ANNIHILATING ELEMENTS IN GENERIC DEFORMED PREPROJECTIVE ALGEBRAS

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Let K be a field and let R be a commutative K-algebra. If Q is a quiver with vertex set I then the *deformed preprojective algebra* of weight $\lambda \in R^I$ is

$$\Pi^{R,\lambda}(Q) = R\overline{Q}/(\sum_{a\in Q}[a,a^*] - \sum_{i\in I}\lambda_i e_i).$$

Here $\overline{Q}$ is the *double* of Q, obtained by adjoining a reverse arrow $a^* : j \to i$ for each arrow $a : i \to j$ in Q, we write $R\overline{Q}$ for the path algebra of $\overline{Q}$ over R, and e_i is the trivial path at vertex i. The *generic deformed preprojective algebra* is obtained by taking R to be the polynomial ring $K[x_i : i \in I]$ and the components of λ to be the indeterminates. We prove the following result, conjectured by Rump [5]

Theorem 0.1. *If $\lambda \in R^I$ and Q is a Dynkin quiver, then the product $\prod_\alpha \lambda\cdot\alpha$ over all positive roots α for Q, is equal to zero in $\Pi^{R,\lambda}(Q)$.*

Thus $\Pi^{R,\lambda}(Q)$ is really an algebra over the commutative ring $R/(\prod_\alpha \lambda\cdot\alpha)$. For the generic deformed preprojective algebra this is the coordinate ring of the union of walls in K^I perpendicular to the positive roots.

Recall that if Q is a Dynkin quiver then the algebra KQ has finite representation type, and all KQ-modules, even infinite dimensional ones, are direct sums of finite-dimensional indecomposables. The theorem is therefore a special case of the result below.

Henceforth Q is an arbitrary quiver without oriented cycles. Thus the path algebra KQ is finite-dimensional, and there is the notion of a preprojective KQ-module. Let C be a subset of $\mathbb{N}^I$. We say that a $\Pi^{R,\lambda}(Q)$-module is of *type C* if its restriction to KQ is a direct sum of finite-dimensional indecomposables, each of which has dimension vector in C.

Theorem 0.2. *If C is a finite set of dimension vectors of indecomposable preprojective KQ-modules, then every $\Pi^{R,\lambda}(Q)$-module of type C is annihilated by the product $\prod_{\alpha\in C}\lambda\cdot\alpha$.*

For the proof of this we need some reflection functors $\sigma_i^\pm$, introduced for representations of quivers by Bernstein, Gelfand and Ponomarev [1], and adapted to deformed preprojective algebras by Rump [5]. (But note that Rump used different sign conventions.) For any vertex i in Q let S_i be the one-dimensional KQ-module at vertex i, and let $M(Q,\lambda,i)$ be the category of $\Pi^{R,\lambda}(Q)$-modules whose restriction to KQ does not have S_i as a direct summand. If i is a vertex in Q which is a sink (that is, no arrows have tail at i), let s_iQ be the quiver obtained from Q by reversing all arrows incident at i. Let s_i and r_i be the corresponding reflections on $\mathbb{Z}^I$ and K^I.

Lemma 0.3. *If i is a sink in Q then there are inverse equivalences*

$$M(Q,\lambda,i) \underset{\sigma_i^+}{\overset{\sigma_i^-}{\rightleftarrows}} M(s_iQ, r_i\lambda, i),$$

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and these functors act as the usual reflection functors on the underlying KQ- and Ks_iQ-modules.

Proof of the Theorem. Let $\overline{C}$ be set of indecomposable modules consisting of the preprojective indecomposables with dimension vector in C, and all predecessors of these modules in the Auslander-Reiten quiver of KQ.

We prove the theorem for all orientations of Q and all sets C, by induction on the size of $\overline{C}$. If $\overline{C}$ is empty, there is nothing to prove, so suppose that $\overline{C} \neq \emptyset$. Choose a simple projective module in $\overline{C}$. It is of the form S_i, with i a sink in Q.

Let X be a $\Pi^{R,\lambda}(Q)$-module. We wish to prove that $\prod_{\alpha\in C} \lambda \cdot \alpha$ annihilates X. Let

$$Y = \sum_{j\neq i} e_j X + \sum_{\substack{a\in Q \\ h(a)=i}} aX.$$

This is clearly a submodule of X, and we have $Y \cong \sigma_i^+(X')$ for some $\Pi^{R,r_i\lambda}(s_iQ)$-module X'. Now X' has type $C' = \{s_i(\alpha) \mid \epsilon_i \neq \alpha \in C\}$, and clearly $\overline{C'}$ has one less element than $\overline{C}$. Thus by induction X', and hence also Y, is annihilated by the element

$$\prod_{\beta\in C'} (r_i\lambda) \cdot \beta = \prod_{\beta\in C'} \lambda \cdot (s_i\beta) = \prod_{\epsilon_i\neq\alpha\in C} \lambda \cdot \alpha.$$

If $Y = X$, the assertion follows. Thus suppose that $Y \neq X$. As X/Y is supported only at the vertex i, as a KQ-module it is isomorphic to a direct sum of a (possibly infinite) direct sum of copies of S_i. Thus it is projective, so the exact sequence $0 \to Y \to X \to X/Y \to 0$ is split as a sequence of KQ-modules. Thus S_i is a direct summand of X, so ϵ_i belongs to C. Also, since X/Y is supported only at i it is annihilated by $\lambda_i e_i$ because of the relation (recall that i is a sink in Q)

$$\lambda_i e_i = \sum_{\substack{a\in Q \\ h(a)=i}} aa^*,$$

and hence also by λ_i. The assertion follows. □

References

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