particular similarity class C is an $\mathrm{O}_n(K)$-orbit of $M_n^\Sigma(K)$ (the vector space of all n by n symmetric matrices), therefore $\dim(C^\Sigma) = \dim(\mathrm{O}_n(K)) - \dim(\mathrm{Stab}_{\mathrm{O}_n}(A))$ (for some $A \in C^\Sigma$). The dimension of $\mathrm{O}_n(K)$ is $n(n-1)/2$, so we compute the dimension of the stabilizer.

Recall the Lie algebra $\mathfrak{o}_n(K)$ of the orthogonal group $\mathrm{O}_n(K)$ is the set of skew-symmetric matrices, (i.e. $\mathfrak{o}_n(K) = \{A \in M_n(K) \colon A^T = -A\}$). The following lemma simplifies the calculation of $\dim(\mathrm{Stab}_{\mathrm{O}_n(K)}(A))$.

Lemma 6.1.18. Let $A \in M_n(K)$. The dimension of the stabilizer of A in $\mathrm{O}_n(K)$ is the same as the dimension of the stabilizer of A in $\mathfrak{o}_n(K)$, that is $\dim(\mathrm{Stab}_{\mathrm{O}_n(K)}(A)) = \dim\{X \in \mathrm{O}_n(K) \colon X^T A X = A\} = \dim\{X \in \mathfrak{o}_n(K) \colon AX = XA\} = \dim(\mathrm{Stab}_{\mathfrak{o}_n(K)}(A))$.

Proof. We show $\mathrm{Stab}_{\mathfrak{o}_n(K)}(A)$ is the Lie algebra of $\mathrm{Stab}_{\mathrm{O}_n(K)}(A)$ by computing the tangent space at the identity. The tangent space of $\mathrm{Stab}_{\mathrm{O}_n(K)}(A)$ at I_n is the space of $X \in M_n(K)$ such that $A(I_n + \epsilon X) = (I_n + \epsilon X)A$ and $(I_n + \epsilon X)^T(I_n + \epsilon X) = I_n$, where ϵ is small. Expanding the first one gives $AX = XA$, and expanding the second gives $\epsilon(X + X^T) + \epsilon^2 X^T X = 0$. As ϵ is small, this implies $X + X^T = 0$ so $X \in \mathfrak{o}_n(K)$. Therefore $\mathrm{Stab}_{\mathfrak{o}_n(K)}(A)$ is the Lie algebra of $\mathrm{Stab}_{\mathrm{O}_n(K)}(A)$, which implies their dimensions are equal. □

The next lemma shows that every Jordan block is similar to a particular symmetric matrix which we describe below. We use the fact that every Jordan block is similar to a symmetric matrix in Theorem 6.1.22.

The skew-diagonal of a square matrix is the bottom-left to top-right diagonal. Let E_n be the n by n *exchange matrix*, that is the matrix in $M_n(K)$ with ones along the bottom-left to top-right diagonal, and zeroes everywhere else, that is:

$$E_n = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{6.4}$$

Simple computation shows that $E_n \mathrm{J}_n(0)$ is $\mathrm{J}_n(0)$ reflected vertically and $\mathrm{J}_n(0)E_n$ is $\mathrm{J}_n(0)$ reflected horizontally.

Lemma 6.1.19. Let $\imath \in K$ such that $\imath^2 = -1$. The Jordan block $\mathrm{J}_n(\xi)$, where $\xi \in K$, is similar to the symmetric matrix

$$S_n(\xi) = \xi I_n + \frac{1}{2}\Big(\mathrm{J}_n(0) + \mathrm{J}_n(0)^T + \imath(\mathrm{J}_n(0)E_n - E_n \mathrm{J}_n(0))\Big).$$

Proof. [Gan59, Vol. II, Chap. XI, Sec. 3].[1] □

We can think of $S_n(\xi)$ as being a symmetric analogue to a Jordan block. Any matrix is similar to a direct sum of such matrices.

[1]Gantmacher considers complex symmetric matrices but the results generalize to the case of algebraically closed fields of characteristic zero by fixing a value for $\imath \in K$ where $\imath^2 = -1$

Lemma 6.1.20. Let A be a symmetric nonderogatory matrix. The only skew-symmetric matrix which commutes with A is the zero matrix.

Proof. By [Gan59, Vol. I, Chap. VIII, Sec. 2, Cor. 1] all matrices commuting with A can be expressed as polynomials in A. This is due to the fact $\mathrm{char}_A(x) = \mathrm{min}_A(x)$ as A is nonderogatory. Any polynomial in a symmetric matrix must also be symmetric and the only skew-symmetric matrix which is also symmetric is the zero matrix. □

Lemma 6.1.21. Let $A \in M_n(K)$ and $B \in M_m(K)$. We have

$$\dim(V(A,B)) = \dim(V(P^{-1}AP, Q^{-1}BQ))$$

for all $P \in \mathrm{GL}_n(K), Q \in \mathrm{GL}_m(K)$.

Proof. We show there is a bijective map from $V(A,B)$ to $V(P^{-1}AP, Q^{-1}BQ)$ given by $X \mapsto Q^{-1}XP$. If $X, Y \in V(A,B)$, then $Q^{-1}XP = Q^{-1}YP$ clearly implies $X = Y$, so the map is injective. If $Z \in V(P^{-1}AP, Q^{-1}BQ)$, then $QZP^{-1} \in V(A,B)$ as $QZP^{-1}A = BQZP^{-1}$ is equivalent to $Z(P^{-1}AP) = (Q^{-1}BQ)Z$. As $Q^{-1}(QZP^{-1})P = Z$, we see the map is surjective. □

We calculate the number of parameters necessary to describe the stabilizer (over the orthogonal group) of a symmetric matrix.

Theorem 6.1.22. Let $A \in M_n^{\Sigma}$. We have

$$\dim(\mathrm{Stab}_{\mathrm{O}_n(K)}(A)) = \sum_{\xi \in \Psi(A)} \sum_{j_1=1}^{\mathcal{L}(\mu_A(\xi))} \sum_{j_2=j_1+1}^{\mathcal{L}(\mu_A(\xi))} \min(\mu_A(\xi)_{j_1}, \mu_A(\xi)_{j_2}).$$

Proof. Without loss of generality we assume A is a block diagonal symmetric matrix with blocks of the form given in Lemma 6.1.19. As in Theorem 6.1.16 we can reduce to the case where A has a single eigenvalue, by Lemma 6.1.12. So let $A = \xi I_n + N$ where N is nilpotent symmetric (i.e. a block diagonal nilpotent matrix with blocks of the form given by Lemma 6.1.19). As in the general case the problem reduces to parameterizing matrices which commute with N. By Lemma 6.1.18 we can consider skew-symmetric matrices commuting with A rather than orthogonal ones. Let $N = N_1 \oplus \cdots \oplus N_p$ where N_i is nilpotent of size $\mu_N(0)_i$, and of the form given in Lemma 6.1.19. Note that, by Lemma 6.1.19, N_i is similar to a nilpotent Jordan block of size $\mu_N(0)_i$.

Let X be an arbitrary skew-symmetric matrix, let us partition X into blocks $X_{i,j}$ conformal with N, that is $X_{i,j}$ is of size $\mu_N(0)_i \times \mu_N(0)_j$. As X is skew-symmetric we also have that

$X_{i,j} = -X_{j,i}$, and in particular $X_{i,i}$ is skew-symmetric. To say X commutes with N gives p^2 matrix relations of the form $X_{i,j}N_j = N_iX_{i,j}$. If $i \neq j$, then transposing the relation for (j,i), gives $N_i^TX_{j,i}^T = X_{j,i}^TN_j^T$ which is $-N_iX_{i,j} = -X_{i,j}N_j$, i.e. the same relation as for (i,j). So in fact only $p(p+1)/2$ of these matrix relations are independent. If $i = j$, then $X_{i,i}N_i = N_iX_{i,i}$ states that the skew-symmetric matrix $X_{i,i}$ commutes with the symmetric matrix N_i by Lemma 6.1.20 this only happens when $X_{i,i}$ is zero. So, as all diagonal relations are trivial, we get $p(p-1)/2$ independent matrix relations.

If $i \neq j$ then the dimension of the set of possible matrices $X_{i,j}$ satisfying $X_{i,j}N_j = N_iX_{i,j}$ is $\dim(V(N_j, N_i))$. By Lemma 6.1.21 we have $\dim(V(N_j, N_i)) = \dim(V(J_{\mu_N(0)_j}(0), J_{\mu_N(0)_i}(0)))$. Using Lemma 6.1.14 the number of parameters needed to describe $V(J_{\mu_N(0)_j}(0), J_{\mu_N(0)_i}(0))$ is $\min(\mu_N(0)_j, \mu_N(0)_i)$. This is equal to $\min(\mu_A(\xi)_i, \mu_A(\xi)_j)$. So the number of degrees of freedom are given by

$$\sum_{j_1=1}^{\mathcal{L}(\mu_A(\xi))} \sum_{j_2=j_1+1}^{\mathcal{L}(\mu_A(\xi))} \min(\mu_A(\xi)_{j_1}, \mu_A(\xi)_{j_2}).$$

To generalize to the multiple eigenvalue case we sum over all eigenvalues. □

So we have, by Theorem 6.1.22, for $A \in M_n^\Sigma$

$$\dim(\mathrm{Orb}_{\mathrm{O}_n(K)}(A)) = n(n-1)/2 - \sum_{\xi\in\Psi(A)} \sum_{j_1=1}^{\mathcal{L}(\mu_A(\xi))} \sum_{j_2=j_1+1}^{\mathcal{L}(\mu_A(\xi))} \min(\mu_A(\xi)_{j_1}, \mu_A(\xi)_{j_2}). \tag{6.5}$$

Theorem 6.1.23. If $\tilde{\sigma}_T^{-1}(0)$ is nonempty then

$$\nu_{\mathrm{O}_n(K)}(\tilde{\sigma}_T^{-1}(0)) \leq (k-1)n(n-1)/2 - \sum_{i=1}^{k} \sum_{\xi\in\Psi(C_i)} \sum_{r_1=1}^{\mathcal{L}(\mu_{C_i}(\xi))} \sum_{r_2=r_1+1}^{\mathcal{L}(\mu_{C_i}(\xi))} \min(\mu_{C_i}(\xi)_{r_1}, \mu_{C_i}(\xi)_{r_2}).$$

Proof. Substitute (6.5) into (6.2). □

6.1.6 Comparing the Inequalities

In Theorems 6.1.17 and 6.1.23 we substituted the formulas for the dimensions of the similarity and symmetric similarity classes respectively into the inequalities at the end of Section 6.1.1. The inequalities now give us a condition upon which if there are irreducible solutions to the additive matrix problem, then there are irreducible solutions which are not symmetrizable. We consider the inequalities in full generality first then consider them in more specific cases.

General Case

Suppose $C_1, \dots, C_k$ are such that there exists an irreducible solution to the additive matrix problem. There exists irreducible solutions which are not symmetrizable if $\nu_{\mathrm{GL}_n(K)}(\tilde{\sigma}^{-1}(0)) > \nu_{\mathrm{GL}_n(K)}(\tilde{\sigma}_T^{-1}(0))$. This inequality is satisfied if

$$(k-2)n^2 + 2 - \sum_{i=1}^{k} \sum_{\xi \in \Psi(C_i)} \sum_{r_1=1}^{\mathcal{L}(\mu_{C_i}(\xi))} \sum_{r_2=1}^{\mathcal{L}(\mu_{C_i}(\xi))} \min(\mu_{C_i}(\xi)_{r_1}, \mu_{C_i}(\xi)_{r_2})$$
$$> (k-1)n(n-1)/2 - \sum_{i=1}^{k} \sum_{\xi \in \Psi(C_i)} \sum_{r_1=1}^{\mathcal{L}(\mu_{C_i}(\xi))} \sum_{r_2=r_1+1}^{\mathcal{L}(\mu_{C_i}(\xi))} \min(\mu_{C_i}(\xi)_{r_1}, \mu_{C_i}(\xi)_{r_2}).$$

We simplify this to get:

$$(k-3)n^2 + (k-1)n + 4 > 2\sum_{i=1}^{k} \sum_{\xi \in \Psi(C_i)} \sum_{r_1=1}^{\mathcal{L}(\mu_{C_i}(\xi))} \sum_{r_2=1}^{r_1} \min(\mu_{C_i}(\xi)_{r_1}, \mu_{C_i}(\xi)_{r_2}). \tag{6.6}$$

Diagonalizable Matrices

As a special case suppose the similarity classes are closed. In this case $\min(\mu_{C_i}(\xi)_{r_1}, \mu_{C_i}(\xi)_{r_2}) = 1$ and $\mathcal{L}(\mu_{C_i}(\xi)) = \mathrm{algr}_{C_i}(\xi)$. (6.6) reduces to:

$$(k-3)n^2 + (k-1)n + 4 > \sum_{i=1}^{k} \sum_{\xi \in \Psi(C_i)} \mathrm{algr}_{C_i}(\xi)(\mathrm{algr}_{C_i}(\xi) + 1).$$

Nonderogatory Matrices

Suppose the similarity classes are nonderogatory. In this case $\mu_{C_i}(\xi)_1 = \mathrm{algr}_{C_i}(\xi)$ and $\mu_{C_i}(\xi)_i = 0$ for $i > 1$, so $\mathcal{L}(\mu_{C_i}(\xi)) = 1$. So $\min(\mu_{C_i}(\xi)_1, \mu_{C_i}(\xi)_1) = \mu_{C_i}(\xi)_1 = \mathrm{algr}_{C_i}(\xi)$. (6.6) reduces to:

$$(k-3)n^2 + (k-1)n + 4 > 2\sum_{i=1}^{k} \sum_{\xi \in \Psi(C_i)} \mathrm{algr}_{C_i}(\xi).$$

Matrices with Distinct Eigenvalues

Suppose the similarity classes have n distinct eigenvalues, so they are both diagonalizable and nonderogatory. In this case $\mathcal{L}(\mu_{C_i}(\xi)) = 1$ and $\mu_{C_i}(\xi)_1 = 1$. So (6.6) reduces to $(k-3)n^2 - (k+1)n + 4 > 0$.

Theorem 6.1.24. The inequality $(k-3)n^2 - (k+1)n + 4 > 0$ is satisfied if and only if $n > 1$ and $k > 4/n + 3$.

Proof. We have

$$
\begin{aligned}
(k-3)n^2 - (k+1)n + 4 &= (k-3)n^2 - (k-3)n - 4n + 4 \\
&= (k-3)(n^2 - n) - 4(n-1) \\
&= ((k-3)n - 4)(n-1).
\end{aligned}
$$

So the inequality $(k-3)n^2 - (k+1)n + 4 > 0$ is satisfied if and only if $(k-3)n - 4 > 0$ and $n-1 > 0$, that is $k > 4/n+3$ and $n > 1$ as required (the case where both factors are negative does not arrise since $n \geq 1$). □

Conclusion

The result of this section assumes that an irreducible solution to the additive matrix problem exists. When such an assumption can be made the results of this section give a condition which, when satisfied, implies there are solutions which are not symmetrizable. More precisely if (6.6) is satisfied and there exists an irreducible solution to the additive matrix problem, then $\nu_{\mathrm{GL}_n(K)}(\tilde{\sigma}^{-1}(0)) > \nu_{\mathrm{GL}_n(K)}(\tilde{\sigma}_T^{-1}(0))$, that is the number of parameters of the kernel of $\tilde{\sigma}$ is strictly greater than the number of parameters of the kernel of $\tilde{\sigma}_T$, that is there are strictly more irreducible solutions to the additive matrix problem than there are symmetrizable irreducible solutions.

The implication does not work the other way however, if the inequality is not satisfied this does not imply the number of parameters of irreducible solutions is equal to the number of parameters of symmetrizable irreducible solutions, nor does it imply every irreducible solution is symmetrizable.

Using the language of Section 5.3, this result shows that in general there are fewer isomorphism classes in the category of strict symmetric representations, denoted $\widetilde{\mathrm{Rep}}_K^{\Sigma}(\Pi^\lambda(Q))$, than in the category of strict representations, denoted $\widetilde{\mathrm{Rep}}_K(\Pi^\lambda(Q))$, where Q is a star-shaped quiver and $\lambda \in K^{Q_0}$.

6.2 Existence of Symmetric Representations of a Given Dimension Vector

Given a positive dimension vector $\alpha \in \mathbb{Z}^{Q_0}$ of a given quiver Q and a K-vector $\lambda \in K^{Q_0}$ such that $\lambda \cdot \alpha = 0$, does there always exist a symmetric (or symmetrizable) representation of $\Pi^\lambda(Q)$ of dimension vector α? We prove in this section that where Q is of Dynkin type or extended Dynkin type $\tilde{A}_n$ or $\tilde{D}_n$ and α is a root there does in fact exist such a symmetric representation of $\Pi^\lambda(Q)$. We conjecture that the result holds for extended Dynkin type $\tilde{E}_n$ and discuss how this might be proved.

We say a quiver is *a Dynkin quiver* if its underlying graph is a simply laced Dynkin diagram, that is from one of the families A_n (for $n \geq 1$), D_n (for $n \geq 4$) or E_n (for $n = 6, 7, 8$). We say a quiver is an *extended Dynkin quiver* is its underlying graph is a simply laced extended Dynkin diagram, that is one of the families $\tilde{A}_n$ (for $n \geq 1$), $\tilde{D}_n$ (for $n \geq 4$) or $\tilde{E}_n$ (for $n = 6, 7, 8$). The facts assumed in this section are well-known and found in references such as [ASS06] and [DR76]. See [ASS06, Chap. VII, Sec. 2] for a description of the simply laced Dynkin and extended Dynkin diagrams.

Lemma 6.2.1. Let Q be a quiver, $\alpha \in \mathbb{Z}^{Q_0}$ a positive root of Q and $\lambda \in K^{Q_0}$ such that $\lambda \cdot \alpha = 0$. There exist simple representations $X_1, \ldots, X_r$ of $\Pi^\lambda(Q)$ such that $\mathbf{dim}(X_i) \in \Sigma_\lambda$ and $\mathbf{dim}(X_1) + \cdots + \mathbf{dim}(X_r) = \alpha$.

Proof. By Kac's theroem, as α is a root, there exists an indecomposable representation Y of Q of dimension vector α. By [CB01, Thm. 3.3] this lifts to a representation of $\Pi^\lambda(Q)$, as $\lambda \cdot \alpha = 0$. This representation may or may not be simple but in either case it has a composition series

$$0 = Y_0 \subset Y_1 \subset Y_2 \subset \cdots \subset Y_{r-1} \subset Y_r = Y$$

where $Y_0, \ldots, Y_r$ are subrepresentations such that $X_i = Y_i/Y_{i-1}$ is simple for $i = 1, \ldots, r$. It is clear that $\mathbf{dim}(X_1) + \cdots + \mathbf{dim}(X_r) = \alpha$. As the X_i are simple Theorem 1.2.9 gives $\mathbf{dim}(X_i) \in \Sigma_\lambda$. □

6.2.1 Dynkin Quivers

Let Q be a Dynkin quiver and $\alpha \in \mathbb{Z}^{Q_0}$ a positive root. It is well known that the only roots of Dynkin quivers are real. Let $\lambda \in K^{Q_0}$ such that $\lambda \cdot \alpha = 0$. Theorem 6.2.2 shows there exists a symmetric representation of $\Pi^\lambda(Q)$ of dimension vector α.

Theorem 6.2.2. There exists a symmetric representation of $\Pi^\lambda(Q)$ of dimension vector α.

Proof. By Lemma 6.2.1 there are positive roots $\beta_1, \ldots, \beta_r \in \Sigma_\lambda$ such that $\beta_1 + \cdots + \beta_r = \alpha$. By Theorem 4.1.14, for each $i = 1, \ldots, r$, there is an irreducible solution to the additive matrix problem (with each collection of conjugacy classes chosen to be compatible with λ and each β_i, see Remark 4.1.5). As Q is Dynkin each of the β_i is a real root, so each irreducible solution is rigid. By Theorem 5.3.7 each of the rigid irreducible solutions is symmetrizable. By using the functor $\mathcal{G}^{\Sigma}$ in Section 5.3.1 we see there is a symmetric representation of $\Pi^{\lambda}(Q)$ of dimension vector β_i. We take the direct sum of these representations to get a symmetric representation of $\Pi^{\lambda}(Q)$ of dimension vector α. □

6.2.2 Extended Dynkin Quivers

Let Q be an extended Dynkin quiver, let $\alpha \in \mathbb{Z}^{Q_0}$ be a positive root and let $\lambda \in K^{Q_0}$ such that $\lambda \cdot \alpha = 0$. In this case the simple representations $X_1, \ldots, X_r$ of $\Pi^{\lambda}(Q)$, such that $\alpha = \mathbf{dim}(X_1) + \cdots + \mathbf{dim}(X_r)$, given by Lemma 6.2.1 might be of imaginary dimension vector. When using the lemma we wish to show that for each of the $X_1, \ldots, X_r$ there exists at least one symmetrizable representation of the same dimension vector. We do this on a case by case basis (i.e. we look at each of the finite number of families of extended Dynkin quivers). By [ASS06, Chap. VII, Lem 4.2] all imaginary roots of extended Dynkin quivers are integer multiples of the *minimal imaginary root* (each extended Dynkin diagram has its own minimal imaginary root), so we need only find symmetric representations of dimension vector equal to the minimal imaginary root.

Due to time limitations it has not been possible to complete the proof in all cases, the following theorem applies when Q is of underlying type $\tilde{A}_n$ or $\tilde{D}_n$. The remaining unproven cases are when Q is of underlying type $\tilde{E}_6$, $\tilde{E}_7$ or $\tilde{E}_8$, however it is conjectured that the proof holds in these cases also, we describe one possible method of proving this at the end of the section. A further question is whether the result holds in the case where Q is neither Dynkin nor extended Dynkin. The methods used in this section do not extend to fully to general quivers. Given that there exists a representation of $\Pi^{\lambda}(Q)$ of dimension vector α whenever α is a root and $\lambda \cdot \alpha = 0$ (by [CB06, Thm. 2]), one might be tempted to conjecture that this holds for symmetric representations also, however the results of Section 6.1 show that the behaviour of strict symmetric representations is not always analogous to the behavious of strict representations. For this reason it would be unsafe to conjecture the result holds for general quivers.

Theorem 6.2.3. Suppose Q is extended Dynkin of underlying type $\tilde{A}_n$ or $\tilde{D}_n$. There exists a symmetric representation of $\Pi^{\lambda}(Q)$ of dimension vector α.

Proof. By Lemma 6.2.1 there are positive roots $\beta_1, \dots, \beta_r \in \Sigma_\lambda$ such that $\beta_1 + \dots + \beta_r = \alpha$, and simple representations $X_1, \dots, X_r$ of $\Pi^\lambda(Q)$ such that $\mathbf{dim}(X_i) = \beta_i$. If β_i is a real root, then by the proof of Theorem 6.2.2 X_i is symmetrizable. If β_i is imaginary, then $\beta_i = m_i\delta$ for some $m_i \geq 1$. By the results in the subsections below we see there exists a symmetrizable representation Z of $\Pi^\lambda(Q)$ of dimension vector δ so the direct sum of m copies of Z, i.e. $Z \oplus \dots \oplus Z$, is a symmetric representation of dimension vector β_i.

So there exists symmetric representations of $\Pi^\lambda(Q)$ of dimension vector $\beta_1, \dots, \beta_r$. We take the direct sum to get a symmetric representation of dimension vector α. □

We complete the proof of the above theorem in the following subsections in which we go through the $\tilde{A}_n$ and $\tilde{D}_n$ families of extended Dynkin quivers and show there exists a symmetric representation of $\Pi^\lambda(Q)$ of dimension vector δ, where Q is of the appropriate underlying type and δ is the minimal imaginary root of Q. The orientation we choose for Q does not matter as $\Pi^\lambda(Q)$ is constructed from $\overline{Q}$. We assume $\lambda \cdot \delta = 0$, this follows as $m\delta \in \Sigma_\lambda$ implies $\lambda \cdot (m\delta) = 0$. We end with a few remarks about the $\tilde{E}_n$ case.

$\tilde{A}_n$ type quivers

We consider quivers of underlying type $\tilde{A}_n$. This is the easiest case to deal with and the minimal imaginary root δ has ones at every vertex. Let Q have underlying graph $\tilde{A}_n$ with each arrow oriented in the same direction.

$$Q = \begin{tikzcd} & & 0 \arrow[drr, "a_n"] & & \\ 1 \arrow[urr, "a_0"] & 2 \arrow[l, "a_1"'] & \cdots \arrow[l, "a_2"'] & n-1 \arrow[l, "a_{n-2}"'] & n \arrow[l, "a_{n-1}"'] \end{tikzcd}$$

Theorem 6.2.4. There exists a symmetric representation of $\Pi^\lambda(Q)$ of dimension vector δ.

Proof. Let V be a representation of $\overline{Q}$ defined as follows: $V_i = K$ for each $i \in Q_0$, the map $V_{a_0} \in K$ is arbitrary and $V_{a_0^*} = V_{a_0}$, for each $i = 1, \dots, k$ the map $V_{a_i} \in K$ is arbitrary such that $V_{a_i}^2 = (\sum_{j=1}^{i} \lambda_j + V_{a_0}^2)$ (as K is algebraically closed it is always possible to define V_{a_i}) and $V_{a_i^*} = V_{a_i}$. This representation is clearly symmetric with the standard inner product at each vertex, we show it satisfies the deformed preprojective relations. For the vertex $i \in \overline{Q}_0$ such that $i \neq 0$ we have:

$$V_{a_i}V_{a_i^*} - V_{a_{i-1}^*}V_{a_{i-1}} = V_{a_i}^2 - V_{a_{i-1}}^2 = \Big(\sum_{j=1}^{i} \lambda_j + V_{a_0}^2\Big) - \Big(\sum_{j=1}^{i-1} \lambda_j + V_{a_0}^2\Big) = \sum_{j=1}^{i} \lambda_j - \sum_{j=1}^{i-1} \lambda_j = \lambda_i.$$

For the vertex $i = 0$ we have:

$$V_{a_0}V_{a_0^*} - V_{a_n^*}V_{a_n} = V_{a_0}^2 - V_{a_n}^2 = V_{a_0}^2 - \Big(\sum_{j=1}^{n}\lambda_j + V_{a_0}^2\Big) = -\sum_{j=1}^{n}\lambda_j = \lambda_0.$$

The last equation follows because $\lambda \cdot \delta = 0$ implies $\lambda_0 + \lambda_1 + \cdots + \lambda_n = 0$. So all deformed preprojective relations are satisfied, therefore V is a symmetric representation of $\Pi^\lambda(Q)$.

□

$\tilde{D}_n$-type quivers

We consider the case where Q has underlying type $\tilde{D}_n$. Let Q^m be the quiver

$$\begin{array}{ccccccc} u_1 & & & & & & w_1 \\ & \searrow^{a_1} & & & & \swarrow^{c_1} & \\ & & v_1 \xrightarrow{b_1} \cdots \xrightarrow{b_{m-1}} v_m & & & & \\ & \nearrow_{a_2} & & & & \nwarrow_{c_2} & \\ u_2 & & & & & & w_2 \end{array}$$

The parameter $m \geq 1$ is the number of vertices on the "middle bar". Note that Q^m is of type $\tilde{D}_{m+3}$. The minimal imaginary root of Q^m is $\delta^m = \begin{array}{ccc} 1 & & 1 \\ & 2 - \cdots - 2 & \\ 1 & & 1 \end{array}$. Given a representation X of $\Pi^\lambda(Q^m)$ for some $\lambda \in K^{Q_0^m}$, let $M(X) = X_{b_{m-1}}X_{b_{m-1}^*}$ if $m > 1$ and $M(X) = X_{a_1}X_{a_1^*} + X_{a_2}X_{a_2^*}$ if $m = 1$. So $M(X)$ returns the endomorphism of X_{v_m} obtained by "traversing" whichever arrows are to the immediate left of v_m (in the diagram).

We prove the existence of a symmetric representation by induction on m. Let $m \geq 2$ and suppose we are given a K-vector $\lambda \in K^{Q_0^m}$ of Q^m such that $\lambda \cdot \delta^m = 0$. Let $\lambda' \in K^{Q_0^{m-1}}$ be defined by $\lambda'_i = \lambda_i$, for $i \in \{u_1, u_2, v_1, \ldots, v_{m-2}, w_1, w_2\}$ and $\lambda'_{v_{m-1}} = \lambda_{v_{m-1}} + \lambda_{v_m}$, so λ' inherits all of its entries from the corresponding entries of λ with the exception of $\lambda'_{v_{m-1}}$ which is equal to the sum of λ_{m-1} and λ_m. That is

$$\lambda' = \begin{array}{ccc} \lambda_{u_1} & & \lambda_{w_1} \\ & \lambda_{v_1} - \cdots - \lambda_{v_{m-2}} - (\lambda_{v_{m-1}} + \lambda_{v_m}) & \\ \lambda_{u_2} & & \lambda_{w_2} \end{array}$$

Note that $\lambda' \cdot \delta^{m-1} = 0$ follows from $\lambda \cdot \delta^m = 0$. Theorem 6.2.5 is an induction argument which allows us to reduce the problem to the case where $m = 1$. Theorem 6.2.6 establishes the existence of symmetric representations in the $m = 1$ case. We work with matrix representations from hereon, we say a matrix representation X of $\overline{Q^m}$ is a *symmetric matrix* *representation* of $\overline{Q^m}$ if $X_{a^*} = X_a^T$. This is a symmetric representation with the standard inner product at each vertex.

Theorem 6.2.5. Let $m \geq 2$. A symmetric matrix representation Y of $\Pi^\lambda(Q^m)$ exists of dimension vector δ^m with $M(Y)$ diagonal whenever a symmetric matrix representation X of $\Pi^{\lambda'}(Q^{m-1})$ exists of dimension vector δ^{m-1} with $M(X)$ diagonal.

Proof. Let us assume such a symmetric matrix representation X of $\Pi^{\lambda'}(Q^{m-1})$ exists. We exhibit a symmetric matrix representation Y of $\Pi^\lambda(Q^m)$ with the required properties.

Let Y inherit the linear maps of X for $a_1, a_2, b_1, \ldots, b_{m-2}, c_1, c_2$ (and for the adjoined arrows), that is let $Y_a = X_a$ and $Y_{a^*} = X_{a^*}$ for $a \in \{a_1, a_2, b_1, \ldots, b_{m-2}, c_1, c_2\}$. What remains is to define $Y_{b_{m-1}}$ and $Y_{b^*_{m-1}}$. We need $Y_{b_{m-1}}$ and $Y_{b^*_{m-1}}$ to satisfy:

$$M(X) - Y_{b^*_{m-1}} Y_{b_{m-1}} = \lambda_{v_{m-1}} I_2, \tag{6.7}$$

$$Y_{b_{m-1}} Y_{b^*_{m-1}} + Y_{c_1} Y_{c_1^*} + Y_{c_2} Y_{c_2^*} = \lambda_{v_m} I_2. \tag{6.8}$$

Whilst we have that X satisfies:

$$M(X) + X_{c_1} X_{c_1^*} + X_{c_2} X_{c_2^*} = \lambda'_{v_{m-1}} I_2 = (\lambda_{v_{m-1}} + \lambda_{v_m}) I_2. \tag{6.9}$$

Let $\begin{pmatrix} a & b \\ b & c \end{pmatrix} = M(X) - \lambda_{v_{m-1}} I_2$ and $\begin{pmatrix} p & q \\ q & r \end{pmatrix} = Y_{b_{m-1}}$, note that a, b, c are fixed and p, q, r are to be determined, also note that we are defining $Y_{b_{m-1}}$ to be symmetric. Let $Y_{b^*_{m-1}} = Y^T_{b_{m-1}} = Y_{b_{m-1}}$. By the induction hypothesis $M(X)$ is diagonal so $b = 0$. We need to satisfy

$$\begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} = \begin{pmatrix} p & q \\ q & r \end{pmatrix} \begin{pmatrix} p & q \\ q & r \end{pmatrix} = \begin{pmatrix} p^2 + q^2 & q(p+r) \\ q(p+r) & q^2 + r^2 \end{pmatrix}.$$

Let $q = 0$ and choose p, r such that $p^2 = a$ and $r^2 = c$. So $M(Y) = Y_{b^*_{m-1}} Y_{b_{m-1}} = Y^T_{b_{m-1}} Y_{b_{m-1}} = \begin{pmatrix} p^2 & 0 \\ 0 & q^2 \end{pmatrix}$. By definition $Y_{b_{m-1}}$ and $Y_{b^*_{m-1}}$ satisfy (6.7), we now show they satisfy (6.8).

$$\begin{aligned}
\lambda_{v_m} I_2 &= \lambda'_{v_{m-1}} I_2 - \lambda_{v_{m-1}} I_2 \\
&= M(X) - \lambda_{v_{m-1}} I_2 + X_{c_1} X_{c_1^*} + X_{c_2} X_{c_2^*} && \text{(by substituting (6.9))} \\
&= M(X) - \lambda_{v_{m-1}} I_2 + Y_{c_1} Y_{c_1^*} + Y_{c_2} Y_{c_2^*}, \\
&= Y_{b^*_{m-1}} Y_{b_{m-1}} + Y_{c_1} Y_{c_1^*} + Y_{c_2} Y_{c_2^*} \\
&= Y_{b_{m-1}} Y_{b^*_{m-1}} + Y_{c_1} Y_{c_1^*} + Y_{c_2} Y_{c_2^*} && \text{(as } Y_{b_{m-1}} \text{ is symmetric).}
\end{aligned}$$

So (6.8) is satisfied. Clearly $M(Y)$ is diagonal so the theorem is proved. $\square$

So if we can show there exists a symmetric matrix representation X when $m = 1$ with $M(X)$ diagonal, then the same follows for any m.

Theorem 6.2.6. Let $\lambda \in K^{Q_0^1}$ such that $\lambda \cdot \delta^1 = 0$. There exists a symmetric representation X of $\Pi^\lambda(Q^1)$ of dimension vector δ^1 such that $M(X)$ is diagonal.

Proof. We exhibit a symmetric matrix representation X of $\Pi^\lambda(Q^1)$. Let

$$X_{a_1} = \begin{pmatrix} p \\ 0 \end{pmatrix}, \qquad X_{a_2} = \begin{pmatrix} q \\ 0 \end{pmatrix}, \qquad X_{c_1} = \begin{pmatrix} x \\ y \end{pmatrix}, \qquad X_{c_2} = \begin{pmatrix} z \\ w \end{pmatrix},$$

where $p, q, x, y, z, w \in K$ are to be determined. By the deformed preprojective relations at the extremities (i.e. at vertices u_1, u_2, w_1 and w_2), we require that the unknowns satisfy: $\lambda_{u_1} = -p^2$, $\lambda_{u_2} = -q^2$, $\lambda_{w_1} = -x^2 - y^2$, $\lambda_{w_2} = -z^2 - w^2$. So we choose $p, q, x, z \in K$ such that

$$p^2 = -\lambda_{u_1}, \qquad q^2 = -\lambda_{u_2}, \qquad x^2 = -(y^2 + \lambda_{w_1}), \qquad z^2 = -(w^2 + \lambda_{w_2}).$$

Note that we obtain an equally valid choice of p, q, x, z by flipping the signs. The central relation is:

$$\begin{aligned} \begin{pmatrix} \lambda_{v_1} & 0 \\ 0 & \lambda_{v_1} \end{pmatrix} &= \begin{pmatrix} p^2 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} q^2 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} x^2 & xy \\ xy & y^2 \end{pmatrix} + \begin{pmatrix} z^2 & zw \\ zw & w^2 \end{pmatrix} \\ &= \begin{pmatrix} p^2 + q^2 + x^2 + z^2 & xy + zw \\ xy + zw & y^2 + w^2 \end{pmatrix} \end{aligned}$$

So the remaining unknowns (y and w) need to satisfy

$$\lambda_{v_1} = -\lambda_{u_1} - \lambda_{u_2} - \lambda_{w_1} - \lambda_{w_2} - (y^2 + w^2), \tag{6.10}$$

$$\lambda_{v_1} = y^2 + w^2, \tag{6.11}$$

$$0 = xy + zw. \tag{6.12}$$

We now fix $y \in K$ such that $(y^2 + \lambda_{w_1})y^2 = (\lambda_{v_1} + \lambda_{w_2} - y^2)(\lambda_{v_1} - y^2)$ and $w \in K$ such that $w^2 = \lambda_{v_1} - y^2$ (satisfing (6.11)). These choices imply $(y^2 + \lambda_{w_1})y^2 = (w^2 + \lambda_{w_2})w^2$ which imply $x^2y^2 = z^2w^2$ which implies either $xy = zw$ or $xy = -zw$. As we have a choice of sign when choosing z we can flip the sign if necessary to ensure $xy = -zw$, which satisfies (6.12). As $\lambda \cdot \delta = 0$ implies $\lambda_{u_1} + \lambda_{u_2} + 2\lambda_{v_1} + \lambda_{w_1} + \lambda_{w_2} = 0$, we have that $w^2 = \lambda_{v_1} - y^2$ implies (6.10).

We have shown the existence of a symmetric representation of $\overline{Q}$ of dimension vector δ which satisfies the deformed preprojective relations. Lastly, we see that $M(X)$ is diagonal, as

$$M(X) = \begin{pmatrix} p \\ 0 \end{pmatrix} \begin{pmatrix} p & 0 \end{pmatrix} + \begin{pmatrix} q \\ 0 \end{pmatrix} \begin{pmatrix} q & 0 \end{pmatrix} = \begin{pmatrix} p^2+q^2 & 0 \\ 0 & 0 \end{pmatrix}.$$

□

$\tilde{E}_n$-type quivers

Time restrictions prevent us from demonstrating the validity of the theorem in the case where Q has underlying type $\tilde{E}_6$, $\tilde{E}_7$ or $\tilde{E}_8$. However we conjecture that the theorem holds in these cases. We describe one possible method for showing this and report that it has succeeded in the $\tilde{E}_6$ case, although we do not prove this due to time restrictions. Our method for exhibiting symmetric representations is to exhibit a triple of symmetric matrices which sum to zero and have similarity types which (by Section 5.3) imply the existence of a symmetric representation of $\Pi^\lambda(Q)$ of dimension vector δ. In this setting we have three similarity classes $C_1, C_2, C_3 \subseteq M_n(K)$ where $n = 3, 4, 6$ for $\tilde{E}_6, \tilde{E}_7, \tilde{E}_8$ respectively. We are able to reduce to the case where the roots of the minimal polynomials are distinct by the following argument.

If δ has a root decomposition $(\beta_1, \ldots, \beta_r)$, then as δ is the minimal imaginary root it follows that the $\beta_1, \ldots, \beta_r$ are real roots. If $(\beta_1, \ldots, \beta_r)$ is compatible with λ, then it follows from the proof of Theorem 6.2.3 that there exist symmetric representations of dimension vectors $\beta_1, \ldots, \beta_r$, so the direct sum yields a symmetric representation of dimension vector δ. We now assume there is no root decomposition of δ compatible with λ. Suppose for C_i where $i \in \{1, 2, 3\}$ the minimal polynomial $\min_{C_i}$ has a repeating zero, this would imply there was a root decomposition of δ compatible with λ. Let $\xi_{i,1}, \ldots, \xi_{i,d_i}$ be a list of the zeros of the minimal polynomial (with multiplicities), for some $j \neq j'$ we have $\xi_{i,j} - \xi_{i,j'} = 0$. Without loss of generality we can assume $j = 1$ and $j' = 2$. The equality can be expressed as $\lambda \cdot \epsilon_{i,1} = 0$ where $\epsilon_{i,1}$ is the simple root at the $[i, 1]$ vertex (recall simple roots defined in Section 1.1.2). We ensure $\delta - \epsilon_{i,1}$ is a positive root by checking through each of the finite number of possible cases (there are 6 cases to check). So $(\epsilon_{i,1}, \delta - \epsilon_{i,1})$ is a positive root decomposition of δ compatible with λ, contradicting our assumption that no such root decomposition of δ exists. Therefore all minimal polynomials have only nonrepeating zeros, that is C_1, C_2, C_3 are diagonalizable.

In the $\tilde{E}_6$ case all three classes have three distinct eigenvalues as $\delta = \begin{array}{lll} & & 2-1 \\ & \diagup & \\ 3 & - & 2-1 \\ & \diagdown & \\ & & 2-1 \end{array}$. In the $\tilde{E}_7$ case two classes have four distinct eigenvalues and the third has two distinct eigenvalues of algebraic multiplicity two as $\delta = \begin{array}{lll} & & 3-2-1 \\ & \diagup & \\ 4 & - & 3-2-1 \\ & \diagdown & \\ & & 2 \end{array}$. In the $\tilde{E}_8$ case one class has six distinct eigenvalues, one class has three distinct eigenvalues of algebraic multiplicity two and the third has two distinct eigenvalues of algebraic multiplicity three as $\delta = \begin{array}{lll} & & 5-4-3-2-1 \\ & \diagup & \\ 6 & - & 4-2 \\ & \diagdown & \\ & & 3 \end{array}$.

To exhibit three symmetric matrices A_1, A_2, A_3, in prescribed similarity classes, which sum to zero, it is enough to exhibit two matrices A_1, A_2 and prove $A_3 = -(A_1 + A_2)$ is in the third similarity class. As similarity does not change under orthogonal transformation we can assume one of the symmetric matrices, say A_1, is diagonal. By Theorem 6.1.7 as A_2 is symmetric it must be orthogonally similar to a diagonal matrix. So we have $A_1 = \operatorname{diag}(a_1, \dots, a_n) \in C_1$ and $A_2 = Q^T \operatorname{diag}(b_1, \dots, b_n) Q \in C_2$, where $a_1, \dots, a_n, b_1, \dots, b_n \in K$ are fixed by the similarity classes and where $Q \in \mathrm{O}_n(K)$ is to be determined. We need to exhibit some $Q \in \mathrm{O}_n(K)$ such that $A_3 = -(A_1 + A_2) \in C_3$.

In each case we have one similarity class with distinct eigenvalues, suppose this class is C_3 so we can determine the similarity type of $A_3 = -(A_1 + A_2)$ from the characteristic polynomial $\operatorname{char}_{C_3}$. So we have to show there exists an orthogonal matrix Q such that the coefficients of

$$\operatorname{char}_{C_3}(x) = \det\left(xI_n + \begin{pmatrix} a_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & a_n \end{pmatrix} + Q^T \begin{pmatrix} b_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & b_n \end{pmatrix} Q \right), \tag{6.13}$$

can take any value (except for those coefficients which are fixed by $a_1, \dots, a_n, b_1, \dots, b_n$ and do not depend on Q). Obviously the coefficent of x^n is always one in the characteristic polynomial and the coefficent of x^{n-1} is always $-\operatorname{trace}(C_3)$ (in fact we have $\operatorname{trace}(C_3) = -a_1 - \dots - a_n - b_1 - \dots - b_n$). All other coefficients in (6.13) depend on Q, so we need to show the coefficients of $1, x, \dots, x^{n-2}$ can take any arbitrary value. As a further simplification, if necessary we can assume the matrices each have trace zero without loss of generality.

Unfortunately time restrictions prevent us from using this method to completion, however we have had success in the $\tilde{E}_6$ case using orthogonal matrices of the form

$$Q = \begin{pmatrix} p_1 & q_1 & 0 \\ -q_1 & p_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & p_2 & q_2 \\ 0 & -q_2 & p_2 \end{pmatrix}.$$

where $p_1, p_2, q_1, q_2 \in K$ are to be determined such that $p_1^2 + q_1^2 = p_2^2 + q_2^2 = 1$. Though there are four coefficients of $\mathrm{char}_{A_3}(x)$, only two are arbitrary: the coefficients of 1 and x. When we expand $\mathrm{char}_{A_3}(x) = \det(xI_3 + \mathrm{diag}(a_1, a_2, a_3) + Q^T \mathrm{diag}(b_1, b_2, b_3)Q)$, we see it is possible to choose $p_1, p_2 \in K$ such that the coefficients of 1 and x take on any value (q_1 and q_2 are easily removed from the calculation using $p_1^2 + q_1^2 = p_2^2 + q_2^2 = 1$). Given more time we would show the calculation does in fact yield the result and show whether it works for $\tilde{E}_7$ and $\tilde{E}_8$.

Bibliography

[ARS95] Maurice Auslander, Idun Reiten, and Sverre O. Smalø. *Representation theory of Artin algebras*, volume 36 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1995.

[Art91] Michael Artin. *Algebra*. Prentice-Hall, 1991.

[ASS06] Ibrahim Assem, Daniel Simson, and Andrzej Skowroński. *Elements of the representation theory of associative algebras. Vol. 1*, volume 65 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2006. Techniques of representation theory.

[BRW96] Desiree A. Beck, Jeffrey B. Remmel, and Tamsen Whitehead. The combinatorics of transition matrices between the bases of the symmetric functions and the b_n analogues. *Discrete Mathematics*, 153:3–27, 1996.

[CB93] William Crawley-Boevey. Geometry of representations of algebras. Technical report, Mathematical Institute, Oxford University, December 1993.

[CB01] William Crawley-Boevey. Geometry of the moment map for representations of quivers. *Compositio Mathematica*, 126:257–293, 2001.

[CB03] William Crawley-Boevey. On matrices in prescribed conjugacy classes with no common invariant subspace and sum zero. *Duke Mathematical Journal*, 118:339–352, 2003.

[CB04] William Crawley-Boevey. Indecomposable parabolic bundles and the existence of matrices in prescribed conjugacy class closures with product equal to the identity. *Publications Mathématiques de L'IHÉS*, 100(1):171–207, 2004.

[CB06] William Crawley-Boevey. Quiver algebras, weighted projective lines, and the Delingne-Simpson problem. In *Proceedings of the ICM, Madrid, Spain, 2006*, 2006.

[CBH98] William Crawley-Boevey and Martin P. Holland. Noncommutative deformations of Kleinian singularities. *Duke Mathematical Journal*, 92(3), 1998.

[DR76] Vlastimil Dlab and Claus Michael Ringel. Indecomposable representations of graphs and algebras. *Memoirs of the Americal Mathematical Society*, 6(173), 1976.

[DW02] Harm Derksen and Jerzy Weyman. Generaliszed quivers associated to reductive groups. *Colloquium Mathematicum*, 94(2):151–173, 2002.

[Fin78] Daniel T. Finkbeiner, II. *Introduction to Matrices and Linear Transformations*. W. H. Freeman and Company, 3rd edition, 1978.

[Gan59] Feliks Ruvimovich Gantmacher. *The Theory of Matrices*. Chelsea Pub. Co, 1959.

[Ger59] Murray Gerstenhaber. On nilalgebras and linear varieties of nilpotent matrices, iii. *Annals of Mathematics*, 70:167–205, 1959.

[Hal58] Paul R. Halmos. *Finite Dimensional Vector Spaces*. D. Van Nostrand Company, Inc., 2 edition, 1958.

[Hum75] James E. Humphreys. *Linear Algebraic Groups*. Springer Verlag, 1975.

[Kac90] Victor G. Kac. *Infinite Dimensional Lie Algebras*. Cambridge University Press, third edition, 1990.

[Kap03] Irving Kaplansky. *Linear Algebra and Geometry, A Second Course*. Dover Publications, Inc, 2003.

[Lam05] T.Y. Lam. *Introduction to Quadratic Forms over Fields*. American Mathematical Society, 2005.

[Lan02] Serge Lang. *Algebra*. Springer, 3rd edition, 2002.

[Mac95] I. G. Macdonald. *Symmetric Functions and Hall Polynomials*. Oxford Science Publications, 1995.

[Mum88] David Mumford. *The Red Book of Varieties and Schemes*, volume 1358 of *Lecture Notes in Mathematics*. SPringer-Verlag, 1988.

[NS99] O. Neto and F. Silva. On the recognition and rigidity problems for sums of matrices. *Linear Algebra and its Applications*, 293:73–84, 1999.

[Shm06] D. A. Shmelkin. Signed quivers, symmetric quivers and root systems. *J. London Math. Soc.*, 73(2):586–606, 2006.

[Sil90] F. Silva. Sums and products of matrices with prescribed similarity classes. *Linear and Multilinear Algebra*, 27:317–323, 1990.