# On real root representations of quivers

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"Gott gebe mir die Gelassenheit, Dinge hinzunehmen, die ich nicht ändern kann; den Mut, Dinge zu ändern, die ich ändern kann; und die Weisheit, das eine vom anderen zu unterscheiden."

Reinhold Niebuhr

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# Abstract


Let $Q$ be a quiver and let $\alpha$ be a positive real root of the associated root system. A theorem of Kac states that there exits a unique indecomposable representation (up to isomorphism) of $Q$ of dimension vector $\alpha$, called a *real root representation*.

We study real root representations and focus on the following question.

*How can one "construct" real root representations and what are their "properties"?*

We introduce the notion of maximal rank type for representations of quivers, which requires certain collections of maps involved in the representation to be of maximal rank, and we show that real root representations have the maximal rank type property.

Using the maximal rank type property and the universal extension functors introduced by Ringel we construct all real root representations of the quiver

$$\begin{tikzcd} Q(f,g,h)\colon\ 1 \arrow[r, shift left=3, "\lambda_1"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=3, "\lambda_f"'] & 2 \arrow[r, bend left=60, "\mu_1"] \arrow[r, bend left=20, "\mu_g"'] & 3 \arrow[l, bend left=20, "\nu_1"'] \arrow[l, bend left=60, "\nu_h"] \end{tikzcd},$$

with $f, g, h \geq 1$. This shows in particular that real root representations of $Q(f,g,h)$ are tree representations. Moreover, formulae given by Ringel can be applied to compute the dimension of the endomorphism ring of a given real root representation.

We ask whether this construction process involving universal extension functors generalises to all quivers and discuss examples of representations which cannot be constructed using universal extension functors.

# Contents

# Chapter 1

# Introduction and outline of thesis

Algebras are objects of fundamental importance in mathematics as well as in physics. A particularly nice example of an algebra is given by the path algebra of a quiver which is simply a directed graph. Here is an example of a quiver

$$\begin{tikzcd} \bullet \arrow[dr] & & \\ \bullet \arrow[r] & \bullet \arrow[r] & \bullet \\ \bullet \arrow[ur] & & \end{tikzcd} .$$

Over the last few decades quivers have played an important role in many areas of mathematics, including representation theory of finite dimensional algebras, the theory of Hall algebras and quantum groups, to name but a few.

One is particularly interested in representations of an algebra; that is, the ways in which a given algebra operates on a vector space. One of the reasons why quivers are very fruitful objects in mathematics is because many mathematical problems, especially many problems in linear algebra, can be reformulated in terms of representations of quivers.

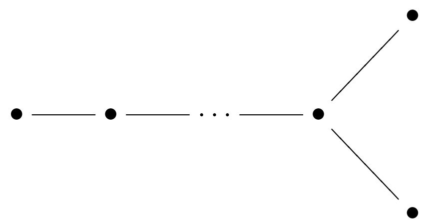
In Chapter 2 we discuss quivers, representations of quivers and related results.

One of the first milestones in the the representation theory of quivers was the work by Gabriel [11] which classified quivers of finite representation type; that is, quivers with only finitely many non-isomorphic indecomposable representations. Gabriel showed that a connected quiver is of finite representation type if and only if the underlying graph is included in the following list, where the subscript $n$ denotes the number of vertices.

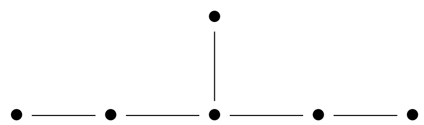
$A_n,\ n \geq 0:$



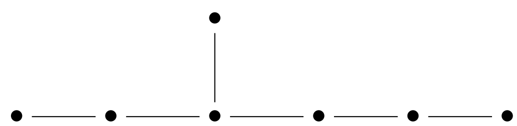
$D_n,\ n \geq 4:$



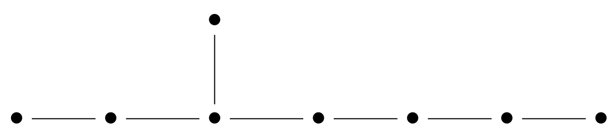
$E_6:$



$E_7:$



$E_8:$



The above diagrams are called Dynkin diagrams; they appear in many different areas of mathematics, for instance the classification of finite dimensional simple Lie algebras. In [3] Bernstein, Gelfand and Ponomarev constructed the indecomposable representations of finite representation type quivers using certain reflection functors. The reflection functors of Bernstein, Gelfand and Ponomarev were one of the first tools invented to study representations of quivers. Gabriel showed that the indecomposable representations are in bijection with the positive roots of the corresponding finite dimensional simple Lie algebra. This bijection is provided by a combinatorial gadget, namely root systems which are relevant to quivers as well as Lie algebras.

In the early 1980's Kac [13] generalised Gabriel's work to all quivers, showing that for an arbitrary quiver the indecomposable representations correspond to the positive roots of the associated root system. To a quiver one can also associate a Lie algebra, namely the Kac-Moody algebra. In this way one obtains a bijection between the indecomposable representations and the positive roots of the Kac-Moody algebra, which generalises Gabriels results. The root system of a quiver consists of two types of roots: real roots (which can be obtained by reflections from simple roots) and imaginary roots (which can be obtained by reflections from roots in the fundamental region). Kac showed that for each positive real root there exists a unique indecomposable representation of the corresponding dimension vector and for each positive imaginary root there exists a family (depending on the size of the ground field) of non-isomorphic indecomposable representations of the corresponding dimension vector. We call the unique indecomposable representation corresponding to a positive real root $\alpha$ a *real root representation* and denote it by $X_\alpha$.

Representations of quivers can be decomposed into indecomposable representations by the Krull-Remak-Schmidt Theorem. Hence, in order to understand the representation theory of a given quiver, one only needs to understand the indecomposable representations. In this thesis we focus on certain indecomposable representations for quivers: the real root representations. We will be concerned with the following question.

**Question** (†). *How can one "construct" real root representations and what are their "properties" ?*

The above question is, of course, formulated rather vaguely but does nevertheless address the central problem of this thesis. However, the terms "construct" and "properties" need to be discussed further. What do we mean by "construct"?

So far it is not known how to construct real root representations representations in general. We explain below certain special situations in which construction processes are known.

We mentioned earlier that Bernstein, Gelfand and Ponomarev constructed the indecomposable representations of finite representation type quivers (which are all real root representations) using certain reflection functors. Hence, there are tools which give an answer to Question (†) for finite representation type quivers.

Over an algebraically closed field of characteristic zero, Crawley-Boevey [7] gave a method to construct real root representations of a given arbitrary quiver using deformed preprojective algebras, discussed in Section 2.3. He showed that real root representations arise as simple representations of deformed preprojective algebras and gave functors to construct these simple representations. This answers part one of Question (†) over algebraically closed fields of characteristic zero. We note that characteristic zero is essential for this construction. Moreover, constructing real root representations in this way does not give many insights into their properties.

A different approach to the problem of constructing real root representations was given by Ringel [18] for the following class of quivers

$$Q''(g,h): \qquad \begin{tikzcd} 2 \arrow[r, bend left=60, "\mu_1"] \arrow[r, bend left=20, "\mu_g"'] & 3 \arrow[l, bend left=20, "\nu_1"'] \arrow[l, bend left=60, "\nu_h"] \end{tikzcd},$$

with $g, h \geq 1$. Ringel showed that all real root representations of this class of quivers can be constructed using universal extension functors, introduced in [18]. A detailed exposition of these functors is given in Chapter 3; here we just give a brief description. Let $Q$ be a quiver and let $k$ be a field. Let $S$ be a real Schur representation of $Q$, that is a real root representation with trivial endomorphism ring, then by [18, Section 1, Proposition 2] there exists an equivalence $\sigma_S$ (called *universal extension functor*) defined on certain full subcategories of the category of finite dimensional representations of $Q$, namely

$$\sigma_S \quad : \quad \mathfrak{M}^{-S}_{-S} \xrightarrow{\sim} \mathfrak{M}^{S}_{S}/S,$$

where $\mathfrak{M}^{-S}_{-S}$ denotes the finite dimensional representations $X$ of $Q$ with no homomorphisms between $X$ and $S$ in either way, and $\mathfrak{M}^{S}_{S}$ denotes the finite dimensional representations $X$ of

$Q$ with no extensions between $X$ and $S$ in either way and moreover, such that no direct summand of $X$ is generated or cogenerated by $S$. Moreover, $\mathfrak{M}_S^S/S$ denotes the quotient category of $\mathfrak{M}_S^S$ modulo the maps factoring through direct sums of copies of $S$. On a given representation $X$ in $\mathfrak{M}_{-S}^{-S}$ the functor $\sigma_S$ operates as follows: it extends $X$ by copies of $S$ at the top and at the bottom until no non-trivial further extension is possible. Moreover, we have for $X \in \mathfrak{M}_{-S}^{-S}$

$$\begin{aligned} \underline{\dim}\, \sigma_S(X) &= \underline{\dim}\, X - (\underline{\dim}\, X, \underline{\dim}\, S) \cdot \underline{\dim}\, S, \quad \text{and} \\ \dim \operatorname{End}_{kQ} \sigma_S(X) &= \dim \operatorname{End}_{kQ} X + \langle \underline{\dim}\, X, \underline{\dim}\, S \rangle \cdot \langle \underline{\dim}\, S, \underline{\dim}\, X \rangle, \end{aligned}$$

where $\underline{\dim}\, M$ denotes the dimension vector of a given representation $M$ (defined in Section 2.2), $\langle -, - \rangle$ denotes the Ringel form and $(-, -)$ its symmetrisation (defined in Section 2.1), and $\operatorname{End}_{kQ} M$ denotes the endomorphism ring of a given representation $M$ (defined in Section 2.2).

The description of the functor $\sigma_S$ already shows one of the main difficulties when applying it; if we want to apply $\sigma_S$ to a given representation $X$ we have to make sure that the representation $X$ is in the subcategory $\mathfrak{M}_{-S}^{-S}$, that is we have no homomorphisms between $X$ and $S$ in either way.

In [18, Section 2] Ringel constructed real root representations of $Q'' = Q''(g, h)$ as follows. Let $\alpha$ be a positive real root for $Q''$. Write $\alpha = s_{i_n} \cdot \ldots \cdot s_{i_1}(e_j)$ with $i_k, j \in \{2, 3\}$ and $n$ minimal, where $s_i$ denotes the simple reflection at vertex $i \in \{2, 3\}$ (defined in Section 2.1) and $e_j$ $(j \in \{2, 3\})$ denotes the simple root at vertex $j$ (defined in Section 2.1). Then we have

$$X_\alpha = \sigma_{S(i_n)} \cdot \ldots \cdot \sigma_{S(i_1)}(S(j)),$$

where $S(i)$ $(i = 2, 3)$ denotes the simple representation at vertex $i$ (defined in Section 2.2). This result and a generalisation are discussed Section 3.2.

The above construction of the real root representation $X_\alpha$ does not just give a way to obtain the representation $X_\alpha$, but also gives insights into the properties of $X_\alpha$ based on the following fact. Namely, the functor $\sigma_S$ allows one to keep track of certain properties of representations. For instance, when the functor $\sigma_S$ is applied to an indecomposable tree representation (for definition see Section 2.2.1) the resulting representation will also be a tree representation by

Theorem 3.1.8. Or, if we apply the functor $\sigma_S$ to a representation of known endomorphism ring dimension, we can easily compute the dimension of the endomorphism ring of the resulting representation by Formula 3.1. This leads us to the second part of Question (†): *'... what are their "properties" ?'*. What would we like to know about real root representations? Here are just two questions we might ask. What is the structure of the endomorphism ring, or simpler, what is the dimension of the endomorphism ring of a given real root representation? Is a given real root representation a tree representation?

A partial answer to the second question follows from a result of Ringel [19, Theorem]. Ringel showed that exceptional representations are tree representations, and hence real Schur representations are tree representations. However, in general it is still open whether real root representations are tree representations.

These questions can now be answered for an arbitrary real root representation of the quiver $Q''(g,h)$, since the simple representation $S(j)$ is trivially a tree representation (and this property is preserved by the functors $\sigma_{S(i_k)}$) and has one-dimensional endomorphism ring (and hence we can compute the dimension of the endomorphism ring after applying the functors $\sigma_{S(i_k)}$).

The heart of this thesis is the following observation which suggests a generalisation of the above construction process for real root representations of the quiver $Q''(g,h)$ to other classes of quivers. Numerical experiments with real root representations, using Crawley-Boevey's method to construct real root representations over algebraically closed fields of characteristic zero, show that for certain classes of quivers we have the following behavior. Let $\alpha$ be a positive non-Schur real root (a real Schur root is a positive real root such that $X_\alpha$ is a real Schur representation), then there exists a real Schur root $\beta$ and a positive real root $\gamma$ such that

$$X_\gamma \in \mathfrak{M}_{-X_\beta}^{-X_\beta}, \quad \text{and} \quad \sigma_{X_\beta}(X_\gamma) = X_\alpha.$$

Moreover, it follows from the theory of universal extension functors (described in Chapter 3) that is this case we have

$$\alpha = -(\beta,\gamma)\cdot\beta+\gamma, \qquad \text{with} \quad (\beta,\gamma)\leq 0, \quad \text{and hence} \quad \gamma<\alpha, \quad \text{and}$$

$$\dim \operatorname{End}_{kQ} X_\alpha \quad = \quad \dim \operatorname{End}_{kQ} X_\gamma + \langle\beta,\gamma\rangle\cdot\langle\beta,\gamma\rangle.$$

An indication of the numerical data supporting this claim and a description of the algorithms used to obtain this data is given in Appendix A.

Now, if $\gamma$ is not a real Schur root, we can continue this process with $\gamma$ in place of $\alpha$, and reduce further in the way described above. We can do this until we arrive at a real Schur representation; in this way we obtain for a given positive real root $\alpha$ a sequence $\beta_n,\dots,\beta_1$ $(n\geq 2)$ of real Schur roots such that

$$X_\alpha = \sigma_{X_{\beta_n}}\cdot\ldots\cdot\sigma_{X_{\beta_2}}(X_{\beta_1}),$$

and hence we can answer the questions about $X_\alpha$ we are interested in (namely the question of the dimension of its endomorphism ring, and whether it is a tree representation) once we have answered these questions for the real Schur representation $X_{\beta_1}$. But for real Schur representations the answers to these questions are known: the dimension of the endomorphism ring is one and real Schur representations are tree representations by Ringel's result [19, Theorem].

In this way we have not only found a way to construct the representation $X_\alpha$ over an arbitrary field but also gained an insight into certain properties of it. Concrete examples of this construction process can be found in Appendix A.

In this thesis we approach Question (†) by focusing on the question whether the construction process described above works for arbitrary quivers. We ask the following question.

**Question** (††). *Let $Q$ be a quiver and let $k$ be a field. Let $\alpha$ be a positive non-Schur real root. Does there exist a sequence $\beta_n,\dots,\beta_1$ $(n\geq 2)$ of real Schur roots such that*

$$X_\alpha = \sigma_{X_{\beta_n}}\cdot\ldots\cdot\sigma_{X_{\beta_2}}(X_{\beta_1})?$$

Note, for a real Schur root $\beta$ we can take the trivial sequence $\beta$, and hence we are basically asking the question of whether all real root representations can be constructed from real Schur representations using universal extension functors; the real Schur representations are considered the "bricks" of this construction process. Our motivation for the above question is based on numerical experiments with real root representations, as indicated in Appendix A. It was, however, brought to our attention that a positive answer to Question (††) had also been conjectured by Ringel.

In order to investigate this question further we have to develop tools to decide whether the functor $\sigma_S$ can be applied to a given representation, that is we have to be able to determine whether a given representation is in the subcategory $\mathfrak{M}^{-S}_{-S}$. In Chapter 4 we discuss the maximal rank type property, as introduced in [26]; this property can help to decide this question. The maximal rank type property requires certain collections of maps involved in the representation to be of maximal rank. The collections of maps considered are the following: for a given vertex and a collection of incoming vertices we can form the sum of these maps and equally for a collection of outgoing vertices. A representation is said to be of maximal rank type if all these maps have maximal rank.

The maximal rank type property of real root representations is one of the authors main results.

**Theorem** ([26, Theorem A]). *Let $Q$ be a quiver and let $\alpha$ be a positive real root for $Q$. The unique indecomposable representation of dimension vector $\alpha$ is of maximal rank type.*

In Section 4.3 we discuss implications of the maximal rank type property of real root representations. One implication is that the dimension of homomorphism spaces and extension spaces between real root representations and simple representations are determined by the Ringel form, and hence are completely combinatorial. In particular, this gives a way to decide whether a real root representation is in the subcategory $\mathfrak{M}^{-S}_{-S}$, for $S$ a simple representation associated to a vertex of a given quiver $Q$.

In Section 4.4 we use the maximal rank type property to construct real root representations of the quiver

$$Q(f,g,h)\colon\ \begin{tikzcd} 1 \arrow[r, shift left=2, "\lambda_1"] \arrow[r, shift right=2, "\lambda_f"'] & 2 \arrow[r, bend left=70, "\mu_1"] \arrow[r, bend left=35, "\mu_g"'] & 3 \arrow[l, bend left=70, "\nu_h"] \arrow[l, bend left=35, "\nu_1"'] \end{tikzcd}\ ,$$

with $f, g, h \geq 1$.

The quiver $Q(1,1,1)$ was considered by Jensen and Su in [12], where all real root representations were constructed explicitly. Moreover, it was shown that all real root representations are tree representations, and formulae to compute the dimensions of the endomorphism rings were given. In [20] Ringel extended their results to the quiver $Q(1,g,h)$ $(g,h \geq 1)$ using universal extension functors.

The main result of Section 4.4 is the following.

**Theorem** ([26, Theorem B])**.** *Let $\alpha$ be a positive real root for the quiver $Q(f,g,h)$. The unique indecomposable representation of dimension vector $\alpha$ can be constructed by using universal extension functors starting from simple representations and real Schur representations of the quiver $Q'(f)$ $(f \geq 1)$, where $Q'(f)$ denotes the following subquiver of $Q(f,g,h)$*

$$Q'(f)\colon\ \begin{tikzcd} 1 \arrow[r, shift left=2, "\lambda_1"] \arrow[r, shift right=2, "\lambda_f"'] & 2 \end{tikzcd}\ .$$

The sequence of real Schur roots $\beta_n, \ldots, \beta_1$ can be obtained in a purely combinatorial way: one only needs to write the real root $\alpha$ in a certain admissible form (described in Section 4.4). We consider an example. Let $\alpha$ be the following real root for $Q(f,g,h)$ with $f \geq 2$.

$$\alpha = s_1\, s_2\, s_1\, s_3\, s_2\, s_1\, s_2\, s_1\, s_2\, s_3\, s_1\, s_2\, (e_3).$$

The real root $\alpha$ is already written in admissible form. The real Schur roots $\beta_n, \ldots, \beta_1$ can be obtained as follows. Each occurrence of $s_3$ gives the real Schur root $e_3$. The symmetric expressions not involving $s_3$ give a real Schur root of the subquiver $Q'(f)$. It can be obtained by taking the left half of the expression and replacing the reflection in the middle by the corresponding simple root. The real Schur root corresponding to the rightmost non-symmetric expression not involving $s_3$, here $s_1\, s_2$, is the one corresponding to the symmetrisation of it. This process gives the following real Schur roots for the above example

$$\begin{tikzcd}[column sep=small]
\alpha = & s_1\, s_2\, s_1 \arrow[dd, squiggly] & s_3 \arrow[dd, squiggly] & s_2\, s_1\, s_2\, s_1\, s_2 \arrow[dd, squiggly] & s_3 \arrow[dd, squiggly] & s_1\, s_2 \arrow[d, squiggly] & (e_3), \\
& & & & & s_1\, s_2\, s_1 \arrow[d, squiggly] & \\
& s_1(e_2) & e_3 & s_2\, s_1(e_2) & e_3 & s_1(e_2) &
\end{tikzcd}$$

and hence

$$\beta_6 = s_1(e_2),\, \beta_5 = e_3,\, \beta_4 = s_2\, s_1(e_2),\, \beta_3 = e_3,\, \beta_2 = s_1(e_2),\, \beta_1 = e_3.$$

The real root representation $X_\alpha$ of dimension vector $\alpha$ can be constructed as follows

$$X_\alpha = \sigma_{X_{\beta_6}}\, \sigma_{X_{\beta_5}}\, \sigma_{X_{\beta_4}}\, \sigma_{X_{\beta_3}}\, \sigma_{X_{\beta_2}}\, (X_{\beta_1}).$$

The class $Q(f,g,h)$, $(f,g,h \geq 1)$ of quivers provides another example for which Question (††) can be answered positively. Moreover, it provides further motivation to study Question (††) in general.

In Chapter 5 we discuss Question (††) in full generality and show that the answer is negative in general. The main result of Chapter 5 is the following example of a real root representation which cannot be constructed in this way.

Consider the quiver

$$\begin{tikzcd}
Q:\ 1 \arrow[r, "a"'] \arrow[rrr, bend left, "b"] & 2 \arrow[r, "c"'] \arrow[rr, bend right, "d"'] & 3 \arrow[r, "e"'] & 4\ ,
\end{tikzcd}$$

and the real root representation $X_\alpha$ with dimension vector $\alpha = (1, 8, 6, 4)$. Our main result is the following.

**Theorem.** *The real root representation $X_\alpha$ is not Schur and there does not exist a sequence of real Schur roots $\beta_n, \ldots, \beta_1$ such that*

$$X_\alpha = \sigma_{X_{\beta_n}} \cdot \ldots \cdot \sigma_{X_{\beta_2}}(X_{\beta_1}).$$

Moreover, we discuss further questions arising from this representation.

The main results of this thesis form the content of the paper [26] and the preprint [25], which is available on the Mathematics ArXiv website. Both research articles are included in Appendix B.

# Chapter 2

# Background

In this chapter we discuss background material and related results. We start with an overview of quivers and representations of quivers. A more detailed exposition of these topics can be found in [1, Chapter II], [2, Chapter III] and [5, 6]. The references given usually refer to the article in which the result first appeared. For standard results we just give a textbook reference.

In the last section of this chapter we discuss Crawley-Boevey's method to construct real root representations in characteristic zero.

## 2.1 Quivers, path algebras and roots

**Definition 2.1.1.** A *quiver* $Q = (Q_0, Q_1, h, t)$ is a finite directed graph with vertex set $Q_0$, arrow set $Q_1$ and two maps $h, t : Q_1 \to Q_0$ which assign to each arrow $a \in Q_1$ its head $h(a) \in Q_0$ and its tail $t(a) \in Q_0$. In particular, we write $a : t(a) \to h(a)$ for $a \in Q_1$.

A quiver $Q = (Q_0, Q_1, h, t)$ is usually just denoted by $Q$. The *opposite quiver* $Q^{\mathrm{op}} = (Q_0, Q_1, h^{\mathrm{op}} = t, t^{\mathrm{op}} = h)$ of a quiver $Q$ is obtained by reversing all arrows. A *subquiver* of a quiver $Q = (Q_0, Q_1, h, t)$ is a quiver $Q' = (Q'_0, Q'_1, h', t')$ with $Q'_0 \subset Q_0$, $Q'_1 \subset Q_1$, $h' = h|_{Q'_1}$

and $t' = t|_{Q'_1}$. A subquiver $Q'$ of $Q$ is called *full* if

$$Q'_1 = \{a \in Q_1 : h(a) \in Q'_0, t(a) \in Q'_0\}.$$

A *loop* is an arrow $a \in Q_1$ with $h(a) = t(a)$. A *path* $p$ of $Q$ is a sequence $p = a_1 \dots a_n$ of arrows $a_i \in Q_1$ such that $t(a_i) = h(a_{i+1})$ for $i = 1, \dots, n-1$. We define $h(p) = h(a_1)$ and $t(p) = t(a_n)$. Moreover, for each vertex $i \in Q_0$ we have a trivial path $\epsilon_i$ with $h(\epsilon_i) = t(\epsilon_i) = i$.

The *path algebra* $kQ$ of a quiver $Q = (Q_0, Q_1, h, t)$ is defined as follows. It has as $k$-basis all the paths (including the trivial paths) of $Q$. The multiplication of two paths $p_1$ and $p_2$ is given by the concatenation $p_1 p_2$ if $h(p_2) = t(p_1)$ and zero otherwise. This defines an associative $k$-algebra with $\sum_{i \in Q_0} \epsilon_i$ being the identity element.

The *Ringel form* on $\mathbb{Z}^{Q_0}$ is the bilinear form defined by

$$\langle \alpha, \beta \rangle := \sum_{i \in Q_0} \alpha[i]\beta[i] - \sum_{a \in Q_1} \alpha[t(a)]\beta[h(a)], \quad \alpha, \beta \in \mathbb{Z}^{Q_0}.$$

Let $(\alpha, \beta) := \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle$ be its symmetrisation.

The Ringel form is represented in the natural basis of $\mathbb{Z}^{Q_0}$ by the *Euler matrix* $E = (b_{ij})_{i,j \in Q_0}$, defined by

$$b_{ij} = \delta_{i,j} - \text{card}\,\{a \in Q_1 : t(a) = i, h(a) = j\},$$

where $d_{i,j}$ is the Kronecker delta symbol.

The *root lattice* of a quiver $Q$ is given by $\mathbb{Z}^{Q_0}$ together with the quadratic form $q(\alpha) = \frac{1}{2}(\alpha, \alpha)$. We write $e_i \in \mathbb{Z}^{Q_0}$ for the coordinate vector at vertex $i$ and by $\alpha[i]$, $i \in Q_0$, we denote the $i$-th coordinate of $\alpha \in \mathbb{Z}^{Q_0}$. The *support* of $\alpha \in \mathbb{Z}^{Q_0}$ is the full subquiver of $Q$ with vertex set $\{i \in Q_0 : \alpha[i] \neq 0\}$. An element $\alpha \in \mathbb{Z}^{Q_0}$ is said to be *sincere* if $\alpha[i] > 0$ for all $i \in Q_0$.

Let $i \in Q_0$ be a loop-free vertex of $Q$; then there is a simple reflection $s_i : \mathbb{Z}^{Q_0} \to \mathbb{Z}^{Q_0}$ defined by

$$s_i(\alpha) := \alpha - (\alpha, e_i)e_i.$$

The *Weyl group*, denoted by $W$, is the subgroup of $\mathrm{GL}(\mathbb{Z}^{Q_0})$ generated by the $s_i$.

A *simple root* is a vector $e_i$ for $i \in Q_0$. We denote by $\Pi$ the set of simple roots $e_i$ with $i$ loop-free. By $\Delta^+_{\mathrm{re}}(Q) := \{\alpha \in W(\Pi) : \alpha \in \mathbb{Z}^{Q_0}, \alpha > 0\}$ we denote the set of (positive) *real roots* for $Q$. For a positive real root $\alpha$ we define the following reflection $s_\alpha : \mathbb{Z}^{Q_0} \to \mathbb{Z}^{Q_0}$

$$s_\alpha(\beta) := \beta - (\beta, \alpha) \cdot \alpha.$$

The *fundamental region* is defined as follows

$$M := \{\alpha \in \mathbb{N}^{Q_0} : \alpha \neq 0,\ \alpha \text{ has connected support and } (\alpha, e_i) \leq 0 \text{ for all } i \in Q_0\}.$$

By $\Delta^+_{\mathrm{im}}(Q) := \bigcup_{w \in W} w(M)$ we denote the set of (positive) *imaginary roots* for $Q$.

Moreover, we define by $\Delta^+(Q) := \Delta^+_{\mathrm{re}}(Q) \cup \Delta^+_{\mathrm{im}}(Q)$ the set of *positive roots*. We have the following fundamental lemma.

**Lemma 2.1.2** ([13, Lemma 2.1])**.** *Let $Q$ be a quiver without loops. For $\alpha \in \Delta^+(Q)$ one has*

*(i)* $\alpha \in \Delta^+_{re}(Q)$ *if and only if* $\langle \alpha, \alpha \rangle = 1$,

*(ii)* $\alpha \in \Delta^+_{im}(Q)$ *if and only if* $\langle \alpha, \alpha \rangle \leq 0$.

## 2.2 Representations of quivers and Kac's Theorem

In this section we fix a quiver $Q$ and a field $k$.

**Definition 2.2.1.** A *$k$-linear representation* or, simply, a *representation* $X = (X_i, X_a)_{i \in Q_0, a \in Q_1}$ of $Q$ is defined by the following data:

(i) a finite dimensional $k$-vector space $X_i$ for each $i \in Q_0$ and

(ii) a $k$-linear map $X_a : X_{t(a)} \to X_{h(a)}$ for each $a \in Q_1$.

Let $i, j \in Q_0$ and let $n_{i,j}$ be the number of arrows from $i$ to $j$. The above definition of a representation $X$ of $Q$ is equivalent to the following definition.

**Definition 2.2.2.** A *$k$-linear representation* or, simply, a *representation* $X = (X_i, X_{i,j})_{i,j \in Q_0}$ of $Q$ is defined by the following data:

(i)' a finite dimensional $k$-vector space $X_i$ for each $i \in Q_0$ and

(ii)' a $k$-linear map $X_{i,j} : X_i \otimes_k k^{n_{i,j}} \to X_j$ for all $i, j \in Q_0$.

Here the maps for the individual arrows from $i$ to $j$ are encoded in the vector space $k^{n_{i,j}}$. One can pass from the second definition to first by choosing a basis of $k^{n_{i,j}}$. Moreover, we have the following natural vector-space isomorphisms

$$\mathrm{Hom}_k(X_i \otimes k^{n_{i,j}}, X_j) \cong \mathrm{Hom}_k(X_i, \mathrm{Hom}_k(k^{n_{i,j}}, X_j)) \cong \mathrm{Hom}_k(X_i, X_j \otimes k^{n_{i,j}}).$$

In this way, given a map $X_{i,j} : X_i \otimes_k k^{n_{i,j}} \to X_j$ one obtains a map $X'_{i,j} : X_i \to X_j \otimes k^{n_{i,j}}$.

We shall usually work with Definition 2.2.1; only in Chapter 4 we work with Definition 2.2.2.

**Example 2.2.3** (Simple representation)**.** We denote by $S(i)$ the representation

$$S(i)_j = \begin{cases} k, & i = j, \\ 0, & i \neq j, \end{cases} \quad \forall\, j \in Q_0, \quad S(i)_a = 0 \quad \forall\, a \in Q_1.$$

**Example 2.2.4** (Dual representation)**.** Let $X$ be a representation of $Q$. We denote by $X^*$ the representation of $Q^{\mathrm{op}}$, defined as follows:

$$(X^*)_i = (X_i)^* \quad \forall\, i \in Q_0, \quad (X^*)_a = (X_a)^* \quad \forall\, a \in Q_1.$$

The *dimension vector* $\underline{\dim}\, X$ of a representation $X$ is defined by

$$\underline{\dim}\, X = (\dim X_i)_{i \in Q_0} \in \mathbb{Z}^{Q_0}.$$

Let $X = (X_i, X_a)_{i \in Q_0, a \in Q_1}$ and $Y = (Y_i, Y_a)_{i \in Q_0, a \in Q_1}$ be two representations of $Q$. A *homomorphism* $\psi : X \to Y$ is defined by a family $\psi = (\psi_i)_{i \in Q_0}$ of $k$-linear maps $\psi_i : X_i \to Y_i\, (i \in Q_0)$ such that for each arrow $a \in Q_1$ the square

$$\begin{tikzcd} X_{t(a)} \arrow[r, "X_a"] \arrow[d, "\psi_{t(a)}"'] & X_{h(a)} \arrow[d, "\psi_{h(a)}"] \\ Y_{t(a)} \arrow[r, "Y_a"'] & Y_{h(a)} \end{tikzcd}$$

commutes.

The *direct sum* $X \oplus Y$ of two representations $X$ and $Y$ is defined by

$$\begin{aligned} (X \oplus Y)_i &= X_i \oplus Y_i, \quad \forall\, i \in Q_0, \\ (X \oplus Y)_a &= \begin{pmatrix} X_a & 0 \\ 0 & Y_a \end{pmatrix}, \quad \forall\, a \in Q_1. \end{aligned}$$

A representation $Z$ is called *decomposable* if $Z$ is isomorphic to $X \oplus Y$ for non-zero representations $X$ and $Y$.

We denote by $\operatorname{rep}_k Q$ the *category of finite dimensional representations of* $Q$.

There is an equivalence between the category $\operatorname{rep}_k Q$ and the category of finite dimensional left $kQ$-modules, given by the following construction. A representation $X = (X_i, X_a)_{i \in Q_0, a \in Q_1}$ defines a $kQ$-module $M$ as follows: $M := \oplus_{i \in Q_0} X_i$ with $\epsilon_i$ acting as the projection onto $X_i$ and with an arrow $a : i \to j$ acting as the following composition $M \xrightarrow{\epsilon_i} X_i \xrightarrow{X_a} X_j \xrightarrow{\text{incl.}} M$. This defines a finite dimension left $kQ$-module. On the other hand, a finite dimensional $kQ$-module $M$ defines a representation of $Q$ as follows: $X_i := \epsilon_i \cdot M$ for each $i \in Q_0$ and $X_a : \epsilon_{t(a)} \cdot M \to \epsilon_{h(a)} \cdot M, m \mapsto a \cdot m$ for each $a \in Q_1$. This construction is functorial.

**Theorem 2.2.5** ([2, Chapter III, Theorem 1.5]). *The category of finite dimensional left $kQ$-modules is equivalent to the category* $\operatorname{rep}_k Q$.

In particular it follows that $\operatorname{rep}_k Q$ is an abelian $k$-category.

One may also want to consider representations of quivers with relations. Namely, let $I \subset kQ$ be an ideal with generators $u_i \in kQ \,(i = 1, \ldots, n)$ such that $u_i \in \epsilon_{k_i} kQ \epsilon_{l_i}$ for $k_i, l_i \in Q_0$. Let $X = (X_i, X_a)_{i \in Q_0, a \in Q_1}$ be representation of $Q$. A path $p = a_1 \ldots a_n$ in $Q$ defines a linear map

$X_p = X_{a_1} \circ \ldots \circ X_{a_n} : X_{t(p)} \to X_{h(p)}$. In this way every generator $u_i \in I$ defines a linear map $X_{u_i} : X_{l_i} \to X_{k_i}$. We denote by $\operatorname{rep}_k(Q, I)$ the full subcategory of $\operatorname{rep}_k Q$ consisting of representations $X = (X_i, X_a)_{i \in Q_0, a \in Q_1}$ of $Q$ such that $X_{u_i} = 0$ for all $i = 1, \ldots, n$.

In the same way as above, the category of finite dimensional left $kQ/I$-modules can be identified with the category $\operatorname{rep}_k(Q, I)$

**Theorem 2.2.6** ([2, Chapter III, Proposition 1.7])**.** *The category of finite dimensional left $kQ/I$-modules is equivalent to the category* $\operatorname{rep}_k(Q, I)$.

We only consider quivers with relations in Section 2.3, where we discuss deformed preprojective algebras. Our main focus is on the category $\operatorname{rep}_k Q$.

One of the key properties of the category $\operatorname{rep}_k Q$ is the Krull-Remak-Schmidt property.

**Theorem 2.2.7** ([16, Part two, Chapter X, Section 7, Theorem 7.5])**.** *Let $X$ be a representation of $Q$. Then*

$$X = \bigoplus_{i=1}^{n} X_i,$$

*with $X_i$ indecomposable for $i = 1, \ldots, n$. This decomposition is unique up to isomorphism and reordering of the summands.*

We will not discuss homological properties of the category $\operatorname{rep}_k Q$; we refer the reader to [1, Section VII.1] and [2, Chapter III] for a discussion of relevant homological properties of $\operatorname{rep}_k Q$. An overview of basic notions of homological algebra can be found in [1, Appendix A.4]. We only mention that the category $\operatorname{rep}_k Q$ is *hereditary*, which implies that $\operatorname{Ext}^2_{kQ}(X, Y) = 0$ for any two representations $X$ and $Y$ of $Q$, see [2, Chapter III, Proposition 1.4 and following comments] or use the standard resolution [5, §1, The standard resolution].

The construction of the extension space $\operatorname{Ext}^1_{kQ}(X, Y)$ of two representations $X$ and $Y$ of a quiver $Q$, as discussed in [19, Section 2] and [17, Section 2.1], can be done in the following elementary way.

Let $X$ and $Y$ be representations of $Q$ and let

$$\begin{aligned} C^0(X,Y) &:= \bigoplus_{i\in Q_0} \mathrm{Hom}_k(X_i, Y_i), \\ C^1(X,Y) &:= \bigoplus_{a\in Q_1} \mathrm{Hom}_k(X_{t(a)}, Y_{h(a)}). \end{aligned}$$

We define the map

$$\begin{aligned} \delta_{XY} : C^0(X,Y) &\to C^1(X,Y) \\ (\phi_i)_i &\mapsto (\phi_j X_a - Y_a \phi_i)_{a:i\to j}. \end{aligned}$$

The importance of $\delta_{XY}$ is given by the following lemma.

**Lemma 2.2.8** ([17, Section 2.1, Lemma])**.** *We have* $\ker \delta_{XY} = \mathrm{Hom}_{kQ}(X,Y)$ *and* $\operatorname{coker} \delta_{XY} = \mathrm{Ext}^1_{kQ}(X,Y)$.

It follows from the above lemma that for two representations $X$ and $Y$ we have the following useful formula

$$\dim \mathrm{Hom}_{kQ}(X,Y) - \dim \mathrm{Ext}^1_{kQ}(X,Y) = \langle \underline{\dim}\, X, \underline{\dim}\, Y\rangle.$$

The heredity of the category $\mathrm{rep}_k\, Q$ implies the following two facts. Let $0 \to X \to Y \to Z \to 0$ be a short exact sequence of representations of $Q$, for a representation $M$ of $Q$ we have the following long exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}_{kQ}(M,X) \arrow[r] & \mathrm{Hom}_{kQ}(M,Y) \arrow[r] & \mathrm{Hom}_{kQ}(M,Z) \arrow[dll, rounded corners] \\ & \mathrm{Ext}^1_{kQ}(M,X) \arrow[r] & \mathrm{Ext}^1_{kQ}(M,Y) \arrow[r] & \mathrm{Ext}^1_{kQ}(M,Z) \arrow[r] & 0, \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}_{kQ}(Z,M) \arrow[r] & \mathrm{Hom}_{kQ}(Y,M) \arrow[r] & \mathrm{Hom}_{kQ}(X,M) \arrow[dll, rounded corners] \\ & \mathrm{Ext}^1_{kQ}(Z,M) \arrow[r] & \mathrm{Ext}^1_{kQ}(Y,M) \arrow[r] & \mathrm{Ext}^1_{kQ}(X,M) \arrow[r] & 0. \end{tikzcd}$$

When studying representations of $Q$ our main interest lies in indecomposable representations, as suggested by Theorem 2.2.7. As already mentioned in the introduction, the indecomposable representations of $Q$ correspond to the positive roots $\Delta^+(Q)$, see [13, 14, 15]. We have the following remarkable theorem, generally known as Kac's Theorem.

**Theorem 2.2.9** (Kac [13, Theorem 1 and 2], Schofield [22, Theorem 9])**.** *Let $k$ be a field, let $Q$ be a quiver without loops and let $\alpha \in \mathbb{N}^{Q_0}$.*

*(i) For $\alpha \notin \Delta^+(Q)$ all representations of $Q$ of dimension vector $\alpha$ are decomposable.*

*(ii) For $\alpha \in \Delta^+_{re}(Q)$ there exists one and only one (up to isomorphism) indecomposable representation of dimension vector $\alpha$.*

For finite fields and algebraically closed fields the theorem is due to Kac [13, Theorem 1 and 2]. As pointed out in the introduction of [22], Kac's method of proof showed that the above theorem holds for fields of characteristic $p$. The proof for fields of characteristic zero is due to Schofield ([22], Theorem 9).

**Remark 2.2.10.** In Section 2.3 we discuss an alternative proof of part (ii) of this result over an algebraically closed field of characteristic zero, given by Crawley-Boevey.

We are now able to precisely define the objects discussed in the introduction: real root representations. The work presented in the later chapters focuses on these objects, motivated by Question (†) and Question (††).

**Definition 2.2.11** (Real root representation)**.** Let $Q$ be a quiver and let $\alpha$ be a positive real root. The unique indecomposable representation (up to isomorphism) of dimension vector $\alpha$ is called a *real root representation* and denoted by $X_\alpha$.

We finish this section with some further definitions. A *Schur representation* is a representation $X$ with $\mathrm{End}_{kQ}(X) = k$. By a *real Schur representation* we mean a real root representation which is also a Schur representation. A positive real root $\alpha$ is called a *real Schur root* if $X_\alpha$ is a real

Schur representation. Note that $\operatorname{Ext}^1_{kQ}(X_\alpha, X_\alpha) = 0$ for $\alpha$ a real Schur root. An indecomposable representation $X$ is called *exceptional* provided $\operatorname{Ext}^1_{kQ}(X, X) = 0$.

By $X = Y$ for two given representations $X$ and $Y$ we mean that $X$ and $Y$ are isomorphic.

### 2.2.1 Tree representations

In the introduction the question of properties of real root representations was raised. One of the questions was whether real root representations are tree representations? We use this section to discuss this notion and relevant results. Our elaborations are mainly based on the article [19] by Ringel.

Let $Q = (Q_0, Q_1, h, t)$ be a quiver and let $k$ be a field. Moreover, let $X \in \operatorname{rep}_k Q$ be a representation of $Q$ with $\underline{\dim}\, X = d$. We denote by $\mathfrak{B}_i$ a fixed basis of the vector space $X_i$ $(i \in Q_0)$ and set $\mathfrak{B} = \cup_{i \in Q_0} \mathfrak{B}_i$. The set $\mathfrak{B}$ is called a basis of $X$. We fix a basis $\mathfrak{B}$ of $X$. For a given arrow $a : i \to j$ we can write $X_a$ as a $d[j] \times d[i]$-matrix $X_{a,\mathfrak{B}}$ with rows indexed by $\mathfrak{B}_j$ and with columns indexed by $\mathfrak{B}_i$. We denote by $X_{a,\mathfrak{B}}(x, x')$ the corresponding matrix entry, where $x \in \mathfrak{B}_i, x' \in \mathfrak{B}_j$; the entries $X_{a,\mathfrak{B}}(x, x')$ are defined by $X_a(x) = \sum_{x' \in \mathfrak{B}_j} X_{a,\mathfrak{B}}(x, x')\, x'$. The *coefficient quiver* $\Gamma(X, \mathfrak{B})$ of $X$ with respect to $\mathfrak{B}$ is defined as follows: the vertex set of $\Gamma(X, \mathfrak{B})$ is the set $\mathfrak{B}$ of basis elements of $X$; there is an arrow $(a, x, x')$ between two basis elements $x \in \mathfrak{B}_i$ and $x' \in \mathfrak{B}_j$ provided $X_{a,\mathfrak{B}}(x, x') \neq 0$ for $a : i \to j$.

Let $\mathfrak{B}$ and $\mathfrak{B}'$ be two bases of $X$. $\mathfrak{B}$ and $\mathfrak{B}'$ are said to proportional if for every $b \in \mathfrak{B}$ there exists a non-zero $\lambda(b) \in k$ such that $\lambda(b) \cdot b \in \mathfrak{B}'$. In this case the coefficient quivers $\Gamma(X, \mathfrak{B})$ and $\Gamma(X, \mathfrak{B}')$ may be identified.

**Definition 2.2.12** (Tree module, see [19])**.** We call an indecomposable representation $X$ of $Q$ a *tree representation* provided there exists a basis $\mathfrak{B}$ of $X$ such that the coefficient quiver $\Gamma(X, \mathfrak{B})$ is a tree.

**Example 2.2.13.** Consider the quiver $A_3$ and the following representation $X$ together with the corresponding coefficient quiver $\Gamma(X, \mathfrak{B})$, where $\mathfrak{B}$ is given by the canonical basis of $k^2$ and $k$.

$$\begin{tikzcd} & & k \\ X: & k^2 \arrow[ur, "{[1\ 1]}"] \arrow[dr, "{[0\ 1]}"'] & \\ & & k \end{tikzcd} \quad , \quad \begin{tikzcd} & & & \bullet \\ \Gamma(X, \mathfrak{B}): & \bullet \arrow[urr] & \bullet \arrow[ur] \arrow[dr] & \\ & & & \bullet \end{tikzcd}$$

Note, the coefficient quiver $\Gamma(X, \mathfrak{B})$ is connected. For the following matrix presentation of $X$ the coefficient quiver is not connected.

$$\begin{tikzcd} & & k \\ X: & k^2 \arrow[ur, "{[1\ 0]}"] \arrow[dr, "{[0\ 1]}"'] & \\ & & k \end{tikzcd} \quad , \quad \begin{tikzcd} & & & \bullet \\ \Gamma(X, \mathfrak{B}): & \bullet \arrow[urr] & \bullet \arrow[dr] & \\ & & & \bullet \end{tikzcd}$$

We see that the coefficient quiver $\Gamma(X, \mathfrak{B})$ may be connected even if $X$ is decomposable.

The coefficient quiver $\Gamma(X, \mathfrak{B})$ has the following properties.

**Lemma 2.2.14** ([19, Section 2, Property 1])**.** *If $X$ is indecomposable and $\mathfrak{B}$ is a basis of $X$ then $\Gamma(X, \mathfrak{B})$ is connected. If $X$ is decomposable then there exists a basis $\mathfrak{B}$ of $X$ such that $\Gamma(X, \mathfrak{B})$ is not connected.*

**Lemma 2.2.15** ([19, Section 2, Property 2])**.** *Let $\mathfrak{B}$ be a basis of $X$ such that $\Gamma(X, \mathfrak{B})$ is a tree. Then there is a basis $\mathfrak{B}'$ of $X$ which is proportional to $\mathfrak{B}$ such that all non-zero coefficients $X_{a,\mathfrak{B}'}(x, x')$ are equal to 1.*

The following remarkable theorem is due to Ringel.

**Theorem 2.2.16** ([19, Theorem])**.** *Let $k$ be a field and let $Q$ be a quiver. Any exceptional representation of $Q$ over $k$ is a tree representation.*

In particular, real Schur representations are tree representations. The importance of this fact was already mentioned in the introduction. Since the property of being a tree representation is preserved under universal extension functors (see Chapter 3), we see that if a real root representation can be constructed from a real Schur representation using universal functors, this property will be preserved.

## 2.3 Deformed preprojective algebras and reflection functors

We mentioned in the introduction that Crawley-Boevey constructed real root representations for an arbitrary quiver over algebraically closed fields of characteristic zero. In this section we give a detailed description of Crawley-Boevey's results. We use the results of this section in Chapter 5 and in Appendix A.

We start by describing deformed preprojective algebras, a generalization of preprojective algebras, and discuss reflection functors for these algebras. In the second part we relate real root representations to simple representations of deformed preprojective algebras, which can be constructed using the reflection functors.

Let $Q = (Q_0, Q_1, h, t)$ be a quiver and let $k$ be a field. The *double quiver* $\overline{Q}$ is obtained by adjoining a reverse arrow $a^* : j \to i$ for each arrow $a : i \to j$ in $Q$. For $\lambda \in k^{Q_0}$ the *deformed preprojective algebra* $\Pi^\lambda(Q)$, introduced by Crawley-Boevey and Holland in [8], is the algebra defined by

$$\Pi^\lambda(Q) \quad = \quad k\overline{Q}/\left(\sum_{a\in Q_1}[a, a^*] - \sum_{i\in Q_0}\lambda[i]\epsilon_i\right)$$

where $[a, a^*] = aa^* - a^*a$. If $Q'$ is obtained from $Q$ by reversing the arrow $a$, then there is an isomorphism $\Pi^\lambda(Q) \to \Pi^\lambda(Q')$ which sends $a$ to $a^*$ and $a^*$ to $-a$. Hence, it is clear that $\Pi^\lambda(Q)$ does not depend on the orientation of $Q$. A representation $X$ of $\Pi^\lambda(Q)$ is given by a representation $X$ of $\bar{Q}$, say, with vector space $X_i$ at vertex $i \in Q_0$ and linear map $X_a : X_{t(a)} \to X_{h(a)}$ for each arrow $a \in \bar{Q}$, which satisfies

$$\sum_{\substack{a\in Q\\ h(a)=i}} X_a X_{a^*} - \sum_{\substack{a\in Q\\ t(a)=i}} X_{a*} X_a = \lambda[i]\,\mathrm{id}_{X_i},$$

for each vertex $i \in Q_0$.

### 2.3.1 Reflection functors for representations of $\Pi^\lambda(Q)$

Let $i \in Q_0$ be a loop-free vertex. The dual reflection $r_i : k^{Q_0} \to k^{Q_0}$ to $s_i$ is defined by

$$r_i(\lambda)[j] := \lambda[j] - (e_i, e_j)\lambda[i],$$

and it satisfies $r_i(\lambda) \cdot \alpha = \lambda \cdot s_i(\alpha)$ for all $\alpha \in \mathbb{Z}^{Q_0}$ and all $\lambda \in k^{Q_0}$. Let $\lambda \in k^{Q_0}$. We recall a theorem from [8].

**Theorem 2.3.1** ([8, Theorem 5.1]). *If $i \in Q_0$ is a loop-free vertex, $\lambda[i] \neq 0$, then there is an equivalence*

$$E_i : \Pi^\lambda(Q)\text{-}modules \to \Pi^{r_i(\lambda)}(Q)\text{-}modules$$

*which acts as the simple reflection $s_i$ on dimension vectors.*

We use the rest of this section to recall the construction of this functor. Let $X$ be a representation of $\Pi^\lambda(Q)$ and let $i$ be a loop-free vertex of $Q$ with $\lambda[i] \neq 0$. We define $T(i) = \{a \in \bar{Q} : t(a) = i\}$ and $X_\oplus = \bigoplus_{a \in T(i)} X_{h(a)}$. For $a \in T(i)$ we define the following canonical projection and inclusion maps

$$\begin{aligned} \pi_a &: X_\oplus \to X_{h(a)}, \\ \mu_a &: X_{h(a)} \to X_\oplus. \end{aligned}$$

Moreover, we define $\mu : X_i \to X_\oplus$ and $\pi : X_\oplus \to X_i$ by

$$\begin{aligned} \mu &= \sum_{a \in T(i)} \mu_a X_a, \\ \pi &= \frac{1}{\lambda[i]} \sum_{a \in T(i)} -\epsilon(a) X_{a^*} \pi_a \end{aligned}$$

where $\epsilon$ is defined as follows: $\epsilon(a) = 1$ for $a \in Q_1$, and $\epsilon(a) = -1$ for $a \in \bar{Q}_1 - Q_1$; in addition to this we are using the following convention: for $a \in Q_1$ we define

$(a^*)^* := a$. The relations for $\Pi^\lambda(Q)$ ensure that $\pi\mu = 1_{X_i}$, and hence $\mu\pi$ is an idempotent endomorphism of $X_\oplus$.

We define a representation $X'$ of $\Pi^{r_i(\lambda)}(Q)$ as follows:

$$\begin{aligned} X'_j &= X_j, \quad \text{for } j \neq i, \\ X'_i &= \ker \pi = \operatorname{im}(1 - \mu\pi), \end{aligned}$$

together with the following maps: $X'_a = X_a$ for $a \in \bar{Q}_1$ with $t(a) \neq i \neq h(a)$, and

$$\begin{aligned} X'_a &= -\epsilon(a)\lambda[i](1-\mu\pi)\mu_{a^*} : X'_{t(a)} \to X'_i, \quad \text{if } h(a) = i, \\ X'_a &= \pi_a|_{X'_i} : X'_i \to X'_{h(a)}, \quad \text{if } t(a) = i. \end{aligned}$$

**Lemma 2.3.2.** *$X'$ is a representation of $\Pi^{r_i(\lambda)}(Q)$.*

*Proof.* See proof of [8, Theorem 5.1].

□

The reflection functor $E_i : \Pi^\lambda(Q) \to \Pi^{r_i(\lambda)}(Q)$ sends a representation $X$ to $E_i(X) := X'$ and operates on homomorphisms in the natural way.

### 2.3.2 Real root representations of $Q$ via representations of $\Pi^\lambda(Q)$

In this section we discuss two of Crawley-Boevey's results from [7] which relate real root representations of $Q$ to simple representations of $\Pi^\lambda(Q)$. This gives an algorithm to construct real root representations over algebraically closed fields of characteristic zero.

The first result we need concerns the question of lifting representations of $Q$ to representations of $\Pi^\lambda(Q)$.

**Theorem 2.3.3** ([7, Theorem 3.3])**.** *Let $k$ be an algebraically closed field. If $\lambda \in k^{Q_0}$ then a representation of $Q$ lifts to a representation of $\Pi^\lambda(Q)$ if and only if the dimension vector $\beta$ of any direct summand satisfies $\lambda \cdot \beta = 0$.*

**Remark 2.3.4.** No assumption for the field is needed for the "$\Longrightarrow$" direction.

This result and the reflection functors described in Section 2.3.1 can be used to construct real root representations of $Q$ over an algebraically closed field of characteristic zero. We give Crawley-Boevey's proof, since it is very instructive.

**Proposition 2.3.5** ([7, Proposition A.4]). *Let $k$ be an algebraically closed field of characteristic zero, and let $\alpha$ be a real root for $Q$. Choose a reflection series*

$$\alpha = s_{i_n} \dots s_{i_1}(e_j)$$

*such that $\alpha^{(k)} = s_{i_k} \dots s_{i_1}(e_j)$ is not a coordinate vector for $1 \leq k \leq n$. Let $\lambda^{(0)} \in k^{Q_0}$ be the vector with $\lambda^{(0)}[j] = 0$ and $\lambda^{(0)}[i] = 1$ for all $i \neq j$, and define $\lambda^{(k)} = r_{i_k}(\lambda^{(k-1)})$ for $1 \leq k \leq n$. Then $\lambda^{(k)}[i_{k+1}] \neq 0$ for all $k$. Moreover, there is a unique indecomposable representation of $Q$ of dimension $\alpha$, and it may be obtained from the simple representation of $\Pi^{\lambda^{(0)}}(Q)$ of dimension $e_j$ by applying successively the reflection functors at the vertices $i_k$, and then restricting the resulting representation of $\Pi^{\lambda^{(n)}}(Q)$ to $Q$.*

*Proof.* Since $k$ has characteristic zero, $\lambda^{(0)} \cdot \beta \neq 0$ for any root $\beta$ which is not equal to $\pm e_j$. Using the formula $r_i(\lambda) \cdot \alpha = \lambda \cdot s_i(\alpha)$, it follows that $\lambda^{(t)} \cdot \beta \neq 0$ for any root $\beta$ which is not equal to $\pm\alpha^{(t)}$. In particular, $\lambda^{(t)} \cdot e_{i_{t+1}} \neq 0$ for $t \geq 0$. Thus $\lambda^{(k)}[i_{k+1}] \neq 0$ for all $k$.

Using the equivalence $E_i$ in Theorem 2.3.1, we get an equivalence between representations of $\Pi^{\lambda^{(0)}}(Q)$ of dimension vector $e_j$, of which there is only one, and representations of $\Pi^{\lambda^{(m)}}(Q)$ of dimension $\alpha$. Thus, up to isomorphism, there is a unique representation $\hat{X}_\alpha$ of $\Pi^{\lambda^{(m)}}(Q)$, of dimension vector $\alpha$. Note that this representation is simple. The restriction of $\hat{X}_\alpha$ to $Q$ is indecomposable, for if it had an indecomposable direct summand of dimension $\beta$, Theorem 2.3.3 would imply $\lambda^{(m)} \cdot \beta = 0$ (here were are using the "$\Longrightarrow$" direction). This is impossible since $\beta$ is a root not equal to $\pm\alpha$.

Finally, observe that any indecomposable representation of $Q$ of dimension $\alpha$ lifts to a representation of $\Pi^{\lambda^{(m)}}(Q)$ of dimension $\alpha$, because $\lambda^{(m)} \cdot \alpha = 0$. Up to isomorphism there is only one representation of $\Pi^{\lambda^{(m)}}(Q)$ of dimension $\alpha$, hence it follows that there is only one indecomposable representation of $Q$ of dimension $\alpha$, up to isomorphism. $\square$

**Remark 2.3.6.** Characteristic zero is essential for this construction method.

**Remark 2.3.7.** In Section 2.2 we discussed Theorem 2.2.9, a remarkable Theorem due to Kac and Schofield. The previous proposition gives an alternative proof of this result over algebraically closed fields of characteristic zero.

# Chapter 3

# Universal extension functors

This chapter is devoted to a detailed discussion of the universal extension functors $\sigma_S$ which were introduced by Ringel in [18]. We recall all necessary definitions and results from [18] which are needed to describe these functors. Moreover, we prove that universal extension functors preserve indecomposable tree representations. In the last part of this section we discuss Ringel's construction of real root representations of the quiver

$$Q''(g,h):\ \begin{tikzcd} 2 \arrow[r, bend left=60, "\mu_1"] \arrow[r, bend left=30, "\mu_g"'] & 3 \arrow[l, bend left=30, "\nu_1"'] \arrow[l, bend left=60, "\nu_h"] \end{tikzcd},$$

with $g, h \geq 1$ and a generalisation of this result.

In this chapter we assume some familiarity with the concepts of categories and functors. For a basic account of category theory, functors and related notions we refer the reader to [1, Appendix A.1 and A.2].

## 3.1 Construction and properties of universal extension functors

Let $Q$ be a quiver and let $k$ be a field. We fix a real Schur representation $S$ of $Q$; that is, a representation $S$ with $\operatorname{End}_{kQ} S = k$ and $\operatorname{Ext}^1_{kQ}(S, S) = 0$.

For a full subcategory $\mathfrak{C}$ of $\operatorname{rep}_k Q$ we define by $\mathfrak{C}/S$ the quotient category of $\mathfrak{C}$ modulo all maps which factor thourgh direct sums of copies of $S$.

In analogy to [18, Section 1], we define the following subcategories of $\operatorname{rep}_k Q$. Let $\mathfrak{M}^S$ be the full subcategory of all modules $X$ with $\operatorname{Ext}^1_{kQ}(S, X) = 0$ such that, in addition, $X$ has no direct summand which can be embedded into some direct sum of copies of $S$. Similarly, let $\mathfrak{M}_S$ be the full subcategory of all modules $X$ with $\operatorname{Ext}^1_{kQ}(X, S) = 0$ such that, in addition, no direct summand of $X$ is a quotient of a direct sum of copies of $S$. Finally, let $\mathfrak{M}^{-S}$ be the full subcategory of all modules $X$ with $\operatorname{Hom}_{kQ}(X, S) = 0$, and let $\mathfrak{M}_{-S}$ be the full subcategory of all modules $X$ with $\operatorname{Hom}_{kQ}(S, X) = 0$. Moreover, we consider

$$\mathfrak{M}^S_S = \mathfrak{M}^S \cap \mathfrak{M}_S, \quad \mathfrak{M}^{-S}_{-S} = \mathfrak{M}^{-S} \cap \mathfrak{M}_{-S}.$$

For a given module $X$ we define by $X^{-S}$ the intersection of the kernels of all maps $X \to S$. Moreover, we define $X_{-S} = X/X'$, where $X'$ is the sum of the images of all maps $S \to X$.

**Remark 3.1.1.** Let $X, Y \in \operatorname{rep}_k Q$ and let $f : X \to Y$. For $x \in X^{-S}$ we have to have that $f(x) \in Y^{-S}$, and hence $f(X^{-S}) \subset Y^{-S}$. Thus, we obtain the following functor $\operatorname{rep}_k Q \to \operatorname{rep}_k Q, X \mapsto X^{-S}$, which operates on homomorphisms by restriction. Moreover, if $f : X \to Y$ factors through a direct sum of copies of $S$ then we have $X^{-S} \subset \ker f$ and the restriction of $f$ to $X^{-S}$ is zero.

Dually, we obtain a functor $\operatorname{rep}_k Q \to \operatorname{rep}_k Q, X \mapsto X_{-S}$.

Let $X \in \mathfrak{M}^S$. Since $X$ does not split off a copy of $S$ (because $\operatorname{End}_{kQ} S = k$ and $X \in \mathfrak{M}^S$, and hence no direct summand of $X$ embeds into a sum of copies of $S$) we have to have that any $f : S \to X$ maps into $X^{-S}$, and hence the natural map $\operatorname{Hom}_{kQ}(S, X^{-S}) \to \operatorname{Hom}_{kQ}(S, X)$ is an isomorphism. Similarly, it follows for $X \in \mathfrak{M}_S$ that the natural map $\mathrm{Hom}_{kQ}(X_{-S}, S) \to \mathrm{Hom}_{kQ}(X, S)$ is an isomorphism.

**Lemma 3.1.2** ([18, Lemma 1])**.** *For any $X \in \mathrm{rep}_k Q$, we have $X^{-S} \in \mathfrak{M}^{-S}$.*

**Dual-Lemma 3.1.2.** *For any $Y \in \mathrm{rep}_k Q$, we have $Y_{-S} \in \mathfrak{M}_{-S}$.*

**Lemma 3.1.3** ([18, Lemma 2])**.** *Let $X \in \mathfrak{M}^S$ and let $\phi_1, \dots, \phi_r$ be a basis of the $k$-vector space $\mathrm{Hom}_{kQ}(X, S)$. Then the sequence*

$$0 \to X^{-S} \to X \overset{(\phi_1,\dots,\phi_r)^t}{\longrightarrow} \bigoplus_{i=1}^{r} S \to 0$$

*is exact and the induced sequences $E_1, \dots, E_r \in \mathrm{Ext}_{kQ}(S, X^{-S})$ form a basis of the $k$-vector space $\mathrm{Ext}^1_{kQ}(S, X^{-S})$.*

**Dual-Lemma 3.1.3.** *Let $Y \in \mathfrak{M}_S$ and let $\phi'_1, \dots, \phi'_u$ be a basis of the $k$-vector space $\mathrm{Hom}_{kQ}(S, Y)$. Then the sequence*

$$0 \to \bigoplus_{i=1}^{u} S \overset{(\phi'_1,\dots,\phi'_u)}{\longrightarrow} Y \to Y_{-S} \to 0$$

*is exact and the induced sequences $E'_1, \dots, E'_u \in \mathrm{Ext}_{kQ}(Y_{-S}, S)$ form a basis of the $k$-vector space $\mathrm{Ext}^1_{kQ}(Y_{-S}, S)$.*

**Lemma 3.1.4** ([18, Lemma 3])**.** *Let $X \in \mathfrak{M}^{-S}$ and let $E_1, \dots, E_s$ be a basis of the $k$-vector space $\mathrm{Ext}^1_{kQ}(S, X)$. Consider the exact sequence $E$*

$$E : 0 \to X \to Z \to \bigoplus_{i=1}^{s} S \to 0$$

*given by the elements $E_j$ $(j = 1, \dots, s)$. Then $Z \in \mathfrak{M}^S$ and $Z^{-S} = X$.*

**Dual-Lemma 3.1.4.** *Let $Y \in \mathfrak{M}_{-S}$ and let $E'_1, \dots, E'_v$ be a basis of the $k$-vector space $\mathrm{Ext}^1_{kQ}(Y, S)$. Consider the exact sequence $E'$*

$$E' : 0 \to \bigoplus_{i=1}^{v} S \to U \to Y \to 0$$

*given by the elements $E'_j$ $(j = 1, \dots, v)$. Then $U \in \mathfrak{M}_S$ and $U_{-S} = Y$.*

**Proposition 3.1.5** ([18, Proposition 1]). *The functor $\overline{\psi}_S : \mathfrak{M}^S/S \to \mathfrak{M}^{-S}, X \mapsto X^{-S}$ defines an equivalence.*

It follows from Remark 3.1.1 and Lemma 3.1.2 that $\overline{\psi}_S$ defines indeed a functor between the respective categories. Moreover, it is clear by Remark 3.1.1 that maps in $\mathfrak{M}^S$ which factor through a direct sum of copies of $S$ get sent to the zero map. Hence, in order to obtain an equivalence we certainly have to factor out all the maps factoring through a direct sum of copies of $S$. This, however, is already enough by [18, Proposition 1] and makes the functor $\overline{\psi}_S$ full and faithful, meaning that it is an isomorphism on homomorphism spaces. The functor is dense, meaning that for every $Y \in \mathfrak{M}^{-S}$ there exists a $X \in \mathfrak{M}^S$ such that $X^{-S} = Y$, by Lemma 3.1.4. Hence, the functor $\overline{\psi}_S$ is full, faithful and dense. It is a well known fact (e.g. see Theorem [1, Theorem 2.5, Appendix A.2]) that this implies that the functor is an equivalence. We denote a quasi-inverse of this equivalence by $\overline{\sigma}_S : \mathfrak{M}^{-S} \to \mathfrak{M}^S/S$; it operates on objects as follows (up to isomorphism). Let $X \in \mathfrak{M}^{-S}$ and let $Z \in \mathfrak{M}^S$ be the representation constructed in Lemma 3.1.4, then we have $\overline{\sigma}_S(X) = Z$. We remark that the construction of the representation $Z \in \mathfrak{M}^S$ depends on the choice of a basis of the $k$-vector space $\mathrm{Ext}^1_{kQ}(S, X)$. Different choices however, give isomorphic representations by Lemma 3.1.4. Since we are only studying representations up to isomorphisms this description of a quasi-inverse is sufficient for us.

We have the following dual result.

**Dual-Proposition 3.1.5** ([18, Proposition 1*]). *The functor $\underline{\psi}_S : \mathfrak{M}_S/S \to \mathfrak{M}_{-S}, Y \mapsto Y_{-S}$ defines an equivalence.*

We denote a quasi-inverse of this equivalence by $\underline{\sigma}_S : \mathfrak{M}_{-S} \to \mathfrak{M}_S/S$; it operates on objects as follows. Let $Y \in \mathfrak{M}_{-S}$ and let $U \in \mathfrak{M}_S$ be the representation constructed in Dual-Lemma 3.1.4, then we have $\underline{\sigma}_S(Y) = U$.

**Proposition 3.1.6** ([18, Proposition 2]). *The functor $\psi_S : \mathfrak{M}^S_S/S \to \mathfrak{M}^{-S}_{-S}, X \mapsto (X^{-S})_{-S}$ defines an equivalence.*

*Idea of proof.* The functor $\psi_S$ can be constructed as follows. Firstly, we restrict the equivalence $\overline{\psi}_S$ to the following equivalence

$$\overline{\psi}_S : \mathfrak{M}_S^S/S \to \mathfrak{M}_S^{-S}, X \mapsto X^{-S}.$$

This can be done by the following arguments. Let $X \in \mathfrak{M}_S^S = \mathfrak{M}^S \cap \mathfrak{M}_S$, by Lemma 3.1.3 we get

$$0 \to X^{-S} \to X \to \bigoplus_{i=1}^{r} S \to 0, \quad r = \dim \operatorname{Hom}_{kQ}(X, S),$$

and hence, using long exact sequences (described in Section 2.2) and the fact that $\operatorname{Ext}^1_{kQ}(S, S) = 0$, it follows that

$$\operatorname{Ext}^1_{kQ}(X, S) = 0 \longleftrightarrow \operatorname{Ext}^1_{kQ}(X^{-S}, S) = 0.$$

Moreover, we must show that (by the second part of definition $\mathfrak{M}_S$)

$$\begin{aligned} & X = X_1 \oplus X_2, \quad X_1 \text{ is a quotient of a direct sum of copies of } S \\ \longleftrightarrow \quad & X^{-S} = Y_1 \oplus Y_2, \quad Y_1 \text{ is a quotient of a direct sum of copies of } S. \end{aligned}$$

If $X = X_1 \oplus X_2$ and $\bigoplus S \overset{\mu}{\twoheadrightarrow} X_1$ then we have

$$0 \to X^{-S} \to X_1 \oplus X_2 \overset{[\nu_1,\nu_2]}{\longrightarrow} \bigoplus_{i=1}^{r} S \to 0,$$

with $\nu_1 = 0$, since $X$ does not split off a copy of $S$ (see Remark 3.1.1). This implies $X^{-S} = X_1 \oplus \ker \nu_2$. If $X^{-S} = Y_1 \oplus Y_2$ and $\bigoplus S \overset{\mu}{\twoheadrightarrow} Y_1$ then we get (use long exact sequence) $\operatorname{Ext}^1_{kQ}(S, \bigoplus S) \twoheadrightarrow \operatorname{Ext}^1_{kQ}(S, Y_1)$, and hence $\operatorname{Ext}^1_{kQ}(S, Y_1) = 0$, since $\operatorname{Ext}^1_{kQ}(S, S) = 0$. Since we have

$$0 \to X^{-S} \to X \to \bigoplus_{i=1}^{r} S \to 0, \quad r = \dim \operatorname{Hom}_{kQ}(X, S),$$

this implies that $X \cong Y_1 \oplus Z$ with $0 \to Y_2 \to Z \to \bigoplus_{i=1}^{r} S \to 0$.

Secondly, we restrict the equivalence $\underline{\psi}_S$ to the following equivalence

$$\underline{\psi}_S : \mathfrak{M}_S^{-S} \to \mathfrak{M}_{-S}^{-S}, Y \mapsto Y_{-S}.$$

This can be done by the following arguments. By definition, in the category $\mathfrak{M}_S^{-S}$ there are no homomorphisms factoring through direct sums of copies of $S$. Hence, we do not need to consider a quotient category. Now, let $Y \in \mathfrak{M}_S$. By Remark 3.1.1 the natural map $\mathrm{Hom}_{kQ}(Y^{-S}, S) \to \mathrm{Hom}_{kQ}(Y, S)$ is an isomorphism, and thus

$$\mathrm{Hom}_{kQ}(Y^{-S}, S) = 0 \longleftrightarrow \mathrm{Hom}^1_{kQ}(Y, S) = 0.$$

The functor $\psi_S$ is defined to be $\overline{\psi_S} \circ \underline{\psi_S}$; this makes sense by the above elaborations. In the same way we could define $\psi_S$ to be $\underline{\psi}_S \circ \overline{\psi}_S$ (using arguments dual to the ones given above). In the following we show that the order in which we apply $\underline{\psi}_S$ and $\overline{\psi}_S$ does not matter (up to isomorphism). That is, for a given representation in $X \in \mathfrak{M}_S^S$ we need to show that $\underline{\psi}_S \circ \overline{\psi}_S(X) = (X^{-S})_{-S} = (X_{-S})^{-S} = \overline{\psi_S} \circ \underline{\psi_S}(X)$. Let $X \in \mathfrak{M}_S^S$, then we have by Lemma 3.1.3 and Dual-Lemma 3.1.3

$$0 \to X^{-S} \to X \to \bigoplus_{i=1}^{r} S \to 0, \quad r = \dim \mathrm{Hom}_{kQ}(X, S),$$
$$0 \to \bigoplus_{i=1}^{u} S \to X \to X_{-S} \to 0, \quad u = \dim \mathrm{Hom}_{kQ}(S, X).$$

By Remark 3.1.1 the natural maps $\mathrm{Hom}_{kQ}(S, X^{-S}) \to \mathrm{Hom}_{kQ}(S, X)$ and $\mathrm{Hom}_{kQ}(X_{-S}, S) \to \mathrm{Hom}_{kQ}(X, S)$ are isomorphisms. Thus, we get the following commuting diagram.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \bigoplus_{i=1}^{u} S \arrow[r] \arrow[d] & \bigoplus_{i=1}^{u} S \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X^{-S} \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & \bigoplus_{i=1}^{r} S \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (X^{-S})_{-S} \arrow[r] \arrow[d] & X_{-S} \arrow[r] \arrow[d] & \bigoplus_{i=1}^{r} S \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The top and the middle rows are exact, and hence so is the bottom row. (This is the $3 \times 3$ lemma, which can be found in [24, Exercise 1.3.2]). This, however, is the exact sequence as constructed in Lemma 3.1.3, and hence $(X^{-S})_{-S} = (X_{-S})^{-S}$. $\square$

We denote a quasi-inverse of the equivalence $\psi_S$ by $\sigma_S : \mathfrak{M}^{-S}_{-S} \to \mathfrak{M}^S_S/S$ and call the functor $\sigma_S$ *universal extension functor*. The above proof shows that the functor $\sigma_S$ operates on objects by applying the constructions for $\overline{\sigma}_S$ and $\underline{\sigma}_S$ successively. Moreover, the order in which we apply $\overline{\sigma}_S$ and $\underline{\sigma}_S$ does not matter, which follows from the fact that in the construction of $\psi_S$ the order of $\overline{\psi}_S$ and $\underline{\psi}_S$ did not matter. By [18, Proposition 2] we have

$$\begin{aligned} \underline{\dim}\,\sigma_S(X) &= s_{\underline{\dim}\,S}(\underline{\dim}\,X), \quad \text{and} \\ \underline{\dim}\,\psi_S(X) &= s_{\underline{\dim}\,S}(\underline{\dim}\,X). \end{aligned}$$

One of the main problems when applying the functor $\sigma_S$ was already mentioned in the introduction and becomes clear now: if we want to apply the equivalence $\sigma_S$ to a representation $X$, we must have $X \in \mathfrak{M}^{-S}_{-S}$. Whether $X \in \mathfrak{M}^{-S}_{-S}$ or $X \notin \mathfrak{M}^{-S}_{-S}$ is one of the main questions we are concerned with. In Chapter 4 we discuss the maximal rank type property which may be used to decide this question for real root representations.

In the introduction we raised two questions about properties of real root representations. What is the dimension of the endomorphism ring of a given real root representation? Is a given real root representation a tree representation? Since we are concerned with the question of the constructibility of real root representations using universal extension functors, it is essential for us to know how these properties behave under the functor $\sigma_S$. This is discussed in the following two results.

**Proposition 3.1.7** ([18, Proposition 3 & 3*]). *Let $X \in \mathfrak{M}^{-S}_{-S}$. Then*

$$\dim \operatorname{End}_{kQ} \sigma_S(X) = \dim \operatorname{End}_{kQ}(X) + \langle \underline{\dim}\,X, \underline{\dim}\,S \rangle \cdot \langle \underline{\dim}\,S, \underline{\dim}\,X \rangle. \tag{3.1}$$

*Let $Y \in \mathfrak{M}^S_S$. Then*

$$\dim \operatorname{End}_{kQ} \psi_S(Y) = \dim \operatorname{End}_{kQ}(Y) - \langle \underline{\dim}\,Y, \underline{\dim}\,S \rangle \cdot \langle \underline{\dim}\,S, \underline{\dim}\,Y \rangle.$$

The following result shows that indecomposable tree representations are preserved under the functors $\overline{\sigma}_S, \underline{\sigma}_S$ and $\sigma_S$. The proof follows closely the arguments given in [19, Section 3 and Section 6].

**Theorem 3.1.8** ([26, Lemma 3.16]). *Let* $X \in \mathfrak{M}^{-S}$ $(resp.,\ X \in \mathfrak{M}_{-S})$ *be an indecomposable tree representation. Then the representation* $\overline{\sigma}_S(X)$ $(resp.,\ \underline{\sigma}_S(X))$ *is an indecomposable tree representation. In particular, let* $X \in \mathfrak{M}^{-S}_{-S}$ *be an indecomposable tree representation, then* $\sigma_S(X)$ *is an indecomposable tree representation.*

*Proof.* We consider only the situation for the functor $\overline{\sigma}_S$. The situation for $\underline{\sigma}_S$ is analogous. Since $\sigma_S$ is given by applying $\overline{\sigma}_S$ and $\underline{\sigma}_S$ successively, the second assertion follows from the first.

We recall the construction of $\overline{\sigma}_S(X)$. Let $E_1, \ldots, E_s$ be a basis of the $k$-vector space $\mathrm{Ext}^1_{kQ}(S, X)$. Consider the exact sequence $E$ given by the elements $E_1, \ldots, E_s$

$$E : 0 \to X \to Z \to \bigoplus_{i=1}^{s} S \to 0; \tag{+}$$

then we have $\overline{\sigma}_S(X) = Z$. First of all, we note that $Z$ is indecomposable since $\overline{\sigma}_S : \mathfrak{M}^{-S} \to \mathfrak{M}^S/S$ defines an equivalence of categories by Proposition 3.1.5. It follows from Theorem 2.2.16 that the representation $S$ is a tree representation. Thus, we can choose a basis $\mathfrak{B}_X$ of $X$ and a basis $\mathfrak{B}_S$ of $S$ such that the corresponding coefficient quivers $\Gamma(X, \mathfrak{B}_X)$ and $\Gamma(S, \mathfrak{B}_S)$ are trees. We set $d_X := \sum_{i \in Q_0} \dim X_i$ (dimension of $X$) and $d_S := \sum_{i \in Q_0} \dim S_i$ (dimension of $S$). It follows that $\Gamma(X, \mathfrak{B}_X)$ has $d_X - 1$ arrows and $\Gamma(S, \mathfrak{B}_S)$ has $d_S - 1$ arrows.

Let $a \in Q_1$. For given $1 \leq s \leq \underline{\dim}\, S[t(a)]$ and $1 \leq t \leq \underline{\dim}\, X[h(a)]$ we denote by

$$M_{SX}(a, s, t) \in \mathrm{Hom}_k(S_{t(a)}, X_{h(a)})$$

the matrix unit with entry one in the column with index $s$ and the row with index $t$, and zeros elsewhere. The set

$$H_{SX} := \{M_{SX}(a, s, t) : a \in Q_1, 1 \leq s \leq \underline{\dim}\, S[t(a)], 1 \leq t \leq \underline{\dim}\, X[h(a)]\}$$

is clearly a basis of $C^1(S, X)$. Hence, we can choose a subset

$$\Phi := \{M_{SX}(a_i, s_i, t_i) : 1 \leq i \leq r\} \subset H_{SX}$$

such that $\operatorname{span}\Phi \oplus \operatorname{im}\delta_{SX} = C^1(S, X)$, which implies that the residue classes $\phi + \operatorname{im}\delta_{SX}$ $(\phi \in \Phi)$ form a basis of $\operatorname{Ext}^1_{kQ}(S, X)$; these elements are responsible for obtaining the extension $(+)$.

We are now able describe the matrices of the representation $Z$ with respect to the basis $\mathfrak{B}_X \cup \bigcup_{d=1}^r \mathfrak{B}_S$. Let $b \in Q_1$. The matrix $Z_b$ has the following form

$$Z_b = \begin{bmatrix} X_b & N(b,1) & \dots & N(b,r) \\ & S_b & & \\ & & \ddots & \\ & & & S_b \end{bmatrix}$$

with all other entries equal to zero and

$$N(b, i) = \begin{cases} M(a_i, s_i, t_i), & \text{if } b = a_i \\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

where $\mathbf{0}$ denotes the zero matrix of the appropriate size. This explicit description allows us to count the overall number of non-zero entries in the matrices of the representation $Z$ with respect to the basis $\mathfrak{B}_X \cup \bigcup_{d=1}^r \mathfrak{B}_S$: this number equals the number of arrows of the coefficient quiver $\Gamma(Z, \mathfrak{B}_X \cup \bigcup_{d=1}^r \mathfrak{B}_S)$. We easily see that there are

$$(d_X - 1) + r(d_S - 1) + |\Phi| = d_X + rd_S - 1 = \sum_{i \in Q_0} \dim Z_i - 1$$

non-zero entries.

Now, since $Z$ is indecomposable, the coefficient quiver $\Gamma(Z, \mathfrak{B}_X \cup \bigcup_{d=1}^r \mathfrak{B}_S)$ is connected, and hence $\Gamma(Z, \mathfrak{B}_X \cup \bigcup_{d=1}^r \mathfrak{B}_S)$ is a tree.

□

We fix the following notation.

**Definition 3.1.9.** Let $\alpha$ be a real Schur root for $Q$. We define

$$\mathfrak{M}_{-\alpha}^{-\alpha} := \mathfrak{M}_{-X_\alpha}^{-X_\alpha}, \quad \mathfrak{M}_{\alpha}^{\alpha} \; := \; \mathfrak{M}_{X_\alpha}^{X_\alpha}, \quad \text{and} \quad \sigma_\alpha := \sigma_{X_\alpha}.$$

The following lemma shall be used frequently throughout this thesis.

**Lemma 3.1.10.** *Let $k$ be a field and let $Q$ be a quiver. Let $\beta$ be a real Schur root and let $\gamma$ be a real root such that $X_\gamma \in \mathfrak{M}_{-\beta}^{-\beta}$. Then we have $X_\alpha = \sigma_\beta(X_\gamma)$ with $\alpha = s_\beta(\gamma)$.*

*Proof.* Since $X_\gamma \in \mathfrak{M}_{-\beta}^{-\beta}$ the functor $\sigma_\beta$ can be applied to $X_\gamma$ and we set $Z = \sigma_\beta(X_\gamma)$. The representation $Z$ is indecomposable, since the representation $X_\gamma$ is indecomposable. Moreover, we get $\dim Z = \alpha$ by formula 3.1. By Kac's Theorem 2.2.9, however, there exists only one indecomposable representation (up to isomorphism) of dimension vector $\alpha$. Hence, we get the desired result $X_\alpha = Z$. $\square$

We demonstrate the functor $\sigma_S$ in an example.

**Example 3.1.11.** Let $k$ be a field. Consider the quiver $Q: \; 1 \rightrightarrows 2$ . Clearly, $S(2) \in \mathfrak{M}_{-e_1}^{-e_1}$ since both representations have disjoint support, and hence the functor $\sigma_{e_1}$ can be applied to $S(2)$. Moreover, we have

$$\begin{aligned} \dim \operatorname{Ext}^1_{kQ}(S(1), S(2)) &= 2 \quad \text{(given by the two arrows)}, \\ \dim \operatorname{Ext}^1_{kQ}(S(2), S(1)) &= 0. \end{aligned}$$

Hence, we get

$$0 \to S(2) \to \sigma_{e_1}(S(2)) \to \bigoplus_{i=1}^{2} S(1) \to 0,$$

where $\sigma_{e_1}(S(2))$ is the real root representation with dimension vector $\alpha = (2,1)$, by Lemma 3.1.10. By construction, the representation $X_\alpha$ has the following matrix form.

$$\begin{tikzcd} k^2 \arrow[d, shift right, "{[1,0]}"'] \arrow[d, shift left, "{[0,1]}"] \\ k \end{tikzcd}$$

Moreover, $X_\alpha$ is a tree representation by the previous theorem, and we get by Formula 3.1

$$\begin{aligned} \dim \operatorname{End}_{kQ} X_\alpha &= \dim \operatorname{End}_{kQ} S(2) + \langle e_2, e_1 \rangle \cdot \langle e_1, e_2 \rangle \\ &= 1 + 0 = 1. \end{aligned}$$

We give a first example of a class of quivers for which Question $(\dagger\dagger)$ can be answered affirmatively.

**Example 3.1.12.** Let $k$ be a field and let $Q$ be an extended Dynkin quiver. For a discussion of the representation theory of extended Dynkin quivers see [5] (over algebraically closed fields) and [9] (over arbitrary fields, using the more general species approach).

Let $\alpha$ be a non-Schur real root. It follows that $X_\alpha$ is a non-homogeneous regular representation and has the following regular composition factors (where $\tau$ denotes the Auslander-Reiten translate)

$$T, \tau T, \tau^2 T, \tau^3 T, \ldots, \tau^r T, \quad r \geq 1$$

with $T$ a regular simple representation and $\tau^{r+1} T \neq T$, since $\alpha$ is a real root. We consider two cases. Firstly, let $T \neq \tau^r T$. Consider the real root representation $X_\gamma$ with the following composition factors

$$T, \tau T, \tau^2 T, \tau^3 T, \ldots, \tau^{r-1} T.$$

Thus $X_\gamma \in \mathfrak{M}_{-X_\beta}^{-X_\beta}$ with $X_\beta = \tau^r T$ (which is a regular simple), since the representation $X_\gamma$ is uniserial (see [9]), that is it has a unique composition series, and $T \neq \tau^r T$ and $\tau^{r-1} T \neq \tau^r T$ (because $T \neq \tau T$, since we are in a tube of length greater than one). Namely, a map $\phi : X_\gamma \to X_\beta$ would imply that $X_\gamma / \ker \phi = X_\beta$, and hence $T = \tau^r T$ (by uniseriality) which is impossible. On the other hand, a map $\phi : X_\beta \to X_\gamma$ would imply (by uniseriality) that $\tau^{r-1} T = \tau^r T$ which is impossible. Hence $X_\alpha = \sigma_{X_\beta}(X_\gamma)$ by Lemma 3.1.10.

Secondly, let $T = \tau^r T$. Consider the real root representation $X_\gamma$ with the following composition factors

$$\tau T, \tau^2 T, \tau^3 T, \ldots, \tau^{r-1} T.$$

Thus $X_\gamma \in \mathfrak{M}^{-X_\beta}_{-X_\beta}$ with $X_\beta = T$ (which is a regular simple), since the representation $X_\gamma$ is uniserial and $T \neq \tau T$ (as above) and $T \neq \tau^{r-1} T$ (because $\tau^r T = T = \tau^{r-1} T$ implies that $T = \tau T$); together with a similar reasoning as above. Hence, $X_\alpha = \sigma_{X_\beta}(X_\gamma)$, by Lemma 3.1.10.

## 3.2 Real root representations of $Q''(g,h)$ and a generalisation

We now consider the following quiver

$$Q''(g,h):\ \begin{tikzcd} 2 \arrow[r, bend left=60, "\mu_1"] \arrow[r, bend left=20, "\mu_g"'] & 3 \arrow[l, bend left=20, "\nu_1"'] \arrow[l, bend left=60, "\nu_h"] \end{tikzcd},$$

with $g, h \geq 1$. As mentioned in the introduction, Question (††) can be answered affirmatively for $Q''(g,h),\ (g,h) \geq 1$. This is due to the following key lemma.

**Lemma 3.2.1** ([18, Lemma 4])**.** *Let $k$ be a field and let $Q$ be a quiver. Let $S, T$ be representations of $Q$, where $T$ is simple.*

*(i) If $\operatorname{Ext}^1_{kQ}(S,T) \neq 0$, then $\mathfrak{M}^S \subset \mathfrak{M}^{-T}$.*

*(ii) If $\operatorname{Ext}^1_{kQ}(T,S) \neq 0$, then $\mathfrak{M}_S \subset \mathfrak{M}_{-T}$.*

We have the following immediate corollary for the quiver $Q''(g,h),\ (g,h \geq 1)$.

**Corollary 3.2.2.** *We have*

$$\begin{aligned} \mathfrak{M}^{e_2}_{e_2} &\subset \mathfrak{M}^{-e_3}_{-e_3}, \\ \mathfrak{M}^{e_3}_{e_3} &\subset \mathfrak{M}^{-e_2}_{-e_2}. \end{aligned}$$

The above corollary and Lemma 3.1.8 together with the trivial fact that

$$S(2) \in \mathfrak{M}_{-e_3}^{-e_3} \quad \text{and} \quad S(3) \in \mathfrak{M}_{-e_2}^{-e_2},$$

imply the following theorem, already mentioned in the introduction.

**Theorem 3.2.3** ([18, Section 2])**.** *Let $k$ be a field and let $\alpha$ be a positive real root for $Q''(g,h)$, $(g,h \geq 1)$. Write $\alpha = s_{i_n} \cdot \ldots \cdot s_{i_1}(e_j)$ with $i_k, j \in \{2,3\}$ and $n$ minimal. Then we have*

$$X_\alpha = \sigma_{e_{i_n}} \cdot \ldots \cdot \sigma_{e_{i_1}}(S(j)),$$

*and hence the representation $X_\alpha$ is a tree representation and formula (3.1) can be used to compute $\dim \operatorname{End}_{kQ} X_\alpha$.*

The above result can be generalized to quivers with the following property (#) for all $i, j \in Q_0$: if there exists $a \in Q_1$ with $a : i \to j$ then there exists $a' \in Q_1$ with $a' : j \to i$.

**Example 3.2.4.** Here is an example of a quiver $Q$ with the above property (#).

$$\begin{tikzcd} Q: 1 \arrow[r, bend left] \arrow[rr, bend right=60] & 2 \arrow[l, bend left] \arrow[r, bend left] & 3 \arrow[l, bend left] \arrow[ll, bend right=60] \end{tikzcd}$$

**Theorem 3.2.5.** *Let $k$ be a field. Let $Q$ be a quiver with the property (#) and let $\alpha$ be a positive real root for $Q$. Write $\alpha = s_{i_n} \cdot \ldots \cdot s_{i_1}(e_j)$ with $i_k, j \in Q_0$ and $n$ minimal. Then we have*

$$X_\alpha = \sigma_{e_{i_n}} \cdot \ldots \cdot \sigma_{e_{i_1}}(S(j)),$$

*and hence the representation $X_\alpha$ is a tree representation and formula (3.1) can be used to compute $\dim \operatorname{End}_{kQ} X_\alpha$.*

*Proof.* We prove the assertion by induction on $n$. The induction base $n = 1$ is trivial, since $S(j) \in \mathfrak{M}_{-e_{i_1}}^{-e_{i_1}}$, and hence $\sigma_{e_{i_1}}$ can be applied. Thus $X_\alpha = \sigma_{e_{i_1}}(S(j))$ by Lemma 3.1.10.

Let $n > 1$ and consider $\alpha = s_{i_n} \cdot \ldots \cdot s_{i_1}(e_j)$ with $n$ minimal. We set $i_0 = j$ and $\alpha' = s_{i_{n-1}} \cdot \ldots \cdot s_{i_1}(e_j)$. Assume that $i_p \neq i_n$ for $0 \leq p \leq n-2$. In this case we clearly have $X_{\alpha'} \in \mathfrak{M}_{-e_{i_n}}^{-e_{i_n}}$ and, thus $X_\alpha = \sigma_{e_{i_n}}(X_{\alpha'})$ by Lemma 3.1.10 and the assertion follows by induction.

Now, assume there exists $0 \leq p \leq n-2$ such that $i_n = i_p$ and choose $p$ maximal. Since $n$ is minimal and from the assumption on the quiver it follows that there exists $p < q < n$ and $a, a' \in Q_1$ such that $a : i_q \to i_n$ and $a' : i_n \to i_q$. (If not, then $s_{i_n}$ and $s_{i_q}$ commute for $p < q < n$, and hence $n$ is not minimal.) In particular, using Lemma 3.2.1, we get

$$\mathfrak{M}_{e_{i_q}}^{e_{i_q}} \subset \mathfrak{M}_{-e_{i_n}}^{-e_{i_n}}.$$

By induction we have $\sigma_{e_{i_q}} \cdot \ldots \cdot \sigma_{e_{i_1}}(S(j)) \in \mathfrak{M}_{e_{i_q}}^{e_{i_q}} \cap \mathfrak{M}_{-e_{i_{q+1}}}^{-e_{i_{q+1}}}$, and hence $\sigma_{e_{i_q}} \cdot \ldots \cdot \sigma_{e_{i_1}}(S(j)) \in \mathfrak{M}_{-e_{i_n}}^{-e_{i_n}} \cap \mathfrak{M}_{-e_{i_{q+1}}}^{-e_{i_{q+1}}}$. Now, let $Y \in \mathfrak{M}_{-e_{i_n}}^{-e_{i_n}} \cap \mathfrak{M}_{-e_{i_t}}^{-e_{i_t}}$ with $i_t \neq i_n$. We can apply $\sigma_{e_{i_t}}$ to $Y$ and we get (by construction of the universal extension functor)

$$\begin{aligned} 0 &\to Y \to \overline{\sigma}_{S(i_t)}(Y) \to \bigoplus_{i=1}^{r} S(i_t) \to 0, \quad r = \dim \mathrm{Ext}^1_{kQ}(S(i_t), X) \\ 0 &\to \bigoplus_{i=1}^{u} S(i_t) \to \sigma_{e_{i_t}}(Y) \to \overline{\sigma}_{S(i_t)}(Y) \to 0, \quad u = \dim \mathrm{Ext}^1_{kQ}(\overline{\sigma}_{S(i_t)}(Y), S(i_t)). \end{aligned}$$

Using long exact sequences (as described in Section 2.2) and the fact that $S(i_n) \not\cong S(i_t)$ (since $i_n \neq i_t$), we see that $\sigma_{e_{i_t}}(Y) \in \mathfrak{M}_{-e_{i_n}}^{-e_{i_n}}$.

Now, by induction we know that $\sigma_{e_{i_q}} \cdot \ldots \cdot \sigma_{e_{i_1}}(S(j)) \in \mathfrak{M}_{-e_{i_n}}^{-e_{i_n}}$, and from the choice of $p$ it follows that $i_r \neq i_n$ for all $p < r < n$. Hence, the above elaborations imply that

$$X_{\alpha'} = \sigma_{e_{i_{n-1}}} \cdot \ldots \cdot \sigma_{e_{i_1}}(S(j)) \in \mathfrak{M}_{-e_{i_n}}^{-e_{i_n}},$$

and thus $X_\alpha = \sigma_{e_{i_n}}(X_{\alpha'})$ by Lemma 3.1.10, and the assertion follows.

The representation $X_\alpha$ is a tree representation by Lemma 3.1.8.

$\square$

# Chapter 4

# Representations of maximal rank type and applications

In the last chapter it became clear that if we want to work with the equivalence $\sigma_S$, we need to be able to decide for a given representation $X$ whether $X \in \mathfrak{M}_{-S}^{-S}$ or $X \notin \mathfrak{M}_{-S}^{-S}$. In this chapter we discuss the maximal rank type property which may be used to answer this question.

In order to obtain a more general result than in [26, Theorem A] we shall work with Definition 2.2.2 of a representation in this chapter.

## 4.1 Representations of maximal rank type

Let $Q = (Q_0, Q_1, h, t)$ be a quiver and let $k$ be a field. For $i \in Q_0$ we define the sets $H^Q(i) := \{a \in Q_1 : h(a) = i\}$ and $T^Q(i) := \{a \in Q_1 : t(a) = i\}$.

**Definition 4.1.1** (Maximal rank type)**.** A representation $X$ of $Q$ is said to be of *maximal rank type*, provided it satisfies the following conditions.

(i) For every vertex $i \in Q_0$ and for every family of vector subspaces $U_{j,i} \subset k^{n_{j,i}}$ $(j \in Q_0)$ the map

$$\bigoplus_{j \in Q_0} X_j \otimes_k U_{j,i} \xrightarrow{X_{j,i}} X_i$$

is of maximal rank.

(ii) For every vertex $i \in Q_0$ and for every family of vector subspaces $U_{i,j} \subset k^{n_{i,j}}$ $(j \in Q_0)$ the map

$$X_i \xrightarrow{X'_{i,j}} \bigoplus_{j \in Q_0} X_j \otimes U_{i,j}$$

is of maximal rank.

**Remark 4.1.2.** In the paper [26] the author introduced the following definition. **Definiton** (Maximal rank type, [26, Definition]). A representation $X$ of $Q$ is said to be of *maximal rank type*, provided it satisfies the following conditions.

(i) For every vertex $i \in Q_0$ and for every subset $A \subseteq H^Q(i)$ the map

$$\bigoplus_{a \in A} X_{t(a)} \xrightarrow{(X_a)_a} X_i$$

is of maximal rank.

(ii) For every vertex $i \in Q_0$ and for every subset $B \subseteq T^Q(i)$ the map

$$X_i \xrightarrow{(X_b)_b} \bigoplus_{b \in B} X_{h(b)}$$

is of maximal rank.

We remark that Definition 4.1.1 is stronger than the above definition, since we are not only allowing arrows but also independent linear combinations of arrows.

Clearly not every representation of $Q$ is of maximal rank type. The following example shows that even indecomposable representations of $Q$ might not be of maximal rank type:

$$k \underset{0}{\overset{1}{\rightrightarrows}} k\ .$$

## 4.2 Real root representations are of maximal rank type

In this section we show that real root representations are of maximal rank type. The main idea of the proof is to insert an extra vertex and to attach to it the image of the considered map. Analysing this modified representation yields the desired result. The technicalities for inserting an extra vertex are discussed in the first part of this section followed by the proof of the main result.

Let $Q = (Q_0, Q_1, h, t)$ be a quiver without loops and let $k$ be a field. Moreover, let $i \in Q_0$ be a vertex of $Q$ and let $X$ be a representation of $Q$. For a given family $U = (U_{j,i})_{j \in Q_0}$ of vector subspaces $U_{j,i} \subset k^{n_{j,i}}$ $(j \in Q_0)$ we define the quiver $Q^i_U$ and the representation $X^i_U$ (of the quiver $Q^i_U$) as follows

$$\begin{aligned}(Q^i_U)_0 := Q_0 \,\dot\cup\, \{z\}, \quad (Q^i_U)_1 := \quad & (Q_1 - H^Q(i)) \\ \dot\cup \quad & \{\gamma^p_{U_{j,i}} : j \in Q_1, 1 \le p \le \dim U_{j,i}\} \\ \dot\cup \quad & \{\gamma^p_{j,i} : j \in Q_1, 1 \le p \le n_{j,i} - \dim U_{j,i}\} \\ \dot\cup \quad & \{\delta\}\end{aligned}$$

with

$$\begin{aligned} t(\gamma^p_{U_{j,i}}) := j, \qquad & h(\gamma^p_{U_{j,i}}) := z \quad \forall\, j \in Q_0, 1 \le p \le \dim U_{(j,i)}, \\ t(\gamma^p_{j,i}) := j, \qquad & h(\gamma^p_{j,i}) := i \quad \forall\, j \in Q_0, 1 \le p \le n_{j,i} - \dim U_{(j,i)}, \\ t(\delta) := z, \qquad & h(\delta) := i, \end{aligned}$$

(heads and tails for all arrows in $Q_1 - H^Q(i)$ remain unchanged) and

$$(X^i_U)_j := X_j \quad \forall\, j \in Q_0, \quad (X^i_U)_z := \operatorname{im}\left(\bigoplus_{j \in Q_0} X_j \otimes_k U_{j,i} \xrightarrow{X_{j,i}} X_i\right) \subset X_i,$$

with maps

$$\begin{aligned} (X^i_U)_{f,g} \quad &:= \quad X_{f,g} \quad \forall\, f, g \in Q_0 - \{i, j\}, \\ (X^i_U)_{z,i} \quad &:= \quad \text{inclusion}, \\ (X^i_U)_{j,i} \quad &:= \quad \text{restriction of } X_{j,i} \text{ to direct complement of } U_{j,i} \quad \forall\, j \in Q_0, \\ (X^i_U)_{j,z} \quad &:= \quad \hat{X}_{j,z} \quad \forall\, j \in Q_0, \end{aligned}$$

where $\hat{X}_{j,z} : X_j \otimes_k U_{i,j} \to (X_U^i)_z$ is the unique linear map with $(X_U^i)_{z,i} \circ \hat{X}_{j,z} = X_{j,i}$; and all other maps zero.

The construction above gives a functor $F_U^i : \operatorname{rep}_k Q \to \operatorname{rep}_k Q_U^i$, defined as follows:

$$\begin{array}{rcl} F_U^i : \mathsf{Ob}(\operatorname{rep}_k Q) & \to & \mathsf{Ob}(\operatorname{rep}_k Q_U^i) \\ X & \mapsto & X_U^i, \end{array}$$

with the natural definition for morphisms. Moreover, there is a natural functor ${}_U^iG : \operatorname{rep}_k Q_U^i \to \operatorname{rep}_k Q$, defined by

$$\begin{array}{rcl} {}_U^iG : \mathsf{Ob}(\operatorname{rep}_k Q_U^i) & \to & \mathsf{Ob}(\operatorname{rep}_k Q) \\ X & \mapsto & {}_U^iG(X), \end{array}$$

with

$$({}_U^iG(X))_j := X_j \quad \forall\, j \in Q_0,$$

and maps

$$\begin{array}{rcl} ({}_U^iG(X))_{f,g} & := & X_{f,g} \quad \forall\, f, g \in Q_0 - \{i, j\}, \\ ({}_U^iG(X))_{j,i} & := & \begin{cases} X_{j,i}, & \text{if } \dim U_{j,i} = 0 \\ X_{z,i} \circ X_{j,z}, & \text{otherwise} \end{cases} \qquad , \forall\, j \in Q_0 \end{array}$$

and all other maps zero, together with the natural definition for morphisms. The functor ${}_U^iG$ is left-adjoint to the functor $F_U^i$, and ${}_U^iG \circ F_U^i$ is naturally isomorphic to the identity functor on $\operatorname{rep}_k Q$.

We get the following useful lemma.

**Lemma 4.2.1.** *Let $i \in Q_0$ and let $X$ be a representation of $Q$. If $X$ is indecomposable, then so is $F_U^i(X) = X_U^i$ for every family $U = (U_{j,i})_{j \in Q_0}$.*

*Proof.* Assume that $X^i_U = F^i_U(X) \cong V \oplus W$, then $X \cong {}^i_U G \circ F^i_U(X) \cong {}^i_U G(V) \oplus {}^i_U G(W)$. By assumption $X$ is indecomposable, so w.l.o.g. we can assume that ${}^i_A G(V) = 0$. Hence,

$$0 = \mathrm{Hom}_{kQ}({}^i_U G(V), X) = \mathrm{Hom}_{kQ^i_U}(V, F^i_U X) = \mathrm{Hom}_{kQ^i_U}(V, V \oplus W),$$

which is only possible in case $V = 0$. This proves the assertion.

□

We are now able to prove the maximal rank type property of real root representations.

**Theorem 4.2.2.** *Let $Q$ be a quiver and let $\alpha$ be a positive real root for $Q$. The unique indecomposable representation of dimension vector $\alpha$ is of maximal rank type.*

*Proof.* Let $\alpha$ be a real root for $Q$, and let $X = X_\alpha$ be the unique indecomposable representation of $Q$ of dimension vector $\alpha$. Moreover, let $i \in Q_0$ and let $U = (U_{j,i})_{j \in Q_0}$ be a family of vector subspaces $U_{j,i} \subset k^{n_{j,i}}$ $(j \in Q_0)$. We have to show that the map

$$\bigoplus_{j \in Q_0} X_j \otimes_k U_{j,i} \xrightarrow{X_{j,i}} X_i$$

has maximal rank. This is equivalent to showing that

$$\dim(X^i_U)_z = \min\left\{\sum_{j \in Q_0} \alpha[j] \dim U_{j,i}, \alpha[i]\right\},$$

The representation $X^i_U$ of $Q^i_U$ is indecomposable by Lemma 4.2.1. It follows from Theorem 2.2.9 that $\underline{\dim}\, X^i_U \in \Delta^+(Q^i_U)$. Hence, by Lemma 2.1.2, $\langle \hat\alpha, \hat\alpha \rangle \leq 1$, where $\hat\alpha := \underline{\dim}\, X^i_U$. We have

$$\begin{aligned}
\langle \hat\alpha, \hat\alpha \rangle &= \underbrace{\langle \alpha, \alpha \rangle}_{=1} + \sum_{j \in Q_0} \alpha[j](\dim U_{j,i})\alpha[i] + \hat\alpha[z]^2 - \hat\alpha[z]\hat\alpha[i] - \sum_{j \in Q_0} \hat\alpha[j](\dim U_{j,i})\hat\alpha[z] \\
&= 1 + \left(\hat\alpha[z] - \sum_{j \in Q_0} \alpha[j] \dim U_{j,i}\right) \cdot (\hat\alpha[z] - \alpha[i]) \leq 1,
\end{aligned}$$

and hence

$$\left(\hat\alpha[z] - \sum_{j \in Q_0} \alpha[j] \dim U_{j,i}\right) \cdot (\hat\alpha[z] - \alpha[i]) \leq 0.$$

However, we clearly have $\hat{\alpha}[z] \leq \min\left\{\sum_{j\in Q_0} \alpha[j] \dim U_{j,i}, \alpha[i]\right\}$, by definition of $X^i_U$. This implies that

$$\left(\hat{\alpha}[z] - \sum_{j\in Q_0} \alpha[j] \dim U_{j,i}\right) \cdot (\hat{\alpha}[z] - \alpha[i]) = 0;$$

that is, $\hat{\alpha}[z] = \min\left\{\sum_{j\in Q_0} \alpha[j] \dim U_{j,i}, \alpha[i]\right\}$, and hence

$$\dim(X^i_U)_z = \min\left\{\sum_{j\in Q_0} \alpha[j] \dim U_{j,i}, \alpha[i]\right\}.$$

This shows that the map $\bigoplus_{j\in Q_0} X_j \otimes_k U_{j,i} \xrightarrow{X_{j,i}} X_i$ has maximal rank.

Dually, for a given family $U = (U_{j,i})_{j\in Q_0}$ to show that the map

$$X_i \xrightarrow{X'_{i,j}} \bigoplus_{j\in Q_0} X_j \otimes U_{i,j}$$

has maximal rank is equivalent to showing that the map

$$\bigoplus_{j\in Q_0} (X \otimes U_{i,j})^* \xrightarrow{(X'_{i,j})^*} X_i^*$$

has maximal rank, where $^*$ denotes the vector space dual. This follows from what is proved above by considering the dual $X^*$ as a representation of the opposite quiver of $Q$. Remark that the representation $X^*$ is also indecomposable.

$\square$

We demonstrate how Theorem 4.2.2 can be used in practice.

**Example 4.2.3.** We describe one of the decompositions discussed in the Introduction. Consider the following quiver $Q$

$$Q: \qquad 1 \underset{b}{\overset{a}{\rightrightarrows}} 2 \underset{d}{\overset{c}{\rightrightarrows}} 3 \underset{f}{\overset{e}{\rightrightarrows}} 4$$

and the real root $\alpha = (2, 37, 20, 4)$, which can be decomposed as follows:

$$\alpha = 18 \cdot \beta + \gamma = (-\langle \beta, \gamma \rangle - \langle \gamma, \beta \rangle) \cdot \beta + \gamma,$$

with $\beta = (0, 2, 1, 0)$ and $\gamma = (2, 1, 2, 4)$. In Remark 5.0.13 we discuss how such decompositions can be obtained combinatorially. Using Theorem 4.2.2, we see that $\mathrm{Hom}_{kQ}(X_\beta, X_\gamma) = \mathrm{Hom}_{kQ}(X_\gamma, X_\beta) = 0$, and thus $X_\gamma \in \mathfrak{M}_{-X_\beta}^{-X_\beta}$. Namely, let $\phi = (0, \phi_2, \phi_3, 0) \in \mathrm{Hom}_{kQ}(X_\beta, X_\gamma)$. Since the map $(X_\gamma)_3 \overset{(X_\gamma)_e}{\to} (X_\gamma)_4$ is injective it follows that $\phi_3 = 0$. This implies that $\phi_2 = 0$, since the map $(X_\gamma)_2 \overset{(X_\gamma)_c}{\to} (X_\gamma)_3$ is injective. Now, let $\phi = (0, \phi_2, \phi_3, 0) \in \mathrm{Hom}_{kQ}(X_\gamma, X_\beta)$. Since the map $(X_\gamma)_1 \overset{(X_\gamma)_a}{\to} (X_\gamma)_2$ is surjective it follows that $\phi_2 = 0$. This implies that $\phi_3 = 0$, since the map $(X_\gamma)_2 \oplus (X_\gamma)_2 \overset{[(X_\gamma)_c, (X_\gamma)_d]}{\longrightarrow} (X_\gamma)_3$ is surjective. Hence, we have $X_\gamma \in \mathfrak{M}_{-X_\beta}^{-X_\beta}$ and by Lemma 3.1.10

$$\begin{aligned} X_\alpha &= \sigma_{X_\beta}(X_\gamma), \quad \text{and} \\ \dim \mathrm{End}_{kQ} X_\alpha &= \dim \mathrm{End}_{kQ} X_\gamma + \langle \beta, \gamma \rangle \cdot \langle \gamma, \beta \rangle, \\ &= 1 + (-6) \cdot (-12) = 73. \end{aligned}$$

Moreover, by Lemma 3.1.8 the representation $X_\alpha$ is a tree representation, since $\gamma$ is a real Schur root.

## 4.3 Relation to homomorphism and extension spaces

In this section we discuss how the maximal rank type property can be used to compute the dimensions of homomorphism spaces and extension spaces between representations of maximal rank type and simple representations. We will see that the dimensions of these spaces are completely determined by the Ringel form. This gives a nice combinatorial result to decide for a representation $X$ of maximal rank type whether $X \in \mathfrak{M}_{-S(i)}^{-S(i)}$ or $X \notin \mathfrak{M}_{-S(i)}^{-S(i)}$ for $i \in Q_0$.

Let $Q = (Q_0, Q_1, h, t)$ be a quiver without loops and let $k$ be a field. The following two facts relate the maximal rank type property to the dimension of homomorphism spaces and extension spaces between representations of maximal rank type and simple representations.

**Lemma 4.3.1.** *Let $X$ be a representation of $Q$. For every vertex $i \in Q_0$ we have the following isomorphisms*

$$\begin{aligned}\mathrm{Hom}_{kQ}(X, S(i)) &\cong \left[\mathrm{coker}\left(\bigoplus_{a\in H^Q(i)} X_{t(a)} \xrightarrow{(X_a)_a} X_i\right)\right]^* \\ \mathrm{Hom}_{kQ}(S(i), X) &\cong \ker\left(X_i \xrightarrow{(X_b)_b} \bigoplus_{b\in T^Q(i)} X_{h(b)}\right).\end{aligned}$$

*Proof.* Follows from Lemma 2.2.8.

□

We immediately get the following corollary.

**Corollary 4.3.2.** *Let $X$ be a representation of $Q$. Then for every vertex $i \in Q_0$ we have the following equivalences*

$$\begin{aligned}\bigoplus_{a\in H^Q(i)} X_{t(a)} \xrightarrow{(X_a)_a} X_i \text{ is injective} &\iff \mathrm{Ext}^1_{kQ}(X, S(i)) = 0, \\ \bigoplus_{a\in H^Q(i)} X_{t(a)} \xrightarrow{(X_a)_a} X_i \text{ is surjective} &\iff \mathrm{Hom}_{kQ}(X, S(i)) = 0, \\ X_i \xrightarrow{(X_b)_b} \bigoplus_{b\in T^Q(i)} X_{h(b)} \text{ is surjective} &\iff \mathrm{Ext}^1_{kQ}(S(i), X) = 0, \\ X_i \xrightarrow{(X_b)_b} \bigoplus_{b\in T^Q(i)} X_{h(b)} \text{ is injective} &\iff \mathrm{Hom}_{kQ}(S(i), X) = 0.\end{aligned}$$

*Proof.* Immediate by the previous lemma.

□

This gives the following proposition.

**Proposition 4.3.3.** *Let $X$ be a representation of maximal rank type and let $i \in Q_0$. We have*

$$\dim \mathrm{Hom}_{kQ}(X, S(i)) = \begin{cases} \langle \underline{\dim}\, X, e_i \rangle, & \text{if } \langle \underline{\dim}\, X, e_i \rangle > 0 \,, \\ 0, & \text{otherwise,} \end{cases}$$

$$\dim \mathrm{Ext}^1_{kQ}(X, S(i)) = \begin{cases} -\langle \underline{\dim}\, X, e_i \rangle, & \text{if } \langle \underline{\dim}\, X, e_i \rangle < 0 \,, \\ 0, & \text{otherwise,} \end{cases}$$

*and*

$$\dim \mathrm{Hom}_{kQ}(S(i), X) = \begin{cases} \langle e_i, \underline{\dim}\, X \rangle, & \text{if } \langle e_i, \underline{\dim}\, X \rangle > 0, \\ 0, & \text{otherwise,} \end{cases}$$

$$\dim \mathrm{Ext}^1_{kQ}(S(i), X) = \begin{cases} -\langle e_i, \underline{\dim}\, X \rangle, & \text{if } \langle e_i, \underline{\dim}\, X \rangle < 0, \\ 0, & \text{otherwise,} \end{cases}$$

*Proof.* Follows immediately from Corollary 4.3.2 and the formulae

$$\begin{aligned} \dim \mathrm{Hom}_{kQ}(X, S(i)) - \dim \mathrm{Ext}^1_{kQ}(X, S(i)) &= \langle \underline{\dim}\, X, e_i \rangle, \\ \dim \mathrm{Hom}_{kQ}(S(i), X) - \dim \mathrm{Ext}^1_{kQ}(S(i), X) &= \langle e_i, \underline{\dim}\, X \rangle. \end{aligned}$$

□

It follows from Theorem 4.2.2 that the dimension of the homomorphism space and the extension space between a simple representation and a real root representation is given by the Ringel form. In particular, we see that in this situation we have either homomorphisms or extensions, never both.

One may ask whether Proposition 4.3.3 can be generalized to other representations in place of simple representations? For instance, if we replace the simple representations $S(i)$ by real Schur representations, will the dimensions of the homomorphism space and the extension space still be given by the Ringel form? In the following we discuss an example which shows that this is not the case.

We consider the following quiver

$$
\begin{tikzcd}
Q: 1 \arrow[r, "a"'] \arrow[rrr, bend left, "b"] & 2 \arrow[r, "c"'] \arrow[rr, bend right, "d"'] & 3 \arrow[r, "e"'] & 4
\end{tikzcd}
$$

and the real Schur roots $\alpha = (2,1,3,4)$ and $\beta = (0,1,1,0)$. The representations $X_\alpha$ and $X_\beta$ are given as follows

$$
\begin{tikzcd}
X_\alpha: k^2 \arrow[r, "X_a"'] \arrow[rrr, bend left, "X_b"] & k \arrow[r, "X_c"'] \arrow[rr, bend right, "X_d"'] & k^3 \arrow[r, "X_e"'] & k^4
\end{tikzcd}
$$

with

$$
X_a = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad X_b = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad X_c = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad X_d = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad X_e = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},
$$

and

$$
\begin{tikzcd}
X_\beta: 0 \arrow[r] \arrow[rrr, bend left] & k \arrow[r, "{[1]}"] \arrow[rr, bend right] & k \arrow[r] & 0 .
\end{tikzcd}
$$

It is easy to see that $\operatorname{End}_{kQ} X_\alpha = k$ and $\operatorname{End}_{kQ} X_\beta = k$ which shows that $\alpha$ and $\beta$ are indeed real Schur roots. Moreover, we get

$$
\operatorname{Hom}_{kQ}(X_\alpha, X_\beta) = \{(0, 0, [\,x_1,\, x_2,\, 0\,], 0) : x_1, x_2 \in k\},
$$

and hence $\langle \alpha, \beta \rangle = 1 \neq \dim \operatorname{Hom}_{kQ}(X_\alpha, X_\beta) = 2$. This shows that Proposition 4.3.3 does not generalize to the case of real Schur representations in place of simple representations.

The following lemma demonstrates how Proposition 4.3.3 can be used in practice.

**Lemma 4.3.4.** *Let $k$ be a field. Let $Q$ be a quiver and let $\alpha = s_{i_n} \cdot \ldots \cdot s_{i_1}(e_j)$ be a positive real root. Assume that $\langle e_{i_1}, e_j \rangle \leq 0$, $\langle e_j, e_{i_1} \rangle \leq 0$,*

$$\langle e_{i_{p+1}}, s_{i_p} \cdot \ldots \cdot s_{i_1}(e_j) \rangle \leq 0 \quad \textit{and} \quad \langle s_{i_p} \cdot \ldots \cdot s_{i_1}(e_j), e_{i_{p+1}} \rangle \leq 0$$

*for $1 \leq p \leq n-1$. Then we have*

$$X_\alpha = \sigma_{e_{i_n}} \cdot \ldots \cdot \sigma_{e_{i_1}}(S(j)).$$

*In particular, $X_\alpha$ is a tree representation and formula 3.1 can be used to compute $\dim \operatorname{End}_{kQ} X_\alpha$.*

*Proof.* It follows from Proposition 4.3.3 that $S(j) \in \mathfrak{M}_{-e_{i_1}}^{-e_{i_1}}$ and

$$\sigma_{e_{i_p}} \cdot \ldots \cdot \sigma_{e_{i_1}}(S(j)) \in \mathfrak{M}_{-e_{i_{p+1}}}^{-e_{i_{p+1}}},$$

for $1 \leq p \leq n-1$, and hence the functors $\sigma_{e_{i_p}}$ can be applied successively and the assertion follows from Lemma 3.1.10. It follows that $\dim \operatorname{End}_{kQ} X_\alpha$ can be computed using Formula 3.1. By Lemma 3.1.8 the representation $X_\alpha$ is a tree representation. $\square$

**Example 4.3.5.** Consider the following quiver $Q$

$$\begin{tikzcd}
2 \arrow[dr, shift left] \arrow[dr, shift right] & & & \\
& 3 \arrow[r, shift left] \arrow[r, shift right] & 4 \arrow[r, shift left] \arrow[r, shift right] & 5 \\
1 \arrow[ur, shift left] \arrow[ur, shift right] & & &
\end{tikzcd}$$

and the real root $\alpha = (1,4,16,44,8) = s_4s_3s_5s_4s_2s_3(e_1)$. We get

$$\begin{aligned}
\alpha &= (1,4,16,44,8) = s_4s_3s_5s_4s_2s_3(e_1) \\
\alpha_1 &= (1,4,16,4,8) = s_3s_5s_4s_2s_3(e_1), \quad \langle e_4,\alpha_1\rangle \le 0, \langle \alpha_1, e_4\rangle \le 0, \\
\alpha_2 &= (1,4,2,4,8) = s_5s_4s_2s_3(e_1), \quad \langle e_3,\alpha_2\rangle \le 0, \langle \alpha_2, e_3\rangle \le 0, \\
\alpha_3 &= (1,4,2,4,0) = s_4s_2s_3(e_1), \quad \langle e_5,\alpha_3\rangle \le 0, \langle \alpha_3, e_5\rangle \le 0, \\
\alpha_4 &= (1,4,2,0,0) = s_2s_3(e_1), \quad \langle e_4,\alpha_4\rangle \le 0, \langle \alpha_4, e_4\rangle \le 0, \\
\alpha_5 &= (1,0,2,0,0) = s_3(e_1), \quad \langle e_2,\alpha_5\rangle \le 0, \langle \alpha_5, e_2\rangle \le 0, \\
\alpha_6 &= (1,0,0,0,0) = e_1, \quad \langle e_3,\alpha_6\rangle \le 0, \langle \alpha_6, e_3\rangle \le 0,
\end{aligned}$$

and hence, using the previous lemma, we get

$$X_\alpha = \sigma_{e_4}\sigma_{e_3}\sigma_{e_5}\sigma_{e_4}\sigma_{e_2}\sigma_{e_3}(S(1)),$$

and, using Formula (3.1),

$$\begin{aligned}
\dim \operatorname{End}_{kQ} X_\alpha &= 1 + \langle e_3,\alpha_6\rangle \cdot \langle \alpha_6, e_3\rangle + \langle e_2,\alpha_5\rangle \cdot \langle \alpha_5, e_2\rangle + \langle e_4,\alpha_4\rangle \cdot \langle \alpha_4, e_4\rangle \\
&\quad + \langle e_5,\alpha_3\rangle \cdot \langle \alpha_3, e_5\rangle + \langle e_3,\alpha_2\rangle \cdot, \langle \alpha_2, e_3\rangle + \langle e_4,\alpha_1\rangle \cdot \langle \alpha_1, e_4\rangle \\
&= 1 + 0\cdot(-2) + (-4)\cdot 0 + 0\cdot(-4) \\
&\quad + 0\cdot(-8) + (-6)\cdot(-8) + (-12)\cdot(-28) \\
&= 385.
\end{aligned}$$

## 4.4 Application: representations of the quiver $Q(f,g,h)$

We fix an arbitrary field $k$. In this section we consider the following quiver

$$\begin{tikzcd}
Q(f,g,h)\colon & 1 \arrow[r, shift left=2, "\lambda_1"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2, "\lambda_f"'] & 2 \arrow[r, bend left=70, "\mu_1"] \arrow[r, bend left=20, "\mu_g"'] \arrow[r, phantom, shift left=3, "\vdots"] & 3 \arrow[l, bend left=20, "\nu_1"'] \arrow[l, bend left=70, "\nu_h"] \arrow[l, phantom, shift left=3, "\vdots"]
\end{tikzcd},$$

with $f, g, h \geq 1$, which was already mentioned in the introduction. The main result of this section is that Question $(\dagger\dagger)$ can be answered affirmatively for $Q(f, g, h)$.

We define the following subquivers

$$Q'(f)\colon\ 1 \underset{\lambda_f}{\overset{\lambda_1}{\underset{\vdots}{\rightrightarrows}}} 2\,,$$

and

$$Q''(g,h)\colon\ \begin{tikzcd} 2 \arrow[r, bend left=60, "\mu_1"] \arrow[r, bend left=20, "\mu_g"'] & 3 \arrow[l, bend left=20, "\nu_1"'] \arrow[l, bend left=60, "\nu_h"] \end{tikzcd}\,.$$

The quiver $Q(1,1,1)$ was considered by Jensen and Su in [12], where an explicit construction of all real root representations was given. Moreover, it was shown that all real root representations are tree representations and formulae to compute the dimensions of the endomorphism rings were given. In [20] Ringel extended their results to the quiver $Q(1, g, h)$ $(g, h \geq 1)$ by using universal extension functors. In this section we consider the general case with $f, g, h \geq 1$.

We briefly discuss the situation for the subquivers $Q'(f)$ and $Q''(g, h)$. The real root representations of the subquiver $Q'(f)$ are preprojective or preinjective representations (see [2, Section VIII.7] for $f = 2$ or [21, Section 1] for the general case), for the path algebra $kQ'(f)$, and can be constructed using BGP reflection functors (see [3]). It follows that the endomorphism ring of a real root representation of the subquiver $Q'(f)$ is isomorphic to the ground field $k$, and hence real root representations of $Q'(f)$ are real Schur representations.

The subquiver $Q''(g, h)$ was considered in Section 3.2. The main result was that all real root representations of $Q''(g, h)$ can be constructed using universal extension functors. In particular, formula (3.1) can be used to compute the dimension of the endomorphism ring of a real root representation.

We see that the situation is very well understood for the subquivers $Q'(f)$ and $Q''(g, h)$. Therefore we focus on real root representations with sincere dimension vectors.

### 4.4.1 The Weyl group of $Q = Q(f, g, h)$

Let $W$ be the Weyl group of $Q$. It is generated by the reflections $s_1, s_2$, and $s_3$ subject to the following relations

$$\begin{aligned} s_i^2 &= 1, \quad i = 1, 2, 3, \\ s_1 s_3 &= s_3 s_1, \\ s_1 s_2 s_1 &= s_2 s_1 s_2, \quad \text{if } f = 1. \end{aligned}$$

We define the following elements of the Weyl group ($n \geq 0$):

$$\begin{aligned} \zeta_1(n) &= (s_1 s_2)^n s_1, \\ \zeta_2(n) &= (s_2 s_1)^n s_2, \\ \rho_1(n) &= (s_1 s_2)^n, \\ \rho_2(n) &= (s_2 s_1)^n, \end{aligned}$$

and we set $E := \{\zeta_1(n), \zeta_2(n), \rho_1(n), \rho_2(n) : n \geq 0\}$.

**Lemma 4.4.1.** *Every element $w \in W - E$ can be written in the following form*

$$w = \chi_m s_3 \chi_{m-1} s_3 \chi_{m-2} s_3 \dots s_3 \chi_2 s_3 \chi_1, \qquad (*)$$

*for some $m \geq 2$, where*

$$\begin{aligned} \chi_m &\in \{\zeta_1(n) : n \geq 1\} \cup \{\zeta_2(n) : n \geq 0\} \cup \{1\}, \\ \chi_j &\in \{\zeta_1(n) : n \geq 1\} \cup \{\zeta_2(n) : n \geq 0\}, \quad j = 2, \dots, m-1, \\ \chi_1 &\in E. \end{aligned}$$

*If $f = 1$ then $w$ can be written in the form $(*)$ with only $\zeta_1(1) = \zeta_2(1), \rho_1(1)$, and $\rho_2(1)$ occurring.*

*Proof.* Let $w \in W - E$. Clearly, we can write $w$ in the form

$$w = \chi'_m s_3 \chi'_{m-1} s_3 \chi'_{m-2} s_3 \dots s_3 \chi'_2 s_3 \chi'_1,$$

with $m \geq 2$, $\chi'_j \in E$ for $j = 1, \ldots, m$, $\chi'_{m-1}, \ldots, \chi'_2 \notin \{1, s_1\}$, and $\chi'_m \neq s_1$. We modify the elements $\chi'_j$ to get a word of the form $(*)$. Let $2 \leq j \leq m$; we consider five cases and modify $\chi'_j$ appropriately.

(i) $\chi'_j = 1$. We set $\chi_j := \chi'_j$ and $\chi''_{j-1} = \chi'_{j-1}$. This case requires $j = m$.

(ii) $\chi'_j = \zeta_1(n)$ for $n \geq 1$. We set $\chi_j := \chi'_j$ and $\chi''_{j-1} := \chi'_{j-1}$.

(iii) $\chi'_j = \zeta_2(n)$ for $n \geq 0$. We set $\chi_j := \chi'_j$ and $\chi''_{j-1} := \chi'_{j-1}$.

(iv) $\chi'_j = \rho_1(n)$ for $n \geq 1$. We set $\chi_j := \zeta_1(n)$ and $\chi''_{j-1} := s_1\chi'_{j-1}$.

(v) $\chi'_j = \rho_2(n)$ for $n \geq 1$. We set $\chi_j := \zeta_2(n-1)$ and $\chi''_{j-1} := s_1\chi'_{j-1}$.

Now we have

$$\begin{aligned} w &= \chi'_m s_3 \chi'_{m-1} s_3 \ldots s_3 \chi'_j s_3 \chi'_{j-1} s_3 \ldots s_3 \chi'_2 s_3 \chi'_1 \\ &= \chi'_m s_3 \chi'_{m-1} s_3 \ldots s_3 \chi_j s_3 \chi''_{j-1} s_3 \ldots s_3 \chi'_2 s_3 \chi'_1, \end{aligned}$$

with $\chi_j$ of the desired form and $\chi''_{j-1} \in E$. The result follows by descending induction on $j$.

□

**Remark 4.4.2.** (i) For a given $w \in W$ the previous proof gives an algorithm to rewrite $w$ in the form $(*)$.

(ii) We adhere to the following convention: in case $f = 1$ we assume that $n \leq 1$ in every occurrence of $\zeta_1(n), \zeta_2(n), \rho_1(n)$, and $\rho_2(n)$. Cases in which $n \geq 2$ is assumed do not apply to the case $f = 1$.

### 4.4.2 Application of the maximal rank type property

To construct real root representations of $Q = Q(f, g, h)$ we will reflect with respect to the following modules $S$: the simple representation $S(3)$ and the real root representations of $Q$

corresponding to certain positive real roots for the subquiver $Q'(f)$. Hence, we will use the following functors

$$\sigma_{e_3} \; : \; \mathfrak{M}_{-e_3}^{-e_3} \to \mathfrak{M}_{e_3}^{e_3}/S(3)$$

and

$$\sigma_\gamma : \mathfrak{M}_{-\gamma}^{-\gamma} \to \mathfrak{M}_\gamma^\gamma / X_\gamma$$

where $\gamma$ denotes a positive real root for the subquiver $Q'(f)$. In order to use these functors, we have to make sure that $\sigma_{e_3}$ and $\sigma_\gamma$ can be applied successively. We remark that the inclusions

$$\begin{aligned} \mathfrak{M}_\gamma^\gamma &\subset \mathfrak{M}_{-e_3}^{-e_3}, \\ \mathfrak{M}_{e_3}^{e_3} &\subset \mathfrak{M}_{-\gamma}^{-\gamma}, \end{aligned}$$

do not hold in general. The following lemmas, however, show that under certain assumptions the functors can be applied successively. We recall a key lemma and a corollary from Section 3.2

**Lemma 4.4.3** ([18, Lemma 4])**.** *Let $k$ be a field and let $Q$ be a quiver. Let $S, T$ be representations of $Q$, where $T$ is simple.*

*(i) If $\operatorname{Ext}^1_{kQ}(S,T) \neq 0$, then $\mathfrak{M}^S \subset \mathfrak{M}^{-T}$.*

*(ii) If $\operatorname{Ext}^1_{kQ}(T,S) \neq 0$, then $\mathfrak{M}_S \subset \mathfrak{M}_{-T}$.*

**Corollary 4.4.4.** *We have*

$$\begin{aligned} \mathfrak{M}_{e_2}^{e_2} &\subset \mathfrak{M}_{-e_3}^{-e_3}, \\ \mathfrak{M}_{e_3}^{e_3} &\subset \mathfrak{M}_{-e_2}^{-e_2}. \end{aligned}$$

The previous corollary shows that $\sigma_{e_2}$ and $\sigma_{e_3}$ can be applied successively. In the following two lemmas we consider the situation when $\gamma$ is a sincere real root for $Q' = Q'(f)$. The maximal rank type property of real root representations ensures that the situation is suitably well-behaved.

**Lemma 4.4.5** ([26, Lemma 3.7])**.** *Let $\gamma$ be a sincere real root for $Q'$. Then we have $\mathfrak{M}_\gamma^\gamma \subset \mathfrak{M}_{-e_3}^{-e_3}$.*

*Proof.* We have $\langle \gamma, e_3\rangle = -g\cdot\gamma[2] < 0$ and $\langle e_3, \gamma\rangle = -h\cdot\gamma[2] < 0$. Thus, Lemma 4.4.3 applies and we deduce $\mathfrak{M}_\gamma^\gamma \subset \mathfrak{M}_{-e_3}^{-e_3}$.

$\square$

**Lemma 4.4.6** ([26, Lemma 3.8]). *Let $\gamma$ be a sincere real root for $Q'$ and let $Y \in \mathfrak{M}_{e_3}^{e_3} - \{S(1)\}$ be a real root representation. Then we have $Y \in \mathfrak{M}_{-\gamma}^{-\gamma}$.*

*Proof.* Let $Y \in \mathfrak{M}_{e_3}^{e_3} - \{S(1)\}$ be a real root representation. Since

$$\mathrm{Ext}^1_{kQ}(Y, S(3)) = 0 = \mathrm{Ext}^1_{kQ}(S(3), Y)$$

we get $\langle \underline{\dim}\, Y, e_3\rangle \geq 0$ and $\langle e_3, \underline{\dim}\, Y\rangle \geq 0$. This implies that

$$\begin{aligned}\langle \underline{\dim}\, Y, e_3\rangle &= -g\cdot \underline{\dim}\, Y[2] + \underline{\dim}\, Y[3] \geq 0,\\ \langle e_3, \underline{\dim}\, Y\rangle &= -h\cdot \underline{\dim}\, Y[2] + \underline{\dim}\, Y[3] \geq 0,\end{aligned}$$

and thus

$$\begin{aligned}\underline{\dim}\, Y[3] &\geq g\cdot \underline{\dim}\, Y[2],\\ \underline{\dim}\, Y[3] &\geq h\cdot \underline{\dim}\, Y[2];\end{aligned}$$

in particular, $\underline{\dim}\, Y[3] \geq \underline{\dim}\, Y[2]$. Since $\underline{\dim}\, Y$ is a positive real root we can apply Theorem 4.2.2, which implies that the maps $Y_{\mu_i}$ $(i = 1, \ldots, g)$ (of the representation $Y$) are injective and the maps $Y_{\nu_i}$ $(i = 1, \ldots, h)$ are surjective.

Now, let $\phi : X_\gamma \to Y$ be a homomorphism. Clearly, $\phi_3 = 0$. The injectivity of the maps $Y_{\mu_i}$ implies that $\phi_2 = 0$. This, however, implies that $\phi_1 = 0$ since otherwise the intersection of the kernels of the maps $Y_{\lambda_j}$ $(j = 1, \ldots, f)$ would be non-zero. This is nonsense since $Y$ is indecomposable and $Y \neq S(1)$. Hence, $\phi = 0$.

Now, let $\psi : Y \to X_\gamma$ be a homomorphism. Clearly, $\psi_3 = 0$. The surjectivity of the maps $Y_{\nu_i}$ implies that $\psi_2 = 0$. This, however, implies that $\psi_1 = 0$ since otherwise the intersection of the kernels of the maps $(X_\gamma)_{\lambda_j}$ $(j = 1, \ldots, f)$ would be non-zero. This is nonsense since $X_\gamma$ is indecomposable and $\gamma$ is sincere for $Q'$. Hence, $\psi = 0$.

This completes the proof.

$\square$

The previous lemma shows the following. Let $X \in \mathfrak{M}_{-e_3}^{-e_3} - \{S(1)\}$ be a real root representation; then we have $\sigma_{e_3}(X) \in \mathfrak{M}_{-\gamma}^{-\gamma}$, where $\gamma$ is a sincere real root for $Q'$.

### 4.4.3 Construction of real root representations for $Q = Q(f, g, h)$

The results of the last section allow us to construct real root representations of $Q$ using universal extension functors.

For $n \geq 1$ we define the functors

$$\sigma_{1,n} := \begin{cases} \sigma_{\rho_1(\frac{n}{2})(e_1)}, & \text{if } n \text{ is even,} \\ \sigma_{\zeta_1(\frac{n-1}{2})(e_2)}, & \text{if } n \text{ is odd,} \end{cases}$$

and for $n \geq 0$ we define the functors

$$\sigma_{2,n} := \begin{cases} \sigma_{\rho_2(\frac{n}{2})(e_2)}, & \text{if } n \text{ is even,} \\ \sigma_{\zeta_2(\frac{n-1}{2})(e_1)}, & \text{if } n \text{ is odd.} \end{cases}$$

**Remark 4.4.7.** For $n \geq 1$ we clearly have

(i) $\rho_1(n)(e_3) = \zeta_1(n)(e_3)$,

(ii) $\rho_2(n)(e_3) = \zeta_2(n-1)(e_3)$.

**Lemma 4.4.8** ([26, Lemma 3.10])**.** *Let $\alpha$ be a positive non-simple real root of $Q$ the following form:*

*(i) $\alpha = \chi(e_j)$ with $j \in \{1, 2\}$ and $\chi \in E$, or*

*(ii)* $\alpha = \chi(e_3)$ *with* $\chi \in E$.

*Then the unique indecomposable representation of dimension vector* $\alpha$ *has the following properties.*

*(i)* $X_\alpha$ *is an indecomposable representation of the subquiver* $Q'(f)$*, and hence can be constructed using BGP reflection functors. Moreover,* $\operatorname{End}_{kQ} X_\alpha = k$ *and* $X_\alpha \in \mathfrak{M}_{-e_3}^{-e_3}$.

*(ii)* $X_\alpha$ *can be constructed using the functors* $\sigma_{i,n}$ $(i = 1, 2)$ *and* $X_\alpha \in \mathfrak{M}_{-e_3}^{-e_3}$.

*Proof.* (i) The statement is clear.

(ii) If $\alpha = \zeta_i(n)(e_3)$ $(i = 1, 2)$ then $X_\alpha = \sigma_{i,n}S(3)$ by Kac's Theorem 2.2.9, and $X_\alpha \in \mathfrak{M}_{-e_3}^{-e_3}$ by Lemma 4.4.5 or Corollary 4.4.4 in case $\alpha = \zeta_2(0)$. If $\alpha = \rho_i(n)(e_3)$ $(i = 1, 2)$ we use the previous Remark 4.4.7 to reduce to the case we have just considered.

□

We are now able to construct all real root representations of $Q = Q(f, g, h)$ with sincere dimension vectors.

**Theorem 4.4.9** ([26, Theorem 3.11])**.** *Let* $\alpha$ *be a sincere real root for* $Q$*. Then* $\alpha$ *is of the form*

*(i)* $\alpha = \zeta_i(n)(e_3)$ *with* $i \in \{1, 2\}$ *and* $n \geq 1$*, or*

*(ii)* $\alpha = w(e_j)$ *with* $j \in \{1, 2, 3\}$ *and* $w = \chi_m s_3 \chi_{m-1} s_3 \chi_{m-2} s_3 \dots s_3 \chi_2 s_3 \chi_1$ *of the form* $(*)$ *(see Section 4.4.1) with* $\chi_1(e_j) \neq e_1$.

*The corresponding unique indecomposable representation of dimension vector* $\alpha$ *can be constructed as follows:*

*(i)* $X_{\zeta_i(n)(e_3)} = \sigma_{i,n}S(3)$,

*(ii)* $X_\alpha = \sigma_{i_m,n_m}\sigma_{e_3}\sigma_{i_{m-1},n_{m-1}}\dots\sigma_{i_2,n_2}\sigma_{e_3}X_{\chi_1(e_j)}$, *where* $X_{\chi_1(e_j)}$ *denotes the unique indecomposable of dimension vector* $\chi_1(e_j)$*: constructed in Lemma 4.4.8; and the indicies* $i_q, n_q$ *are given by* $\chi_q = \zeta_{i_q}(n_q)$ *for* $2 \leq q \leq m$.

*In particular, formula (3.1) can be used to compute* $\dim \operatorname{End}_{kQ} X_\alpha$.

*Proof.* (i) Follows from Lemma 4.4.8.

(ii) It follows from Lemma 4.4.8 that $X_{\chi_1(e_j)} \in \mathfrak{M}^{-e_3}_{-e_3}$, and hence $\sigma_{e_3}$ can be applied. Moreover, by Corollary 4.4.4, Lemma 4.4.5, and Lemma 4.4.6 we have

$$X_\beta \in \mathfrak{M}^{e_3}_{e_3} - \{S(1)\},\ \beta \text{ real root } \implies X_\beta \in \mathfrak{M}^{-\gamma}_{-\gamma},$$
$$\mathfrak{M}^{\gamma}_{\gamma} \subset \mathfrak{M}^{-e_3}_{-e_3},$$

where $\gamma$ is a positive real root for the subquiver $Q'(f)$ not equal to $e_1$. Hence, the functors can be applied successively and the assertion follows from Lemma 3.1.10. This completes the proof.

□

The previous theorem and Lemma 3.1.8 give the following result.

**Proposition 4.4.10** ([26, Proposition 3.17])**.** *Let* $\alpha$ *be a positive real root for* $Q = Q(f,g,h)$ $(f,g,h \geq 1)$. *Then the representation* $X_\alpha$ *is a tree representation.*

*Proof.* Representations of the subquiver $Q' = Q'(f)$ $(f \geq 1)$ are exceptional representations, and hence are tree representations by Theorem 2.2.16.

Now, let $X$ be a representation of $Q$ with $\underline{\dim}\, X[3] \neq 0$. Then, by Theorem 4.4.9 (or the results in Section 3.2 if $X$ is not sincere), $X$ can be constructed by using universal extension functors starting from a simple representation or a real root representation of the subquiver $Q'$, which is a tree representation.

By Lemma 3.1.8 the image of a tree representation under the functor $\sigma_S$ is again a tree representation. This proves the claim.

$\square$

### 4.4.4 Further observations and comments

In Theorem 4.4.9 we constructed real root representations of the quiver $Q(f,g,h)$ $(f,g,h \geq 1)$. The key result for the construction process described in the last section is Lemma 4.4.6 which relied on the fact that that for an indecomposable representation $Y$ of $Q(f,g,h)$ with $Y \neq S(1)$ we have that $\bigcap_{j=1}^{f} \ker Y_{\lambda_i} = 0$, since otherwise we could split of copies of $S(1)$. This is basically a condition on the representation $Y$ restricted to the generalised Kronecker subquiver. If we restrict an indecomposable representation of the following quiver

$$\begin{tikzcd} 1 \arrow[r, bend left] \arrow[r, bend left=60] & 2 \arrow[l, bend left] \arrow[l, bend left=60] \arrow[r, shift left] \arrow[r, shift right] & 3 \arrow[r, bend left] \arrow[r, bend left=60] & 4 \arrow[l, bend left] \arrow[l, bend left=60] \end{tikzcd} .$$

to the generalized Kronecker subquiver, we do not have this behavior in general. This shows that our method does not work for this quiver.

However, the Lemmas 4.4.5 and 4.4.6 still hold true for both generalised Kronecker subquivers of the following class of quivers

$$\begin{tikzcd} 1 \arrow[r, shift left] \arrow[r, shift right] & 2 \arrow[r, bend left] \arrow[r, bend left=60] & 3 \arrow[l, bend left] \arrow[l, bend left=60] \arrow[r, bend left] \arrow[r, bend left=60] & 4 \arrow[l, bend left] \arrow[l, bend left=60] & \cdots & m \arrow[r, bend left] \arrow[r, bend left=60] & m+1 \arrow[l, bend left] \arrow[l, bend left=60] & m+2 \arrow[l, shift left] \arrow[l, shift right] \end{tikzcd} ,$$

with $m \geq 2$. This fact combined with the proof of Theorem 3.2.5 shows that real root representations of the above class of quivers can be constructed with universal extension functors.

In the same way as above we can construct real root representations of the following class of quivers


with $m \geq 2$.

# Chapter 5

# Two examples answering Question $(\dagger\dagger)$ negatively

In this chapter we turn our attention back to Question $(\dagger\dagger)$, stated in the introduction. The results described in this Chapter form the content of the preprint [25].

Our research started with the observation that real root representations of quivers may be constructed with universal extension functors starting from real Schur representations. This observation led us to Question $(\dagger\dagger)$ stated in the Introduction. We saw in Section 3.2 and Section 4.4 that Question $(\dagger\dagger)$ can be answered affirmatively for certain classes of quivers.

Using the notation from Chapter 3, Question $(\dagger\dagger)$ can be stated as follows.

**Question** $(\dagger\dagger)$**.** *Let $k$ be a field. Let $Q$ be a quiver and let $\alpha$ be a positive non-Schur real root for $Q$. Does there exist a sequence of real Schur roots $\beta_1, \ldots, \beta_n$ $(n \geq 1)$ such that*

$$X_\alpha = \sigma_{\beta_n} \cdot \ldots \cdot \sigma_{\beta_2}(X_{\beta_1}) \quad ?$$

**Remark 5.0.11.** For a real Schur root $\alpha$ we can take the trivial sequence $\beta_1 = \alpha$. Moreover, the sequence of real Schur roots $\beta_1, \ldots, \beta_n$ is not necessarily unique; examples are given in Appendix A.

**Remark 5.0.12.** We remark that in the case that $X_\alpha$ can be constructed in the above way we have $\beta_i < \alpha\,(i = 1, \dots, n)$ and we get the following filtration of $X_\alpha$:

$$X_\alpha = \left\{ \begin{array}{c} \bigoplus_{t=1}^{t_n} X_{\beta_n} \\ \hline \vdots \\ \hline \bigoplus_{t=1}^{t_2} X_{\beta_2} \\ \hline X_{\beta_1} \\ \hline \bigoplus_{b=1}^{b_2} X_{\beta_2} \\ \hline \vdots \\ \hline \bigoplus_{b=1}^{b_n} X_{\beta_n} \end{array} \right. .$$

Moreover, it is clear from the discussion in Chapter 3 that

- $\dim \operatorname{End}_{kQ} X_\alpha$ can be computed using formula (3.1), and
- $X_\alpha$ is a tree representation.

**Remark 5.0.13.** It is not difficult to determine all potential real Schur roots $\beta_n$ which may be used for a reflection. Assume that $X_\alpha = \sigma_{X_{\beta_n}} \cdot \ldots \cdot \sigma_{X_{\beta_2}}(X_{\beta_1})$. Then we have $X_\alpha \in \mathfrak{M}_{X_{\beta_n}}^{X_{\beta_n}}$, and hence

$$\operatorname{Ext}^1_{kQ}(X_\alpha, X_{\beta_n}) = 0 = \operatorname{Ext}^1_{kQ}(X_{\beta_n}, X_\alpha),$$

and in particular $\langle \alpha, \beta_n \rangle \geq 0$ and $\langle \beta_n, \alpha \rangle \geq 0$. Moreover, it follows from Lemma 3.1.3 that $\beta_n < \alpha$.

Hence, the real roots $\beta$ with the following properties:

(i) $\beta < \alpha$,

(ii) $\langle \alpha, \beta \rangle \geq 0$ and $\langle \beta, \alpha \rangle \geq 0$, are potential candidates for $\beta_n$. Using the arguments given in [23, Section 6], it is easy to determine the real roots $\beta$ which satisfy (i) and (ii): both conditions imply that $s_\alpha(\beta) < 0$, and hence if $s_\alpha = s_{i_1} \dots s_{i_n}$ we get $s_\alpha(\beta) = s_{i_1} \dots s_{i_n}(\beta) < 0$ if and only if $\beta = s_{i_n} \dots s_{i_{m+1}}(e_{i_m})$ for some $m$. Thus, once we have written $s_\alpha$ as a product of the generators $s_i$ it is straightforward to find the real roots $\beta$ satisfying (i) and (ii). A decomposition of $s_\alpha$ into a product of the generators $s_i$ can be achieved as follows: if $s_i(\alpha) = \alpha' < \alpha$ then $s_\alpha = s_i s_{\alpha'} s_i$; this gives an algorithm to find a shortest expression of $s_\alpha$ in terms of the $s_i$.

We saw that being able to construct a real root representation by using universal extension functors in the above way reveals some interesting information about the representation.

The general answer to Question (††), however, is negative. We discuss two explicit examples of non-Schur real root representations which cannot be constructed with the universal extension functors $\sigma_S$, as in Question (††).

## 5.1 Example one

We consider the following quiver

$$Q:\ \begin{tikzcd} 1 \arrow[r, "a"'] \arrow[rrr, bend left, "b"] & 2 \arrow[r, "c"'] \arrow[rr, bend right, "d"'] & 3 \arrow[r, "e"'] & 4 \end{tikzcd}$$

and the real root $\alpha = (1, 8, 6, 4) = s_2 s_3 s_4 s_2 s_3 s_4(e_1)$.

For the convenience of the reader we give an explicit description of the representation $X_\alpha$ and the endomorphism ring $\operatorname{End}_{kQ} X_\alpha$.

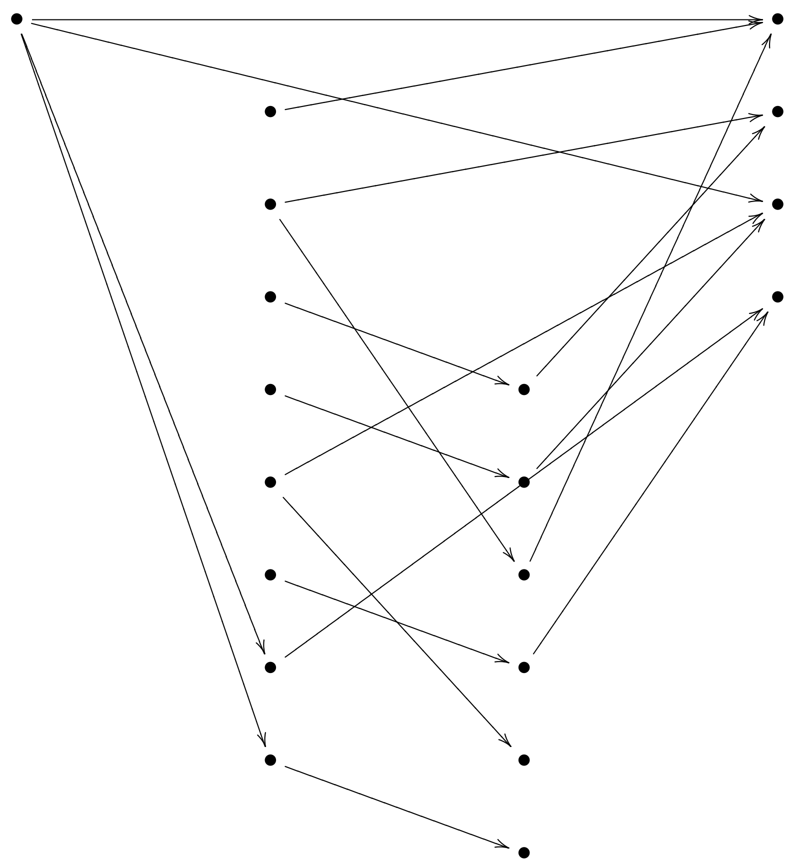
We start by considering the representation $X_\alpha$ over the field $k = \mathbb{C}$. In this case, one can use the results from Section 2.3 (Proposition 2.3.5) to construct the representation $X_\alpha$; we get

$$X_\alpha : \begin{tikzcd} \mathbb{C} \arrow[r, "X_a"'] \arrow[rrr, bend left, "X_b"] & \mathbb{C}^8 \arrow[r, "X_c"'] \arrow[rr, bend right, "X_d"'] & \mathbb{C}^6 \arrow[r, "X_e"'] & \mathbb{C}^4 \end{tikzcd}$$

with

$$
\begin{aligned}
X_a &= \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^t, \\
X_b &= \begin{bmatrix} 0 & 1 & 0 & 1 \end{bmatrix}^t, \\
X_c &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \\
X_d &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \\
X_e &= \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.
\end{aligned}
$$

With respect to the basis $\mathfrak{B}$ consisting of the standard bases at each vertex we get the following coefficient quiver $\Gamma(X_\alpha, \mathfrak{B})$.



In particular, we see that $X_\alpha$ is a tree representation. This can also be seen as follows. Since $X_\alpha$ is indecomposable, the coefficient quiver is connected by Lemma 2.2.14. Moreover, it has $1+8+6+4=19$ vertices and $18$ arrows (number of non-zero entries in the matrices of $X_\alpha$), and hence must be a tree. From this explicit description of the representation it is not difficult to compute the endomorphism algebra of $X_\alpha$; we get

$$\operatorname{End}_{\mathbb{C}Q} X_\alpha = \left\{ (\phi_1(\underline{x}), \phi_2(\underline{x}), \phi_3(\underline{x}), \phi_4(\underline{x})) : \underline{x} \in \mathbb{C}^9 \right\},$$

with

$$\begin{aligned}
\phi_1(\underline{x}) &= \begin{bmatrix} x_1 \end{bmatrix}, \\
\phi_2(\underline{x}) &= \begin{bmatrix} x_1 & 0 & x_2 & x_3 & x_4 & x_5 & x_6 & 0 \\ 0 & x_1 & 0 & x_8 & 0 & 0 & x_9 & -x_8 \\ 0 & 0 & x_1 & 0 & x_8 & x_9 & -x_8 & 0 \\ 0 & 0 & 0 & x_1 & 0 & 0 & x_7 & 0 \\ 0 & 0 & 0 & 0 & x_1 & x_7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1 \end{bmatrix}, \\
\phi_3(\underline{x}) &= \begin{bmatrix} x_1 & x_3 & x_2 & x_6 & x_4 & x_5 \\ 0 & x_1 & 0 & x_7 & 0 & 0 \\ 0 & 0 & x_1 & -x_8 & x_8 & x_9 \\ 0 & 0 & 0 & x_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_1 & x_7 \\ 0 & 0 & 0 & 0 & 0 & x_1 \end{bmatrix}, \\
\phi_4(\underline{x}) &= \begin{bmatrix} x_1 & x_8 & x_9 & -x_8 \\ 0 & x_1 & x_7 & 0 \\ 0 & 0 & x_1 & 0 \\ 0 & 0 & 0 & x_1 \end{bmatrix},
\end{aligned}$$

and, in particular $\dim \operatorname{End}_{\mathbb{C}Q}(X_\alpha) = 9$ so that $X_\alpha$ is not a real Schur representation.

The representation $X_\alpha$, as given above, is defined over every field $k$. Moreover, the endomorphism algebra $\operatorname{End}_{kQ} X_\alpha$ is local over every field $k$. Namely, the description of the endomorphism ring only involves triangular matrices with $k$ on the diagonal which implies that $\operatorname{End}_{kQ} X_\alpha$ is local, see for instance [1, Lemma 4.6]. Hence, the representation $X_\alpha$ is the unique indecomposable representation of dimension vector $\alpha$ over every field $k$.

**Theorem 5.1.1.** *There exists no real Schur root $\beta < \alpha$ with the property*

$$X_\alpha \in \mathfrak{M}_{X_\beta}^{X_\beta}.$$

If we had a sequence of real Schur roots $\beta_1, \dots, \beta_n$ $(n \geq 2)$ such that

$$X_\alpha = \sigma_{X_{\beta_n}} \dots \sigma_{X_{\beta_2}}(X_{\beta_1})$$

then $X_\alpha \in \mathfrak{M}_{X_\beta}^{X_\beta}$. Thus, once we have established the claim it is clear that $X_\alpha$ provides an example which answers Question (††) negatively.

We use the rest of this section to prove the above theorem.

*Proof of Theorem 5.1.1.* We start by determining all potential candidates for $\beta$, using Remark 5.0.13. A shortest expression of $s_\alpha$ with $\alpha = (1, 8, 6, 4) = s_2 s_3 s_4 s_2 s_3 s_4(e_1)$ is given by $s_\alpha = s_2 s_3 s_4 s_2 s_3 s_4 s_1 s_4 s_3 s_2 s_4 s_3 s_2$. Now, applying the algorithm given in Remark 5.0.13 to the real root $\alpha$ it is easy to see that the only possibilities are $\beta_1 = (0, 1, 1, 0)$ and $\beta_2 = (0, 2, 1, 1)$. Note, that $\beta_1$ and $\beta_2$ are real Schur roots, and hence are indeed potential candidates for a reflection. However, we establish in the following that $\beta_1$ and $\beta_2$ do not have the desired property stated in the theorem.

(I) $X_\alpha \notin \mathfrak{M}_{X_{\beta_1}}^{X_{\beta_1}}$. Assume to the contrary that $X_\alpha \in \mathfrak{M}_{X_{\beta_1}}^{X_{\beta_1}}$. Then $\sigma_{X_{\beta_1}}^{-1}(X_\alpha) \in \mathfrak{M}_{-X_{\beta_1}}^{-X_{\beta_1}}$; that is,

$$\mathrm{Hom}_{kQ}(\sigma_{X_{\beta_1}}^{-1}(X_\alpha), X_{\beta_1}) = 0 = \mathrm{Hom}_{kQ}(X_{\beta_1}, \sigma_{X_{\beta_1}}^{-1}(X_\alpha)).$$

Using formula (3.1), we get $\gamma_1 := \underline{\dim}\, \sigma_{X_{\beta_1}}^{-1}(X_\alpha) = (1, 3, 1, 4)$.

The following diagram, however, shows that $\mathrm{Hom}_{kQ}(X_{\gamma_1}, X_{\beta_1}) \neq 0$. The representation $X_{\gamma_1}$ can be constructed using the results from Section 2.3 (Proposition 2.3.5) together with the same reasoning as for $X_\alpha$ to pass to any field $k$.

$$\begin{tikzcd}[row sep=large, column sep=huge]
X_{\gamma_1} & X_{\beta_1} \\
k \arrow[r] \arrow[d, "{\begin{bmatrix}1\\0\\0\end{bmatrix}}"] \arrow[ddd, bend right=60, swap, "{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}}"] & 0 \arrow[d] \arrow[ddd, bend left=40] \\
k^3 \arrow[r, "{\begin{bmatrix}0&1&0\end{bmatrix}}"] \arrow[d, "{\begin{bmatrix}0&1&0\end{bmatrix}}"] \arrow[dd, bend right=30, swap, "{\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\\1&0&0\end{bmatrix}}"] & k \arrow[d, swap, "{[1]}"] \arrow[dd, bend left=30] \\
k \arrow[r, "{[1]}"] \arrow[d, "{\begin{bmatrix}1\\1\\0\\0\end{bmatrix}}"] & k \arrow[d] \\
k^4 \arrow[r] & 0
\end{tikzcd}$$

This is a contradiction, and hence $X_\alpha \notin \mathfrak{M}_{X_{\beta_1}}^{X_{\beta_1}}$.

(II) $X_\alpha \notin \mathfrak{M}_{X_{\beta_2}}^{X_{\beta_2}}$. Assume to the contrary that $X_\alpha \in \mathfrak{M}_{X_{\beta_2}}^{X_{\beta_2}}$. Then $\sigma_{X_{\beta_2}}^{-1}(X_\alpha) \in \mathfrak{M}_{-X_{\beta_2}}^{-X_{\beta_2}}$; that is,

$$\mathrm{Hom}_{kQ}(\sigma_{X_{\beta_2}}^{-1}(X_\alpha), X_{\beta_2}) = 0 = \mathrm{Hom}_{kQ}(X_{\beta_2}, \sigma_{X_{\beta_2}}^{-1}(X_\alpha)).$$

Using formula (3.1), we get $\gamma_2 := \underline{\dim}\, \sigma_{X_{\beta_2}}^{-1}(X_\alpha) = (1,2,3,1)$.

The following diagram, however, shows that $\mathrm{Hom}_{kQ}(X_{\beta_2}, X_{\gamma_2}) \neq 0$.

$$
\begin{tikzcd}[row sep=large, column sep=huge]
X_{\gamma_2} & X_{\beta_2} \\
k \arrow[d, "\begin{bmatrix}1\\0\end{bmatrix}"] \arrow[ddd, bend right=60, "\begin{bmatrix}1\end{bmatrix}"'] & 0 \arrow[l] \arrow[d] \arrow[ddd, bend left=60] \\
k^2 \arrow[d, "\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}"] \arrow[dd, bend right=30, "\begin{bmatrix}0&1\end{bmatrix}"'] & k^2 \arrow[l, "\begin{bmatrix}0&1\\0&0\end{bmatrix}"'] \arrow[d, "\begin{bmatrix}0&1\end{bmatrix}"'] \arrow[dd, bend left=30, "\begin{bmatrix}1&0\end{bmatrix}"] \\
k^3 \arrow[d, "\begin{bmatrix}0&0&1\end{bmatrix}"] & k \arrow[l, "\begin{bmatrix}1\\0\\0\end{bmatrix}"'] \arrow[d, "\begin{bmatrix}1\end{bmatrix}"'] \\
k & k \arrow[l, "\begin{bmatrix}0\end{bmatrix}"']
\end{tikzcd}
$$

This is a contradiction, and hence $X_\alpha \notin \mathfrak{M}_{X_{\beta_2}}^{X_{\beta_2}}$.

This completes the proof of the theorem and we see indeed that the representation $X_\alpha$ answers Question (††) negatively in general.

□

The following was pointed out to the author by C. Ringel. Let $\tilde{Q}$ denote the universal cover of $Q$ (for the definition of “universal cover” we refer the reader to [10, Section 2]). Consider the following indecomposable representation $\tilde{X}$ (given by its dimension vector) of $\tilde{Q}$.

$$\begin{tikzcd}
0 \arrow[r] & 2 \arrow[r] \arrow[dr] & 1 \arrow[r] & 0 & 0 \arrow[l, bend left] \arrow[r] & 0 \arrow[r] \arrow[dr] & 0 \arrow[r] & 0 \\
0 \arrow[r] & 2 \arrow[r] \arrow[dr] & 2 \arrow[r] & 2 & 1 \arrow[l, bend left] \arrow[r] & 2 \arrow[r] \arrow[dr] & 1 \arrow[r] & 0 \\
0 \arrow[r] & 1 \arrow[r] & 1 \arrow[r] & 1 & 0 \arrow[l, bend left] \arrow[r] & 1 \arrow[r] & 1 \arrow[r] & 1
\end{tikzcd}$$

The push-down of $\tilde{X}$ is $X_\alpha$ (see [4, Section 3.2] and [10, Section 3.5]). If we identify vertices connected by arrows which have isomorphisms attached to them, we obtain the following real Schur representation of $\tilde{D}_6$.

$$\begin{tikzcd}
1 & & & & 1 \\
& 2 \arrow[ul] \arrow[dl] & 1 \arrow[l] \arrow[r] & 2 \arrow[ur] \arrow[dr] & \\
1 & & & & 1
\end{tikzcd}$$

This shows that the representation $X_\alpha$ can be constructed by employing the universal cover $\tilde{Q}$ of $Q$.

The universal cover of a quiver is always a tree and it remains to be seen whether Question (††) can be answered affirmatively for quivers which are trees. In the next section we show that the answer is also negative in this case.

## 5.2 Example two

We consider the following quiver $Q$

$$Q: \quad \begin{tikzcd} 1 \arrow[dr, "a"'] & 2 \arrow[d, "b"] & 3 \arrow[dl, "c"] \\ & 4 \arrow[d, "d"] & \\ & 5 \arrow[dl, "e"'] \arrow[d, "f"] \arrow[dr, "g"] & \\ 6 & 7 & 8 \end{tikzcd}$$

and the real root

$$\alpha = (1, 1, 1, 8, 12, 2, 7, 7) = s_8 s_7 s_5 s_4 s_8 s_7 s_5 s_8 s_7 s_5 s_6 s_4 s_5 s_4 s_1 s_2 s_3 (e_4).$$

We apply the same method as in Section 5.1 to construct the representation $X_\alpha$ over an arbitrary field $k$. The representation $X_\alpha$ is given by

$$X_\alpha: \quad \begin{tikzcd} k \arrow[dr, "X_a"'] & k \arrow[d, "X_b"] & k \arrow[dl, "X_c"] \\ & k^8 \arrow[d, "X_d"] & \\ & k^{12} \arrow[dl, "X_e"'] \arrow[d, "X_f"] \arrow[dr, "X_g"] & \\ k^2 & k^7 & k^7 \end{tikzcd}$$

with

$$
\begin{aligned}
X_a &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}^t, \\
X_b &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}^t, \\
X_c &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}^t, \\
X_d &= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}, \\
X_e &= \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0
\end{bmatrix},
\end{aligned}
$$

$$X_f = \begin{bmatrix} 0&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&0&0&1 \end{bmatrix},$$

$$X_g = \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&1&0&0&0&0&0 \\ 0&0&0&0&0&1&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&1&0&0&1 \end{bmatrix}.$$

As in the last section, we see that $X_\alpha$ is a tree representation. The endomorphism algebra of $X_\alpha$ is given as follows

$$\mathrm{End}_{kQ}(X_\alpha) = \left\{(\phi_1(\underline{x}), \phi_2(\underline{x}), \phi_3(\underline{x}), \phi_4(\underline{x}), \phi_5(\underline{x}), \phi_6(\underline{x}), \phi_7(\underline{x}), \phi_8(\underline{x})) : \underline{x} \in k^9\right\},$$

with

$$\phi_1(\underline{x}) = \begin{bmatrix} x_1 \end{bmatrix}, \quad \phi_2(\underline{x}) = \begin{bmatrix} x_1 \end{bmatrix}, \quad \phi_3(\underline{x}) = \begin{bmatrix} x_1 \end{bmatrix},$$

$$\phi_4(\underline{x}) = \begin{bmatrix} x_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_1 & 0 & 0 & 0 & 0 \\ x_2+x_6 & x_3+x_7 & x_8 & x_9 & x_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x_1 & 0 \\ x_8-x_4-x_6 & x_9-x_5-x_7 & -x_4 & -x_5 & 0 & 0 & 0 & x_1 \end{bmatrix},$$

$$\phi_5(\underline{x}) = \begin{bmatrix} x_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1 & 0 & 0 & 0 & 0 \\ x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1 & 0 \\ 0 & 0 & -x_4 & -x_5 & x_8-x_4-x_6 & x_9-x_5-x_7 & -x_4 & -x_5 & 0 & 0 & 0 & x_1 \end{bmatrix},$$

$$\phi_6(\underline{x}) = \begin{bmatrix} x_1 & 0 \\ 0 & x_1 \end{bmatrix},$$

$$\phi_7(\underline{x}) = \begin{bmatrix} x_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_1 & 0 \\ x_8-x_4-x_6 & x_9-x_5-x_7 & -x_4 & -x_5 & 0 & 0 & x_1 \end{bmatrix},$$

$$\phi_8(\underline{x}) = \begin{bmatrix} x_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_1 & 0 \\ x_2 & x_3 & x_8-x_4 & x_9-x_5 & 0 & 0 & x_1 \end{bmatrix},$$

and, in particular $\dim \operatorname{End}_{kQ}(X_\alpha) = 9$ so that $X_\alpha$ is not a real Schur representation.

**Theorem 5.2.1** ([25, Theorem 2.1])**.** *There exists no real Schur root $\beta < \alpha$ with the property*

$$X_\alpha \in \mathfrak{M}_{X_\beta}^{X_\beta}.$$

*Proof.* We proceed as in the proof of Theorem 5.1.1. We start by determining all potential candidates for $\beta$, using Remark 5.0.13. The only possibilities are

$$\begin{aligned} \beta_1 &= (0,0,0,1,2,0,1,1), \\ \beta_2 &= (0,1,1,4,7,1,4,4), \\ \beta_3 &= (1,0,1,4,7,1,4,4), \quad \text{and} \\ \beta_4 &= (1,1,0,4,7,1,4,4). \end{aligned}$$

We see that $\langle \beta_i, \alpha \rangle = 0 = \langle \alpha, \beta_i \rangle$ for $i = 2, 3, 4$, and hence the only reflection candidate is $\beta_1$, which is a real Schur root.

As in the proof of Theorem 5.1.1, we show that $\beta_1$ does not have the desired property; that is, $X_\alpha \notin \mathfrak{M}_{X_{\beta_1}}^{X_{\beta_1}}$. Assume to the contrary that $X_\alpha \in \mathfrak{M}_{X_{\beta_1}}^{X_{\beta_1}}$. Then $\sigma_{X_{\beta_1}}^{-1}(X_\alpha) \in \mathfrak{M}_{-X_{\beta_1}}^{-X_{\beta_1}}$; that is

$$\mathrm{Hom}_{kQ}(\sigma_{X_{\beta_1}}^{-1}(X_\alpha), X_{\beta_1}) = 0 = \mathrm{Hom}_{kQ}(X_{\beta_1}, \sigma_{X_{\beta_1}}^{-1}(X_\alpha)).$$

Using formula (3.1), we get $\gamma_1 := \underline{\dim}\, \sigma_{X_{\beta_1}}^{-1}(X_\alpha) = (1, 1, 1, 3, 2, 2, 2, 2)$. The following diagram, however, shows that $\mathrm{Hom}_{kQ}(X_{\beta_1}, X_{\gamma_1}) \neq 0$.

$$\begin{tikzcd}
 & X_{\beta_1} & & & & X_{\gamma_1} & \\
0 \arrow[dr] & 0 \arrow[d] & 0 \arrow[dl] & & k \arrow[dr, "\begin{bmatrix}1\\1\\1\end{bmatrix}"'] & k \arrow[d, "\begin{bmatrix}0\\1\\0\end{bmatrix}"] & k \arrow[dl, "\begin{bmatrix}0\\0\\1\end{bmatrix}"] \\
 & k \arrow[d, "\begin{bmatrix}1\\1\end{bmatrix}"] \arrow[rrrr, "\begin{bmatrix}1\\0\\0\end{bmatrix}"] & & & & k^3 \arrow[d, "\begin{bmatrix}0&1&0\\0&0&1\end{bmatrix}"] & \\
 & k^2 \arrow[dl] \arrow[d, "{[1\ 0]}"] \arrow[dr, "{[0\ 1]}"] \arrow[rrrr, "\begin{bmatrix}0&0\\0&0\end{bmatrix}"] & & & & k^2 \arrow[dl, "\mathrm{id}"'] \arrow[d, "\mathrm{id}"] \arrow[dr, "\mathrm{id}"] & \\
0 & k \arrow[rrrr, bend right, "\begin{bmatrix}0&0\end{bmatrix}"'] & k \arrow[rrrr, bend right, "\begin{bmatrix}0&0\end{bmatrix}"'] & & k^2 & k^2 & k^2
\end{tikzcd}$$

This is a contradiction, and hence $X_\alpha \notin \mathfrak{M}_{X_{\beta_1}}^{X_{\beta_1}}$ which completes the proof of the theorem.

$\square$

# Chapter 6

# Conclusion

In this thesis we have studied real root representations of quivers, motivated by the following question.

**Question** (†). *How can one "construct" real root representations and what are their "properties" ?*

Our main observation was that this question may be approached using universal extension functors, as described in Chapter 3. Numerical experiments with real root representations, as indicated in Appendix A, suggest that real root representations can be constructed using universal extension functors. This approach to Question (†) was independently made by Ringel, who conjectured that this process of constructing real root representations works for arbitrary quivers. The advantage of this construction process is that it gives an insight into the properties of real root representations.

Our approach to Question (†) was as follows.

**Question** (††). *Let $Q$ be a quiver and let $k$ be a field. Let $\alpha$ be a positive non-Schur real root. Does there exist a sequence $\beta_n, \ldots, \beta_1$ $(n \geq 2)$ of real Schur roots such that*

$$X_\alpha = \sigma_{\beta_n} \cdot \ldots \cdot \sigma_{\beta_2}(X_{\beta_1})?$$

In Chapter 4 we introduced the notion of "maximal rank type". One of our main results was that real root representations have this property. We used this property to construct all real root representations of the quiver

$$Q(f,g,h)\colon\ \begin{tikzcd} 1 \arrow[r, shift left=2, "\lambda_1"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2, "\lambda_f"'] & 2 \arrow[r, bend left=60, "\mu_1"] \arrow[r, bend left=30, "\mu_g"'] & 3 \arrow[l, bend left=30, "\nu_1"'] \arrow[l, bend left=60, "\nu_h"] \end{tikzcd},$$

with $f, g, h \geq 1$, proving in particular that all real root representations of $Q(f, g, h)$ are tree representations. Moreover, the dimension of the endomorphism ring can be computed easily.

The pivotal result of this thesis however, is that despite the results for the class of quivers $Q(f, g, h)$ and the empirical data indicated in Appendix A the answer to Question $(\dagger\dagger)$ is negative in general.

Even though the answer to Question $(\dagger\dagger)$ is negative in general, the approach to Question $(\dagger)$ using universal extension functors should not be abandoned altogether. Our numerical data suggests that this approach does indeed work for a large class of quivers. We feel confident to conjecture that all real root representations for the quivers in Appendix A can be constructed in this way. One example considered in Appendix A is the 6-subspace quiver. We conjecture moreover, that this construction process works indeed for all $n$-subspace quivers.

However, the main problem remains the following: further tools have to be developed to decided whether a given representation $X$ is in the subcategory $\mathfrak{M}^{-S}_{-S}$, that is that there are no homomorphisms between $X$ and $S$ in either way; so the functor $\sigma_S$ can be applied. The maximal rank type property is only a first step, and generalisations of this property need to be developed.

The maximal rank type property only involves certain collections and independent combinations of arrows and of the quiver. One may study more general situations. For instance, one could restrict real root representations to subquivers and study these. This may yield results which allow one to determine whether a given real root representation is in $\mathfrak{M}^{-S}_{-S}$ for more complicated representations $S$.

# Appendix

# Appendix A

# More quivers

In this appendix we consider further quivers and present an indication of the numerical data which led to posing Question (††). The results of this chapter give an indication that the functors $\sigma_S$ can be used in many cases to construct real root representations. This suggests that the two real root representations and the quivers discussed in Chapter 5, which answered Question (††) negatively in general, are singular examples.

Throughout the appendix we fix the field $k = \mathbb{C}$. The author has used the results of Section 2.3, namely the construction of real root representations over algebraically closed fields of characteristic zero using deformed preprojective algebras, to write a `matlab` package `construct_real_root_rep(`$E$`,`$\alpha$`)` (with input: $E$ - Euler matrix of the quiver $Q$; $\alpha$ - positive real root for $Q$) to construct real root representations (over $k = \mathbb{C}$) for a given quiver $Q$. This package is based on Proposition 2.3.5 and basically relies on implementing the functors $E_i$, as described in Theorem 2.3.1; this only involves matrix operations.

The author has also written the following `matlab` programs. A program `homomorphisms(`$E, X, Y$`)` (with input: $E$ - Euler matrix of the quiver $Q$; $X, Y$ - representations of the quiver $Q$ in matrix form), to compute the dimensions of homomorphism spaces between representations of a given quiver $Q$. This program is based on Lemma 2.2.8 and basically relies on determining the rank of the map $\delta_{XY}$, described in

Section 2.2; this only involves matrix operations. Moreover, the author has written a program `decomposition(`$E,\alpha$`)` (with input: $E$ - Euler matrix of the quiver $Q$, $\alpha$ - positive real root for $Q$) which calculates for a given real root $\alpha$ all possible decompositions

$$\alpha = -(\langle\beta,\gamma\rangle + \langle\gamma,\beta\rangle)\cdot\beta + \gamma$$

with $\beta$ a real Schur root, $\gamma$ a real root, and $X_\gamma \in \mathfrak{M}_{-\beta}^{-\beta}$. These are exactly those decompositions of $\alpha$ such that $X_\alpha = \sigma_\beta(X_\gamma) \in \mathfrak{M}_\beta^\beta$. The decompositions are obtained as follows. Based on Remark 5.0.13 we compute for a given positive real root $\alpha$ all positive real roots $\beta$ such that $\beta \leq \alpha$, $\langle\alpha,\beta\rangle \geq 0$ and $\langle\beta,\alpha\rangle \geq 0$. This is a straightforward algorithm. Then, using `construct_real_root_rep(`$E,\beta$`)` and `homomorphisms(`$E,X_\beta,X_\beta$`)`, we single out the real roots which are real Schur roots. In case there are no real Schur roots amongst the $\beta$'s the program stops. The corresponding $\gamma$ for each $\beta$ is obtained by taking $s_\beta(\alpha)$. We have now obtained all decompositions of $\alpha$ of the following form

$$\alpha = -(\langle\gamma,\beta\rangle + \langle\gamma,\beta\rangle)\cdot\beta + \gamma,$$

with $\langle\beta,\gamma\rangle \leq 0$ and $\langle\gamma,\beta\rangle \leq 0$ (which is a necessary condition for $X_\gamma \in \mathfrak{M}_{-\beta}^{-\beta}$), and $\beta$ a real Schur root, $\gamma$ a real root. Now, using `construct_real_root_rep(`$E,\beta$`)`, `construct_real_root_rep(`$E,\gamma$`)`, `homomorphisms(`$E,X_\beta,X_\gamma$`)` and `homomorphisms(`$E,X_\gamma,X_\beta$`)` we determine whether $X_\gamma \in \mathfrak{M}_{-\beta}^{-\beta}$. In this way the program `decomposition(`$E,\alpha$`)` obtains all desired decompositions of $\alpha$. In case there are no such decompositions, the program produces no output.

The program `decomposition(`$E,\alpha$`)` can be used to study real root representations for given quivers in view of Question (††). We have used it to produce the tables presented in this appendix.

The data presented in the subsequent sections is arranged as follows. At the beginning of each section the considered quiver $Q$ is given together with an upper bound $b \in \mathbb{N}^{Q_0}$ for the dimension vectors of real roots. The tables contain all sincere non-Schur real roots $\alpha$ with $\alpha[i] \leq b[i]$ for all $i \in Q_0$, which can be obtained in the following way. Firstly, one searches through all dimension vectors up to a given upper bound and singles out all real roots, that is one checks whether a given dimension vector can be obtained from a simple root by a sequence of reflections. Here we use the fact that for a given real root there always exists a simple reflection $s_i$ such that $s_i(\alpha)$ is strictly smaller than $\alpha$. Secondly, using the program `homomorphisms`$(E, X_\alpha, X_\alpha)$ one singles out all the non-Schur real roots $\alpha$.

The tables contain the following information. For each non-Schur real root $\alpha$ we give the dimension of the endomorphism ring together with all possible decompositions

$$\alpha = (-\langle \beta, \gamma \rangle - \langle \gamma, \beta \rangle) \cdot \beta + \gamma$$

such that $\beta$ is a real Schur root, $\gamma$ is a real root, and $X_\gamma \in \mathfrak{M}_{-\beta}^{-\beta}$. In this case we have $\sigma_\beta(X_\gamma) = X_\alpha$. Hence, the sequence $\beta_n, \ldots, \beta_1$ $(n \geq 2)$ as sought in Question (††) for a non-Schur real root which is contained in the table can be obtained easily. Note that for the quivers considered in this chapter all non-Schur real root representations with dimension vectors below the given upper bound possess such a sequence.

The tables presented in this chapter may also be used by the reader to check further conjectures about real root representations.

## A.1 The quiver $Q_1$

$$Q_1\colon\ 1 \longrightarrow 2 \rightrightarrows 3\,, \quad b = (40, 40, 40)$$

| $\underline{\dim}\, X_\alpha$ | $\dim \mathrm{End}_{kQ}\, X_\alpha$ | decomposition(s) |
|---|---|---|
| (2,14,9) | 6 | (5+1)·(0,2,1)+(2,2,3) |
| (2,26,19) | 13 | (6+2)·(0,3,2)+(2,2,3) |
| (2,26,33) | 6 | (1+5)·(0,4,5)+(2,2,3) |
| (3,19,12) | 13 | (6+2)·(0,2,1)+(3,3,4) |
| (3,36,26) | 29 | (7+4)·(0,3,2)+(3,3,4) |
| (4,16,9) | 6 | (1+5)·(0,2,1)+(4,4,3) |

| $\underline{\dim}\, X_\alpha$ | $\dim \mathrm{End}_{kQ}\, X_\alpha$ | decomposition(s) |
|---|---|---|
| (4,24,15) | 22 | (7+3)·(0,2,1)+(4,4,5) |
| (5,21,12) | 13 | (2+6)·(0,2,1)+(5,5,4) |
| (5,29,18) | 33 | (8+4)·(0,2,1)+(5,5,6) |
| (6,26,15) | 22 | (3+7)·(0,2,1)+(6,6,5) |
| (6,34,21) | 46 | (9+5)·(0,2,1)+(6,6,7) |
| (7,31,18) | 33 | (4+8)·(0,2,1)+(7,7,6) |
| (7,39,24) | 61 | (10+6)·(0,2,1)+(7,7,8) |
| (8,36,21) | 46 | (5+9)·(0,2,1)+(8,8,7) |
| (12,14,19) | 6 | (1+5)·(2,2,3)+(0,2,1) |
| (12,16,23) | 6 | (5+1)·(2,2,3)+(0,4,5) |
| (12,36,19) | 13 | (2+6)·(1,4,2)+(4,4,3) |
| (16,19,26) | 13 | (2+6)·(2,2,3)+(0,3,2) |
| (16,21,30) | 13 | (6+2)·(2,2,3)+(0,5,6) |
| (20,24,33) | 22 | (3+7)·(2,2,3)+(0,4,3) |
| (20,26,37) | 22 | (7+3)·(2,2,3)+(0,6,7) |
| (24,26,19) | 6 | (5+1)·(4,4,3)+(0,2,1) |
| (24,26,33) | 13 | (2+6)·(3,3,4)+(0,2,1) |
| (24,29,40) | 33 | (4+8)·(2,2,3)+(0,5,4) |
| (30,34,23) | 6 | (1+5)·(4,4,3)+(6,10,5) |
| (33,36,26) | 13 | (6+2)·(4,4,3)+(1,4,2) |

## A.2 The quiver $Q_2$

In this section we consider the following quiver

$$Q_2\colon\; \begin{tikzcd} 1 \arrow[r, bend left] & 2 \arrow[l, bend left] \arrow[r, bend left] & 3 \arrow[l, bend left] \arrow[ll, bend right=60] \end{tikzcd}, \quad b = (20, 20, 20).$$

Since there is no arrow going from 1 to 3 Theorem 3.2.5 does not apply. However, as the data below shows, universal extension functors may still be used; but the real Schur representations used for a reflection are no longer just $S(1), S(2)$ and $S(3)$. This shows how the construction problem gets more complicated. The further we move away from the ideal situation described in Theorem 3.2.5, were we only reflected with respect to simples corresponding to vertices, the more complicated the reflection process seems to get.

| $\underline{\dim}\, X_\alpha$ | $\dim \operatorname{End}_{kQ} X_\alpha$ | decomposition(s) |
|---|---|---|
| (1,2,5) | 8 | (3+2)·(0,0,1)+(1,2,0) |
| (1,4,1) | 5 | (2+2)·(0,1,0)+(1,0,1) |
| (1,4,8) | 17 | (4+3)·(0,0,1)+(1,4,1) |
| (1,10,5) | 24 | (4+4)·(0,1,0)+(1,2,5) |
| (1,10,16) | 54 | (6+5)·(0,0,1)+(1,10,5) |
| (1,14,8) | 42 | (5+5)·(0,1,0)+(1,4,8) |
| (2,1,4) | 5 | (3+1)·(0,0,1)+(2,1,0) |
| (2,3,8) | 18 | (5+3)·(0,0,1)+(2,3,0) |
| (2,11,4) | 30 | (5+5)·(0,1,0)+(2,1,4) |
| (2,11,20) | 93 | (9+7)·(0,0,1)+(2,11,4) |
| (2,17,8) | 67 | (7+7)·(0,1,0)+(2,3,8) |
| (3,2,7) | 13 | (5+2)·(0,0,1)+(3,2,0) |
| (3,4,11) | 32 | (7+4)·(0,0,1)+(3,4,0) |
| (3,18,7) | 77 | (8+8)·(0,1,0)+(3,2,7) |
| (4,1,2) | 5 | (1+3)·(1,0,0)+(0,1,2) |
| (4,1,4) | 5 | (2+2)·(1,0,1)+(0,1,0) |
| (4,3,10) | 25 | (7+3)·(0,0,1)+(4,3,0) |
| (4,5,14) | 50 | (9+5)·(0,0,1)+(4,5,0) |
| (4,11,2) | 30 | (5+5)·(0,1,0)+(4,1,2) |
| (4,15,4) | 54 | (7+7)·(0,1,0)+(4,1,4) |

| $\underline{\dim}\, X_\alpha$ | $\dim \mathrm{End}_{kQ} X_\alpha$ | decomposition(s) |
|---|---|---|
| (5,2,1) | 8 | (2+3)·(1,0,0)+(0,2,1) |
| (5,2,8) | 14 | (6+1)·(0,0,1)+(5,2,1) |
| (5,4,13) | 41 | (9+4)·(0,0,1)+(5,4,0) |
| (5,6,17) | 72 | (11+6)·(0,0,1)+(5,6,0) |
| (5,10,1) | 24 | (4+4)·(0,1,0)+(5,2,1) |
| (6,5,16) | 61 | (11+5)·(0,0,1)+(6,5,0) |
| (6,7,20) | 98 | (13+7)·(0,0,1)+(6,7,0) |
| (7,2,3) | 13 | (2+5)·(1,0,0)+(0,2,3) |
| (7,2,8) | 14 | (3+4)·(1,0,1)+(0,2,1) |
| (7,6,19) | 85 | (13+6)·(0,0,1)+(7,6,0) |
| (7,18,3) | 77 | (8+8)·(0,1,0)+(7,2,3) |
| (8,2,5) | 14 | (1+6)·(1,0,0)+(1,2,5) |
| (8,2,7) | 14 | (4+3)·(1,0,1)+(1,2,0) |
| (8,3,2) | 18 | (3+5)·(1,0,0)+(0,3,2) |
| (8,3,12) | 27 | (9+1)·(0,0,1)+(8,3,2) |
| (8,4,1) | 17 | (3+4)·(1,0,0)+(1,4,1) |
| (8,4,15) | 50 | (11+3)·(0,0,1)+(8,4,1) |
| (8,14,1) | 42 | (5+5)·(0,1,0)+(8,4,1) |
| (8,17,2) | 67 | (7+7)·(0,1,0)+(8,3,2) |
| (10,3,4) | 25 | (3+7)·(1,0,0)+(0,3,4) |
| (10,3,12) | 27 | (4+6)·(1,0,1)+(0,3,2) |
| (11,4,3) | 32 | (4+7)·(1,0,0)+(0,4,3) |
| (11,4,16) | 44 | (12+1)·(0,0,1)+(11,4,3) |
| (12,3,8) | 27 | (1+9)·(1,0,0)+(2,3,8) |
| (12,3,10) | 27 | (6+4)·(1,0,1)+(2,3,0) |
| (13,4,5) | 41 | (4+9)·(1,0,0)+(0,4,5) |

| $\underline{\dim}\, X_\alpha$ | $\dim \mathrm{End}_{kQ} X_\alpha$ | decomposition(s) |
|---|---|---|
| (13,4,16) | 44 | (5+8)·(1,0,1)+(0,4,3) |
| (14,5,4) | 50 | (5+9)·(1,0,0)+(0,5,4) |
| (14,5,20) | 65 | (15+1)·(0,0,1)+(14,5,4) |
| (15,4,8) | 50 | (3+11)·(1,0,0)+(1,4,8) |
| (15,4,15) | 54 | (7+7)·(1,0,1)+(1,4,1) |
| (16,4,11) | 44 | (1+12)·(1,0,0)+(3,4,11) |
| (16,4,13) | 44 | (8+5)·(1,0,1)+(3,4,0) |
| (16,5,6) | 61 | (5+11)·(1,0,0)+(0,5,6) |
| (16,5,20) | 65 | (6+10)·(1,0,1)+(0,5,4) |
| (16,10,1) | 54 | (5+6)·(1,0,0)+(5,10,1) |
| (17,6,5) | 72 | (6+11)·(1,0,0)+(0,6,5) |
| (19,6,7) | 85 | (6+13)·(1,0,0)+(0,6,7) |
| (20,5,14) | 65 | (1+15)·(1,0,0)+(4,5,14) |
| (20,5,16) | 65 | (10+6)·(1,0,1)+(4,5,0) |
| (20,7,6) | 98 | (7+13)·(1,0,0)+(0,7,6) |
| (20,11,2) | 93 | (7+9)·(1,0,0)+(4,11,2) |

## A.3 The six-subspace quiver

The data for the six-subspace quiver below may give some suggestions for more general tools needed to decide whether a given representation $X$ is in $\mathfrak{M}_{-S}^{-S}$, where $S$ is a real Schur representation.

One suggestion is the following. For a given real root representation for a quiver $Q$ one may restrict this representation to a subquiver of $Q$. Then one could ask what are the homomorphisms between the representation obtained and certain real Schur representations? Tools similar to this may be useful to be able to approach $n$-subspace quivers in the way stated in Question $(\dagger\dagger)$.

$$Q_3: \quad \begin{tikzcd} & & & 7 & & & \\ 1 \arrow[urrr] & 2 \arrow[urr] & & 3 \arrow[u] & 4 \arrow[ul] & 5 \arrow[ull] & 6 \arrow[ulll] \end{tikzcd} \quad , \quad b = (5,5,5,5,5,5,10)$$

The tables only contain real roots $\alpha$ with the property

$$\alpha[1] \leq \alpha[2] \leq \alpha[3] \leq \alpha[4] \leq \alpha[5] \leq \alpha[6].$$

| $\underline{\dim}\, X_\alpha$ | $\dim \mathrm{End}_{kQ}\, X_\alpha$ | decomposition(s) |
|---|---|---|
| (1,1,1,1,4,4,5) | 4 | (1+3)·(0,0,0,0,1,1,1)+(1,1,1,1,0,0,1) |
| (1,1,1,1,4,4,7) | 4 | (3+1)·(0,0,0,0,1,1,1)+(1,1,1,1,0,0,3) |
| (1,1,1,2,5,5,7) | 7 | (2+3)·(0,0,0,0,1,1,1)+(1,1,1,2,0,0,2) |
| (1,1,1,2,5,5,8) | 7 | (3+2)·(0,0,0,0,1,1,1)+(1,1,1,2,0,0,3) |
| (1,1,1,3,3,4,4) | 3 | (4+0)·(0,0,0,0,0,1,0)+(1,1,1,3,3,0,4) |
| | | (1+2)·(0,0,0,1,1,1,1)+(1,1,1,0,0,1,1) |
| (1,1,1,3,3,4,9) | 3 | (0+4)·(0,0,0,0,0,1,1)+(1,1,1,3,3,0,5) |
| | | (2+1)·(0,0,0,1,1,1,2)+(1,1,1,0,0,1,3) |
| (1,1,1,3,3,5,5) | 3 | (5+0)·(0,0,0,0,0,1,0)+(1,1,1,3,3,0,5) |
| | | (2+1)·(0,0,0,1,1,1,1)+(1,1,1,0,0,2,2) |
| (1,1,1,3,3,5,9) | 3 | (0+5)·(0,0,0,0,0,1,1)+(1,1,1,3,3,0,4) |
| | | (1+2)·(0,0,0,1,1,1,2)+(1,1,1,0,0,2,3) |
| (1,1,1,3,3,6,7) | 5 | (1+2)·(0,0,0,1,0,1,1)+(1,1,1,0,3,3,4) |
| | | (1+2)·(0,0,0,0,1,1,1)+(1,1,1,3,0,3,4) |
| (1,1,1,3,3,6,8) | 5 | (2+1)·(0,0,0,1,0,1,1)+(1,1,1,0,3,3,5) |
| | | (2+1)·(0,0,0,0,1,1,1)+(1,1,1,3,0,3,5) |
| (1,1,1,3,4,6,7) | 4 | (1+3)·(0,0,0,0,1,1,1)+(1,1,1,3,0,2,3) |
| | | (0+2)·(0,0,0,1,0,1,1)+(1,1,1,1,4,4,5) |

| $\underline{\dim}\, X_\alpha$ | $\dim \mathrm{End}_{kQ} X_\alpha$ | decomposition(s) |
|---|---|---|
| (1,1,1,3,4,6,9) | 4 | (2+0)·(0,0,0,1,0,1,1)+(1,1,1,1,4,4,7)<br>(3+1)·(0,0,0,0,1,1,1)+(1,1,1,3,0,2,5) |
| (1,1,1,3,5,6,8) | 7 | (2+3)·(0,0,0,0,1,1,1)+(1,1,1,3,0,1,3)<br>(0+1)·(0,0,0,1,0,1,1)+(1,1,1,2,5,5,7) |
| (1,1,1,3,5,6,9) | 7 | (3+2)·(0,0,0,0,1,1,1)+(1,1,1,3,0,1,4)<br>(1+0)·(0,0,0,1,0,1,1)+(1,1,1,2,5,5,8) |
| (1,1,1,4,4,4,5) | 5 | (2+2)·(0,0,0,1,1,1,1)+(1,1,1,0,0,0,1) |
| (1,1,1,4,4,4,10) | 5 | (2+2)·(0,0,0,1,1,1,2)+(1,1,1,0,0,0,2) |
| (1,1,1,4,6,6,9) | 7 | (2+3)·(0,0,0,0,1,1,1)+(1,1,1,4,1,1,4)<br>(0+1)·(0,0,0,1,1,0,1)+(1,1,1,3,5,6,8)<br>(0+1)·(0,0,0,1,0,1,1)+(1,1,1,3,6,5,8) |
| (1,1,1,4,6,6,10) | 7 | (3+2)·(0,0,0,0,1,1,1)+(1,1,1,4,1,1,5)<br>(1+0)·(0,0,0,1,1,0,1)+(1,1,1,3,5,6,9)<br>(1+0)·(0,0,0,1,0,1,1)+(1,1,1,3,6,5,9) |
| (1,1,1,5,5,5,7) | 5 | (4+1)·(0,0,0,1,1,1,1)+(1,1,1,0,0,0,2) |
| (1,1,1,5,6,6,9) | 4 | (1+3)·(0,0,0,0,1,1,1)+(1,1,1,5,2,2,5)<br>(0+2)·(0,0,0,1,0,1,1)+(1,1,1,3,6,4,7)<br>(0+2)·(0,0,0,1,1,0,1)+(1,1,1,3,4,6,7) |
| (1,1,1,6,6,6,10) | 7 | (1+2)·(0,0,0,0,1,1,1)+(1,1,1,6,3,3,7)<br>(1+2)·(0,0,0,1,0,1,1)+(1,1,1,3,6,3,7)<br>(1+2)·(0,0,0,1,1,0,1)+(1,1,1,3,3,6,7) |
| (1,1,2,2,2,6,7) | 4 | (1+1)·(0,0,1,0,0,1,1)+(1,1,0,2,2,4,5)<br>(1+1)·(0,0,0,1,0,1,1)+(1,1,2,0,2,4,5)<br>(1+1)·(0,0,0,0,1,1,1)+(1,1,2,2,0,4,5) |
| (1,1,2,2,3,3,3) | 2 | (3+0)·(0,0,0,0,1,0,0)+(1,1,2,2,0,3,3)<br>(3+0)·(0,0,0,0,0,1,0)+(1,1,2,2,3,0,3) |

| $\underline{\dim}\, X_\alpha$ | $\dim \operatorname{End}_{kQ} X_\alpha$ | decomposition(s) |
|---|---|---|
| | | (1+1)·(0,0,1,1,1,1,1)+(1,1,0,0,1,1,1) |
| (1,1,2,2,3,3,9) | 2 | (0+3)·(0,0,0,0,1,0,1)+(1,1,2,2,0,3,6) |
| | | (0+3)·(0,0,0,0,0,1,1)+(1,1,2,2,3,0,6) |
| | | (1+1)·(0,0,1,1,1,1,3)+(1,1,0,0,1,1,3) |
| (1,1,2,2,3,6,6) | 2 | (6+0)·(0,0,0,0,0,1,0)+(1,1,2,2,3,0,6) |
| | | (0+3)·(0,0,0,0,1,1,1)+(1,1,2,2,0,3,3) |
| | | (1+1)·(0,0,1,1,1,2,2)+(1,1,0,0,1,2,2) |
| (1,1,2,2,3,6,9) | 2 | (0+6)·(0,0,0,0,0,1,1)+(1,1,2,2,3,0,3) |
| | | (3+0)·(0,0,0,0,1,1,1)+(1,1,2,2,0,3,6) |
| | | (1+1)·(0,0,1,1,1,2,3)+(1,1,0,0,1,2,3) |
| (1,1,2,2,4,5,5) | 3 | (5+0)·(0,0,0,0,0,1,0)+(1,1,2,2,4,0,5) |
| | | (1+1)·(0,0,1,0,1,1,1)+(1,1,0,2,2,3,3) |
| | | (1+1)·(0,0,0,1,1,1,1)+(1,1,2,0,2,3,3) |
| (1,1,2,2,4,5,10) | 3 | (0+5)·(0,0,0,0,0,1,1)+(1,1,2,2,4,0,5) |
| | | (1+1)·(0,0,1,0,1,1,2)+(1,1,0,2,2,3,6) |
| | | (1+1)·(0,0,0,1,1,1,2)+(1,1,2,0,2,3,6) |
| (1,1,2,2,5,6,7) | 5 | (1+4)·(0,0,0,0,1,1,1)+(1,1,2,2,0,1,2) |
| (1,1,2,2,5,6,10) | 5 | (4+1)·(0,0,0,0,1,1,1)+(1,1,2,2,0,1,5) |
| (1,1,2,2,6,6,9) | 11 | (3+3)·(0,0,0,0,1,1,1)+(1,1,2,2,0,0,3) |
| (1,1,2,3,4,5,5) | 3 | (5+0)·(0,0,0,0,0,1,0)+(1,1,2,3,4,0,5) |
| | | (1+2)·(0,0,0,1,1,1,1)+(1,1,2,0,1,2,2) |
| | | (0+1)·(0,0,1,0,1,1,1)+(1,1,1,3,3,4,4) |
| (1,1,2,3,4,6,6) | 3 | (6+0)·(0,0,0,0,0,1,0)+(1,1,2,3,4,0,6) |
| | | (1+0)·(0,0,1,0,1,1,1)+(1,1,1,3,3,5,5) |
| | | (2+1)·(0,0,0,1,1,1,1)+(1,1,2,0,1,3,3) |
| (1,1,2,5,5,6,7) | 7 | (3+2)·(0,0,0,1,1,1,1)+(1,1,2,0,0,1,2) |

| $\underline{\dim}\, X_\alpha$ | $\dim \operatorname{End}_{kQ} X_\alpha$ | decomposition(s) |
|---|---|---|
| (1,1,3,3,3,4,4) | 3 | (4+0)·(0,0,0,0,0,1,0)+(1,1,3,3,3,0,4)<br>(2+1)·(0,0,1,1,1,1,1)+(1,1,0,0,0,1,1) |
| (1,1,3,3,6,6,7) | 5 | (2+1)·(0,0,1,0,1,1,1)+(1,1,0,3,3,3,4)<br>(2+1)·(0,0,0,1,1,1,1)+(1,1,3,0,3,3,4) |
| (1,1,3,4,4,6,6) | 3 | (6+0)·(0,0,0,0,0,1,0)+(1,1,3,4,4,0,6)<br>(1+2)·(0,0,0,1,1,1,1)+(1,1,3,1,1,3,3)<br>(0+1)·(0,0,1,1,0,1,1)+(1,1,2,3,4,5,5)<br>(0+1)·(0,0,1,0,1,1,1)+(1,1,2,4,3,5,5) |
| (1,1,3,4,6,6,7) | 6 | (2+2)·(0,0,0,1,1,1,1)+(1,1,3,0,2,2,3)<br>(1+1)·(0,0,1,0,1,1,1)+(1,1,1,4,4,4,5) |
| (1,1,4,4,4,4,5) | 4 | (3+1)·(0,0,1,1,1,1,1)+(1,1,0,0,0,0,1) |
| (1,1,4,4,6,6,7) | 5 | (1+2)·(0,0,0,1,1,1,1)+(1,1,4,1,3,3,4)<br>(1+2)·(0,0,1,0,1,1,1)+(1,1,1,4,3,3,4) |
| (1,2,2,3,5,6,6) | 3 | (6+0)·(0,0,0,0,0,1,0)+(1,2,2,3,5,0,6)<br>(0+1)·(0,1,0,0,1,1,1)+(1,1,2,3,4,5,5)<br>(0+1)·(0,0,1,0,1,1,1)+(1,2,1,3,4,5,5)<br>(1+2)·(0,0,0,1,1,1,1)+(1,2,2,0,2,3,3) |
| (1,2,2,5,6,6,7) | 7 | (2+3)·(0,0,0,1,1,1,1)+(1,2,2,0,1,1,2) |
| (1,2,3,4,4,5,5) | 3 | (5+0)·(0,0,0,0,0,1,0)+(1,2,3,4,4,0,5)<br>(2+1)·(0,0,1,1,1,1,1)+(1,2,0,1,1,2,2)<br>(1+0)·(0,1,0,1,1,1,1)+(1,1,3,3,3,4,4) |
| (1,2,3,4,5,6,6) | 2 | (6+0)·(0,0,0,0,0,1,0)+(1,2,3,4,5,0,6)<br>(0+3)·(0,0,0,1,1,1,1)+(1,2,3,1,2,3,3)<br>(3+0)·(0,0,1,1,1,1,1)+(1,2,0,1,2,3,3)<br>(1+1)·(0,1,1,1,2,2,2)+(1,0,1,2,1,2,2) |
| (1,2,5,5,6,6,7) | 5 | (4+1)·(0,0,1,1,1,1,1)+(1,2,0,0,1,1,2) |

| $\underline{\dim}\, X_\alpha$ | $\dim \operatorname{End}_{kQ} X_\alpha$ | decomposition(s) |
|---|---|---|
| (1,3,3,4,4,5,5) | 3 | (5+0)·(0,0,0,0,0,1,0)+(1,3,3,4,4,0,5)<br>(1+1)·(0,0,1,1,1,1,1)+(1,3,1,2,2,3,3)<br>(1+1)·(0,1,0,1,1,1,1)+(1,1,3,2,2,3,3) |
| (1,3,4,4,5,6,6) | 3 | (6+0)·(0,0,0,0,0,1,0)+(1,3,4,4,5,0,6)<br>(2+1)·(0,0,1,1,1,1,1)+(1,3,1,1,2,3,3)<br>(1+0)·(0,1,1,0,1,1,1)+(1,2,3,4,4,5,5)<br>(1+0)·(0,1,0,1,1,1,1)+(1,2,4,3,4,5,5) |
| (1,3,4,6,6,6,7) | 4 | (3+1)·(0,0,1,1,1,1,1)+(1,3,0,2,2,2,3)<br>(2+0)·(0,1,0,1,1,1,1)+(1,1,4,4,4,4,5) |
| (1,4,4,6,6,6,7) | 5 | (2+1)·(0,0,1,1,1,1,1)+(1,4,1,3,3,3,4)<br>(2+1)·(0,1,0,1,1,1,1)+(1,1,4,3,3,3,4) |
| (1,5,5,5,6,6,7) | 4 | (1+1)·(0,0,1,1,1,1,1)+(1,5,3,3,4,4,5)<br>(1+1)·(0,1,0,1,1,1,1)+(1,3,5,3,4,4,5)<br>(1+1)·(0,1,1,0,1,1,1)+(1,3,3,5,4,4,5) |
| (2,2,2,6,6,6,7) | 5 | (1+4)·(0,0,0,1,1,1,1)+(2,2,2,1,1,1,2) |
| (2,2,3,3,5,5,5) | 3 | (5+0)·(0,0,0,0,1,0,0)+(2,2,3,3,0,5,5)<br>(5+0)·(0,0,0,0,0,1,0)+(2,2,3,3,5,0,5)<br>(1+1)·(0,0,1,1,1,1,1)+(2,2,1,1,3,3,3) |
| (2,2,3,5,5,6,6) | 3 | (6+0)·(0,0,0,0,0,1,0)+(2,2,3,5,5,0,6)<br>(1+0)·(1,0,0,1,1,1,1)+(1,2,3,4,4,5,5)<br>(1+0)·(0,1,0,1,1,1,1)+(2,1,3,4,4,5,5)<br>(2+1)·(0,0,1,1,1,1,1)+(2,2,0,2,2,3,3) |
| (2,2,4,4,4,5,5) | 3 | (5+0)·(0,0,0,0,0,1,0)+(2,2,4,4,4,0,5)<br>(1+2)·(0,0,1,1,1,1,1)+(2,2,1,1,1,2,2) |
| (2,2,5,6,6,6,7) | 7 | (3+2)·(0,0,1,1,1,1,1)+(2,2,0,1,1,1,2) |
| (2,3,4,5,5,6,6) | 3 | (6+0)·(0,0,0,0,0,1,0)+(2,3,4,5,5,0,6) |

| $\underline{\dim}\, X_\alpha$ | $\dim \operatorname{End}_{kQ} X_\alpha$ | decomposition(s) |
|---|---|---|
| | | (1+2)·(0,0,1,1,1,1,1)+(2,3,1,2,2,3,3) |
| | | (0+1)·(0,1,0,1,1,1,1)+(2,2,4,4,4,5,5) |
| (3,3,6,6,6,6,7) | 4 | (1+3)·(0,0,1,1,1,1,1)+(3,3,2,2,2,2,3) |
| (3,4,4,5,5,6,6) | 2 | (6+0)·(0,0,0,0,0,1,0)+(3,4,4,5,5,0,6) |
| | | (3+0)·(0,1,1,1,1,1,1)+(3,1,1,2,2,3,3) |
| | | (1+1)·(1,1,1,2,2,2,2)+(1,2,2,1,1,2,2) |

## A.4 The quiver $Q_4$

$$Q_4 : \quad \begin{tikzcd} 1 \arrow[dd] \arrow[dr] & & \\ & 3 \arrow[r, shift left] \arrow[r, shift right] & 4, \quad b = (10, 10, 10, 10) \\ 2 \arrow[ur] & & \end{tikzcd}$$

| $\underline{\dim}\, X_\alpha$ | $\dim \operatorname{End}_{kQ} X_\alpha$ | decomposition(s) |
|---|---|---|
| (1,2,1,2) | 2 | (1+1)·(0,1,0,0,)+(1,0,1,2) |
| | | (0+2)·(0,0,0,1)+(1,2,1,0) |
| (1,2,6,2) | 8 | (3+2)·(0,0,1,0)+(1,2,1,2) |
| (1,2,9,4) | 7 | (6+1)·(0,0,1,0)+(1,2,2,4) |
| (1,5,4,2) | 5 | (4+1)·(0,1,0,0,)+(1,0,4,2) |
| (1,5,4,6) | 5 | (4+1)·(0,1,0,0)+(1,0,4,6) |
| (1,5,6,2) | 5 | (4+1)·(0,1,1,0,)+(1,0,1,2) |
| | | (0+2)·(0,0,1,0)+(1,5,4,2) |
| (1,10,9,6) | 10 | (9+1)·(0,1,0,0,)+(1,0,9,6) |
| (2,1,2,4) | 2 | (0+4)·(0,0,0,1,)+(2,1,2,0) |

| $\underline{\dim}\, X_\alpha$ | $\dim \operatorname{End}_{kQ} X_\alpha$ | decomposition(s) |
|---|---|---|
| | | (1+1)·(1,0,1,2)+(0,1,0,0) |
| (2,1,6,2) | 7 | (3+2)·(0,0,1,0)+(2,1,1,2) |
| (2,1,6,10) | 2 | (1+1)·(1,0,1,2,)+(0,1,4,6) |
| | | (0+2)·(0,0,2,3)+(2,1,2,4) |
| (2,1,9,4) | 8 | (6+1)·(0,0,1,0)+(2,1,2,4) |
| (2,2,7,2) | 10 | (3+3)·(0,0,1,0)+(2,2,1,2) |
| (2,3,2,4) | 3 | (1+1)·(0,1,0,0,)+(2,1,2,4) |
| | | (0+4)·(0,0,0,1)+(2,3,2,0) |
| (2,4,2,1) | 5 | (2+2)·(0,1,0,0,)+(2,0,2,1) |
| (2,4,2,3) | 5 | (2+2)·(0,1,0,0,)+(2,0,2,3) |
| (2,4,6,1) | 5 | (0+4)·(0,0,1,0,)+(2,4,2,1) |
| | | (2+2)·(0,1,1,0)+(2,0,2,1) |
| (2,4,10,3) | 21 | (4+4)·(0,0,1,0)+(2,4,2,3) |
| (2,7,6,2) | 12 | (5+1)·(0,1,0,0,)+(2,1,6,2) |
| (2,7,6,10) | 7 | (5+1)·(0,1,0,0)+(2,1,6,10) |
| (2,7,7,2) | 10 | (5+0)·(0,1,0,0,)+(2,2,7,2) |
| | | (3+3)·(0,1,1,0)+(2,1,1,2) |
| (2,8,6,3) | 13 | (6+2)·(0,1,0,0,)+(2,0,6,3) |
| (2,8,6,9) | 13 | (6+2)·(0,1,0,0)+(2,0,6,9) |
| (2,8,10,3) | 13 | (6+2)·(0,1,1,0,)+(2,0,2,3) |
| | | (0+4)·(0,0,1,0)+(2,8,6,3) |
| (2,10,9,4) | 16 | (8+1)·(0,1,0,0,)+(2,1,9,4) |
| (3,2,3,6) | 3 | (0+6)·(0,0,0,1,)+(3,2,3,0) |
| | | (1+1)·(1,0,1,2)+(1,2,1,2) |
| (3,4,3,6) | 4 | (1+1)·(0,1,0,0,)+(3,2,3,6) |
| | | (0+6)·(0,0,0,1)+(3,4,3,0) |

| $\underline{\dim}\, X_\alpha$ | $\dim \mathrm{End}_{kQ} X_\alpha$ | decomposition(s) |
|---|---|---|
| (3,6,3,2) | 10 | (3+3)·(0,1,0,0,)+(3,0,3,2) |
| (3,6,3,4) | 10 | (3+3)·(0,1,0,0,)+(3,0,3,4) |
| (3,6,10,2) | 16 | (1+6)·(0,0,1,0,)+(3,6,3,2) |
| (3,7,4,2) | 13 | (4+3)·(0,1,0,0)+(3,0,4,2) |
| (3,7,4,6) | 13 | (4+3)·(0,1,0,0,)+(3,0,4,6) |
| (3,7,10,2) | 13 | (4+3)·(0,1,1,0,)+(3,0,3,2)<br>(0+6)·(0,0,1,0)+(3,7,4,2) |
| (4,2,10,3) | 17 | (4+4)·(0,0,1,0)+(4,2,2,3) |
| (4,3,4,8) | 4 | (0+8)·(0,0,0,1,)+(4,3,4,0)<br>(1+1)·(1,0,1,2)+(2,3,2,4) |
| (4,5,4,8) | 5 | (1+1)·(0,1,0,0,)+(4,3,4,8)<br>(0+8)·(0,0,0,1)+(4,5,4,0) |
| (4,8,4,3) | 17 | (4+4)·(0,1,0,0,)+(4,0,4,3) |
| (4,8,4,5) | 17 | (4+4)·(0,1,0,0,)+(4,0,4,5) |
| (4,8,6,1) | 9 | (4+2)·(0,1,0,0,)+(4,2,6,1) |
| (4,10,6,3) | 25 | (6+4)·(0,1,0,0)+(4,0,6,3) |
| (4,10,6,9) | 25 | (6+4)·(0,1,0,0,)+(4,0,6,9) |
| (5,1,6,10) | 5 | (1+4)·(1,0,1,2)+(0,1,1,0) |
| (5,4,5,10) | 5 | (0+10)·(0,0,0,1,)+(5,4,5,0)<br>(1+1)·(1,0,1,2)+(3,4,3,6) |
| (5,6,5,10) | 6 | (1+1)·(0,1,0,0,)+(5,4,5,10)<br>(0+10)·(0,0,0,1)+(5,6,5,0) |
| (5,8,4,2) | 13 | (3+4)·(0,1,0,0,)+(5,1,4,2) |
| (5,8,4,6) | 13 | (3+4)·(0,1,0,0)+(5,1,4,6) |
| (5,10,5,4) | 26 | (5+5)·(0,1,0,0,)+(5,0,5,4) |
| (5,10,5,6) | 26 | (5+5)·(0,1,0,0,)+(5,0,5,6) |

| $\underline{\dim}\, X_\alpha$ | $\dim \operatorname{End}_{kQ} X_\alpha$ | decomposition(s) |
|---|---|---|
| (5,10,6,2) | 21 | (5+4)·(0,1,0,0,)+(5,1,6,2) |
| (5,10,6,10) | 25 | (5+4)·(0,1,0,0,)+(5,1,6,10) |
| (6,3,10,2) | 7 | (1+6)·(0,0,1,0)+(6,3,3,2) |
| (7,2,6,2) | 5 | (4+1)·(1,0,1,0)+(2,2,1,2) |
| (7,2,7,2) | 7 | (5+1)·(1,0,1,0)+(1,2,1,2) |
| (7,8,4,2) | 5 | (1+4)·(0,1,0,0)+(7,3,4,2) |
| (7,8,4,6) | 5 | (1+4)·(0,1,0,0,)+(7,3,4,6) |
| (8,1,8,4) | 5 | (2+2)·(2,0,2,1)+(0,1,0,0) |
| (8,1,9,4) | 5 | (0+1)·(0,0,1,0,)+(8,1,8,4)<br>(2+2)·(2,0,2,1)+(0,1,1,0) |
| (8,4,6,1) | 5 | (2+2)·(1,0,1,0,)+(4,4,2,1) |
| (8,4,8,1) | 13 | (4+2)·(1,0,1,0,)+(2,4,2,1) |
| (8,10,6,1) | 13 | (2+4)·(0,1,0,0,)+(8,4,6,1) |
| (10,5,6,2) | 2 | (1+1)·(1,0,1,0,)+(8,5,4,2)<br>(2+0)·(4,2,2,1)+(2,1,2,0) |
| (10,5,6,10) | 7 | (2+3)·(2,1,1,2)+(0,0,1,0) |

# Appendix B

# Research papers

In this appendix we include the paper [26] and the preprint [25], as required by the University of Leeds regulations for the presentation of theses for higher degrees.

## B.1 M. Wiedemann, *Representations of maximal rank type and an application to representations of a quiver with three vertices*, Bull. London Math. Soc. 40 (2008), 479-492

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# Quiver representations of maximal rank type and an application to representations of a quiver with three vertices

Marcel Wiedemann

Abstract

We introduce the notion of 'maximal rank type' for representations of quivers, which requires certain collections of maps involved in the representation to be of maximal rank. We show that real root representations of quivers are of maximal rank type. By using the maximal rank type property and universal extension functors we construct all real root representations of a particular wild quiver with three vertices. From this construction it follows that real root representations of this quiver are tree modules. Moreover, formulae given by Ringel can be applied to compute the dimension of the endomorphism ring of a given real root representation.

## *Introduction*

Throughout this paper we fix an arbitrary field $k$. Let $Q$ be a (finite) *quiver*, that is, an oriented graph with finite vertex set $Q_0$ and finite arrow set $Q_1$ together with two functions $h, t : Q_1 \to Q_0$ assigning a head and a tail to each arrow $a \in Q_1$. For $i \in Q_0$ we define the sets $H^Q(i) := \{a \in Q_1 : h(a) = i\}$ and $T^Q(i) := \{a \in Q_1 : t(a) = i\}$.

A *representation* $X$ of $Q$ is given by a vector space $X_i$ (over $k$) for each vertex $i \in Q_0$ together with a linear map $X_a : X_{t(a)} \to X_{h(a)}$ for each arrow $a \in Q_1$.

Definition (Maximal rank type). A representation $X$ of $Q$ is said to be of *maximal rank type*, provided that it satisfies the following conditions.

(i) For every vertex $i \in Q_0$ and for every subset $A \subseteq H^Q(i)$ the map

$$\bigoplus_{a\in A} X_{t(a)} \xrightarrow{(X_a)_a} X_i$$

is of maximal rank.

(ii) For every vertex $i \in Q_0$ and for every subset $B \subseteq T^Q(i)$ the map

$$X_i \xrightarrow{(X_b)_b} \bigoplus_{b\in B} X_{h(b)}$$

is of maximal rank.

Clearly not every representation of $Q$ is of maximal rank type. The following example shows that even indecomposable representations of $Q$ might not be of maximal rank type:

$$k \underset{0}{\overset{1}{\rightrightarrows}} k\ .$$

However, if $k$ is algebraically closed, a general representation for a given dimension vector $d$ is of maximal rank type. In particular, real Schur representations have this property; but clearly not all real root representations are real Schur representations.

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Let $\alpha$ be a positive real root for $Q$. Recall that there is a unique indecomposable representation of dimension vector $\alpha$ (see Section 1 for details).

The main result of this paper is the following.

THEOREM A. *Let $Q$ be a quiver, and let $\alpha$ be a positive real root for $Q$. The unique indecomposable representation of dimension vector $\alpha$ is of maximal rank type.*

In the second part of this paper we use Theorem A to construct all real root representations of the quiver

$$Q(f,g,h): \begin{tikzcd} 1 \arrow[r, shift left, "\lambda_1"] \arrow[r, shift right, "\lambda_f"'] & 2 \arrow[r, bend left=60, "\mu_1"] \arrow[r, bend left=20, "\mu_g"] & 3 \arrow[l, bend left=20, "\nu_1"] \arrow[l, bend left=60, "\nu_h"'] \end{tikzcd}$$

with $f, g, h \geqslant 1$.

The quiver $Q(1,1,1)$ is considered by Jensen and Su in [**2**], where all real root representations are constructed explicitly. In [**7**] Ringel extends their results to the quiver $Q(1,g,h)$ $(g,h \geqslant 1)$ by using universal extension functors. In this paper we consider the general case and obtain the following result.

THEOREM B. *Let $\alpha$ be a positive real root for the quiver $Q(f,g,h)$. The unique indecomposable representation of dimension vector $\alpha$ can be constructed by using universal extension functors starting from simple representations and real Schur representations of the quiver $Q'(f)$ $(f \geqslant 1)$, where $Q'(f)$ denotes the following subquiver of $Q(f,g,h)$.*

$$Q'(f): \begin{tikzcd} 1 \arrow[r, shift left, "\lambda_1"] \arrow[r, shift right, "\lambda_f"'] & 2 \end{tikzcd}$$

The paper is organized as follows. In Section 1 we discuss further notation and background results. In Section 2 we prove Theorem A after discussing the constructions needed for the proof. To prove that real root representations are of maximal rank type, we have to show that certain collections of maps have maximal rank. The main idea of the proof is to insert an extra vertex and to attach to it the image of the map under consideration. Analysing this modified representation yields the desired result.

In Section 3 we use the maximal rank type property of real root representations to prove Theorem B. It follows that real root representations of $Q(f,g,h)$ are tree modules. Moreover, using the formulae given in [**5**, Section 1] we can compute the dimension of the endomorphism ring for a given real root representation.

## 1. *Further notation and background results*

Let $Q$ be a quiver with vertex set $Q_0$ and arrow set $Q_1$. Let $X$ and $Y$ be two representations of $Q$. A *homomorphism* $\phi : X \to Y$ is given by linear maps $\phi_i : X_i \to Y_i$ such that for each arrow $a \in Q_1$, $a : i \to j$ say, the following square commutes.

$$\begin{tikzcd} X_i \arrow[r, "X_a"] \arrow[d, "\phi_i"] & X_j \arrow[d, "\phi_j"] \\ Y_i \arrow[r, "Y_a"] & Y_j \end{tikzcd}$$

The morphism $\phi$ is said to be an isomorphism if $\phi_i$ is an isomorphism for all $i \in Q_0$. The direct sum $X \oplus Y$ of two representations $X$ and $Y$ is defined by

$$(X \oplus Y)_i = X_i \oplus Y_i \qquad \forall\, i \in Q_0,$$
$$(X \oplus Y)_a = \begin{pmatrix} X_a & 0 \\ 0 & Y_a \end{pmatrix} \qquad \forall\, a \in Q_1.$$

A representation $Z$ is called *decomposable* if $Z \cong X \oplus Y$ for non-zero representations $X$ and $Y$. In this way one obtains a category of representations, denoted by $\operatorname{Rep}_k Q$.

A *dimension vector* for $Q$ is given by an element of $\mathbb{N}^{Q_0}$. We will write $e_i$ for the coordinate vector at vertex $i$ and by $d[i]$, $i \in Q_0$, we denote the $i$th coordinate of $d \in \mathbb{N}^{Q_0}$. A dimension vector $d \in \mathbb{N}^{Q_0}$ is said to be *sincere*, provided that $d[i] > 0$ for all $i \in Q_0$. If $X$ is a finite-dimensional representation, meaning that all vector spaces $X_i$ $(i \in Q_0)$ are finite-dimensional, then $\underline{\dim}\, X = (\dim X_i)_{i \in Q_0}$ is the dimension vector of $X$. Throughout this paper we consider only finite-dimensional representations. We denote by $\operatorname{rep}_k Q$, the full subcategory with the finite-dimensional representations of $Q$ as objects.

The *Ringel form* on $\mathbb{Z}^{Q_0}$ is defined by

$$\langle \alpha, \beta \rangle = \sum_{i \in Q_0} \alpha[i]\beta[i] - \sum_{a \in Q_1} \alpha[t(a)]\beta[h(a)].$$

Moreover, let $(\alpha, \beta) = \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle$ be its symmetrization.

We say that a vertex $i \in Q_0$ is *loop-free* if there are no arrows $a : i \to i$. By a quiver without loops we mean a quiver with only loop-free vertices. In this paper we consider only quivers without loops. For a loop-free vertex $i \in Q_0$ the simple reflection $s_i : \mathbb{Z}^{Q_0} \to \mathbb{Z}^{Q_0}$ is defined by

$$s_i(\alpha) := \alpha - (\alpha, e_i)e_i.$$

A *simple root* is a vector $e_i$ for $i \in Q_0$. The set of simple roots is denoted by $\Pi$. The *Weyl group*, denoted by $W$, is the subgroup of $\mathrm{GL}(\mathbb{Z}^n)$, where $n = |Q_0|$, generated by the $s_i$. By $\Delta^+_{\mathrm{re}}(Q) := \{\alpha \in W(\Pi) : \alpha > 0\}$ we denote the set of (positive) *real root* for $Q$. Let

$$M := \{\beta \in \mathbb{N}^{Q_0} : \beta \text{ has connected support and } (\beta, e_i) \leqslant 0 \text{ for all } i \in Q_0\}.$$

By $\Delta^+_{\mathrm{im}}(Q) := \bigcup_{w \in W} w(M)$ we denote the set of (positive) *imaginary roots* for $Q$. Moreover, we define $\Delta^+(Q) := \Delta^+_{\mathrm{re}}(Q) \cup \Delta^+_{\mathrm{im}}(Q)$. We have the following lemma.

Lemma 1.1 [**3**, Lemma 2.1]. *For $\alpha \in \Delta^+(Q)$ one has*
(i) $\alpha \in \Delta^+_{\mathrm{re}}(Q)$ *if and only if* $\langle \alpha, \alpha \rangle = 1$,
(ii) $\alpha \in \Delta^+_{\mathrm{im}}(Q)$ *if and only if* $\langle \alpha, \alpha \rangle \leqslant 0$.

As mentioned in the introduction we have the following remarkable theorem.

Theorem 1.2 (Kac [**3**, Theorems 1 and 2] and Schofield [**8**, Theorem 9]). *Let $k$ be a field and $Q$ a quiver, and let $\alpha \in \mathbb{N}^{Q_0}$.*
(i) *For $\alpha \notin \Delta^+(Q)$ all representations of $Q$ of dimension vector $\alpha$ are decomposable.*
(ii) *For $\alpha \in \Delta^+_{\mathrm{re}}(Q)$ there exists one and only one indecomposable representation of dimension vector $\alpha$.*

For finite fields and algebraically closed fields the theorem is due to Kac [**3**, Theorems 1 and 2]. As pointed out in the introduction of [**8**], Kac's method of proof shows that the above theorem holds for fields of characteristic $p$. The proof for fields of characteristic zero is due to Schofield [**8**, Theorem 9].

For a given positive real root $\alpha$ for $Q$ the unique indecomposable representation (up to isomorphism) of dimension vector $\alpha$ is denoted by $X_\alpha$. By a *real root representation* we mean an $X_\alpha$ for $\alpha$ a positive real root. The simple representation at vertex $i \in Q_0$ is denoted by $S(i)$. By a *simple representation* we always mean an $S(i)$ for some vertex $i \in Q_0$. A *Schur representation* is a representation with $\mathrm{End}_{kQ}(X) = k$. By a *real Schur representation* we mean a real representation that is also a Schur representation. A positive real root is called a *real Schur root* if $X_\alpha$ is a real Schur representation. An indecomposable representation $X$ is called *exceptional* if $\mathrm{Ext}^1_{kQ}(X, X) = 0$.

We complete this section with the following useful formula: if $X, Y$ are representations of $Q$ then we have

$$\dim \mathrm{Hom}_{kQ}(X, Y) - \dim \mathrm{Ext}^1_{kQ}(X, Y) = \langle \underline{\dim}\, X, \underline{\dim}\, Y \rangle.$$

It follows that $\mathrm{Ext}^1_{kQ}(X_\alpha, X_\alpha) = 0$ for $\alpha$ a real Schur root.

## 2. *Proof of Theorem* A

Let $Q$ be a quiver with vertex set $Q_0$ and arrow set $Q_1$. Moreover, let $i \in Q_0$ be a vertex of $Q$ and let $X$ be a representation of $Q$. Note that we consider only quivers without loops. For a given subset $A \subseteq H^Q(i)$ we define the quiver $Q^i_A$ and the representation $X^i_A$ (of the quiver $Q^i_A$) as follows:

$$(Q^i_A)_0 := Q_0 \,\dot\cup\, \{z\} \quad (Q^i_A)_1 := (Q_1 - A) \,\dot\cup\, \{\gamma_a : a \in A\} \,\dot\cup\, \{\delta\}$$

with

$$\begin{aligned} t(\gamma_a) &:= t(a), & h(\gamma_a) &:= z \qquad \forall\, a \in A, \\ t(\delta) &:= z, & h(\delta) &:= i, \end{aligned}$$

(heads and tails for all arrows in $Q_1 - A$ remain unchanged) and

$$(X^i_A)_j := X_j \quad \forall\, j \in Q_0, \qquad (X^i_A)_z := \mathrm{im}\left( \bigoplus_{a \in A} X_{t(a)} \xrightarrow{(X_a)_a} X_i \right) \subset X_i,$$

with maps

$$\begin{aligned} (X^i_A)_q &:= X_q \quad \forall\, q \in Q_1 - A, \\ (X^i_A)_\delta &:= \text{inclusion}, \\ (X^i_A)_{\gamma_a} &:= \hat{X}_a \quad \forall\, a \in A, \end{aligned}$$

where $\hat{X}_a : X_{t(a)} \to (X^i_A)_z$ is the unique linear map with $(X^i_A)_\delta \circ \hat{X}_a = X_a$.

The construction above gives a functor $F^i_A : \mathrm{rep}_k\, Q \to \mathrm{rep}_k\, Q^i_A$, defined as follows:

$$\begin{aligned} F^i_A : \mathrm{Ob}(\mathrm{rep}_k\, Q) &\longrightarrow \mathrm{Ob}(\mathrm{rep}_k\, Q^i_A), \\ X &\longmapsto X^i_A, \end{aligned}$$

with the obvious definition on morphisms. Moreover, there is a natural functor ${}^i_A G : \mathrm{rep}_k\, Q^i_A \to \mathrm{rep}_k\, Q$, defined by

$$\begin{aligned} {}^i_A G : \mathrm{Ob}(\mathrm{rep}_k\, Q^i_A) &\longrightarrow \mathrm{Ob}(\mathrm{rep}_k\, Q) \\ X &\longmapsto {}^i_A G(X), \end{aligned}$$

with

$$({}^i_A G(X))_j := X_j \quad \forall\, j \in Q_0,$$

and maps

$$\begin{aligned}({}^{i}_{A}G(X))_q &:= X_q \qquad \forall\, q \in Q_1 - A,\\ ({}^{i}_{A}G(X))_a &:= X_\delta X_{\gamma_a} \quad \forall\, a \in A,\end{aligned}$$

together with the obvious definition on morphisms. The functor ${}^{i}_{A}G$ is left-adjoint to the functor $F^i_A$, and ${}^{i}_{A}G \circ F^i_A$ is naturally isomorphic to the identity functor on $\mathrm{rep}_k\, Q$.

We get the following useful lemma.

LEMMA 2.1. *Let $Q$ be a quiver with vertex set $Q_0$ and arrow set $Q_1$. Moreover, let $i \in Q_0$ be a vertex, and let $X$ be a representation of $Q$. If $X$ is indecomposable, then so is $F^i_A(X) = X^i_A$ for every subset of $A \subseteq H^Q(i)$.*

*Proof.* Assume that $X^i_A = F^i_A(X) \cong U \oplus V$, then $X \cong {}^{i}_{A}G \circ F^i_A(X) \cong {}^{i}_{A}G(U) \oplus {}^{i}_{A}G(V)$. By assumption $X$ is indecomposable, and so without loss of generality we can assume that ${}^{i}_{A}G(U) = 0$. Hence,

$$0 = \mathrm{Hom}_{kQ}({}^{i}_{A}G(U), X) = \mathrm{Hom}_{kQ^i_A}(U, F^i_A X) = \mathrm{Hom}_{kQ^i_A}(U, U \oplus V),$$

which is possible only in the case $U = 0$. This proves the assertion. $\square$

We are now able to prove the main theorem of this paper.

THEOREM A. *Let $Q$ be a quiver, and let $\alpha$ be a positive real root for $Q$. The unique indecomposable representation of dimension vector $\alpha$ is of maximal rank type.*

*Proof.* Let $\alpha$ be a real root for $Q$, and let $X_\alpha$ be the unique indecomposable representation of $Q$ of dimension vector $\alpha$. Moreover, let $i \in Q_0$ and let $A \subset H^Q(i)$. We have to show that the map

$$\bigoplus_{a\in A} X_{t(a)} \xrightarrow{(X_a)_a} X_i$$

has maximal rank. This is equivalent to showing that

$$\dim(X^i_A)_z = \min\left\{\sum_{a\in A}\alpha[t(a)], \alpha[i]\right\}.$$

The representation $X^i_A$ of $Q^i_A$ is indecomposable by Lemma 2.1. It follows from Theorem 1.2 that $\underline{\dim}\, X^i_A \in \Delta^+(Q^i_A)$. Hence, by Lemma 1.1, $\langle\hat\alpha,\hat\alpha\rangle \leqslant 1$, where $\hat\alpha := \underline{\dim}\, X^i_A$. We have

$$\begin{aligned}\langle\hat\alpha,\hat\alpha\rangle &= \underbrace{\langle\alpha,\alpha\rangle}_{=1} + \sum_{a\in A}\alpha[t(a)]\alpha[i] + \hat\alpha[z]^2 - \hat\alpha[z]\hat\alpha[i] - \sum_{a\in A}\hat\alpha[t(\gamma_a)]\hat\alpha[z]\\ &= 1 + \left(\hat\alpha[z] - \sum_{a\in A}\alpha[t(a)]\right)\cdot(\hat\alpha[z] - \alpha[i]) \leqslant 1,\end{aligned}$$

and hence

$$\left(\hat\alpha[z] - \sum_{a\in A}\alpha[t(a)]\right)\cdot(\hat\alpha[z] - \alpha[i]) \leqslant 0.$$

However, we clearly have $\hat{\alpha}[z] \leqslant \min\left\{\sum_{a\in A}\alpha[t(a)], \alpha[i]\right\}$, by definition of $X^i_A$. This implies that

$$\left(\hat{\alpha}[z] - \sum_{a\in A}\alpha[t(a)]\right)\cdot(\hat{\alpha}[z] - \alpha[i]) = 0;$$

that is, $\hat{\alpha}[z] = \min\left\{\sum_{a\in A}\alpha[t(a)], \alpha[i]\right\}$, and hence

$$\dim(X^i_A)_z = \min\left\{\sum_{a\in A}\alpha[t(a)], \alpha[i]\right\}.$$

This shows that the map $\bigoplus_{a\in A} X_{t(a)} \to X_i$ has maximal rank.

Dually, given the subset $B \subset T^Q(i)$, we wish to show that the map

$$X_i \xrightarrow{(X_b)_b} \bigoplus_{b\in B} X_{h(b)}$$

has maximal rank, this is equivalent to showing that the map

$$\bigoplus_{b\in B} X^*_{h(b)} \xrightarrow{(X^*_b)_b} X^*_i$$

has maximal rank, where $^*$ denotes the vector space dual. This follows from what we have proved above by considering the dual $X^*$ as a representation of the opposite quiver of $Q$. □

## 3. *Application(s): representations of a quiver with three vertices*

In this section we consider the quiver

$$\begin{tikzcd} Q(f,g,h):\ 1 \arrow[r, shift left=2, "\lambda_1"] \arrow[r, shift right=2, "\lambda_f"', "\vdots" description] & 2 \arrow[r, bend left=60, "\mu_1"] \arrow[r, bend left=20, "\mu_g"'] & 3 \arrow[l, bend left=20, "\nu_1"'] \arrow[l, bend left=60, "\nu_h"] \end{tikzcd}$$

with $f, g, h \geqslant 1$.

We define the following subquivers:

$$\begin{tikzcd} Q'(f):\ 1 \arrow[r, shift left=2, "\lambda_1"] \arrow[r, shift right=2, "\lambda_f"', "\vdots" description] & 2 \end{tikzcd}$$

and

$$\begin{tikzcd} Q''(g,h):\ 2 \arrow[r, bend left=60, "\mu_1"] \arrow[r, bend left=20, "\mu_g"'] & 3 \arrow[l, bend left=20, "\nu_1"'] \arrow[l, bend left=60, "\nu_h"] \end{tikzcd}.$$

The quiver $Q(1,1,1)$ is considered by Jensen and Su in [2], where an explicit construction of all real root representations is given. Moreover, it is shown that all real root representations are tree modules, and formulae to compute the dimensions of the endomorphism rings are given. In [7] Ringel extends their results to the quiver $Q(1,g,h)$ $(g,h \geqslant 1)$ by using the universal extension functors introduced in [5].

In this section we consider the general case with $f, g, h \geqslant 1$. We use Ringel's universal extension functors to construct the real root representations of $Q = Q(f,g,h)$.

We briefly discuss the situation for the subquivers $Q'(f)$ and $Q''(g,h)$. The real root representations of the subquiver $Q'(f)$ are preprojective or preinjective modules, for the path algebra $kQ'(f)$, and can be constructed using BGP reflection functors (see [**1**]). It follows that the endomorphism ring of a real root representation of the subquiver $Q'(f)$ is isomorphic to the ground field $k$, and hence real root representations of $Q'(f)$ are real Schur representations.

The subquiver $Q''(g,h)$ is considered by Ringel in [**5**]. It is shown that all real root representations of $Q''(g,h)$ can be constructed using the universal extension functors defined in [**5**, Section 1]. Moreover, formulae to compute the dimensions of the endomorphism rings are given.

We see that the situation is very well understood for the subquivers $Q'(f)$ and $Q''(g,h)$. Therefore we will focus on real root representations with sincere dimension vectors.

## 3.1. *The Weyl group of $Q = Q(f,g,h)$*

Let $W$ be the Weyl group of $Q$. It is generated by the reflections $s_1, s_2$, and $s_3$ subject to the following relations

$$\begin{aligned} s_i^2 &= 1, && i = 1,2,3,\\ s_1s_3 &= s_3s_1, \\ s_1s_2s_1 &= s_2s_1s_2 && \text{if } f = 1. \end{aligned}$$

We define the following elements of the Weyl group ($n \geqslant 0$):

$$\begin{aligned} \zeta_1(n) &= (s_1s_2)^n s_1,\\ \zeta_2(n) &= (s_2s_1)^n s_2,\\ \rho_1(n) &= (s_1s_2)^n,\\ \rho_2(n) &= (s_2s_1)^n, \end{aligned}$$

and we set $E := \{\zeta_1(n), \zeta_2(n), \rho_1(n), \rho_2(n) : n \geqslant 0\}$.

LEMMA 3.1. *Every element $w \in W - E$ can be written in the form*

$$w = \chi_m s_3 \chi_{m-1} s_3 \chi_{m-2} s_3 \cdot \ldots \cdot s_3 \chi_2 s_3 \chi_1, \qquad (*)$$

*for some $m \geqslant 2$, where*

$$\begin{aligned} &\chi_m \in \{\zeta_1(n) : n \geqslant 1\} \cup \{\zeta_2(n) : n \geqslant 0\} \cup \{1\},\\ &\chi_j \in \{\zeta_1(n) : n \geqslant 1\} \cup \{\zeta_2(n) : n \geqslant 0\}, && j = 2, \ldots, m-1,\\ &\chi_1 \in E. \end{aligned}$$

*If $f = 1$ then $w$ can be written in the form $(*)$ with only $\zeta_1(1) = \zeta_2(1)$, $\rho_1(1)$, and $\rho_2(1)$ occurring.*

*Proof.* Let $w \in W - E$. Clearly, we can write $w$ in the form

$$w = \chi'_m s_3 \chi'_{m-1} s_3 \chi'_{m-2} s_3 \cdot \ldots \cdot s_3 \chi'_2 s_3 \chi'_1,$$

with $m \geqslant 2$, $\chi'_j \in E$ for $j = 1, \ldots, m$, $\chi'_{m-1}, \ldots, \chi'_2 \notin \{1, s_1\}$, and $\chi'_m \neq s_1$. We modify the elements $\chi'_j$ to get a word of the form $(*)$. Let $2 \leqslant j \leqslant m$; we consider five cases and modify $\chi'_j$ appropriately.

(i) $\chi'_j = 1$. We set $\chi_j := \chi'_j$ and $\chi''_{j-1} = \chi'_{j-1}$. This case requires $j = m$.
(ii) $\chi'_j = \zeta_1(n)$ for $n \geqslant 1$. We set $\chi_j := \chi'_j$ and $\chi''_{j-1} := \chi'_{j-1}$.
(iii) $\chi'_j = \zeta_2(n)$ for $n \geqslant 0$. We set $\chi_j := \chi'_j$ and $\chi''_{j-1} := \chi'_{j-1}$.

(iv) $\chi'_j = \rho_1(n)$ for $n \geqslant 1$. We set $\chi_j := \zeta_1(n)$ and $\chi''_{j-1} := s_1\chi'_{j-1}$.
(v) $\chi'_j = \rho_2(n)$ for $n \geqslant 1$. We set $\chi_j := \zeta_2(n-1)$ and $\chi''_{j-1} := s_1\chi'_{j-1}$.

Now we have

$$\begin{aligned} w &= \chi'_m s_3 \chi'_{m-1} s_3 \cdot \ldots \cdot s_3 \chi'_j s_3 \chi'_{j-1} s_3 \cdot \ldots \cdot s_3 \chi'_2 s_3 \chi'_1 \\ &= \chi'_m s_3 \chi'_{m-1} s_3 \cdot \ldots \cdot s_3 \chi_j s_3 \chi''_{j-1} s_3 \cdot \ldots \cdot s_3 \chi'_2 s_3 \chi'_1, \end{aligned}$$

with $\chi_j$ of the desired form and $\chi''_{j-1} \in E$. The result follows by descending induction on $j$. □

Remark 3.2. (i) For a given $w \in W$ the previous proof gives an algorithm to rewrite $w$ in the form $(*)$.

(ii) We adhere to the following convention: in the case $f = 1$ we assume that $n \leqslant 1$ in every occurrence of $\zeta_1(n)$, $\zeta_2(n)$, $\rho_1(n)$, and $\rho_2(n)$. Cases in which $n \geqslant 2$ is assumed do not apply to the case $f = 1$.

### 3.2. *Universal extension functors*

In this section we recall some of the results from [**5**] and prove the key lemmas, which will be used in the next section to construct the real root representations of $Q$.

We fix a representation $S$ with $\operatorname{End}_{kQ} S = k$ and $\operatorname{Ext}^1_{kQ}(S,S) = 0$. In analogy to [**5**, Section 1], we define the following subcategories of $\operatorname{rep}_k Q$. Let $\mathfrak{M}^S$ be the full subcategory of all modules $X$ with $\operatorname{Ext}^1_{kQ}(S,X) = 0$ such that, in addition, $X$ has no direct summand that can be embedded into some direct sum of copies of $S$. Similarly, let $\mathfrak{M}_S$ be the full subcategory of all modules $X$ with $\operatorname{Ext}^1_{kQ}(X,S) = 0$ such that, in addition, no direct summand of $X$ is a quotient of a direct sum of copies of $S$. Finally, let $\mathfrak{M}^{-S}$ be the full subcategory of all modules $X$ with $\operatorname{Hom}_{kQ}(X,S) = 0$, and let $\mathfrak{M}_{-S}$ be the full subcategory of all modules $X$ with $\operatorname{Hom}_{kQ}(S,X) = 0$. Moreover, we consider

$$\mathfrak{M}^S_S = \mathfrak{M}^S \cap \mathfrak{M}_S, \quad \mathfrak{M}^{-S}_{-S} = \mathfrak{M}^{-S} \cap \mathfrak{M}_{-S}.$$

According to [**5**, Propositions 1 and 1* and Proposition 2], we have the following equivalences of categories:

$$\begin{aligned} \overline{\sigma}_S &: \mathfrak{M}^{-S} \longrightarrow \mathfrak{M}^S/S, \\ \underline{\sigma}_S &: \mathfrak{M}_{-S} \longrightarrow \mathfrak{M}_S/S, \\ \sigma_S &: \mathfrak{M}^{-S}_{-S} \longrightarrow \mathfrak{M}^S_S/S, \end{aligned}$$

where $\mathfrak{M}^S/S$ denotes the quotient category of $\mathfrak{M}^S$ modulo the maps that factor through direct sums of copies of $S$, and similarly for $\mathfrak{M}_S/S$ and $\mathfrak{M}^S_S/S$.

In the following, we briefly discuss how these functors and their inverses operate on objects. The functor $\overline{\sigma}_S$ is given by the following construction. Let $X \in \mathfrak{M}^{-S}$ and let $E_1, \ldots, E_r$ be a basis of the $k$-vector space $\operatorname{Ext}^1_{kQ}(S,X)$. Consider the exact sequence $E$ given by the elements $E_1, \ldots, E_r$:

$$E : 0 \longrightarrow X \longrightarrow Z \longrightarrow \bigoplus_r S \longrightarrow 0.$$

According to [**5**, Lemma 3], we have $Z \in \mathfrak{M}^S$ and we define $\overline{\sigma}_S(X) := Z$. Now, let $Y \in \mathfrak{M}_{-S}$, and let $E'_1, \ldots, E'_s$ be a basis of the $k$-vector space $\operatorname{Ext}^1_{kQ}(Y,S)$. Consider the exact sequence $E'$ given by $E'_1, \ldots, E'_s$:

$$E' : 0 \longrightarrow \bigoplus_s S \longrightarrow U \longrightarrow Y \longrightarrow 0.$$

Then we have $U \in \mathfrak{M}_S$ and we set $\underline{\sigma}_S(Y) := U$. The functor $\sigma_S$ is given by applying both constructions successively.

The inverse $\overline{\sigma}_S^{-1}$ is constructed as follows. Let $X \in \mathfrak{M}^S$ and let $\phi_1, \dots, \phi_r$ be a basis of the $k$-vector space $\mathrm{Hom}_{kQ}(X, S)$. Then by [5, Lemma 2], the sequence

$$0 \longrightarrow X^{-S} \longrightarrow X \xrightarrow{(\phi_i)_i} \bigoplus_r S \longrightarrow 0$$

is exact, where $X^{-S}$ denotes the intersection of the kernels of all maps $X \to S$. We set $\overline{\sigma}_S^{-1}(X) := X^{-S}$. Now, let $Y \in \mathfrak{M}_S$. The inverse $\underline{\sigma}_S^{-1}$ is given by $\underline{\sigma}_S^{-1}(Y) := Y/Y'$, where $Y'$ is the sum of the images of all maps $S \to Y$. The inverse $\sigma_S^{-1}$ is given by applying both constructions successively.

Both construction show that

$$\underline{\dim}\, \sigma_S^{\pm 1}(X) = \underline{\dim}\, X - (\underline{\dim}\, X, \underline{\dim}\, S)\, \underline{\dim}\, S.$$

We have the following proposition.

Proposition 3.3 [5, Propositions 3 and 3*]. *Let $X \in \mathfrak{M}_S^S$. Then*

$$\dim \mathrm{End}_{kQ}\, \sigma_S^{-1}(X) = \dim \mathrm{End}_{kQ}(X) - \langle \underline{\dim}\, X, \underline{\dim}\, S\rangle \cdot \langle \underline{\dim}\, S, \underline{\dim}\, X\rangle.$$

*Let $Y \in \mathfrak{M}_{-S}^{-S}$. Then*

$$\dim \mathrm{End}_{kQ}\, \sigma_S(Y) = \dim \mathrm{End}_{kQ}(Y) + \langle \underline{\dim}\, Y, \underline{\dim}\, S\rangle \cdot \langle \underline{\dim}\, S, \underline{\dim}\, Y\rangle. \tag{1}$$

Definition 3.4. Let $\alpha$ be a real Schur root for $Q$. We define

$$\mathfrak{M}_{-\alpha}^{-\alpha} := \mathfrak{M}_{-X_\alpha}^{-X_\alpha}, \quad \mathfrak{M}_{\alpha}^{\alpha} := \mathfrak{M}_{X_\alpha}^{X_\alpha}, \quad \text{and} \quad \sigma_\alpha := \sigma_{X_\alpha}.$$

To construct real root representations of $Q$ we will reflect, with respect to the following modules $S$: the simple representation $S(3)$ and the real root representations of $Q$ corresponding to certain positive real roots for the subquiver $Q'(f)$. Hence, we will use the functors

$$\sigma_{e_3} : \mathfrak{M}_{-e_3}^{-e_3} \longrightarrow \mathfrak{M}_{e_3}^{e_3}/S(3)$$

and

$$\sigma_\chi : \mathfrak{M}_{-\chi}^{-\chi} \longrightarrow \mathfrak{M}_{\chi}^{\chi}/X_\chi,$$

where $\chi$ denotes a positive real root for the subquiver $Q'(f)$. In order to use these functors, we have to make sure that $\sigma_{e_3}$ and $\sigma_\chi$ can be applied successively; that is, we have to show that

$$\mathfrak{M}_{\chi}^{\chi} \subset \mathfrak{M}_{-e_3}^{-e_3},$$
$$\mathfrak{M}_{e_3}^{e_3} \subset \mathfrak{M}_{-\chi}^{-\chi}.$$

In general these inclusions do not hold. The following lemmas, however, show that under certain assumptions the functors can be applied successively. We recall a key lemma from [5].

Lemma 3.5 [5, Lemma 4]. *Let $S, T$ be modules, where $T$ is simple.*
(i) *If $\mathrm{Ext}_{kQ}^1(S, T) \neq 0$, then $\mathfrak{M}^S \subset \mathfrak{M}^{-T}$.*
(ii) *If $\mathrm{Ext}_{kQ}^1(T, S) \neq 0$, then $\mathfrak{M}_S \subset \mathfrak{M}_{-T}$.*

Corollary 3.6. *We have*

$$\mathfrak{M}_{e_2}^{e_2} \subset \mathfrak{M}_{-e_3}^{-e_3},$$
$$\mathfrak{M}_{e_3}^{e_3} \subset \mathfrak{M}_{-e_2}^{-e_2}.$$

Corollary 3.6 shows that $\sigma_{e_2}$ and $\sigma_{e_3}$ can be applied successively. In the following two lemmas we consider the situation when $\chi$ is a sincere real root for $Q' = Q'(f)$. The maximal rank type property of real root representations ensures that the situation is suitably well behaved.

Lemma 3.7. *Let $\chi$ be a sincere real root for $Q'$. Then we have $\mathfrak{M}_{\chi}^{\chi} \subset \mathfrak{M}_{-e_3}^{-e_3}$.*

*Proof.* We have $\langle \chi, e_3 \rangle = -g \cdot \chi[2] < 0$ and $\langle e_3, \chi \rangle = -h \cdot \chi[2] < 0$. Thus, Lemma 3.5 applies and we deduce that $\mathfrak{M}_{\chi}^{\chi} \subset \mathfrak{M}_{-e_3}^{-e_3}$. $\square$

Lemma 3.8. *Let $\chi$ be a sincere real root for $Q'$, and let $Y \in \mathfrak{M}_{e_3}^{e_3} - \{S(1)\}$ be a real root representation. Then we have $Y \in \mathfrak{M}_{-\chi}^{-\chi}$.*

*Proof.* Let $Y \in \mathfrak{M}_{e_3}^{e_3} - \{S(1)\}$ be a real root representation. Since

$$\mathrm{Ext}_{kQ}^1(Y, S(3)) = 0 = \mathrm{Ext}_{kQ}^1(S(3), Y)$$

we get $\langle \underline{\dim}\, Y, e_3 \rangle \geqslant 0$ and $\langle e_3, \underline{\dim}\, Y \rangle \geqslant 0$. This implies that

$$\langle \underline{\dim}\, Y, e_3 \rangle = -g \cdot \underline{\dim}\, Y[2] + \underline{\dim}\, Y[3] \geqslant 0,$$
$$\langle e_3, \underline{\dim}\, Y \rangle = -h \cdot \underline{\dim}\, Y[2] + \underline{\dim}\, Y[3] \geqslant 0,$$

and thus

$$\underline{\dim}\, Y[3] \geqslant g \cdot \underline{\dim}\, Y[2],$$
$$\underline{\dim}\, Y[3] \geqslant h \cdot \underline{\dim}\, Y[2];$$

in particular, $\underline{\dim}\, Y[3] \geqslant \underline{\dim}\, Y[2]$. Since $\underline{\dim}\, Y$ is a positive real root we can apply Theorem A, which implies that the maps $Y_{\mu_i}$ $(i = 1, \ldots, g)$ (of the representation $Y$) are injective and the maps $Y_{\nu_i}$ $(i = 1, \ldots, h)$ are surjective.

Now, let $\phi : X_\chi \to Y$ be a morphism. Clearly, $\phi_3 = 0$. The injectivity of the maps $Y_{\mu_i}$ implies that $\phi_2 = 0$. This, however, implies that $\phi_1 = 0$ since otherwise the intersection of the kernels of the maps $Y_{\lambda_j}$ $(j = 1, \ldots, f)$ would be non-zero. This is nonsense since $Y$ is indecomposable and $Y \neq S(1)$. Hence, $\phi = 0$.

Now, let $\psi : Y \to X_\chi$ be a morphism. Clearly, $\psi_3 = 0$. The surjectivity of the maps $Y_{\nu_i}$ implies that $\psi_2 = 0$. This, however, implies that $\psi_1 = 0$ since otherwise the intersection of the kernels of the maps $(X_\chi)_{\lambda_j}$ $(j = 1, \ldots, f)$ would be non-zero. This is nonsense since $X_\chi$ is indecomposable and $\chi$ is sincere for $Q'$. Hence, $\psi = 0$.

This completes the proof. $\square$

The previous lemma shows the following. Let $X \in \mathfrak{M}_{-e_3}^{-e_3} - \{S(1)\}$ be a real root representation; then we have $\sigma_{e_3}(X) \in \mathfrak{M}_{-\chi}^{-\chi}$, where $\chi$ is a sincere real root for $Q'$.

### 3.3. *Construction of real root representations for $Q = Q(f, g, h)$*

In this section we construct the real root representations for $Q$ by using universal extension functors together with the results of the last section.

For $n \geqslant 1$ we define the functors

$$\sigma_{\zeta_1(n)} := \begin{cases} \sigma_{\rho_1(\frac{n}{2})(e_1)} & \text{if } n \text{ is even,} \\ \sigma_{\zeta_1(\frac{n-1}{2})(e_2)} & \text{if } n \text{ is odd,} \end{cases}$$

and for $n \geqslant 0$ we define the functors

$$\sigma_{\zeta_2(n)} := \begin{cases} \sigma_{\rho_2(\frac{n}{2})(e_2)} & \text{if } n \text{ is even,} \\ \sigma_{\zeta_2(\frac{n-1}{2})(e_1)} & \text{if } n \text{ is odd.} \end{cases}$$

REMARK 3.9. For $n \geqslant 1$ we clearly have
(i) $\rho_1(n)(e_3) = \zeta_1(n)(e_3)$,
(ii) $\rho_2(n)(e_3) = \zeta_2(n-1)(e_3)$.

LEMMA 3.10. *Let $\alpha$ be a positive non-simple real root of the following form:*
(i) *$\alpha = \chi(e_j)$ with $j \in \{1, 2\}$ and $\chi \in E$;*
(ii) *$\alpha = \chi(e_3)$ with $\chi \in E$.*
*Then the unique indecomposable representation of dimension vector $\alpha$ has the following properties.*
(i) *$X_\alpha$ is an indecomposable representation of the subquiver $Q'(f)$, and hence can be constructed using* BGP *reflection functors. Moreover,* $\mathrm{End}_{kQ}\, X_\alpha = k$ *and* $X_\alpha \in \mathfrak{M}^{-e_3}_{-e_3}$;
(ii) *$X_\alpha$ can be constructed using the functors $\sigma_{\zeta_i(n)}$ $(i = 1, 2)$ and $X_\alpha \in \mathfrak{M}^{-e_3}_{-e_3}$.*

*Proof.* (i) The statement is clear.
(ii) If $\alpha = \zeta_i(n)(e_3)$ $(i = 1, 2)$ then $X_\alpha = \sigma_{\zeta_i(n)} S(3)$ and $X_\alpha \in \mathfrak{M}^{-e_3}_{-e_3}$ by Lemma 3.7 or Corollary 3.6 in the case $\alpha = \zeta_2(0)$. If $\alpha = \rho_i(n)(e_3)$ $(i = 1, 2)$ we use the previous remark to reduce to the case that we have just considered. □

We are now able to state and prove a more explicit version of Theorem B.

THEOREM 3.11. *Let $\alpha$ be a sincere real root for $Q$. Then $\alpha$ is of the form*
(i) *$\alpha = \zeta_i(n)(e_3)$ with $i \in \{1, 2\}$ and $n \geqslant 1$, or*
(ii) *$\alpha = w(e_j)$ with $j \in \{1, 2, 3\}$ and $w = \chi_m s_3 \chi_{m-1} s_3 \chi_{m-2} s_3 \cdot \ldots \cdot s_3 \chi_2 s_3 \chi_1$ of the form $(*)$ with $\chi_1(e_j) \neq e_1$.*
*The corresponding unique indecomposable representation of dimension vector $\alpha$ can be constructed as follows:*
(i) *$X_{\zeta_i(n)(e_3)} = \sigma_{\zeta_i(n)} S(3)$;*
(ii) *$X_\alpha = \sigma_{\chi_m} \sigma_{e_3} \sigma_{\chi_{m-1}} \ldots \sigma_{\chi_2} \sigma_{e_3} X_{\chi_1(e_j)}$, where $X_{\chi_1(e_j)}$ denotes the unique indecomposable of dimension vector $\chi_1(e_j)$: constructed in Lemma* 3.10.

*Proof.* (i) This follows from Lemma 3.10.
(ii) It follows from Lemma 3.10 that $X_{\chi_1(e_j)} \in \mathfrak{M}^{-e_3}_{-e_3}$, and hence $\sigma_{e_3}$ can be applied. Moreover, by Corollary 3.6, Lemma 3.7, and Lemma 3.8 we have

$$\begin{aligned} X_\beta \in \mathfrak{M}^{e_3}_{e_3} - \{S(1)\},\ \beta \text{ real root} \quad &\Longrightarrow \quad X_\beta \in \mathfrak{M}^{-\chi}_{-\chi}, \\ \mathfrak{M}^{\chi}_{\chi} \quad &\subset \quad \mathfrak{M}^{-e_3}_{-e_3}, \end{aligned}$$

where $\chi$ is a positive real root for the subquiver $Q'(f)$ not equal to $e_1$. This completes the proof. □

REMARK 3.12. Using formula (1) together with Theorem B one can easily compute the dimension of the endomorphism ring of a sincere real root representation of $Q$.

### 3.4. *Real root representations of $Q = Q(f, g, h)$ are tree modules*

In this section we show that real root representations of $Q = Q(f, g, h)$ are tree modules. We recall some definitions from [6]. Let $Q$ be an arbitrary quiver with vertex set $Q_0$ and arrow set $Q_1$. Moreover, let $X \in \operatorname{rep}_k Q$ be a representation of $Q$ with $\underline{\dim}\, X = d$. We denote by $\mathfrak{B}_i$ a fixed basis of the vector space $X_i$ ($i \in Q_0$) and we set $\mathfrak{B} = \bigcup_{i \in Q_0} \mathfrak{B}_i$. The set $\mathfrak{B}$ is called *a basis* of $X$. We fix a basis $\mathfrak{B}$ of $X$. For a given arrow $a : i \to j$ we can write $X_a$ as a $d[j] \times d[i]$-matrix $X_{a,\mathfrak{B}}$ with rows indexed by $\mathfrak{B}_j$ and with columns indexed by $\mathfrak{B}_i$. We denote by $X_{a,\mathfrak{B}}(x, x')$ the corresponding matrix entry, where $x \in \mathfrak{B}_i$, $x' \in \mathfrak{B}_j$; the entries $X_{a,\mathfrak{B}}(x, x')$ are defined by $X_a(x) = \sum_{x' \in \mathfrak{B}_j} X_{a,\mathfrak{B}}(x, x')\, x'$. The *coefficient quiver* $\Gamma(X, \mathfrak{B})$ of $X$ with respect to $\mathfrak{B}$ is defined as follows: the vertex set of $\Gamma(X, \mathfrak{B})$ is the set $\mathfrak{B}$ of basis elements of $X$, and there is an arrow $(a, x, x')$ between two basis elements $x \in \mathfrak{B}_i$ and $x' \in \mathfrak{B}_j$, provided that $X_{a,\mathfrak{B}}(x, x') \neq 0$ for $a : i \to j$.

DEFINITION 3.13 (Tree module; see [6]). We call an indecomposable representation $X$ of $Q$ a *tree module* if there exists a basis $\mathfrak{B}$ of $X$ such that the coefficient quiver $\Gamma(X, \mathfrak{B})$ is a tree.

The following remarkable theorem is due to Ringel.

THEOREM 3.14 [6]. *Let $k$ be a field and let $Q$ be a quiver. Any exceptional representation of $Q$ over $k$ is a tree module.*

We briefly recall the construction of extensions of representations of quivers, as discussed in [6, Section 3; 4, Section 2.1].

Let $Q$ be a quiver with vertex set $Q_0$ and arrow set $Q_1$. Moreover, let $X$ and $X'$ be representations of $Q$. The group $\operatorname{Ext}^1_{kQ}(X, X')$ can be constructed as follows. Let

$$C^0(X, X') := \bigoplus_{i \in Q_0} \operatorname{Hom}_k(X_i, X'_i),$$
$$C^1(X, X') := \bigoplus_{a \in Q_1} \operatorname{Hom}_k(X_{t(a)}, X'_{h(a)}).$$

We define the map

$$\begin{aligned} \delta_{XX'} : C^0(X, X') &\longrightarrow C^1(X, X'), \\ (\phi_i)_i &\longmapsto (\phi_j X_a - X'_a \phi_i)_{a:i \to j}. \end{aligned}$$

The importance of $\delta_{XX'}$ is given by the following lemma.

LEMMA 3.15 [4, Section 2.1, Lemma]. *We have* $\ker \delta_{XX'} = \operatorname{Hom}_{kQ}(X, X')$ *and* $\operatorname{coker} \delta_{XX'} = \operatorname{Ext}^1_{kQ}(X, X')$.

The following proof follows closely the arguments given in [6, Sections 3 and 6].

LEMMA 3.16. *Let $Q$ be a quiver. Let $S$ be a representation with $\operatorname{End}_{kQ} S = k$ and $\operatorname{Ext}^1_{kQ}(S, S) = 0$. Moreover, let $X \in \mathfrak{M}^{-S}$ (resp. $X \in \mathfrak{M}_{-S}$) be a tree module. Then the* *representation* $\overline{\sigma}_S(X)$ (resp. $\underline{\sigma}_S(X)$) *is a tree module. In particular, let* $X \in \mathfrak{M}_{-S}^{-S}$ *be a tree module; then* $\sigma_S(X)$ *is a tree module.*

*Proof.* We consider only the situation for the functor $\overline{\sigma}_S$. The situation for $\underline{\sigma}_S$ is analogous. Since $\sigma_S$ is given by applying $\overline{\sigma}_S$ and $\underline{\sigma}_S$ successively, the second assertion follows from the first.

We recall the construction of $\overline{\sigma}_S(X)$. Let $E_1, \dots, E_r$ be a basis of the $k$-vector space $\mathrm{Ext}^1_{kQ}(S, X)$. Consider the exact sequence $E$ given by the elements $E_1, \dots, E_r$:

$$E : 0 \longrightarrow X \longrightarrow Z \longrightarrow \bigoplus_r S \longrightarrow 0; \tag{+}$$

then we have $\overline{\sigma}_S(X) = Z$. First of all, we note that $Z$ is indecomposable since $\overline{\sigma}_S : \mathfrak{M}^{-S} \to \mathfrak{M}^S/S$ defines an equivalence of categories. Moreover, by Theorem 3.14 the representation $S$ is a tree module. Thus, we can choose a basis $\mathfrak{B}_X$ of $X$ and a basis $\mathfrak{B}_S$ of $S$ such that the corresponding coefficient quivers $\Gamma(X, \mathfrak{B}_X)$ and $\Gamma(S, \mathfrak{B}_S)$ are trees. We set $d_X := \sum_{i \in Q_0} \dim X_i$ (dimension of $X$) and $d_S := \sum_{i \in Q_0} \dim S_i$ (dimension of $S$). Since $X$ and $S$ are indecomposable representations the corresponding coefficient quivers are connected, and hence $\Gamma(X, \mathfrak{B}_X)$ has $d_X - 1$ arrows and $\Gamma(S, \mathfrak{B}_S)$ has $d_S - 1$ arrows.

Let $a \in Q_1$. For given $1 \leqslant s \leqslant t(a)$ and $1 \leqslant t \leqslant h(a)$ we denote by

$$M_{SX}(a, s, t) \in \mathrm{Hom}_k(S_{t(a)}, X_{h(a)})$$

the matrix unit with entry 1 in the column with index $s$ and the row with index $t$, and zeros elsewhere. The set

$$H_{SX} := \{M_{SX}(a, s, t) : a \in Q_1, 1 \leqslant s \leqslant t(a), 1 \leqslant t \leqslant h(a)\}$$

is clearly a basis of $C^1(S, X)$. Hence, we can choose a subset

$$\Phi := \{M_{SX}(a_i, s_i, t_i) : 1 \leqslant i \leqslant r\} \subset H_{SX}$$

such that $\Phi \oplus \mathrm{im}\, \delta_{SX} = C^1(S, X)$, which implies that the residue classes $\phi + \mathrm{im}\, \delta_{SX}$ ($\phi \in \Phi$) form a basis of $\mathrm{Ext}^1_{kQ}(S, X)$; these elements are responsible for obtaining the extension (+).

We are now able to describe the matrices of the representation $Z$ with respect to the basis $\mathfrak{B}_X \cup \mathfrak{B}_S$. Let $b \in Q_1$. The matrix $Z_b$ has the form

$$Z_b = \begin{bmatrix} X_b & N(b,1) & \cdots & N(b,r) \\ & S_b & & \\ & & \ddots & \\ & & & S_b \end{bmatrix}$$

with all other entries equal to zero and

$$N(b, i) = \begin{cases} M(a_i, s_i, t_i) & \text{if } b = a_i, \\ \mathbf{0} & \text{otherwise,} \end{cases}$$

where $\mathbf{0}$ denotes the zero matrix of the appropriate size. This explicit description allows us to count the overall number of non-zero entries in the matrices of the representation $Z$ with respect to the basis $\mathfrak{B}_X \cup \mathfrak{B}_S$: this number equals the number of arrows of the coefficient quiver $\Gamma(Z, \mathfrak{B}_X \cup \mathfrak{B}_S)$. We easily see that there are

$$(d_X - 1) + r(d_S - 1) + |\Phi| = d_X + r d_S - 1 = \sum_{i \in Q_0} \dim Z_i - 1$$

non-zero entries.

Now, since $Z$ is indecomposable, the coefficient quiver $\Gamma(Z, \mathfrak{B}_X \cup \mathfrak{B}_S)$ is connected, and hence $\Gamma(Z, \mathfrak{B}_X \cup \mathfrak{B}_S)$ is a tree. $\square$

The previous lemma and Theorem B give the following result.

Proposition 3.17. *Let $\alpha$ be a positive real root for $Q = Q(f, g, h)$ $(f, g, h \geqslant 1)$. Then the representation $X_\alpha$ is a tree module.*

*Proof.* Representations of the subquiver $Q' = Q'(f)$ $(f \geqslant 1)$ are exceptional representations; that is, they have no self-extensions, and hence are tree modules by Theorem 3.14.

Now, let $X$ be a representation of $Q$ with $\underline{\dim}\, X[3] \neq 0$. Then, by Theorem B (or the results in [**5**] if $X$ is not sincere), $X$ can be constructed by using universal extension functors starting from a simple representation or a real root representation of the subquiver $Q'$, which is a tree module.

By Lemma 3.16 the image of a tree module under the functor $\sigma_S$ is again a tree module. This proves the claim. □

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## B.2 M. Wiedemann, *A remark on the constructibility of real root representations using universal extension functors*, Preprint, arXiv:0802.2803 [math.RT]

# A REMARK ON THE CONSTRUCTIBILITY OF REAL ROOT REPRESENTATIONS OF QUIVERS USING UNIVERSAL EXTENSION FUNCTORS

MARCEL WIEDEMANN

Abstract. In this paper we consider the following question: Is it possible to construct all real root representations of a given quiver $Q$ by using universal extension functors, starting with a real Schur representation? We give a concrete example answering this question negatively.

## 0. Introduction

Let $k$ be a field and let $Q$ be a (finite) quiver. We fix a representation $S$ with $\mathsf{End}_{kQ}\, S = k$ and $\mathsf{Ext}^1_{kQ}(S,S) = 0$. In analogy to [3, Section 1] we consider the following subcategories of $\mathsf{rep}_k\, Q$. Let $\mathfrak{M}^S$ be the full subcategory of all modules $X$ with $\mathsf{Ext}^1_{kQ}(S,X) = 0$ such that, in addition, $X$ has no direct summand which can be embedded into some direct sum of copies of $S$. Similarly, let $\mathfrak{M}_S$ be the full subcategory of all modules $X$ with $\mathsf{Ext}^1_{kQ}(X,S) = 0$ such that, in addition, no direct summand of $X$ is a quotient of a direct sum of copies of $S$. Finally, let $\mathfrak{M}^{-S}$ be the full subcategory of all modules $X$ with $\mathsf{Hom}_{kQ}(X,S) = 0$, and let $\mathfrak{M}_{-S}$ be the full subcategory of all modules $X$ with $\mathsf{Hom}_{kQ}(S,X) = 0$. Moreover, we consider

$$\mathfrak{M}^S_S = \mathfrak{M}^S \cap \mathfrak{M}_S, \quad \mathfrak{M}^{-S}_{-S} = \mathfrak{M}^{-S} \cap \mathfrak{M}_{-S}.$$

According to [3, Proposition 1 & 1* and Proposition 2], we have the following equivalences of categories

$$\begin{aligned}
\overline{\sigma}_S &: \mathfrak{M}^{-S} \to \mathfrak{M}^S/S,\\
\underline{\sigma}_S &: \mathfrak{M}_{-S} \to \mathfrak{M}_S/S,\\
\sigma_S &: \mathfrak{M}^{-S}_{-S} \to \mathfrak{M}^S_S/S,
\end{aligned}$$

where $\mathfrak{M}^S/S$ denotes the quotient category of $\mathfrak{M}^S$ modulo the maps which factor through direct sums of copies of $S$, similarly for $\mathfrak{M}_S/S$ and $\mathfrak{M}^S_S/S$. We call the functor $\sigma_S$ *universal extension functor*. A brief description of these functors is given in Section 1. This paper is dedicated to the following question.

**Question** ($\star$)**.** *Let $\alpha$ be a positive non-Schur real root for $Q$ and let $X_\alpha$ be the unique indecomposable representation of dimension vector $\alpha$.*

*Does there exist a sequence of real Schur roots $\beta_1, \ldots, \beta_n$ ($n \geq 2$) such that*

$$X_\alpha = \sigma_{X_{\beta_n}} \cdot \ldots \cdot \sigma_{X_{\beta_2}}(X_{\beta_1}) \quad ?$$

*Here, $X_{\beta_i}$ denotes the unique indecomposable representation of dimension vector $\beta_i$.*

One might reformulate the above question as follows. Is it possible to construct all real root representations of $Q$ using universal extension functors, starting with a real Schur representation?

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One of the nice facts about the universal extension functor $\sigma_S$ is that it allows one to keep track of certain properties of representations. For instance, the functor $\sigma_S$ preserves indecomposable tree representations [7, Lemma 3.16] (for a definition of "tree representation" and background results we refer the reader to [4, Introduction]) and, moreover, if we apply the functor $\sigma_S$ to a representation of known endomorphism ring dimension, we can easily compute the dimension of the endomorphism ring of the resulting representation [3, Proposition 3 & 3*]. Hence, if $X_\alpha = \sigma_{X_{\beta_n}} \cdot \ldots \cdot \sigma_{X_{\beta_2}}(X_{\beta_1})$ with $\beta_i$ $(i = 1, \ldots, n)$ real Schur roots, then $X_\alpha$ is a tree representation and one can easily compute $\dim \mathsf{End}_{kQ} X_\alpha$.

Question $(\star)$ was first answered affirmatively by Ringel [3, Section 2] for the quiver

$$Q(g,h):\ \begin{tikzcd} 1 \arrow[r, bend left=60, "\mu_1"] \arrow[r, bend left=20, "\mu_g"'] & 2 \arrow[l, bend left=20, "\nu_1"'] \arrow[l, bend left=60, "\nu_h"] \end{tikzcd},$$

with $g, h \geq 1$. In [7, Theorem B] Question $(\star)$ was answered affirmatively for the quiver

$$Q(f,g,h):\ \begin{tikzcd} 1 \arrow[r, shift left, "\lambda_1"] \arrow[r, shift right, "\lambda_f"'] & 2 \arrow[r, bend left=60, "\mu_1"] \arrow[r, bend left=20, "\mu_g"'] & 3 \arrow[l, bend left=20, "\nu_1"'] \arrow[l, bend left=60, "\nu_h"] \end{tikzcd},$$

with $f, g, h \geq 1$. More examples of real root representations which can be constructed using universal extension functors can be found in [8, Appendix].

Hence, there are quivers for which Question $(\star)$ can be answered affirmatively. The question is, can it be answered affirmatively in general? Unfortunately the answer is negative in general.

**Answer** (to Question $(\star)$)**.** *In Section 2 we give a concrete example answering Question $(\star)$ negatively.*

This paper is organized as follows. In Section 1 we discuss further notation and background results and in Section 2 we describe an example answering Question $(\star)$ negatively.

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## 1. Further Notation and Background Results

Let $k$ be a field. Let $Q$ be a finite quiver, i.e. an oriented graph with finite vertex set $Q_0$ and finite arrow set $Q_1$ together with two functions $h, t : Q_1 \to Q_0$ assigning head and tail to each arrow $a \in Q_1$. A representation $X$ of $Q$ is given by a vector space $X_i$ (over $k$) for each vertex $i \in Q_0$ together with a linear map $X_a : X_{t(a)} \to X_{h(a)}$ for each arrow $a \in Q_1$. Let $X$ and $Y$ be two representations of $Q$. A homomorphism $\phi : X \to Y$ is given by linear maps $\phi_i : X_i \to Y_i$ such that for each arrow $a \in Q_1$, $a : i \to j$ say, the square

$$\begin{tikzcd} X_i \arrow[r, "X_a"] \arrow[d, "\phi_i"'] & X_j \arrow[d, "\phi_j"] \\ Y_i \arrow[r, "Y_a"] & Y_j \end{tikzcd}$$

commutes.

A dimension vector for $Q$ is given by an element of $\mathbb{N}^{Q_0}$. We will write $e_i$ for the coordinate vector at vertex $i$ and by $\alpha[i]$, $i \in Q_0$, we denote the $i$-th coordinate of $\alpha \in \mathbb{N}^{Q_0}$. We can partially order $\mathbb{N}^{Q_0}$ via $\alpha \geq \beta$ if $\alpha[i] \geq \beta[i]$ for all $i \in Q_0$. We define $\alpha > \beta$ to mean $\alpha \geq \beta$ and $\alpha \neq \beta$. If $X$ is a finite dimensional representation, meaning that all vector spaces $X_i$ $(i \in Q_0)$ are finite dimensional, then $\underline{\dim} X = (\dim X_i)_{i \in Q_0}$ is the dimension vector of $X$. Throughout this paper we only consider finite dimensional representations. We denote by $\mathsf{rep}_k\, Q$ the full subcategory with objects the finite dimensional representations of $Q$. The Ringel form on $\mathbb{Z}^{Q_0}$ is defined by

$$\langle \alpha, \beta \rangle = \sum_{i \in Q_0} \alpha[i]\beta[i] - \sum_{a \in Q_1} \alpha[t(a)]\beta[h(a)]$$

Moreover, let $(\alpha, \beta) = \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle$ be its symmetrization.

We say that a vertex $i \in Q_0$ is loop-free if there are no arrows $a : i \to i$. By a quiver without loops we mean a quiver with only loop-free vertices. For a loop-free vertex $i \in Q_0$ the simple reflection $s_i : \mathbb{Z}^{Q_0} \to \mathbb{Z}^{Q_0}$ is defined by

$$s_i(\alpha) := \alpha - (\alpha, e_i)e_i.$$

A simple root is a vector $e_i$ for $i \in Q_0$. The set of simple roots is denoted by $\Pi$. The Weyl group, denoted by $W$, is the subgroup of $\mathrm{GL}(\mathbb{Z}^n)$, where $n = |Q_0|$, generated by the $s_i$. By $\Delta^+_{\mathrm{re}}(Q) := \{\alpha \in W(\Pi) : \alpha > 0\}$ we denote the set of (positive) real roots for $Q$.

We have the following remarkable theorem.

**Theorem 1.1** (Kac [2, Theorem 1 and 2], Schofield [6, Theorem 9])**.** *Let $k$ be a field, $Q$ be a quiver and let $\alpha \in \Delta^+_{re}(Q)$. There exists a unique indecomposable representation (up to isomorphism) of dimension vector $\alpha$.*

For finite fields and algebraically closed fields the theorem is due to Kac [2, Theorem 1 and 2]. As pointed out in the introduction of [6], Kac's method of proof showed that the above theorem holds for fields of characteristic $p$. The proof for fields of characteristic zero is due to Schofield [6, Theorem 9].

For a given positve real root $\alpha$ for $Q$ the unique indecomposable representation (up to isomorphism) of dimension vector $\alpha$ is denoted by $X_\alpha$. By a real root representation we mean an $X_\alpha$ for $\alpha$ a positive real root. A Schur representation is a representation with $\mathsf{End}_{kQ}(X) = k$. By a real Schur representation we mean a real representation which is also a Schur representation. A positive real root is called a real Schur root if $X_\alpha$ is a real Schur representation.

We have the following useful formula: if $X, Y$ are representations of $Q$ then we have

$$\dim \mathsf{Hom}_{kQ}(X, Y) - \dim \mathsf{Ext}^1_{kQ}(X, Y) = \langle \underline{\dim} X, \underline{\dim} Y \rangle.$$

It follows that $\mathsf{Ext}^1_{kQ}(X_\alpha, X_\alpha) = 0$ for $\alpha$ a real Schur root.

1.1. **Universal Extension Functors.** We use this section to describe briefly how the functors

$$\begin{aligned} \overline{\sigma}_S &: \mathfrak{M}^{-S} \to \mathfrak{M}^S/S, \\ \underline{\sigma}_S &: \mathfrak{M}_{-S} \to \mathfrak{M}_S/S, \\ \sigma_S &: \mathfrak{M}^{-S}_{-S} \to \mathfrak{M}^S_S/S, \end{aligned}$$

operate on objects.

The functor $\overline{\sigma}_S$ is given by the following construction: Let $X \in \mathfrak{M}^{-S}$ and let $E_1, \ldots, E_r$ be a basis of the $k$-vector space $\mathsf{Ext}^1_{kQ}(S, X)$. Consider the exact sequence $E$ given by the elements $E_1, \ldots, E_r$

$$E : 0 \to X \to Z \to \bigoplus_r S \to 0.$$

According to [3, Lemma 3] we have $Z \in \mathfrak{M}^S$ and we define $\overline{\sigma}_S(X) := Z$. Now, let $Y \in \mathfrak{M}_{-S}$ and let $E'_1, \ldots, E'_s$ be a basis of the $k$-vector space $\mathsf{Ext}^1_{kQ}(Y, S)$. Consider the exact sequence $E'$ given by $E'_1, \ldots, E'_s$

$$E' : 0 \to \bigoplus_s S \to U \to Y \to 0.$$

Then we have $U \in \mathfrak{M}_S$ and we set $\underline{\sigma}_S(Y) := U$. The functor $\sigma_S$ is given by applying both constructions successively.

The inverse $\overline{\sigma}_S^{-1}$ is constructed as follows: Let $X \in \mathfrak{M}^S$ and let $\phi_1, \ldots, \phi_r$ be a basis of the $k$-vector space $\mathsf{Hom}_{kQ}(X, S)$. Then by [3, Lemma 2] the sequence

$$0 \to X^{-S} \to X \xrightarrow{(\phi_i)_i} \bigoplus_r S \to 0$$

is exact, where $X^{-S}$ denotes the intersection of the kernels of all maps $X \to S$. We set $\overline{\sigma}_S^{-1}(X) := X^{-S}$. Now, let $Y \in \mathfrak{M}_S$. The inverse $\underline{\sigma}_S^{-1}$ is given by $\underline{\sigma}_S^{-1}(Y) := Y/Y'$, where $Y'$ is the sum of the images of all maps $S \to Y$. The inverse $\sigma_S^{-1}$ is given by applying both constructions successively.

Both constructions show that

$$(\dagger) \qquad \underline{\dim}\, \sigma_S^{\pm 1}(X) = \underline{\dim}\, X - (\underline{\dim}\, X, \underline{\dim}\, S) \cdot \underline{\dim}\, S.$$

Moreover, we have the following proposition.

**Proposition 1.2** ([3, Proposition 3 & 3*])**.** *Let $X \in \mathfrak{M}^{-S}_{-S}$. Then*

$$\dim \mathsf{End}_{kQ}\, \sigma_S(X) = \dim \mathsf{End}_{kQ}(X) + \langle \underline{\dim}\, X, \underline{\dim}\, S\rangle \cdot \langle \underline{\dim}\, S, \underline{\dim}\, X\rangle.$$

*Let $Y \in \mathfrak{M}^S_S$. Then*

$$\dim \mathsf{End}_{kQ}\, \sigma_S^{-1}(Y) = \dim \mathsf{End}_{kQ}(Y) - \langle \underline{\dim}\, Y, \underline{\dim}\, S\rangle \cdot \langle \underline{\dim}\, S, \underline{\dim}\, Y\rangle.$$

## 2. A negative and unpleasant example

Let $k$ be a field and let $Q$ be a quiver. We recall Question ($\star$) stated in the introduction.

**Question** ($\star$)**.** *Let $\alpha$ be a positive non-Schur real root for $Q$ and let $X_\alpha$ be the unique indecomposable representation of dimension vector $\alpha$.*

*Does there exist a sequence of real Schur roots $\beta_1, \ldots, \beta_n$ ($n \geq 2$) such that*

$$X_\alpha = \sigma_{X_{\beta_n}} \cdot \ldots \cdot \sigma_{X_{\beta_2}}(X_{\beta_1}) \quad ?$$

We remark that in the case that $X_\alpha$ can be constructed in the above way we have $\beta_i < \alpha$ for $i = 1, \ldots, n$.

In the following we give an explicit example of a non-Schur real root representations which cannot be constructed using universal extension functors.

We consider the quiver $Q$

$$Q : \quad \begin{tikzcd}
1 \arrow[dr, "a"'] & 2 \arrow[d, "b"] & 3 \arrow[dl, "c"] \\
& 4 \arrow[d, "d"] & \\
& 5 \arrow[dl, "e"'] \arrow[d, "f"] \arrow[dr, "g"] & \\
6 & 7 & 8
\end{tikzcd}$$

and the real root $\alpha = (1, 1, 1, 8, 12, 2, 7, 7) = s_8 s_7 s_5 s_4 s_8 s_7 s_5 s_8 s_7 s_5 s_6 s_4 s_5 s_4 s_1 s_2 s_3 (e_4)$.

For the convenience of the reader we give an explicit description of the representation $X_\alpha$.

We start by considering the representation $X_\alpha$ over the field $k = \mathbb{Q}$. In this case, one can use the result [1, Proposition A.4] to construct the representation $X_\alpha$; we get

$$X_\alpha : \quad \begin{tikzcd}
k \arrow[dr, "X_a"'] & k \arrow[d, "X_b"] & k \arrow[dl, "X_c"] \\
& k^8 \arrow[d, "X_d"] & \\
& k^{12} \arrow[dl, "X_e"'] \arrow[d, "X_f"] \arrow[dr, "X_g"] & \\
k^2 & k^7 & k^7
\end{tikzcd}$$

with

$$X_a = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}^t,$$

$$X_b = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}^t,$$

$$X_c = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}^t,$$

$$X_d = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix},$$

$$X_e = \begin{bmatrix} 0&0&0&0&0&0&1&0&0&1&0&0 \\ 0&0&0&0&0&0&0&1&0&0&1&0 \end{bmatrix},$$

$$X_f = \begin{bmatrix} 0&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&0&0&1 \end{bmatrix},$$

$$X_g = \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&1&0&0&0&0&0 \\ 0&0&0&0&0&1&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&1&0&0&1 \end{bmatrix}.$$

In particular, we see that $X_\alpha$ is a tree representation.

The representation $X_\alpha$, as given above, is defined over every field $k$. Moreover, it is not difficult to see that $\mathsf{End}_{kQ}(X_\alpha)$ is local. Hence, the representation $X_\alpha$ is the unique indecomposable representation of dimension vector $\alpha$ over every field $k$.

Moreover, $\dim \mathsf{End}_{kQ}(X_\alpha) = 9$ so that $X_\alpha$ is not a real Schur representation.

**Theorem 2.1.** *There exists no real Schur root $\beta$ with the following properties:*

*(i)* $X_\alpha \in \mathfrak{M}_{X_\beta}^{X_\beta}$, *and*
*(ii)* $\mathsf{Hom}_{kQ}(X_\alpha, X_\beta) \neq 0$ *or* $\mathsf{Hom}_{kQ}(X_\beta, X_\alpha) \neq 0$.

If we had a sequence of real Schur roots $\beta_1, \ldots, \beta_n$ $(n \geq 2)$ such that $X_\alpha = \sigma_{X_{\beta_n}} \cdot \ldots \cdot \sigma_{X_{\beta_2}}(X_{\beta_1})$ then $\beta_n$ would have to satisfy conditions (i) and (ii). Note that condition (ii) merely states that $\sigma^{-1}_{X_{\beta_n}}(X_\alpha) \neq X_\alpha$. Thus, once we have established the claim it is clear that $X_\alpha$ provides an example which answers Question ($\star$) negatively.

We use the rest of this section to prove the above theorem. We show that there are no real Schur roots satisfying (i).

*Proof of Theorem 2.1.* Condition (i) requires $\beta < \alpha$ by [3, Lemma 2] and

$$\mathsf{Ext}^1_{kQ}(X_\alpha, X_\beta) = 0 = \mathsf{Ext}^1_{kQ}(X_\beta, X_\alpha),$$

which implies that $\langle \alpha, \beta \rangle \geq 0$ and $\langle \beta, \alpha \rangle \geq 0$. Hence, we start by determining the set of real roots $\beta$ with the following properties:

(i') $\beta < \alpha$,
(ii') $\langle \alpha, \beta \rangle \geq 0$ and $\langle \beta, \alpha \rangle \geq 0$.

These roots are potential candidates for a reflection. Using the arguments given in [5, Section 6], it is easy to determine the real roots $\beta$ which satisfy (i') and (ii'): both conditions imply that $s_\alpha(\beta) < 0$ and, hence, if $s_\alpha = s_{i_1} \ldots s_{i_n}$ we get $s_\alpha(\beta) = s_{i_1} \ldots s_{i_n}(\beta) < 0$ if and only if $\beta = s_{i_n} \ldots s_{i_{m+1}}(e_{i_m})$ for some $m$. Thus, once we have written $s_\alpha$ as a product of the generators $s_i$ it is straightforward to find the real roots $\beta$ satisfying (i') and (ii'). A decomposition of $s_\alpha$ into a product of the generators $s_i$ can be achieved as follows: if $s_i(\alpha) = \alpha' < \alpha$ then $s_\alpha = s_i s_{\alpha'} s_i$; this gives an algorithm to find a shortest expression of $s_\alpha$ in terms of the $s_i$.

Applying the above algorithm to the real root $\alpha$, we get the following potential candidates for a reflection

$$\begin{aligned}\beta_1 &= (0,0,0,1,2,0,1,1),\\ \beta_2 &= (0,1,1,4,7,1,4,4),\\ \beta_3 &= (1,0,1,4,7,1,4,4), \quad \text{and}\\ \beta_4 &= (1,1,0,4,7,1,4,4).\end{aligned}$$

We see that $\langle \beta_i, \alpha \rangle = 0 = \langle \alpha, \beta_i \rangle$ for $i = 2, 3, 4$, and hence the only reflection candidate is $\beta_1$. Note that $\beta_1$ is a real Schur root, and hence indeed a candidate for a reflection. However, $\beta_1$ does not satisfy condition (i), that is $X_\alpha \notin \mathfrak{M}_{X_{\beta_1}}^{X_{\beta_1}}$. Assume to the contrary that $X_\alpha \in \mathfrak{M}_{X_{\beta_1}}^{X_{\beta_1}}$. Then $\sigma_{X_{\beta_1}}^{-1}(X_\alpha) \in \mathfrak{M}_{-X_{\beta_1}}^{-X_{\beta_1}}$, that is

$$\mathsf{Hom}_{kQ}(\sigma_{X_{\beta_1}}^{-1}(X_\alpha), X_{\beta_1}) = 0 = \mathsf{Hom}_{kQ}(X_{\beta_1}, \sigma_{X_{\beta_1}}^{-1}(X_\alpha)).$$

Using formula (†) from Section 1.1, we get $\gamma_1 := \underline{\dim}\, \sigma_{X_{\beta_1}}^{-1}(X_\alpha) = (1,1,1,3,2,2,2,2)$. The following diagram, however, shows that $\mathsf{Hom}_{kQ}(X_{\beta_1}, X_{\gamma_1}) \neq 0$. The representation $X_{\gamma_1}$ can be constructed using the result [1, Proposition A.4] together with the same reasoning as for $X_\alpha$ to pass to any field $k$.

$$\begin{tikzcd}
 & X_{\beta_1} & & & X_{\gamma_1} & \\
0 \arrow[dr] & 0 \arrow[d] & 0 \arrow[dl] & k \arrow[dr, "\begin{bmatrix}1\\1\\1\end{bmatrix}"'] & k \arrow[d, "\begin{bmatrix}0\\1\\0\end{bmatrix}"] & k \arrow[dl, "\begin{bmatrix}0\\0\\1\end{bmatrix}"] \\
 & k \arrow[rrr, "\begin{bmatrix}1\\0\\0\end{bmatrix}"] \arrow[d, "\begin{bmatrix}1\\1\end{bmatrix}"] & & & k^3 \arrow[d, "\begin{bmatrix}0&1&0\\0&0&1\end{bmatrix}"] & \\
 & k^2 \arrow[rrr, "\begin{bmatrix}0&0\\0&0\end{bmatrix}"] \arrow[dl] \arrow[d, "{[1\,0]}"] \arrow[dr, "{[0\,1]}"] & & & k^2 \arrow[dl, "\mathrm{id}"'] \arrow[d, "\mathrm{id}"] \arrow[dr, "\mathrm{id}"] & \\
0 & k \arrow[rrr, bend right, "{\begin{bmatrix}0&0\end{bmatrix}}"'] & k \arrow[rrr, bend right, "{\begin{bmatrix}0&0\end{bmatrix}}"] & k^2 & k^2 & k^2
\end{tikzcd}$$

This is a contradiction, and hence $X_\alpha \notin \mathfrak{M}_{X_{\beta_1}}^{X_{\beta_1}}$ which completes the proof of the theorem and we see that, indeed, the representation $X_\alpha$ answers Question ($\star$) negatively.

□

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