Hyperfinite families, amenable representation type and non-amenability of controlled wild algebras

Dissertation zur Erlangung des Doktorgrades
der Fakultät für Mathematik
der Universität Bielefeld

betreut von
Prof. Dr. William Crawley-Boevey

vorgelegt von
Sebastian Eckert
sebastian.eckert@uni-bielefeld.de

Januar 2022

Abstract

Motivated by the introduction of the notion of amenable representation type by Elek [Ele17], we study the conjecture that finite dimensional algebras are of tame representation type if and only if they are of amenable representation type. While string algebras were shown to be amenable in [Ele17], in this thesis we prove that tame hereditary algebras, in particular path algebras of extended Dynkin quivers, are of amenable type.

Further results concern the amenability of tame concealed algebras as well as partial positive results for indecomposable modules of integral slope for tubular canonical algebras and the preprojective and postinjective component for path algebras of generalised Kronecker quivers.

To prove the other direction of the conjecture for algebraically closed fields, one may show that wild algebras are not of amenable type. Employing the notion of dimension expanders, we give a tangible example of a family of modules for (generalised) Kronecker quivers over arbitrary fields that preclude their amenability. This result is then extended to hereditary wild and strictly wild algebras. Finally, a weak notion of amenable representation type is used to show that no finitely controlled wild algebra over an algebraically closed field is of amenable type.

Acknowledgements

Firstly, I would like to express my gratitude to my supervisor, Professor William Crawley-Boevey, for his continuous guidance, many fruitful discussions, his advice and patience during my PhD studies and the preparation of this thesis, and for suggesting the interesting topic of this thesis.

I am also very thankful to Dr. Andrew Hubery for helpful discussions and valuable suggestions. I would like to thank the other members of the BIREP group for all the mathematical and non-mathematical discussions. The group provided a great atmosphere for doing mathematics in Bielefeld.

During my PhD studies, I was supported by the Alexander von Humboldt Foundation in the framework of an Alexander von Humboldt Professorship endowed by the German Federal Ministry of Education and Research. In the summer of 2021, during the last period of my PhD studies, I was also supported by Bielefeld University and the Faculty of Mathematics with a Doctorate Scholarship.

Contents

Introduction

As one of the main objects of representation theory of algebras, algebras of tame representation type are widely studied. Elek [Ele17] has suggested a new characterisation of this tameness: Instead of checking if the indecomposable modules in every dimension occur in a finite number of one-parameter families, one should check whether every indecomposable module is "almost" the direct sum of modules of bounded dimension.

Conjecture (after [Ele17])**.** *Let k be a field and R a finite dimensional algebra of infinite representation type over k. Then R is of tame representation type if and only if R is of amenable representation type.*

This notion of amenable representation type for finite dimensional algebras and the related notion of hyperfiniteness will be at the heart of the considerations in this thesis, as we investigate well-known classes of algebras with respect to these properties. Roughly speaking, an algebra is of amenable type if for all $\varepsilon > 0$, every finite dimensional module has a submodule which is a direct sum of modules which are small with respect to ε such that the quotient is also small in that respect. Families of modules having this property are said to be hyperfinite. We will give the precise definition and deduce some general results in Chapter 1.

String algebras are shown to be amenable in [Ele17, Proposition 10.1], using the classification of their modules as string and band modules. While this result covers extended Dynkin quivers of type $\tilde{A}_n$, the corresponding result for the other tame path algebras was not known. The first class of algebras we consider in this thesis therefore are path algebras of extended Dynkin quivers. They are a staple of representation theory of algebras and their module structure is well understood. The generalisation to tame hereditary algebras is more technical.

Main Theorem A. *Let Q be an acyclic quiver of extended Dynkin type $\widetilde{A}_n$, $\widetilde{D}_n$, $\widetilde{E}_6$, $\widetilde{E}_7$ or $\widetilde{E}_8$. Let k be any field. Then the path algebra kQ of Q is of amenable representation type.*

Main Theorem B. *Let k be a field. If A is a tame hereditary, finite dimensional k-algebra, then A is of amenable representation type. Moreover, any tame concealed k-algebra is of amenable representation type.*

To keep the result for path algebras of quivers straightforward and to serve as an exposition, this situation is treated in its own Chapter 2. The proof employs the concept of universal localisation from [GL91; Sch86]. En route we will give an elementary proof for the 2-Kronecker quiver in Section 2.2. This result will then be used to prove the first main result in Section 2.3, using a descent argument.

In Chapter 3, we generalise the result to tame hereditary algebras, which also include non-simply-laced extended Dynkin diagrams. Here, we show that the family of indecomposable modules in the rank two-cases is hyperfinite. Again, a descent argument extends this to higher ranks. By an application of the Brenner–Butler tilting theorem and a result from [HR81], the amenability of tame concealed algebras follows.

Next, we show in Chapter 4 that for tubular canonical algebras, an infinite collection of tubes from the regular component is hyperfinite—precisely those of integral slope. Here, we use a classification of Dowbor, Meltzer and Mróz [DMM14b]. As tubular canonical algebras are tame, this is work towards the above conjecture, yet the question if they are of amenable type remains open.

Theorem. *Let A be a tubular canonical algebra. Then the families $\mathcal{P}_0$ of preprojective modules, $\mathcal{Q}_\infty$ of postinjective modules and the family $\bigvee_{\mu\in\mathbb{Z}^\infty\setminus(0,\underline{p})} \mathcal{X}_\mu$ of all indecomposable regular modules of integral slope are hyperfinite.*

We further study the related notion of fragmentability from graph theory introduced by Edwards and McDiarmid [EM94] in Chapter 5. We utilise it to show that the preprojective and postinjective components of generalised Kronecker quivers form hyperfinite families. In this context, knowledge about the class of the underlying graph and the maximum degrees of coefficient quivers of indecomposable modules proves helpful.

To approach the other direction of the conjecture, we turn to study wild algebras in Chapter 6. The case of path algebras of wild acyclic quivers can be reduced to wild Kronecker quivers, thus proving that they are not of amenable representation type. In the finite field case (and stated more generally), this is due to Elek showing the non-amenability of wild generalised Kronecker path algebras. Yet, by employing the notion of dimension expanders, we can give a different argument for the existence of a family of non-hyperfinite modules for (generalised) Kronecker quivers over arbitrary fields in Section 6.1. For fields of characteristic zero, we also give a tangible example supplementing the family constructed by Elek. In this general setting, the amenability of finite dimensional wild hereditary and strictly wild algebras can be reduced to that of wild Kronecker algebras, too.

A thorough analysis then suggests to modify the original definition, yielding a weak notion. Finally, combining this weaker notion with a result of Gregory and Prest [GP16], we show that no finitely controlled wild algebra is of amenable representation type.

Main Theorem C. *Let k be any field. Then there exists some $d \geq 3$ such that the path algebra of the wild generalised Kronecker quiver $k\Theta(d)$ is not of amenable representation type and thus no strictly wild k-algebra is of amenable type. Moreover, if k is algebraically closed, finitely controlled wild (finite dimensional) k-algebras are not of amenable representation type.*

As it has been conjectured [Rin02] (and announced by Y. Drozd in 2007) that all wild algebras are finitely controlled wild, Main Theorem C may settle one direction of the motivating conjecture.

1 Hyperfiniteness and Amenability

Elek [Ele17] has introduced the notion of hyperfiniteness for countable sets of modules and that of amenable representation type for algebras. In doing so, he was ultimately motivated by the classical notion of amenable groups and continuing work from [Ele06], where the amenability of skew fields was defined, and keeping up the notion of hyperfinite collections of graphs from [Ele07].

We deviate slightly from his notion and give the following definition.

Definition 1.1. Let k be a field, A a finite dimensional k-algebra and $\mathcal{M} \subseteq \operatorname{mod} A$ a family of A-modules. We say that $\mathcal{M}$ is **hyperfinite** provided for every $\varepsilon > 0$ there exists $L_\varepsilon > 0$ such that for every $M \in \mathcal{M}$ there exists a submodule $N \subseteq M$ such that

$$\dim_k N \geq (1-\varepsilon) \dim_k M, \tag{1.1}$$

and there are modules

$$N_1, N_2, \dots, N_t \in \operatorname{mod} A, \text{ with } \dim_k N_i \leq L_\varepsilon, \tag{1.2}$$

such that

$$N \cong \bigoplus_{i=1}^{t} N_i. \tag{1.3}$$

A k-algebra A is said to be of **amenable representation type** provided $\operatorname{mod} A$ itself is a hyperfinite family.

We use $\operatorname{mod} A$ to denote the class of all finite dimensional (left) A-modules. If A is a finite dimensional k-algebra, these are precisely the finitely generated A-modules.

Remark. Since N is a submodule of M, the condition (1.1) is equivalent to

$$\dim_k (M/N) \leq \varepsilon \dim_k M, \tag{1.4}$$

for $\dim_k(M/N) = \dim_k M - \dim_k N$.

Remark. Finite sets are hyperfinite, for we can take L_ε to be the maximum of the dimensions and $N = M$.

For the same reason, families of modules of bounded dimension are hyperfinite. Moreover, if $\mathcal{M}$ and $\mathcal{M}'$ are hyperfinite families, so is $\mathcal{M} \cup \mathcal{M}'$: We can choose L_ε to be the maximum of $L_\varepsilon^{\mathcal{M}}$ and $L_\varepsilon^{\mathcal{M}'}$, corresponding to $\mathcal{M}$ and to $\mathcal{M}'$, respectively. Similarly, any finite union of hyperfinite families is hyperfinite.

Proposition 1.2. *Let $\mathcal{M}$ be a family of A-modules. If $\mathcal{M}$ is hyperfinite, so is the family of all finite direct sums of modules in $\mathcal{M}$.*

Proof. Let $\mathcal{M}$ be hyperfinite and let $\varepsilon > 0$. Then there exists L_ε satisfying the conditions in the definition. Now assume $M = \bigoplus_{i=1}^n M_i$ with $M_i \in \mathcal{M}$. For each $1 \leq i \leq n$, choose

$$\bigoplus_{j=1}^{t_i} N_{i,j} = N_i \subseteq M_i,$$

as for the hyperfiniteness of $\mathcal{M}$. Then

$$N := \bigoplus_{i=1}^{n} N_i \subseteq \bigoplus_{i=1}^{n} M_i = M,$$

as direct sums respect submodule inclusions. Also

$$\dim_k N = \sum_{i=1}^{n} \dim_k N_i \geq \sum_{i=1}^{n} (1-\varepsilon) \dim_k M_i = (1-\varepsilon) \sum_{i=1}^{n} \dim_k M_i = (1-\varepsilon) \dim_k M.$$

Moreover, $\dim_k N_{i,j} \leq L_\varepsilon$. □

This shows that to check amenability, it is enough to check the criterion on all indecomposable modules.

Example 1.3. Since for a representation finite algebra A, there are only finitely many isomorphism classes of (finitely generated) indecomposable modules, and the k-dimensions of indecomposable modules are therefore bounded by

$$\max_{M \in \operatorname{ind} A} \{\dim_k M\},$$

such an algebra A is of amenable representation type.
A non-example is given in [Ele17] by the wild Kronecker algebras (see also Theorem 6.12), while string algebras were shown to be of amenable representation type, including the 2-Kronecker algebra (see also Theorem 2.9).

Proposition 1.4. *Let A be a finite dimensional k-algebra and $\mathcal{M}, \mathcal{N} \subseteq \operatorname{mod} A$ where $\mathcal{N}$ is hyperfinite. If there is some $H \geq 0$ such that for all $M \in \mathcal{M}$, there exists a submodule $N \subseteq M$ with $N \in \mathcal{N}$, of codimension less than or equal to H, then $\mathcal{M}$ is also hyperfinite.*

Proof. Let $H \geq 0$ and $\varepsilon > 0$. If the dimension of the modules in $\mathcal{M}$ was bounded, say by K, we can set $L_\varepsilon := K$ and choose $N = M$ for all $M \in \mathcal{M}$, and we are done. On the other hand, if the dimension is not bounded, there is $M \in \mathcal{M}$ with $\dim_k M > \frac{2H}{\varepsilon}$. We choose a submodule $N \in \mathcal{N}$ of codimension bounded by H. Since $\mathcal{N}$ is hyperfinite, there is some submodule $Y \subseteq N$ such that $\dim Y \geq (1 - \frac{\varepsilon}{2}) \dim N$, while Y decomposes into direct summands of dimension less than or equal to $L^{\mathcal{N}}_{\frac{\varepsilon}{2}}$.

We thus have that

$$\begin{aligned}\dim Y &\geq \left(1-\frac{\varepsilon}{2}\right)\dim N = \dim N - \frac{\varepsilon}{2}\dim N \\ &\geq (\dim M - H) - \frac{\varepsilon}{2}\dim M \\ &\geq \dim M - \frac{\varepsilon}{2}\dim M - \frac{\varepsilon}{2}\dim M \\ &= (1-\varepsilon)\dim M,\end{aligned}$$

using that we have $H \leq \frac{\varepsilon}{2}\dim M$. What is more, Y decomposes into direct summands of dimension less than or equal to $L^{\mathcal{N}}_{\frac{\varepsilon}{2}}$. If we therefore choose $L^{\mathcal{M}}_{\varepsilon}$ to be the maximum of $L^{\mathcal{N}}_{\frac{\varepsilon}{2}}$ and $\frac{2H}{\varepsilon}$, we have shown that $\mathcal{M}$ is hyperfinite. ☐

Proposition 1.5. *Let k be a field and let A, B be two finite dimensional k-algebras. Let $F\colon \operatorname{mod} A \to \operatorname{mod} B$ be an additive, left-exact functor such that there exists $K_1, K_2 > 0$ with*

$$K_1 \dim X \leq \dim F(X) \leq K_2 \dim X, \tag{1.5}$$

for all $X \in \operatorname{mod} A$. If $\mathcal{N} \in \operatorname{mod} A$ is a hyperfinite family, then the family

$$\mathcal{M} := \{F(X)\colon X \in \mathcal{N}\} \subseteq \operatorname{mod} B$$

is also hyperfinite.

Proof. By the hypothesis, for any $\tilde{\varepsilon}$ we can find some $L^{\mathcal{N}}_{\tilde{\varepsilon}} > 0$ to exhibit the hyperfiniteness of the family $\mathcal{N}$. Let $M \in \operatorname{mod} B$ such that $F(N) = M$ for some $N \in \mathcal{N}$. Then there is a submodule $P \subseteq N$ such that $P = \bigoplus_{i=1}^{t} P_i$ with $\dim P_i \leq L^{\mathcal{N}}_{\tilde{\varepsilon}}$ and $\dim P \geq (1-\tilde{\varepsilon})\dim N$. Since F is additive, we have that $F(P) = \bigoplus_{i=1}^{t} F(P_i)$, and by the right-hand side of (1.5),

$$\dim F(P_i) \leq K_2 \dim P_i \leq K_2 L^{\mathcal{N}}_{\tilde{\varepsilon}}.$$

Moreover, the sequence

$$0 \to F(P) \to M \to F(N/P)$$

is exact, so $F(P)$ is (isomorphic to) a submodule of M, and by the rank-nullity theorem,

$$\begin{aligned}\dim F(P) &\geq \dim M - \dim F(N/P) \\ &\geq \dim M - K_2 \dim N/P = \dim M - K_2(\dim N - \dim P) \\ &\geq \dim M - K_2 \dim N + K_2(1-\tilde{\varepsilon})\dim N = \dim M - K_2\tilde{\varepsilon}\dim N \\ &\geq \dim M - \frac{K_2}{K_1}\tilde{\varepsilon}\dim M = (1-\varepsilon)\dim M,\end{aligned}$$

if we choose $\tilde{\varepsilon} = \frac{K_1}{K_2}\varepsilon$. We can therefore choose $L^{\mathcal{M}}_{\varepsilon}$ to be $K_2 L^{\mathcal{N}}_{\tilde{\varepsilon}}$ to prove the hyperfiniteness of $\mathcal{M}$. ☐

Remarks. A functor fulfilling the hypothesis of Proposition 1.5 may be called hyperfiniteness preserving or HF-preserving. Moreover, inspection of the proof shows that the left inequality of (1.5) need only hold for $X \in \mathcal{N}$.

Example 1.6. We give some examples of functors preserving hyperfiniteness and show how they may be applied.

1. Equivalences are HF-preserving functors: they are left exact and the fact that simple modules are mapped to simple modules ensures the existence of suitable constants K_1 and K_2.

2. If A is the path algebra of a quiver Q of amenable representation type, and $i \in Q_0$ is a sink, then we can take the reflection functor $F = S_i^+$ (see, e.g., [ASS10, Section VII.5]) to show that kQ', where $Q' = \sigma_i(Q)$ is the quiver obtained from Q by reversing all arrows starting or ending in i, is also amenable. To see this, let $\mathcal{C} = \mathcal{C}_i$ be the full subcategory of $\operatorname{mod} kQ'$ of objects having no direct summand isomorphic to the simple module $S(i)$. Then every indecomposable object of $\operatorname{mod} kQ'$ is either contained in $\mathcal{C}$ or is isomorphic to $S(i)$. As $\mathcal{C}$ is in the essential image of F, the family of indecomposable modules of $\operatorname{mod} kQ'$ is hyperfinite, proving the claim. We will study this situation explicitly in Proposition 2.2.

3. Let $L|k$ be a separable field extension of finite degree $[L : k] = n$. Let A be a finite dimensional k-algebra. Then restriction of scalars

$$A^L \operatorname{mod} \to A \operatorname{mod},\ N \mapsto \operatorname{Hom}_{A^L}(A^L, N)$$

is a hyperfiniteness preserving functor, where $A^L = A \otimes_k L$.

2 Extended Dynkin quivers

As a first step towards proving the conjecture, we will consider those finite dimensional k-algebras which come from quivers of extended Dynkin type. As Elek succeeded in proving a result for string algebras (see [Ele17, Proposition 10.1]) that include the affine quivers $\tilde{A_n}$, this approach lends itself as a suitable next step.

We keep this treatise separate and in its own chapter. On the one hand this is done to account for the fact that quiver representations have been more widely studied than the representation theory of tame hereditary algebras in their full generality. We will turn to this more general situation in Chapter 3. On the other hand, this chapter bears witness to the fact that the study of extended Dynkin quivers, especially the tame Kronecker quiver $\Theta(2)$ and the four-subspace quiver $\S(4)$, were the starting point of this thesis.

2.1 Setup

Recall that a *quiver* is a quadruple $Q = (Q_0, Q_1, s, t)$ of a finite set of vertices Q_0, a finite set of arrows Q_1 between these vertices and two mappings s and t associating to every arrow $\rho \in Q_1$ its starting vertex respectively its terminating vertex. We may concatenate arrows α and β provided $t(\alpha) = s(\beta)$ and write $\beta \circ \alpha$ for this *path*. A quiver Q is said to be *connected* if there is such a sequence of arrows connecting every pair of vertices. Vertices which are not the terminating vertex of an arrow are said to be *sources*, while vertices at which no arrow starts are said to be *sinks*.

Recall further that a *representation M of a quiver Q over a field k* is given by a family of vector spaces $\big(M(i)\big)_{i\in Q_0}$ and k-linear maps $M(\rho)\colon M_{s(\rho)} \to M_{t(\rho)}$ for each arrow $\rho \in Q_1$. We denote such a representation by

$$M = \big((M(i))_{i\in Q_0}, (M(\rho))_{\rho\in Q_1}\big).$$

The class of k-representations of a quiver Q along with its morphisms gives rise to the category of k-representations of Q that we denote by $\mathrm{Rep}_k(Q)$. The full subcategory of all finite dimensional representations will be denoted by $\mathrm{rep}_k(Q)$.

This category of representations $\mathrm{rep}_k(Q)$ is equivalent to the module category of finite dimensional left modules over the path algebra kQ that we denote by $\mathrm{mod}\, kQ$. By ϵ_i, for $i \in Q_0$, we denote the idempotent elements of this algebra corresponding to the vertices of Q. Note that $1 = \sum_{i\in Q_0} \epsilon_i$.

We also recall that the path algebra of a quiver without oriented cycles (vel *acyclic* quiver) is *hereditary*, that is the submodules of projective modules are also projective.

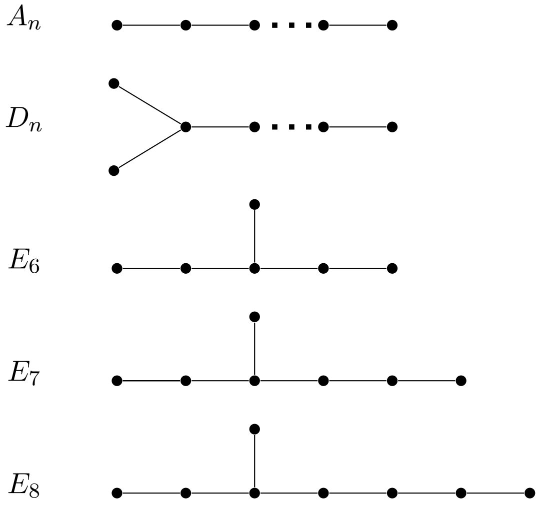


Figure 2.1: The (simply-laced) Dynkin diagrams of types A, D and E

Key results for path algebras of quivers concern their representation type. Recall that given a finite dimensional k-algebra A, we say that it is of *finite representation type* provided there are only finitely many isomorphism classes of finite dimensional, indecomposable A-modules. Otherwise, we say that A is *representation-infinite.* Moreover, a representation-infinite algebra A is of *tame representation type* if the indecomposable modules in each dimension come in finitely many one-parameter families with only finitely many exceptions.

Results due to work of Gabriel [Gab72] as well as Nazarova [Naz73] and Donovan and Freislich [DF73] for the algebraically closed case and Dlab and Ringel [DR76] for arbitrary fields and valued quivers, provide a classification of those path algebras that are of finite respective tame representation type.

Theorem 2.1 (Gabriel, Donovan–Freislich, Nazarova)**.** *[DW05, Theorems 9+10]*

a) The path algebra of a (connected) quiver Q is of finite representation type if and only if the underlying graph is a Dynkin diagram (listed in Figure 2.1).

b) The path algebra of a (connected) quiver Q is of tame representation type if and only if the underlying graph is an extended Dynkin diagram (listed in Figure 2.2).

As a first result about quivers and the amenability of their path algebras, we revisit a previous example of hyperfiniteness-preserving functors.

Proposition 2.2. *Let Q be a quiver such that the path algebra kQ is of amenable representation type. Let $i \in Q_0$ be a sink. Let $Q' = \sigma_i(Q)$ be the quiver obtained from Q by reversing all arrows which start or end in i. Then kQ' is also amenable.*

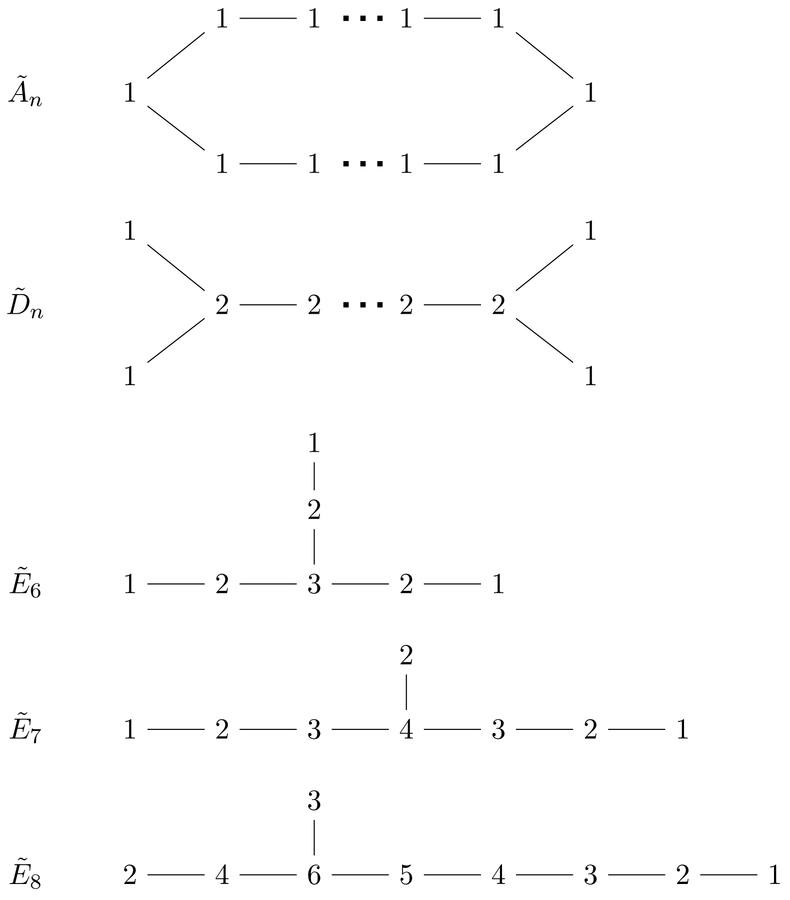


Figure 2.2: The extended (vel affine) Dynkin diagrams of types $\tilde{A}$, $\tilde{D}$ and $\tilde{E}$, also showing the minimal radical vector $\underline{h}_Q$. They are also called Euclidean diagrams.

Proof. Let $\varepsilon > 0$ and set $K := |Q_1| + 1$. By hypothesis, for all $\tilde{\varepsilon} > 0$, there exists $L_{\tilde{\varepsilon}}$ exhibiting the hyperfiniteness of $\operatorname{mod} kQ$. Let $M' \in \operatorname{mod} kQ'$ be indecomposable. If $M' \cong S(i)$ is the simple at vertex i, we choose $N' = M'$. Then, $\dim N' \geq (1-\tilde{\varepsilon}) \dim M'$, and $\dim N' = \dim S(i) \leq L'_\varepsilon$ for $L'_\varepsilon := \max\{L_{\tilde{\varepsilon}}, 1\}$.

Thus, we may assume that $M' \ncong S(i)$. Then $M := S_i^- M'$ is indecomposable as a representation of kQ, where S_i^- is the reflection functor $\operatorname{mod} kQ' \to \operatorname{mod} kQ$ (see, e.g., [ASS10, Section VII.5]). By the hypothesis, for every $\tilde{\varepsilon}$ there exists a submodule $N \subseteq M$ such that $\dim N \geq (1-\tilde{\varepsilon}) \dim M$ and $N = \bigoplus_{l=1}^m N^{(l)}$ with $\dim N^{(l)} \leq L_{\tilde{\varepsilon}}$. We choose $\tilde{\varepsilon} = \frac{\varepsilon}{K^2} > 0$,

Now, $S_i^+ S_i^- M' \cong M'$, so the short exact sequence

$$0 \to N \to M \to M/N \to 0,$$

yields an exact sequence

$$0 \to S_i^+ N \to M' \to S_i^+(M/N) \to Z \to 0.$$

Note that the definition of S_i^+ and the Snake Lemma imply that Z is a direct sum of copies of $S(i)$. We let $N' := S_i^+ N = \bigoplus_{l=1}^m S_i^+ N^{(l)}$. Now, by the definition of the reflection functor,

$$\begin{aligned} \dim S_i^+ N^{(l)} = \sum_{j\in Q_0} \dim \left(S_i^+ N^{(l)}\right)_j &= \sum_{\substack{j\in Q_0,\\ j\neq i}} \dim N_j^{(l)} + \left(\sum_{\substack{\alpha\in Q_1,\\ t(\alpha)=i}} \dim N_{s(\alpha)}^{(l)} - \dim N_i^{(l)}\right) \\ &\leq \sum_{\substack{j\in Q_0,\\ j\neq i}} \dim N_j^{(l)} + |Q_1| \underbrace{\sum_{j\in \mathbb{Q}_0} \dim N_j^{(l)}}_{=\dim N^{(l)}} \leq (|Q_1|+1) \dim N^{(l)} \leq (|Q_1|+1)\, L_{\tilde{\varepsilon}}. \end{aligned}$$

On the other hand, as we put $K := |Q_1| + 1$,

$$\begin{aligned} \dim N' = \dim S_i^+ N &= \dim M' - \dim S_i^+(M/N) + \dim Z \\ &\geq \dim M' - K \dim(M/N) \\ &= \dim M' - K(\dim M - \dim N) \\ &\geq \dim M' - K \dim M + K(1-\tilde{\varepsilon}) \dim M \\ &\geq \dim M' - K^2 \tilde{\varepsilon} \dim M'. \end{aligned}$$

Thus, letting $L'_\varepsilon := \max\{1, L_{\tilde{\varepsilon}}\}$, we are done. □

Remark. This result will be generalised in Theorem 3.19.

In the remainder of this chapter, let Q be an acyclic, extended Dynkin quiver, that is a quiver of type $\widetilde{A}_n$, $\widetilde{D}_n$, $\widetilde{E}_6$, $\widetilde{E}_7$ or $\widetilde{E}_8$ (see Figure 2.2) which has no oriented cycles. It is well known that the module categories of the associated path algebras kQ are tame and hereditary (see Theorem 2.1 above and, e.g., [ASS10, Theorem VII.1.7]). We shall write $\underline{\dim}\, M$ for the *dimension vector* of some representation M, where

$$(\underline{\dim}\, M)_i = \dim_k M(i)$$

for each vertex $i \in Q_0$.

Recall the *Euler bilinear form*, which can be defined on the dimension vectors of the representations by

$$\langle \underline{\dim}\, X, \underline{\dim}\, Y \rangle := \sum_{t \geq 0} (-1)^t \dim_k \operatorname{Ext}^t_{kQ}(X, Y),$$

the *Tits form* $q\colon \mathbb{Z}^{|Q_0|} \to \mathbb{Z}$, that is, the corresponding quadratic form, $q(x) = \langle x, x \rangle$, and its radical

$$\operatorname{rad} q = \{x \in \mathbb{Q}^{|Q_0|} \colon q(x) = 0\}.$$

We shall denote the minimal (integer) generator of $\operatorname{rad} q$, the *minimal radical vector* by $\underline{h}_Q$. Furthermore, there is a uniquely determined, normalised q-invariant linear form ∂, such that $\partial(X) = \partial(\underline{\dim}\, X) = \langle \underline{h}_Q, \underline{\dim}\, X \rangle$, called the *defect* of X. This form allows us to distinguish between *preprojective* ($\partial < 0$), *regular* ($\partial = 0$) and *postinjective* ($\partial > 0$) indecomposables of $\operatorname{mod} kQ$. We say that a module M is *exceptional* provided it is indecomposable and has no self-extensions, that is, $\operatorname{Ext}^1_{kQ}(M, M) = 0$.

Also recall that $\operatorname{mod} kQ$ has Auslander–Reiten (AR) sequences, giving rise to the Auslander–Reiten translate τ and its inverse. The category $\operatorname{mod} kQ$ may thus be described by its Auslander–Reiten quiver Γ_{kQ}, a translation quiver having as vertices the isomorphism classes of indecomposable modules and arrows for the so-called *irreducible* morphisms. Moreover, there exists a transformation c, called the *Coxeter transformation*, such that for any module X without projective direct summand, we have

$$\underline{\dim}\, \tau(X) = c(\underline{\dim}\, X).$$

In what follows, we may also frequently use some elementary properties of the Coxeter transformation and the Euler bilinear form, see for instance [ASS10, Lemma III.3.16].

Recall that the *defect number* d_Q is the smallest positive integer d such that

$$c^d(x) - x \in \operatorname{rad} q \text{ for each } x \in \mathbb{Z}^{|Q_0|}.$$

Next, recall that the regular component $\mathcal{R}$ of Γ_{kQ} is an Abelian category comprising pairwise orthogonal[1] stable[2] tubes, of which at most three are *inhomogeneous* (vel exceptional), consisting of more than one regular simple module, that is, simple with respect to the Abelian subcategory $\mathcal{R}$. As the regular simple modules in each tube form a cycle under τ, we may therefore use a triple (p, q, r) of positive integers to list the cycle lengths of the inhomogeneous tubes and call it the *tubular type* of Q.

---

[1] i.e. there are no non-zero homomorphisms between objects in different tubes

[2] i.e. they do not contain projective or injective objects

We will further use the notion of a *perpendicular category*, defined for some module $X \in \operatorname{mod} kQ$ by

$$X^{\perp} := \{Y \in \operatorname{mod} kQ \colon \operatorname{Hom}_{kQ}(X, Y) = 0 = \operatorname{Ext}^1_{kQ}(X, Y)\}.$$

To proceed with the proof of the first major theorem, we gather some technical lemmas. The first will be a result on the sum of the dimension vectors of the simple regular modules in inhomogeneous tubes of Γ_{kQ}.

Lemma 2.3. *Let k be any field. Let $\mathbb{T}$ be an inhomogeneous tube of rank m in Γ_{kQ}. Let us denote the isomorphism classes of regular simple modules on the mouth of $\mathbb{T}$ by $S_1, \dots, S_m$ such that $\tau S_i = S_{i-1}$ for $i = 2, \dots, m$ and $\tau S_1 = S_m$. Then we have*

$$\sum_{i=1}^{m} \underline{\dim}\, S_i = \underline{h}_Q.$$

Proof. For the case of an algebraically closed field, we may argue as follows: By [Rin84, Theorem 3.6.(5)] in connection with [Rin84, Section 3.4], the regular component of Γ_{kQ} is given by certain tubes $\mathbb{T}(\rho)$, which are each generated by orthogonal indecomposable modules $E_1^{(\rho)}, \dots, E_{m_\rho}^{(\rho)}$ with endomorphism ring k. As these lie on the mouth of a stable tube of rank m_ρ, the isoclasses $S_1, \dots, S_m$ correspond to the $E_i^{(\rho)}$ for some ρ. Now it follows from an argument in the proof of [Rin84, 3.4.(10)] that $\sum_{i=1}^{m} \underline{\dim}\, S_i = \underline{h}_Q$.

For the general case, one may inspect the relevant tables of [DR76, Chapter 6], where the dimension vectors of the regular simple modules are listed. □

Lemma 2.4. *Let $\mathbb{T}$ be a tube of rank $m \geq 2$ in Γ_{kQ}. Let X be an indecomposable regular module in $\mathbb{T}$. Then there exists a submodule $Y \subseteq X$ of codimension bounded by the sum of the entries of $\underline{h}_Q$ and a regular simple module $T \in \mathbb{T}$ such that $Y \in T^{\perp}$.*

Proof. Let us denote the isoclasses of regular simples on the mouth of $\mathbb{T}$ by $T_1, \dots, T_m$ such that $\tau T_i = T_{i-1}$ for $i = 2, \dots, m$ and $\tau T_1 = T_m$. Following [Rin84, Chapter 3], we define the objects $T_i[\ell]$. First, let $T_i[1] := T_i$ for each $1 \leq i \leq m$. Now, for $\ell \geq 2$, recursively define $T_i[\ell]$ to be the indecomposable module in $\mathbb{T}$ with $T_i[1]$ as a submodule such that $T_i[\ell]/T_i[1] \cong T_{i+1}[\ell - 1]$. Thus $T_i[\ell]$ is the regular module of regular length ℓ with regular socle T_i. Now, any regular indecomposable in $\mathbb{T}$ will be given as some $T_i[\ell]$. We may define $T_i[\ell]$ for all $i \in \mathbb{Z}$ by letting $T_i[\ell] \cong T_j[\ell]$ iff $i \equiv j \mod m$. Note that

$$\underline{\dim}\, T_i[\ell] = \sum_{j=i}^{i+\ell-1} \underline{\dim}\, T_j. \tag{2.1}$$

By [Rin84, 3.1.(3')], we have that

$$\langle \underline{\dim}\, T_i, \underline{\dim}\, T_j \rangle = \begin{cases} 1, & i \equiv j \mod m, \\ -1, & i \equiv j + 1 \mod m, \\ 0, & \text{else.} \end{cases}$$

This implies that for any $j \in \mathbb{Z}$, $\langle \underline{\dim}\, T_j, \underline{\dim}\, T_i[\ell] \rangle = 0$, provided $\ell \equiv 0 \mod m$. Since $T_i[\ell]$ is uniserial, we have that $\operatorname{Hom}(T_j, T_i[\ell]) = 0$ if and only if $j \not\equiv i \mod m$. As $m \geq 2$, this implies that for all $i \in \mathbb{Z}$ and $\ell \equiv 0 \mod m$, $T_i[\ell]$ is contained in the perpendicular category of some regular simple, that is in $T_{i+1}^{\perp}$.

If $\ell \not\equiv 0 \mod m$, then write $\ell = n \cdot m + r$, where $0 < r < m$. Then there is a short exact sequence

$$0 \to T_i[nm] \to T_i[\ell] \to Z \to 0,$$

where $\underline{\dim}\, Z \leq \underline{h}_Q$, using (2.1) and Lemma 2.3. Thus, we have found a suitable submodule $T_i[nm] \in T_{i+1}^{\perp}$. ☐

Lemma 2.5. *Let $X = \tau^{-r}P(i)$ be some indecomposable preprojective kQ-module of defect $\partial(X) = -d < 0$. Let $\mathbb{T}$ be a tube of rank $m > d$. Then there is a simple regular module $S \in \mathbb{T}$ such that $X \in S^{\perp}$.*

Proof. Clearly,

$$-d = \langle \underline{h}_Q, \underline{\dim}\, X \rangle = -\langle \underline{\dim}\, X, \underline{h}_Q \rangle = -\langle \underline{\dim}\, P(i), \underline{h}_Q \rangle = -(\underline{h}_Q)_i,$$

so $\underline{h}_Q$ has a component equal to d. Let $S_1, \ldots, S_m$ be the regular simple modules on the mouth of $\mathbb{T}$. By Lemma 2.3, $\sum_{j=1}^{m} \underline{\dim}\, S_j = \underline{h}_Q$. Now, since $m > d$, only d out of the m modules can have a vector space at vertex i which is non-zero. Let j_0 be such that $(\underline{\dim}\, S_{j_0})_i = 0$. Let $1 \leq j \leq m$. Then

$$\begin{aligned} \langle \underline{\dim}\, S_j, \underline{\dim}\, \tau^{-r}P(i) \rangle &= -\langle \underline{\dim}\, \tau^{-r}P(i), c(\underline{\dim}\, S_j) \rangle = -\langle \underline{\dim}\, P(i), c^{r+1}(\underline{\dim}\, S_j) \rangle \\ &= -\langle \underline{\dim}\, P(i), \underline{\dim}\, S_{j-r-1} \rangle = -(\underline{\dim}\, S_{j-r-1})_i. \end{aligned}$$

Now, we can choose j such that $j - r - 1 \equiv j_0 \mod m$. Since $\operatorname{Hom}_{kQ}(S_j, X) = 0$, for S_j is regular and X is preprojective, we must have that $\operatorname{Ext}^1_{kQ}(S_j, X) = 0$. Thus, $X \in S_j^{\perp}$. ☐

Indeed, a slightly stronger result can be shown.

Lemma 2.6. *Let $X = \tau^{-r}P(i)$ be some indecomposable preprojective kQ-module of defect $\partial(X) = -d < 0$ and $r > d_Q$. Let $\mathbb{T}$ be an inhomogeneous tube of rank $m > d/2$. Then one of the following holds:*

(1) There exists a simple regular module $S \in \mathbb{T}$ such that $X \in S^{\perp}$.

(2) There exists a submodule $Y \subseteq X$ and regular simple modules $S, T \in \mathbb{T}$ such that $0 \to Y \to X \to T \to 0$ is exact and $Y \in S^{\perp}$.

Proof. Clearly,

$$-d = \langle \underline{h}_Q, \underline{\dim}\, X \rangle = -\langle \underline{\dim}\, X, \underline{h}_Q \rangle = -\langle \underline{\dim}\, P(i), \underline{h}_Q \rangle = -(\underline{h}_Q)_i,$$

so $\underline{h}_Q$ has a component equal to d. Let $S_1, \ldots, S_m$ be the regular simple modules on the mouth of $\mathbb{T}$. Then $\sum_{j=1}^{m} \underline{\dim}\, S_j = \underline{h}_Q$ by Lemma 2.3. We will write $d_j = (\underline{\dim}\, S_j)_i$ and have $\sum_{j=1}^{m} d_j = d$.

Now, for $1 \leq j \leq m$,

$$\begin{aligned}\dim \operatorname{Hom}_{kQ}(X, S_j) - \dim \operatorname{Ext}^1_{kQ}(X, S_j) &= \langle \underline{\dim}\, X, \underline{\dim}\, S_j \rangle = \langle \underline{\dim}\, \tau^{-r} P(i), \underline{\dim}\, S_j \rangle \\ &= \langle \underline{\dim}\, P(i), \underline{\dim}\, S_{j-r} \rangle = (\underline{\dim}\, S_{j-r})_i = d_{j-r},\end{aligned}$$

where $\dim \operatorname{Ext}^1_{kQ}(X, S_j) = \dim D\operatorname{Hom}_{kQ}(S_{j+1}, X) = 0$ by [Rin84, 2.4.(6*)] and the fact that there are no non-zero maps from the regular to the preprojective component.

Thus, if there is some j such that $d_{j-r} > 0$, we can choose some non-zero map

$$\theta \colon X \to S_j.$$

The image $\operatorname{im} \theta \subseteq S_j$ must be regular or has a preprojective summand. If there was a preprojective summand Z, it must be to the right of X in Γ_{kQ}. But for any preprojective module M in the r-th τ-translate of the projectives or further to the right in Γ_{kQ}, we know that $\underline{\dim}\, M = \underline{\dim}\, \tau^{d_Q} M - \partial(M) \underline{h}_Q > \underline{h}_Q$ by the definition of the defect. On the other hand, $\underline{\dim} \operatorname{im} \theta \leq \underline{\dim}\, S_j \leq \underline{h}_Q$, a contradiction. Thus, $\operatorname{im} \theta$ must be regular. Since S_j is a regular simple, this implies that θ is surjective. We therefore have an exact sequence

$$0 \to Y \to X \to S_j \to 0,$$

by letting $Y := \ker \theta$. Applying $\operatorname{Hom}_{kQ}(-, S_j)$, we get an exact sequence

$$\begin{aligned}\xi \colon 0 \to \operatorname{Hom}_{kQ}(S_j, S_j) \to \operatorname{Hom}_{kQ}(X, S_j) \to \operatorname{Hom}_{kQ}(Y, S_j)& \\ \to \operatorname{Ext}^1_{kQ}(S_j, S_j) \to \operatorname{Ext}^1_{kQ}(X, S_j).&\end{aligned}$$

Since S_j is an inhomogeneous regular simple, there are no self-extensions, and we have $\operatorname{Ext}^1_{kQ}(S_j, S_j) = 0$. Hence, ξ becomes the short exact sequence

$$\xi' \colon 0 \to \operatorname{End}_{kQ}(S_j) \to \operatorname{Hom}_{kQ}(X, S_j) \to \operatorname{Hom}_{kQ}(Y, S_j) \to 0.$$

Now, assume $d_{j-r} > 1$ for all j. Using the hypothesis, we would have

$$2m \leq \sum_{j=1}^{m} d_j = d < 2m,$$

a contradiction. Therefore, there is some j_0 with $d_{j_0-r} \leq 1$.

If $d_{j_0-r} = 1$, we have that $\dim \operatorname{Hom}_{kQ}(X, S_{j_0}) = 1$, so the exact sequence ξ' implies that $\operatorname{Hom}_{kQ}(Y, S_{j_0})$ must be zero by dimension arguments, for $\dim \operatorname{End}_{kQ}(S_{j_0}) \geq 1$. Hence, using the Auslander–Reiten formulae (see, e.g., [ASS10, Corollary IV.2.14]), we have

$$\operatorname{Ext}^1_{kQ}(S_{j_0+1}, Y) = D\operatorname{Hom}_{kQ}(Y, S_{j_0}) = 0.$$

Along with the fact that $\operatorname{Hom}_{kQ}(S_{j_0+1}, Y) = 0$, since there are no maps from regular to preprojective modules, this implies that $Y \in S_{j_0+1}^{\perp}$ and we are in case (2).

If $d_{j_0-r} = 0$, we have that $\dim \operatorname{Hom}_{kQ}(X, S_{j_0}) = 0$. So, similarly,

$$\operatorname{Ext}^1_{kQ}(S_{j_0+1}, X) = 0, \qquad \operatorname{Hom}_{kQ}(S_{j_0+1}, X) = 0,$$

and $X \in S_{j_0+1}^{\perp}$, showing that we are in case (1). □

To prove hyperfiniteness results for postinjective indecomposables, we will employ a descent argument based on the following lemma.

Lemma 2.7. *Let $X = \tau^r I(i)$ be some indecomposable postinjective module of defect $\partial(X) = d$. Then there is an injective module $I(j)$ such that there exists a non-zero homomorphism $\theta\colon X \to I(j)$ and for any direct summand Z of $\ker\theta$, we have that defect $\partial(Z) < d$.*

Proof. Let $E(X)$ be the injective envelope of X, and take $I(j)$ to be some direct summand of $E(X)$. This yields a non-zero homomorphism $\theta\colon X \to E(X) \to I(j)$. Consider the exact sequence

$$0 \to \ker\theta \to X \to \operatorname{im}\theta \to 0.$$

Since there is a non-zero map from a postinjective module to $\operatorname{im}\theta$, the latter must be postinjective or zero. Yet, $\operatorname{im}\theta \neq 0$, since θ is non-zero. Thus, $\operatorname{im}\theta$ has positive defect, implying that $\partial(\ker\theta) < \partial(X)$. If $\ker\theta$ only had preprojective or regular summands Z, we are done, for then $\partial(Z) \leq 0$. Thus, we may assume that there is some postinjective direct summand Z. Since Z embeds into $\ker\theta$ and the kernel embeds into X, we get a short exact sequence

$$0 \to Z \to X \to X/Z \to 0,$$

using the fact that $\operatorname{mod} kQ$ is Abelian. Since X is postinjective, again X/Z must be postinjective or zero. If X/Z was zero, then $Z \cong \ker\theta \cong X$, a contradiction, since $\partial(\ker\theta) \neq \partial(X)$. Thus, $\partial(X/Z) > 0$, and hence we may conclude that the defect $\partial(Z) < \partial(X) = d$. ☐

2.2 The special case of the 2-Kronecker quiver

In this section, we will prove that the tame Kronecker quiver $\Theta(2)$, which has two vertices connected by two equi-oriented arrows, is of amenable representation type. This quiver is of type $\widetilde{A}_1$. The theorem in this section will be used as the base case in the proof of Main Theorem A. It follows from the results on string algebras in [Ele17, Proposition 10.1], but we give a direct and independent proof here for illustration purposes and for the convenience of the reader.

Lemma 2.8. *Let A be a tame hereditary algebra. Assume that the preprojective indecomposable modules* $\operatorname{ind}\mathcal{P}$ *and the regular indecomposable modules* $\operatorname{ind}\mathcal{R}$ *form hyperfinite families. Then* $\operatorname{ind}\mathcal{Q}$ *is also hyperfinite.*

Proof. By Lemma 2.7, for each indecomposable postinjective module X, we can find a submodule $Y := \ker\theta$ of strictly smaller defect. Moreover, if Y had a postinjective summand Z, it must have defect $\partial(Z) < \partial(X)$. We proceed by an induction on the defect d. If $d = 1$, then we can choose the hyperfinite family $\mathcal{N}_0 = \mathcal{P} \cup \mathcal{R}$ of all preprojective and regular modules. For all postinjective indecomposables of defect $d = 1$, the submodule Y must be in $\operatorname{add}\mathcal{N}_0$, since there are no non-zero postinjective modules Z with defect $\partial(Z) < 1$. The family $\mathcal{N}_0$ is hyperfinite by the hypothesis and Proposition 1.2. Moreover, the codimension of Y is bounded by the dimension of the indecomposable injectives, of which there are only finitely many. Hence, we can use Proposition 1.4 to prove the hyperfiniteness of the indecomposable postinjectives of defect one. We recursively define

$$\mathcal{N}_d := \mathcal{N}_{d-1} \cup \{\text{indecomposable postinjectives of defect } d\}.$$

Note that the base case implies that $\mathcal{N}_1$ is hyperfinite. For the induction, note that Lemma 2.7 also yields a submodule in $\operatorname{add}\mathcal{N}_d$ for every indecomposable postinjective of defect $d+1$ of bounded codimension. Assuming the hyperfiniteness of $\mathcal{N}_d$, Proposition 1.4 yields that $\mathcal{N}_{d+1}$ is hyperfinite. This concludes the induction, as the defect of the indecomposable modules is bounded. ☐

Theorem 2.9. *Let k be any field. Then the path algebra of the 2-Kronecker quiver $\Theta(2)$ is of amenable representation type.*

Proof. We fix notation for the vertices and arrows as follows.

$$\begin{tikzcd} 1 \arrow[r, bend left, "a"] \arrow[r, bend right, "b"'] & 2. \end{tikzcd}$$

It is well-known (see, e.g., [Ben98, Theorem 4.3.2] or [Bur86]) that the indecomposable preprojective and postinjective k-representations of Q can be parametrised by

$$P_n : \quad \begin{tikzcd} k^n \arrow[r, bend left, "\begin{bmatrix} \mathrm{id} \\ 0 \end{bmatrix}"] \arrow[r, bend right, "\begin{bmatrix} 0 \\ \mathrm{id} \end{bmatrix}"'] & k^{n+1}, \end{tikzcd} \qquad \text{and} \quad Q_n : \quad \begin{tikzcd} k^{n+1} \arrow[r, bend left, "\begin{bmatrix} \mathrm{id} & 0 \end{bmatrix}"] \arrow[r, bend right, "\begin{bmatrix} 0 & \mathrm{id} \end{bmatrix}"'] & k^n, \end{tikzcd} \qquad \text{respectively,}$$

both for $n \geq 0$, while the indecomposable regular representations can be parametrised by

$$R_n(\phi, \psi) : \quad \begin{tikzcd} k^n \arrow[r, bend left, "\phi"] \arrow[r, bend right, "\psi"'] & k^n, \end{tikzcd}$$

where either ϕ is the identity and ψ is given by the companion matrix of a power of a monic irreducible polynomial over k, or ψ is the identity and ϕ is given by the companion matrix of a monomial.

We will show that the preprojective component $\mathcal{P}$, the regular component $\mathcal{R}$ and the postinjective component $\mathcal{Q}$ are each hyperfinite families to conclude the amenability of

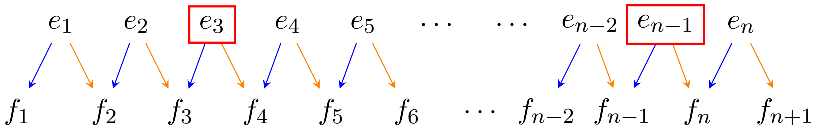


Figure 2.3: The coefficient quiver of P_n, showing a decomposition for $n \equiv 1 \mod 3$ and $K_\varepsilon = 3$.

mod kQ. We will give an argument for the indecomposable objects in each component and then apply Proposition 1.2 to extend the result.

We start with the preprojectives and let $\varepsilon > 0$. Set $K_\varepsilon := \lceil \frac{1}{2\varepsilon} \rceil + 1$ and $L_\varepsilon = \frac{1}{\varepsilon} + 3$. Let $X = P_n$ be some indecomposable preprojective. If $\dim X \leq L_\varepsilon$, there is nothing to show. We may thus assume that $\dim X > L_\varepsilon$, implying $n \geq K_\varepsilon$, and write $n = j \cdot K_\varepsilon + r$, where $0 \leq r < K_\varepsilon$. Now consider the standard basis $\{e_1, e_2, \ldots, e_n\}$ of k^n. Let U be the submodule of X generated by the subset

$$\{e_1, \ldots, e_{K_\varepsilon - 1}\} \cup \{e_{K_\varepsilon+1}, \ldots, e_{2K_\varepsilon-1}\} \cup \cdots \cup \{e_{(j-1)K_\varepsilon+1}, \ldots, e_{jK_\varepsilon-1}\} \cup \{e_{jK_\varepsilon+1}, \ldots, e_n\},$$

dropping every K_ε-th basis vector. Then U decomposes into j direct summands of type $P_{K_\varepsilon - 1}$ and a smaller rest term in case $r \neq 0$. All summands will thus be of k-dimension smaller than $2(K_\varepsilon - 1) + 1 < L_\varepsilon$. Moreover,

$$\begin{aligned} \dim U = \dim X - j = \dim X - \frac{n-r}{K_\varepsilon} &= \dim X - \frac{\dim X - 1}{2K_\varepsilon} + \frac{r}{K_\varepsilon} \\ &\geq \dim X - \varepsilon(\dim X - 1) > (1-\varepsilon)\dim X. \end{aligned}$$

This shows that the family of indecomposable preprojective modules $\mathcal{P}(kQ)$ is hyperfinite. We exemplify this process in Figure 2.3.

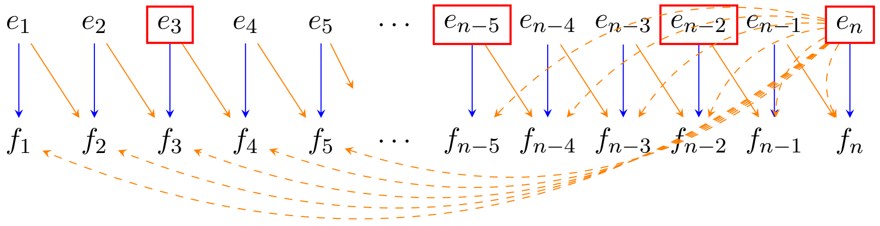


Figure 2.4: The coefficient quiver of some $R_n(\mathrm{id}, \varphi)$, exhibiting a way to find a suitable submodule for $n \equiv 2 \mod 3$, $K_\varepsilon = 3$.

If $X = R_n(\phi, \psi)$ is an indecomposable regular module, we may consider the submodule Y generated by the basis vectors $\{e_1, \ldots, e_{n-1}\}$ of the vector space at vertex 1. Note that we assume that ψ corresponds to the Frobenius companion matrix of a power of a monic polynomial. Then $Y \cong P_{n-1}$, so by the above it belongs to the hyperfinite family $\mathcal{P}(kQ)$. We have that $\dim Y = \dim X - 1$. Thus, an application of Proposition 1.4 with $H = 1$ gives the hyperfiniteness of the indecomposable regular modules. See Figure 2.4 for an example.

To deal with the postinjective modules, we apply Lemma 2.8. Now apply Proposition 1.2 to $\mathcal{P}(kQ) \cup \mathcal{R}(kQ) \cup \mathcal{Q}(kQ)$ to see that $\operatorname{mod} kQ$ is hyperfinite, and thus kQ is amenable. $\square$

2.3 Proof of Main Theorem A

Having prepared some technical results and having established the base case for the Kronecker quiver, we now move to prove a proposition that will help in proving Main Theorem A by induction.

Proposition 2.10. *Let Q be an acyclic quiver.*

(1) If $T \in \operatorname{mod} kQ$ is an exceptional module without preprojective summands, $T^{\perp}$ is equivalent to $\operatorname{mod} kQ'$ for some quiver Q'.

(2) Assume Q is of tubular type (p, q, r), where $p > 1$, and all extended Dynkin quivers of type $(p-1, q, r)$ are amenable. If T is an inhomogeneous simple regular module belonging to a tube of rank p in Γ_{kQ}, then $T^{\perp}$ is hyperfinite.

Proof. By [GL91, Proposition 1.1], in both cases, $T^{\perp}$ is an exact Abelian subcategory of $\operatorname{mod} kQ$ closed under the formation of kernels, cokernels and extensions. What is more, [GL91, Theorem 4.16] yields that $T^{\perp} = \operatorname{mod} \Lambda$ for some finite dimensional hereditary algebra Λ, along with a homological epimorphism $\varphi \colon kQ \to \Lambda$, which induces a functor $\varphi_* \colon \operatorname{mod} \Lambda \to \operatorname{mod} kQ$. By Morita equivalence, we may assume that Λ is basic.

Now, if S is any simple Λ-module, then $S \cong P/\operatorname{rad}(P)$, where P is a principal indecomposable Λ-module. By [GL91, Theorem 4.4], the natural maps

$$\operatorname{End}_{\Lambda}(P) \to \operatorname{End}_{kQ}(\varphi_* P) \quad \text{and} \quad \operatorname{Ext}^1_{\Lambda}(P, P) \to \operatorname{Ext}^1_{kQ}(\varphi_* P, \varphi_* P)$$

are isomorphisms, so φ_* maps exceptional modules to exceptional modules. It follows from [Rin94, Corollary 1] that

$$\operatorname{End}_{kQ}(\varphi_* P) \cong \operatorname{End}_{kQ}(E),$$

for some simple kQ-module E. But the simple kQ-modules all have trivial endomorphism ring k. Next, note that by [AF92, Corollary 17.12],

$$\operatorname{End}_{\Lambda}(S) \cong \operatorname{End}_{\Lambda}(P)/\mathcal{J}(\operatorname{End}_{\Lambda}(P)),$$

where $\mathcal{J}$ denotes the Jacobson radical. Recall that $\operatorname{End}_{\Lambda}(P) \cong \operatorname{End}_{kQ}(\varphi_* P) \cong k$, thus $\mathcal{J}(\operatorname{End}_{\Lambda}(P)) = 0$. Hence it follows that $\operatorname{End}_{\Lambda}(S) \cong k$. At large,

$$\operatorname{End}(\Lambda)/\mathcal{J}(\operatorname{End}(\Lambda)) \cong k \times \cdots \times k.$$

Finally, [ARS95, Proposition III.1.13] shows that Λ is isomorphic to kQ' for some quiver Q'.

It remains to prove the additional statements of (2). The proof of [Sch86, Theorem 13] implies that Λ is tame and hence the path algebra of an extended Dynkin quiver, and has tubular type $(p-1, q, r)$. By the hypothesis, it is amenable.

Now, if $F\colon \operatorname{mod} kQ' \to \operatorname{mod} \Lambda \to T^{\perp}$ is an equivalence, the simples $S(i)$ of kQ' get mapped to certain modules B_i in $\operatorname{mod} kQ$. The k-dimension of any module M over a path algebra is determined by the length of any composition series. Such a series for M in kQ' gets mapped to a composition series in the perpendicular category, and thus a series in $\operatorname{mod} kQ$, such that the factor modules are isomorphic to some B_i. Letting $K_2 := \max\{\dim B_i\}$, we thus know that

$$\dim_k F(M)_{kQ} \leq K_2 \dim_k M_{kQ'}.$$

On the other hand, if $F(M) \in T^{\perp}$, any submodule of $F(M)$ in $T^{\perp}$ is also a submodule in $\operatorname{mod} kQ$, so a composition series of $F(M)$ in $\operatorname{mod} kQ$ is at least as long as one in $T^{\perp}$. Thus,

$$\dim_k M_{kQ'} \leq \dim_k F(M)_{kQ},$$

using the fact that the length of M in $\operatorname{mod} kQ'$ equals the length of $F(M)$ considered as an object of $T^{\perp}$. Hence by Proposition 1.5, we have that each $T^{\perp}$ is a hyperfinite family. ☐

Remark. The above proposition shows a slight improvement of [GL91, Theorem 10.1(3)], by removing the condition on k to be algebraically closed.

We have now gathered the necessary accessories to prove Main Theorem A.

Theorem 2.11. *Let Q be an acyclic quiver of extended Dynkin type $\widetilde{A}_n$, $\widetilde{D}_n$, $\widetilde{E}_6$, $\widetilde{E}_7$ or $\widetilde{E}_8$. Let k be any field. Then the path algebra kQ of Q is of amenable representation type.*

Proof. Recall the tubular types and minimal radical vectors $\underline{h}_Q$ of the extended Dynkin diagrams, see Table 2.1. Note that $\widetilde{A}_{p,q}$ is a quiver of type $\widetilde{A}_n$ with $p+q = n+1$ vertices, where there are p arrows in clockwise and q arrows in anti-clockwise orientation.

We will prove the claim by induction on n for the case of the acyclic $\widetilde{A}_n$ and for $\widetilde{D}_n$, and use the case of $\widetilde{D}_5$ to prove it for the $\widetilde{E}$-family, stepping from 6 to 7 to 8.

Case $\widetilde{A}_n$. For $\widetilde{A}_1$, the only acyclic orientation is the 2-Kronecker quiver, for which its path algebra has been shown to be of amenable type in Theorem 2.9.

Now assume all acyclic quivers $\widetilde{A}_n$, for some $n \geq 1$, are of amenable representation type. Let $\widetilde{A}_{p,q}$ be of type $\widetilde{A}_{n+1}$. Then $p+q = n+2 \geq 3$. We may thus assume that $p \geq 2$, and choose a tube $\mathbb{T}$ of rank $m := p$, and denote the isoclasses of regular simples in this tube by $T_1, \ldots, T_m$. From the minimal radical vector, we see that all indecomposable preprojective kQ-modules X have defect $\partial(X) = -1$ like in the proof of Lemma 2.5. Hence Lemma 2.5 implies that every indecomposable preprojective is contained in the perpendicular category $T_i^{\perp}$ for some $1 \leq i \leq m$. By Proposition 2.10,

Q	(m_i)	$\underline{h}_Q$
$\widetilde{A}_{p,q}$	(p, q)	$\begin{smallmatrix} & 1\ldots1 & \\ 1 & & 1 \\ & 1\ldots1 & \end{smallmatrix}$
$\widetilde{D}_n$	$(2, 2, n-2)$	$\begin{smallmatrix} 1 & & & & 1 \\ & 2 & \ldots & 2 & \\ 1 & & & & 1 \end{smallmatrix}$
$\widetilde{E}_6$	$(2, 3, 3)$	$\begin{smallmatrix} & & 1 & & \\ & & 2 & & \\ 1 & 2 & 3 & 2 & 1 \end{smallmatrix}$
$\widetilde{E}_7$	$(2, 3, 4)$	$\begin{smallmatrix} & & & 2 & & & \\ 1 & 2 & 3 & 4 & 3 & 2 & 1 \end{smallmatrix}$
$\widetilde{E}_8$	$(2, 3, 5)$	$\begin{smallmatrix} & & 3 & & & & & \\ 2 & 4 & 6 & 5 & 4 & 3 & 2 & 1 \end{smallmatrix}$

Table 2.1: Tubular types and minimal radical vector of the acyclic, extended Dynkin diagrams (see, e.g., [Rin79, p. 335]).

each $T_i^{\perp}$ is hyperfinite. This shows that the preprojectives form a hyperfinite family, using Proposition 1.2.

Next, we consider the regular modules. Indecomposable regular modules in a tube other than $\mathbb{T}$ will be contained in $T_1^{\perp}$ by [Rin84, 3.1.(3')]. By Lemma 2.4, any regular indecomposable in $\mathbb{T}$ either is contained in the perpendicular category of some regular simple in $\mathbb{T}$ or has a submodule of globally bounded codimension that is in the perpendicular category of some regular simple in $\mathbb{T}$. But by the above argument, the perpendicular categories are hyperfinite. In the latter case, we can apply Proposition 1.4 to show the hyperfiniteness of these indecomposable regular modules.

For the postinjective modules, we apply Lemma 2.8.

Case $\widetilde{D}_n$. For the case of $\widetilde{D}_4$, choose a tube $\mathbb{T}$ of rank 2, and denote the regular simple modules in $\mathbb{T}$ by S and T. In this case, as an extended Dynkin quiver of tubular type $(1, 2, 2)$ is one of type $\widetilde{A}_{2,2}$, which is known by the above to have an amenable path algebra, Proposition 2.10 implies that $S^{\perp}$ and $T^{\perp}$ are hyperfinite.

All preprojective modules X of defect $\partial(X) = -1$ are in $S^{\perp}$ or $T^{\perp}$ by Lemma 2.5. Using Lemma 2.6, we can find a submodule Y for all but finitely many indecomposable preprojectives X of defect $\partial(X) = -2$, which are not themselves in $S^{\perp}$ or $T^{\perp}$. Since the dimension vector of a regular simple in $\mathbb{T}$ is bounded, the conditions of Proposition 1.4 are satisfied for all but finitely many indecomposable preprojectives of defect -2. This shows that the preprojectives form a hyperfinite family.

Moreover, the regular modules are hyperfinite: If they are in a tube other than $\mathbb{T}$, they will be contained in $S^{\perp}$ by [Rin84, 3.1.(3')]. Choosing a second inhomogeneous tube $\mathbb{T}'$ and a regular simple $U \in \mathbb{T}'$, we know that $\mathbb{T} \subset U^{\perp}$, which is also hyperfinite.

We are left to deal with the postinjective modules. Here, we again apply Lemma 2.8. This proves the claim for $\widetilde{D}_4$, using Proposition 1.2.

Now assume the case of $\widetilde{D}_n$ has been established for some $n \geq 4$. To prove the amenability of $\widetilde{D}_{n+1}$, choose $\mathbb{T}$ to be the unique tube of maximal rank $n-1$. Similarly to the base case $\widetilde{D}_4$, $S^{\perp}$ is amenable for $S \in \mathbb{T}$, since the tubular type $(2, 2, (n+1) - 2 - 1)$

belongs to $\widetilde{D}_n$. By inspection of Table 2.1 (and using an argument from the proof of Lemma 2.5), we see that the indecomposable preprojectives have negative defect one or two. Hence, they are in a hyperfinite family by Lemma 2.5. The regular indecomposables are hyperfinite by an argument similar to that of the base case. To deal with the indecomposable postinjectives, we again apply Lemma 2.8.

Case $\widetilde{E}_n$. We proceed with $\widetilde{E}_n$ for $n = 6, 7, 8$. Assume the path algebras of tubular type $(2, 3, n-4)$ have already been shown to be of amenable type. By choosing $\mathbb{T}$ to be a tube of maximal rank $m = n - 3$, we find regular simple modules S such that $S^{\perp}$ is hyperfinite, for the argument of Proposition 2.10 shows that the perpendicular category is of tubular type $(2, 3, m-1)$. Inspection tells us that any indecomposable preprojective module will have negative defect less than $2m$. Thus we can use Lemma 2.6—if needed in connection with Proposition 1.4—to show that all but finitely many, and hence all preprojective indecomposables form a hyperfinite family. For the indecomposable regular and postinjective modules, use the same arguments as for $\widetilde{D}_n$. $\square$

3 Tame hereditary algebras

3.1 Setup

In dealing with tame hereditary algebras, we will follow the notation, conventions and theory laid out for example in [BGL87, Section 1] and [Rin79, Section 1]. We start by recalling the important notions used below.

In the following, let k be a (commutative) field and A a finite dimensional k-algebra. Since the representation type is invariant under Morita equivalence, we may assume that A is basic. Further assume that A is a *hereditary* algebra, that is, an algebra such that all submodules of projective A-modules are projective again, or equivalently such that $\operatorname{gl.dim} A \leq 1$.

For $M \in \operatorname{mod} A$, the dimension vector of M is given by

$$\underline{\dim}(M) = \Big(\dim_{\operatorname{End}(P_i)} \operatorname{Hom}_A(P_i, M)\Big)_{i=1,\dots,n},$$

where $(P_1, \dots, P_n)$ is a complete system of pairwise non-isomorphic indecomposable projective modules. We call n the *rank of A*. Recall that $\underline{\dim}$ can serve as a natural map from $\operatorname{mod} A$ to its *Grothendieck group* $K_0(A)$. We denote the element of $K_0(A)$ determined by the A-module M by $[M]$. Recall further that $K_0(A)$ is the Abelian group generated by the isomorphism classes of indecomposable objects of $\operatorname{mod} A$ subject to the relation that $[Y] = [X] + [Z]$ whenever $0 \to X \to Y \to Z \to 0$ is a short exact sequence in $\operatorname{mod} A$. In the present situation, $K_0(A)$ hence coincides with the free Abelian group with n generators, $\mathbb{Z}^n$.

By the formula

$$\langle \underline{\dim}\, M, \underline{\dim}\, N \rangle_A := \dim_k \operatorname{Hom}_A(M, N) - \dim_k \operatorname{Ext}^1_A(M, N),$$

we define a bilinear form on $K_0(A)$ and call it the *Euler form of A*. We denote the associated quadratic form by $q_A = \langle -, - \rangle_A$ and call it the *Tits-form of A*. The homological bilinear form can also be expressed combinatorially: On the standard basis e_i given by the dimension vectors of the simple modules, put

$$\langle e_i, e_j \rangle = \delta_{ij} \dim_k \operatorname{End}_A(P_i) - \dim_k \epsilon_i \, \mathcal{J} \,/\, \mathcal{J}^2 \, \epsilon_j,$$

where δ_{ij} is the Kronecker delta, $\mathcal{J}$ is the Jacobson radical of A and ϵ_i is the idempotent associated to P_i such that $A\epsilon_i = P_i$. Note that the endomorphism rings are division rings over k. Further note that this is the bilinear form from [Rin76] and leads to the quadratic form as in [Rin79, Section 1].

In addition to this bilinear form, we can also associate to a basic, finite dimensional k-algebra a valued diagram in the following way: The vertices correspond to the isomorphism classes of simple modules S_i of A, while there is an edge $i \to j$ if $\operatorname{Ext}^1_A(S_i, S_j) \neq 0$. Note that this Ext-space is a left-End(S_i)-right-End(S_j)-bimodule. We label the edge by (d_{ij}, d'_{ij}), where

$$d_{ij} = \dim_{\operatorname{End}(S_i)} \operatorname{Ext}^1_A(S_i, S_j)$$

and

$$d'_{ij} = \dim D\operatorname{Ext}^1_A(S_i, S_j)_{\operatorname{End}(S_j)}.$$

However, in case $d_{ij} = d'_{ij} = 1$, we may omit the label. For more details, also see [DR76, Section 1; Rin79, Section 1] or [Ben98, Section 4.1].
Note that a related combinatorial description is achieved by generalised Cartan matrices or a Cartan datum (see, e.g., [Moo69] and [Kac90, §1.1; HK16, Sections 3-4]).

For the ensuing definition, we will follow [Cra91; Rin79, Section 1].

Definition 3.1. A connected, finite dimensional k-algebra A is called **tame hereditary** provided A is hereditary and A is of tame representation type, that is its quadratic form q_A is semidefinite but not positive definite.

Given the first two terms of a minimal projective resolution $P_1 \xrightarrow{f} P_2 \to M$ of a left A-module M, we denote by $\operatorname{Tr} M$ the cokernel of $\operatorname{Hom}(f, {}_AA)$, a right A-module. By $\tau(M) := D \operatorname{Tr} M$ we denote the k-dual of this transpose and call it the *Auslander–Reiten translation.* Similarly, we call $\tau^{-1} = \operatorname{Tr} D$ the inverse Auslander–Reiten translation. For hereditary A, τ is functorial, vanishes on the projective modules and induces an equivalence with inverse τ^{-1} from the category of all finite dimensional A-modules without non-trivial projective summands into the category of all finite dimensional A-modules without non-trivial injective summands. Restriction to the former subcategory gives an exact functor. We further define a group isomorphism $c\colon K_0(A) \to K_0(A)$ such that $c([M]) = [\tau M]$ for nonprojective indecomposable A-modules M and call it the *Coxeter transformation.* Note that we have

$$\langle x, y \rangle = -\langle y, c(x) \rangle = \langle c(x), c(y) \rangle,$$

and hence the quadratic form q_A is invariant under this transformation.

Using τ, we can distinguish between up to three kinds of indecomposable modules in $\operatorname{mod} A$: The *preprojective* indecomposable modules are of the form $\tau^{-m}(P_i)$ for some non-negative integer m and some $i \in \{1, \ldots, n\}$. The *postinjective* indecomposable modules are of the form $\tau^m(I_i)$ for some non-negative integer m and some $i \in \{1, \ldots, n\}$. Here, the I_i denote the indecomposable, injective modules. The remaining indecomposable modules and their finite direct sums are called *regular.*

Let us now assume that A is tame. The radical $\{x \in K_0(A) \colon q_A(x) = 0\}$ of the associated semidefinite quadratic form q_A is then generated by a vector $\underline{h}_A$ whose entries are positive integers and at least one of them is one. We call this the *minimal*

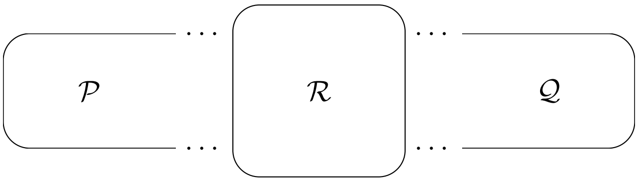


Figure 3.1: Structure of the module category mod A of a tame hereditary algebra A.

positive radical element of A. Further recall that there exists an indecomposable regular module S with $\underline{\dim}(S) = \underline{h}_A$. Any indecomposable A-module M with $\underline{\dim}\, M$ a multiple of $\underline{h}_A$ will be called *homogeneous*. Related with S is the linear form $\langle \underline{\dim}\, S, - \rangle$ on $K_0(A)$. We put $\partial M := \partial(\underline{\dim}\, M) := \frac{1}{r}\langle \underline{\dim}\, S, \underline{\dim}\, M\rangle$ for the normalised form such that $\partial P_i = -1$ for some $i \in \{1, \dots, n\}$. We will consider this form as a function $\partial\colon \operatorname{mod} A \to \mathbb{Z}$ and call it the *defect of* A. Note that it is also invariant under c. One of the main properties of the defect is that we can use it to characterise the preprojective, regular and postinjective indecomposables. Namely, $\partial M < 0$ [$\partial M = 0$, $\partial M > 0$] if and only if M is preprojective [regular, postinjective, respectively]. The closure of the preprojective, regular and postinjective indecomposable modules under finite direct sums will be denoted by $\mathcal{P}(A)$, $\mathcal{R}(A)$ and $\mathcal{Q}(A)$, respectively.

Using these facts, we can establish that in the tame hereditary representation type case, the module category can be depicted as in Figure 3.1. Within in the picture, non-zero morphisms only exist from left to right.

In the tame case, the finite dimensional regular modules $\mathcal{R} = \mathcal{R}(A)$ also form an exact Abelian subcategory. As this category is Abelian, we consider its simple objects and call them *regular simple*. If M is in $\mathcal{R}$, the sum of the regular simple submodules of M is called the *regular socle* of M, and the length n of a regular composition series

$$0 = M_0 \subset M_1 \subset \cdots \subset M_{n-1} \subset M_n = M,$$

with M_i/M_{i-1} regular simple for all i, is called the *regular length of* M. Given a regular simple module S and $n \in \mathbb{N}$, there is a unique indecomposable regular module $S^{[n]}$ with regular socle S and of regular length n, and every indecomposable regular module is of this form.

It is therefore useful to consider the set of (isomorphism classes of) regular simple modules. On this set, τ operates with finite orbits, and all but at most three orbits are one element sets. Let $\mathbb{X}$ be the set of these orbits. If S and S' are regular simple, then $\operatorname{Ext}^1_A(S, S') \neq 0$ if and only if $S' = \tau S$. Thus, the category $\mathcal{R}$ decomposes as the direct sum of categories R_t, where t runs through the set $\mathbb{X}$, and an indecomposable regular module with regular composition series given by $\{M_i\}$ belongs to R_t if and only if one—and therefore all—of the regular composition factors M_i/M_{i-1} belongs to R_t. We have thus decomposed $\mathcal{R} \cong \coprod_{t\in\mathbb{X}} R_t(A)$ into uniserial subcategories $R_t(A)$, where each $R_t(A)$ has only finitely many simple objects. We call the set of (isomorphism classes of) indecomposable objects in such a uniserial category R_t a *tube* and use $\mathbb{T}$ to denote it for a fixed $t \in \mathbb{X}$. If a tube $\mathbb{T}$ does not contain projective or injective objects, we say that it is *stable*. The regular simple modules of R_t are said to form the *mouth* of the corresponding tube $\mathbb{T}$.

Recall further that the tubular $\mathbb{X}$-family $\mathcal{R}$ is *separating*, as $\mathcal{R}$ is standard,

$$\operatorname{Hom}(\mathcal{Q},\mathcal{P}) = \operatorname{Hom}(\mathcal{Q},\mathcal{R}) = \operatorname{Hom}(\mathcal{R},\mathcal{P}) = 0,$$

and given $t \in \mathbb{X}$, any map $\mathcal{P} \to \mathcal{Q}$ can be factored through R_t. In this situation, as the indecomposable regular modules belong to tubes, we may write $\mathcal{T}$ instead of $\mathcal{R}$ and $\mathcal{T}_t$ instead of R_t.

For S regular simple in R_t, let n_t denote the smallest positive integer such that

$$\tau^{n_t} S \cong S.$$

We call n_t the *rank of the tube* corresponding to $R(t)$. Note that $S^{[n_t]}$ is always homogeneous, whereas modules $S^{[i]}$, with $1 \leq i < n_t$ are not homogeneous. For S regular simple in R_t, S itself is homogeneous if and only if $n_t = 1$, if and only if all modules in R_t are homogeneous. In this case, we say that R_t itself is homogeneous. Otherwise, we call R_t *inhomogeneous* or *exceptional*. Recall that there are at most three exceptional tubes. We collect the tuple of the n_t for which R_t is inhomogeneous as a triple (n_1, n_2, n_3). It determines the *tubular type of* A. Describing $\mathbb{X}$ in general may be difficult, but $\mathbb{X}$ is always an infinite set. In the case that k is algebraically closed, we can identify $\mathbb{X}$ with $\mathbb{P}_1(k)$, the points of the projective line over k.

A classification We now want to recall a classification of the tame hereditary algebras. We start by introducing two classes of algebras.

Given a k-algebra A and an A-bimodule M, on which k acts centrally, in general, we can form the *tensor algebra*

$$T_A(M) := \bigoplus_{n \geq 0} M^{\otimes n} = A \oplus M \oplus (M \otimes_A M) \oplus (M \otimes_A M \otimes_A M) \oplus \dots.$$

Note that if A is semisimple, $T_A(M)$ is hereditary. Those hereditary tensor algebras that are tame have been studied closely for instance by [DR76]. The underlying graphs that occur in this case (excluding the finite representation type cases) are the extended Dynkin diagrams of Figure 3.2.

We also recall the following from [DR78, Section 1]: For a field k, a k-automorphism ε and an $(\varepsilon, 1)$-derivation δ of k, we define the k-k-bimodule $M(\varepsilon, \delta)$ which as a left k-vector space is ${}_k k \oplus {}_k k$, while the right k-action is given by

$$(a, b) \cdot \lambda = (a\lambda + b\delta(\lambda), b\varepsilon(\lambda)) \quad \text{for} \quad a, b, \lambda \in k.$$

We then denote by $\tilde{A}_n(\varepsilon, \delta)$ the $(n+1) \times (n+1)$-matrix ring

$$\begin{pmatrix} k & & & & \\ k & k & & 0 & \\ \vdots & \vdots & \ddots & & \\ k & k & \dots & k & \\ M(\varepsilon,\delta) & k & \dots & k & k \end{pmatrix}.$$

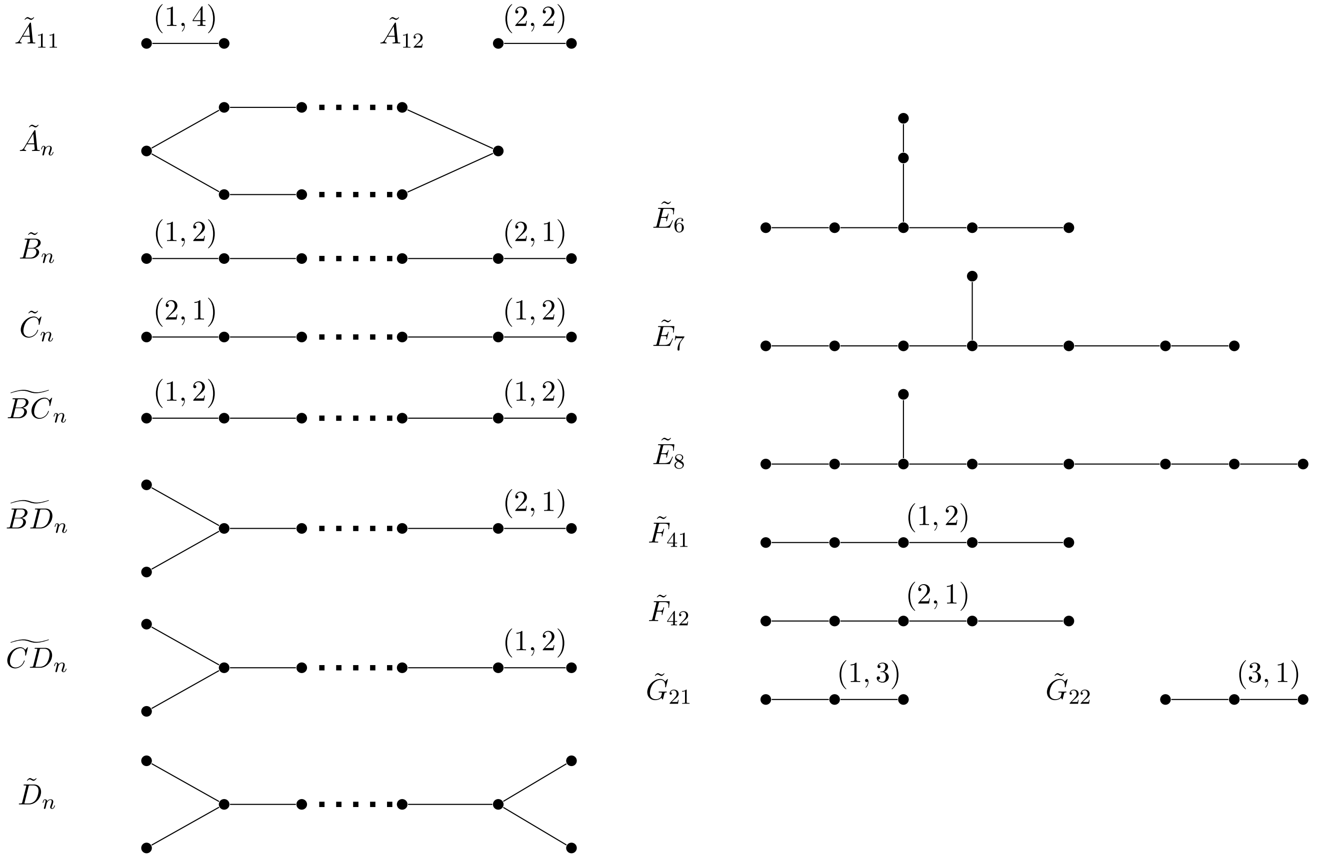


Figure 3.2: (not necessarily simply-laced) extended Dynkin diagrams of types A through G (c.f. [DR76])

We can now recall the desired classification:

Theorem 3.2. *[DR78, Corollary 2] A hereditary finite dimensional k-algebra A is of tame representation type if and only if it is Morita equivalent to the product of a tame tensor algebra and a finite number of algebras of the form $\tilde{A}_n(\varepsilon, \delta)$.*

We further note that $\tilde{A}_1(\varepsilon, \delta)$ is indeed a tensor algebra. Thus, all connected tame hereditary algebras of rank two are tensor algebras.

A first lemma pertaining the existence of preprojective submodules of bounded codimension of regular modules will be used in the next section but also later on.

Lemma 3.3. *Let A be a finite dimensional tame hereditary algebra. Denote by q the maximum of the dimensions of the indecomposable injective modules. Let R be an indecomposable regular module. Then there exists a preprojective submodule $U \subseteq R$ such that*

$$\dim_k R - \dim_k U \leq q.$$

Proof. Let S be the regular socle of R. Then there exists a non-zero map $f \colon S \to I$, where I is an indecomposable injective module. Since I is injective and $\iota \colon S \to R$ is injective, this map lifts to a map $\bar{f} \colon R \to I$. Denote $U := \ker \bar{f}$ and $K := \ker f$. Using the Snake Lemma, we get an exact commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & K \arrow[r] \arrow[d] & U \arrow[r] \arrow[d] & R/S \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & S \arrow[r, hook, "\iota"] \arrow[d, "f"'] & R \arrow[r] \arrow[d, "\bar{f}"'] & R/S \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I \arrow[r, equal] \arrow[d] & I \arrow[r] \arrow[d] & 0 & \\
 & 0 & 0 & &
\end{tikzcd}$$

Clearly, K and U as subobjects of regular modules cannot have postinjective summands, as there are no non-zero homomorphisms $\mathcal{Q} \to \mathcal{R}$. Assume that U had a regular summand. Then this summand must contain S respectively $\iota(S)$, for this is the smallest regular submodule of R, using the fact that the regular modules form a uniserial category and no other tubes map to R. But then we have

$$0 = \bar{f}(U) \supset \bar{f}(\iota S) = f(S) \neq 0,$$

a contradiction. Hence U can only have preprojective summands and is therefore preprojective.

□

3.2 Finding large submodules for rank two algebras

As was the case for path algebras of affine quivers, we begin our study by considering algebras of rank two, the non-algebraically closed analogue of the 2-Kronecker quiver case. As the matrix ring $\tilde{A}_1(\varepsilon, \delta)$ is a tensor algebra, the representation theory of all tame hereditary algebras of rank two is covered by the methods of [DR76].

Lemma 3.4. *Let A be a tame hereditary algebra of rank two. Then there is $g \in \mathbb{N}^+$ such that*

$$c^{\pm}(x) = x \pm g\underline{h}_A \partial(x), \qquad \textit{for all } x \in K_0(A).$$

Proof. Recall that the defect is given by $\partial(x) = \frac{1}{r}\langle \underline{h}_A, x\rangle$ for some $r \in \mathbb{N}^+$. We show the claim by a case-by-case analysis. There are two types of algebras of rank two, namely $\tilde{A}_{12}$ with label $(2, 2)$ and $\tilde{A}_{11}$. For the latter, we also take into consideration the two possible orientations with labels $(1, 4)$ and $(4, 1)$.

label $(2, 2)$. Here, we have $\underline{h}_A = (1, 1)$ and the Coxeter transformation (respectively its inverse) on $\mathbb{Z}^2$ is given by

$$c = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}.$$

Now

$$\begin{aligned} c(x_1, x_2) = (-x_1 + 2x_2, -2x_1 + 3x_2) &= (x_1, x_2) + (-2x_1 + 2x_2, -2x_1 + 2x_2) \\ &= (x_1, x_2) + (2, 2)(-x_1 + x_2). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \partial(x) &= \langle \underline{h}_A, x\rangle = \langle e_1, x\rangle + \langle e_2, x\rangle \\ &= \Big(x_1 \dim_k \mathrm{End}_A P_1 - x_2 \dim_k \varepsilon_1 \mathcal{J} \,/\, \mathcal{J}^2 \varepsilon_2\Big) \\ &\quad + \Big(x_2 \dim_k \mathrm{End}_A P_2 - x_1 \dim_k \varepsilon_2 \mathcal{J} \,/\, \mathcal{J}^2 \varepsilon_1\Big) \\ &= x_1 + x_2 - 2x_1 = -x_1 + x_2. \end{aligned}$$

label $(1, 4)$. Here, we have $\underline{h}_A = (2, 1)$ and the Coxeter transformation (respectively its inverse) on $\mathbb{Z}^2$ is given by

$$c = \begin{pmatrix} -1 & 4 \\ -1 & 3 \end{pmatrix}.$$

Now

$$\begin{aligned} c(x_1, x_2) = (-x_1 + 4x_2, -x_1 + 3x_2) &= (x_1, x_2) + (-2x_1 + 4x_2, -x_1 + 2x_2) \\ &= (x_1, x_2) + (2, 1)(-x_1 + 2x_2). \end{aligned}$$

On the other hand, we have

$$\begin{aligned}\partial(x) &= \frac{1}{2}\langle \underline{h}_A, x\rangle = \frac{1}{2}\left(2\langle e_1, x\rangle + \langle e_2, x\rangle\right) \\ &= (x_1) + \frac{1}{2}\left(4x_2 - 4x_1\right) = -x_1 + 2x_2.\end{aligned}$$

label $(4,1)$. Here, we have $\underline{h}_A = (1,2)$ and the Coxeter transformation (respectively its inverse) on $\mathbb{Z}^2$ is given by

$$c = \begin{pmatrix} -1 & 1 \\ -4 & 3 \end{pmatrix}.$$

Now

$$\begin{aligned}c(x_1, x_2) = (-x_1 + x_2, -4x_1 + 3x_2) &= (x_1, x_2) + (-2x_1 + x_2, -4x_1 + 2x_2) \\ &= (x_1, x_2) + (1,2)(-2x_1 + x_2).\end{aligned}$$

On the other hand, similar to the other orientation, we have $\partial(x) = -2x_1 + x_2$. □

Proposition 3.5. *Let P be an indecomposable projective module of defect $\partial(P) = -1$. For $i \geq 0$, in type $\tilde{A}_{11}$, consider the modules $P[i] := \tau^{-i}P$ and in type $\tilde{A}_{12}$ consider the indecomposable preprojective modules $P[i]$ with $\underline{\dim}\, P[i] = \underline{\dim}\, P + i\underline{h}_A$. Choose some non-zero homomorphism $P \to P[1]$. Let R be the regular module given as the cokernel of this map and denote by $R^{[m]}$ the unique indecomposable regular module with regular socle R and of regular length m. Then for any $n > 0$ and $m \leq n$, there exists an epimorphism $\phi_{n,m} \colon P[n] \twoheadrightarrow R^{[m]}$ with kernel $P[n-m]$.*

Proof. Let P be an indecomposable projective module of defect $\partial(P) = -1$. For type $\tilde{A}_{11}$, let $P[1] := \tau^{-1}P$ be its inverse Auslander–Reiten-translate. For type $\tilde{A}_{12}$, let $P[1]$ be the uniquely determined indecomposable preprojective module with dimension vector $\dim P + \underline{h}_A$. Then $\langle P, P[1]\rangle > 0$, so there is a non-zero map $\iota \colon P \to P[1]$. Since $\partial(P) = -1$, this map must be injective. Consider the short exact sequence

$$\eta \colon 0 \to P \xrightarrow{\iota} P[1] \xrightarrow{\pi} R \to 0,$$

where $R = \operatorname{coker} \iota$. Since $\partial(P) = \partial(P[1]) = -1$, we have $\partial(R) = 0$. If R had a postinjective summand Q of defect ≥ 1, we would get the pullback

$$\begin{tikzcd}
0 \arrow[r] & P \arrow[r, hook] \arrow[d, equal] & T \arrow[r, two heads] \arrow[d, hook] & Q \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & P \arrow[r, hook, "\iota"] & P[1] \arrow[r, two heads, "\pi"] & R \arrow[r] & 0,
\end{tikzcd}$$

where $\partial(T) = \partial(P) + \partial(Q) \geq 0$, but T is also a submodule of $P[1]$, hence preprojective. This implies that $T = 0$. Thus, R cannot have postinjective and hence neither preprojective summands. It is therefore regular. By construction, as $\partial(P) = -1$ and due to Lemma 3.4, we have that $\underline{\dim} R = \underline{h}_A$. As every (indecomposable) submodule of R would have dimension vector belonging to an indecomposable preprojective or postinjective module, we see that R must be simple regular.

Next, consider the regular module $R^{[2]}$ such that

$$\xi\colon 0 \to R \to R^{[2]} \to R \to 0,$$

is an almost split sequence, that is $0 \neq \xi \in \operatorname{Ext}^1_A(R,R)$. By applying $\operatorname{Hom}_A(R,-)$ to η, we get a long exact sequence

$$0 \to (R,P) \xrightarrow{\iota_*} (R,P[1]) \xrightarrow{\pi_*} (R,R) \xrightarrow{\eta_*} {}^1(R,P) \xrightarrow{\iota_*} {}^1(R,P[1]) \xrightarrow{\pi_*} {}^1(R,R) \to 0,$$

as A is hereditary. This shows that the map $\pi_*\colon \operatorname{Ext}^1_A(R,P[1]) \to \operatorname{Ext}^1_A(R,R)$ is surjective. Hence, there exists $\zeta \in \operatorname{Ext}^1_A(R,P[1])$ s.t. $\pi_*(\zeta) = \xi$, where π_* is the push-out map. Now,

$$\begin{tikzcd}
& 0 \arrow[r] & P \arrow[r] \arrow[d, hook, "\iota"'] & \ker\phi \arrow[r] \arrow[d, hook] & 0 \arrow[d] \arrow[dddll, dashed] & \\
\zeta\colon & 0 \arrow[r] & P[1] \arrow[r, hook, "\sigma"] \arrow[d, two heads, "\pi"'] & E \arrow[r, two heads, "\theta"] \arrow[d, two heads, "\phi"'] & R \arrow[r] \arrow[d, equal] & 0 \\
\xi = \pi_*(\zeta)\colon & 0 \arrow[r] & R \arrow[r, hook, "\alpha"] \arrow[d] & R^{[2]} \arrow[r, two heads, "\beta"] \arrow[d] & R \arrow[r] \arrow[d] & 0 \\
& & \operatorname{coker}\pi \arrow[r] & \operatorname{coker}\pi' \arrow[r] & 0 &
\end{tikzcd}$$

is an exact commutative diagram, using the Snake Lemma. Altogether, we get the exact commutative diagram

$$\begin{tikzcd}
& & & 0 \arrow[d] & 0 \arrow[d] & \\
\eta\colon & 0 \arrow[r] & P \arrow[r, hook, "\iota"] \arrow[d, "\cong"'] & P[1] \arrow[r, two heads, "\pi"] \arrow[d, hook, "\sigma"'] & R \arrow[r] \arrow[d, hook, "\alpha"'] & 0 \\
& 0 \arrow[r] & \ker\phi \arrow[r, hook, "\gamma"] & E \arrow[r, two heads, "\phi"] \arrow[d, two heads, "\theta"'] & R^{[2]} \arrow[r] \arrow[d, two heads, "\beta"'] & 0 \\
& & & R \arrow[r, equal] \arrow[d] & R \arrow[d] & \\
& & & 0 & 0 &
\end{tikzcd}$$

where $\ker\phi \cong P$ via the Snake Lemma, for $\ker(R \xrightarrow{\sim} R) = 0$.

Now assume that E has a regular summand, say $E = R' \oplus E'$ for regular R'. Since the regular component is uniserial, we must have

i) $\phi(R') \subseteq \alpha(R)$ or

ii) $\alpha(R) \subset \phi(R') \subseteq R^{[2]}$.

Since $\alpha(R)$ is a maximal submodule of $R^{[2]}$, ii) implies $\phi(R') = R^{[2]}$.

i). Applying β yields $\theta(R') = \beta\phi(R') \subseteq \beta\alpha(R) = 0$, so $R' \subseteq \ker\theta \cong \operatorname{im}\sigma \cong P[1]$, thus R' is preprojective for A is hereditary. A contradiction.

ii). This implies that $\phi_{|R'} \colon R' \to R^{[2]}$ is surjective. On the other hand,

$$\ker(\phi_{|R'}) = \ker(\phi) \cap R' \subseteq \ker(\phi) \cong P,$$

so the kernel is preprojective. But $\phi_{|R'}$ is a map between regular modules, thus has a regular kernel. Hence $\ker(\phi_{|R'}) = 0$, and $\phi_{|R'} \colon R' \to R^{[2]}$ is bijective. But then there exists a section $\phi' \colon R^{[2]} \to R' \subset E$ such that $\phi \circ \phi' = \mathrm{id}_{R^{[2]}}$. This is equivalent to the existence of a retraction $\gamma' \colon E \to P$ such that $\gamma' \circ \gamma = \mathrm{id}_P$. But then

$$\gamma' \circ \sigma \circ \iota = \gamma' \circ \gamma = \mathrm{id}_P,$$

so η splits. A contradiction.

Altogether, E must be preprojective. Note that as an extension of a preprojective and a regular module it cannot be postinjective. Moreover,

$$\partial(E) = \partial(P[1]) + \partial(R) = \partial(P[1]) = -1,$$

so it is indecomposable. Besides,

$$\underline{\dim}\, E = \underline{\dim}\, P[1] + \underline{\dim}\, R = \underline{\dim}\, P[1] + (\underline{\dim}\, P[1] - \underline{\dim}\, P) = \underline{\dim}\, P[2].$$

Hence $E \cong P[2]$, since both are rigid modules and thus determined by their dimension vectors. We have therefore constructed a surjective map $\phi \colon P[2] \to R^{[2]}$ with kernel P.

Now assume we have already constructed a surjective map $\pi_m \colon P[m] \to R^{[m]}$ for some $m \in \mathbb{N}$ with kernel P. Consider the exact sequence

$$\eta_m \colon 0 \to P \xrightarrow{\iota_m} P[m] \xrightarrow{\pi_m} R^{[m]} \to 0.$$

As in the base case, the pushout map $\pi_* \colon \operatorname{Ext}^1_A(R, P[m]) \to \operatorname{Ext}^1_A(R, R^{[m]})$ is surjective. Thus, given the standard AR-sequence (as it appears for instance in [Rin84, 3.1.(2)(a)])

$$\xi_m \colon 0 \to R^{[m]} \xrightarrow{\alpha_m} R^{[m+1]} \xrightarrow{\beta_m} R \to 0,$$

there exists some $\zeta_m \in \operatorname{Ext}^1_A(R, P[m])$,

$$\zeta_m \colon 0 \to P[m] \xrightarrow{\sigma_m} E_m \xrightarrow{\theta_m} R \to 0,$$

such that we get the following exact commutative diagram.

$$\begin{tikzcd}
 & & & 0 \arrow[d] & 0 \arrow[d] & \\
\eta_m\colon & 0 \arrow[r] & P \arrow[r, hook, "\iota_m"] \arrow[d, "\cong"'] & P[m] \arrow[r, two heads, "\pi_m"] \arrow[d, hook, "\sigma_m"'] & R^{[m]} \arrow[r] \arrow[d, hook, "\alpha_m"'] & 0 \\
 & 0 \arrow[r] & \ker\phi_m \arrow[r, hook, "\gamma_m"] & E_m \arrow[r, two heads, "\phi_m"] \arrow[d, two heads, "\theta_m"'] & R^{[m+1]} \arrow[r] \arrow[d, two heads, "\beta_m"'] & 0 \\
 & & & R \arrow[r, equal] \arrow[d] & R \arrow[d] & \\
 & & & 0 & 0 &
\end{tikzcd}$$

As in the base case, $\ker\phi_m \cong P$. Assume that E_m has a regular summand R', say $E_m \cong R' \oplus E'$. Since the regular component is uniserial, we must have

i) $\phi_m(R') \subseteq \alpha_m(R^{[m]})$ or

ii) $\alpha_m(R^{[m]}) \subset \phi_m(R') \subseteq R^{[m+1]}$.

Since $\alpha_m(R^{[m]})$ is a maximal submodule of $R^{[m+1]}$, ii) implies $\phi_m(R') = R^{[m+1]}$.

i). Applying β_m yields $\theta_m(R') = \beta_m\phi_m(R') \subseteq \beta_m\alpha_m(R^{[m]}) = 0$, so

$$R' \subseteq \ker\theta_m \cong \operatorname{im}\sigma_m \cong P[m],$$

thus R' is preprojective as A is hereditary. A contradiction.

ii). This implies that $\phi_{m|R'}\colon R' \to R^{[m+1]}$ is surjective. On the other hand,

$$\ker(\phi_{m|R'}) = \ker(\phi_m) \cap R' \subseteq \ker(\phi_m) \cong P,$$

so the kernel is preprojective. But $\phi_{m|R'}$ is a map between regular modules, thus has a regular kernel. Hence $\ker(\phi_{m|R'}) = 0$, and $\phi_{m|R'}\colon R' \to R^{[m+1]}$ is bijective. But then there exists a section $\phi'_m\colon R^{[m+1]} \to R' \subset E_m$ such that $\phi_m \circ \phi'_m = \mathrm{id}_{R^{[m+1]}}$. This is equivalent to the existence of a retraction $\gamma'_m\colon E_m \to P$ such that $\gamma'_m \circ \gamma_m = \mathrm{id}_P$. But then $\gamma'_m \circ \sigma_m \circ \iota_m = \gamma'_m \circ \gamma_m = \mathrm{id}_P$, so η_m splits. A contradiction.

All in all, E_m must be preprojective. Note that it cannot be postinjective as an extension of preprojective and regular modules. Moreover,

$$\partial(E_m) = \partial(P[m]) + \partial(R) = \partial(P[m]) = -1,$$

so E_m is indecomposable. Besides,

$$\underline{\dim}\, E_m = \underline{\dim}\, P[m] + \underline{\dim}\, R = \underline{\dim}\, P[m] + (\underline{\dim}\, P[1] - \underline{\dim}\, P) = \underline{\dim}\, P[m+1].$$

Hence $E \cong P[m+1]$. We have therefore constructed a surjective map

$$\phi\colon P[m+1] \to R^{[m+1]}$$

with kernel P.

Thus, we may assume that there exists a surjective map $\pi_n \colon P[n] \to R^{[n]}$ for all $n > 0$ which has kernel P. Since there is also a surjective map $R^{[n]} \to R^{[m]}$ for all $m \leq n$ by the uniseriality, we have established the existence of a surjective homomorphism $\pi_{n,m} \colon P[n] \to R^{[m]}$. This morphism fits into a short exact sequence

$$0 \to \ker \pi_{n,m} \hookrightarrow P[n] \twoheadrightarrow R^{[m]} \to 0.$$

Since $P[n]$ is preprojective, so is $\ker \pi_{n,m}$. Since $\partial(\ker \pi_{n,m}) = \partial(P_n) - \partial(R^{[m]}) = -1$, it is indecomposable. Moreover,

$$\begin{aligned} \underline{\dim} \ker \pi_{n,m} &= \underline{\dim}\, P[n] - \underline{\dim}\, R^{[m]} \\ &= \underline{\dim}\, P[m] + (m-n)\,(\underline{\dim}\, P[1] - \underline{\dim}\, P) - (\underline{\dim}\, P[m] - \underline{\dim}\, P) \\ &= \underline{\dim}\, P[n-m], \end{aligned}$$

using the recursion formula $\underline{\dim}\, P[i+1] = \underline{\dim}\, P[i] + (\underline{\dim}\, P[1] - \underline{\dim}\, P)$ along with $\underline{\dim}\, R^{[m]} = \underline{\dim}\, P[m] - \underline{\dim}\, P$. Hence $\ker \pi_{n,m} \cong P[n-m]$. ☐

Algorithm for preprojectives of defect -1 for rank 2 examples

Let A be a tame hereditary algebra of rank two, with minimal radical vector $\underline{h}_A$. Put $h_A := \sum_{j=1}^{n} f_j \, (\underline{h}_A)_j$ to be the k-weighted sum of the entries, where $f_j = \dim_k \operatorname{End}(S_j)$ is the k-dimension of the endomorphism ring of the simple module S_j. Let Q be the indecomposable injective module of maximal k-dimension and put $\underline{q} = \underline{\dim}\, Q$ and

$$q = \sum_{j=1}^{n} \dim_k Q_j.$$

Let $1 > \varepsilon > 0$. Choose $L_\varepsilon := \max\left\{\frac{2q}{\varepsilon}, 2gh_A\right\}$ where g is from Lemma 3.4.

Now let $X = P[i_0]$ be any indecomposable preprojective module of defect $\partial(X) = -1$, where P is an indecomposable projective of defect -1. Denote the corresponding regular modules as in Proposition 3.5 by $R^{[i]}$ respectively. If $\dim_k X \leq L_\varepsilon$, choosing $Y = X$ we have found a suitable submodule to prove hyperfiniteness. Hence we may assume that $\dim_k X > L_\varepsilon$.

We will now give an iterative construction involving i_j, $T_j = P[i_j]$ and t_{j+1}. We start with $j = 0$.

For each j we proceed as follows: Choose $t_{j+1} \leq i_j$ such that the regular indecomposable $R^{[t_{j+1}]}$ fulfils

$$L_\varepsilon - gh_A \leq \dim_k R^{[t_{j+1}]} \leq L_\varepsilon.$$

This is possible, since by construction $\underline{\dim}\, R^{[1]} = g\underline{h}_A$ and $P[i_j]$ surjects onto all $R^{[m]}$ with $m \leq i_j$. Then by Proposition 3.5, we get a short exact sequence

$$0 \to P[i_j - t_{j+1}] \to P[i_j] \to R^{[t_{j+1}]} \to 0.$$

By Lemma 3.3, there exists a preprojective submodule $U_{j+1} \subset R^{[t_{j+1}]}$ such that

$$\underline{\dim}\, R^{[t_{j+1}]} - \underline{\dim}\, U_{j+1} \leq \underline{q}.$$

Consider the commutative diagram given by the pullback $T_j \times_{R^{[t_{j+1}]}} U_{j+1}$.

$$\begin{tikzcd}
0 \arrow[r] & P[i_j - t_{j+1}] \arrow[r, hook] \arrow[d, equal] & E_{j+1} \arrow[r, two heads] \arrow[d, hook] & U_{j+1} \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & P[i_j - t_{j+1}] \arrow[r, hook] & T_j \arrow[r, two heads] & R^{[t_{j+1}]} \arrow[r] & 0.
\end{tikzcd}$$

If the upper of the two sequences is non-split, we have

$$0 \neq \mathrm{Ext}^1_A(S, P[i_j - t_{j+1}]) \cong D\mathrm{Hom}_A(P[i_j - t_{j+1}], \tau S),$$

for some indecomposable direct summand $S \mid U_{j+1}$ and a non-zero homomorphism

$$P[i_j - t_{j+1}] \to \tau S,$$

which must be a monomorphism, since $\partial(P[i_j - t_{j+1}]) = -1$. Hence,

$$\dim P[i_j - t_{j+1}] \leq \dim \tau S \leq \dim S \leq L_\varepsilon. \tag{3.1}$$

Here, we use Lemma 3.4 to ensure that the AR-translation decreases the dimension of an indecomposable, preprojective module. But now the iteration terminates at $N = j + 1$. We choose $Y := E_N \oplus \bigoplus_{\ell=1}^{N-1} U_\ell$ and have found a suitable submodule: Its summands' dimensions are bounded by $2L_\varepsilon$ due to (3.1) and the fact that all $R^{[t_\ell]}$ have bounded dimension. Also, $\dim Y$ is large enough using

$$\underline{\dim}\, X = \underline{\dim}\, E_N + \sum_{\ell=1}^{N-1} \underline{\dim}\, U_\ell + \sum_{\ell=1}^{N} e_\ell,$$

where e_ℓ is an error vector bounded by $\underline{q}$.

Thus we may assume that the upper sequence splits and $E_{j+1} \cong P[i_j - t_{j+1}] \oplus U_{j+1}$. Now let $i_{j+1} := i_j - t_{j+1}$ and proceed with step $j + 1$. This process terminates since the dimension of T_j compared to T_{j-1} decreases in each "splitting" step by at least $L_\varepsilon - gh_A$ until it is smaller than L_ε. Note that the number of steps N is bounded, as

$$N \leq \frac{\dim_k X}{L_\varepsilon - gh_A} \leq \frac{\dim_k X}{(\frac{q}{\varepsilon} + gh_A) - gh_A} = \frac{\dim_k X\varepsilon}{q}.$$

Moreover, in each step we have

$$\underline{\dim}\, T_j = \underline{\dim}\, T_{j+1} + \underline{\dim}\, R^{[t_{j+1}]} \quad \text{and} \quad \underline{\dim}\, R^{[t_j]} = \underline{\dim}\, U_j + e_j,$$

where e_j is an error vector bounded by $\underline{q}$. Combining this in a telescope sum yields

$$\underline{\dim}\, X = \underline{\dim}\, T_0 = \cdots = \underline{\dim}\, T_N + \sum_{\ell=1}^{N} \left(\underline{\dim}\, U_\ell + e_\ell\right). \tag{3.2}$$

Define the submodule $Y \subset X$ to be

$$Y := T_N \oplus \bigoplus_{\ell=1}^{N} U_\ell.$$

Here, each summand has k-dimension bounded by L_ε by the construction of N respectively of $R^{[t_\ell]}$. By (3.2), we have

$$\underline{\dim}\, Y = \underline{\dim}\, T_0 - \sum_{\ell=1}^{N} e_\ell \geq \underline{\dim}\, T_0 - N \cdot \underline{q} \geq \underline{\dim}\, X - \frac{\dim_k X\varepsilon}{q}\underline{q},$$

implying that $\dim_k Y \geq (1-\varepsilon) \dim_k X$.

This construction thus shows the following

Lemma 3.6. *Let A be a tame hereditary algebra of rank two. Let $\varepsilon > 0$. Then there exists some $L_\varepsilon > 0$ such that for all indecomposable preprojective modules X of defect -1 there exists a submodule Y such that $\dim_k Y \geq (1-\varepsilon) \dim_k X$ and each indecomposable direct summand of Y has dimension bounded by L_ε.*

Before we continue, let us recall some more Auslander–Reiten theory. Given a morphism $g\colon B \to C$, we say that g is a *split epimorphism* provided it has a right inverse, that is there exists some $f\colon C \to B$ such that $g \circ f = \mathrm{id}_B$. Dually, we say that $f\colon A \to B$ is a *split monomorphism* if f has a left inverse. A morphism $g\colon B \to C$ is then said to be *irreducible* if g is neither a split monomorphism nor a split epimorphism and a factorisation $g = s \circ t$ implies that s is a split epimorphism or t is a split monomorphism.

Moreover, g is said to be *right almost split* if it is not a split epimorphism and for any morphism $f\colon X \to C$ which is not a split epimorphism there is $f'\colon X \to B$ such that $g \circ f' = f$. Dually, $f\colon A \to B$ is *left almost split* if it is not a split monomorphism and any $A \to Y$ not a split monomorphism factors through f. Finally, a morphism is *minimal right almost split* if it is both right almost split and right minimal. Dually, we define *minimal left almost split* morphisms.

Further, an exact sequence $0 \to A \xrightarrow{g} B \xrightarrow{f} C \to 0$ is said to be an *almost split sequence* (vel Auslander–Reiten sequence) provided g is left almost split and f is right almost split.

Also recall that given a finite dimensional algebra Λ, the graph $\Gamma_\Lambda = \Gamma(\mathrm{mod}\,\Lambda)$ with vertices the isomorphism classes of indecomposable modules denoted by $[M]$ for some indecomposable Λ-module M and arrows the irreducible morphisms between the indecomposable modules is called the *Auslander–Reiten quiver of* Γ. Γ_Λ along with the Auslander–Reiten translation τ is a *translation quiver*, that is, at each vertex of Γ_Λ, only finitely many arrows start and end, there are no multiple arrows, the quiver has no loops and the arrows ending at a vertex x are precisely those starting at $\tau(x)$ if $\tau(x)$ is defined (see [ARS95, Section VII.4]).

To now deal with the remaining indecomposable preprojective modules, we use the following well known theorem due to Auslander and Reiten.

Theorem 3.7. *[ARS95, Theorem V.5.3],[ASS10, Theorem IV.1.10] Let C be an indecomposable module and let $h\colon B \to C$ be a minimal right almost split morphism. If $E \neq 0$ is some direct summand of B, the induced map $g\colon E \to C$ with $g = h_{|E}$ is irreducible.*

Proof. We fix notation denoting the decomposition $B = E \oplus E'$ and we put $g' = h_{|E'}$, that is $h = \begin{pmatrix} g & g' \end{pmatrix}$.

First we assume that g is a split monomorphism, thus there exists some $f\colon C \to E$ such that $f \circ g = \mathrm{id}_E$ and $C \cong E \oplus \ker f$. Now, C is indecomposable and $E \neq 0$, so $\ker f = 0$ and f must be a monomorphism. But f is also surjective. Thus, f is an isomorphism, and so must be g. By considering $\begin{pmatrix} g & g' \end{pmatrix} \circ \begin{pmatrix} g^{-1} \\ 0 \end{pmatrix} = \mathrm{id}_C$ we see that h is a split epimorphism, a contradiction.

Second, assume that g is a split epimorphism. Then $\begin{pmatrix} g & g' \end{pmatrix} \circ \begin{pmatrix} f' \\ 0 \end{pmatrix} = \mathrm{id}_C$, which also implies that h is a split epimorphism.

Now assume that $g = s \circ t$, where $t\colon E \to X$ and $s\colon X \to C$ and assume that s is not a split epimorphism. We want to show that t is a split monomorphism. Since h is right almost split, there exists some $\eta = \begin{pmatrix} u \\ v \end{pmatrix}\colon X \to B = E \oplus E'$ with $s = h \circ \eta = \begin{pmatrix} g & g' \end{pmatrix} \circ \begin{pmatrix} u \\ v \end{pmatrix}$. We obtain the following commutative diagram.

$$\begin{tikzcd}
E \oplus E' \arrow[r, "{\begin{pmatrix} t & 0 \\ 0 & \mathrm{id}_{E'} \end{pmatrix}}"] \arrow[dr, "{\begin{pmatrix} g & g' \end{pmatrix}}"'] & X \oplus E' \arrow[r, "{\begin{pmatrix} u & 0 \\ v & \mathrm{id}_{E'} \end{pmatrix}}"] \arrow[d, "{\begin{pmatrix} s & g' \end{pmatrix}}"] & E \oplus E' \arrow[dl, "{\begin{pmatrix} g & g' \end{pmatrix}}"] \\
& C &
\end{tikzcd}$$

Since $\begin{pmatrix} g & g' \end{pmatrix}$ is right minimal, we have that

$$\begin{pmatrix} u & 0 \\ v & \mathrm{id}_{E'} \end{pmatrix} \circ \begin{pmatrix} t & 0 \\ 0 & \mathrm{id}_{E'} \end{pmatrix} = \begin{pmatrix} u \circ t & 0 \\ v \circ t & \mathrm{id}_{E'} \end{pmatrix}$$

is an isomorphism. Hence $u \circ t\colon E \to X \to E$ is an isomorphism, thus we have shown the existence of some $u\colon X \to E$ such that $u \circ t = \mathrm{id}_E$, so t is a split monomorphism. This finishes the proof. ☐

Lemma 3.8. *Let A be a tame hereditary algebra of rank two. Let M be an indecomposable preprojective module of defect -2 which is not projective. Then there exists a monomorphism $f\colon N \to M$ such that N is a direct sum of preprojective modules of defect -1 and we have that $\dim_k \operatorname{coker} f \leq h_A$.*

Proof. First, as M is an indecomposable non-projective module, it is well-known (see, e.g., [ARS95, Theorem V.1.15]) that there exists a right minimal almost split morphism $h\colon E \to M$. As this map is the right-hand side of an almost split sequence, it is surjective. For rank two algebras, such indecomposable preprojective modules only exist in type $\widetilde{A}_{11}$ with label $(1, 4)$.

Case 1: $(1,4)$. We have $\underline{h}_A = (2,1)$. We may assume $\underline{m} = \underline{\dim} M = (4k, 2k-1)$ for some $k > 1$. Then $E = {P_{2k-1}}^{\oplus 4}$, where P_{2k-1} is the unique indecomposable preprojective module of dimension vector $(2k-1, k-1)$. Now, we pick N to be the direct sum of two indecomposable summands of E, so $N = P_{2k-1} \oplus P_{2k-1}$. Then the induced morphism $f = h_{|N}$ is irreducible by Theorem 3.7. Thus, f is either surjective or injective. But f cannot be surjective for dimension reasons, so f must be injective. Moreover,

$$\underline{\dim}\operatorname{coker} f = \underline{\dim} M - \underline{\dim} N = (4k, 2k-1) - 2(2k-1, k-1) = (2,1) = \underline{h}_A.$$

Also, $\partial(\operatorname{coker} f) = 0$.

Case 2: $(4,1)$. We have $\underline{h}_A = (1,2)$. We may assume $\underline{m} = \underline{\dim} M = (2k+1, 4k)$ for some $k \geq 1$. Then $E = {P_{2k}}^{\oplus 4}$, where P_{2k} is the unique indecomposable preprojective module of dimension vector $(k, 2k-1)$. Now, we pick N to be the direct sum of two indecomposable summands of E, so $N = P_{2k} \oplus P_{2k}$. Then the induced morphism $f = h_{|N}$ is irreducible by Theorem 3.7. Thus, f is either surjective or injective. But f cannot be surjective for dimension reasons, so f must be injective. Moreover,

$$\underline{\dim}\operatorname{coker} f = \underline{\dim} M - \underline{\dim} N = (2k+1, 4k) - 2(k, 2k-1) = (1,2) = \underline{h}_A.$$

Also, $\partial(\operatorname{coker} f) = 0$.

Thus we have found a submodule N of M which is a direct sum of preprojective modules of defect -1 and has codimension bounded by h_A. ☐

Proposition 3.9. *Let A be a finite dimensional tame hereditary k-algebra of rank two. Then the family of (isomorphism classes of) preprojective modules $\mathcal{P}(A)$ is hyperfinite.*

Proof. By Proposition 1.2, it is enough to check this for the indecomposable preprojective modules that have defect either -1 or -2. By Lemma 3.6, the family of indecomposable preprojective modules of defect -1 is hyperfinite. For a preprojective of defect -2, Lemma 3.8 yields the existence of a submodule that lies in a hyperfinite family and is of bounded codimension. By Proposition 1.4, the indecomposable preprojectives of defect -2 thus form a hyperfinite family. This concludes the proof. ☐

Lemma 3.10. *Let R be a sincere, indecomposable regular module of regular length $\ell(R) = r$. Let q be the maximal dimension of the simple injective modules. Then there is a submodule $S \subseteq R$ of codimension at most q such that S decomposes into a preprojective summand P and a regular L summand with $\ell(L) < \ell(R)$, with the possibility that $L = 0$.*

Proof. R is sincere (this is the case if R is homogeneous or has regular length $r \geq 2$). Then there exists a simple injective module I and a non-zero map $\theta \colon R \to I$. Since I is simple, θ must be surjective. Consider the short exact sequence

$$0 \to \ker\theta \to R \to I \to 0.$$

Since $\partial(I) \geq 1$, $\ker\theta$ must have a preprojective summand $P \neq 0$. If $L \mid \ker\theta$ is a regular summand, those also embed into R, and must have smaller regular length, for $\underline{\dim}\, L < \underline{\dim}\, R$ implies that $L \ncong R$, but both R and L have the same regular socle, and regular socle and length determine indecomposable regular modules uniquely. □

Lemma 3.11. *Let A be a finite dimensional tame hereditary k-algebra of rank two. Then the family of (isomorphism classes of) regular simple modules is hyperfinite.*

Proof. Since for rank two tame hereditary algebras, all regular simple modules are homogeneous, Lemma 3.10 yields the existence of some preprojective submodule N for each regular simple module R of bounded codimension, as the submodule constructed in that lemma cannot have a regular summand. Thus, Proposition 1.4 in connection with Proposition 3.9 shows that they form a hyperfinite family. □

Proposition 3.12. *Let A be a finite dimensional tame hereditary k-algebra. Assume that $\mathcal{P}(A)$ is hyperfinite. Then the family of (isomorphism classes of) regular modules is also hyperfinite.*

Proof. By Proposition 1.2, it is enough to show that the set of all indecomposable regular modules is a hyperfinite family. So let T be a regular indecomposable module. By Lemma 3.3, there exists a submodule $P \subset T$ of codimension bounded by the maximum of the dimensions of the indecomposable injective modules with P preprojective. Thus, an application of Proposition 1.4 along with the hypothesis shows that $\mathcal{R}(A)$ is hyperfinite. □

Remark. This approach will be generalised later, see Proposition 4.16.

Theorem 3.13. *Let k be any field. Let A be a finite dimensional tame hereditary k-algebra of rank two. Then A is of amenable representation type.*

Proof. That the family of (isoclasses of) preprojective modules $\mathcal{P}(A)$ is hyperfinite has been shown in Proposition 3.9. Moreover, the hyperfiniteness of the regular component then follows from Proposition 3.12.

We are left to deal with the postinjective modules. Yet, Lemmas 2.7 and 2.8 generalise to tame hereditary algebras mutatis mutandis. This finishes the proof. □

3.3 Amenability and perpendicular categories of regular simples

Having shown that tame hereditary algebras of small rank are of amenable representation type, we want to lift this result to algebras of larger rank. As in the quiver case, we will do so by universal localisation and perpendicular calculus.

Recall that we have the perpendicular categories

$$X^{\perp} := \{Y \in \operatorname{mod} A \colon \operatorname{Hom}_A(X,Y) = 0 = \operatorname{Ext}^1_A(X,Y)\}$$

and

$$^{\perp}X := \{Y \in \operatorname{mod} A\colon \operatorname{Hom}_A(Y,X) = 0 = \operatorname{Ext}^1_A(Y,X)\}.$$

Proposition 3.14. *Let A be a connected finite dimensional tame hereditary k-algebra of rank $n > 2$. Suppose all connected finite dimensional tame hereditary k-algebras of rank $n-1$ are of amenable representation type. If T is an inhomogeneous regular simple A-module, then $T^{\perp}$ is hyperfinite.*

Proof. By [GL91, Proposition 1.1], $T^{\perp}$ is an exact Abelian subcategory of $\operatorname{mod} A$ closed under the formation of kernels, cokernels and extensions. What is more, [GL91, Theorem 4.16] yields that $T^{\perp} = \operatorname{mod} \Lambda$ for some finite dimensional hereditary algebra Λ of rank $n-1$, along with a homological epimorphism $\varphi\colon A \to \Lambda$, which induces a functor $\varphi_*\colon \operatorname{mod} \Lambda \to \operatorname{mod} A$. Moreover, [GL91, Theorem 10.1] shows that Λ is tame and connected.

Now, if $F\colon \operatorname{mod} \Lambda \to T^{\perp}$ is an equivalence, the simple Λ-modules $S(i)$ get mapped to certain modules B_i in $\operatorname{mod} A$. The k-dimension of any module M over a finite dimensional k-algebra is determined by the length of any composition series. Such a series for M in Λ gets mapped to a composition series in the perpendicular category, and thus a series in $\operatorname{mod} A$, such that the factor modules are isomorphic to some B_i. Letting $K_2 := \max_{i=1,\dots,n-1}\{\dim B_i\}$, we thus know that

$$\dim_{k\,A} F(M) \leq K_2 \dim_{k\,\Lambda} M.$$

On the other hand, if $F(M) \in T^{\perp}$, any submodule of $F(M)$ in $T^{\perp}$ is also a submodule in $\operatorname{mod} A$, so a composition series of $F(M)$ in $\operatorname{mod} A$ is at least as long as one in $T^{\perp}$. Thus,

$$\dim_{k\,\Lambda} M \leq \dim_{k\,A} F(M),$$

using the fact that the length of M in $\operatorname{mod} \Lambda$ equals the length of $F(M)$ considered as an object of $T^{\perp}$. Hence by Proposition 1.5, we have that each $T^{\perp}$ is a hyperfinite family. ☐

We will introduce notation for the indecomposable modules of a given inhomogeneous tube $\mathbb{T}$ of a finite dimensional tame hereditary algebra A. We start by denoting the isoclasses of regular simples on the mouth of $\mathbb{T}$ by $T_1, \dots, T_m$ such that $\tau T_i = T_{i-1}$ for $i = 2, \dots, m$ and $\tau T_1 = T_m$. Here, m is the rank of $\mathbb{T}$. Similar to [Rin84, Section 3.1], we then define the objects $T_i[\ell]$. First, let $T_i[1] := T_i$ for each $1 \leq i \leq m$. Now, for $\ell \geq 2$, recursively define $T_i[\ell]$ to be the indecomposable module in $\mathbb{T}$ with $T_i[1]$ as a submodule such that $T_i[\ell]/T_i[1] \cong T_{i+1}[\ell-1]$. Thus $T_i[\ell]$ is the regular module of regular length ℓ with regular socle T_i. Any regular indecomposable in $\mathbb{T}$ will be given as some $T_i[\ell]$ since $\mathbb{T}$ is uniserial. We may define $T_i[\ell]$ for all $i \in \mathbb{Z}$ by letting $T_i[\ell] \cong T_j[\ell]$ iff $i \equiv j \mod m$. Note that

$$\underline{\dim}\, T_i[\ell] = \sum_{j=i}^{i+\ell-1} \underline{\dim}\, T_j. \tag{3.3}$$

Lemma 3.15. *Let $\mathbb{T}$ be an inhomogeneous tube of rank m. Then $\sum_{i=1}^{m} \underline{\dim} T_i = g_{\mathbb{T}} \underline{h}_A$, and $g_{\mathbb{T}}$ is globally bounded by some g across all tubes.*

Proof. Given an inhomogeneous tube of rank $m \geq 2$ with regular simples $T_1, \dots, T_m$, recall that all $T_i[m]$ are homogeneous, that is $\underline{\dim} T_i[m]$ is a multiple of $\underline{h}_A$. By (3.3), this implies that the sum of the dimension vectors of the regular simples in each tube is a multiple of $\underline{h}_A$. As there are only finitely many inhomogeneous tubes, there is a global bound on this multiple. ☐

Lemma 3.16. *Let $\mathbb{T}$ be a tube of rank $m \geq 2$. Let X be an indecomposable regular module in $\mathbb{T}$. Then there exists a submodule $Y \subseteq X$ of codimension bounded by the sum of the k-valued entries of $g\underline{h}_A$, and a regular simple module $T \in \mathbb{T}$ such that $Y \in T^{\perp}$.*

Proof. By adapting the proof of [Rin84, 3.1.(3')] to the fact that $\dim_k \operatorname{Hom}_A(T_i, T_i) = e$ for all i, we have that

$$\langle \underline{\dim} T_i, \underline{\dim} T_j \rangle = \begin{cases} e, & i \equiv j \mod m, \\ -e, & i \equiv j+1 \mod m, \\ 0, & \text{else.} \end{cases}$$

This implies that for any $j \in \mathbb{Z}$, $\langle \underline{\dim} T_j, \underline{\dim} T_i[\ell] \rangle = 0$, provided $\ell \equiv 0 \mod m$. As $T_i[\ell]$ is uniserial, we also have that $\operatorname{Hom}_A(T_j, T_i[\ell]) = 0$ if and only if $j \not\equiv i \mod m$.

We write $\ell = n \cdot m + r$, where $0 \leq r < m$. By construction, there is a short exact sequence

$$0 \to T_i[nm] \to T_i[\ell] \to Z \to 0,$$

where $\underline{\dim} Z \leq g\underline{h}_A$ using (3.3) and Lemma 3.15. Thus, we have found a suitable submodule $T_i[nm] \in T_{i+1}^{\perp}$. ☐

Lemma 3.17. *Let A be a finite dimensional tame hereditary algebra. Let S be some regular simple inhomogeneous module in a tube $\mathbb{T}$, then there is a constant $c > 0$. If $X = \tau^{-p}P(i)$ is some indecomposable preprojective A-module, there exists a submodule $Y \subseteq X$ and a module Q with $\underline{\dim} Q \leq c \underline{\dim} S$ such that $0 \to Y \to X \to Q \to 0$ is exact and $Y \in \tau^{-}S^{\perp}$.*

Proof. Let $f \colon X \to S^c$ be a minimal left $\operatorname{add}(S)$-approximation of X, in particular, the induced map

$$f^* \colon \operatorname{Hom}_A(S^c, S) \to \operatorname{Hom}_A(X, S), \quad \phi \mapsto \phi \circ f$$

is surjective. Now, let $Y = \ker f$ and $Q = \operatorname{im} f$. Note that $f = g \circ h$ factors through $h \colon X \to Q$ and $g \colon Q \to S^c$. We have a short exact sequence

$$\xi \colon 0 \to Y \xrightarrow{\iota} X \xrightarrow{h} Q \to 0,$$

to which we apply $\mathrm{Hom}_A(-,S)$ and get the long exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}_A(Q,S) \xrightarrow{h^*} \mathrm{Hom}_A(X,S) \xrightarrow{\iota^*} \mathrm{Hom}_A(Y,S) \xrightarrow{\xi^*} \mathrm{Ext}^1_A(Q,S) \\ \to \mathrm{Ext}^1_A(X,S) \to \mathrm{Ext}^1_A(Y,S) \to 0, \end{aligned}$$

for A is hereditary. First, by [Rin84, 2.4.(6*)],

$$\mathrm{Ext}^1_A(X,S) \cong D\mathrm{Hom}_A(\tau^- S, X) = 0, \tag{3.4}$$

since there are no maps from the regular to the preprojective component. By exactness, it follows that $\mathrm{Ext}^1_A(Y,S) = 0$. Second, h^* is injective. But $f^* = (g \circ h)^* = h^* \circ g^*$ is surjective, so h^* is also surjective, hence bijective. Since Q can only have preprojective summands or regular summands isomorphic to copies of S, we must have that $\mathrm{Ext}^1_A(Q,S) = 0$, implying $\mathrm{Hom}_A(Y,S) = 0$. This shows that $Y \in {}^\perp S = (\tau^- S)^\perp$, using the identity from (3.4) to move between the two categories.

By the construction of the (minimal) left $\mathrm{add}(S)$-approximation of X, we see that

$$c = \frac{\dim_k \mathrm{Hom}_A(X,S)}{\dim_k \mathrm{End}(S)}.$$

Denoting the regular simples on the mouth of $\mathbb{T}$ by $T_1, \dots, T_m$, we may assume that $S \cong T_j$ for some $1 \le j \le m$. Then, since $\mathrm{Ext}^1_A(X,S) = 0$,

$$\begin{aligned} \dim_k \mathrm{Hom}_A(X,S) &= \langle \underline{\dim}\, X, \underline{\dim}\, T_j \rangle = \langle \underline{\dim}\, \tau^{-p} P(i), \underline{\dim}\, T_j \rangle \\ &= \langle \underline{\dim}\, P(i), \underline{\dim}\, T_{j-p} \rangle = f_i (\underline{\dim}\, T_{j-p})_i, \end{aligned}$$

where $f_i = \dim_k \mathrm{End}(P(i))$. Now, $\underline{\dim}\, Q \le \underline{\dim}\, S^c \le c\, \underline{\dim}\, S$. Thus, the codimension of Y as a submodule of X is bounded by the constant c solely depending on the finitely many regular simple inhomogeneous modules and their finite k-dimensions.

□

Theorem 3.18. *Let k be any field. Let A be a finite dimensional tame hereditary k-algebra. Then A is of amenable representation type.*

Proof. We prove the theorem by an induction on the rank of A. There are no (finite dimensional, non-finite) tame hereditary k-algebras of rank one. The case $n = 2$ is the subject of Theorem 3.13.

Let A be some finite dimensional tame hereditary algebra of rank $n > 2$. Now assume that it has been shown that all finite dimensional tame hereditary connected k-algebras of rank $n - 1$ are of amenable type.

We will show that the indecomposable preprojective modules form a hyperfinite family first.

By an application of Lemma 3.17, all indecomposable preprojective modules P have a submodule of (globally) bounded codimension which lies in the perpendicular category of a fixed regular simple module T_1 in an inhomogeneous tube $\mathbb{T}$. Then Proposition 3.14

shows that the indecomposable preprojectives lie in a hyperfinite family, additionally applying Proposition 1.4. This uses the fact that there are only finitely many inhomogeneous tubes, each of finite rank.

Next, we consider the regular modules. Indecomposable regular modules in a tube other than $\mathbb{T}$ will be contained in $T_1^{\perp}$ by an analogue of [Rin84, 3.1.(3')]. By Lemma 3.16, any regular indecomposable in $\mathbb{T}$ either is contained in the perpendicular category of some regular simple in $\mathbb{T}$ or has a submodule of bounded codimension that is in the perpendicular category of some regular simple in $\mathbb{T}$. But by Proposition 3.14, the perpendicular categories are hyperfinite. In the latter case, we can therefore apply Proposition 1.4 to show the hyperfiniteness of the family of these indecomposable regular modules.

For the postinjective modules, we may apply the argument used for the postinjectives in the proof of Theorem 3.13. This completes the induction and finishes the proof. □

3.4 Tilted Algebras

One way to obtain algebras which are not hereditary from tame hereditary algebras is via tilting theory, introduced by Auslander, Platzeck and Reiten [APR79] and generalised by Brenner and Butler [BB80]. We show that under certain conditions on the tilting module, this preserves amenability.

Let A be a finite dimensional, hereditary algebra. Recall that a module T is a *tilting module* for A provided

- T is *rigid*, that is, $\operatorname{Ext}^1_A(T,T)=0$, and
- the number of isoclasses of indecomposable direct summands of T is equal to the number of isomorphism classes of simple A-modules.

Now, an algebra of the form $B=\operatorname{End}_A(T)$ for some tilting module T is called a *tilted algebra* (see, e.g., [HR82]).

Theorem 3.19. *Let A be a finite dimensional tame hereditary algebra. Let T be a tilting module without a postinjective summand. Then $B=\operatorname{End}_A(T)$ is of amenable representation type.*

Proof. First note that T cannot be regular by [HR81, Lemma 3.1] since A is tame. Thus, T must have a non-zero preprojective summand.

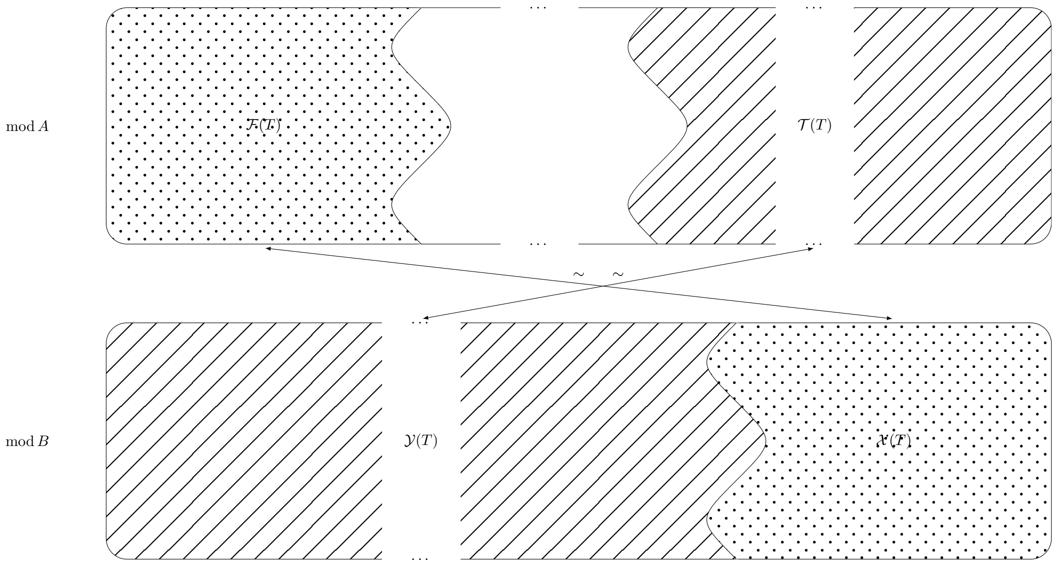


Figure 3.3: Module classes in mod A and mod B as in the proof of Theorem 3.19.

Now, by [HR81, Proposition 3.2], the torsion class $\mathcal{T}(T)$ is infinite. It is in bijection with the torsion free class $\mathcal{Y}(T)$ via $G=\operatorname{Hom}(T,-)$ of the Brenner–Butler Tilting

Theorem [BB80, Theorem III]. The functor G is hyperfiniteness preserving, since we can apply Proposition 1.5 as T is finite dimensional. As $\operatorname{mod} A$ is of amenable type by Theorem 3.18, it follows that $\mathcal{Y}(T)$ is hyperfinite. Since T is a splitting tilting module (see, e.g., [ASS10, Corollary VI.5.7]), any indecomposable module of $\operatorname{mod} B$ either lies in $\mathcal{Y}(T)$ or $\mathcal{X}(T)$, the torsion class in $\operatorname{mod} B$. But $\mathcal{F}(T)$, which is in bijection with $\mathcal{X}(T)$ via the Tilting Theorem, is finite by [HR81, Proposition 3.2*][1]. □

3.4.1 Concealed algebras

For the following, let us restrict to algebras over algebraically closed fields. Recall that a module $M \in \operatorname{mod} A$ is preprojective if each indecomposable summand of M lies in some preprojective component of Γ_A, the AR-quiver of A. Now, let A be hereditary and $T \in \operatorname{mod} A$ be some preprojective tilting module. Then the tilted algebra $B = \operatorname{End}_A(T)$ is called a *concealed algebra*.

Let us recall the following result on their module structure, which will be of importance in the next chapter.

Theorem 3.20. *[Rin84, Theorem 4.3.(3)] An algebra B as above is tame and satisfies $\operatorname{mod} B = \mathcal{P} \vee \mathcal{T} \vee \mathcal{Q}$, where $\mathcal{T} = (\mathcal{T}_\rho)_{\rho \in \mathbb{P}^1(k)}$ is a stable separating tubular $\mathbb{P}^1(k)$-family of regular modules, $\mathcal{P}$ is a preprojective and $\mathcal{Q}$ is a postinjective component of Γ_B.*

Now we can conclude from Theorem 3.19 a result on the amenability of tame concealed algebras.

Corollary 3.21. *Let $B = \operatorname{End}_A(T)$ be a tame concealed algebra, that is, let the algebra A be of tame but not finite representation type. Then B is of amenable representation type.*

---

[1] Note that condition (vii) in the cited proposition should read "T_A has no non-zero preprojective direct summand", as it is dual to Proposition 3.2 ibid.

4 Tubular canonical algebras

Having proven amenability for tame hereditary and tame concealed algebras, we turn to another class of algebras for which similar results can be expected. While the tame concealed algebras were obtained from the hereditary ones by tilting, the tubular algebras of this chapter will be obtained by so-called tubular extensions to be recalled below. They have been introduced and studied by Ringel [Rin84]. Moreover, there is some connection to the category of coherent sheaves on the weighted projective line, see Geigle and Lenzing [GL87] and Lenzing and Meltzer [LM93].

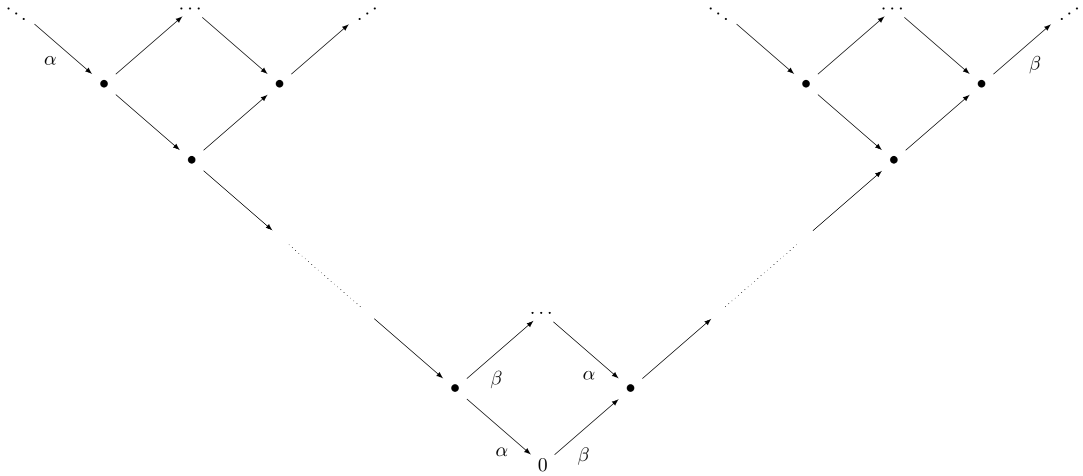
For simplicity, in this chapter, we will always work over algebraically closed fields k.

4.1 Setup

We will first sketch the construction of tubular algebras following [Rin84] and supplemented by [SS07].

4.1.1 Tubular extensions

By a *branch* $\mathscr{L} = (B, J)$, we mean a finite connected full bound subquiver B of the following binary infinite tree



containing the lowest vertex 0, denoted the *germ of* $\mathscr{L}$, bound by all possible zero relations J of the form $\beta\alpha = 0$. $\mathscr{L}$ has **length** $|\mathscr{L}| = n$ provided B has n vertices. The empty quiver is called a branch of length 0.

Recall that a path in a translation quiver (for example in the AR-quiver $\Gamma(\operatorname{mod} A)$

of the module category of an algebra A),

$$X_1 \to X_2 \to \cdots \to X_{n-1} \to X_n \to \ldots,$$

is called *sectional* provided $\tau X_{i+1} \neq X_{i-1}$ for all possible i.

A vertex v in a translation quiver Γ is said to be a *ray vertex* provided there exists an infinite sectional path

$$v = v_{[1]} \xrightarrow{\nu_1} v_{[2]} \xrightarrow{\nu_2} \cdots \to v_{[i]} \xrightarrow{\nu_i} v_{[i+1]} \xrightarrow{\nu_{i+1}} \ldots$$

with pairwise different $v_{[i]}$, $i \in \mathbb{N}$, such that for any i, the path $(v[i]|\nu_1, \ldots, \nu_i|v[i+1])$ is the only sectional path of length i starting at v. An indecomposable A-module N in a standard component $\mathcal{C}$ of $\operatorname{mod} A$ is said to be a *ray module* provided the corresponding vertex $[N]$ is a ray vertex in $\Gamma(\mathcal{C})$.

Example. Given a tame hereditary algebra A, modules on the mouth of standard stable tubes in $\Gamma(\operatorname{mod} A)$ are ray modules.

Recall that for a k-algebra A and a module E, we denote by

$$A[E] = \begin{bmatrix} A & E \\ 0 & k \end{bmatrix}$$

the *one-point extension of A by E*. Here, addition and multiplication are the usual matrix operations. For $A = kQ/I$ a quiver algebra with relations I, the quiver of $A[E]$ contains Q as a full subquiver along with the extension vertex ω, a source. Modules for $A[E]$ can be identified with triples (M, V, φ), where M is an A-module, V is a k-vector space and $\varphi \colon V \to \operatorname{Hom}_A(E, M)$ is a k-linear map.

Given a bound quiver algebra $A = kQ/I$ and a branch $\mathscr{L} = (B, J)$, we may consider the path algebra having as vertices those of Q and B, only identifying $0 \in B$ with some $\omega \in Q$, with relations generated by I and J. In this way we construct an algebra from A by *adding the branch $\mathscr{L}$ in ω*.

Now, we inductively define, for $E_1, \ldots, E_t \in \operatorname{mod} A$ and $\mathscr{L}_1, \ldots, \mathscr{L}_t$ branches, the algebra $A[E_i, \mathscr{L}_i]_{i=1}^t$, where $A[E_i, \mathscr{L}_i]$ is obtained from the one-point extension $A[E_i]$ with extension vertex ω_i by adding the branch $\mathscr{L}_i$ in ω_i, and put

$$A[E_i, \mathscr{L}_i]_{i=1}^s = \left(A[E_i, \mathscr{L}_i]_{i=1}^{s-1}\right)[E_s, \mathscr{L}_s]$$

for all $s \leq t$.

Definition 4.1. [Rin84, Section 4.7] Assume there is a standard stable tubular family $\mathcal{T}$ in $\operatorname{mod} A$, separating $\mathcal{P}$ from $\mathcal{Q}$. Let the modules $E_1, \ldots, E_t$ be pairwise orthogonal ray modules from $\mathcal{T}$. Then $A[E_i, \mathscr{L}_i]_{i=1}^t$ is called a **tubular extension of A using modules from $\mathcal{T}$.**

Example. Let $A = k\,(1 \rightrightarrows 2)$ be the Kronecker algebra, $E = k \underset{1}{\overset{1}{\rightrightarrows}} k$ and branch $\mathscr{L} = (B, I)$ with $B = - \xrightarrow{\alpha} 0 \xrightarrow{\beta} +$ and $I = \langle \beta\alpha \rangle$. Then $A[E, \mathscr{L}]$ is the path algebra of

$$\begin{tikzcd}
- \arrow[dr, "\alpha"] & & & \\
 & 0 \arrow[r, "\sigma"] \arrow[dl, "\beta"] & 1 \arrow[r, shift left, "\eta"] \arrow[r, shift right, "\nu"'] & 2 \\
+ & & &
\end{tikzcd}$$

bound by the relations $\langle \eta\sigma - \nu\sigma, \beta\alpha \rangle$.

Remarks. For a tame concealed algebra A_0, there is a unique separating tubular family (cf. Theorem 3.20). If $\mathcal{T} = (\mathbb{T}_\rho)_{\rho \in I}$ is a stable I-family, and $E_i \in \mathcal{T}$, the *extension* or *tubular type* of A over A_0 is given by the function

$$m\colon I \to \mathbb{N},\ m(\rho) = \operatorname{rk}(\rho) + \sum_{E_i \in \mathbb{T}(\rho)} |\mathscr{L}_i|,$$

where $\operatorname{rk}(\rho)$ is the rank of the tube $\mathbb{T}(\rho)$. We usually drop all values of m where $m(\rho) = 1$ and write the tubular type as a finite tuple $(m_1, \dots, m_t)$. Note that

$$\operatorname{rk} A = 2 + \left(\sum_{i=1}^{t} m_i \right) - t.$$

We will now recall some results on the module structure and the Auslander–Reiten quiver of these tubular extensions.

Given a separating tubular family $\mathcal{T}$ in $\operatorname{mod} A_0$, a branch $\mathscr{L}$ and a vertex v in the AR-quiver $\Gamma(\mathcal{T})$, we construct a new translation quiver $\Gamma(\mathcal{T})(v, \mathscr{L})$ as follows: As a first step, we cut along the ray starting at v, split it into "sinks" $(v, i, 0)$ and "sources" $(v, i, |\mathscr{L}|)$, and add $|\mathscr{L}| - 1$ additional rays

$$(v, i, 1) \to (v, i, 2) \to \dots, \text{with}\, 0 < i < |\mathscr{L}|,$$

connected by arrows $(v, i, j) \to (v, i, j+1)$. Next, we glue to it the AR-quiver of a specific, full subcategory of $\operatorname{mod} k\vec{A}_{|\mathscr{L}|}$, i.e. of the representations of the linearly oriented quiver of type A_n with $|\mathscr{L}|$ vertices, by identifying the injective indecomposable modules of the latter subcategory with $(v, 1, 0), \dots, (v, 1, |\mathscr{L}|)$. Here, the subcategory is constructed using the tilting module T corresponding to the branch $\mathscr{L}$ (see [Rin84, Subsections 4.4.(2)-(3)]). Indeed, it has as objects just all indecomposable modules $Y \in \operatorname{mod} k\vec{A}_{|\mathscr{L}|}$ such that $\operatorname{Hom}(Y, \tau X) = 0$ for all direct summands $X \mid T$.

Theorem 4.2. *[Rin84, Sections 4.4–4.7] Let A_0 be an algebra with separating tubular family $\mathcal{T}$ (separating $\mathcal{P}$ from $\mathcal{Q}$). Let $A = A_0[E_i, \mathscr{L}_i]_{i=1}^{t}$ be a tubular extension using modules from $\mathcal{T}$. Define in* $\operatorname{mod} A$*,*

- *$\mathcal{P}_0 = \mathcal{P}$, extending the representations by the zero vector space outside the support of A_0,*

- $\mathcal{T}_0$, *a module class where the indecomposable modules* M *are either such that* $M_{|A_0}$ *is non-zero and in* $\mathcal{T}$ *or* $\operatorname{supp} M \subset \mathscr{L}_i$ *and* $\langle \ell_{\mathscr{L}_i}, \underline{\dim}\, M \rangle < 0$, *for some* $i \in \{1, \ldots, t\}$,
- $\mathcal{Q}_0$, *a module class where the indecomposable modules* M *are either such that* $M_{|A_0}$ *is non-zero and in* $\mathcal{Q}$ *or* $\operatorname{supp} M \subset \mathscr{L}_i$ *and* $\langle \ell_{\mathscr{L}_i}, \underline{\dim}\, M \rangle > 0$, *for some* $i \in \{1, \ldots, t\}$,

where

$$\ell_{\mathscr{L}_i} = (|B(a)|)_{a \in \mathscr{L}_i},$$

and $B(a)$ *is the restriction of* $\mathscr{L}_i$ *to the vertices which are dependants of* a.

Then $\operatorname{mod} A = \mathcal{P}_0 \vee \mathcal{T}_0 \vee \mathcal{Q}_0$, *and* $\mathcal{T}_0$ *is a separating tubular family, separating* $\mathcal{P}_0$ *from* $\mathcal{Q}_0$. *Moreover,* $\Gamma(\mathcal{T}_0) = \Gamma(\mathcal{T})[e_i, \mathscr{L}_i]_{i=1}^t$, *with* $e_i = [E_i]$. *Here,*

$$\Gamma(\mathcal{T})[e_i, \mathscr{L}_i]_{i=1}^s = \left(\Gamma(\mathcal{T})[e_i, \mathscr{L}_i]_{i=1}^{s-1} \right) [e_s, \mathscr{L}_s],$$

is inductively defined.

4.1.2 Tubular algebras

Following Ringel [Rin84], we can now give the necessary definition and recall further notions ibid.

Definition 4.3. Let A_0 be a tame concealed algebra and let $A = A_0[e_i, \mathscr{L}_i]_{i=1}^t$ be a tubular extension of A_0 of extension type $(m_1, \ldots, m_t)$ by A_0-modules E_i and branches $\mathscr{L}_i$. If the extension type of A is one of the types $(2,2,2,2)$, $(3,3,3)$, $(2,4,4)$ or $(2,3,6)$, then A is said to be a **tubular algebra**.

Remarks. The rank of $K_0(A)$ is $6, 8, 9$ or 10 for a tubular algebra.
An algebra A is said to be cotubular provided A^{op} is tubular.

Example 4.4. Consider the algebra given by the quiver

$$\begin{tikzcd}
 & & & a_1 \arrow[dl, "\alpha_1"'] & a_2 \arrow[l, "\alpha_2"'] & \\
c' & c \arrow[l, "\gamma'"'] & b_1 \arrow[l, "\gamma"'] & & & a_3 \arrow[ul, "\alpha_3"'] \arrow[dl, "\beta_3"] \\
 & & & b_2 \arrow[ul, "\beta_1"] & b_3 \arrow[l, "\beta_2"'] &
\end{tikzcd}$$

and bound by the relation

$$\gamma \circ (\alpha_1 \circ \alpha_2 \circ \alpha_3 - \beta_1 \circ \beta_2 \circ \beta_3) = 0.$$

This is a tubular extension of a path algebra of a quiver of type $\tilde{E}_6$ and a cotubular coextension (of length two) of a path algebra of a quiver of type $\tilde{A}_{3,3}$. It has extension type $(3,3,3)$.

M in	$\iota_0(\underline{\dim} M)$	$\iota_\infty(\underline{\dim} M)$	index(M)
$\mathcal{P}_0$	< 0	≤ 0	negative or ∞
$\mathcal{T}_0$	$= 0$	< 0	0
$\mathcal{Q}_0 \cap \mathcal{P}_\infty$	> 0	< 0	positive
$\mathcal{T}_\infty$	> 0	$= 0$	∞
$\mathcal{Q}_\infty$	≥ 0	> 0	negative or 0

Table 4.1: Signs of the linear forms and the index depending on the module class.

Let A be both, a tubular and cotubular algebra, respectively assume that A is a tubular extension of A_0 and A is a tubular coextension of some A_∞. Denote by $\underline{h}_0$ and $\underline{h}_\infty$ the minimal positive radical elements of $K_0(A_0)$ and $K_0(A_\infty)$, respectively. Extend them to elements of $K_0(A)$ by adding zeros and call them the *canonical radical elements.* We define two linear forms on $K_0(A)$,

$$\iota_0 = \langle \underline{h}_0, - \rangle, \quad \text{and} \quad \iota_\infty = \langle \underline{h}_\infty, - \rangle,$$

where $\langle -, - \rangle$ is the Euler bilinear form on $K_0(A)$.

Now, given $x \in K_0(A)$ such that not both $\iota_0(x)$ and $\iota_\infty(x)$ are zero, the *index of x* is defined by

$$\operatorname{index}(x) = -\frac{\iota_0(x)}{\iota_\infty(x)} \in \mathbb{Q} \cup \{\infty\}.$$

We write $\operatorname{index}(M) = \operatorname{index}(\underline{\dim} M)$ when $M \in \operatorname{mod} A$.

Now, for any $\gamma \in \mathbb{Q}_0^\infty = \mathbb{Q}^+ \cup \{0, \infty\}$, we want to define module classes $\mathcal{P}_\gamma$, $\mathcal{T}_\gamma$ and $\mathcal{Q}_\gamma$ in $\operatorname{mod} A$. For $\gamma = 0$, these were defined in the previous subsection, and dually one can define them for $\gamma = \infty$. We first recall a proposition that enables us to proceed with this idea.

Proposition 4.5. *[Rin84, 5.2.(1)] The module classes $\mathcal{P}_0, \mathcal{T}_0, \mathcal{Q}_0 \cap \mathcal{P}_\infty, \mathcal{T}_\infty$ and $\mathcal{Q}_\infty$ are pairwise disjoint and give all of* $\operatorname{mod} A$. *Indeed,*

$$\mathcal{Q}_0 = (\mathcal{Q}_0 \cap \mathcal{P}_\infty) \vee \mathcal{T}_\infty \vee \mathcal{Q}_\infty \quad \text{and} \quad \mathcal{P}_\infty = \mathcal{P}_0 \vee \mathcal{T}_0 \vee (\mathcal{Q}_0 \cap \mathcal{P}_\infty),$$

and for each indecomposable A-module M, one of $\iota_0(M)$ or $\iota_\infty(M)$ is non-zero.

The situation is hence as in Table 4.1 and the index can be defined for all indecomposable modules. We decompose

$$\mathcal{Q}_0 \cap \mathcal{P}_\infty = \bigvee_{\gamma \in \mathbb{Q}^+} \mathcal{T}_\gamma,$$

where each $\mathcal{T}_\gamma$ denotes the module class of all indecomposable A-modules of index γ. Further, given an index $\gamma = \frac{\gamma_\infty}{\gamma_0}$, with $\gamma_\infty \in \mathbb{Z}, \gamma_0 \in \mathbb{N}_0$ coprime, we define

$$\underline{h}_\gamma := \gamma_0 \underline{h}_0 + \gamma_\infty \underline{h}_\infty.$$

We recall two further results:

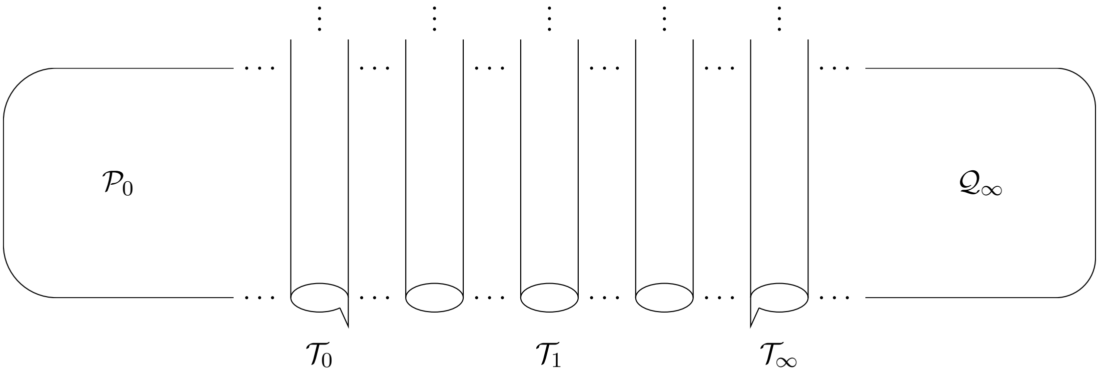


Figure 4.1: Structure of the module category $\operatorname{mod} A$ of a tubular algebra A.

Theorem 4.6. *[Rin84, Theorem 5.2.(3)] Any tubular algebra is also cotubular.*

Theorem 4.7. *[Rin84, Theorem 5.2.(4); SS07, Theorem XIX.3.20] Let A be a tubular algebra of type $(p_1, \dots, p_t)$. Then* $\operatorname{mod} A$ *has the following components:*

- *a preprojective component $\mathcal{P}_0$,*
- *a $\mathbb{P}_1(k)$-family $\mathcal{T}_0$ of pairwise orthogonal standard (ray) tubes of type $(p_1, \dots, p_t)$, containing at least one indecomposable projective A-module,*
- *for every $\gamma \in \mathbb{Q}^+$, a $\mathbb{P}_1(k)$-family $\mathcal{T}_\gamma$, of pairwise orthogonal, standard stable tubes of type $(p_1, \dots, p_t)$,*
- *a $\mathbb{P}_1(k)$-family $\mathcal{T}_\infty$ of pairwise orthogonal standard (coray) tubes of type $(p_1, \dots, p_t)$, containing at least one indecomposable injective A-module, and*
- *a postinjective component $\mathcal{Q}_\infty$.*

For each $\gamma \in \mathbb{Q}^+ \cup \{0, \infty\}$, the $\mathbb{P}_1(k)$-family $\mathcal{T}_\gamma$ separates

$$\mathcal{P}_\gamma := \mathcal{P}_0 \cup \bigcup_{0 \leq \beta < \gamma} \mathcal{T}_\beta \quad \textit{from} \quad \mathcal{Q}_\gamma := \bigcup_{\gamma < \beta \leq \infty} \mathcal{T}_\beta \cup \mathcal{Q}_\infty.$$

Moreover, $\operatorname{gl.dim} A = 2$ *and* $\operatorname{p.dim} X \leq 1$ *for any indecomposable module $X \in \mathcal{P}_\infty$ and* $\operatorname{i.dim} Y \leq 1$ *for any indecomposable module $Y \in \mathcal{Q}_0$.*

4.1.3 Tubular canonical algebras

Next, we recall a different class of algebras, the canonical algebras $C(p, \lambda)$.

Definition 4.8. For $t \geq 2$ and $(p_1, p_2, \dots, p_t)$ with $p_i \geq 2$ for all i consider the quiver $Q = Q(p_1, \dots, p_t)$ of Figure 4.2.

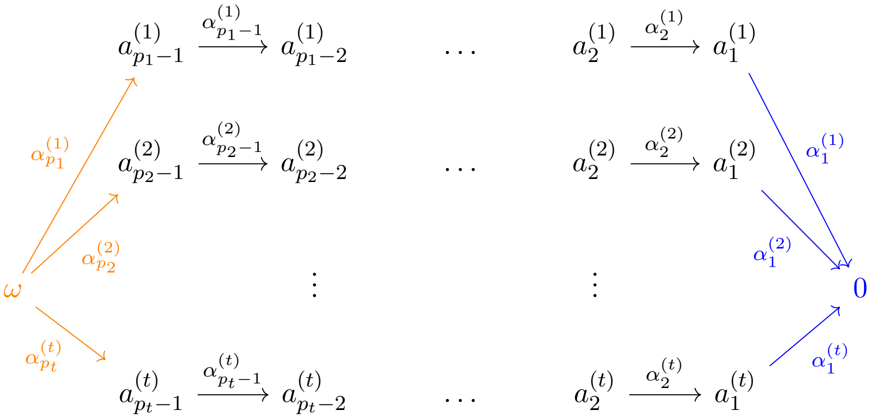


Figure 4.2: Quiver $Q(p_1, \dots, p_t)$ of a canonical algebra of type $p = (p_1, \dots, p_t)$.

- If $t \geq 3$, let $\lambda = (\lambda_3, \dots, \lambda_t) \in k^{t-2}$. Assume the λ_i are non-zero and pairwise different. Without loss of generality, we assume that $\lambda_3 = 1$.
- In case $t = 2$, we set $\lambda = 0$.

The algebra $C(p, \lambda) = kQ/I$, where the (generic) ideal I is generated by the relations

$$\rho_j := \alpha_1^{(1)} \dots \alpha_{p_1}^{(1)} + \lambda_j \alpha_1^{(2)} \dots \alpha_{p_2}^{(2)} - \alpha_1^{(j)} \dots \alpha_{p_j}^{(j)},$$

for $3 \leq j \leq t$, is a **canonical algebra of type** $p = (p_1, p_2, \dots, p_t)$.
We denote by $\underline{p}$ the least common multiple of the p_i.

We will refer to the arrows $\alpha_{p_j}^{(j)}, \dots, \alpha_1^{(j)}$ as belonging to the *j-th arm*, and may denote the arm by $\alpha^{(j)}$. In case that $t = 3$, we might also speak of the top ($j = 1$), central ($j = 2$) and lower ($j = 3$) arm.

Examples 4.9.

- The path algebra given by the quiver

$$\begin{tikzcd}
 & a_2^{(1)} \arrow[r, "\alpha_2^{(1)}"] & a_1^{(1)} \arrow[dr, "\alpha_1^{(1)}"] & \\
\omega \arrow[ur, "\alpha_3^{(1)}"] \arrow[dr, "\alpha_3^{(2)}"] & & & 0 \\
 & a_2^{(2)} \arrow[r, "\alpha_2^{(2)}"] & a_1^{(2)} \arrow[ur, "\alpha_1^{(2)}"] &
\end{tikzcd}$$

  (without any relations) is a canonical algebra of type $(3, 3)$. It is the extended Dynkin path algebra of type $\tilde{A}_{3,3}$.

- The path algebra of the quiver

$$\begin{tikzcd}
 & & a_1^{(1)} \arrow[drr, bend left, "\alpha_1^{(1)}"] & & \\
\omega \arrow[urr, bend left, "\alpha_2^{(1)}"] \arrow[r, "\alpha_3^{(2)}"] \arrow[dr, "\alpha_4^{(3)}"'] & a_2^{(2)} \arrow[rr, "\alpha_2^{(2)}"] & & a_1^{(2)} \arrow[r, "\alpha_1^{(2)}"] & 0 \\
 & a_3^{(3)} \arrow[r, "\alpha_3^{(3)}"] & a_2^{(3)} \arrow[r, "\alpha_2^{(3)}"] & a_1^{(3)} \arrow[ur, "\alpha_1^{(3)}"] &
\end{tikzcd}$$

  bound by the relation

$$\rho_3 := \alpha_1^{(1)} \circ \alpha_2^{(1)} + \alpha_1^{(2)} \circ \alpha_2^{(2)} \circ \alpha_3^{(2)} - \alpha_1^{(3)} \circ \alpha_2^{(3)} \circ \alpha_3^{(3)} \circ \alpha_4^{(3)}$$

  is a canonical algebra of type $(2,3,4)$. Moreover, there exists a tilting module $T \in \operatorname{mod} k\tilde{E}_7$ such that this canonical algebra is isomorphic to its endomorphism ring $\operatorname{End}_{k\tilde{E}_7}(T)$.

- The path algebra of the quiver

$$\begin{tikzcd}
 & a_2^{(1)} \arrow[r, "\alpha_2^{(1)}"] & a_1^{(1)} \arrow[dr, "\alpha_1^{(1)}"] & \\
\omega \arrow[ur, "\alpha_3^{(1)}"] \arrow[r, "\alpha_3^{(2)}"] \arrow[dr, "\alpha_3^{(3)}"] \arrow[ddr, bend right, "\alpha_3^{(4)}"] & a_2^{(2)} \arrow[r, "\alpha_2^{(2)}"] & a_1^{(2)} \arrow[r, "\alpha_1^{(2)}"] & 0 \\
 & a_2^{(3)} \arrow[r, "\alpha_2^{(3)}"] & a_1^{(3)} \arrow[ur, "\alpha_1^{(3)}"] & \\
 & a_2^{(4)} \arrow[r, "\alpha_2^{(4)}"] & a_1^{(4)} \arrow[uur, bend right, "\alpha_1^{(4)}"'] &
\end{tikzcd}$$

  bound by the relations

$$\rho_3 := \alpha_1^{(1)} \circ \alpha_2^{(1)} \circ \alpha_3^{(1)} + \alpha_1^{(2)} \circ \alpha_2^{(2)} \circ \alpha_3^{(2)} - \alpha_1^{(3)} \circ \alpha_2^{(3)} \circ \alpha_3^{(3)}$$
$$\rho_4 := \alpha_1^{(1)} \circ \alpha_2^{(1)} \circ \alpha_3^{(1)} + \lambda_4 \alpha_1^{(2)} \circ \alpha_2^{(2)} \circ \alpha_3^{(2)} - \alpha_1^{(4)} \circ \alpha_2^{(4)} \circ \alpha_3^{(4)},$$

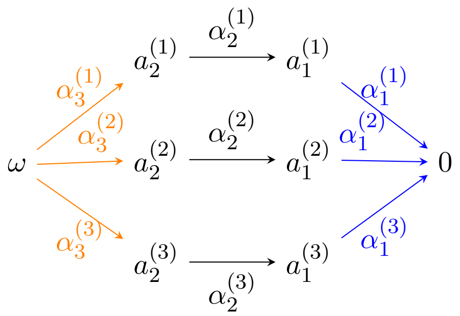
  for some $\lambda_4 \in k$, is a canonical algebra of type $(3,3,3,3)$. It is of wild representation type.

Definition 4.10. A **tubular canonical algebra** is a canonical algebra of type $(3,3,3)$, $(2,4,4)$, $(2,3,6)$ or $(2,2,2,2)$.

type	$\underline{h}_0$	$\underline{h}_\infty$	type of A_0 resp. A_∞
$(2,2,2,2)$	$\begin{smallmatrix} & 1 & \\ & 1 & \\ 0 & 1 & 2 \\ & 1 & \end{smallmatrix}$	$\begin{smallmatrix} & 1 & \\ & 1 & \\ 2 & 1 & 0 \\ & 1 & \end{smallmatrix}$	$\tilde{D}_4$
$(3,3,3)$	$\begin{smallmatrix} & 1 & 2 & \\ 0 & 1 & 2 & 3 \\ & 1 & 2 & \end{smallmatrix}$	$\begin{smallmatrix} & 2 & 1 & \\ 3 & 2 & 1 & 0 \\ & 2 & 1 & \end{smallmatrix}$	$\tilde{E}_6$
$(2,4,4)$	$\begin{smallmatrix} & & 2 & & \\ 0 & 1 & 2 & 3 & 4 \\ & 1 & 2 & 3 & \end{smallmatrix}$	$\begin{smallmatrix} & & 2 & & \\ 4 & 3 & 2 & 1 & 0 \\ & 3 & 2 & 1 & \end{smallmatrix}$	$\tilde{E}_7$
$(2,3,6)$	$\begin{smallmatrix} & & & & & & 3 & & & & & & \\ 0 & & & & 2 & & & & 4 & & & & 6 \\ & & 1 & & 2 & & 3 & & 4 & & 5 & & \end{smallmatrix}$	$\begin{smallmatrix} & & & & & & 3 & & & & & & \\ 6 & & & & 4 & & & & 2 & & & & 0 \\ & & 5 & & 4 & & 3 & & 2 & & 1 & & \end{smallmatrix}$	$\tilde{E}_8$

Table 4.2: Tubular types of tubular canonical algebras, minimal positive radical elements and type of the corresponding tame concealed algebras.

Example 4.11. The path algebra of the following quiver



bound by the relation

$$\rho_3 := \alpha_1^{(1)} \circ \alpha_2^{(1)} \circ \alpha_3^{(1)} + \alpha_1^{(2)} \circ \alpha_2^{(2)} \circ \alpha_3^{(2)} - \alpha_1^{(3)} \circ \alpha_2^{(3)} \circ \alpha_3^{(3)}$$

is a tubular canonical algebra of type $(3,3,3)$. It is the extension and coextension of an algebra of type $\tilde{E}_6$.

As the name suggests, tubular canonical algebras are just those canonical algebras that are tubular.

For a tubular canonical algebra A, we define several linear forms. Given a module M, we have the **rank of** M,

$$\operatorname{rk} M := \operatorname{rk} \underline{\dim} M := \dim M(0) - \dim M(\omega),$$

the **degree of** M,

$$\deg M := \deg \underline{\dim} M := \sum_{j=1}^{t} \frac{\underline{p}}{p_j} \sum_{i=1}^{p_j - 1} \dim M\left(a_i^{(j)}\right) - \underline{p} \dim M\left(\omega\right),$$

and the **slope of** M,

$$\operatorname{slope} M := \mu(M) := \begin{cases} \frac{\deg M}{\operatorname{rk} M}, & \operatorname{rk} M \neq 0, \\ \infty, & \text{else.} \end{cases}$$

We also recall the **index of** M, introduced in the previous section,

$$\operatorname{index} M := \begin{cases} \frac{-\langle \underline{h}_0, \underline{\dim} M \rangle}{\langle \underline{h}_\infty, \underline{\dim} M \rangle}, & \langle \underline{h}_\infty, \underline{\dim} M \rangle \neq 0, \\ \infty, & \text{else,} \end{cases}$$

where $\langle -, - \rangle$ is the Euler bilinear form of the algebra A. By Proposition 4.5, the index is defined for all modules, as not both the denominator and numerator are zero at the same time.

The module class of all indecomposable A-modules of index γ will be denoted by $\mathcal{T}_\gamma$. We will denote the full subcategory of $\operatorname{mod} A$ formed by the additive closure of all indecomposable modules isomorphic to those in $\mathcal{T}_\gamma$ by $\langle \mathcal{T}_\gamma \rangle$.

We will also denote by $\mathcal{X}_q$ the full subcategory of $\operatorname{mod} A$ formed by all indecomposable A-modules with $\mu(M) = q$.

Also note that we have the following correspondence between slope $\mu(M)$ and index $\gamma(M)$ of an indecomposable module M, if the slope is integral and does lie in the open interval $(0, \underline{p})$:

$$\gamma(M) = \begin{cases} 1 - \frac{\underline{p}}{\underline{p}+i}, & \mu(M) = \underline{p} + i \geq \underline{p}, \\ 1 + \frac{\underline{p}}{i}, & \mu(M) = -i < 0. \end{cases}$$

While $\langle \mathcal{T}_1 \rangle$ corresponds to $\mathcal{X}_\infty$, this implies that for $\underline{p} \leq q < \infty$ respectively $q \leq 0$, we have that $\mathcal{X}_q$ coincides with $\langle \mathcal{T}_\gamma \rangle$. We collect this information in Table 4.3. In what follows, we will mostly be concerned with modules having slope in these two intervals. Further note that an indecomposable module M with positive [negative] rank lies in $\mathcal{P}_1$ [$\mathcal{Q}_1$].

Remark. Given a canonical algebra A, we construct a contravariant functor

$$F\colon \operatorname{mod} A \to \operatorname{mod} A$$

in the following way: If $M = ((M_i), (M_\alpha))$ is a representation of Q, we define $F(M)$ as a representation on the vertices by

$$F(M)(0) = M(\omega)^*,\ F(M)(\omega) = M(0)^*,\ F(M)\left(a_i^{(j)}\right) = M\left(a_{p_j - i}^{(j)}\right)^*$$

and on the arrows by

$$F(M)\left(\alpha_i^{(j)}\right) = M\left(\alpha_{p_j - i + 1}^{(j)}\right)^*,$$

where $-^*$ denotes the dual of a vector space respectively the transposed map. F is an anti-equivalence and maps modules of slope m to those of slope $\underline{p} - m$.

component	$\mathcal{P}_0$	$\mathcal{T}_0 \equiv \mathcal{X}_p$	$\mathcal{T}_\gamma \equiv \mathcal{X}_\mu$	$\mathcal{T}_1 \equiv \mathcal{X}_\infty$	$\mathcal{T}_\gamma \equiv \mathcal{X}_\mu$	$\mathcal{T}_\infty \equiv \mathcal{X}_0$	$\mathcal{Q}_\infty$
slope μ		$\underline{p}$	$\underline{p} < \mu < \infty$	∞	$-\infty < \mu < 0$	0	
index γ	$-$ve or ∞	0	$0 < \gamma < 1$	1	$1 < \gamma < \infty$	∞	$-$ve or 0

Table 4.3: Slope and index for the different components of the module category of a tubular canonical algebra

4.2 First hyperfiniteness results

4.2.1 Tubular canonical algebras as one-point extensions

We are now in the position to prove a first hyperfiniteness result for tubular canonical algebras using their structure as a one-point extension of a tubular algebra.

Proposition 4.12. *Let $B = C(p, \lambda)$ be tubular canonical algebra. Then the family of all preprojective B-modules is hyperfinite.*

Proof. By [Rin84, Section 3.7], $C(p, \lambda)$ is given as the one-point extension

$$A[M] = \begin{bmatrix} A & M \\ 0 & k \end{bmatrix},$$

where the quiver of A is obtained from that of $C(p, \lambda)$ by the deletion of the unique source ω and the so-called coordinate module $M \in \operatorname{mod} A$ is given as

$$M = \begin{array}{ccccc} U^{(1)} & U^{(1)} & \dots & U^{(1)} & U^{(1)} \\ U^{(2)} & U^{(2)} & \dots & U^{(2)} & U^{(2)} \\ & & \vdots & & \\ U^{(t)} & U^{(t)} & \dots & U^{(t)} & U^{(t)} \end{array} \quad k^2,$$

where the $U^{(i)}$ are pairwise different one-dimensional subspaces of k^2. Clearly, A is a tame hereditary algebra of type $\widetilde{D}_4$, $\widetilde{E}_6$, $\widetilde{E}_7$ or $\widetilde{E}_8$, with the quiver given in subspace orientation. Moreover, it is easy to see that M is indecomposable, and thus it is easy to check that M is a regular A-module.

Denote by $e = e_A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ the idempotent of B corresponding to A. B-modules can then be described as a triple (V, W, f), where V is an A-module, W is a k-vector space and $f \in \operatorname{Hom}_A(M \otimes_k W, V)$. Note that by [ARS95, III.2.5(b)], the module $(P, 0, 0)$ is projective for a projective A-module P.

Now, the right adjoint to the restriction functor $\operatorname{res}_{e_A} \colon \operatorname{mod} B \to \operatorname{mod} A, Y \mapsto e_A Y$ is given by the left exact functor

$$L = L_{e_A} = \operatorname{Hom}_A(e_A B, -) \colon \operatorname{mod} A \to \operatorname{mod} B.$$

For $X \in \operatorname{mod} A$, the module $L_{e_A}(X) = (V, W, f)$ is given by the A-module

$$V = e_A \mathrm{Hom}_A(e_A B, X) = \mathrm{Hom}_A(e_A B e_A, X) \cong \mathrm{Hom}_A(A, X) \cong X,$$

the k-module

$$W = (1 - e_A)\mathrm{Hom}_A(e_A B, X) = \mathrm{Hom}_A(e_A B(1 - e_A), X) \cong \mathrm{Hom}_A(M, X),$$

and the A-linear map f induced by the multiplication map

$$m \otimes \varphi \mapsto \varphi(m) \text{ for } \varphi \in \mathrm{Hom}_A(M, X) \text{ and } m \in M.$$

This implies that if X is a preprojective A-module, $L(X) = (X, 0, 0)$, since M is regular. Thus, the projective modules in the preprojective component $\mathcal{P}(B)$ lie in the essential image of L. What is more, by considering [SS07, Corollary XV.1.7], we know that all indecomposable preprojective modules are in the essential image of L.

Now, $\dim_k L(X) \leq \dim_k e_A B \cdot \dim_K X$ by the definition of L, as k acts centrally on $\mathrm{Hom}_A(e_A B, X)$. Moreover, since L is fully faithful,

$$\dim_k X = \dim_k(\mathrm{res} \circ L)(X) = \dim_k e_A L(X) \leq \dim_k L(X).$$

As L is also left-exact, the conditions of Proposition 1.5 are fulfilled, so $L(\operatorname{mod} A)$ is hyperfinite, as $\operatorname{mod} A$ is by Theorem 3.18. This implies the result. □

Proposition 4.13. *Let $B = C(p, \lambda)$ be a tubular canonical algebra. Let $n \in \mathbb{N}$. Let $\mathcal{M}$ be a family of modules such that for all modules in $\operatorname{ind} \mathcal{M}$, the vector space at vertex ω has dimension at most n. Assume that for every $\varepsilon > 0$ there are only finitely many indecomposable modules $M \in \mathcal{M}$ such that $\dim M + n < \frac{2n}{\varepsilon}$. Then $\mathcal{M}$ is hyperfinite.*

Proof. By [Rin84, Section 3.7], B is given as a one-point extension $A[X]$, where the quiver of A is obtained from that of B by the deletion of the unique source ω. Clearly, A is a tame hereditary algebra of type $\widetilde{D}_4$, $\widetilde{E}_6$, $\widetilde{E}_7$ or $\widetilde{E}_8$, with the quiver given in subspace orientation. By Proposition 1.2, it suffices to check the hyperfiniteness condition on the indecomposable modules in $\mathcal{M}$. Let $\varepsilon > 0$. Let $L^A_{\frac{\varepsilon}{2}}$ be the bound for the associated tame hereditary algebra A established for $\frac{\varepsilon}{2}$. Let $M \in \operatorname{ind} \mathcal{M}$. Denote by $\overline{M}$ the restriction of M to the (sub-)quiver of A. As such, by Theorem 2.11, there exists an A-submodule $\overline{N} \subseteq \overline{M}$ such that $\dim \overline{N} \geq (1 - \frac{\varepsilon}{2}) \dim \overline{M}$ and $\overline{N} \cong \bigoplus_{i=1}^s \overline{N_i}$ with $\dim_k \overline{N_i} \leq L^A_{\frac{\varepsilon}{2}}$. Now, extend $\overline{N}$ to a B-module N by choosing the zero vector space at vertex ω and the linear map for each arrow $\alpha^{(j)}_{p_j} : \omega \to a^{(j)}_{p_j - 1}$ as $1 \leq j \leq t$ to be the zero map. Then N is a B-submodule of M. The decomposition lifts as well, so $N = \bigoplus_{i=1}^s N_i$ and $\dim_k N_i \leq L^A_{\frac{\varepsilon}{2}}$. Moreover, we have that

$$\begin{aligned}
\dim_k N = \dim_k \overline{N} &\geq \left(1 - \frac{\varepsilon}{2}\right) \dim_k \overline{M} \\
&\geq \left(1 - \frac{\varepsilon}{2}\right) (\dim_k M - n) = \left(1 - \frac{\varepsilon}{2}\right) \dim_k M + \frac{\varepsilon n}{2} - n \\
&= (1 - \varepsilon) \dim_k M + \frac{\varepsilon}{2} \dim M + \frac{\varepsilon n}{2} - n \\
&\geq (1 - \varepsilon) \dim_k M,
\end{aligned}$$

since

$$\varepsilon \dim M + \varepsilon n - 2n \geq 2n - 2n = 0$$

for all but finitely many indecomposable M by the hypothesis. Thus, by choosing

$$L_\varepsilon := \max \left\{ L^A_{\frac{\varepsilon}{2}}, \max \left\{ \dim M \;\middle|\; M \in \mathcal{M}\colon\ \dim M + n < \frac{2n}{\varepsilon} \right\} \right\},$$

we can show the hyperfiniteness of $\mathcal{M}$. ☐

4.2.2 Stable and right stable tubular families

Next, we discuss consequences for the right stable family $\mathcal{T}_0$ of semiregular modules, that is a component containing a projective but no injective module, and individual, stable tubular families.

Corollary 4.14. *For a tubular canonical algebra B, the right stable family $\mathcal{T}_0$ of semiregular modules, containing the unique indecomposable projective module $P(\omega)$ which is not preprojective, is hyperfinite.*

Proof. The indecomposable projective module $P(\omega)$ at the extending vertex is one-dimensional at ω, while its radical $\operatorname{rad} P(\omega)$ is zero-dimensional at the extending vertex. As this ray tube $\mathcal{T}_0(\rho)$ just has one ray, all indecomposable modules in it are inverse Auslander–Reiten translates of $P(\omega)$. Now, a combinatorial argument shows that the dimension of the vector space at vertex ω is at most one for all modules in $\operatorname{ind} \mathcal{T}_0(\rho)$. Moreover, for each $\varepsilon > 0$, the number of modules such that $\dim M + 1 < \frac{2}{\varepsilon}$ is finite, as there are only finitely many indecomposable modules for each dimension in $\operatorname{ind} \mathcal{T}_0(\rho)$. Now apply the previous Proposition. For the remaining indecomposables in $\mathcal{T}_0$, nothing needs to be shown as they are just A_0-modules. ☐

We can also prove this last result more directly.

Lemma 4.15. *Let B be a tubular canonical algebra. Then the right stable family $\mathcal{T}_0$ of semiregular modules containing the unique indecomposable projective module which is not preprojective is hyperfinite.*

Proof. We know that B is a tubular extension of A of branch length one using a single ray module E from the tubular family $\mathcal{T}$ of the tame hereditary algebra A. By [Rin84, Theorem 4.7.(1)] and its proof, $\mathcal{T}_0 \subset \operatorname{mod} B$ consists of all the tubes of $\mathcal{T}$ except for the one containing E. As the indecomposable modules in these tubes of B are just the ones for A extended by a zero vector space at ω, we know how to find submodules exhibiting their hyperfiniteness. The remaining component $\mathcal{T}_0(\rho)$ is a standard ray tube. It consists of all the indecomposable modules of $\mathcal{T}(\rho)$ not belonging to the ray starting at E, the modules in the ray $E[i]$ (complemented by the zero vector space) and modules $\overline{E[i]} = (E[i], \operatorname{Hom}(E, E[i]), \varepsilon_M)$ with $\varepsilon_M\colon E \otimes \operatorname{Hom}(E, E[i]) \to E[i]$, the evaluation map, as spelled out in the proof of [Rin84, Proposition 4.5.(1)]. Even more is known, as there are irreducible maps $\iota_i\colon E[i] \to \overline{E[i]}$ (also see Figure 4.3). It follows

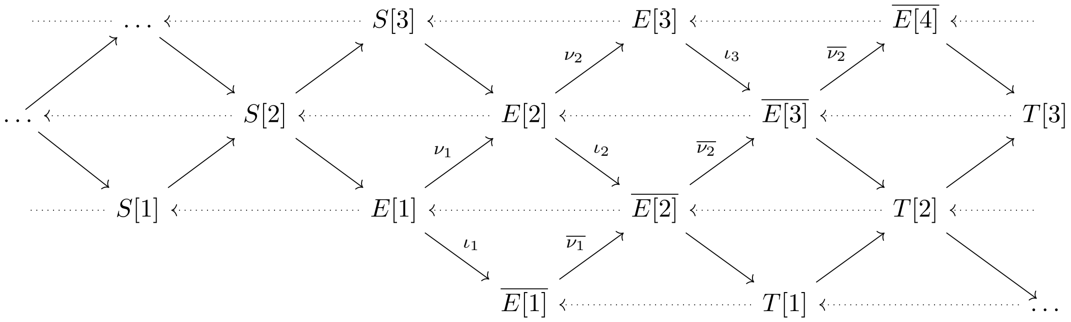


Figure 4.3: Schematics of the translation quiver of the ray tube of $\mathcal{T}_0$ in $\operatorname{mod} B$, where $S[1], E[1]$ and $T[1]$ denote regular simples on the mouth of the corresponding tube in $\operatorname{mod} A$. The dashed arrows indicate the action of τ.

that all indecomposables modules in $\mathcal{T}_0(\rho)$ are either just A-modules or have such a module as a submodule of codimension one, as we note that $\dim \operatorname{Hom}(E, E[i]) = 1$. This concludes the proof. ☐

Let us now consider a way of obtaining hyperfiniteness for a single stable tubular family based on such a result for its $\mathcal{P}$-class.

Proposition 4.16. *Let A be a finite dimensional k-algebra such that the k-dimension of the indecomposable injective modules is bounded. Let $\mathcal{T}$ be a stable tubular family separating $\mathcal{P}$ from $\mathcal{Q}$. If $\mathcal{P}$ is hyperfinite, so is $\mathcal{T}$.*

Proof. By Proposition 1.2, it is enough to check this for indecomposable regular modules R in $\mathcal{T}$. Let S be the $\mathcal{T}$-socle of R. Then there exists a non-zero map $f\colon S \to I$, where I is an indecomposable injective module not in $\mathcal{T}$. Since I is injective and $\iota\colon S \to T$ is injective, this map lifts to a map $\overline{f}\colon T \to I$. Denote $L := \ker \overline{f}$ and $K := \ker f$. Using the Snake Lemma, we get the following exact commutative diagram.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & K \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & T/S \arrow[d, equal] & \\
0 \arrow[r] & S \arrow[r, hook, "\iota"] \arrow[d, "f"'] & T \arrow[r] \arrow[d, "\overline{f}"'] & T/S \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I \arrow[r, equal] \arrow[d] & I \arrow[r] \arrow[d] & 0 & \\
 & 0 & 0 & &
\end{tikzcd}$$

Clearly, K and L as subobjects of modules in $\mathcal{T}$ cannot have a summand from $\mathcal{Q}$ by the separation property. Assume that L had a summand in $\mathcal{T}$. Then this summand must contain S respectively $\iota(S)$, for this is the smallest $\mathcal{T}$-submodule of T, using the fact that every indecomposable object in $\mathcal{T}$ has a unique composition series by [Rin84, 3.1.(3)]. But then we have

$$0 = \overline{f}(L) \supset \overline{f}(\iota S) = f(S) \neq 0,$$

a contradiction. Hence L can only have summands from $\mathcal{P}$, a hyperfinite family. Now the fact that I is of globally bounded dimension can be used in Proposition 1.4 to show that $\mathcal{T}$ is hyperfinite. □

We would like to use this result to achieve hyperfiniteness for tubular families $\mathcal{T}_\gamma$. While we know that both $\mathcal{P}_0$ and $\mathcal{T}_0$ are hyperfinite, there is no “next largest” rational number γ with a hyperfiniteness result for $\mathcal{P}_\gamma$. We will therefore use the following result to obtain hyperfiniteness for a single, arbitrary stable tubular family.

Proposition 4.17. *Let A be a tubular canonical algebra. Let $\mathcal{T}$ be a stable tubular family in* $\operatorname{mod} A$. *Then $\mathcal{T}$ is hyperfinite.*

Proof. Assume that A has tubular type $(p_1, \dots, p_t)$. By the definition of tubular algebras, we know that $p_1 \geq 2$. Consider the corresponding tube $\mathbb{T}$ of rank p_1 in $\mathcal{T}$. Let $T \in \mathbb{T}$ be a $\mathcal{T}$-simple module on the mouth of $\mathbb{T}$. By [GL91, Theorem 10.3], the module class $T^\perp$ is given by the module category of a canonical algebra Λ of type $(p_1 - 1, p_2, \dots, p_t)$. For all canonical tubular types, this algebra Λ is tame concealed by the proof of [Rin84, 4.3.(5)], as one can check from the (reduced) tubular type. Thus $T^\perp$, being equivalent to an algebra of amenable representation type (see Corollary 3.21), is hyperfinite. Similarly, we can show that $S^\perp$ is hyperfinite, for some $\mathcal{T}$-simple module S on the mouth of a second tube of rank $p_2 \geq 2$. Thus, we can cover all of $\mathcal{T}$ by two hyperfinite families, as two distinct tubes in the same tubular family are orthogonal, showing the hyperfiniteness of $\mathcal{T}$. □

While this proposition enables us to show hyperfiniteness for a single tubular family, unless we can control the change of the L_εs involved in the definition of hyperfiniteness and the HF-preserving functors, we can only achieve results for finitely many tubular families: Each equivalence $T^\perp \cong \operatorname{mod} \Lambda$ for a tame concealed algebra Λ has its own pair of (possibly) different constants K_1, K_2 (see Proposition 1.5 and the proof of Proposition 2.10). Given an HF-preserving functor F and fixed $\varepsilon > 0$, we have for the “new” bound that

$$L_\varepsilon = K_2 L^{\mathcal{N}}_{\frac{K_1}{K_2}\varepsilon}.$$

Hence, this constant might be different and even unbounded for $K_2 \to \infty$. Indeed, if $F\colon \operatorname{mod} \Lambda \to T^\perp$ is such an equivalence, the associated K_2 grows like the dimension of the $\mathcal{T}$-simple modules in $\mathbb{T}$, as made precise by the following lemma.

Lemma 4.18. *Let A be a tubular canonical algebra. For stable tubular families $\mathcal{T}$ of index γ, let T be a $\mathcal{T}$-simple module in an inhomogeneous tube $\mathbb{T}$. Then the dimensions of the simple objects of $T^{\perp}$ grow like h_γ.*

Proof. By [GL91, Theorem 10.3], the module class $T^{\perp}$ is given by the module category of a canonical algebra Λ and Λ is tame concealed by the proof of [Rin84, 4.3.(5)]. If S is a second $\mathcal{T}$-simple module in the same tube as T, we have $S \in T^{\perp}$. If $F\colon \operatorname{mod}\Lambda \to T^{\perp}$ is an equivalence, there is a regular simple Λ-module R such that $F(R) \cong S$. The length of R is bounded (and only depends on the type of Λ, which is one of finitely many). Now, the length of S as an object of the perpendicular category equals that of R over Λ. This implies that there is a simple object in $T^{\perp}$ with k-dimension at least

$$\frac{\dim S}{\text{length of }{}_{\Lambda}R}.$$

The sum of the dimension vectors of the indecomposable modules on the mouth of $\mathbb{T}$ adds up to a multiple of $\underline{h}_\gamma$. Now, h_γ grows with γ (c.f. Subsection 4.1.2). As the dimension vector determines index, this growth must happen across all indecomposable modules on the mouth. The claim then follows. □

4.3 Explicit construction of modules for tubular canonical algebras

As we have seen, to exhibit hyperfiniteness for infinitely many stable tubular families, we need to further understand indecomposable modules and their submodules and use information on their ranks and slopes. One way to achieve this is to provide an explicit description in terms of linear algebra.

We begin by studying the linear maps involved in these representations.

Lemma 4.19. *Let A be a tubular canonical algebra. Let M be an indecomposable module of positive rank. Then the maps $M(\alpha)$ are injective for all $\alpha \in Q_1$.*

Proof. First, let $\alpha = \alpha_i^{(j)}$ for $1 \leq i \leq \underline{p}_j - 1$. Then $s(\alpha) \neq \omega$. Assume that $\ker M(\alpha)$ is non-zero, that is there exists some $0 \neq x \in M(s(\alpha))$ such that $M(\alpha)(x) = 0$. Consider the submodule of M generated by x. Since the only arrow starting in $s(\alpha)$ maps x to zero, this submodule is the simple module $S_{s(\alpha)}$. Yet, $S_{s(\alpha)} \in \mathcal{X}_\infty$, so it cannot map to M, since $\operatorname{Hom}(\mathcal{T}_1, \mathcal{P}_1) = 0$ (see Table 4.3 and Theorem 4.7). A contradiction. Hence $M(\alpha)$ must be injective.

Now, assume $\alpha = \alpha_{p_j}^{(j)}$, thus $s(\alpha) = \omega$. Assume again that $\ker M(\alpha) \neq 0$, that is, there exists some $0 \neq x \in M(\omega)$ such that $M(\alpha)(x) = 0$ and consider the submodule $N \subset M$ generated by x. It will be one-dimensional in ω and have dimension at most one at vertex 0, as the map along arm (j) is zero and the relations J imply that the homomorphism space generated by the concatenation of maps along the arms is at most one-dimensional. This shows that N has rank $\operatorname{rk} N \in \{0, -1\}$. So N cannot have a non-zero map to M. A contradiction. Thus $M(\alpha)$ must be injective. □

Lemma 4.20. *Let A be a tubular canonical algebra. Let M be an indecomposable module of negative rank. Then the maps M_α are surjective for all $\alpha \in Q_1$.*

Proof. Let M be an indecomposable module of negative rank, say of slope $\underline{p} - m$. Let F be as in Subsection 4.1.3. Then $F(M)$ is an indecomposable module of slope $\underline{p}-(\underline{p}-m) = m$ and thus has positive rank. Given $M(\alpha)\colon M(s(\alpha)) \to M(t(\alpha))$, the map $M^*(\alpha)\colon M^*(t(\alpha)) \to M^*(s(\alpha))$ is injective, since by the construction of F, this is an arrow map for $F(M)$, to which Lemma 4.19 applies. We write $M(t(\alpha)) \cong \operatorname{im} M(\alpha) \oplus V$, and define $\varphi \in M^*(t(\alpha))$ to be the linear extension of $\varphi(e_i) = 0$ and $\varphi(f_j) = 1$, where $\{e_i\colon 1 \leq i \leq \dim \operatorname{im} M(\alpha)\}$ forms a basis of $\operatorname{im} M(\alpha)$ and $\{f_j\colon 1 \leq i \leq \dim V\}$ is a basis of V. Then $\varphi_{|\operatorname{im} M(\alpha)} = 0$, so $M^*(\alpha)(\varphi) = \varphi \circ M(\alpha) = 0$, and by the above injectivity, $\varphi = 0$, a contradiction, unless $V = 0$. Thus $M(\alpha)$ is surjective. $\square$

4.3.1 Rank one modules parametrised by Meltzer

We continue by discussing exceptional indecomposable modules of rank one, relying on a result by Meltzer [Mel07].

Proposition 4.21. *[Mel07, Proposition 4.3] Let C be a canonical algebra of arbitrary representation type and M an exceptional C-module of rank one. Then M is isomorphic to one of the following modules.*

$$\begin{tikzcd}
 & \cdots \arrow[r] & M\left(a^{(1)}_{r_1+1}\right) \arrow[r] & M\left(a^{(1)}_{r_1}\right) \arrow[r] & \cdots \arrow[ddr] & \\
 & \cdots \arrow[r] & M\left(a^{(2)}_{r_2+1}\right) \arrow[r] & M\left(a^{(2)}_{r_2}\right) \arrow[r] & \cdots \arrow[dr] & \\
k^m = M(\omega) \arrow[uur] \arrow[ur] \arrow[dr] & & \vdots & \vdots & & M(0) = k^{m+1}, \\
 & \cdots \arrow[r] & M\left(a^{(t)}_{r_t+1}\right) \arrow[r] & M\left(a^{(t)}_{r_t}\right) \arrow[r] & \cdots \arrow[ur] &
\end{tikzcd}$$

where the r_j are integers such that $0 \leq r_j < p_j$, for each $j = 1, \ldots, t$, (we stipulate that $a^{(j)}_0 = 0$), and

$$M\left(a^{(j)}_s\right) = k^{m+1} \text{ for } 0 \leq s \leq r_j,$$

whereas

$$M\left(a^{(j)}_s\right) = k^m \text{ for } r_j + 1 \leq s < p_j.$$

Further, the matrices of M are given as follows:

$$M\left(\alpha^{(1)}_{r_1+1}\right) = X_m := \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ 0 & \dots & 0 \end{pmatrix} \in \operatorname{Mat}_{m+1\times m}(k),$$

$$M\left(\alpha^{(2)}_{r_2+1}\right) = Y_m := \begin{pmatrix} 0 & \dots & 0 \\ 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} \in \operatorname{Mat}_{m+1\times m}(k),$$

$M\left(\alpha^{(j)}_s\right) = I_{m+1}$, *for* $1 \leq s \leq r_j$, *and* $M\left(\alpha^{(j)}_s\right) = I_m$ *for* $r_j + 1 < s \leq p_j$, *for both* $j = 1, 2$.
For $j = 3, \dots, t$, *we distinguish two cases:*

a) If $r_j = 0$, *we put*

$$M\left(\alpha^{(j)}_1\right) = X_m + \lambda_i Y_m \text{ and } M\left(\alpha^{(j)}_s\right) = I_m \text{ for } 1 < s \leq p_j.$$

b) If $r_j > 0$, *we put*

$$M\left(\alpha^{(j)}_1\right) = \begin{pmatrix} 1 & & & \\ \lambda_i & 1 & & \\ & \ddots & \ddots & \\ & & \lambda_i & 1 \end{pmatrix} \in \operatorname{Mat}_{m+1\times m+1}(k), \quad M\left(\alpha^{(j)}_{r_j+1}\right) = X_m,$$

$$M\left(\alpha^{(j)}_s\right) = I_{m+1} \text{ for } 1 < s \leq r_j \text{ and } M\left(\alpha^{(j)}_s\right) = I_m \text{ for } r_j + 1 < s \leq p_j.$$

This insight is then sufficient to prove a result independent of index respectively slope but dependent on rank.

Proposition 4.22. *Let A be a tubular canonical algebra. Then the family of all exceptional modules of rank one is hyperfinite.*

Proof. Let A be of tubular type $(p_1, \dots, p_t)$. Let $\varepsilon > 0$ and set $L_\varepsilon := |A|^{\frac{2(|A|-1)}{\varepsilon}}$. Let M be an exceptional indecomposable A-module of rank one. By Proposition 4.21, we know that $\underline{\dim}\, M = (m, \dots, m+1)$ and that the path maps $M_{\alpha^{(j)}}$ along the arms correspond to linear maps $k^m \to k^{m+1}$ given by X_m for the first arm and Y_m for the second arm, while for $3 \leq j \leq t$, the path map is given as $X_m + \lambda_j Y_m$. Assume that $\dim M > L_\varepsilon$, as otherwise we can choose the submodule to be M itself. This situation just corresponds to the regular modules over a generalised t-Kronecker quiver $\Theta(t)$ having arrows $\{\alpha^{(j)}\}_{j=1}^t$ bound by the relations

$$\alpha^{(j)} = \alpha^{(1)} + \lambda_j \alpha^{(2)} \text{ for } 3 \leq j \leq t.$$

Thus similar to the proof of Theorem 2.9, setting $K_\varepsilon := \left\lceil \frac{2(|A|-1)}{\varepsilon} \right\rceil$, we know how to remove every K_εth basis element from the vector space M_ω at vertex ω to make this subspace decompose. As the maps along the arms are injective by Lemma 4.19, this decomposition can be pushed along the arms to yield a decomposition of a submodule $N \subset M$ into summands of dimension at most $|A|(K_\varepsilon - 1) + 1 \leq L_\varepsilon$, while

$$\begin{aligned} \dim N &= (|A|-1)(m-s) + m + 1 \geq (\dim M - |A| + 2) - (|A|-1)s \\ &= \dim M - (|A|-1)\frac{\dim M - r}{K_\varepsilon} - |A| + 2 \geq \dim M - \frac{\varepsilon}{2}\dim M - |A| \\ &= (1-\varepsilon)\dim M + \frac{\varepsilon}{2}\dim M - |A| > (1-\varepsilon)\dim M, \end{aligned}$$

for $\dim M = s \cdot K_\varepsilon + r$, where $0 \leq r < K_\varepsilon$, as $\dim M > L_\varepsilon$. □

4.3.2 Explicit description of integral slope modules based on Dowbor–Meltzer–Mróz

For modules of higher rank, we turn to the work of Dowbor, Meltzer and Mróz [DMM14b]. They give a complete description of the homogeneous indecomposable modules of integral slope over tubular canonical algebras. This is done by constructing matrix bimodules over a localisation of a polynomial algebra. Using tensor products, a parametrisation of all indecomposable modules in homogeneous tubes of integral slope is then attained.

In particular, [DMM14b, Theorem 4.1, Remark 4.1a], gives a detailed description (in terms of matrices) of a representation isomorphic to each indecomposable module $M(\mu, l, \xi)$ in homogeneous tubes of integral slope μ and parameter ξ, of regular length l. In what follows, we will only use this description for integral slopes $\mu \neq 0, \underline{p}, \infty$. We will mostly rely on this explicit matrix description and not use the bimodule construction. Given an indecomposable module $E = M(\mu, 1, \xi)$, we may write $E^{\odot l}$ for the corresponding indecomposable $M(\mu, l, \xi)$.

It is this explicit description—slightly extended—that we will exploit to give further hyperfiniteness results for tubular canonical algebras.

Example 4.23. Consider the canonical tubular algebra of type $(3, 3, 3)$. We describe an indecomposable homogeneous module M of slope $3m + 4$ for $m \in \mathbb{N}$ and of regular length one. The module M has dimension vector

$$\underline{\dim}(M) = \begin{pmatrix} & 1+3m+1 & 2+3m+1 & \\ 3m+1 & 3m+2 & 1+3m+2 & 3m+4 \\ & 3m+2 & 3m+3 & \end{pmatrix}.$$

The matrices of the first arm, for example, are given by the block matrices

$$M(\alpha_3^{(1)}) = \left(\begin{array}{c|c|c} 0_{1,m} & 0_{1,m} & 0_{1,m+1} \\ \hline I_m & & \\ \hline & I_m & \\ \hline & & I_{m+1} \end{array}\right), \quad M(\alpha_2^{(1)}) = \left(\begin{array}{c|c|c|c} 1 & 0_{1,m} & 0_{1,m} & 0_{1,m+1} \\ 0 & 0_{1,m} & 0_{1,m} & 0_{1,m+1} \\ \hline 0_{m,1} & I_m & & \\ \hline 0_{m,1} & & I_m & \\ \hline 0_{m+1,1} & & & I_{m+1} \end{array}\right),$$

$$\text{and } M(\alpha_1^{(1)}) = \left(\begin{array}{cc|c|c|c}
0 & 0 & & & \\
\vdots & \vdots & X_m & & \\
0 & 0 & & & \\
1 & \xi & & & \\
\hline
0 & 0 & & & \\
\vdots & \vdots & & X_m & \\
0 & 0 & & & \\
1 & 1 & & & \\
\hline
0 & 0 & & & \\
\vdots & \vdots & & & \\
0 & 0 & & & X_{m+1} \\
1 & 0 & & & \\
1 & 0 & & &
\end{array}\right)$$

respectively. We denote this module by $M(3m+4, 1, \xi)$.
Note that for $M^{\odot l}$, the entry ξ will be replaced by a Jordan-block $J_l(\xi)$ of eigenvalue ξ, while the ones will be replaced by the identity matrix I_l.

The full list of the matrices can be found in [DMM14b, pp. 345–354].

We will further group the basis elements of the vector spaces at vertices 0 and ω to ease referring to the blocks appearing. Let us consider some $M(\mu, l, \xi)$ of slope $\mu = m\underline{p} + r$ with $0 \leq r < \underline{p}$ for positive rank. The vector space at vertex ω is given as

$$k^{l(m)}{}^{\oplus \underline{p}-r} \oplus k^{l(m+1)}{}^{\oplus r},$$

while the vector space at vertex 0 is given as

$$k^{l(m+1)}{}^{\oplus \underline{p}-r} \oplus k^{l(m+2)}{}^{\oplus r}.$$

The basis vectors of each vector space $k^{l(m)}$ respectively $k^{l(m+1)}$ respectively $k^{l(m+2)}$ will be said to form a *block*. These $\underline{p}$ blocks will be ordered from left to right and then be grouped according to Table 4.4.

type	# in first group	# in middle group	# in last group
$(2,2,2,2)$	0	2	0
$(3,3,3)$	1	1	1
$(2,4,4)$	1	2	1
$(2,3,6)$	1	3	2

Table 4.4: Number of so-called blocks in each group when describing the block matrices for $M(\mu, l, \xi)$.

In denoting the (block) matrices, we will use the matrices

$$X_m := \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ 0 & \dots & 0 \end{pmatrix}, \quad Y_m := \begin{pmatrix} 0 & \dots & 0 \\ 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} \in \mathrm{Mat}_{m+1 \times m}(k),$$

and $Z_m^\kappa = X_m + \kappa Y_m$. Here, we may omit κ if it is equal to one. ote that they already appeared in Proposition 4.21. We will also write $X_m^{\odot l}$ and $Y_m^{\odot l}$ to denote the matrix obtained from X_m respectively Y_m by replacing 1 and 0 by the identity matrix I_l respectively the zero matrix $0_{l,l}$.

4.3.3 Special values of ξ and exceptional tubes

The construction of Dowbor, Meltzer and Mróz [DMM14b] also works for the special values zero, one and λ of ξ. Moreover, an analogous construction can give a sensible meaning to $M(\mu, l, \infty)$: morally speaking, one just makes the manifest changes in the relevant columns as one would in the 2-Kronecker case. In more detail, we start with $M(\mu, l, 1)$ and make changes in the block matrices describing the linear maps along the arms. We replace the Jordan block of eigenvalue $\xi = 1$ by the identity matrix I_l. Further, in the same block matrix, we replace the identity matrices in the block column corresponding to the last block of the middle group by the Jordan block of eigenvalue zero, $J_l(0)$.

Example 4.24. For the tubular algebra of type $(3, 3, 3)$, the indecomposable module $M(6, 1, \infty)$ has dimension vector

$$\underline{\dim}(M) = \left(\begin{smallmatrix} & 4 & 5 & \\ 3 & 4 & 5 & 6 \\ & 4 & 4 & \end{smallmatrix} \right),$$

and the linear maps for the first arm are given by the matrices

$$M(\alpha_3^{(1)}) = \left(\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 1 & & \\ \hline & 1 & \\ \hline & & 1 \end{array} \right), M(\alpha_2^{(1)}) = \left(\begin{array}{c|c|c|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 1 & & \\ \hline 0 & & 1 & \\ \hline 0 & & & 1 \end{array} \right),$$

$$\text{and } M(\alpha_1^{(1)}) = \left(\begin{array}{cc|c|c|c} 0 & 0 & 1 & & \\ 1 & 1 & 0 & & \\ \hline 0 & 0 & & 1 & \\ 1 & 0 & & 0 & \\ \hline 0 & 0 & & & 1 \\ 1 & 0 & & & 0 \end{array} \right)$$

respectively. All other matrices remain unchanged from $M(6, 1, 1)$. This module is then—in the sense of [DMM10]—isomorphic to the module in the second inhomogeneous tube of regular length 3, having as regular socle the first regular simple in that tube, that is, having tubular coordinates $[6, 2, 1, 3]$.

Remark. Note that for these special values of ξ, l does not equal the regular length of $M(\mu, l, \xi)$. These modules are indeed indecomposable representations of the tubular canonical algebra, as we will see from the following lemma.

Lemma 4.25. *Let A be a canonical tubular algebra. Let $M(\mu, l, \xi)$ be the A-module of slope μ, positive rank and (regular) length l constructed for parameter ξ as above. Then*

$$\operatorname{Hom}_A\left(M(\mu,l,\xi), M(\mu,l',\xi')\right) \cong \begin{cases} k[x]/(x^{\min(l,l')}), & \xi = \xi', \\ 0, & \xi \neq \xi'. \end{cases}$$

Proof. Fix a canonical tubular algebra A and let Q be the underlying quiver. Let $M = M(\underline{p}(m+1)+r, l, \xi)$ and $N = M(\underline{p}(m+1)+r, l', \xi')$ be two A-modules of the given form with the same slope $\underline{p}(m+1)+r$, where $0 \leq r < \underline{p}$. Let $f \in \operatorname{Hom}(M, N)$. Then f is given as a $|Q_0|$-tuple of linear maps $\operatorname{Hom}\left(M(i), N(i)\right)$ for $i \in Q_0$. We have $M(\omega) = {k^{lm}}^{\oplus \underline{p}-r} \oplus {k^{l(m+1)}}^{\oplus r}$ and $M(0) = {k^{l(m+1)}}^{\oplus \underline{p}-r} \oplus {k^{l(m+2)}}^{\oplus r}$. A similar description holds for N. We consider the induced representation of the 2-Kronecker quiver $\Theta(2)$ given as

$$\tilde{M} = M(\omega) \overset{M(\alpha^{(1)})}{\underset{M(\alpha^{(2)})}{\rightrightarrows}} M(0), \quad \text{where} \quad M\left(\alpha^{(j)}\right) = M\left(\alpha_1^{(j)}\right) \circ \cdots \circ M\left(\alpha_{p_j-1}^{(j)}\right),$$

and $\tilde{N}$, constructed in the same manner. Using the given structure of the $M(\alpha_i^{(j)})$, we see that

$$\tilde{M} \cong {P_m}^{\oplus l(\underline{p}-r)} \oplus {P_{m+1}}^{\oplus lr},$$

while

$$\tilde{N} \cong {P_m}^{\oplus l'(\underline{p}-r)} \oplus {P_{m+1}}^{\oplus l'r}.$$

We may conclude that $\operatorname{Hom}_{k\Theta(2)}(\tilde{M}, \tilde{N})$ is a $\underline{p}^2 ll'$-dimensional algebra. Indeed, we fully understand the induced homomorphism (f_ω, f_0): The k-linear map $f_0 = (B^{(ij)})$ for vertex 0 is a $\underline{p} \times \underline{p}$-block matrix, where each block is itself an (almost) diagonal block matrix,

$$B^{(ij)} = \begin{cases} \begin{pmatrix} \lambda^{(ij)} & & \\ & \ddots & \\ & & \lambda^{(ij)} \end{pmatrix} \in \operatorname{Mat}_{m+1 \times m+1}\left(\operatorname{Mat}_{l' \times l}(k)\right), & 1 \leq i,j \leq \underline{p}-r, \\ \begin{pmatrix} 0 & & \\ & \ddots & \\ & & 0 \end{pmatrix} \in \operatorname{Mat}_{m+1 \times m+2}\left(\operatorname{Mat}_{l' \times l}(k)\right), & 1 \leq i \leq \underline{p}-r < j \leq \underline{p}, \\ \begin{pmatrix} \mu^{(ij)} & & \\ \nu^{(ij)} & & \\ & \ddots & \\ & & \mu^{(ij)} \\ & & \nu^{(ij)} \end{pmatrix} \in \operatorname{Mat}_{m+2 \times m+1}\left(\operatorname{Mat}_{l' \times l}(k)\right), & 1 \leq j \leq \underline{p}-r < i \leq \underline{p}, \\ \begin{pmatrix} \lambda^{(ij)} & & \\ & \ddots & \\ & & \lambda^{(ij)} \end{pmatrix} \in \operatorname{Mat}_{m+2 \times m+2}\left(\operatorname{Mat}_{l' \times l}(k)\right), & \underline{p}-r < i,j \leq \underline{p}, \end{cases}$$

and

$$\lambda^{(ij)} = \begin{pmatrix} \lambda_{11}^{(ij)} & \dots & \lambda_{1l}^{(ij)} \\ \vdots & \ddots & \vdots \\ \lambda_{l'1}^{(ij)} & \dots & \lambda_{l'l}^{(ij)} \end{pmatrix}.$$

Similarly, $f_\omega = (A^{(ij)})$ is a $\underline{p} \times \underline{p}$-block matrix of blocks themselves being block matrices. We will stick to this convention of writing block matrices of block matrices for the linear maps at other vertices, that is, for all other $f_{a_i^{(j)}}$.

We shall now collect some consequences of matrix identities which we will use in the ensuing case analysis quite frequently and may refer to them as *basic considerations.*

Note that for some $m \times w$ block matrix $E = \left(e^{(u,v)}\right)$ where $e^{(u,v)} = \left(e_{ij}^{(u,v)}\right) \in \mathrm{Mat}_{l' \times l}(k)$, we have

$$Z_m^\kappa \circ E = \begin{pmatrix} \left(e_{ij}^{(1,1)}\right) & \dots & \left(e_{ij}^{(1,w)}\right) \\ \kappa(e_{ij}^{(1,1)}) + (e_{ij}^{(2,1)}) & \dots & \kappa(e_{ij}^{(1,w)}) + (e_{ij}^{(2,w)}) \\ \vdots & \ddots & \vdots \\ \kappa(e_{ij}^{(m-1,1)}) + (e_{ij}^{(m,1)}) & \dots & \kappa(e_{ij}^{(m-1,w)}) + (e_{ij}^{(m,w)}) \\ \kappa e^{(m,1)} & \dots & \kappa e^{(m,w)} \end{pmatrix}, \tag{4.1}$$

thus $Z_m^\kappa \circ E = B^{(ij)}$ for non-zero $B^{(ij)}$ (as above) either implies $\lambda^{(ij)} = 0$ or $\kappa\mu^{(ij)} = \nu^{(ij)}$. A similar result holds for Z_{m+1}^κ and suitable E'.

From $X_m^{\odot l'} \circ D = B$ respectively $Y_m^{\odot l'} \circ D = B$ it follows that the first row respectively the last row of B must consist of zero matrices. (4.2)

Clearly, from $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)^{\odot l'} F = C \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)^{\odot l}$, it follows that C_{21} must be the zero matrix, where $C = (C_{ij})$ and suitable F. (4.3)

Moreover, for $C \in \mathrm{Mat}_{l' \times l}(k)$, we have that $J_{l'}(\xi') \circ C = C \circ J_l(\xi)$ implies that

$$C = 0 \text{ whenever } \xi \neq \xi', \quad \text{and otherwise } C = \begin{pmatrix} c_{11} & 0 & 0 & \dots & 0 \\ c_{21} & c_{11} & 0 & \dots & 0 \\ c_{31} & c_{21} & c_{11} & \dots & 0 \\ \vdots & & \ddots & \ddots & \\ c_{l',1} & c_{l'-1,1} & \dots & c_{21} & c_{11} \end{pmatrix}. \tag{4.4}$$

Finally, for $C, \lambda \in \mathrm{Mat}_{l' \times l}(k)$, it follows from $C = \lambda \circ J_l(\xi)$ and $J_{l'}(0) \circ C = \lambda$ (respectively from $J_{l'}(0) \circ C = \lambda$ and $C = \lambda \circ J_l(0)$) that both C and λ are the zero matrix. (4.5)

We introduce notation for some of the occurring matrices, such as

$$f_{a_1^{(1)}} = \left(C^{(ij)}\right)_{i,j=0,\dots,\underline{p}}.$$

Next, start by considering the situation for $\xi, \xi' \neq \infty$ for slope $\mu = \underline{p}(m+1)+r$, where $0 \leq r < \underline{p}$.

Type $(2,2,2,2)$*,* $r=0$*.* Using relation (4.1) for $\kappa = 1$ and $\kappa = \lambda$ on arms (3) and (4) shows that $\lambda^{(21)}, \lambda^{(12)} = 0$. Along the second arm, the relation for $\alpha_1^{(2)}$ implies that $\lambda^{(11)} = \lambda^{(22)}$. Now we turn to the first arm. Here, the relation for $\alpha_1^{(1)}$ shows that $J_{l'}(\xi') \circ \lambda^{(22)} = \lambda^{(11)} \circ J_l(\xi)$.

Type $(2,2,2,2)$*,* $r=1$*.* We invoke type (4.1) occurring in the relations for $\alpha_1^{(3)}$ and $\alpha_1^{(4)}$ to see that $\mu^{(21)} = \nu^{(21)} = \lambda\mu^{(21)}$. Since the parameter $\lambda \neq 1$, it follows that $\mu^{(21)} = \nu^{(21)} = 0$. Recall that $B^{(12)}$ is zero by the above considerations from the Kronecker module. As in the even case $r=0$, the relation for $\alpha_1^{(2)}$ then shows that $\lambda^{(11)} = \lambda^{(22)}$, while relations along the first arm imply that $J_{l'}(\xi') \circ \lambda^{(22)} = J_l(\xi) \circ \lambda^{(11)}$, as the parameter $\lambda \neq 0$.

Type $(3,3,3)$*,* $r=0$*.* Relations along the third arm of type (4.1) show that $\lambda^{(12)}, \lambda^{(13)}$ and $\lambda^{(23)}$ are zero matrices. We employ the relations along the second arm next: First, we use the one for $\alpha_2^{(2)}$ to conclude that $D^{(21)}$ must be zero, where $f_{a_1^{(2)}} = \left(D^{(ij)}\right)$. Now, from the relation for $\alpha_1^{(2)}$, three type (4.2) relations follow, from which we may conclude that $\lambda^{(31)}, \lambda^{(32)}$ and $\lambda^{(21)}$ are zero. Turning to relations of the first arm, we see that the relation for $\alpha_2^{(1)}$ via type (4.3) shows that $C_{21}^{(00)} = 0$. Eventually, the relation for $\alpha_1^{(1)}$ implies that $\lambda^{(11)} = \lambda^{(22)} = \lambda^{(33)}$, while $J_{l'}(\xi') \circ \lambda^{(22)} = \lambda^{(11)} \circ J_l(\xi)$.

Type $(3,3,3)$*,* $r=1$*.* Relations along the lower arm of type (4.1) show that $\lambda^{(12)} = 0$, $\mu^{(31)} = \nu^{(31)}$ and $\mu^{(32)} = \nu^{(32)}$. The central arm further shows via type (4.2) that $\mu^{(31)} = 0$ and $\mu^{(32)} = 0$. Similarly to the previous case we also see that $\lambda^{(21)} = 0$. Acknowledging the Kronecker considerations, we also have $B^{(13)}, B^{(23)} = 0$. Thus, we have the same results from the top arm as in the previous case.

Type $(3,3,3)$*,* $r=2$*.* Here, basic considerations for the lower arm yield $\lambda^{(23)} = 0$, $\mu^{(21)} = \nu^{(21)}$ and $\mu^{(31)} = \nu^{(31)}$, while the central arm gives $\lambda^{(32)} = 0$, $\mu^{(21)} = 0$ and $\mu^{(31)} = 0$. Together with the considerations from the Kronecker module situation, we may draw the same conclusions from the relations along the top arm as in the previous two cases.

Type $(2,4,4)$*,* $r=0$*.* Relations along the lower arm show via type (4.1) identities that $\lambda^{(12)}, \lambda^{(13)}, \lambda^{(14)}, \lambda^{(23)}, \lambda^{(24)}$ and $\lambda^{(34)}$ must be zero matrices. The commutativity relation for $\alpha_3^{(2)}$ shows that $F^{(21)}$ must be zero where $f_{a_2^{(2)}} = \left(F^{(ij)}\right)$. The relation for $\alpha_2^{(2)}$ further implies that $D^{(21)}$ must be zero and $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)^{\odot l'} F_{22} = D^{(22)} \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)^{\odot l}$, where $f_{a_1^{(2)}} = (D^{(ij)})$. Now, from the relation for $\alpha_1^{(2)}$, four type (4.2) relations follow, which lead us to conclude that $\lambda^{(41)}, \lambda^{(42)}, \lambda^{(43)}, \lambda^{(31)}$ and $\lambda^{(21)}$ are zero. It follows as well that $\lambda^{(32)} = 0$, by a modified type (4.2) equality invoking $d_{21}^{(22)} = 0$, where $D^{(22)} = \left(d_{ij}^{(22)}\right)$. Eventually, the relation for $\alpha_1^{(1)}$ implies that $\lambda^{(11)} = \lambda^{(22)} = \lambda^{(33)} = \lambda^{(44)}$, while $J_{l'}(\xi') \circ \lambda^{(22)} = \lambda^{(11)} J_l(\xi)$.

Type $(2,4,4)$*,* $r \neq 0$*.* As for type $(3,3,3)$, relations for the lower and central arms and the basic considerations combine to show that $B^{(ij)} = 0$ for $i \neq j$. Thus we have the same results from the top arm as in the previous case.

Type $(2,3,6)$*,* $r = 0$*.* Commutativity relations for arrows of the lower arm will show via type (4.1) identities that $\lambda^{(12)}, \ldots, \lambda^{(16)}; \lambda^{(23)}, \ldots, \lambda^{(26)}; \lambda^{(34)}, \lambda^{(35)}, \lambda^{(36)}; \lambda^{(45)}, \lambda^{(46)}$ and $\lambda^{(56)}$ are zero. From relations of the central arm, as in the above cases, (modified) type (4.2) relations and preliminary results giving zero entries of $D^{(21)}$ and $D^{(22)}$, where $f_{a_1^{(2)}} = \left(D^{(ij)}\right)$, show that $\lambda^{(61)}, \ldots, \lambda^{(64)}; \lambda^{(51)}, \ldots, \lambda^{(54)}; \lambda^{(41)}, \lambda^{(42)}; \lambda^{(31)}, \lambda^{(32)}$ are zero. Using this information, we turn to the relation for $\alpha_1^{(1)}$ and deduce that $\lambda^{(56)}, \lambda^{(43)}$ and $\lambda^{(21)}$ are zero. It then follows from the same commutativity relation that $\lambda^{(11)} = \lambda^{(22)} = \lambda^{(33)} = \lambda^{(44)} = \lambda^{(55)} = \lambda^{(66)}$. We also conclude $J_{l'}(\xi') \circ \lambda^{(22)} = \lambda^{(66)} \circ J_l(\xi)$.

Type $(2,3,6)$*,* $r \neq 0$*.* We proceed as in the case for $(2,4,4)$: Using relations for the lower and central arms and the Kronecker considerations to show that most $B^{(ij)}$ are zero. The remaining off-diagonal blocks are forced to be zero by relations of the first arm. Finally, we have the same results from the top arm as in the previous case.

We note that the above considerations also work for if we put ξ and ξ' to be infinity, that is when $M = M(\mu, l, \infty)$ and $N = M(\mu, l', \infty)$, since the affected equalities can still be inferred, as the Jordan block and the identity matrix block switch their places in $M_{a_1^{(1)}}$ respectively $N_{a_1^{(1)}}$, while setting $\xi = 0$ (c.f. the above remark on the construction of the infinity case). If $\xi = \infty \neq \xi'$ or vice versa, we make some modifications to the above arguments:

Type $(2,2,2,2)$*.* We deduce that the off-diagonal entries of f_0 are zero as above. From $\alpha_1^{(2)}$ it still follows that $\lambda^{(11)} = \lambda^{(22)}$, while the first arm now gives an identity of type (4.5) for matrices $C_{11}^{(00)}$ respectively $\lambda^{(11)} = \lambda^{(22)}$.

Type $(3,3,3)$*.* We deduce that the off-diagonal entries of f_0 are zero as above. Note that the description of the module for parameter ∞ does not affect the commutativity relation for $\alpha_1^{(1)}$, which still shows that $\lambda^{(11)} = \lambda^{(22)} = \lambda^{(33)}$. Yet, by the modification for the infinity case, we also deduce an identity of type (4.5) for matrices $C_{22}^{(00)}$ respectively $\lambda^{(11)} = \lambda^{(22)}$.

Type $(2,4,4)$*.* We deduce that the off-diagonal entries of f_0 are zero as above. Avoiding effects of describing the infinity case, the commutativity relation for $\alpha_1^{(1)}$ still shows that $\lambda^{(11)} = \lambda^{(33)} = \lambda^{(44)}$. Yet, by the modification for the infinity case, we also deduce an identity of type (4.5) for matrices $C_{11}^{(00)}$ respectively $\lambda^{(11)} = \lambda^{(33)}$.

Type $(2,3,6)$*.* We deduce that most off-diagonal entries of f_0 are zero as above. From the relation for $\alpha_1^{(1)}$ then follows an identity of type (4.5) for matrices $C_{11}^{(00)}$ respectively $\lambda^{(44)} = \lambda^{(66)}$.

In each case, we now apply matrix identity consequence (4.4) or (4.5) to see that $f = 0$ if $\xi \neq \xi'$ or that f is determined by a matrix of the same shape as C above. This concludes the proof. $\square$

Remark. The previous lemma shows that for each positive slope μ there is a fully faithful functor $\operatorname{reg} k\Theta(2) \to \mathcal{X}_\mu$.

Corollary 4.26. *The modules $M(\mu, l, \xi)$ are pairwise non-isomorphic and indecomposable. For fixed integral slope μ and fixed l, there is one in each tube $\mathbb{T}$, the tube determined by the parameter ξ. They have regular length pl, where $p = \operatorname{rk}\mathbb{T}$ is the tube rank.*

Proof. We will discuss the positive rank case first. Here, by the previous Lemma 4.25, the endomorphism ring of a given $M(\mu, l, \xi)$ is a local algebra of dimension l. Moreover, the homomorphism space is zero for modules of the same slope if they do not have the same parameter ξ. As modules with distinct l' have different dimension vectors, they cannot be isomorphic.

By [DMM14b], for all but finitely many values of ξ, the $M(\mu, 1, \xi)$ give regular simple indecomposables in (different) homogeneous tubes. We fix μ (and the respective γ) and consider one of these homogeneous tubes, denoting it by $\mathcal{T}_\gamma(\rho)$. Using the notation of [Rin84, Chapter 5], $\mathcal{T}_\gamma(\rho)$ contains a module having dimension vector $h(\mathcal{T}_\gamma(\rho))$. Similarly, by [Rin84, 5.3.(5)], every inhomogeneous tube $\mathcal{T}_\gamma(\zeta)$ contains an indecomposable module with dimension vector

$$h(\mathcal{T}_\gamma(\zeta)) := \sum_{i=1}^{\operatorname{rk}\mathcal{T}_\gamma(\zeta)} \underline{\dim}\, E_i.$$

Here, we use a notation similar to Section 3.3 to denote the indecomposable objects in a fixed tube. Yet, $h(\mathcal{T}_\gamma(\xi))$ is a multiple of h_γ for all values of ξ (c.f. [Rin84, Proof of 5.5.(1)]). Since all $h(\mathcal{T}_\gamma(\xi))$ are primitive by [Rin84, 5.3.(3)], it follows that

$$h(\mathcal{T}_\gamma(\zeta)) = \underline{h}_\gamma = h(\mathcal{T}_\gamma(\rho)).$$

This shows that

$$\underline{\dim}\, M(\mu, 1, \xi) = \sum_{i=1}^{\operatorname{rk}\mathcal{T}_\gamma(\xi)} \underline{\dim}\, E_i,$$

establishing that $M(\mu, l, \xi)$ has regular length $l \operatorname{rk}\mathcal{T}_\gamma(\xi)$.

Now assume that $M(\mu, l, \xi)$ and $M(\mu, l', \xi')$ lie in the same tube $\mathbb{T}$. Then the corresponding modules $E_1[r] = M(\mu, 1, \xi)$ and $E_j[r] = M(\mu, 1, \xi')$ must also lie in the same tube. $E_1[r]$ and $E_j[r]$ have the same regular length $r = \operatorname{rk}\mathbb{T}$. If $r = 1$, they must both be regular simple, and hence isomorphic, as there is just one regular simple in a homogeneous tube. If $r \neq 1$, the coray of $E_1[r]$ consists of irreducible epimorphisms and it will pass through $E_j[s]$ for some $1 \leq s < r$. On the other hand, the ray of $E_j[r]$ consists of irreducible monomorphisms and contains all $E_j[s]$. Now, the composition

$E_1[r] \twoheadrightarrow E_j[s] \hookrightarrow E_j[r]$ is a non-zero homomorphism, hence $\mathrm{Hom}(E_1[r], E_j[r]) \neq 0$. But $\mathrm{Hom}(E_1[r], E_j[r]) = 0$ unless $\xi = \xi'$ by Lemma 4.25, thus these modules can only lie in the same tube if they have the same parameter ξ.

Finally, to get the negative rank case, we make use of the duality F from Subsection 4.1.3. This will map the indecomposables from a tube of slope m to one of slope $\underline{p} - m$ while preserving the tubular structure (see also [DMM14b, Section 4.2]). It follows that $F(M(\mu, l, \xi)) \cong M(\underline{p} - \mu, l, \xi)$. □

Remark. Since [DMM14b] have shown that $M(\mu, l, \xi)$ for $\xi \neq 0, 1, \lambda, \infty$ give all the indecomposable modules in the homogeneous tubes, we know that the special values for ξ give indecomposable modules of regular length each multiple of the tube rank for all the exceptional tubes of a given tubular family associated to slope μ.

4.4 Hyperfiniteness for integral slope modules

We now turn to apply the explicit descriptions to achieve hyperfiniteness results.

4.4.1 Homogeneous modules

The first class of modules to be discussed will be the homogeneous modules of integral slope.

Proposition 4.27. *Let A be a tubular canonical algebra. Then the family of all homogeneous modules of integral slope and positive rank is hyperfinite. Moreover, the family of (isoclasses of) indecomposable modules isomorphic to some $M(\mu, l, \xi)$ is hyperfinite.*

Proof. We prove the more general statement. Let c depend on the tubular type with $c = 2$ if $\underline{p} = 2$ and $c = \underline{p} + 1$ otherwise. Let $\varepsilon > 0$. Choose L_ε as the maximum of those bounds appearing in Proposition 4.22—appearing for $\frac{\varepsilon}{2}$—and those coming from Proposition 4.17 for the finite number of tubular families of integral slope containing all the regular simple (homogeneous) modules of dimension less than $\frac{2c}{\varepsilon}$.

Let $M = M(\mu, l, \xi)$ be some indecomposable module of integral slope μ, of regular length l and parameter ξ. If M is homogeneous, denote its regular socle by S. More generally, put $S = M(\mu, 1, \xi)$. Note that all homogeneous modules are isomorphic to such a representation by [DMM14b, Theorem 4.1, Remark 4.1a].

In case $\dim S < \frac{2c}{\varepsilon}$, M lies in one of the finitely many tubular families mentioned above, having tubular index $1 - \frac{\underline{p}}{\mu}$. Indeed, we make the argument that $\underline{\dim}\, S = \underline{h}_\mu$. Thus, Proposition 4.17 yields a suitable submodule with its summands' dimensions bounded by L_ε.

In case $\dim S \geq \frac{2c}{\varepsilon}$, remove the cl basis elements corresponding to the non-block entries of all the $M_{\alpha_i^{(j)}}$, i.e. all basis elements not belonging to a block with some I_m, X_m, Y_m or Z_m appearing already for $l = 1$. Thus, we arrive at a submodule $N \subset M$. Note that this is indeed a submodule, as all paths passing through the basis elements removed also start in basis elements which are removed. Moreover, $\dim M - \dim N = cl$. Thus, we have

$$\begin{aligned}\dim N &= \dim M - cl = l(\dim S - c)\\ &\geq l\left(\dim S - \frac{\dim S\varepsilon}{2}\right) = \left(1 - \frac{\varepsilon}{2}\right) l \dim S\\ &= \left(1 - \frac{\varepsilon}{2}\right) \dim M\end{aligned}$$

Moreover, by the block structure of the matrices N_α, N decomposes into $\underline{p}l$ summands, which are of rank one: for each vertex space of dimension lm, use the l distinct embeddings

$$\iota_j \colon k^m \hookrightarrow k^{lm}, \quad e_i \mapsto e_{li-j+1} \text{ for } j \in \{1, \dots, l\}.$$

Inspection of the corresponding matrices of the summands and a comparison with Proposition 4.21 further shows that these are exceptional modules. Now, Proposition 4.22 yields the existence of a submodule $N' \subseteq N$ with $\dim N' \geq (1 - \frac{\varepsilon}{2}) \dim N$, while N' decomposes into summands of dimension less than or equal to L_ε. A calculation as in the proof of Proposition 1.4 then shows that $\dim N' \geq (1 - \varepsilon) \dim M$. In this way, we have found a submodule $N' \subset M$ which is nearly as big as M and decomposes into summands of dimension at most L_ε.

□

Example 4.28. Consider the canonical tubular algebra $(3, 3, 3)$ and an indecomposable homogeneous module M of slope $3m + 4$ for $m \in \mathbb{N}$ and of regular length l. Then M has dimension vector

$$\underline{\dim}(M) = l \left(3m{+}1 \begin{smallmatrix} 1{+}3m{+}1 \\ 3m{+}2 \\ 3m{+}2 \end{smallmatrix} \begin{smallmatrix} 2{+}3m{+}1 \\ 1{+}3m{+}2 \\ 3m{+}3 \end{smallmatrix} 3m{+}4 \right).$$

We will examine the first arm. The corresponding linear maps are given by the block matrices

$$M_{\alpha_3^{(1)}} = \left(\begin{array}{c|c|c} 0_{l,lm} & 0_{l,lm} & 0_{l,l(m+1)} \\ \hline I_{lm} & & \\ \hline & I_{lm} & \\ \hline & & I_{l(m+1)} \end{array}\right), \quad M_{\alpha_2^{(1)}} = \left(\begin{array}{c|c|c|c} I_l & 0_{l,lm} & 0_{l,lm} & 0_{l,l(m+1)} \\ 0_{l,l} & 0_{l,lm} & 0_{l,lm} & 0_{l,l(m+1)} \\ \hline 0_{lm,l} & I_{lm} & & \\ \hline 0_{lm,l} & & I_{lm} & \\ \hline 0_{l(m+1),l} & & & I_{l(m+1)} \end{array}\right),$$

$$\text{and} M_{\alpha_1^{(1)}} = \left(\begin{array}{cc|c|c|c}
0_{l,l} & 0_{l,l} & & & \\
\vdots & \vdots & X_{lm} & & \\
0_{l,l} & 0_{l,l} & & & \\
I_l & J_l(\xi) & 0_{l-1,lm} & & \\ \hline
0_{l,l} & 0_{l,l} & & & \\
\vdots & \vdots & & X_{lm} & \\
0_{l,l} & 0_{l,l} & & & \\
I_l & I_l & & 0_{l-1,lm} & \\ \hline
0_{l,l} & 0_{l,l} & & & \\
\vdots & \vdots & & & \\
0_{l,l} & 0_{l,l} & & & X_{l(m+1)} \\
I_l & 0_{l,l} & & & \\
I_l & 0_{l,l} & & & 0_{l-1,l(m+1)}
\end{array}\right)$$

respectively. We will remove the basis elements corresponding to the highlighted rows and columns, that is, we remove the first l basis elements at vertex $a_2^{(1)}$ and the first $2l$ basis elements at vertex $a_1^{(1)}$. We proceed similarly for the other arms, hence removing the first l basis elements at vertex $a_1^{(2)}$. Thus we remove $4l$ basis elements in total.

Now it is easy to see that the submodule N generated by the remaining basis elements decomposes into (at least) three summands, corresponding to each of the three large non-zero blocks. Moreover, at each vertex, the vector space embeddings

$$\iota_j \colon k^m \hookrightarrow k^{lm}, \quad e_i \mapsto e_{li-j+1} \text{ for } j \in \{1, \dots, l\},$$

are compatible with this decomposition and further exhibit the decomposition into $3l$ indecomposable summands of rank one.

Now, the maps $(N_i)(\alpha^{(J)})$ for each summand corresponding to the full path $\alpha^{(J)}$ along each of the arms are given by X_m, Y_m and $Z_m^\kappa = X_m + \kappa Y_m$ respectively, so they are exceptional by Proposition 4.21.

There is a dual result for negative rank. Yet, it will not be used in proving the final result. Let us introduce the appropriate notation before we state it, though. By the duality, the vector space at vertex ω of some $M(\mu, l, \xi)$ of slope $\mu = -m\underline{p} - r$ with $0 \leq r < \underline{p}$ for negative rank is given as

$$M(\omega) = k^{l(m+1)^{\oplus \underline{p}-r}} \oplus k^{l(m+2)^{\oplus r}}.$$

We have the same blocks as for positive rank.

Proposition 4.29. *Let A be a tubular canonical algebra. Then the family of (isomorphism classes of) indecomposable modules isomorphic to some $M(\mu, l, \xi)$ for negative μ is hyperfinite.*

Proof. Let c be a constant depending on the tubular type, determined later. Choose L_ε to be the maximum of those bounds appearing in Proposition 4.22—appearing for $\frac{\varepsilon}{2}$—and those coming from Proposition 4.17 for the finite number of tubular families containing all the regular modules $M(\mu, l, \xi)$ of dimension less than $\frac{2c}{\varepsilon}$. We now let $M = M(\mu, l, \xi)$ be some indecomposable module of integral slope $\mu = -\underline{p}m - r$, where $0 \le r < \underline{p}$, of regular length l and parameter ξ. Put $S = M(\mu, 1, \xi)$. In case $\dim S < \frac{2c}{\varepsilon}$, M lies in one of finitely many tubular families mentioned above, having tubular index $1 - \frac{\underline{p}}{\mu}$. Indeed, we know that $\underline{\dim}\, S = h_\mu$. Thus, Proposition 4.17 yields a suitable submodule with its summands' dimensions bounded by L_ε. In case $\dim S \ge \frac{2c}{\varepsilon}$, consider the submodule $N \subset M$ generated in vertex ω by the following basis elements:

- All but the last l basis elements in the first block (note that there is no such block for for case $(2, 2, 2, 2)$). This block always has $l(m + 1)$ basis elements.
- The central basis elements (all but the first/last l respectively $2l$) in each middle block (of which there are two respectively one respectively two respectively three).
- All but the last l respectively $2l$ basis elements in the last blocks (of which there are none respectively one respectively one respectively two).

Clearly, $\dim M - \dim N = cl$, for c a constant depending only on the tubular type and the modulus of the slope $\mu \mod \underline{p}$. Now, N is the direct sum of $\underline{p}l$ indecomposable summands of rank zero or one, as can be seen using the block structure of the matrices N_α. Note that the rank one modules are exceptional as in the proof of Proposition 4.27, thus combining Proposition 4.22 (for rank one) respectively Proposition 4.17 (for rank zero) via Proposition 1.2 yields the existence of a submodule $N' \subseteq N$ with

$$N' \ge (1 - \frac{\varepsilon}{2}) \dim N,$$

while N' decomposes into summands of dimension less than or equal to L_ε. A calculation as in the proof of Proposition 1.4 then shows that $\dim N' \ge (1 - \varepsilon) \dim M$. In this way, we have found a submodule $N' \subset M$ which has nearly the same dimension as M and decomposes into summands of dimension at most L_ε. ☐

Example 4.30. Consider the canonical tubular algebra of type $(2, 4, 4)$. We will construct a submodule exhibiting hyperfiniteness for $M(-4m - 1, l, \xi)$, $\xi \ne \infty$. Here,

$$M(\omega) = k^{l(m+1)} \oplus k^{l(m+1)} \oplus k^{l(m+1)} \oplus k^{l(m+2)},$$

has basis

$$\{a_1^1, \ldots, a_1^l, a_2^1, \ldots, a_2^l, \ldots, a_{m+1}^1, \ldots, a_{m+1}^l; b_1^1, \ldots, b_1^l, \ldots, b_{m+1}^1, \ldots, b_{m+1}^l;$$
$$c_1^1, \ldots, c_1^l, \ldots, c_{m+1}^1, \ldots, c_{m+1}^l; d_1^1, \ldots, d_1^l, \ldots, d_{m+2}^1, \ldots, d_{m+2}^l\},$$

and we construct the submodule N generated by the vector space

$$\langle a_1^1, \ldots, a_1^l, \ldots, a_m^1, \ldots, a_m^l; b_2^1, \ldots, b_2^l, \ldots, b_m^1, \ldots, b_m^l;$$
$$c_2^1, \ldots, c_2^l, \ldots, c_m^1, \ldots, c_m^l; d_1^1, \ldots, d_1^l, \ldots, d_m^1, \ldots, d_m^l\rangle$$

at the source vertex ω.

In particular, consider $M(-9,1,1)$. We have

$$\underline{\dim}(M) = l\left(\begin{smallmatrix} & & 11 & & \\ 13 & 12 & 11 & 10 & 9 \\ & 12 & 11 & 10 & \end{smallmatrix}\right).$$

In case $l = 1$, the matrices of N are obtained from those of M by removing the highlighted rows and columns, as the vector space at each subsequent vertex is generated by the image with the said basis elements removed from the source vertex.

$$M(\alpha_2^{(1)}) = \left(\begin{array}{ccc|ccc|ccc|cccc}
0 & 0 & \xi & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
\hline
1 & 0 & 0 & & & & & & & & & & \\
0 & 1 & 0 & & & & & & & & & & \\
\hline
 & & & 1 & 0 & 0 & & & & & & & \\
 & & & 0 & 1 & 0 & & & & & & & \\
\hline
 & & & & & & 1 & 0 & 0 & & & & \\
 & & & & & & 0 & 1 & 0 & & & & \\
\hline
 & & & & & & & & & 1 & 0 & 0 & 0 \\
 & & & & & & & & & 0 & 1 & 0 & 0 \\
 & & & & & & & & & 0 & 0 & 1 & 0
\end{array}\right), \; M(\alpha_1^{(1)}) = \dots,$$

$$M(\alpha_4^{(2)}) = \left(\begin{array}{ccc|ccc|ccc|cccc}
1 & 0 & 0 & & & & & & & & & & \\
0 & 1 & 0 & & & & & & & & & & \\
0 & 0 & 1 & & & & & & & & & & \\
\hline
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
 & & & 0 & 1 & 0 & & & & & & & \\
 & & & 0 & 0 & 1 & & & & & & & \\
\hline
 & & & & & & 0 & 1 & 0 & & & & \\
 & & & & & & 0 & 0 & 1 & & & & \\
\hline
 & & & & & & & & & 0 & 1 & 0 & 0 \\
 & & & & & & & & & 0 & 0 & 1 & 0 \\
 & & & & & & & & & 0 & 0 & 0 & 1
\end{array}\right), \; M(\alpha_3^{(2)}) = \dots,$$

$$M(\alpha_4^{(3)}) = \left(\begin{array}{ccc|ccc|ccc|cccc}
1 & 0 & 0 & & & & & & & & & & \\
0 & 1 & 0 & & & & & & & & & & \\
0 & 0 & 1 & & & & & & & & & & \\
\hline
 & & & 1 & 0 & 0 & & & & & & & \\
 & & & 0 & 1 & 0 & & & & & & & \\
 & & & 0 & 0 & 1 & & & & & & & \\
\hline
 & & & & & & 1 & 0 & 0 & & & & \\
 & & & & & & 0 & 1 & 0 & & & & \\
 & & & & & & 0 & 0 & 1 & & & & \\
\hline
 & & & & & & & & & 1 & 1 & 0 & 0 \\
 & & & & & & & & & 0 & 1 & 1 & 0 \\
 & & & & & & & & & 0 & 0 & 1 & 1
\end{array}\right), \; M(\alpha_3^{(3)}) = \dots .$$

Then

$$\underline{\dim}(N) = l \left(\begin{smallmatrix} & & 6 & & \\ 6 & 5 & 5 & 5 & 8 \\ & 6 & 6 & 6 & \end{smallmatrix} \right).$$

Here, we have constant $c = 46$. Hence note that $\dim(M(-5,1,1)) = 63 < 2c$ would not be an interesting example.

4.4.2 Further indecomposable regular modules

Having studied homogeneous modules in homogeneous and exceptional tubes, we will now approach the remaining indecomposable regular modules.

Lemma 4.31. *Let A be a tubular canonical algebra. Let $\mathbb{T}$ be a tube in a standard stable tubular family $\mathcal{T}$ of integral slope. Then the $\mathcal{T}$-simple modules in $\mathbb{T}$ have rank $\frac{p}{\operatorname{rk}\mathbb{T}}$.*

Proof. Let us denote the $\mathcal{T}$-simple modules in $\mathbb{T}$ by $E_1, \dots, E_{\operatorname{rk}\mathbb{T}}$. By Corollary 4.26, some indecomposable module H in $\mathbb{T}$ of $\mathcal{T}$-length $\operatorname{rk}\mathbb{T}$ is given by $M(\mu, 1, \xi)$ and thus has rank $\pm \underline{p}$ by construction. We also know that $\underline{\dim}\, M = \sum_{i=1}^{\operatorname{rk}\mathbb{T}} \underline{\dim}\, E_i$, as $M \cong E_1[\operatorname{rk}\mathbb{T}]$ (c.f. [Rin84, Chapter 3]). On the other hand, for regular modules, the rank is stable under the translate τ, as

$$\begin{aligned}
\operatorname{rk} \tau M &= \dim(\tau M)_0 - \dim(\tau M)_\omega = \langle \underline{\dim}\, P(0) - \underline{\dim}\, P(\omega), \underline{\dim}\, \tau M \rangle \\
&= \langle \underline{\dim}\, P(0) - \underline{\dim}\, P(\omega), \Phi_A(\underline{\dim}\, M) \rangle = \langle \underline{\dim}\, M, \underline{\dim}\, I(0) - \underline{\dim}\, I(\omega) \rangle \\
&= \dim M_0 - \dim M_\omega = \operatorname{rk} M,
\end{aligned}$$

applying [Rin84, 2.4.(4)], as $\operatorname{p.dim} M \leq 1$ and M does not map to a projective module, and using $\underline{\dim}\, P(\omega) - \underline{\dim}\, P(0) = \underline{\dim}\, I(0) - \underline{\dim}\, I(\omega)$. Since the rank is a linear form on $K_0(A)$, it follows that $\operatorname{rk} E_i = \frac{p}{\operatorname{rk}\mathbb{T}}$ for $1 \leq i \leq \operatorname{rk}\mathbb{T}$. □

Corollary 4.32. *Let A be a canonical tubular algebra. Let $\mathbb{T}$ be a tube of maximal rank in a tubular family $\mathcal{T}$ of positive integral slope. Then the $\mathcal{T}$-simple modules in $\mathbb{T}$ have rank one.*

Proposition 4.33. *Let A be a canonical tubular algebra. Then there is a hyperfinite family of indecomposable modules containing a module of each regular length in each inhomogeneous tube of integral slope $\nu \gg \underline{p}$ and positive rank.*

Proof. Because of Corollary 4.26 and the trailing remark, in a given exceptional tube $\mathbb{T}$ of rank p_s of a given standard stable tubular family $\mathcal{T}_\gamma \equiv \mathcal{X}_\mu$ of slope μ, some indecomposable of $\mathcal{T}_\gamma$-length lp_s is given by $M(\mu, l, \xi)$ for $\xi \in \{0, 1, \infty, \lambda\}$. We denote this module by $E[lp_s]$. For a given $\mathcal{T}_\gamma$-length $(l-1)p_s < h \leq lp_s$, consider the sequence of injective irreducible maps in the tube $\mathbb{T}$

$$E[h] \xrightarrow{\iota^{(l)}_{p_s-h}} E[h+1] \xrightarrow{\iota^{(l)}_{p_s-h-1}} \ldots \xrightarrow{\iota^{(l)}_1} E[lp_s],$$

between indecomposable modules with the same $\mathcal{T}_\gamma$-socle E. We consider the submodule M of $E[h]$ generated by all basis elements of the vector space $E[h]_\omega$ in the source vertex. As $E[h]$ has positive rank, the maps $E[h]_\alpha$ are injective by Lemma 4.19, so denoting $a = \dim E[h]_\omega$, we have

$$\underline{\dim}\, M = \begin{array}{ccc} & a \;\ldots\; a & \\ a & \vdots & a{+}b \\ & a \;\ldots\; a & \end{array}, \text{ for some } b \in \mathbb{N}_0.$$

Note that we may also consider M as the submodule of $E[lp_s]$ generated by $f_\omega(E[h]_\omega)$, where $f = \iota^{(l)}_1 \circ \cdots \circ \iota^{(l)}_{p_s-h}$.

Next, recall that $E[lp_s]$ has slope $\mu = (m+1)\underline{p} + r$, with $0 \leq r < \underline{p}$, and induces a $k\Theta(2)$-module

$$\widetilde{E[lp_s]} = \left(k^{lm^{\oplus \underline{p}-r}} \oplus k^{l(m+1)^{\oplus r}} \underset{\left(Y_m^{\odot l}\right)^{\oplus \underline{p}-r} \oplus \left(Y_{m+1}^{\odot l}\right)^{\oplus r}}{\overset{\left(X_m^{\odot l}\right)^{\oplus \underline{p}-r} \oplus \left(X_{m+1}^{\odot l}\right)^{\oplus r}}{\rightrightarrows}} k^{l(m+1)^{\oplus \underline{p}-r}} \oplus k^{l(m+2)^{\oplus r}} \right),$$

given the composition of the maps along two generic arms respectively. We see that this module decomposes into $\underline{p}l$ summands of type P_m respectively P_{m+1} (where P_m as in Theorem 2.9). Thus, any $k\Theta(2)$-submodule of this module must also be a direct sum of preprojective indecomposables, that is, of modules of type P_i. As the submodule M is generated solely in $E[lp_s]_\omega$, this data is sufficient to understand M as an A-submodule of $E[p_s]$, thus showing that M is a direct sum of rank one modules.

We continue by showing that the codimension of M is suitable. To this end, recall that $\underline{\dim}\, E[h] = \sum_{j=0}^{h-1} \underline{\dim}\, E_{i+j}$, where the $E_j := E_j[1]$ denote the $\mathcal{T}_\gamma$-simple modules on the mouth of $\mathbb{T}$ and we put $E_i := E[1]$. By Lemma 4.31, the $\mathcal{T}_\gamma$-simples all have rank $c = \frac{\underline{p}}{p_s}$. Denote $\overline{p} = \sum_{i=1}^{t} p_i = |A| - 2$. Thus, the dimension of $E[h]$ is at most $a + \overline{p}(a + hc) + (a + hc)$, as the dimension vector is the sum of h dimension vectors of regular simple modules, adding up to a in vertex ω. The dimension of M on the other hand is at least $(2 + \overline{p})a$, as the linear maps along the arms in M are restrictions from injective maps. The codimension of M in $E[h]$ then is at most $(1 + \overline{p})ch$.

Now let $\varepsilon > 0$ and assume that

$$\min\{\dim S \colon S \text{ regular simple in } \mathcal{X}_\mu\} \geq \frac{\overline{p}(1+\overline{p})}{\varepsilon}, \tag{4.6}$$

noting that the minimum exists since there are only finitely many inhomogeneous tubes and the dimension vectors in the homogeneous tubes of fixed slope μ are constant. This yields

$$\begin{aligned}\dim M &\geq \dim E[h] - (1+\overline{p})ch \geq \dim E[h] - \varepsilon \min_j \{\dim E_j\} h \\ &\geq \dim E[h] - \varepsilon \sum_{j=0}^{h-1} \dim E_{i+j} = (1-\varepsilon)\dim E[h].\end{aligned}$$

Finally, we note that the imposed condition (4.6) is one on the tubular family $\mathcal{T}_\gamma$ and will only be false for finitely many tubular families of small slope: For indecomposable modules, the dimension vector determines the slope of the module. For a given dimension (and fixed number of vertices), there are only finitely many dimension vectors of smaller dimension. Thus, the condition requires the smallest regular simple to have one of only finitely many dimension vectors and hence one of finitely many integral slopes. This implies that there is some $\nu \geq \underline{p}$ such that the condition holds for all $\mu \geq \nu$. □

Lemma 4.34. *Let A be a finite dimensional k-algebra. On the modules of projective dimension at most one, the translate τ is given by the functor $D\mathrm{Ext}^1_A(-, A)$, and if $0 \to X \to Y \to Z \to 0$ is an exact sequence of modules of projective dimension at most one, and $\mathrm{Hom}(X, A) = 0$, then the induced sequence $0 \to \tau X \to \tau Y \to \tau Z \to 0$ is exact.*

Proof. Let M be indecomposable and such that $\mathrm{p.dim}\, M \leq 1$. Let

$$\eta\colon 0 \to P_1 \to P_0 \to M \to 0$$

be a minimal projective resolution. Then there exists an exact sequence

$$\epsilon\colon 0 \to \tau M \to \nu P_1 \to \nu P_0 \to \nu M \to 0.$$

On the other hand, applying $\mathrm{Hom}_A(-, A)$ to the first sequence η yields

$$0 \to \mathrm{Hom}_A(M, A) \to \mathrm{Hom}_A(P_0, A) \to \mathrm{Hom}_A(P_1, A) \to \mathrm{Ext}^1_A(M, A) \to 0,$$

showing that $\tau M \cong D\mathrm{Ext}^1_A(M, A)$ for $\nu = D\mathrm{Hom}_A(-, A)$ (see [ASS10][Chapter IV.2]). Now, consider the exact sequence

$$\xi\colon 0 \to X \to Y \to Z \to 0.$$

By applying the above fact, we get an exact sequence

$$0 = D\mathrm{Ext}^2_A(Z, A) \to \tau X \to \tau Y \to \tau Z \to \nu X \to \nu Y \to \nu Z \to 0.$$

Now the result follows, since $\nu X = D\mathrm{Hom}_A(X, A) = 0$ by assumption. □

Corollary 4.35. *Let A be a canonical tubular algebra. Then the family of all indecomposable modules in exceptional tubes of integral slope and positive rank is hyperfinite.*

Proof. By Proposition 4.33, there is a hyperfinite family of indecomposable modules, containing a module of each regular length in each exceptional tube of integral slope and positive rank. Let M be such a module. We can assume that M is not in one of the finitely many exceptional families, for otherwise there would be nothing to show. Let N be the submodule constructed in the first step of the proof of Proposition 4.27 respectively in the proof of Proposition 4.33. Clearly, M and N are of projective dimension at most one (see Theorem 4.7). Moreover, N is the direct sum of indecomposable modules of rank one. If one of these summands was projective, we would either have $m = 0$ or $m = 1$ (in the notation of the proof and subsequent example). But then $\dim S$, the dimension of the regular simple underlying M would be bounded, so we could just add this tubular family to the list of finite exceptions (note that there is just a finite number of such families). Thus assume no summand is projective. By the construction of N, a removal of basis elements only happens in certain vertices. Thus, by inspection of the coefficient quivers in the inclusion sequence $N \hookrightarrow M \twoheadrightarrow C$, we see that the cokernel C is a direct sum of indecomposable summands, either a simple in a vertex of one arm or containing a $k \xrightarrow{\text{id}} k$ stretch in two vertices of one arm. These belong to $\mathcal{T}_1$, as they are indecomposable, regular and of rank 0, so the cokernel C has projective dimension at most one and has no projective direct summands. Thus, applying Lemma 4.34 along with Proposition 1.5 shows the result. Note that neither M, N nor C are in the τ-orbit of projective or injective modules, so one can derive the required constants K_1 and K_2 from the Coxeter matrix of A as it describes the effect of τ on the dimension vectors (c.f. [Rin84, 2.4.(4), 2.4.(4*)]). Indeed, considering the dimension vector of the summands of τN, we see that these are again rank one modules, so we apply Proposition 4.22 as in the proof of Proposition 4.27. □

Proposition 4.36. *Let A be a tubular canonical algebra. Then the family of (isomorphism classes) of all indecomposable regular modules of integral slope and negative rank is hyperfinite.*

Proof. Let $\varepsilon > 0$. Assume that the slope μ is such that

$$\min\{\dim S\colon S \text{ regular simple in } \mathcal{X}_\mu\} \geq \frac{2(|A|-2)\underline{p}}{\varepsilon}.$$

There are only finitely many inhomogeneous tubes for each slope—and they have finite rank. Moreover, the dimension vector of the regular simples in the homogeneous tubes is constant. This yields that the minimum above exists for each μ. Now, the duality F discussed in Section 4.1.3 further implies that the condition on the minimum (similar to the argument in the proof of Proposition 4.33) holds for all but finitely many integral slopes, for which we use Proposition 4.17.

Let M be an indecomposable regular module of slope μ, regular length l and rank $-r$ in some stable tube $\mathbb{T}$. Since we know that the regular simples in $\mathbb{T}$ have rank $-\frac{\underline{p}}{p_s}$, we must have $r = l\frac{\underline{p}}{p_s}$. Let $\alpha^{(1)} = \alpha_1^{(1)} \circ \cdots \circ \alpha_{p_1}^{(1)}$ and $f = M(\alpha^{(1)})\colon M(\omega) \to M(0)$ be the composition of the corresponding linear maps. We have $\dim M(\omega) = \dim M(0) + r$. As

M has negative rank, all the $M(\alpha_i^{(j)})$ are surjective by Lemma 4.20, and so is f. We decompose $M(\omega) = \ker f \oplus M'(\omega)$. As

$$0 \to \ker f \to M(\omega) \xrightarrow{f} M(0) \to 0$$

is exact, we have $\dim M'(\omega) = \dim M(0)$ and $f_{|M'(\omega)}$ is bijective. It follows that the submodule $N \subset M$ generated by $M'(\omega)$ has rank zero.

Assume N had a submodule N' of negative rank. Then the induced submodule $\tilde{N}'$ of the $k\Theta(2)$-module $\tilde{N} = N(\omega) \rightrightarrows N(0)$ (as in the proof of Proposition 4.33) must have an indecomposable postinjective summand $Q_m \hookrightarrow \tilde{N}$, for otherwise $\dim N(0) \geq \dim N(\omega)$. Yet, this postinjective summand has dimension vector $(m+1, m)$. But the map associated to the arrow $\alpha^{(1)}$ in Q_m is a restriction of f, so must be injective, implying that $m + 1 \leq m$, a contradiction.

Similarly, assume that N had a summand N'' of positive rank. Then

$$(\dim N''(\omega), \dim N''(0)) = (m, m'), \text{ where } m < m'.$$

As we have established that N cannot have submodules of negative rank, all other summands of N must have $(\omega, 0)$-dimension vector (u, v) where $u \leq v$. This would imply that $\operatorname{rk} N = (m' + v) - (m + u) > 0$, a contradiction. Thus, we have shown that N can only have summands of rank zero, i.e. $N \in \mathcal{X}_\infty = \langle \mathcal{T}_1 \rangle$ lies in a hyperfinite family.

Next, we want to determine $\dim N$. To this end, assume that

$$\underline{\dim}\, M = \begin{array}{ccccc} & m+r^{(1)}_{p_1-1} & \cdots & m+r^{(1)}_1 & \\ m+r & & \vdots & & m \\ & m+r^{(t)}_{p_t-1} & \cdots & m+r^{(t)}_1 & \end{array}, \text{ where } 0 \leq r_i^{(j)} \leq r.$$

Note that this holds as all the maps $M(\alpha)$ are surjective. We further put $r_{p_j}^{(j)} := r$ and $r_0^{(j)} := 0$ for all j and note that $0 \leq r_1^{(j)} \leq \cdots \leq r_{p_j-1}^{(j)} \leq r$. We have that $\dim \ker M(\alpha_i^{(j)}) = r_i^{(j)} - r_{i-1}^{(j)}$ for $2 \leq j \leq t$ and $1 \leq i \leq p_j$. Similarly, we may write

$$\underline{\dim}\, N = \begin{array}{ccccc} & m & \cdots & m & \\ & n^{(2)}_{p_2-1} & \cdots & n^{(2)}_1 & \\ m & & \vdots & & m \\ & n^{(t)}_{p_t-1} & \cdots & n^{(t)}_1 & \end{array}, \text{ where } 0 \leq n_i^{(j)} \leq m.$$

We also set $n_{p_j}^{(j)} := m =: n_0^{(j)}$ for all j. As N is generated in the source ω, we have that $N(\alpha_i^{(j)})$ is surjective for $2 \leq j \leq t$ and $1 \leq i \leq p_j$. Since the maps $N(\alpha)$ are restrictions of M_α, we have that $\ker N(\alpha) \subseteq \ker M(\alpha)$. It follows that for $2 \leq j \leq t$ and $1 \leq i \leq p_j - 1$

$$\begin{aligned} n_i^{(j)} &= \dim \operatorname{im} N\left(\alpha_{i+1}^{(j)}\right) = \dim N(a_{i+1}^{(j)}) - \dim \ker N(\alpha_{i+1}^{(j)}) \\ &\geq \dim N(a_{i+1}^{(j)}) - \dim \ker M(\alpha_{i+1}^{(j)}) = n_{i+1}^{(j)} - (r_{i+1}^{(j)} - r_i^{(j)}). \end{aligned}$$

Claim: $n_i^{(j)} \geq m - r + r_i^{(j)}$ *for* $2 \leq j \leq t$ *and* $1 \leq i \leq p_j - 1$. We proof this by an inverse induction on i. By the above, the base case for $i = p_j - 1$ holds. Assume the claim holds for some $2 \leq i \leq p_j - 1$. Then by the above

$$n_{i-1}^{(j)} \geq n_i^{(j)} - r_i^{(j)} + r_{i-1}^{(j)} \overset{\text{hyp.}}{\geq} (m - r + r_i^{(j)}) - r_i^{(j)} + r_{i-1}^{(j)} = m - r + r_{i-1}^{(j)}.$$

We now conclude that

$$\dim N = m + (p_1 - 1)m + \sum_{j=2}^{t} \sum_{i=1}^{p_j - 1} n_i^j + m \geq |A|m - r \sum_{j=2}^{t} (p_j - 1).$$

On the other hand,

$$\dim M = |A|m + r + \sum_{j=1}^{t} \sum_{i=1}^{p_j - 1} r_i^{(j)}.$$

This implies that the codimension of N in M is at most $2r(|A| - 2)$. Recalling that $r = l\frac{p}{p_s}$ and that $\underline{\dim}\, M = \sum_{j=0}^{l-1} \underline{\dim}\, E_{i+j}$, where $E_i = \operatorname{soc} M$ and the E_j are the regular simples of $\mathbb{T}$, we have

$$\begin{aligned} \dim N &\geq \dim M - 2l\frac{p}{p_s}(|A| - 2) \geq \dim M - \varepsilon \min_j \{\dim E_j\}\, l \\ &\geq \dim M - \varepsilon \sum_{j=0}^{l-1} \dim E_{i+j} = (1 - \varepsilon) \dim M. \end{aligned}$$

as in the proof of Proposition 4.33. The hyperfiniteness of $\mathcal{X}_\infty$ can thus be lifted to $\mathcal{X}_\mu$. ☐

4.4.3 Regular modules of integral slope, $\mathcal{P}_0$ and $\mathcal{Q}_\infty$

Before we turn to the final result of this chapter, we are left to consider some semiregular families and the postinjective component.

Lemma 4.37. *Let* A *be a canonical tubular algebra. Then the family of postinjective modules* $\mathcal{Q}_\infty$ *is hyperfinite.*

Proof. By the dual of [Rin84, Theorem 4.7.(1)], it is known that the postinjective modules of A are just the postinjective modules of kQ_∞, where Q_∞ is the full subquiver of Q given on $Q_0 \setminus \{0\}$: We have the obvious embedding $\operatorname{mod} kQ_\infty \to \operatorname{mod} kQ$, $M \mapsto \bar{M}$, where $\bar{M}(0) = 0$ and $\bar{M}(\alpha) = 0$ for $\alpha = \alpha_1^{(1)}, \ldots, \alpha_1^{(t)}$. By Theorem 2.11, the modules over kQ_∞ form a hyperfinite family. But a submodule $N \subseteq M$ of $M \in \operatorname{mod} kQ_\infty$ such that $\dim N \geq (1 - \varepsilon) \dim M$ and $N = \bigoplus_{i=1}^{s} N_i$ such that $\dim N_i \leq L_\varepsilon$, will yield a submodule $\bar{N} \subseteq \bar{M}$, exhibiting the same hyperfiniteness properties. ☐

Lemma 4.38. *Let* A *be a canonical tubular algebra. Then the left stable family* $\mathcal{T}_\infty$ *of semiregular modules, containing the unique indecomposable injective module which is not postinjective, is hyperfinite.*

Proof. By construction, A is a tubular extension and coextension of the tame hereditary algebras A_0 respectively A_∞ (see Table 4.2 for the list of underlying affine types). For each tubular type, the branch length is one, leading to just a single coray insertion. It follows from the dual of the proof of [Rin84, Theorem 4.7.(1)] that all but one tube in $\mathcal{T}_\infty$—the coray tube $\mathbb{T}$—consist of the corresponding indecomposable modules of A_∞ complemented by a zero vector space at the coextension vertex 0. It further follows that the coray tube $\mathbb{T}$ has one more coray than the corresponding tube of A_∞, denoting these additional modules by $\overline{[i]E}$, where E is the coray module from $\mathcal{T} \subset \operatorname{mod} A_\infty$, and the maps by $\overline{{}_{i}\nu}$. Indeed, $\overline{[1]E} = I(0)$, the indecomposable injective. What is more, by the proof of the dual of [Rin84, Proposition 4.5.(1)], the indecomposable modules M in this coray tube that do not coincide with $M_{|A_\infty}$ lie in this very coray. They also have $(\underline{\dim}\, M)_0 = 1$. If $M = \overline{[i]E}$ is one of them, we can thus find a surjective map to $I(0)$, namely $\overline{{}_{i}\nu} \circ \overline{{}_{1}\nu}$, exhibiting as kernel a submodule of M which is just an A_∞-module. But for these, we have shown how to find a submodule exhibiting hyperfiniteness, as A_∞ is tame hereditary. Hence, the claim follows by an application of Proposition 1.4 with the codimension being $\dim I(0)$. □

Theorem 4.39. *Let A be a tubular canonical algebra. Then the families $\mathcal{P}_0$ of preprojective modules, $\mathcal{Q}_\infty$ of postinjective modules and the family $\bigvee_{\mu\in(-\infty,0]\cup[\underline{p},\infty]} \mathcal{X}_\mu$ of all indecomposable regular modules of integral slope are hyperfinite.*

Proof. The family of preprojective modules $\mathcal{P}_0$ is hyperfinite by Proposition 4.12, while the family of postinjective modules $\mathcal{Q}_\infty$ is hyperfinite due to Lemma 4.37.

For the family of indecomposable (semi)regular modules of integral slope, the result follows from the results of four subfamilies. The right stable family $\mathcal{T}_0 = \mathcal{X}_{\underline{p}}$ is hyperfinite due to Corollary 4.14 and Lemma 4.15; the family of all indecomposable regular modules of positive integral slopes $\mathcal{X}_\mu$ is hyperfinite by combining the results for the homogeneous tubes from Proposition 4.27 with those for exceptional tubes from Corollary 4.35; the family of all indecomposable regular modules of negative integral slopes $\mathcal{X}_\mu$ is hyperfinite by Proposition 4.36, while the left stable family $\mathcal{T}_\infty = \mathcal{X}_0$ is hyperfinite due to Lemma 4.38. □

5 Hyperfiniteness from fragmentability and exceptional modules over path algebras of generalised Kronecker quivers

We will continue by considering the path algebra $k\Theta(m)$ of wild generalised Kronecker quivers and show that the indecomposable preprojective and postinjective modules for these algebras form hyperfinite families. We start with a result connecting to the notion of fragmentability from graph theory and make use of the tree structure of coefficient quivers of certain modules.

5.1 Graph-theoretic background

Recall that a graph G is given by its set of vertices V and a set of edges E containing ordered pairs $(u, v) \in V^2$, describing an edge starting at u and ending at v.

Now, Edwards and McDiarmid [EM94] have introduced a notion for classes of graphs which is similar to hyperfiniteness for families of modules.

Definition 5.1 ([EF01; EM94])**.** Let ε be a non-negative real number, and C an integer. We say that a graph $G = (V, E)$ is (C, ε)**-fragmentable** provided there is a set $X \subseteq V$, called the fragmenting set, such that

(1) $|X| \leq \varepsilon |V|$, and

(2) every component of $G \setminus X$ has at most C vertices.

Now consider a class Γ of graphs. We will say that Γ is ε**-fragmentable** provided there is an integer C such that for all $G \in \Gamma$, G is (C, ε)-fragmentable. Moreover, a class Γ of graphs is called **fragmentable** if

$$c_f(\Gamma) := \inf\{\varepsilon \colon \Gamma \text{ is } \varepsilon\text{-fragmentable}\} = 0.$$

Remark. We may relax the definition to say that a class Γ of graphs is fragmentable iff for any $\varepsilon > 0$, there are positive integers $n_0, c(\varepsilon)$ such that if $G \in \Gamma$ is a graph with $n \geq n_0$ non-isolated vertices, then there is a set X of vertices, with $|X| \leq \varepsilon n$, such that each component of $G \setminus X$ has $\leq c(\varepsilon)$ vertices.

5.2 Coefficient quivers as fragmentable graphs

In the following, we will consider the path algebra of a given quiver Q. Recall from [Rin98, Section 1] that given a certain basis $\mathcal{B}$ of a representation M of Q (that is, a collection of basis elements from bases for vector spaces at all vertices), we say that the *coefficient quiver* $\Gamma(M, \mathcal{B})$ *of* M *with respect to* $\mathcal{B}$ is the quiver with vertex set $\mathcal{B}$ and having an arrow $b \xrightarrow{\alpha} b'$ provided the entry corresponding to b and b' in the matrix corresponding to $M(\alpha)$ with respect to the chosen basis $\mathcal{B}$ is non-zero.

We can now obtain hyperfiniteness results for a family of modules $\mathcal{M}$ provided a corresponding class of coefficient quivers is fragmentable.

Proposition 5.2. *Let* $d, \ell \in \mathbb{N}$*. Let* A *be the path algebra of a quiver* Q*. Let* $\mathcal{M}$ *be a class of indecomposable tree modules for* A*, that is, of modules* M *such that there exists bases* $\mathcal{B}$ *of* $(M_i)_{i \in Q_0}$ *such that the corresponding coefficient quiver* Γ *is a tree, and additionally assume that the maximal indegree of* Γ *is* d *and the maximal path length of* Q *is* ℓ*. Then* $\mathcal{M}$ *is hyperfinite.*

Proof. Let $M \in \mathcal{M}$. By [EM94, Lemma 3.6], it is enough to show that the removal of at most d^ℓ basis elements decomposes the coefficient quiver into components of size at most half that of M which are a member of $\mathcal{M}$. Since M is a tree module, there is a vertex v (one of the central points of the underlying tree graph) in the coefficient quiver whose removal will result in splitting the quiver into (non-connected) subtrees of size at most half that of M. If this vertex v is a source in the coefficient quiver, it can be removed, and the induced subtrees are submodules of M, which are themselves tree modules in $\mathcal{M}$ (pick the bases given by restriction). If v is not source, at most d arrows map to it. Each of their starting vertices will be removed as well, to each of which again at most d arrows map. Since the path length is bounded by ℓ, we have to remove at most $\sum_{i=1}^{\ell} d^i$ vertices to produce submodules of M of dimension at most half that of M. These are again tree modules in $\mathcal{M}$. $\square$

Proposition 5.3. *Let* $d, \ell \in \mathbb{N}$*. Let* A *be the path algebra of a quiver* Q*. Let* $\mathcal{M}$ *be a class of indecomposable modules, such that the class of underlying graphs of their coefficient quivers* Γ *is fragmentable, their indegree is bounded by* d *and the path length is bounded by* ℓ*. Then* $\mathcal{M}$ *is hyperfinite.*

Proof. Let $\varepsilon > 0$. Let $\tilde{\varepsilon} = \varepsilon \left(\sum_{i=0}^{\ell} d^i \right)^{-1}$ and pick $L_\varepsilon := c(\tilde{\varepsilon})$, the constant from the definition of fragmentability. Let $M \in \mathcal{M}$. By the definition of fragmentability, there exists a set S of vertices of the coefficient quiver of M of cardinality at most $\tilde{\varepsilon} \dim M$, such that the underlying graph G splits into components of size at most L_ε if we remove the $|S|$ vertices. Now, if all the vertices in S are sources, this subgraph describes a submodule. As this might not hold, for each $s \in S$, we also remove all the vertices that map to s, of which there are at most $\sum_{i=1}^{\ell} d^i$. In this way, we arrive at a submodule N of dimension at least $\dim M - \tilde{\varepsilon} \sum_{i=0}^{\ell} d^i \dim M$. This finishes the proof. $\square$

5.3 The preprojective and postinjective components of generalised Kronecker quivers

Let us now turn to the Kronecker algebra $k\Theta(m)$ of the generalised Kronecker quiver $\Theta(m)$ with m equi-oriented arrows between two vertices.

Proposition 5.4. *The family of preprojective $k\Theta(m)$-modules is hyperfinite.*

Proof. By Proposition 1.2 it is enough to consider the indecomposable preprojective modules. Now, [Rin98, Proposition 3] gives a detailed description of these modules, showing that they are tree modules and for each arrow, each basis element at the sink vertex is mapped to from at most one basis element at the source vertex. This shows that the indegree is bounded by m. Note that submodules of preprojective modules are preprojective. Now apply Proposition 5.2 with $d = m$ and $\ell = 1$ to finish the proof. ☐

In the following, we will use the sequence a_t from [Rin98, Section 8] to describe the dimension vectors of the indecomposable preprojective and postinjective $k\Theta(m)$-modules. The sequence is defined recursively by $a_0 = 0, a_1 = 1$ and $a_{t+1} = ma_t - a_{t-1}$ for $t \geq 1$.

Lemma 5.5. *Fix $m \geq 3$. Then the closed-form solution of the recurrence relation for a_t is given by*

$$a_t = \frac{\varphi^t - \psi^t}{\sqrt{m^2-4}} \quad \textit{where } \varphi = \frac{m + \sqrt{m^2-4}}{2} \quad \textit{and } \psi = \frac{m - \sqrt{m^2-4}}{2}.$$

Moreover, the quotient a_t/a_{t+1} of consecutive terms converges to φ^{-1}.

Proof. We check that the closed form satisfies the recurrence relation. It follows at once that $a_0 = 0$. Also note that

$$2(\varphi - \psi) = \Big((m + \sqrt{m^2-4}) - (m - \sqrt{m^2-4})\Big) = 2\sqrt{m^2-4},$$

so $a_1 = 1$. We also have that both $m\varphi - 1 = \varphi^2$ and $m\psi - 1 = \psi^2$. Finally,

$$ma_t - a_{t-1} = \frac{(m\varphi - 1)\varphi^{t-1} - (m\psi - 1)\psi^{t-1}}{\sqrt{m^2-4}} = \frac{\varphi^{t+1} - \psi^{t+1}}{\sqrt{m^2-4}} = a_{t+1}.$$

Next, we note that $\varphi\psi = 1$. We then see that

$$\begin{aligned} \frac{a_t}{a_{t+1}} &= \frac{\varphi^t - \psi^t}{\sqrt{m^2-4}} \frac{\sqrt{m^2-4}}{\varphi^{t+1} - \psi^{t+1}} = \frac{\varphi^t - \psi^t}{\varphi^{t+1} - \psi^{t+1}} \\ &= \frac{\varphi^{2t+1} - \varphi^{t+1}\psi^t}{\varphi^{2t+2} - \psi^{t+1}\varphi^{t+1}} = \varphi \frac{\varphi^{2t} - 1}{\varphi^{2t+2} - 1} \\ &= \varphi \frac{1 - \varphi^{-2t}}{\varphi^2 - \psi\varphi^{-2t}} \xrightarrow{t\to\infty} \frac{1}{\varphi}, \end{aligned}$$

as $\frac{1}{\varphi^2} < 1$. ☐

Lemma 5.6. *Fix $m \geq 2$. Let $Q[t]$ be an indecomposable postinjective module as described in [Rin98, Section 8], with coefficient quiver Γ given there. Then the outdegree of the vertices of Γ is bounded by two, and the indegree is bounded by $(t-1)(m-2)+m$.*

Proof. By the description of the arrow maps for the postinjective indecomposable module $Q[t]$ in the dual of [Rin98, Proposition 3], the matrices of the arrows $1, \ldots, m-1$ have no common non-zero columns, so the outdegree of each source with respect to the arrows α_i, $1 \leq i \leq m-1$ is at most one. On the other hand, each row of one of these arrow matrices contains exactly a single one. Moreover, as the matrix for the last arrow α_m is constructed by concatenating zero matrices or column block matrices containing a single identity matrix block, at most one arrow α_m starts at each source. Indeed, the concatenation involves $t-1$ matrices—the $C(a_{j-1}, a_j)$—containing $m-2$ identity matrices, the $E(a_{j-1})$, of varying size a_{j-1} each, and one additional identity matrix $E(a_t)$. Combining this information yields the desired result. ☐

Lemma 5.7. *Fix $m \geq 3$. Let M be a module of dimension vector (a_{t+1}, a_t). Then we can express*

$$t = \log_\varphi \left(\dim M + \sqrt{\dim M^2 + \frac{4}{m-2}} \right) - \log_\varphi \left(2 \frac{1+\varphi}{\sqrt{m^2-4}} \right).$$

Moreover, for $\dim M \geq 3$, it holds that $t \leq c_m \sqrt{\dim M}$ for some constant c_m.

Proof. Clearly, $\dim M = a_{t+1} + a_t$. Now using the closed form of Lemma 5.5, we have that

$$\begin{aligned}\dim M &= \frac{\varphi^{t+1} - \psi^{t+1}}{\sqrt{m^2-4}} + \frac{\varphi^t - \psi^t}{\sqrt{m^2-4}} = \frac{\varphi^{2t+1} - (\varphi\psi)^t\psi}{\varphi^t\sqrt{m^2-4}} + \frac{\varphi^{2t} - (\varphi\psi)^t}{\varphi^t\sqrt{m^2-4}} \\ &= \frac{\varphi^{2t+1} - \psi + \varphi^{2t} - 1}{\varphi^t\sqrt{m^2-4}} = \varphi^t \frac{1+\varphi}{\sqrt{m^2-4}} - \varphi^{-t} \frac{1+\psi}{\sqrt{m^2-4}}.\end{aligned}$$

By substitution, noting that real powers of positive numbers are positive and using $(1+\varphi)(1+\psi) = m+2$, we get

$$t = \log_\varphi \left(\dim M + \sqrt{\dim M^2 + \frac{4}{m-2}} \right) - \log_\varphi \left(2 \frac{1+\varphi}{\sqrt{m^2-4}} \right).$$

Now it remains to show the estimate. We first note that for $\varphi > 1$ and

$$\frac{2+2\varphi}{\sqrt{m^2-4}} > \frac{m+\sqrt{m^2-4}}{\sqrt{m^2-4}} > 1,$$

the subtrahend is always positive, resulting in its omittance leaving an upper bound. Now, when $\dim M \geq 3$, we have

$$t \leq \log_\varphi(1+\sqrt{2}) + \log_\varphi(\dim M) < \log_\varphi(3) + \log_\varphi(\dim M) \leq 2\log_\varphi(\dim M).$$

Now, as $\varphi > 1$, it is enough to further consider $\ln(\dim M)$. Clearly,

$$\exp\left(2\sqrt{\dim M}\right) > \frac{\left(2\sqrt{\dim M}\right)^2}{2!} = 2\dim M > \dim M,$$

so $2\sqrt{\dim M} > \ln \dim M$. All in all, this combines to the desired inequality

$$t \leq 2\log_\varphi \dim M = 2\frac{1}{\ln \varphi}\ln \dim M < \frac{4}{\ln \varphi}\sqrt{\dim M}.$$

□

Proposition 5.8. *The family of indecomposable postinjective $k\Theta(m)$-modules is hyperfinite.*

Proof. We want to give a proof similar to that of [EM94, Lemma 3.6], but adapt it to coefficient quivers of modules instead of graphs. In a first step towards proving hyperfiniteness, we hence want to find $A > 0$, $0 \leq \lambda < 1$ and $0 < \alpha < 1$ such that for any indecomposable postinjective module of dimension n, there are at most An^λ basis elements that can be removed from the coefficient quiver to leave a submodule for which every indecomposable summand has dimension at most αn, and to each summand, a similar construction can be applied, and so forth.

Let $\varepsilon > 0$. We put $\alpha = \frac{1}{2} + \delta$ for some $0 < \delta < \frac{1}{4}$. Let $Q[t]$ be the indecomposable postinjective module of dimension vector (a_{t+1}, a_t). Let $n = \dim Q[t]$ and assume $n > 5$. We show how to split this module into small components by a sequence of stages. Before each stage i, all components are isomorphic to indecomposable postinjective modules, having at most $\alpha^{i-1}n$ vertices in their coefficient quivers, while the number of components with more than $\alpha^i n$ vertices is at most α^{-1}. Since the coefficient quiver Γ of $Q[s]$ is a tree, there is a vertex whose removal creates subtrees of size at most $\alpha \dim Q[s]$. Note that we can assume that this vertex to remove is a sink: if the vertex to remove was a source—since all sources have outdegree at most two by Lemma 5.6 and all their neighbours are sinks—we can just remove a neighbouring sink. Note that the size of α allows for this modification, as we do not require that the subtrees are at most half the size of $Q[s]$. But a removal of a sink corresponds to passing to the cokernel of an inclusion of $S_1 \hookrightarrow Q[s]$. Yet, this cokernel must have smaller dimension than $Q[s]$, and since $Q[s]$ is postinjective, must also be postinjective. This implies that the cokernel is the direct sum of indecomposable postinjective modules for smaller s, as the dimension of the indecomposable postinjectives strictly increases for growing s. This proves that after stage i, all components are indecomposable postinjective modules with no more than $\alpha^i n$ vertices in their coefficient quivers. The number of stages is the least k such that $\alpha^k n \leq L$, for an L to be determined later. Hence $\alpha^{k-1}n > L$, so that $\alpha^{1-k} < \frac{n}{L}$.

Unfortunately, this process does not create a submodule of $Q[t]$, but a factor module given by the direct sum of many smaller indecomposable postinjective modules. To attain a submodule, we must delete further vertices. Parallel to the above sequence of stages, we conduct a downstream stage to construct a submodule. In each of these stages, we only deal with the components M with

$$\alpha^i n < \dim M \leq \alpha^{i-1} n.$$

Since ${}^{a_t}\!/_2 > a_{t-1}$, not in every stage a reduction takes place. But when a reduction takes place, we create submodules from $Q[s]$ by additionally removing all the vertices adjacent to the deleted sink. By the structure of the canonical coefficient quiver of $Q[s]$, there are at most $s(m-2)+2$ such vertices, and $s \leq c\sqrt{\dim Q[s]}$ by Lemma 5.7. Note that while the dimensions of the submodules left before downstream stage i are smaller than $\dim Q[s]$, as we have removed at least one more source, the operand in this stage i is still $\alpha^{i-1}n \geq \dim Q[s]$, as we base our considerations on the original indecomposable postinjective module. This implies that in downstream stage i, we remove at most $A\sqrt{\alpha^{i-1}n}$ vertices, letting $A = 2 + c(m-2)$. Thus, choose $\lambda = \frac{1}{2}$. Note that in order to apply Lemma 5.7 throughout, we require $\dim Q[s] \geq 3$ in all downstream stages, leading to $\alpha^{k-1}n > 2$

Now, the total number r_i of vertices removed in downstream stage i is at most

$$\alpha^{-i} A(\alpha^{i-1}n)^\lambda = An^\lambda \alpha^{\lambda i - i - \lambda} = \frac{An^\lambda}{\alpha}\alpha^{(1-i)(1-\lambda)}$$

and since $\alpha^{1-k} < n/L$, we have $\alpha^{1-i} < (n/L)\alpha^{k-i}$. Hence,

$$r_i < \frac{An^\lambda}{\alpha}(n/L)^{1-\lambda}\alpha^{(k-i)(1-\lambda)} = \frac{An}{\alpha L^{1-\lambda}}\beta^{k-i},$$

where $\beta = \alpha^{1-\lambda}$, with $0 < \beta < 1$. Then the total number R of vertices removed from the coefficient quiver of $Q[t]$ is

$$\begin{aligned} R = \sum_{i=1}^{k} r_i &< \frac{An}{\alpha L^{1-\lambda}} \sum_{i=1}^{k} \beta^{k-i} \\ &< \frac{An}{\alpha L^{1-\lambda}} \sum_{i\geq 0} \beta^{i} \\ &= \frac{An}{\alpha L^{1-\lambda}} \frac{1}{1-\beta}. \end{aligned}$$

Since we have $1 - \lambda > 0$, it follows that we can choose $L = L_\varepsilon$ independently of n such that $R \leq \varepsilon n$. This then shows the hyperfiniteness of $\{Q[t] \colon t > 0\}$ and thus of the postinjective component. □

Example 5.9. We will illustrate this procedure for a module over $k\Theta(3)$. Here, we can put $\alpha = \frac{5}{8}$, $A = 2 + c$ with $c = \frac{4}{\ln 3 + \sqrt{5} - \ln 2} > 4$ and $\lambda = \frac{1}{2}$. We will consider the indecomposable postinjective module $Q[4]$ with dimension vector $(55, 21)$. Note that this module would not be subject to the algorithm described in the proof, as $L_\varepsilon > 2211$ for any suitable $0 < \varepsilon < 1$.

We draw a coefficient quiver Γ of $Q[4]$ which is a tree, but do not distinguish between the arrows in Γ corresponding to the same arrow of $\Theta(3)$ (see Figure 5.9). The basis elements at the sink vertex are denoted $(1), \dots, (21)$, while the basis elements at the source vertex are denoted $1, \dots, 55$. We show the modifications in the first three stages. In stage i, we remove the sink vertices coloured in red and denoted $(j)^i$, while removing the vertices coloured in orange connected to them in the corresponding downstream stage. That is, in the first step we remove basis element (1), while the second stage sees the removal of vertices (8) and (12) before removing vertices $(3), (9), (10), (16)$ and (20) in stage three. In these upstream stages, we remove at most one vertex per connected component. In the ensuing downstream stages, we remove at most 6 respectively 5 respectively 4 more vertices from the (preceding) connected component. We see that $7 < c\sqrt{76}$, $6 < c\sqrt{29}$ and $5 < c\sqrt{11}$.

Remark. Recall that the logarithm can be bounded above by any radical power: We have $\ln x \leq n \sqrt[n]{x}$. Now the proof of the previous proposition suggests that we have an adaptation of Proposition 5.2 in the case of coefficient quivers that are graphs of genus at most γ for fixed $\gamma \geq 0$ or for rectangular lattices of dimension d for a fixed integer d, provided the indegree has a logarithmic bound with respect to the dimension, as these classes of graphs were shown to be fragmentable using suitable A, λ and α (see [EM94, Corollary 3.7]).

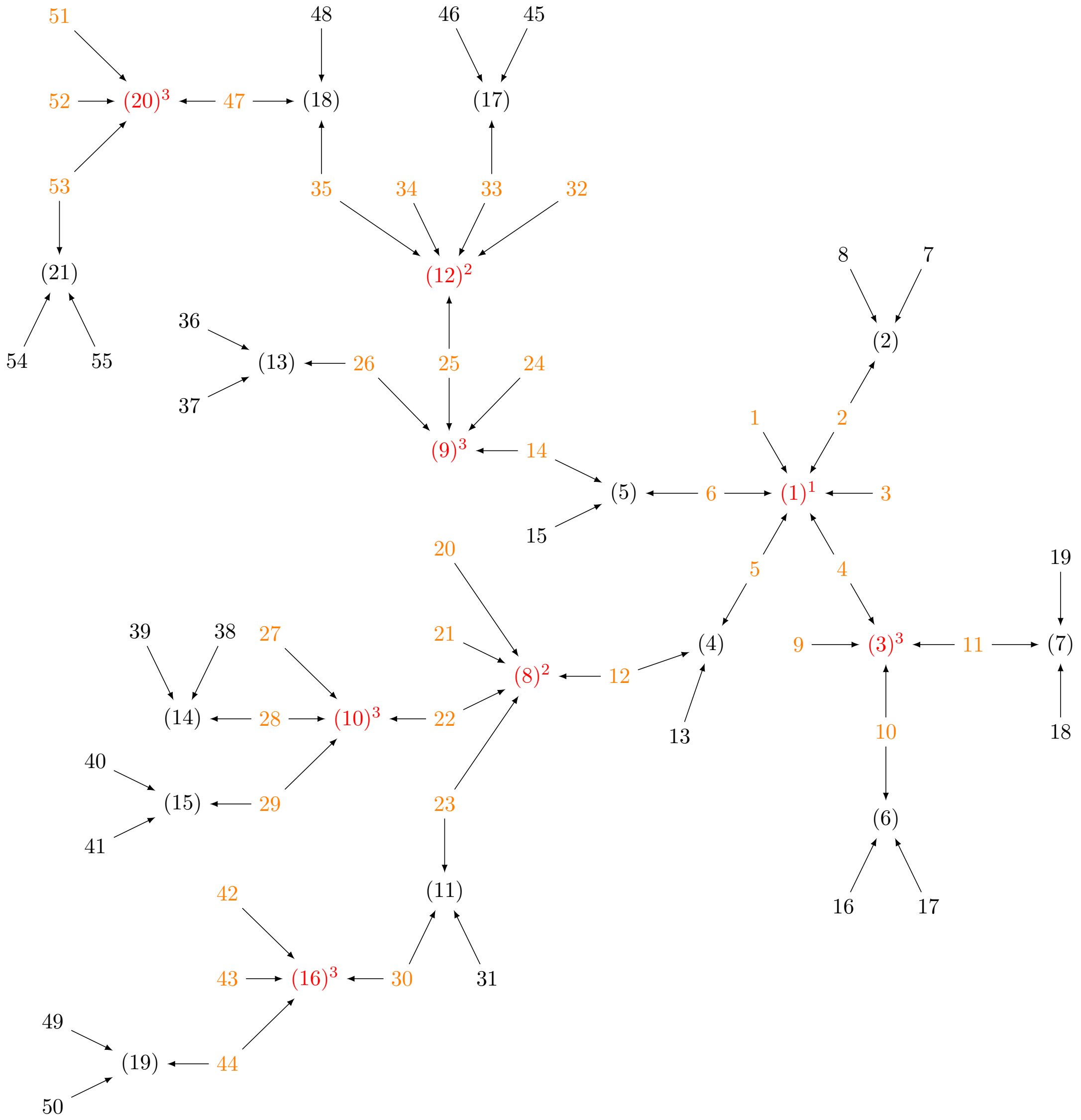


Figure 5.1: The coefficient quiver of $Q[4] \in \operatorname{mod} k\Theta(3)$, with highlighted vertices indicating how to produce a submodule exhibiting hyperfiniteness.

6 Wild phenomena

6.1 A family of non-hyperfinite modules

While Elek [Ele17, Section 8] has given an argument to show that any wild Kronecker algebra is not of amenable representation type by showing that there are non-hyperfinite families of modules over the free algebras $k\langle x_1, \dots, x_r\rangle$ with $r \geq 2$ generators, we are interested in understanding and providing concrete counterexamples of non-hyperfinite families of modules for algebras of non-amenable representation type.

Motivated by a similar notion of graph expanders, Barak, Impagliazzo, Shpilka and Wigderson have introduced the notion of dimension expanders (see [LZ08; DS11; Bou09; DW10]).

Definition 6.1. Let k be a field, $d \in \mathbb{N}$ and $\alpha > 0$. For a vector space V and a set $\{T_1, \dots, T_d\}$ of endomorphisms of V, the pair $(V, \{T_i\}_{i=1}^d)$ is called an **α-dimension expander of degree** d provided for every subspace $W \subset V$ of dimension less than or equal to ${}^{\dim_k V}\!/_2$, we have that

$$\dim_k \left(W + \sum_{i=1}^{d} T_i(W) \right) \geq (1+\alpha) \dim_k W.$$

Indeed, we shall utilise a slightly generalised notion embracing the different versions present in the literature.

Definition 6.2. Let k be a field, $d \in \mathbb{N}$, $0 < \eta \leq 1$ and $\alpha > 0$. Given a vector space V and a set $\{T_1, \dots T_d\}$ of endomorphisms of V, the pair $(V, \{T_i\}_{i=1}^d)$ is called an **(η, α)-dimension quasi-expander of degree** d provided every subspace $W \subset V$ of dimension at most $\eta \dim_k V$, we have that

$$\dim_k \sum_{i=1}^{d} T_i(W) \geq (1+\alpha) \dim_k W.$$

Remark. Every α-dimension expander of degree d along with the identity map id_V is a $(\frac{1}{2}, \alpha)$-dimension quasi-expander of degree $d+1$.

Now, a family of dimension quasi-expanders of degree d of unbounded dimensions gives rise to a non-hyperfinite family for the d-Kronecker algebra $k\Theta(d)$:

Proposition 6.3. *Let k be a field, $d \in \mathbb{N}$ and $\eta, \alpha > 0$. If $\{(V_i, \{T_l^{(i)}\}_{l=1}^d)\}_{i \in I}$ is a family of (η, α)-dimension quasi-expanders of degree d such that $\dim V_i$ is unbounded, then the induced family of $k\Theta(d)$-modules*

$$\begin{tikzcd} V_i \arrow[r, bend left, "T_1^{(i)}"] \arrow[r, phantom, "\vdots"] \arrow[r, bend right, "T_d^{(i)}"'] & V_i \end{tikzcd}$$

is not hyperfinite.

Proof. Let $\alpha > 0$ and $\{(V_i, (T_1^{(i)}, \dots, T_d^{(i)}))\}$ be a family of α-dimension expanders degree d and of unbounded dimension $\dim V_i$. Consider the family

$$\Big\{M_i = \Big((V_i, V_i), (T_1^{(i)}, \dots, T_d^{(i)})\Big)\Big\}_{i \in I} \in \operatorname{mod} k\Theta(d).$$

If this family was hyperfinite, for each $\varepsilon > 0$, there exists an $L_\varepsilon > 0$ and we can find some $M \in \{M_i \colon i \in I\}$—given by an (η, α)-dimension quasi-expander space V—such that $\dim M = 2 \dim V > 2\frac{L_\varepsilon}{\eta}$ with a suitable submodule P exhibiting hyperfiniteness. We will denote the vector space of P_j at vertex $v \in Q_0$ by $P_j(v)$. We have that

$$\dim P_j(1) + \dim P_j(2) = \dim P_j \leq L_\varepsilon < \eta \dim V,$$

also noting that each $P_j(v)$ is a subspace of the vector space V of an (η, α)-dimension quasi-expander. As each P_j is a $k\Theta(d)$-module, thus $T_1(P_j(1)) + \cdots + T_d(P_j(2)) \subseteq P_j(2)$, this implies that

$$\dim P_j(2) \geq (1 + \alpha) \dim P_j(1). \tag{6.1}$$

Moreover,

$$\begin{aligned} 2(1-\varepsilon)\dim V \qquad & \leq \sum_{j=1}^{t} (\dim P_j(1) + \dim P_j(2)) \leq \sum_{j=1}^{t} \dim P_j(1) + \dim V \\ \Leftrightarrow (1-2\varepsilon)\dim V \qquad & \qquad\qquad\qquad\qquad\qquad\quad\;\; \leq \sum_{j=1}^{t} \dim P_j(1), \end{aligned}$$

which in light of inequality (6.1) yields that

$$\begin{aligned} (1-2\varepsilon)\dim V \qquad & \leq \sum_{j=1}^{t} \frac{\dim P_j(2)}{1+\alpha} \leq \frac{\dim V}{1+\alpha} \\ \Leftrightarrow \qquad \varepsilon \qquad & \qquad\qquad\qquad\;\; \geq \frac{\alpha}{2(1+\alpha)}, \end{aligned}$$

contradicting the hyperfiniteness of the family $\{M_i \colon i \in I\}$. $\square$

Remark. If all T_i are such that $T_i \circ T_j = 0$ for any combination, then in general $\Big(V, \{T_i\}_{i=1}^d\Big)$ is neither a dimension expander nor a dimension quasi-expander: In this situation, we have $\operatorname{im} T_j \subset \bigcap_{i=1}^d \ker T_i$ for all $1 \leq j \leq d$. Without loss of generality, we consider $0 \neq v \in \operatorname{im} T_1$ (if all T_is were zero, the claim is obviously true). Let $W = \langle v \rangle$. Then $\sum_{i=1}^{d} T_i(W) = 0$, so the dimension property cannot hold for some non-trivial subspace of dimension one. Thus—unless $\eta \dim V < 1$—the pair $\left(V, \{T_i\}_{i=1}^{d}\right)$ cannot be a dimension (quasi-)expander.

Proposition 6.3 reduces the problem of exhibiting a non-hyperfinite family for the d-Kronecker quiver to finding families of dimension expanders for fixed d and α such that the dimension of the vector spaces is unbounded. This latter question has already been asked by A. Wigderson in 2004. We will make use of results by Lubotzky and Zelmanov in a proposition and theorem in [LZ08] to answer it. They provide several ways of constructing α-dimension expanders of degree two over the complex numbers and generalise to every field of characteristic zero.

We first need another

Definition 6.4. Consider a group Γ generated by a finite set S. Given a Hilbert space H and a unitary representation $\rho\colon \Gamma \to U(H)$, where $U(H)$ denotes the unitary endomorphisms of H, the **Kazhdan constant** is defined as

$$K_\Gamma^S(H, \rho) := \inf_{0 \neq v \in H} \max_{s \in S} \left\{ \frac{||\rho(s)v - v||}{||v||} \right\}.$$

Further, the group Γ has **property** (T) if

$$K_\Gamma^S = \inf_{(H,\rho) \in \mathcal{R}_0(\Gamma)} \{K_\Gamma^S(H, \rho)\} > 0,$$

where $\mathcal{R}_0(\Gamma)$ is the family of all unitary representations of Γ which have no non-trivial Γ-fixed vector. In this case, K_Γ^S is called the **Kazhdan constant of** Γ **with respect to** S.

This Kazhdan constant is now relevant in the following Proposition determining the expansion rate α. In the following, by $U_n(\mathbb{C})$, we denote the group of $n \times n$ unitary matrices over $\mathbb{C}$.

Proposition 6.5. *[LZ08, Proposition 2.1] Let* $\rho\colon \Gamma \to U_n(\mathbb{C})$ *be an irreducible unitary representation of a group* Γ *with finite generating set* S*, then* $(\mathbb{C}^n, \rho(S))$ *is an* α*-dimension expander of degree* $|S|$ *where* $\alpha = \frac{\kappa^2}{12}$, $\kappa = K_\Gamma^S(S\ell_n(\mathbb{C}), \operatorname{adj} \rho)$*, where* $S\ell_n(\mathbb{C})$ *denotes the subspace of all linear transformations of zero trace, and* $\operatorname{adj} \rho$ *is the adjoint representation on* $\operatorname{End}(\mathbb{C}^n)$ *induced by conjugation.*

Remarks. (1) The endomorphism space $\operatorname{End}(\mathbb{C}^n) \cong M_n(\mathbb{C})$ and its subspace $S\ell_n(\mathbb{C})$ become Hilbert spaces via $\langle S, T \rangle = \operatorname{tr}(ST^*)$.

(2) The induced representation $\operatorname{adj} \rho$ on $M_n(\mathbb{C})$, given as

$$\gamma \mapsto \left(T \mapsto \rho(\gamma) T \rho(\gamma)^{-1}\right),$$

is unitary, as $\operatorname{adj}(\rho)(\gamma)$ is surjective and preserves the inner product for each $\gamma \in \Gamma$.

(3) The subspace $S\ell_n(\mathbb{C})$ of trace zero matrices is invariant under $\operatorname{adj}\rho$, since conjugation by invertible matrices preserves the trace. Thus, $(S\ell_n(\mathbb{C}), \operatorname{adj}\rho)$ is a unitary representation.

(4) Note that if ρ is irreducible, then by Schur's Lemma, $S\ell_n(\mathbb{C})$ does not have any non-trivial $\operatorname{adj}\rho(\Gamma)$-fixed vector: If $T \in S\ell_n(\mathbb{C})$ is fixed by $\operatorname{adj}(\rho)$, then $\ker T$ is an invariant subspace of ρ, as $T(v) = 0$ along with $\rho(\gamma)T = T\rho(\gamma)$ implies that $T(\rho(\gamma)(v)) = 0$. By the irreducibility of ρ, $\ker T$ must be a trivial subspace. If $\ker T = 0$, we have that T is invertible, even $T = \lambda \mathrm{id}$ for some eigenvalue λ of T. But $0 = \operatorname{tr} T = n\lambda$, a contradiction. Thus, $\ker T = \mathbb{C}^n$, so T is trivial.

Remark. A quick way to now see that $\mathbb{C}\Theta(3)$ is not of amenable representation type is the following: The groups $\Gamma = SL(n, \mathbb{Z})$ for $n \geq 3$ have generating sets S of size two (see, e.g., [Tro62; GT93]) such that they have property (T) [Kaž67; Sha99]. To this end, note that Γ having property (T) with respect to some compact subset implies the property with respect to any generating set with a modified constant (see [HV89, Proposition 1.15], [BHV08, Proposition 1.3.2]). It then follows that the vector spaces constructed in Proposition 6.5 are α-dimension expanders of degree two for $12\alpha \geq K_\Gamma^S$. Moreover, there are unitary irreducible representations of $SL(3, \mathbb{Z})$ of unbounded dimension, coming from the Steinberg representations ψ_p of $SL(3, \mathbb{F}_p)$ of dimension p^3 for each prime p (see, e.g., [SF73]).

In the following, we will make explicit an example using representations of $SL(2, \mathbb{Z})$. This allows us to describe the Kronecker representations more easily. To this end, we first consider representations of the special linear group $SL(2, p)$ of 2×2-matrices over the finite field of characteristic p, $\mathbb{F}_p$. We recall the following two classical results, fixing some notation.

Lemma 6.6. *For each prime $p \in \mathbb{P}$, there is an irreducible, complex p-dimensional representation of* $\mathrm{SL}(2, p)$.

Proof. Let p be a prime number. Then $\Gamma = \mathrm{SL}(2, p)$ acts on $\mathbb{P}_1(\mathbb{F}_p) = \{0, 1, \ldots, p-1, \infty\}$ by

$$\pi\colon \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \left(z \mapsto \frac{az + b}{cz + d}\right),$$

with the usual conventions that $\frac{x}{0} = \infty$ for $x \neq 0$ and $\frac{a\infty + b}{c\infty + d} = \frac{a}{c}$. This permutation action extends to a permutation representation $\rho\colon \Gamma \to \mathrm{GL}_{p+1}(\mathbb{C})$,

$$g \mapsto \left(\sum_{z \in \mathbb{P}_1(\mathbb{F}_p)} \lambda_z e_z \mapsto \sum_{z \in \mathbb{P}_1(\mathbb{F}_p)} \lambda_z e_{\pi(z)} \right),$$

identifying $\mathbb{C}^{p+1}$ with the free complex vector space on $\mathbb{P}_1(\mathbb{F}_p)$ via

$$e_1 \leftrightarrow f_0,\ \ldots,\ e_p \leftrightarrow f_{p-1} \text{ and } e_{p+1} \leftrightarrow f_\infty,$$

where $\{e_1, \dots, e_{p+1}\}$ is the standard basis of $\mathbb{C}^{p+1}$ and by $f_0, \dots, f_{p-1}, f_\infty$ we denote a standard basis of the free vector space. The character values of χ_ρ can be calculated via the number of fixed points of π on representatives of the conjugacy classes of $SL(2,p)$. Consider the subspace $W = \{v \in \mathbb{C}^{p+1} \colon \sum_{i=1}^{p+1} v_i = 0\}$ of dimension p. It is ρ-invariant and the restriction of ρ to W is the complement of the trivial representation in ρ. Using character theory, this is sufficient information to show that $\rho_{|W}$ is an irreducible complex representation of $\mathrm{SL}(2,p)$ (see also [FH91, Section 5.2]). □

Corollary 6.7. *The group $SL_2(\mathbb{Z})$ has irreducible, unitary representations of unbounded dimension.*

Proof. Consider the natural maps $\pi_p \colon SL_2(\mathbb{Z}) \to SL(2,p)$ mapping each matrix to the matrix of the residue classes of its entries modulo p. Let $\rho \colon SL(2,p) \to GL(V)$ be an irreducible p-dimensional representation. As $SL(2,p)$ is a discrete group, we can endow V with an inner product in such a way to assume that ρ is unitary. Now consider $\rho \circ \pi_p$. This is certainly a group homomorphism. Moreover, a subspace $W \subseteq V$ is $SL(2,p)$-invariant if and only if W is $SL_2(\mathbb{Z})$-invariant, showing that $\rho \circ \pi_p$ is irreducible since ρ is. □

Remark. We may refer to the subgroups $\Gamma(p) := \ker(SL_2(\mathbb{Z}) \to SL(2,p))$ as the *principal congruence subgroups* and have

$$\Gamma(p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \colon \begin{matrix} a \equiv d \equiv 1 \mod p \\ b \equiv c \equiv 0 \mod p \end{matrix} \right\}.$$

Since the projections are surjective, the subgroups have finite index $p^3 - p$ in $SL_2(\mathbb{Z})$.

Definition 6.8. [Lub94, Definition 4.3.1] Let Γ be a finitely generated group generated by a finite symmetric set of generators S. Given a family $\{N_i\}_{i \in I}$ of finite index normal subgroups, Γ is said to have **property (τ) with respect to the family $\{N_i\}_{i \in I}$** provided there exists a $\kappa > 0$ such that if (H, ρ) is a non-trivial unitary irreducible representation of Γ whose kernel contains N_i for some $i \in I$, then $K_\Gamma^S(H, \rho) > \kappa$.

Remark. This definition is equivalent to requiring that the trivial representation is isolated in the set of all unitary representations of Γ whose kernel contains some N_i or to requiring that the non-trivial irreducible representations of Γ factoring through Γ/N_i for some $i \in I$ are bounded away from the trivial representation. Further note that a finitely generated group having property (T) has property (τ) with respect to all finite index normal subgroups.

Theorem 6.9. *[LZ89, Section 1] The group $SL_2(\mathbb{Z})$ has property (τ) with respect to $\{\Gamma(p)\}_{p \in \mathbb{P}}$.*

Proof. By Selberg's $\frac{3}{16}$ Theorem, given a congruence subgroup $\Gamma(p)$ of $SL_2(\mathbb{Z})$, the smallest positive eigenvalue $\lambda_1(\Gamma(p)\backslash\mathbb{H})$ of the Laplacian on the principal modular curve $\Gamma(p)\backslash\mathbb{H}$ is at least $\frac{3}{16}$. Here, $\mathbb{H}$ denotes the hyperbolic plane endowed with the structure of a Riemannian manifold as in the Poincaré half-plane model. Yet, by [Lub94, Theorem 4.3.2], having λ_1 bound away from zero is equivalent to $SL_2(\mathbb{Z})$ having property (τ) with respect to $\{\Gamma(p)\}_{p\in\mathbb{P}}$. $\square$

Theorem 6.10. *Let k be a field of characteristic zero. Then the wild Kronecker algebra $k\Theta(3)$ is not of amenable representation type.*

Proof. By Proposition 6.3, it is sufficient to find a sequence of α-dimension expanders of degree two and of unbounded dimension for some $\alpha > 0$. Now, by an application of Proposition 6.5, it suffices to exhibit a sequence of irreducible, unitary representations $\rho\colon \Gamma \to U_n(\mathbb{C})$ of unbounded dimension for some group Γ with generating set S of cardinality two, such that the Kazhdan constants $K^S_\Gamma(\mathcal{S}\ell_n(\mathbb{C}), \operatorname{adj}\rho)$ are uniformly bounded from below by a constant $\kappa > 0$.

We let $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ with generating set $S = \{\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right), \left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)\}$. For now, we specialise to $k = \mathbb{C}$. By Corollary 6.7, there is a sequence $\rho_p\colon \Gamma \to U_p(\mathbb{C})$ of non-trivial irreducible, unitary representations of unbounded dimension. Moreover, by Theorem 6.9, $\mathrm{SL}_2(\mathbb{Z})$ has property (τ) with respect to $\{\Gamma(p)\}$, that is, there is a constant $\kappa > 0$ such that if (H, σ) is a non-trivial unitary irreducible representation of $\mathrm{SL}_2(\mathbb{Z})$ whose kernel contains $\Gamma(p)$ for some $p \in \mathbb{P}$, then the Kazhdan constant $K^S_\Gamma(H, \sigma) > \kappa$. Yet, by the remarks following Proposition 6.5, the $(\mathcal{S}\ell_p(\mathbb{C}), \operatorname{adj}\rho_p)$ are unitary representations factoring through $\mathrm{SL}(2, p)$, that is, their kernels contain $\Gamma(p)$, and they do not contain non-trivial fixed vectors, so are irreducible. Thus, for their Kazhdan constants we have that $K^S_\Gamma(\mathcal{S}\ell_p(\mathbb{C}), \operatorname{adj}\rho_p) > \kappa$.

The case for general k follows as in [LZ08, comments after Example 3.4]: Since $\operatorname{char} k = 0$, k contains $\mathbb{Q}$, and the representations of Corollary 6.7 are all defined over $\mathbb{Q}$, say $\rho_p\colon \Gamma \to \mathrm{GL}_p(\mathbb{Q})$. If $|k| \leq \aleph$, then k can be embedded into $\mathbb{C}$ and so can $\mathrm{GL}_p(k) \subset \mathrm{GL}_p(\mathbb{C})$. As the ρ_p factor through a finite group, they can be unitarised over $\mathbb{C}$. We have $\mathbb{C}^p = \mathbb{C} \otimes_k k^p$, thus every k-subspace $W \subseteq k^p$ spans a $\mathbb{C}$-subspace $\overline{W} \subset \mathbb{C}^p$ of the same dimension. If $\rho(s) \in \mathrm{GL}_p(k)$, then

$$\dim_k(W + \rho(s)W) = \dim_{\mathbb{C}}(\overline{W} + \rho(s)\overline{W}).$$

Since $(\mathbb{C}^p, \rho_p(S))$ is a dimension expander by the above, so is $(k^p, \rho_p(S))$. Now, if k has large cardinality and $W \subset k^n$ does not have the dimension expansion property, then the entries of a basis of W generate a finitely generated field k_1 of characteristic zero, and we get a counterexample $W \subset k_1^n$. But k_1^n is a dimension expander by the previous argument. $\square$

Remark. This proof does not use the fact that the group $\mathrm{SL}_2(\mathbb{Z})$ has property (τ) with respect to all principal congruence subgroups, let alone all congruence subgroups. Our result follows from property (τ) with respect to infinitely many $\Gamma(p)$ such that p is unbounded. Thus, weaker versions of Selberg's Theorem should suffice in proving this. For these, see e.g., [Tao15, Section 3.3]. Also compare [DSV03, Theorem 4.4.4], where by the use of only elementary methods it is shown that the corresponding construction for graphs gives expander graphs.

Example 6.11. Put $s = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ and $t = \left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$. Then the desired (counter)example for $k\Theta(3)$ is given by the family $\{((k^p, k^p), (\mathrm{id}, \rho_p(s), \rho_p(t)))\}_{p \in \mathbb{P}}$.

Considering the basis $\{e_2 - e_1, \dots, e_p - e_{p-1}, e_{p+1} - e_p\}$ of $W \cong k^p$, we do have

$$\rho_p(s) = \begin{pmatrix} 0 & \dots & 0 & -1 & 1 \\ 1 & & & -1 & 1 \\ & \ddots & & \vdots & \vdots \\ & & 1 & -1 & 1 \\ 0 & \dots & 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}_p(\mathbb{Q}), \quad \rho_3(t) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

$$\rho_5(t) = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 1 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \rho_7(t) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\rho_{11}(t) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & -1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 & -1 & 1 & -1 & 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

While the latter matrices $\rho_p(t)$ share a pattern, the strict rule to construct them ad-hoc is unclear. Note that we do have a certain symmetry $a_{i,j} = a_{p+1-i,p+1-j}$.

For fields of prime characteristic, this construction of dimension expanders cannot be used. Thus, for general k, we rely on of work of Dvir and Shpilka [DS11] and Bourgain [Bou09] to generalise our result.

Theorem 6.12. *Let k be any field. Then there exists some $d \geq 3$ such that $k\Theta(d)$ is not of amenable representation type.*

Proof. By a reduction of [DS11], the result of [Bou09] can be used to construct degree-d dimension expanders over any field k, thus showing that the wild $(d+1)$-Kronecker algebras $k\Theta(d+1)$ are not of amenable representation type (also see [DW10] for an overview of the construction of these dimension expanders). □

6.2 Propagating non-amenability

In this section we will discuss implications of the non-hyperfiniteness of wild generalised Kronecker algebras. In particular, we will show results for other well-behaved wild algebras.

6.2.1 Passing on non-amenability from subquivers

Lemma 6.13. *Let Q be a quiver. If Q has a subquiver Q' such that* $\operatorname{mod} kQ'$ *is not of amenable representation type, then neither is* $\operatorname{mod} kQ$ *of amenable representation type.*

Proof. Let $F\colon \operatorname{mod} kQ' \to \operatorname{mod} kQ$ be the embedding mapping any representation

$$M' = \big((M'(i))_{i\in Q_0}, (M'(\alpha))_{\alpha\in Q_1}\big)$$

of Q' to the representation $M = \big((M(i))_{i\in Q_0}, (M(\alpha))_{\alpha\in Q_1}\big)$ of Q given by

$$M(i) = \begin{cases} M'(i), & i \in Q'_0, \\ 0, & \text{else,} \end{cases} \qquad M(\alpha) = \begin{cases} M'(\alpha), & \alpha \in Q'_1, \\ 0, & \text{else.} \end{cases}$$

Since $\operatorname{mod} kQ'$ is not of amenable representation type, there exists a non-hyperfinite family of modules $\{M'_j \colon j \in J\}$. Put $M_j = FM'_j$ and assume that $\{M_j \colon j \in J\}$ is hyperfinite, for otherwise we have found a non-hyperfinite family exhibiting the non-amenability of $\operatorname{mod} kQ$. Note that any submodule N of some M_j is given by subspaces $N(i) \subseteq M_j(i)$ for each $i \in Q_0$ and linear maps $N(\alpha)$ for each $\alpha \in Q_1$ such that $\operatorname{im} N(\alpha) \subseteq N(t(\alpha))$. Hence,

$$N' := \left((N(i))_{i\in Q'_0}, (N(\alpha))_{\alpha\in Q'_1}\right),$$

is a subrepresentation of M'_j. Moreover, $\dim_k M_j = \dim_k M'_j$ and $\dim_k N = \dim_k N'$. Let S be a direct summand of N, then each $S(i)$ is a direct summand of $N(i)$. This along with $S(i) = 0$ for all $i \in Q_0 \setminus Q'_0$ implies that S also yields a direct summand S' of N' of dimension $\dim_k S' = \dim_k S$. Altogether, this implies that $\{M'_j \colon j \in J\}$ is hyperfinite, a contradiction. ☐

Corollary 6.14. *The wild Kronecker algebras $\mathbb{C}\Theta(d)$ for $d \geq 3$ are not of amenable representation type.*

We will now impose conditions on a pair of functors that allow us to preserve non-hyperfinite families of modules.

Proposition 6.15. *Let k be a field and $L|k$ a finite field extension. Let A be a finite dimensional L-algebra and let B be a finite dimensional k-algebra. Let $\{M_i \colon i \in I\}$ be a non-hyperfinite family of finite dimensional A-modules. Let $K_1, K_2 > 0$. If for each $i \in I$ there exist additive functors $F_i\colon$ $\operatorname{mod} A \to \operatorname{mod} B$ and $G_i\colon$ $\operatorname{mod} B \to \operatorname{mod} A$ such that*

- *$G_iF_i(M_i) \cong M_i$ for all $i \in I$,*
- *all G_i are left exact,*
- *$K_1 \dim_k F_i(M_i) \le \dim_L G_iF_i(M_i)$ for all $i \in I$,*
- *$\dim_L G_i(X) \le K_2 \dim_k X$ for all $X \in \operatorname{mod} B$ and $i \in I$,*

then $\mathcal{N} = \{F_i(M_i) \colon i \in I\} \subseteq \operatorname{mod} B$ is a non-hyperfinite family.

Proof. Assume that $\mathcal{N}$ is hyperfinite. Towards a contradiction we want to show that $\{G_iF_i(M_i) \colon i \in I\}$ is also hyperfinite. For any $\tilde{\varepsilon}$, we can find some $L_{\tilde{\varepsilon}}^{\mathcal{N}} > 0$ to exhibit the hyperfiniteness of the family $\mathcal{N}$. Let $M = M_i$ for some $i \in I$. Denote $N = F_i(M_i)$. We can find a submodule $P \subseteq N$ such that $P = \bigoplus_{j=1}^{t} P_j$ with $\dim_k P_j \le L_{\tilde{\varepsilon}}^{\mathcal{N}}$ and $\dim_k P \ge (1-\tilde{\varepsilon}) \dim_k N$. Since all G_i are additive, we have that $G_i(P) = \bigoplus_{j=1}^{t} G_i(P_j)$, and

$$\dim_L G_i(P_j) \le K_2 \dim_k P_j \le K_2 L_{\tilde{\varepsilon}}^{\mathcal{N}}.$$

Moreover, the sequence

$$0 \to G_i(P) \to G_i(N) \to G_i(N/P)$$

is exact in $\operatorname{mod} A$, so $G_i(P)$ is a submodule of $G_i(N) = G_iF_i(M_i) \cong M_i = M$, and by the rank-nullity theorem,

$$\begin{aligned} \dim_L G_i(P) &\ge \dim_L M - \dim_L G_i(N/P) \\ &\ge \dim_L M - K_2 \dim_k N/P \\ &\ge \dim_L M - K_2 \tilde{\varepsilon} \dim_k N \\ &\ge \dim_L M - \frac{K_2}{K_1} \tilde{\varepsilon} \dim_L G_i(N) = (1-\varepsilon) \dim_L M, \end{aligned}$$

if we put $\tilde{\varepsilon} = \frac{K_1}{K_2}\varepsilon$. We can therefore choose L_ε to be $K_2 L_{\tilde{\varepsilon}}^{\mathcal{N}}$ to show the hyperfiniteness of $\{G_iF_i(M_i) \colon i \in I\} = \{M_i \colon i \in I\}$.

But this is a contradiction, since we assumed that this set was not hyperfinite. Thus $\{F_i(M_i) \colon i \in I\}$ cannot be hyperfinite. □

Remark. This proof uses a slight modification of the proof of Proposition 1.5, adapting to the fact that the functor G might be different for each $i \in I$.

We are now able to directly prove that the path algebra of a wild quiver, that is, of a quiver with some connected component that is neither Dynkin nor extended Dynkin, is not amenable.

Theorem 6.16. *Let k be a field of characteristic zero and Q a wild quiver. Then $\operatorname{mod} kQ$ is not of amenable representation type.*

Proof. If Q contains the wild 3-Kronecker quiver $\Theta(3)$ as a subquiver, Corollary 6.14 implies that $\operatorname{mod} kQ$ is not of amenable type.

If Q is a wild quiver but does not contain a wild Kronecker quiver, it must have $n \geq 3$ vertices, hence $\operatorname{mod} kQ$ has at least three isoclasses of simple modules. By [Bae89, Theorem 2.1], which also holds for arbitrary fields, there exists a regular indecomposable module T without self-extensions. Thus we can apply Proposition 2.10 to see that $T^{\perp}$ is isomorphic to $\operatorname{mod} kQ'$ for some quiver Q'. Note that there is a corresponding homological epimorphism $\varphi\colon kQ \to kQ'$ and the induced functor $F = \varphi_*\colon \operatorname{mod} kQ' \to \operatorname{mod} kQ$ is fully faithful and exact. Indeed, Q' has $n-1$ vertices and [Bae89, Theorem 4.1] shows that kQ' is a wild quiver algebra.

Now, if we have some equivalence $\operatorname{mod} kQ' \xrightarrow{\sim} T^{\perp}$, the simple modules $S(i)$ of kQ' are mapped to certain objects B_i, considered as modules in $\operatorname{mod} kQ$. The k-dimension of any module M over a path algebra is determined by the length of any composition series. Such a series for some M' in kQ' is mapped to a composition series in the perpendicular category, and thus a series in $\operatorname{mod} kQ$, such that the factor modules are isomorphic to some B_i. This shows that

$$\dim_k F(M')_{kQ} \leq \max_{i=1,\dots,n-1} \{\dim_k B_i\} \dim_k M'_{kQ'}.$$

Moreover, by [Ker96, Lemma 11.1], $T^{\perp}$ is closed under kernels, cokernels and extensions. Assuming that Z is a relative projective generator of $\operatorname{mod} kQ'$, [HK16, Proposition A.1] shows that there exists a right adjoint to the inclusion F, which we denote by $G\colon \operatorname{mod} kQ \to \operatorname{mod} kQ'$. Indeed, for $M \in \operatorname{mod} kQ$, $G(M)$ is given as a factor module of a right $\operatorname{add}(Z)$-approximation Z_M of M. Since we may assume that $Z_M \cong Z \otimes_{\operatorname{End}(Z)} \operatorname{Hom}(Z, M)$, this implies that

$$\dim_k G(M) \leq \dim_k Z_M \leq (\dim_k Z)^2 \dim_k M.$$

Moreover, G is left exact. Since F is fully faithful, we have that $X \xrightarrow{\sim} GF(X)$ for all $X \in \operatorname{mod} kQ'$.

To conclude the proof, just note that we have prepared a descent argument leading to a wild quiver with two vertices, which has to include a wild Kronecker quiver $\Theta(m)$ as a subquiver. Given any non-hyperfinite family $\{M_j\colon j \in J\}$ in $\operatorname{mod} k\Theta(m)$, we choose Z as above and let $K_1 = \max\{\dim_k B_i\}^{-1}$ and $K_2 = (\dim_k Z)^2$. We can now choose all F_n as F and all G_n as G and have fulfilled the conditions of Proposition 6.15, which we apply to show the non-hyperfiniteness in each step, until we reach $\operatorname{mod} kQ$. □

Remark. The above theorem can also be proved more directly, by supplying a concrete embedding for each minimal wild quiver Q (see [Ker88, Section 4] for a list). In each case, we need to exhibit exceptional objects $X, Y \in \operatorname{mod} kQ$ such that (X, Y) is an orthogonal exceptional pair, that is,

$$\operatorname{Hom}_{kQ}(Y, X) = \operatorname{Hom}_{kQ}(X, Y) = 0 = \operatorname{Ext}^1_{kQ}(Y, X),$$

and such that $m := \dim_k \operatorname{Ext}^1_{kQ}(X, Y) \geq 3$.

Then, by [Rin76, 1.5 Lemma], there is a full exact embedding

$$F\colon \operatorname{mod} k\Theta(m) \to \operatorname{mod} kQ,$$

such that the simple representations of $\Theta(m)$ are mapped to X and Y respectively. Now, if M is a module for $k\Theta(m)$, any composition series will get mapped to a series in $\operatorname{mod} kQ$, such that the factor modules are isomorphic to either X or Y. This shows that

$$\dim_k F(M) \leq \max\{\dim_k X, \dim_k Y\} \dim_k M.$$

Denoting the closure of the full subcategory of $\operatorname{mod} kQ$ containing X and Y under kernels, images, cokernels and extensions by $\mathcal{C}(X,Y)$, F induces an equivalence $\operatorname{mod} k\Theta(m) \xrightarrow{\sim} \mathcal{C}(X,Y)$, see for instance [Rin76, Section 1] in connection with [Rin94, Corollary 1]. Assuming that Z is a relative projective generator of $\mathcal{C}(X,Y)$, [HK16, Proposition A.1] shows that there exists a right adjoint to the inclusion, which we denote by $G\colon \operatorname{mod} kQ \to \mathcal{C}(X,Y)$. Moreover, if $M \in \operatorname{mod} kQ$, $G(M)$ is given as a factor module of a right $\operatorname{add}(Z)$-approximation Z_M of M. Since we may assume that $Z_M \cong Z \otimes_{\operatorname{End}(Z)} \operatorname{Hom}(Z,M)$, this implies that

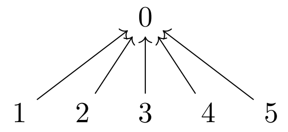
$$\dim_k G(M) \leq \dim_k Z_M \leq (\dim_k Z)^2 \dim_k M.$$

Moreover, G is left exact, and we have $GF(M) \cong M$ for all $M \in \operatorname{mod} k\Theta(m)$.

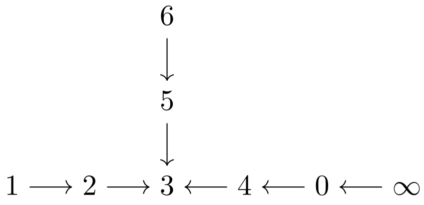
To conclude the proof in this case, we may chose a non-hyperfinite family $\{M_j\colon j \in J\}$ in $\operatorname{mod} k\Theta(m)$, guaranteed to exist by Theorem 6.10 and Lemma 6.13. Choose X, Y, Z as above and let $K_1 = \max\{\dim_k X, \dim_k Y\}^{-1}$ and $K_2 = (\dim_k Z)^2$. We can now choose all F_j as F and all G_j as G and have fulfilled the conditions of Proposition 6.15, which we apply to show the non-hyperfiniteness of $\operatorname{mod} kQ$.

Examples 6.17.

(1)



(2)



(1) Let Q be the five-subspace quiver §(5). Choose the modules $X = S_5$, the simple for vertex 5 and $Y = \tau^{-1}P_5$. Then

$$\operatorname{Hom}(X,Y) = 0 = \operatorname{Hom}(Y,X),$$

since X and Y have disjoint support. Also, $\operatorname{Ext}^1(Y,X) = 0$ since X is injective. On the other hand, $\operatorname{Ext}^1(X,Y) \cong k^3$ for dimension reasons.

(2) Let Q be the one-point extension by a source ∞ at an extending vertex 0 of the subspace-oriented $\tilde{E}_6$. Choose $X = S_\infty$ and Y to be the representation induced by $\tau^{-6}P_1$ of the underlying $\tilde{E}_6$.

(3)
$$\begin{tikzcd}
4 \arrow[dr] & & & 0 & \infty \arrow[l] \\
& 3 \arrow[r] & 2 \arrow[ur] \arrow[dr] & & \\
5 \arrow[ur] & & & 1 &
\end{tikzcd}$$

(4)
$$\begin{tikzcd}
4 \arrow[dr] & & & 0 \arrow[r] & \infty \\
& 3 \arrow[r] & 2 \arrow[ur] \arrow[dr] & & \\
5 \arrow[ur] & & & 1 &
\end{tikzcd}$$

(3) Let Q be the one-point extension at an extending vertex 0 of a linearly oriented $\tilde{D}_5$ by a source ∞. Choose $X = S_\infty$ and Y as the representation induced by $\tau^{-3}P_3$ of the underlying $\tilde{D}_5$.

(4) Let Q be the one-point extension at an extending vertex 0 of a linearly oriented $\tilde{D}_5$ by a sink ∞. Choose $Y = S_\infty$ and X to be the representation induced by $\tau^{-3}P_3$ of the underlying $\tilde{D}_5$.

6.2.2 Strict wildness and related notions

Next, we recall several notions of wildness.

Definition 6.18. [Cra92, Section 8.3] Let A be a finite dimensional hereditary algebra. We call A **hereditary wild** provided the associated quadratic form q_A (see Section 3.1) is indefinite.

Definition 6.19. [Cra92, Section 8.2] Let A be a finite dimensional k-algebra. A is called **strictly wild** provided there exists an orthogonal pair (X, Y) of finite dimensional modules, that is, $\operatorname{Hom}_A(X, Y) = \operatorname{Hom}_A(Y, X) = 0$, which are finitely presented such that their endomorphism rings $\operatorname{End}(X)$ and $\operatorname{End}(Y)$ are division rings and such that

$$p = \dim_{\operatorname{End}_A(Y)} \operatorname{Ext}^1_A(X, Y) \cdot \dim_{\operatorname{End}_A(X)} \operatorname{Ext}^1_A(X, Y) \geq 5.$$

Definition 6.20. [Sim05, Section 2] Let A be finite dimensional k-algebra. A is called **fully wild** provided there exists a finite field extension $L|k$ and a fully faithful exact additive functor $G\colon \operatorname{mod} L\langle x, y\rangle \to \operatorname{mod} A$

A first relation between these notions may follow from the next lemma.

Lemma 6.21. *[Cra92, Lemma 8.2] Let A be a finite dimensional algebra. A is strictly wild if and only if there exists a finite field extension $L|k$ and an A-$L\langle x, y\rangle$-bimodule T, finitely generated and projective over $L\langle x, y\rangle$ such that*

$$T \otimes_{L\langle x,y\rangle} -\colon \operatorname{Mod} L\langle x, y\rangle \to \operatorname{Mod} A$$

is fully faithful.

Remark. Note that the statement of the previous lemma is not about fully wild algebras: While a strictly wild algebra is fully wild since the tensor product functor restricts to a fully faithful exact functor between the finitely generated modules, a functor from the definition of fully wild need not be given as a tensor product. Also compare the necessary conditions in [Sim93, Lemma 2.2] and [Wat60].

However, we do have a different implication.

Theorem 6.22. *[Cra92, Theorem 8.4] Let A be a finite dimensional algebra. If A is hereditary wild, then A is strictly wild.*

Definition 6.23. [Sim03, Definition 2.4] Let A be a finite dimensional k-algebra. A is called **fully k-wild** provided there exists a k-linear functor $G\colon \operatorname{mod} k\langle x,y\rangle \to \operatorname{mod} A$ which is full, respects isomorphism classes and preserves indecomposability.

By [Sim03, Lemma 2.5], A is fully k-wild if and only if there exists a fully faithful exact k-linear functor $H\colon \operatorname{mod} k\langle x,y\rangle \to \operatorname{mod} A$. This implies that in case k is algebraically closed, the notions of fully wildness and fully k-wildness coincide.

As we have dealt with Kronecker algebras so far, we discuss how to make use of their module category in place of $\operatorname{mod} k\langle x,y\rangle$.

Lemma 6.24. *Let A be a finite dimensional k-algebra. Assume that A is strictly wild. Then there exists a finite field extension $L|k$ such that for $d \geq 3$, there is an A-$L\Theta(d)$-bimodule M such that M is of finite L-dimension, projective as an $L\Theta(d)$-module and the functor $F = M \otimes_{L\Theta(d)} -\colon \operatorname{mod} L\Theta(d) \to \operatorname{mod} A$ is full and faithful.*

Proof. By [Cra92, Lemma 8.2], there exists a finite field extension $L|k$ and a fully faithful exact additive functor

$$G = T \otimes_{L\langle x,y\rangle} -\colon \operatorname{Mod} L\langle x,y\rangle \to \operatorname{Mod} A,$$

where T is a left A-right $L\langle x,y\rangle$-bimodule which is finitely generated projective over $L\langle x,y\rangle$.

By [SS07, Thm. XIX.1.7], since $L\Theta(d)$ is finitely generated as an algebra over L, there is a full, faithful and exact functor

$$H = N \otimes_{L\Theta(d)} -\colon \operatorname{Mod} L\Theta(d) \to \operatorname{Mod} L\langle x,y\rangle,$$

where N is a left $L\langle x,y\rangle$-right $L\Theta(d)$-bimodule which is finitely generated and free over $L\Theta(d)$. Note that H restricts to a fully faithful exact functor between the modules of finite L-dimension.

The composition of these functors is then a fully faithful and exact functor

$$G \circ H\colon \operatorname{Mod} L\Theta(d) \to \operatorname{Mod} A,$$

given by tensoring with $T \otimes_{L\langle x,y\rangle} N$. It remains to show that this is a projective $L\Theta(d)$-module of finite L-dimension.

As T is a finitely generated projective module over $L\langle x,y\rangle$, we know that it is a direct summand of a finite number—say of m—copies of $L\langle x,y\rangle$. This implies that $T \otimes_{L\langle x,y\rangle} N$ is a direct summand of $L\langle x,y\rangle^m \otimes_{L\langle x,y\rangle} N$, when viewed as a $L\Theta(d)$-module. As these, we have $L\langle x,y\rangle^m \otimes_{L\langle x,y\rangle} N \cong N^m$. But N is a finitely generated projective module when viewed over $L\Theta(d)$, hence so is N^m. But this already implies that $T \otimes N$ is a finitely generated projective module over $L\Theta(d)$. Since the Kronecker algebra $L\Theta(d)$ is of finite dimension over L, $T \otimes N$ also has finite L-dimension.

□

6.2.3 Hereditary wild implies non-amenability

We will now turn to study hereditary wild algebras with respect to amenability. To apply Proposition 6.15, we prove two technical lemmas first.

Lemma 6.25. *Let $L|k$ be a finite field extension. Let A be a finite dimensional k-algebra and let B be a finite dimensional L-algebra. Assume that $F\colon \operatorname{mod} B \to \operatorname{mod} A$ is an additive, fully faithful functor given as $F = C \otimes_B -$ for some A-B-bimodule C which is finitely generated projective over B. Then there exists a left exact functor $G\colon \operatorname{mod} A \to \operatorname{mod} B$ such that $GF(M) \cong M$ for all $M \in \operatorname{mod} B$ and such that there are $K_1, K_2 > 0$ with $\dim_k F(M) \leq K_1 \dim_L M$ and $\dim_L G(N) \leq K_2 \dim_k N$.*

Proof. We do have

$$\operatorname{Hom}_A(C \otimes X, Y) \cong \operatorname{Hom}_B(X, \operatorname{Hom}_A(C, Y)).$$

We therefore choose $G = \operatorname{Hom}_A(C, -)$. Thus, G is left exact. Moreover, $GF(M) \cong M$ as F is fully faithful (see, e.g., [Par69, Korollar in 2.12]). It remains to show the existence of the inequality constants K_1 and K_2.

First, let M be a B-module. Then $C \otimes_L M$ surjects onto $C \otimes_B M$ as L-vector spaces, so $\dim_L C \otimes_B M \leq \dim_L C \otimes_L M$, implying

$$\dim_k F(M) = \dim_k (C \otimes_B M) \leq [L : k] \cdot \dim_L C \cdot \dim_L M.$$

This shows that we can choose $K_1 = ([L : k] \dim_L C)^{-1}$.

Second, let N be an A-module. Then $\operatorname{Hom}_A(C, N)$ as a set of A-homomorphisms is a k-vector space of dimension bounded by $\dim_k {}_AC \cdot \dim_k N$. But it is also a B-module, and the action of k (via these module structures) is central (cf. [Iva11, Remark 2.5]). Thus we have

$$\dim_L G(N) = \dim_L \operatorname{Hom}_A(C, N) = \frac{\dim_k \operatorname{Hom}_A(C, N)}{[L : k]} \leq \frac{\dim_k C}{[L : k]} \dim_k N.$$

This shows that we can choose $K_2 = \frac{\dim_k C}{[L:k]}$. □

Lemma 6.26. *Let A be a finite dimensional hereditary wild k-algebra of rank two. Then there is $d \geq 3$ such that there exist additive functors $F\colon \operatorname{mod} L\Theta(d) \to \operatorname{mod} A$ and $G\colon \operatorname{mod} A \to \operatorname{mod} L\Theta(d)$ for some finite field extension $L|k$ such that*

(1) $GF(M) \cong M$ for all $M \in \operatorname{mod} L\Theta(d)$,

(2) G is left exact,

(3) there exists $K_1 > 0$ such that

$$K_1 \dim_k F(M) \leq \dim_L GF(M)$$

for all $M \in \operatorname{mod} L\Theta(d)$, *(4) there exists* $K_2 > 0$ *such that*

$$\dim_L G(N) \leq K_2 \dim_k N$$

for all $N \in \operatorname{mod} A$*,*

Proof. We have that B is a bimodule algebra given by two division rings D and E over k and a bimodule ${}_D M_E$ (notation as in [Rin76]). Inspection of the proof of [Rin76, Theorem 2] in Section 5.2 ibid. in connection with [Cra92, Lemma 8.1] shows that there is a full exact embedding $F\colon \operatorname{mod} L\Theta(d) \to \operatorname{mod} A$ for some $d \geq 3$ and $k \subseteq L \subseteq D, E$. Thus, by the Eilenberg–Watts Theorem [Wat60, Theorem 1], F is naturally equivalent to the functor $C \otimes_{L\Theta(d)} -$, where C is the A-$L\Theta(d)$-bimodule $F(L\Theta(d))$, which is finite dimensional over k and projective over $L\Theta(d)$. Now we apply Lemma 6.25 to get

$$G = \operatorname{Hom}_A(C, -)$$

and finish the proof. □

In addition to this base case of rank two, we will now see how to prove stepping from rank n to rank $n+1$.

Proposition 6.27. *Let k be any field and $n \geq 2$. If no finite dimensional connected wild hereditary algebra of rank n is of amenable representation type, then no wild hereditary algebra of rank $n+1$ is of amenable type, either.*

Proof. Let A be a finite dimensional connected wild hereditary algebra of rank $n+1$. By [Rin88, Theorem in §1], A has a regular tilting module and thus there exists an indecomposable regular module T with no self-extensions. By [HR82, Lemma 4.1], $\operatorname{End}(T)$ is a division ring. Now [GL91, Theorem 4.16] yields that $T^{\perp} \cong \operatorname{mod} \Lambda$ for some finite dimensional hereditary algebra Λ, along with a homological epimorphism $\varphi\colon A \to \Lambda$. Note that the induced functor $F = \varphi_*\colon \operatorname{mod} \Lambda \to \operatorname{mod} A$ is fully faithful and exact. We may assume that Λ is basic. Moreover, Λ has n simple modules. Without loss of generality, we may assume that T is regular simple. For if T is not regular simple, we can consider the regular top of T, which also does not have self-extensions by [Hos84, Lemma 2.3, Proposition 2.4]. Thus, [Str91, Proposition 6.2] shows that Λ is connected and by [Str91, Theorem 6.5], Λ is wild as the induced quadratic form on Λ is indefinite, so it is of non-amenable type by hypothesis.

Now, if we have some equivalence $\operatorname{mod} \Lambda \xrightarrow{\sim} T^{\perp}$, the simples S_i of Λ get mapped to certain objects B_i, considered as modules in $\operatorname{mod} A$. The k-dimension of any module M over a finite dimensional k-algebra is determined by the length of any composition series. Such a series for M in $\operatorname{mod} \Lambda$ gets mapped to a composition series in the perpendicular category, and thus a series in $\operatorname{mod} A$, such that the factor modules are isomorphic to some B_i. This shows that

$$\dim_k F(M)_A \leq \max\{\dim B_i\} \dim_k M_\Lambda.$$

Moreover, by [Str91, Lemma 1.2], $T^{\perp}$ is closed under extensions, kernels and cokernels. Assuming that Z is a relative projective generator of $\operatorname{mod}\Lambda$ (we may use the middle term in the universal short exact sequence of Bongartz), [HK16, Proposition A.1] shows that there exists a right adjoint to the inclusion F, which we denote by $G\colon \operatorname{mod} A \to \operatorname{mod}\Lambda$. Indeed, for $M \in \operatorname{mod} A$, $G(M)$ is given as a factor module of a right $\operatorname{add}(Z)$-approximation Z_M of M. We may assume $Z_M \cong Z \otimes_{\operatorname{End}(Z)} \operatorname{Hom}(Z, M)$, implying that

$$\dim_k G(M) \leq \dim_k Z_M \leq (\dim_k Z)^2 \dim_k M.$$

Moreover, G is left exact. Since F is fully faithful, we have that $X \xrightarrow{\sim} GF(X)$ for all $X \in \operatorname{mod}\Lambda$.

By hypothesis, there is a non-hyperfinite family $\{M_j : j \in J\}$ in $\operatorname{mod}\Lambda$. Choose Z as above and let $K_1 = \max\{\dim_k B_i\}^{-1}$ and $K_2 = (\dim_k Z)^2$. We can now choose all F_j as F and all G_j as G and have fulfilled the conditions of Proposition 6.15, which we apply to show the non-hyperfiniteness of $\operatorname{mod} A$. ☐

Theorem 6.28. *Let A be a finite dimensional k-algebra. If A is wild hereditary of rank $n \geq 2$, then A is not of amenable representation type.*

Proof. We prove this by induction on the rank. If A is a finite dimensional connected wild hereditary algebra of rank two, Lemma 6.26 implies the existence of functors F, G as in Proposition 6.15, allowing us to infer that A is not of amenable representation type, since the wild Kronecker algebra $L\Theta(d)$ was shown to be not of amenable representation type in Theorem 6.12. The induction step then is Proposition 6.27. ☐

6.2.4 Strict and full wildness imply non-amenability

Theorem 6.29. *Let A be a finite dimensional k-algebra and let A be fully wild. If we assume that k is algebraically closed or that the functor from the definition of fully wildness is k-linear, then A is not of amenable representation type.*

Proof. We want to apply Proposition 6.15. By the definition, there is a finite field extension $L|k$ and a fully faithful exact functor $F'\colon \operatorname{mod} L\langle x, y\rangle \to \operatorname{mod} A$. Note that F' is k-linear, either because this is assumed or because this follows as k is algebraically closed and we have $L = k$. Now, let $d \geq 3$ be such that $L\Theta(d)$ is not of amenable type (see Theorem 6.12). By [Sim92, Proposition 14.10], there exists an L-linear fully faithful exact functor $F''\colon \operatorname{mod} L\Theta(d) \to \operatorname{mod} L\langle x, y\rangle$. Clearly, their composition,

$$F = F' \circ F''\colon \operatorname{mod} L\Theta(d) \to \operatorname{mod} A,$$

is also fully faithful and exact as well as k-linear. Thus $F \cong F(L\Theta(d)) \otimes -$ by [Iva11, Theorem 2.4]. We will write $C = F(L\Theta(d))$.

We may now apply Lemma 6.25 to yield $G = \operatorname{Hom}_A(C, -)$ fulfilling the conditions of Proposition 6.15.

Now, choosing all F_j as F and all G_j as G and picking some non-hyperfinite family of modules in $\operatorname{mod} L\Theta(d)$, we can show the existence of some non-hyperfinite family of modules in $\operatorname{mod} A$. This proves that A cannot be of amenable representation type. ☐

If we instead assume that A is strictly wild, no additional assumption is necessary.

Theorem 6.30. *Let A be a finite dimensional k-algebra. If A is strictly wild, then A is not of amenable representation type.*

Proof. We want to apply Proposition 6.15. By Lemma 6.24, there is a finite field extension $L|k$. We use Theorem 6.12 to find $d \geq 3$ such that $L\Theta(d)$ is not of amenable representation type. Resorting to Lemma 6.24, there is a fully faithful exact functor $F\colon \operatorname{mod} L\Theta(d) \to \operatorname{mod} A$ given by $C \otimes_{L\Theta(d)} -$, where C is an A-$L\Theta(d)$-bimodule finitely generated projective over $L\Theta(d)$ and thus of finite k-dimension. We may now apply Lemma 6.25 to produce the functor $G = \operatorname{Hom}_A(C, -)$.

Now, choosing all F_j as F and all G_j as G and picking some non-hyperfinite family of modules in $\operatorname{mod} L\Theta(d)$, we can show the existence of some non-hyperfinite family of modules in $\operatorname{mod} A$. This proves that A cannot be of amenable representation type. $\square$

As a corollary, we get a new proof of Theorem 6.28.

Corollary 6.31. *A hereditary wild algebra A is not of amenable representation type.*

Proof. Apply Theorems 6.22 and 6.30. $\square$

6.3 Radical square zero and wild local algebras

As not all wild algebras are strictly wild or fully wild, we want to broaden our approach. In doing so, we now turn to wild local algebras, that is, to algebras that are local and wild (see, e.g., [Rin74, Section 3]). We will especially focus on the local wild algebra $A = k\langle x_1, x_2, x_3\rangle / M_2$, where M_2 is the ideal generated by all monomials of degree two. This algebra has radical square zero, that is, $\mathfrak{r}^2 = 0$, where $\mathfrak{r} = \mathcal{J}$ is the Jacobson ring radical of A. We will use this latter property to discuss its amenability.

Lemma 6.32. *Let A be a finite dimensional k-algebra with radical square zero. Then $M \in \operatorname{mod} A$ has no simple direct summand if and only if $\operatorname{rad} M = \operatorname{soc} M$.*

Proof. We first note that in this situation, for $M \in \operatorname{mod} A$, we have

$$\operatorname{rad} M = \mathfrak{r}M = \left\{ \sum_{i=1}^{n} r_i m_i \colon n \in \mathbb{N}, r_i \in \mathfrak{r}, m_i \in M \right\},$$

and

$$\operatorname{soc} M = \{m \in M \colon \mathfrak{r}m = 0\} = \{m \in M \colon \forall r \in \mathfrak{r} \colon rm = 0\}.$$

Let $M \in \operatorname{mod} A$. If $m \in \operatorname{rad} M$, then $m = \sum r_i m_i$ with $r_i \in \mathfrak{r}, m_i \in M$. But then $\mathfrak{r}m = \mathfrak{r} \sum r_i m_i = 0$, since $\mathfrak{r}^2 = 0$. Thus $m \in \operatorname{soc} M$, proving $\operatorname{rad} M \subseteq \operatorname{soc} M$.

Now, since $M/\operatorname{rad} M$ is semisimple, and $\operatorname{soc} M/\operatorname{rad} M \subseteq M/\operatorname{rad} M$, there exists $C \supset \operatorname{rad} M$ such that

$$M/\operatorname{rad} M = \operatorname{soc} M/\operatorname{rad} M \oplus C/\operatorname{rad} M.$$

Moreover, as $\operatorname{soc} M$ is semisimple and $\operatorname{rad} M \subseteq \operatorname{soc} M$, there exists $S' \in \operatorname{mod} A$ (possibly zero) such that $\operatorname{soc} M = \operatorname{rad} M \oplus S'$. First, we have that $\operatorname{soc} M + C = M$, as $\operatorname{rad} M \subseteq \operatorname{soc} M$. Second, $\operatorname{rad} M \subseteq \operatorname{soc} M \cap C \subseteq \operatorname{rad} M$, as

$$(\operatorname{soc} M/\operatorname{rad} M) \cap (C/\operatorname{rad} M) = 0 \in M/\operatorname{rad} M,$$

thus $\operatorname{soc} M \cap C = 0$. Third, $C + S \supseteq \operatorname{rad} M + S' = \operatorname{soc} M$. It follows that $C + S' = M$. We also have $C \cap S' \subseteq C \cap \operatorname{soc} M = \operatorname{rad} M$, yet $S' \cap \operatorname{rad} M = 0$, showing that $C \cap S' = 0$. After all, we have established that $M = C \oplus S'$, where S' is semisimple as a submodule of $\operatorname{soc} M$, but is also a direct summand of M.

In conclusion, if M has no simple summand, we must have $S' = 0$, so $\operatorname{soc} M = \operatorname{rad} M$. Conversely, if M has a simple summand S, $M = M' \oplus S$, then

$$\operatorname{rad} M = \operatorname{rad} M' \subseteq \operatorname{soc} M' \subset \operatorname{soc} M,$$

where the last inclusion is strict, as $\operatorname{soc} S = S$. It follows that M has no simple summand if and only if $S' \neq 0$ if and only if $\operatorname{soc} M \neq \operatorname{rad} M$. ☐

Lemma 6.33. *For a finite dimensional k-algebra A with radical square zero, the subcategory $\mathcal{C} \subset \operatorname{mod} A$ of all finite dimensional modules M such that $\operatorname{soc} M = \operatorname{rad} M$, is closed under extensions.*

Proof. Let $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be an exact sequence with $X, Z \in \mathcal{C}$. Since $\mathfrak{r}^2 = 0$, it is enough to show that $\operatorname{soc} Y \subseteq \operatorname{rad} Y$. Let $y \in \operatorname{soc} Y$, thus $\mathfrak{r}y = 0$. Then $0 = g(ry) = rg(y)$ for all $r \in \mathfrak{r}$, so $g(y) \in \operatorname{soc} Z = \operatorname{rad} Z$. We can then write $g(y) = \sum r_i z_i$ with $r_i \in \mathfrak{r}, z_i \in Z$. As g is surjective, there are $w_i \in Y$ such that $g(w_i) = z_i$. It follows that $g\left(y - \sum r_i w_i\right) = 0$, so $y - \sum r_i w_i \in \ker g = \operatorname{im} f$. Now, there exists $x \in X$ such that $f(x) = y - \sum r_i w_i$. Thus, $f(rx) = rf(x) = ry - r\sum r_i w_i = 0$ for all $r \in \mathfrak{r}$. As f is injective, this shows that $rx = 0$ for all $r \in \mathfrak{r}$, that is, $x \in \operatorname{soc} X = \operatorname{rad} X$. We may then write $x = \sum s_i v_i$ with $s_i \in \mathfrak{r}, v_i \in X$. We now have that

$$y = \sum r_i w_i + f\left(\sum s_i v_i\right) = \sum r_i w_i + \sum s_i f(v_i) \in \mathfrak{r}Y = \operatorname{rad} Y,$$

showing that $\operatorname{soc} Y \subseteq \operatorname{rad} Y$. ☐

Given a finite dimensional k-algebra A with radical square zero, we associate to it the hereditary, triangular matrix algebra

$$B = \left(\begin{smallmatrix} A/\mathfrak{r} & 0 \\ \mathfrak{r} & A/\mathfrak{r} \end{smallmatrix}\right).$$

Let $F\colon \operatorname{mod} A \to \operatorname{mod} B$ be the functor defined in [ARS95, Section X.2] given on objects as $F(X) = (X/\mathfrak{r}X, \mathfrak{r}X, f)$, where $f\colon \mathfrak{r} \otimes_{A/\mathfrak{r}} X/\mathfrak{r}X \to \mathfrak{r}X$ is induced by the natural multiplication $\mathfrak{r} \otimes_A X \to \mathfrak{r}X$, using $\mathfrak{r}^2 = 0$.

Proposition 6.34. *Let A be a finite dimensional k-algebra with radical square zero and let B and F be as above. If $\mathcal{C} \subset \operatorname{mod} A$ is the subcategory of all modules without simple direct summands, then the functor F is exact when restricted to $\mathcal{C}$. Moreover, if $f\colon N \to M$ is a monomorphism in $\operatorname{mod} A$ and $N, M \in \mathcal{C}$, then $F(f)\colon F(N) \to F(M)$ is a monomorphism in $\operatorname{mod} B$.*

Proof. Let $\eta\colon 0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be a short exact sequence with $X, Y, Z \in \mathcal{C}$. Clearly, for A-modules U and V, if $h\colon U \to V$, then $h(\operatorname{soc} U) \subseteq \operatorname{soc} V$. We will write $\operatorname{soc} h$ or even h instead of $h_{|\operatorname{soc} U}$. We will also write $\overline{h}$ for $\operatorname{top} h$, where

$$\operatorname{top} h(u + \operatorname{rad} U) = h(u) + \operatorname{rad} V.$$

soc *is left exact.* First, let $x \in \operatorname{soc} X \subseteq X$ such that $f(x) = 0 \in \operatorname{soc} Y \subseteq Y$. Since f is injective, we must then have $x = 0$. Second, let $y \in \operatorname{soc} Y \subseteq Y$ such that

$$g(y) = 0 \in \operatorname{soc} Z \subseteq Z.$$

Then $y \in \ker g = \operatorname{im} f$, and there is $x \in X$ such that $f(x) = y$. Since $ry = 0$ for all $r \in \mathfrak{r}$, we also have that $f(rx) = rf(x) = ry = 0$ for all $r \in \mathfrak{r}$. By the injectivity of f, this implies that $rx = 0$ for all $r \in \mathfrak{r}$, showing that $x \in \operatorname{soc} X$, so $y \in \operatorname{im} f_{|\operatorname{soc} X}$. Third, let $y \in \operatorname{im} f_{|\operatorname{soc} X}$, that is, $y = f(x)$ for some $x \in \operatorname{soc} X$. Then $(g_{|\operatorname{soc} Y} \circ f_{|\operatorname{soc} X})(x) = 0$ since $g \circ f = 0$. This shows that

$$0 \to \operatorname{soc} X \xrightarrow{f_{|\operatorname{soc} X}} \operatorname{soc} Y \xrightarrow{g_{|\operatorname{soc} Y}} \operatorname{soc} Z$$

is exact.

exactness at $\operatorname{soc} Z$*.* Let $z \in \operatorname{soc} Z = \operatorname{rad} Z$, and write $z = \sum r_i w_i$ with $r_i \in \mathfrak{r}, w_i \in Z$. As g is surjective, there exist $v_i \in Y$ such that $g(v_i) = w_i$. Now,

$$g\left(\sum r_i v_i\right) = \sum r_i g(v_i) = \sum r_i w_i = z,$$

and $\sum r_i v_i \in \operatorname{rad} Y = \operatorname{soc} Y$. Thus $g_{|\operatorname{soc} Y}$ is surjective.

exactness of top *in this situation.* That top is right exact follows from the duality $D(\operatorname{top} M) = \operatorname{soc} DM$ or can be proven directly: First, let $z + \operatorname{rad} Z \in \operatorname{top} Z$. Since g is surjective, there is $y \in Y$ such that $g(y) = z$. But then $\bar{g}(y + \operatorname{rad} Y) = z + \operatorname{rad} Z$, so $\bar{g}$ is surjective. Second, let $\bar{y} = y + \operatorname{rad} Y \in \operatorname{top} Y$ be such that $\bar{g}(\bar{y}) = 0 \in \operatorname{top} Z$, hence $g(y) \in \operatorname{rad} Z$. Then $g(y) = \sum r_i w_i$ with $r_i \in \mathfrak{r}, w_i \in Z$. As g is surjective, there exist v_i such that $g(v_i) = w_i$. Now $g\left(y - \sum r_i v_i\right) \in \ker g = \operatorname{im} f$, and there is $x \in X$ such that $f(x) = y - \sum r_i v_i$. Now $\bar{f}(x + \operatorname{rad} X) = f(x) + \operatorname{rad} Y = y - \sum r_i v_i + \operatorname{rad} Y = \bar{y}$, showing that $\bar{y} \in \operatorname{im} \bar{f}$.

The exactness at $\operatorname{top} X$ follows from duality, too, once we realise that we also have $D(\operatorname{cosoc} M) = \operatorname{rad} DM$, or can be seen directly: Let $x + \operatorname{rad} X \in \operatorname{top} X$ be such that $\bar{f}(x + \operatorname{rad} X) = 0 \in \operatorname{top} Y$, hence $f(x) \in \operatorname{rad} Y = \operatorname{soc} Y$. Then $0 = rf(x) = f(rx)$ for all $r \in \mathfrak{r}$. As f is injective, this shows that $rx = 0$ for all $r \in \mathfrak{r}$, thus $x \in \operatorname{soc} X = \operatorname{rad} X$. We now have that $x + \operatorname{rad} X = 0 \in \operatorname{top} X$, showing that $\bar{f}$ is injective.

F is exact on $\mathcal{C}$. Given $h\colon U \to V$ in $\operatorname{mod} A$, we have $F(h)\colon F(U) \to F(V)$ in $\operatorname{mod} B$. As $F(M) = (\operatorname{top} M, \operatorname{rad} M, s_M)$, where the structure map $s_M\colon \mathfrak{r} \otimes \operatorname{top} M \to \operatorname{rad} M$, and $F(h) = (\operatorname{rad} h, \operatorname{top} h)$ is a morphism of B-modules and thus commutes with the structure maps s_-, we have a commutative diagram for each $r \in \mathfrak{r}$

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{top} X \arrow[r, "\operatorname{top} f"] \arrow[d, "r\otimes -"] & \operatorname{top} Y \arrow[r, "\operatorname{top} g"] \arrow[d, "r\otimes -"] & \operatorname{top} Z \arrow[r] \arrow[d, "r\otimes -"] & 0 \\
0 \arrow[r] & \operatorname{rad} X \arrow[r, "\operatorname{rad} f"] & \operatorname{rad} Y \arrow[r, "\operatorname{rad} g"] & \operatorname{rad} Z \arrow[r] & 0
\end{tikzcd}$$

of exact rows, establishing the exactness of $0 \to F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \to 0$.

An inspection of the proof shows that $F(f)\colon F(N) \to F(M)$ is a monomorphism provided $f : N \to M$ is and $N, M \in \mathcal{C}$, since only the injectivity of f and the properties of N, M are used. □

Remark. Note that it was shown in [Che09, Lemma 3.3] that given $f\colon N \hookrightarrow M$, an inclusion of indecomposable A-modules, $F(f)$ is a monomorphism provided N is not simple and $N \not\subseteq \mathfrak{r}M$.

Theorem 6.35. *Let k be a field of characteristic zero. Then the local wild algebra $k\langle x_1, x_2, x_3\rangle/M_2$ is not of amenable representation type.*

Proof. Consider the functor F as in Proposition 6.34. Then the algebra B is the wild Kronecker algebra $k\Theta(3)$. We denote its arrows by α, β and γ. Here, F on the objects is given by

$$\begin{tikzcd}
F(M) = \operatorname{top} M \arrow[r, bend left, "x_1\cdot -"] \arrow[r, "x_2\cdot -"] \arrow[r, bend right, "x_3\cdot -"'] & \operatorname{rad} M.
\end{tikzcd}$$

Let $\{\tilde{M}_i\colon i \in I\}$ be a non-hyperfinite family in $\operatorname{ind} B$ with all $\tilde{M}_i$ non-simple and non-projective. We may assume the latter since there are only finitely many projective indecomposables. As F is essentially surjective on the non-projective indecomposables by [ARS95, Theorem X.2.4], there are M_i such that $F(M_i) = \tilde{M}_i$ for all $i \in I$. By [ARS95, Lemma X.2.1], the M_i are indecomposable and not simple since F preserves length.

Towards a contradiction, assume that $\operatorname{mod} A$ is hyperfinite. Then for all $\varepsilon > 0$, there is $L_\varepsilon > 0$ such that for all $i \in I$, there are submodules $N_i \subseteq M_i$ such that $\dim N_i \geq (1-\varepsilon)\dim M_i$ and $N_i = \oplus_{j=1}^{s_i} N_j^{(i)}$ with $\dim N_j^{(i)} \leq L_\varepsilon$. For each $i \in I$, decompose $N_i = N_i' \oplus N_i''$ such that N_i' has no simple summands and N_i'' is semisimple. By Proposition 6.34, $F(N_i')$ is a submodule of $F(M_i) = \tilde{M}_i$. Moreover, let $\{b_1, \ldots, b_{n''}\}$ be a k-basis of N_i'', each $\langle b_\ell\rangle$ a simple summand of N_i and a simple submodule of M_i.

Now, as $\langle b_\ell \rangle$ is simple, $b_\ell \in \ker M_i(\alpha) \cap \ker M_i(\beta) \cap \ker M_i(\gamma)$, thus $b_\ell \in \operatorname{soc} M_i = \operatorname{rad} M_i$, since M_i is indecomposable and not simple. Since $N_i' \cap N_i'' = 0$,

$$b_\ell \notin \operatorname{rad} N_i = \operatorname{rad} N_i' \oplus \operatorname{rad} N_i'' = \operatorname{rad} N_i' \subseteq N_i',$$

as N_i'' is semisimple. This shows that $\dim \operatorname{rad} M - \dim \operatorname{rad} N_i' \geq \dim N_i''$. Now put $\tilde{N}_i = F(N_i') \oplus S(2)^{|N_i''|}$, where $|N_i''|$ is the composition length of N_i''. Then

$$\dim \tilde{N}_i = \dim F(N_i') + |N_i''| = \dim N_i \geq (1-\varepsilon) \dim M_i = (1-\varepsilon) \dim \tilde{M}_i.$$

What is more, $F(N_i')$ has a decomposition into summands of dimension at most L_ε (by Krull-Remak-Schmidt), while $S(2)^{|N_i''|}$ is semisimple. This would thus give submodules exhibiting the hyperfiniteness of $\{\tilde{M}_i \colon i \in I\}$, a contradiction. □

Unfortunately, this method does not work for other locally wild algebras, as it relies on having radical square zero. We use that the functor F is left-exact on a suitable subcategory, but this cannot be expected for instance for $k\langle x, y\rangle/(x^2, yx, xy^2, y^3)$ or $k\langle x, y\rangle/(xy, x^2 - y^2)$.

6.4 Weak notions and finitely controlled wild algebras

In order to approach the problem of disproving hyperfiniteness for further classes of wild algebras, we introduce the notion of weak hyperfiniteness and use it in connection with interpretation functors for (finitely) controlled wild algebras.

Definition 6.36. Let k be a field, let A be a finite dimensional k-algebra and let $\mathcal{M} \subseteq \operatorname{mod} A$ be a family of finite dimensional A-modules. $\mathcal{M}$ is called **weakly hyperfinite** provided for every $\varepsilon > 0$ there exists some $L_\varepsilon > 0$ such that for every $M \in \mathcal{M}$ there is a homomorphism $\theta \colon N \to M$ for some $N \in \operatorname{mod} A$ such that

$$\begin{aligned} \dim_k \ker \theta &\leq \varepsilon \dim M, \\ \dim_k \operatorname{coker} \theta &\leq \varepsilon \dim M, \end{aligned} \tag{6.2}$$

and there are modules $N_1, \dots, N_t \in \operatorname{mod} A$ with $\dim_k N_i \leq L_\varepsilon$ such that $N \cong \bigoplus_{i=1}^t N_i$.

A k-algebra A is said to be of **weakly amenable representation type** provided $\operatorname{mod} A$ itself is a weakly hyperfinite family.

We see that the term “weakly hyperfinite” is suitably chosen:

Proposition 6.37. *Let A be a finite dimensional k-algebra. If $\mathcal{M} \subseteq \operatorname{mod} A$ is hyperfinite, then $\mathcal{M}$ is weakly hyperfinite.*

Proof. Let $\varepsilon > 0$. By the hyperfiniteness, there is some $L_\varepsilon > 0$. Let $M \in \mathcal{M}$. Then there is $N \subseteq M$ with $\dim N \geq (1-\varepsilon) \dim M$ and $N = \bigoplus_{i=1}^t N_i$ with $\dim N_i \leq L_\varepsilon$. Let θ be the inclusion of the submodule $N \hookrightarrow M$. Then $\ker \theta = 0$, and $\operatorname{coker} \theta \cong M/N$, thus $\dim \operatorname{coker} \theta = \dim M - \dim N \leq \varepsilon \dim M$. We have shown weak hyperfiniteness with the same ε and L_ε. □

We next recover previous results respectively state the appropriate analogues for weak hyperfiniteness.

Proposition 6.38. *Let k be a field and A, B be finite dimensional k-algebras. Let $G\colon \operatorname{mod} A \to \operatorname{mod} B$ be an additive, left exact functor such that there exist constants $K_1, K_2, K_4 > 0$ with*

$$K_1 \dim X \leq \dim G(X), \quad \textit{for all } X \in \mathcal{N}, \tag{6.3}$$

$$\dim G(X) \leq \dim K_2 \dim X, \quad \textit{for all } X \in \operatorname{mod} A, \tag{6.4}$$

$$\dim R^1 G(X) \leq K_4 \dim X, \quad \textit{for all } X \in \operatorname{mod} A, \tag{6.5}$$

where R^1G is the first right derived functor of G. If $\mathcal{N} \subseteq \operatorname{mod} A$ is a weakly hyperfinite family, then the family $\mathcal{M} = \{G(X)\colon X \in \mathcal{N}\} \subseteq \operatorname{mod} B$ is also weakly hyperfinite.

Proof. By the hypothesis, for every $\tilde{\varepsilon} > 0$ we can find some $L_{\tilde{\varepsilon}}^{\mathcal{N}} > 0$ to exhibit the weak hyperfiniteness of the family $\mathcal{N}$. Let $N \in \mathcal{N}$, we want to construct a homomorphism showing the weak hyperfiniteness of $G(N)$. To this end let $\theta\colon P \to N$ be such that $P = \bigoplus_{i=1}^{t} P_i$ with $\dim P_i \leq L_{\tilde{\varepsilon}}^{\mathcal{N}}$, $\dim \ker \theta \leq \tilde{\varepsilon} \dim N$ and $\dim \operatorname{coker} \theta \leq \tilde{\varepsilon} \dim N$. We consider the short exact sequences

$$\begin{aligned} \epsilon\colon 0 \to \ker \theta \xrightarrow{\alpha} P \xrightarrow{\beta} \operatorname{im} \theta \to 0, \\ \eta\colon 0 \to \operatorname{im} \theta \xrightarrow{\gamma} N \xrightarrow{\delta} \operatorname{coker} \theta \to 0. \end{aligned}$$

We have $\theta = \gamma \circ \beta$, so $G(\theta) = G(\gamma) \circ G(\beta)$. Applying G, we obtain the exact sequences

$$\begin{aligned} \epsilon'\colon 0 \to G(\ker \theta) \xrightarrow{G(\alpha)} G(P) \xrightarrow{G(\beta)} G(\operatorname{im} \theta) \xrightarrow{\epsilon^*} R^1G(\ker \theta), \\ \eta'\colon 0 \to G(\operatorname{im} \theta) \xrightarrow{G(\gamma)} G(N) \xrightarrow{G(\delta)} G(\operatorname{coker} \theta) \xrightarrow{\eta^*} R^1G(\operatorname{im} \theta). \end{aligned}$$

Now, $G(\ker \theta)$ is a kernel of $G(\beta)$. Moreover,

$$\ker G(\theta) = \ker\bigl(G(\gamma) \circ G(\beta)\bigr) = G(\beta)^{-1}(\ker G(\gamma)) = \ker G(\beta),$$

as $G(\gamma)$ is a monomorphism. Thus

$$\dim \ker G(\theta) = \dim G(\ker \theta) \leq K_2 \dim \ker \theta \leq K_2 \tilde{\varepsilon} \dim N \leq \frac{K_2}{K_1} \tilde{\varepsilon} \dim G(N).$$

On the other hand,

$$\begin{aligned} \dim \operatorname{coker} G(\theta) &= \dim G(N) - \dim \operatorname{im} G(\theta) \\ &= \dim G(N) - \dim \operatorname{im} G(\gamma) + \dim \operatorname{im} G(\gamma) - \dim\bigl(G(\gamma) \circ G(\beta)\bigr)(G(P)) \\ &= \dim \operatorname{coker} G(\gamma) + \dim G(\operatorname{im} \theta) - \dim \operatorname{im} G(\beta), \end{aligned}$$

as $G(\gamma)$ is a monomorphism. Now, by the exactness of η' at $G(N)$ respectively ϵ' at $G(\operatorname{im}\theta)$, we have

$$\begin{aligned}\dim\operatorname{coker} G(\theta) &= \dim G(N) - \dim\ker G(\delta) + \dim G(\operatorname{im}\theta) - \dim\ker\epsilon^* \\ &= \dim\operatorname{im} G(\delta) + \dim\operatorname{im}\epsilon^* \leq \dim G(\operatorname{coker}\theta) + \dim R^1G(\ker\theta) \\ &\leq K_2\tilde{\varepsilon}\dim N + K_4\tilde{\varepsilon}\dim N \leq \frac{K_2+K_4}{K_1}\tilde{\varepsilon}\dim G(N).\end{aligned}$$

Finally, if $\varepsilon > 0$, then by letting $\tilde{\varepsilon} = \varepsilon\frac{K_1}{K_2+K_4}$ and putting $L_\varepsilon = L^{\mathcal{N}}_{\tilde{\varepsilon}}$, we have constructed a map $G(\theta)\colon G(P) \to G(N)$, such that we have $G(P) \cong \bigoplus_{i=1}^t G(P_i)$ by the additivity of G, with $\dim G(P_i) \leq L_\varepsilon$, and we have that $\dim\ker G(\theta) \leq \varepsilon\dim G(N)$ as well as $\dim\operatorname{coker} G(\theta) \leq \varepsilon\dim G(N)$. □

The following generalisation of Proposition 6.15 on preserving non-hyperfiniteness using certain functors is then imminent.

Proposition 6.39. *Let k be a field and $L|k$ a finite field extension. Let A be a finite dimensional L-algebra and let B be a finite dimensional k-algebra. Let $\mathcal{M} = \{M_i\colon i \in I\}$ be family of finite dimensional A-modules which is not weakly hyperfinite. Let $K_1, K_2, K_4 > 0$. If for each $i \in I$ there exist additive functors $F_i\colon \operatorname{mod} A \to \operatorname{mod} B$ and $G_i\colon \operatorname{mod} B \to \operatorname{mod} A$ such that*

- *$G_iF_i(M_i) \cong M_i$ for all $i \in I$,*
- *all G_i are left exact, with right derived functor R^1G_i,*
- *$K_1\dim_k F_i(M_i) \leq \dim_L G_iF_i(M_i)$ for all $i \in I$,*
- *$\dim_L G_i(X) \leq K_2\dim_k X$ for all $X \in \operatorname{mod} B$ and $i \in I$,*
- *$\dim_L R^1G_i(X) \leq K_4\dim_k X$ for all $X \in \operatorname{mod} B$ and $i \in I$,*

then $\{F_i(M_i)\colon i \in I\}$ is not weakly hyperfinite.

Proof. Consider the family $\mathcal{N} = \{F_i(M_i)\colon i \in I\}$ in $\operatorname{mod} B$. Assume that it is weakly hyperfinite. Towards a contradiction we want to show that $\{G_iF_i(M_i)\colon i \in I\}$ is also weakly hyperfinite. For any $\tilde{\varepsilon}$, we can find some $L^{\mathcal{N}}_{\tilde{\varepsilon}} > 0$ to exhibit the hyperfiniteness of the family $\mathcal{N}$. Let $M = M_i$ for some $i \in I$. Denote $N = F_i(M_i)$. We can find a homomorphism $\theta\colon P \to N$ such that $P = \bigoplus_{j=1}^t P_j$ with $\dim_k P_j \leq L^{\mathcal{N}}_{\tilde{\varepsilon}}$ and $\dim_k\ker\theta \leq \tilde{\varepsilon}\dim_k N$ as well as $\dim_k\operatorname{coker}\theta \leq \tilde{\varepsilon}\dim_k N$. Since all G_i are additive, we have that $G_i(P) = \bigoplus_{j=1}^t G_i(P_j)$, and

$$\dim_L G_i(P_j) \leq K_2\dim_k P_j \leq K_2 L^{\mathcal{N}}_{\tilde{\varepsilon}}.$$

Moreover, the sequences

$$\begin{aligned}\epsilon'\colon 0 \to G_i(\ker\theta) &\xrightarrow{G(\alpha)} G_i(P) \xrightarrow{G(\beta)} G_i(\operatorname{im}\theta) \xrightarrow{\epsilon^*} R^1G_i(\ker\theta), \\ \eta'\colon 0 \to G_i(\operatorname{im}\theta) &\xrightarrow{G(\gamma)} G_i(N) \xrightarrow{G(\delta)} G_i(\operatorname{coker}\theta) \xrightarrow{\eta^*} R^1G_i(\operatorname{im}\theta),\end{aligned}$$

are exact. We see that $G_i(\ker\theta) = \ker G_i(\theta)$, since $G_i(\operatorname{im}\theta) \to G_i(N)$ is a monomorphism, showing that

$$\begin{aligned}\dim_L \ker G(\theta) &= \dim_L G(\ker\theta) \leq K_2 \dim_k \ker\theta \leq K_2\tilde{\varepsilon}\dim_k N \\ &= K_2\tilde{\varepsilon}\dim_k F_i(M_i) \leq \frac{K_2}{K_1}\tilde{\varepsilon}\dim_L G_i(N) = \frac{K_2}{K_1}\tilde{\varepsilon}\dim_L M.\end{aligned}$$

Also,

$$\begin{aligned}\dim_L \operatorname{coker} G_i(\theta) &= \dim_L G_i(N) - \dim_L \operatorname{im} G_i(\theta) \\ &= \dim_L G_i(N) - \dim_L \operatorname{im} G_i(\gamma) + \dim_L \operatorname{im} G_i(\gamma) \\ &\qquad - \dim_L\big(G_i(\gamma)\circ G_i(\beta)\big)(G_i(P)) \\ &= \dim_L \operatorname{coker} G_i(\gamma) + \dim_L G_i(\operatorname{im}\theta) - \dim_L \operatorname{im} G_i(\beta),\end{aligned}$$

as $G_i(\operatorname{im}\theta) \to G_i(N)$ is a monomorphism. Now, by the exactness of of the sequences ϵ' and η' above, we have

$$\begin{aligned}\dim_L \operatorname{coker} G_i(\theta) &= \dim_L G_i(N) - \dim_L \ker G_i(\delta) + \dim_L G_i(\operatorname{im}\theta) - \dim_L \ker\epsilon^* \\ &= \dim_L \operatorname{im} G_i(\delta) + \dim_L \operatorname{im}\epsilon^* \leq \dim_L G_i(\operatorname{coker}\theta) + \dim_L R^1G_i(\ker\theta) \\ &\leq K_2\dim_k \operatorname{coker}\theta + K_4\dim_k\ker\theta \leq (K_2+K_4)\tilde{\varepsilon}\dim_k N \\ &\leq \frac{K_2+K_4}{K_1}\tilde{\varepsilon}\dim_L G_i(N) = \frac{K_2+K_4}{K_1}\tilde{\varepsilon}\dim_L M.\end{aligned}$$

This shows that $G_i(\theta)\colon G_i(P) \to G_i(N) \cong G_i(F_i(M)) \cong M$ fulfils

$$\dim_L \ker G_i(\theta) \leq \varepsilon\dim_L M,\ \dim_L \operatorname{coker} G_i(\theta) \leq \varepsilon\dim_L M,$$

if we chose $\tilde{\varepsilon} = \varepsilon\frac{K_1}{K_2+K_4}$, while $G_i(P) = \bigoplus_{j=1}^t G_i(P_j)$ and $\dim_L G_i(P_j) \leq K_2 L_{\tilde{\varepsilon}}^{\mathcal{N}} =: L_\varepsilon$. But this is a contradiction, since by the hypothesis, $\mathcal{M}$ is not weakly hyperfinite. Thus $\{F_i(M_i)\colon i\in I\}$ cannot be weakly hyperfinite. ☐

We next turn to the relation between dimension expanders and examples of non-hyperfinite families and generalise this result to families which are not weakly hyperfinite.

Proposition 6.40. *Let k be a field, $d\in\mathbb{N}$ and $\eta,\alpha > 0$. If $\{(V_i, \{T_l^{(i)}\}_{l=1}^d)\}_{i\in I}$ is a family of (η,α)-dimension quasi-expanders of degree d such that $\dim V_i$ is unbounded, then the induced family of $k\Theta(d)$-modules*

$$\begin{tikzcd} M_i = V_i \arrow[r, bend left, "T_1^{(i)}"] \arrow[r, bend right, "T_d^{(i)}"'] \arrow[r, phantom, "\vdots"] & V_i \end{tikzcd}$$

is not weakly hyperfinite.

Proof. Assume to the contrary that the induced family $\mathcal{M} = \{M_i : i \in I\}$ was weakly hyperfinite. Let $\varepsilon > 0$. Then there exists $L_\varepsilon > 0$ such that for all $M \in \mathcal{M}$ there exists $\theta : P \to M$ such that $\dim \ker \theta \leq \varepsilon \dim M$, $\dim \operatorname{coker} \theta \leq \varepsilon \dim M$ and $P = \bigoplus_{j=1}^{s} P_j$ where $\dim P_j \leq L_\varepsilon$. For some (α, η)-quasi dimension expander $(V, \{T_\ell\})$ of degree d, such that $\dim V \geq \frac{1}{\eta} L_\varepsilon$, consider

$$M = V \underset{\cdots}{\overset{\{T_\ell\}}{\longrightarrow}} V.$$

We have $\theta_j : P_j \xrightarrow{\iota_j} \bigoplus P_j \xrightarrow{\theta} M$. Then

$$\dim \left(\theta_j(P_j)\right)(1) \leq \dim \theta_j(P_j) \leq \dim P_j \leq L_\varepsilon < \eta \dim V,$$

and $\left(\theta_j(P_j)\right)(1)$ is a subspace of V. Moreover, $\theta_j(P_j)$ is a $k\Theta(d)$-module. It follows that

$$\sum_{\ell=1}^{d} T_\ell\big((\theta_j(P_j))(1)\big) \subseteq \left(\theta_j(P_j)\right)(2).$$

By the expander property, we hence have

$$\dim \left(\theta_j(P_j)\right)(2) \geq \dim \sum_{\ell=1}^{d} T_\ell\left((\theta_j(P_j))(1)\right) \geq (1+\alpha) \dim \left(\theta_j(P_j)\right)(1). \tag{6.6}$$

As $0 \to \operatorname{im} \theta \to M \to \operatorname{coker} \theta \to 0$ is exact, we have that

$$\dim \operatorname{im} \theta = \dim M - \dim \operatorname{coker} \theta \geq (1-\varepsilon) \dim M = 2(1-\varepsilon) \dim V.$$

Also,

$$\begin{aligned} \dim \operatorname{im} \theta &\leq \sum_{j=1}^{s} \left[\dim \left(\theta_j(P_j)\right)(1) + \dim \left(\theta_j(P_j)\right)(2)\right] \\ &\overset{(6.6)}{\leq} \sum_{j=1}^{s} \left[\left(1 + \frac{1}{1+\alpha}\right) \dim \left(\theta_j(P_j)\right)(2)\right] = \left(1 + \frac{1}{1+\alpha}\right) \sum_{j=1}^{s} \dim \left(\theta_j(P_j)\right)(2). \end{aligned}$$

Next, note that

$$0 \to \ker \theta(2) \hookrightarrow \left(\bigoplus_{j=1}^{s} P_j\right)(2) \xrightarrow{(\,\theta_1(2)\ \dots\ \theta_s(2)\,)} V$$

is exact, showing that

$$\dim \bigoplus_{j=1}^{s} P_j(2) \leq \dim \ker \theta(2) + \dim V.$$

We have that $0 \to \ker \theta_j \to P_j \to \theta_j(P_j) \to 0$ is exact, so

$$\dim \theta_j(P_j) = \dim P_j - \dim \ker \theta_j \leq \dim P_j,$$

with the analogue statement holding for each vertex. Henceforth,

$$\begin{aligned}
2(1-\varepsilon)\dim V &\le \dim\operatorname{im}\theta \le \left(1+\frac{1}{1+\alpha}\right)\sum_{j=1}^{s}\dim\theta_j(P_j)(2)\\
&\le \left(1+\frac{1}{1+\alpha}\right)\sum_{j=1}^{s}\dim P_j(2) \le \left(1+\frac{1}{1+\alpha}\right)(\dim V+\dim\ker\theta(2))\\
&\le \left(\frac{2+\alpha}{1+\alpha}\right)(1+2\varepsilon)\dim V\\
\Leftrightarrow 2-2\varepsilon &\le \left(\frac{2+\alpha}{1+\alpha}\right)(1+2\varepsilon) \Leftrightarrow 2-\left(\frac{2+\alpha}{1+\alpha}\right)(1+2\varepsilon)\le 2\varepsilon\\
\Leftrightarrow \varepsilon &\ge \frac{2+2\alpha-(2+\alpha)(1+2\varepsilon)}{2(1+\alpha)} = \frac{\alpha-4\varepsilon-2\varepsilon\alpha}{2(1+\alpha)}\\
\Leftrightarrow \frac{\alpha}{2(1+\alpha)} &\le \varepsilon+\frac{2\varepsilon+\varepsilon\alpha}{1+\alpha} = \varepsilon\left(1+\frac{2+\alpha}{1+\alpha}\right) = \varepsilon\left(\frac{1+\alpha+2+\alpha}{1+\alpha}\right)\\
\Leftrightarrow \varepsilon &\ge \frac{\alpha}{2(1+\alpha)}\frac{1+\alpha}{3+2\alpha} = \frac{\alpha}{6+4\alpha} > 0,
\end{aligned}$$

contradicting the weak hyperfiniteness of $\mathcal{M}$. □

Corollary 6.41. *Let k be any field. Then there exists some $d \geq 3$ such that $k\Theta(d)$ is not of weak amenable representation type.*

Remark. It follows from the last two results that all strictly wild finite dimensional k-algebras are not even weakly amenable: We modify the proof of Theorem 6.30 by using Propositions 6.39 and 6.40 instead of Propositions 6.15 and 6.3.

Now we recall yet a different notion of wildness, extending strict wildness, originally due to Ringel [Rin02].

Definition 6.42. [Han01, Definition 2.1; GP16, Section 4] An algebra A is **controlled wild** and **controlled by $\mathcal{C}$** provided there exist a faithful exact functor

$$F\colon \operatorname{mod} k\langle x,y\rangle \to \operatorname{mod} A,$$

and a full subcategory $\mathcal{C}$ of $\operatorname{mod} A$ which is closed under direct sums and direct summands such that for any X and Y in $\operatorname{mod} k\langle x,y\rangle$,

$$\operatorname{Hom}_A(F(X),F(Y)) = F\left(\operatorname{Hom}_{k\langle x,y\rangle}(X,Y)\right)\oplus \operatorname{Hom}_A(F(X),F(Y))_{\mathcal{C}},$$

and

$$\operatorname{Hom}_A(F(X),F(Y))_{\mathcal{C}} \subseteq \operatorname{rad}\operatorname{Hom}_A(F(X),F(Y)).$$

What is more, we say that A is **finitely controlled wild** if it is controlled by $\operatorname{add}(C)$ for some $C \in \operatorname{mod} A$.

Here, $\operatorname{Hom}_A(X,Y)_{\mathcal{C}}$ denotes the set of those A-homomorphisms $X \to Y$ factoring through $\mathcal{C}$.

Remark. Strictly wild algebras are controlled wild and controlled by add(0). Further, all wild algebras with radical square zero (see [Han01, Theorem 4.2]) and all wild local algebras (see [Han01, Theorem 4.4]) are finitely controlled wild.

It has been asked in [Rin02, Problem 10] whether all wild algebras are controlled wild. This result was announced by Y. Drozd in 2007 (even for finitely controlled), but has not yet been published.

We now conclude this chapter by showing that finitely controlled wild algebras are not weakly amenable.

Theorem 6.43. *Let $k = \bar{k}$ be an algebraically closed field and let A be a finite dimensional k-algebra. If A is finitely controlled wild, then A is not of weakly amenable representation type.*

Proof. Let d be as in Corollary 6.41. Since A is finitely controlled wild, by [Han01, Lemma 2.4] there is a faithful and exact functor $F\colon \operatorname{mod} k\Theta(d) \to \operatorname{mod} A$, which is a finitely controlled representation embedding in the sense of [GP16, Section 4] controlled by a full subcategory $\mathcal{C} = \operatorname{add}(C)$ for some $C \in \operatorname{mod} A$. Then by [GP16, Theorem 4.2], there is a functor $G\colon \operatorname{mod} A \to \operatorname{mod} k\Theta(d)$ such that $(G \circ F)(M) \cong M$ for all $M \in \operatorname{mod} k\Theta(d)$. Let us denote by $K = {}_{k\Theta(d)}k\Theta(d)$ the Kronecker algebra as a left module over itself. Now, the functor G is given on the objects by

$$G(X) = {}^{\operatorname{Hom}_A(F(K),X)}\!\big/\!{}_{\operatorname{Hom}_A(F(K),X)_{\mathcal{C}}}.$$

We can find a $\mathcal{C}$-preenvelope of $F(K)$,

$$\Delta\colon F(K) \to \mathcal{C}_{F(K)} \cong C^n,$$

where $n = \dim_k \operatorname{Hom}_A(F(K), C)$, and we have

$$\operatorname{Hom}(\Delta, -)\colon \operatorname{Hom}_A(C^n, -) \to \operatorname{Hom}_A(F(K), -), \quad f \mapsto f \circ \Delta.$$

We note that $\operatorname{Hom}_A(\Delta, C')$ is surjective for each $C' \in \mathcal{C}$ and that every morphism in the image factors through $\mathcal{C}$. It follows that G is the cokernel functor of $\operatorname{Hom}_A(\Delta, -)$, and for all $X \in \operatorname{mod} A$ we have that

$$0 \to \operatorname{Hom}_A(F(K), X)_{\mathcal{C}} = \operatorname{im}\operatorname{Hom}_A(\Delta, X) \hookrightarrow \operatorname{Hom}_A(F(K), X) \twoheadrightarrow G(X) \to 0. \tag{6.7}$$

Assume that $\operatorname{mod} A$ was weakly hyperfinite. Then, for every $\tilde{\varepsilon} > 0$, there exists $L_{\tilde{\varepsilon}}^{\operatorname{mod} A} > 0$ fulfilling the usual conditions. Hence, given $M \in \operatorname{mod} k\Theta(d)$, we can find $\vartheta\colon N \to F(M)$ in $\operatorname{mod} A$ such that $N \cong \bigoplus_{i=1}^{s} N_i$, with $\dim N_i \le L_{\tilde{\varepsilon}}^{\operatorname{mod} A}$ and $\ker\vartheta, \operatorname{coker}\vartheta \le \tilde{\varepsilon} \dim F(M)$. We shall also consider the exact sequences

$$\epsilon\colon 0 \to \ker\vartheta \xrightarrow{\alpha} N \xrightarrow{\beta} \operatorname{im}\vartheta \to 0,$$
$$\eta\colon 0 \to \operatorname{im}\vartheta \xrightarrow{\gamma} M \xrightarrow{\delta} \operatorname{coker}\vartheta \to 0.$$

Connecting two sequences of type (6.7) by $b = \operatorname{Hom}(F(K), \vartheta)$, via an application of the Snake Lemma, we get the following commutative diagram.

$$\begin{tikzcd}
0 \arrow[r, dashed] & \ker a \arrow[r, dashed] \arrow[d] & \ker b \arrow[r, dashed] \arrow[d] & \ker G(\vartheta) \arrow[d] \arrow[dddll, dashed, rounded corners, "d"] & \\
0 \arrow[r] & \operatorname{im}(\Delta, N) \arrow[r, hook, "f"] \arrow[d, "a"] & (F(K), N) \arrow[r, two heads, "g"] \arrow[d, "b"] & G(N) \arrow[r] \arrow[d, "G(\vartheta)"] & 0 \\
0 \arrow[r] & \operatorname{im}(\Delta, F(M)) \arrow[r, hook, "f'"] \arrow[d] & (F(K), F(M)) \arrow[r, two heads, "g'"] \arrow[d] & G(F(M)) \arrow[r] \arrow[d] & 0 \\
 & \operatorname{coker} a \arrow[r, dashed] & \operatorname{coker} b \arrow[r, dashed] & \operatorname{coker} G(\vartheta) \arrow[r, dashed] & 0.
\end{tikzcd}$$

We want to find bounds on $\dim \ker G(\vartheta)$ and $\dim \operatorname{coker} G(\vartheta)$. We do have that

$$\begin{aligned}
\dim \ker G(\vartheta) &= \dim \ker d + \dim \operatorname{im} d \leq \dim \ker b + \dim \operatorname{coker} a, \text{ and} \\
\dim \operatorname{coker} G(\vartheta) &\leq \dim \operatorname{coker} b.
\end{aligned}$$

If $\varphi \in \ker b$, then $\varphi\colon F(K) \to N$ is such that $\vartheta \circ \varphi = 0$ as a map $F(K) \to F(M)$, and by the universal property of the kernel, we have that there is a unique $\varphi'\colon F(K) \to \ker \varphi$ such that $\operatorname{Ker} \varphi \circ \varphi' = \varphi$, showing that $\operatorname{Hom}(F(K), \ker \varphi) \twoheadrightarrow \ker b$, hence

$$\dim \ker b \leq \dim \operatorname{Hom}\left(F(K), \ker \varphi\right).$$

Since $\operatorname{im} b \subseteq \operatorname{im} \operatorname{Hom}(F(K), \gamma)$, $\operatorname{coker} b$ has as submodule $\operatorname{im} \operatorname{Hom}(F(K), \gamma)/\operatorname{im} b$. As a vector space, the quotient by this submodule is

$$\begin{aligned}
\left.{}^{\operatorname{Hom}(F(K), F(M))}\middle/{}_{\operatorname{im} b}\right. \Big/ {}_{\operatorname{im}\operatorname{Hom}(F(K),\gamma)/\operatorname{im} b} &\cong {}^{\operatorname{Hom}(F(K), F(M))}/{}_{\operatorname{im}\operatorname{Hom}(F(K),\gamma)} \\
&= \operatorname{coker} \operatorname{Hom}_A(F(K), \gamma).
\end{aligned}$$

It now follows that

$$\begin{aligned}
\dim \text{quotient} &= \dim(F(K), F(M)) - \dim \operatorname{im}(F(K), \gamma) \\
&= \dim(F(K), F(M)) - \dim \ker(F(K), \delta) \\
&= \dim \operatorname{im}(F(K), \delta) \leq \dim \operatorname{Hom}_A(F(K), \operatorname{coker} \vartheta),
\end{aligned}$$

using the left exactness of $\operatorname{Hom}_A(F(K), -)$. On the other hand, as we recall that $\epsilon^*\colon (F(K), \operatorname{im} \vartheta) \to {}^1(F(K), \ker \vartheta)$,

$$\begin{aligned}
\dim \text{submodule} &= \dim \operatorname{im}(F(K), \gamma) - \dim \operatorname{im} b \\
&= \dim \operatorname{im}(F(K), \gamma) - \dim \operatorname{im}\left((F(K), \gamma) \circ (F(K), \beta)\right) \\
&= \dim(F(K), \operatorname{im} \vartheta) - \dim \operatorname{im}(F(K), \beta) = \dim(F(K), \operatorname{im} \vartheta) - \dim \ker \epsilon^* \\
&= \dim \operatorname{im} \epsilon^* \leq \dim \operatorname{Ext}^1_A(F(K), \ker \vartheta),
\end{aligned}$$

as $\operatorname{Hom}(F(K), \gamma)$ is a monomorphism. Combining these two inequalities shows that

$$\dim \operatorname{coker} b \leq \dim \operatorname{Hom}_A(F(K), \operatorname{coker} \vartheta) + \dim \operatorname{Ext}^1_A(F(K), \ker \vartheta).$$

We are left to deal with $\operatorname{coker} a$. Here, we consider yet another diagram with exact rows which we complete again by an application of the Snake Lemma.

$$\begin{tikzcd}
0 \arrow[r, dashed] & \ker\sigma \arrow[r, dashed] \arrow[d] & \ker\tau \arrow[r, dashed] \arrow[d] & \ker a \arrow[d] \arrow[dddll, dashed, rounded corners] & \\
0 \arrow[r] & \ker(\Delta, N) \arrow[r, hook] \arrow[d, "\sigma"] & (C^n, N) \arrow[r, two heads] \arrow[d, "\tau = (C^n, \vartheta)"] & \operatorname{im}(\Delta, N) \arrow[r] \arrow[d, "a"] & 0 \\
0 \arrow[r] & \ker(\Delta, F(M)) \arrow[r, hook] \arrow[d] & (C^n, F(M)) \arrow[r, two heads] \arrow[d] & \operatorname{im}(\Delta, F(M)) \arrow[r] \arrow[d] & 0 \\
& \operatorname{coker}\sigma \arrow[r, dashed] & \operatorname{coker}\tau \arrow[r, dashed] & \operatorname{coker} a \arrow[r, dashed] & 0.
\end{tikzcd}$$

As above, $\dim \operatorname{coker} a \leq \dim \operatorname{coker} \tau$. As $\tau = \operatorname{Hom}_A(C^n, \vartheta)$, we follow the same line of argument as before to show that

$$\dim \operatorname{coker} \tau \leq \dim \operatorname{Hom}_A(C^n, \operatorname{coker} \vartheta) + \dim \operatorname{Ext}^1_A(C^n, \ker \vartheta).$$

Now consider that given $X \in \operatorname{mod} A$, there are $m \in \mathbb{N}$ and $Y_X = \ker(A^m \twoheadrightarrow X)$ such that $0 \to Y_X \to A^m \to X \to 0$ is exact. Applying $\operatorname{Hom}_A(-, \ker \vartheta)$ then gives the exact sequence

$$0 \to (X, \ker \vartheta) \to (A^m, \ker \vartheta) \to (Y_X, \ker \vartheta) \to {}^1(X, \ker \vartheta) \to 0,$$

from which we deduce that

$$\begin{aligned}
\dim \operatorname{Ext}^1_A(X, \ker \vartheta) &= \dim \operatorname{Hom}_A(X, \ker \vartheta) + \dim \operatorname{Hom}_A(Y_X, \ker \vartheta) \\
&\quad - m \dim \operatorname{Hom}_A(A, \ker \vartheta) \\
&\leq (\dim X + \dim Y_X) \dim \ker \vartheta.
\end{aligned}$$

All in all, it follows that

$$\begin{aligned}
\dim \operatorname{coker} G(\vartheta) &\leq \dim \operatorname{coker} b \leq \dim \operatorname{Hom}_A(F(K), \operatorname{coker} \vartheta) + \dim \operatorname{Ext}^1_A(F(K), \ker \vartheta) \\
&\leq \dim F(K) \tilde{\varepsilon} \dim F(M) + \Big(\dim Y_{F(K)} + \dim F(K)\Big) \tilde{\varepsilon} \dim F(M) \\
&\leq \dim F(K) \Big(2 \dim F(K) + \dim Y_{F(K)}\Big) \tilde{\varepsilon} \dim M.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\dim \ker G(\vartheta) &\leq \ker b + \dim \operatorname{coker} a \\
&\leq \dim \operatorname{Hom}_A(F(K), \ker \vartheta) + \dim \operatorname{Hom}_A(C^n, \operatorname{coker} \vartheta) \\
&\qquad + \dim \operatorname{Ext}^1_A(C^n, \ker \vartheta) \\
&\leq (\dim F(K) + 2n \dim C + \dim Y_{C^n}) \, \tilde{\varepsilon} \dim F(M) \\
&\leq \dim F(K) \Big(\dim F(K) + 2 \dim F(K) (\dim C)^2 + \dim Y_{C^n}\Big) \tilde{\varepsilon} \dim M.
\end{aligned}$$

To conclude the proof, we now choose

$$\tilde{\varepsilon} = \varepsilon\Big(\dim F(K) \cdot \Big(\dim F(K) + 2\dim F(K)(\dim C)^2 + \dim Y_{F(K)} + \dim Y_{C^n}\Big)\Big)^{-1}$$

and put $L_\varepsilon = \dim F(K) L_{\tilde{\varepsilon}}^{\operatorname{mod} A}$. Note that $\tilde{\varepsilon}$ depends only on properties of A and its controlled wildness. Then we have

$$G(\vartheta)\colon G(N) \to G(F(M)) \cong M,$$

such that $G(N) \cong \bigoplus_{i=1}^{s} G(N_i)$ by the addivity of G, with

$$\begin{aligned}\dim_k G(N_i) &\leq \dim_k \operatorname{Hom}_A(F(K), N_i) - \dim_k \operatorname{Hom}_A(F(K), N_i)_{\mathcal{C}} \\ &\leq \dim_k F(K) \dim_k N_i \leq L_\varepsilon,\end{aligned}$$

and

$$\dim_k \ker G(\vartheta),\ \dim_k \operatorname{coker} G(\vartheta) \leq \varepsilon \dim_k M.$$

This shows that $\operatorname{mod} k\Theta(d)$ is weakly hyperfinite, a contradiction to Corollary 6.41. Hence, $\operatorname{mod} A$ cannot be weakly hyperfinite, so A is not of weakly amenable representation type. ☐

Corollary 6.44. *Let $k = \bar{k}$ be an algebraically closed field and let A be a finite dimensional k-algebra. If A is finitely controlled wild, then A is not of amenable representation type.*

Proof. Apply Proposition 6.37 and Theorem 6.43. ☐

Further research suggestions and outlook

Self-duality. Elek [Ele17] introduced the notion of hyperfiniteness and amenability for finite dimensional algebras and we worked with a conjecture based on these notions in this thesis. Yet, the notions itself elicit further studies. It is a natural question to ask whether the notion is self-dual, that is, if we can we check that a family $\mathcal{M}$ of modules is hyperfinite by providing quotient modules nearly the same size instead of submodules. To formalise this, we have the following.

Question. Given a family $\mathcal{M}$ of A-modules such that for every $\varepsilon > 0$, there exists L_ε such that for every $M \in \mathcal{M}$ there exists a quotient module $M \twoheadrightarrow P$ such that

$$\dim_k P \geq (1 - \varepsilon) \dim_k M,$$

and modules $P_1, \dots, P_t \in \operatorname{mod} A$, with $\dim_k P_i \leq L_\varepsilon$, such that $P \cong \bigoplus_{i=1}^t P_i$, is $\mathcal{M}$ hyperfinite?

We did not succeed in proving this. What is more, preserving such a notion in the spirit of Proposition 1.5 would require right-exact functors instead of left-exact functors. Even our weak notion need not be self-dual: It asks for a morphism $\theta \colon N \to M$ but no morphism with domain M. Moreover, preserving weak hyperfiniteness by Proposition 6.38 is not symmetric in this matter: We only require a condition on the right derived functor but not on the left derived functor.

Generalisations A further direction of study would be to generalise the notion of hyperfiniteness. Elek ibid. gives a version for countable dimensional algebras by requiring that the "subobjects" are only nearly submodules, that is, are closed under the action of s generators, where $s = \left\lceil \frac{1}{\varepsilon} \right\rceil$ is the integral part of the reciprocal of ε.

Instead of module categories of finite dimensional algebras, it also seems plausible to also apply the notion to Abelian length categories. Here, the manifest adaptation is to replace the k-dimension of a module by the length of an object. In this way, for module categories of finite dimensional algebras, we would recover the original notion.

A more interesting problem is to adapt the notions to triangulated categories. Yet, it is unclear what should replace the dimension and what notion of length could be used.

Question. How should the notion of hyperfinite families of modules be generalised to objects of a triangulated category?

Such a generalisation would enable us to ask whether the (bounded) derived category of a tame (hereditary) algebra is of amenable type—in a suitable sense. This further allows to discuss the question if amenability is preserved under derived equivalences.

Amenability and classification results As Elek has formulated his conjecture of equivalence of amenable and tame representation type in full generality, a search for strategies to prove the positive part without the need of working along known classifications of tame algebras is desirable. Possibly methods from the proof of the tame–wild dichotomy for algebraically closed fields can be used here. Given a non-hyperfinite family $\mathcal{M}$, a different approach is the construction of a functor with essential image in $\mathcal{M}$ that exhibits wildness.

On the other hand, if one is willing to continue with classes of algebras where a classification of modules is known or within reach, we would like to further understand the (sub)module structure of tubular (canonical) algebras and prove that they are of amenable representation type. Existing strategies from [DMM14a] might be adapted to see how to lift hyperfiniteness from integral slope to arbitrary slope.

Similarly, following [Rin90; Len96; LP99; Kus00], one can define canonical and tubular canonical algebras for non-algebraically closed fields. It would be interesting to see whether hyperfiniteness results hold there as well.

We also recall that the amenability of special biserial algebra can be proved by adapting Elek's proof for string algebras mutatis mutandis.

Theorem. *Let A be a special biserial algebra. Then A is of amenable representation type.*

It is therefore natural to ask if positive results for amenability extend for instance to clannish algebras.

Understanding submodule lattices Related to checking for amenability is understanding the submodule lattices of indecomposable modules. Unfortunately, this readily becomes a wild problem. Yet, as was the case for the motivating question of this thesis, partial results can also be of interest.

Related notions include the Gabriel–Roiter measure and Gabriel–Roiter inclusions. In particular, we would like to study connection to the following results from Ringel [Rin10]. Here, let p and q be the maximal lengths of an indecomposable projective respectively indecomposable injective module.

Corollary. *Let $X \to Y$ be a Gabriel–Roiter inclusion. Then $|Y| \leq pq|X|$.*

Corollary. *Let M be an indecomposable module and $1 \leq a < |M|$ a natural number. Then there exists an indecomposable submodule M' of M with length in the interval $[a+1, pqa]$.*

Corollary. *Let M be an indecomposable module and assume that all indecomposable proper submodules of M are of length at most b. Then $|M| \leq pqb$.*

Question. Is there a connection between a hyperfinite family $\mathcal{N}$ and $\mathcal{N}$-critical modules?

Limits of hyperfinite families Infinite dimensional modules may appear as limits of unbounded hyperfinite families. Given a hyperfinite family $\mathcal{M}$ such that the dimension of the modules $M \in \mathcal{M}$ is not bounded (otherwise hyperfiniteness would be trivial), we can ask about the limit of this family in the sense of Elek [Ele16, Section 2; Ele17, Section 1]. We expect that this limit is an infinite dimensional module.

Question. What are the limits of the families of preprojective, postinjective or regular (indecomposable) modules for a tame hereditary algebra? Is it plausible to assume that Prüfer, adic or generic modules appear here?

We would also like to study a possible connection to [Ele17, Theorem 2].

Dimension expanders and stability Appearing in the latter half of this thesis, dimension expanders play an important role. Their role in connecting expander results for graphs and representation theory of finite dimensional algebras is an interesting topic to study.

Question. How can dimension expanders be used to define a notion of stability for modules? Can such a notion be used to study modules over algebras of wild type?

Consider the Kronecker algebra

$$k\Theta(3) = k\left(1 \substack{\rightarrow \\ \rightarrow \\ \rightarrow} 2\right).$$

Let us define a character $\theta\colon \mathbb{Z}^2 \to \mathbb{R}, x \mapsto x_2 - x_1$, a slope $\mu\colon \mathbb{Z}^2 \to \mathbb{R}, x \mapsto {}^{\theta(x)}/_{x_1+x_2}$, and say that a module M is (μ, β)-*stable* provided $\mu(\underline{\dim}\, M) = 0$ and for all submodules $0 \neq N \subseteq M$ with $\dim N \leq \frac{1}{2} \dim M$, we have $\mu(\underline{\dim}\ N) > \beta$. Note that dimension expanders give modules with this property. Now, a family of (μ, β)-stable modules with $\beta < 1$ is an example of a family of non-hyperfinite modules.

Question. How can this notion of (μ, β)-stability be modified to also apply to further wild algebras?

Changing perspective, this might suggest that expander graphs give certain families of modules for wild Kronecker quivers. It should be investigated whether there are other combinatorial structures that give similar families for more complicated wild algebras.

Further notions Elek's paper [Ele17] has served as a starting point of this thesis. Yet it contains further, interesting notions. For instance, two modules M and N are said to be ε-*close* provided there are submodules $P \subseteq M$ and $Q \subseteq N$ with $P \cong Q$ such that $\dim P \geq (1 - \varepsilon) \dim M$ and $\dim Q \geq (1 - \varepsilon) \dim N$. This can be understood as a way to "approximate" modules.

Question. Which properties of modules are be preserved by being ε-close?

Additionally, one may want to apply this notion to the study of persistence modules in topological data analysis and persistence homology.

Index

Bibliography

[AF92] Frank W. Anderson and Kent R. Fuller. *Rings and Categories of Modules.* Vol. 13. Grad. Texts Math. Springer, 1992.

[APR79] Maurice Auslander, María Inés Platzeck and Idun Reiten. “Coxeter functors without diagrams”. In: *Trans. Am. Math. Soc.* 250 (1979), pp. 1–46. DOI: `10.2307/1998978`.

[ARS95] Maurice Auslander, Idun Reiten and Sverre O. Smalø. *Representation Theory of Artin Algebras.* Vol. 36. Camb. Stud. Adv. Math. Camb. Univ. Press, 1995. DOI: `10.1017/CBO9780511623608`.

[ASS10] Ibrahim Assem, Daniel Simson and Andrzej Skowroński. *Elements of the Representation Theory of Associative Algebras. Volume 1: Techniques of representation theory.* Vol. 65. Lond. Math. Soc. Stud. Texts. Camb. Univ. Press, 2010. DOI: `10.1017/CBO9780511614309`.

[Bae89] Dagmar Baer. “A note on wild quiver algebras and tilting modules”. In: *Commun. Algebra* 17.3 (1989), pp. 751–757. DOI: `10.1080/00927878908823755`.

[BB80] Sheila Brenner and Michael C. R. Butler. “Generalizations of the Bernstein-Gel’fand-Ponomarev reflection functors”. In: *Representation Theory II (Ottawa, ON, 1979).* Vol. 832. Lect. Notes Math. Springer, 1980, pp. 103–169.

[Ben98] David J. Benson. *Representations and Cohomology. Volume I. Basic representation theory of finite groups and associative algebras.* Vol. 30. Camb. Stud. Adv. Math. Camb. Univ. Press, 1998.

[BGL87] Dagmar Baer, Werner Geigle and Helmut Lenzing. “The preprojective algebra of a tame hereditary Artin algebra”. In: *Commun. Algebra* 15.1-2 (1987), pp. 425–457. DOI: `10.1080/00927878708823425`.

[BHV08] Bachir Bekka, Pierre de la Harpe and Alain Valette. *Kazhdan’s Property (T).* Vol. 11. New Math. Monogr. Camb. Univ. Press, 2008. DOI: `10.1017/CBO9780511542749`.

[Bou09] Jean Bourgain. “Expanders and dimensional expansion”. In: *C. R. Acad. Sci.* 347.7-8 (2009), pp. 357–362. DOI: `10.1016/j.crma.2009.02.009`.

[Bur86] Pierre-François Burgermeister. “Classification des représentations de la double flèche”. In: *Enseign. Math. (2)* 32.3-4 (1986), pp. 199–210. DOI: `10.5169/seals-55086`.

[Che09] Bo Chen. “The Gabriel-Roiter measure for radical-square zero algebras”. In: *Bull. Lond. Math. Soc.* 41.1 (2009), pp. 16–26. DOI: `10.1112/blms/bdn091`.

[Cra91] William Crawley-Boevey. “Regular modules for tame hereditary algebras”. In: *Proc. Lond. Math. Soc. (3)* 62.3 (1991), pp. 490–508. DOI: `10.1112/plms/s3-62.3.490`.

[Cra92] William Crawley-Boevey. “Modules of finite length over their endomorphism rings”. In: *Representations of Algebras and Related Topics (Kyoto, 1990)*. Vol. 168. Lond. Math. Soc. Lect. Note Ser. Camb. Univ. Press, 1992, pp. 127–184.

[DF73] Peter Donovan and Mary Ruth Freislich. *The Representation Theory of Finite Graphs and Associated Algebras.* Vol. 5. Carleton Math. Lect. Notes. Carleton University, 1973.

[DMM10] Piotr Dowbor, Hagen Meltzer and Andrzej Mróz. “An algorithm for the construction of exceptional modules over tubular canonical algebras”. In: *J. Algebra* 323.10 (2010), pp. 2710–2734. DOI: `10.1016/j.jalgebra.2009.12.027`.

[DMM14a] Piotr Dowbor, Hagen Meltzer and Andrzej Mróz. “An algorithm for the construction of parametrizing bimodules for homogeneous modules over tubular canonical algebras”. In: *Algebras Represent. Theory* 17.1 (2014), pp. 357–405. DOI: `10.1007/s10468-013-9430-2`.

[DMM14b] Piotr Dowbor, Hagen Meltzer and Andrzej Mróz. “Parametrizations for integral slope homogeneous modules over tubular canonical algebras”. In: *Algebras Represent. Theory* 17.1 (2014), pp. 321–356. DOI: `10.1007/s10468-012-9386-7`.

[DR76] Vlastimil Dlab and Claus Michael Ringel. “Indecomposable representations of graphs and algebras”. In: *Mem. Am. Math. Soc.* 6.173 (1976). DOI: `10.1090/memo/0173`.

[DR78] Vlastimil Dlab and Claus Michael Ringel. “The representations of tame hereditary algebras”. In: *Representation Theory of Algebras (Philadelphia, PA, 1976)*. Ed. by Robert Gordon. Vol. 37. Lect. Notes Pure Appl. Math. Dekker, 1978, pp. 329–353.

[DS11] Zeev Dvir and Amir Shpilka. “Towards dimension expanders over finite fields”. In: *Combinatorica* 31.3 (2011), pp. 305–320. DOI: `10.1007/s00493-011-2540-8`.

[DSV03] Giuliana Davidoff, Peter Sarnak and Alain Valette. *Elementary Number Theory, Group Theory, and Ramanujan Graphs.* Vol. 55. Lond. Math. Soc. Stud. Texts. Camb. Univ. Press, 2003. DOI: `10.1017/CBO9780511615825`.

[DW05] Harm Derksen and Jerzy Weyman. “Quiver representations”. In: *Not. Am. Math. Soc.* 52.2 (2005), pp. 200–206.

[DW10] Zeev Dvir and Avi Wigderson. "Monotone expanders: constructions and applications". In: *Theory Comput.* 6 (2010), pp. 291–308. DOI: `10.4086/toc.2010.v006a012`.

[EF01] Keith Edwards and Graham Farr. "Fragmentability of graphs". In: *J. Comb. Theory. Ser. B* 82.1 (2001), pp. 30–37. DOI: `10.1006/jctb.2000.2018`.

[Ele06] Gábor Elek. "The amenability and non-amenability of skew fields". In: *Proc. Am. Math. Soc.* 134.3 (2006), pp. 637–644. DOI: `10.1090/S0002-9939-05-08128-1`.

[Ele07] Gábor Elek. "The combinatoral cost". In: *Enseign. Math. (2)* 53.3-4 (2007), pp. 225–235.

[Ele16] Gábor Elek. "Convergence and limits of linear representations of finite groups". In: *J. Algebra* 450 (2016), pp. 588–615. DOI: `10.1016/j.jalgebra.2015.11.023`.

[Ele17] Gábor Elek. "Infinite dimensional representations of finite dimensional algebras and amenability". In: *Math. Ann.* 369.1 (2017), pp. 397–439. DOI: `10.1007/s00208-017-1552-0`.

[EM94] Keith Edwards and Colin McDiarmid. "New upper bounds on harmonious colorings". In: *J. Graph Theory* 18.3 (1994), pp. 257–267. DOI: `10.1002/jgt.3190180305`.

[FH91] William Fulton and Joe Harris. *Representation Theory.* Vol. 129. Grad. Texts Math. Springer, 1991. DOI: `10.1007/978-1-4612-0979-9`.

[Gab72] Peter Gabriel. "Unzerlegbare Darstellungen. I". In: *Manuscr. Math.* 6 (1972), pp. 71–103. DOI: `10.1007/BF01298413`.

[GL87] Werner Geigle and Helmut Lenzing. "A class of weighted projective curves arising in representation theory of finite-dimensional algebras". In: *Singularities, Representation of Algebras, and Vector Bundles (Lambrecht, 1985).* Vol. 1273. Lect. Notes Math. Springer, 1987, pp. 265–297. DOI: `10.1007/BFb0078849`.

[GL91] Werner Geigle and Helmut Lenzing. "Perpendicular categories with applications to representations and sheaves." In: *J. Algebra* 144.2 (1991), pp. 273–343. DOI: `10.1016/0021-8693(91)90107-J`.

[GP16] Lorna Gregory and Mike Prest. "Representation embeddings, interpretation functors and controlled wild algebras". In: *J. Lond. Math. Soc. (2)* 94.3 (2016), pp. 747–766. DOI: `10.1112/jlms/jdw055`.

[GT93] Roderick Gow and Maria C. Tamburini. "Generation of $\mathrm{SL}(n, \mathbf{Z})$ by a Jordan unipotent matrix and its transpose". In: *Linear Algebra Its Appl.* 181 (1993), pp. 63–71. DOI: `10.1016/0024-3795(93)90023-H`.

[Han01] Yang Han. "Controlled wild algebras". In: *Proc. Lond. Math. Soc. (3)* 83.2 (2001), pp. 279–298. DOI: `10.1112/plms/83.2.279`.

[HK16] Andrew Hubery and Henning Krause. “A categorification of non-crossing partitions”. In: *J. Eur. Math. Soc.* 18.10 (2016), pp. 2273–2313. DOI: `10.4171/JEMS/641`.

[Hos84] Mitsuo Hoshino. “Modules without self-extensions and Nakayama’s conjecture”. In: *Arch. Math. (Basel)* 43.6 (1984), pp. 493–500. DOI: `10.1007/BF01190950`.

[HR81] Dieter Happel and Claus Michael Ringel. “Construction of tilted algebras”. In: *Representations of Algebras (Puebla, 1980)*. Vol. 903. Lect. Notes Math. Springer, 1981, pp. 125–144.

[HR82] Dieter Happel and Claus Michael Ringel. “Tilted algebras”. In: *Trans. Am. Math. Soc.* 274.2 (1982), pp. 399–443. DOI: `10.2307/1999116`.

[HV89] Pierre de la Harpe and Alain Valette. “La propriété (T) de Kazhdan pour les groupes localement compacts”. In: *Astérisque* 175 (1989). With an appendix by Marc Burger, p. 158.

[Iva11] Sergeĭ Olegovich Ivanov. “Nakayama functors and Eilenberg-Watts theorems”. In: *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)* 388 (2011), pp. 179–188, 311. Trans. by Natal’ya B. Lebedinskaya. In: *J. Math. Sci. (N.Y.)* 183.5 (2012), pp. 675–680. DOI: `10.1007/s10958-012-0831-2`.

[Kac90] Victor G. Kac. *Infinite-dimensional Lie Algebras.* Camb. Univ. Press, 1990. DOI: `10.1017/CBO9780511626234`.

[Kaž67] David A. Každan. “On the connection of the dual space of a group with the structure of its closed subgroups”. In: *Funkcional. Anal. i Priložen.* 1.1 (1967), pp. 71–74. Trans. by unknown. In: *Funct. Anal. Appl.* 1.1 (1967), pp. 63–65. DOI: `10.1007/BF01075866`.

[Ker88] Otto Kerner. “Preprojective components of wild tilted algebras”. In: *Manuscr. Math.* 61.4 (1988), pp. 429–445. DOI: `10.1007/BF01258598`.

[Ker96] Otto Kerner. “Representations of wild quivers”. In: *Representation Theory of Algebras and Related Topics (Mexico City, 1994)*. Vol. 19. CMS Conf. Proc. Am. Math. Soc., 1996, pp. 65–107.

[Kus00] Dirk Kussin. “On the K-theory of tubular algebras”. In: *Colloq. Math.* 86.1 (2000), pp. 137–152. DOI: `10.4064/cm-86-1-137-152`.

[Len96] Helmut Lenzing. “A K-theoretic study of canonical algebras”. In: *Representation Theory of Algebras (Cocoyoc, 1994)*. Vol. 18. CMS Conf. Proc. Am. Math. Soc., 1996, pp. 433–454. DOI: `10.4064/cm-71-2-161-181`.

[LM93] Helmut Lenzing and Hagen Meltzer. “Sheaves on a weighted projective line of genus one, and representations of a tubular algebra”. In: *Representations of Algebras (Ottawa, ON, 1992)*. Vol. 14. CMS Conf. Proc. Am. Math. Soc., 1993, pp. 313–337.

[LP99] Helmut Lenzing and José Antonio de la Peña. "Concealed-canonical algebras and separating tubular families". In: *Proc. Lond. Math. Soc. (3)* 78.3 (1999), pp. 513–540. DOI: `10.1112/S0024611599001872`.

[Lub94] Alexander Lubotzky. *Discrete Groups, Expanding Graphs and Invariant Measures.* Vol. 125. Prog. Math. With an appendix by Jonathan D. Rogawski. Birkhäuser, 1994. DOI: `10.1007/978-3-0346-0332-4`.

[LZ08] Alexander Lubotzky and Efim Zelmanov. "Dimension expanders". In: *J. Algebra* 319.2 (2008), pp. 730–738. DOI: `10.1016/j.jalgebra.2005.12.033`.

[LZ89] Alexander Lubotzky and Robert J. Zimmer. "Variants of Kazhdan's property for subgroups of semisimple groups". In: *Isr. J. Math.* 66.1-3 (1989), pp. 289–299.

[Mel07] Hagen Meltzer. "Exceptional modules for tubular canonical algebras". In: *Algebras Represent. Theory* 10.5 (2007), pp. 481–496. DOI: `10.1007/s10468-007-9067-0`.

[Moo69] Robert V. Moody. "Euclidean Lie algebras". In: *Can. J. Math.* 21 (1969), pp. 1432–1454. DOI: `10.4153/CJM-1969-158-2`.

[Naz73] Ludmila A. Nazarova. "Representations of quivers of infinite type". In: *Izv. Akad. Nauk SSSR Ser. Mat.* 37.4 (1973), pp. 752–791. Trans. by Geoffrey A. Kandall. In: *Math. USSR-Izv.* 7.4 (Aug. 1973), pp. 749–792. DOI: `10.1070/im1973v007n04abeh001975`.

[Par69] Bodo Pareigis. *Kategorien und Funktoren.* Math. Leitf. Teubner, 1969.

[Rin02] Claus Michael Ringel. "Combinatorial representation theory – History and future". In: *Representations of Algebra. Vol. I, II.* Beijing Norm. Univ. Press, 2002, pp. 122–144.

[Rin10] Claus Michael Ringel. "Gabriel-Roiter inclusions and Auslander-Reiten theory". In: *J. Algebra* 324.12 (2010), pp. 3579–3590. DOI: `10.1016/j.jalgebra.2010.09.003`.

[Rin74] Claus Michael Ringel. "The representation type of local algebras". In: *International Conference on Representations of Algebras (Ottawa, ON, 1974).* Ed. by Vlastimil Dlab and Peter Gabriel. Vol. 9. Carleton Math. Lect. Notes. Carleton University. 1974, Paper No. 22, 24 pp.

[Rin76] Claus Michael Ringel. "Representations of K-species and bimodules". In: *J. Algebra* 41.2 (1976), pp. 269–302. DOI: `10.1016/0021-8693(76)90184-8`.

[Rin79] Claus Michael Ringel. "Infinite dimensional representations of finite dimensional hereditary algebras". In: *Abelian Groups and their Relationship to the Theory of Modules (Rome, 1977).* Vol. XXIII. Symp. Math. Acad. Press, 1979, pp. 321–412.

[Rin84] Claus Michael Ringel. *Tame Algebras and Integral Quadratic Forms.* Vol. 1099. Lect. Notes Math. Springer, 1984.

[Rin88] Claus Michael Ringel. "The regular components of the Auslander-Reiten quiver of a tilted algebra". In: *Chin. Ann. Math. Ser. B* 9.1 (1988), pp. 1–18.

[Rin90] Claus Michael Ringel. "The canonical algebras". In: *Topics in Algebra, Part 1 (Warsaw, 1988).* Vol. 26. Banach Center Publ. With an appendix by William Crawley-Boevey. PWN, 1990, pp. 407–432.

[Rin94] Claus Michael Ringel. "The braid group action on the set of exceptional sequences of a hereditary Artin algebra". In: *Abelian Group Theory and Related Topics (Oberwolfach, 1993).* Ed. by Rüdiger Göbel. Vol. 171. Contemp. Math. Am. Math. Soc., 1994, pp. 339–352. DOI: `10.1090/conm/171/01786`.

[Rin98] Claus Michael Ringel. "Exceptional modules are tree modules". In: *Linear Algebra Its Appl.* 275/276 (1998), pp. 471–493. DOI: `10.1016/S0024-3795(97)10046-5`.

[Sch86] Aidan H. Schofield. "Universal localisation for hereditary rings and quivers". In: *Ring Theory (Antwerp, 1985).* Vol. 1197. Lect. Notes Math. Springer, 1986, pp. 149–164. DOI: `10.1007/BFb0076322`.

[SF73] William A. Simpson and J. Sutherland Frame. "The character tables for SL(3, q), SU(3, q^2), PSL(3, q), PSU(3, q^2)". In: *Can. J. Math.* 25 (1973), pp. 486–494. DOI: `10.4153/CJM-1973-049-7`.

[Sha99] Yehuda Shalom. "Bounded generation and Kazhdan's property (T)". In: *Publ. Math. IHÉS* 90 (1999), pp. 145–168. DOI: `10.1007/BF02698832`.

[Sim03] Daniel Simson. "On large indecomposable modules, endo-wild representation type and right pure semisimple rings". In: *Algebra Discrete Math.* 2 (2003), pp. 93–118.

[Sim05] Daniel Simson. "On Corner type endo-wild algebras". In: *J. Pure Appl. Algebra* 202.1-3 (2005), pp. 118–132. DOI: `10.1016/j.jpaa.2005.01.012`.

[Sim92] Daniel Simson. *Linear Representations of Partially Ordered Sets and Vector Space Categories.* Vol. 4. Algebra Log. Appl. Gordon and Breach Science Publishers, 1992.

[Sim93] Daniel Simson. "On representation types of module subcategories and orders". In: *Bull. Pol. Acad. Sci. Math.* 41.2 (1993), pp. 77–93.

[SS07] Daniel Simson and Andrzej Skowroński. *Elements of the Representation Theory of Associative Algebras. Volume 3: Representation-infinite tilted algebras.* Vol. 72. Lond. Math. Soc. Stud. Texts. Camb. Univ. Press, 2007.

[Str91] Hubertus Strauss. "On the perpendicular category of a partial tilting module". In: *J. Algebra* 144.1 (1991), pp. 43–66. DOI: `10.1016/0021-8693(91)90126-S`.

[Tao15] Terence Tao. *Expansion in Finite Simple Groups of Lie Type*. Vol. 164. Grad. Stud. Math. Am. Math. Soc., 2015.

[Tro62] Stanton M. Trott. “A pair of generators for the unimodular group”. In: *Can. Math. Bull.* 5 (1962), pp. 245–252. DOI: `10.4153/CMB-1962-024-x`.

[Wat60] Charles E. Watts. “Intrinsic characterizations of some additive functors”. In: *Proc. Am. Math. Soc.* 11 (1960), pp. 5–8. DOI: `10.2307/2032707`.

gap code

Let us record here some gap code used to better understand examples for tubular canonical algebras and ways to construct submodules showing hyperfiniteness. We will reproduce the code only for the algebra of type $(3, 3, 3)$, but the other types are similar.

We start by constructing the tubular canonical algebra B of type $\underline{p} = (3, 3, 3)$ with $\lambda = 1$ as well as the underlying tame hereditary algebra A of type $\tilde{E}_6$. Also, we construct functions useful to work with these algebras.

```
LoadPackage("qpa");;
k := Rationals;;

Q := Quiver(7, [ [1,2,"a2"], [2,7,"a1"], [3,4,"b2"], [4,7,"b1"],
 ↪  [5,6,"c2"], [6,7,"c1"] ]);
A := PathAlgebra(k,Q);

pp := [3,3,3];
t := 3;
lambda := [];
lambda[3] := One(k);
p := Lcm(pp);

B := CanonicalAlgebra(k,pp,One(k)*[1]);
QB := QuiverOfPathAlgebra(B);

vertices := VerticesOfQuiver(QB);
vertices0 := ShallowCopy(vertices);;
Remove(vertices0,1);;

verticesoo := ShallowCopy(vertices);;
Remove(verticesoo,Size(vertices));;

Q0 := FullSubquiver(QB,vertices0);
h0 := [0,1,2,1,2,1,2,3];

Qoo := FullSubquiver(QB,verticesoo);
hoo := [3,2,1,2,1,2,1,0];
```

```
verticesKronecker := vertices{[1,Size(vertices)]};
QKronecker := Quiver(List(verticesKronecker,String),[
↪  ["v","w","arm1"], ["v","w","arm2"] ]);
C := PathAlgebra(k,QKronecker);

projA := IndecProjectiveModules(A);
AModule := DirectSumOfQPAModules(projA{[1..7]});
injA := IndecInjectiveModules(A);
simplesA := SimpleModules(A);

projB := IndecProjectiveModules(B);
BModule := DirectSumOfQPAModules(projB{[1..8]});
injB := IndecInjectiveModules(B);
simplesB := SimpleModules(B);


# functions for lists and lists of modules
PositionMaximum := function(list)
        return Position(list,MaximumList(list));
end;

ModuleOfMaximalDimension := function(list)
        return  list[PositionMaximum(List(list,x->Dimension(x)))];
end;

maximalDimension := function(list)
        return MaximumList(List(list,x->Dimension(x)));
end;


# general functions for Artinian algebras resp. hereditary path
↪  algebras of quivers
tau := function(M)
        return DTr(M);
end;

tauinverse := function(M)
        return TrD(M);
end;

middleTerm := function(M)
        return Range(AlmostSplitSequence(M,"l")[1]);
end;
```

```
EulerForm := function(x,y,algebra)
        return x*TransposedMat(CartanMatrix(algebra)^-1)*y;
end;

defectOf := function(x,algebra,minRadicalVector)
        return EulerBilinearFormOfAlgebra(algebra)(minRadicalVector,x);
end;

ModuleExtensions := function(M,N)
        local i,ext,list;
        ext := ExtOverAlgebra(M,N);
        list := [];;
        for i in [1..Size(ext[2])] do
                Add(list,Range(PushOut(ext[1], ext[2][i])[1]));
        od;
        return list;
end;

getAuslanderReitenMiddleTerms := function(M)
        return BlockDecompositionOfModule( Range( AlmostSplitSequence(
         ↪  M )[1] ) );
end;

DualOfTransposeOfModuleHomomorphism := function(h)
        return
         ↪  DualOfModuleHomomorphism(TransposeOfModuleHomomorphism(h));
end;

TransposeOfDualOfModuleHomomorphism := function(h)
        return
         ↪  TransposeOfModuleHomomorphism(DualOfModuleHomomorphism(h));
end;

# functions for nicely constructed tubular canonical algebra B

index := function(x)
        local bf;
        bf := EulerBilinearFormOfAlgebra(B);
        if (bf(h0,x) <> 0 and bf(hoo,x) = 0) then
                return "oo";
        elif (bf(h0,x) = 0 and bf(hoo,x) = 0) then
                return false;
```

```
        else
                return -bf(h0,x)/bf(hoo,x);
        fi;
end;

rank := function(x)
        return x[8]-x[1];
end;

degree := function(x)
        local sum;
        sum := 0;
        sum := sum + p/3 * (x[2]+x[3]);
        sum := sum + p/3 * (x[4]+x[5]);
        sum := sum + p/3 * (x[6]+x[7]);
        sum := sum - p*x[1];
        return sum;
end;

slope := function(x)
        if (rank(x) <> 0) then
                return degree(x)/rank(x);
        else
                return "oo";
        fi;
end;

slopeOfModule := function(M)
        return slope(DimensionVector(M));
end;

i0LinearForm := function(x)
        return EulerBilinearFormOfAlgebra(B)(h0,x);
end;

iooLinearForm := function(x)
        return EulerBilinearFormOfAlgebra(B)(hoo,x);
end;

indexOfModule := function(M)
        return index(DimensionVector(M));
end;
```

```
keyArmElementToVertex := function(i,j)
        if (j = 0) then
                return Sum(pp-1)+2;
        elif (j = pp[i]) then
                return 1;
        else
                return 1+Sum(pp{[1..i]}-1)-(j-1);
        fi;
end;

keyArmElementToMatrix := function(i,j)
        return Sum(pp{[1..i]})-(j-1);
end;

# restrictions and inductions

restriction := function(M)
        local N,mats,newmats,dim,arrow;
        mats := MatricesOfPathAlgebraModule(M);
        newmats := [];
        arrow:=1;
        if IsZero(mats[2]) = false then
                newmats[arrow] := ["a2",mats[2]];
                arrow:=arrow+1;
        fi;
        if IsZero(mats[3]) = false then
                newmats[arrow] := ["a1",mats[3]];
                arrow:=arrow+1;
        fi;
        if IsZero(mats[5]) = false then
                newmats[arrow] := ["b2",mats[5]];
                arrow:=arrow+1;
        fi;
        if IsZero(mats[6]) = false then
                newmats[arrow] := ["b1",mats[6]];
                arrow:=arrow+1;
        fi;
        if IsZero(mats[8]) = false then
                newmats[arrow] := ["c2",mats[8]];
                arrow:=arrow+1;
        fi;
        if IsZero(mats[9]) = false then
                newmats[arrow] := ["c1",mats[9]];
```

```
                arrow:=arrow+1;
        fi;
        dim := ShallowCopy(DimensionVector(M));
        Remove(dim,1);;
        N:= RightModuleOverPathAlgebra(A,dim,newmats);
        return N;
end;

restrictionToKronecker := function(M)
  local mats,newmats,dim,arrow;
        mats := MatricesOfPathAlgebraModule(M);
        newmats := [];
        arrow:=1;
        if IsZero(Product(mats{[1..3]})) = false then
                newmats[arrow] := ["arm1",Product(mats{[1..3]})];
                arrow:=arrow+1;
        fi;
        if IsZero(Product(mats{[7..9]})) = false then
                newmats[arrow] := ["arm2",Product(mats{[7..9]})];
                arrow:=arrow+1;
        fi;
        dim := DimensionVector(M){[1,Size(vertices)]};
        return RightModuleOverPathAlgebra(C,dim,newmats);
end;

# now finally create preprojective and postinjective modules for A and
↪  B as well as the semi-regular tubes for B

PPA := [];;
PPB := [];;
PIA := [];;
PIB := [];;
PPA[1] := projA;;
PPB[1] := projB;;
PIA[1] := injA;;
PIB[1] := injB;;
for m in [2..8] do
        PPA[m] := [];;
        PPB[m] := [];;
        PIA[m] := [];;
        PIB[m] := [];;
        for i in [1..Size(VerticesOfPathAlgebra(B))] do
                PPB[m][i] := tauinverse(PPB[m-1][i]);
```

```
                PIB[m][i] := tau(PIB[m-1][i]);
        od;
        for i in [1..Size(VerticesOfPathAlgebra(A))] do
                PPA[m][i] := tauinverse(PPA[m-1][i]);
                PIA[m][i] := tau(PIA[m-1][i]);
        od;
od;

PAlpha := [];
PAlpha[1] := projB[1];
for m in [2..15] do
        PAlpha[m] := tauinverse(PAlpha[m-1]);
od;

IOmega := [];
IOmega[1] := injB[8];
for m in [2..10] do
        IOmega[m] := tau(IOmega[m-1]);
od;

coordinateModule := restriction(PAlpha[1]);
```

We now need some functions to load the adjacency matrix of basis vectors as output from Maple and construct the module over the algebra B from it. We use the Maple package containing the procedures for generating bimodules parametrising all homogeneous modules (of integral slopes) over tubular canonical algebras, constructed in [DMM14b], as well as a package containing the procedures for construction of modules from exceptional tubes over tubular canonical algebras, implementing the algorithms from [DMM10]. Both are available on A. Mróz' homepage. The import process relies on two template files to format and output the dimension vectors and matrices from Maple.

`outputIntSlopeModule.tpl`

```
read "Homogen.src":
with(Homogen):
initialize(3, 3, 3):
B := BimodIntSlope(n):
M := specBimod(B, xi, h):
bD := [ColumnDimension(M[1, 3]), ColumnDimension(M[1, 2]),
 ↪  ColumnDimension(M[2, 2]), ColumnDimension(M[3, 2]),
 ↪  ColumnDimension(M[1, 1]), ColumnDimension(M[2, 1]),
 ↪  ColumnDimension(M[3, 1]), RowDimension(M[1, 1])]:
basisToVertex := convert([seq(Vector(bD[i],i),i=1..8)],Vector):
```

```
adjacencyMatrix := Matrix([[Matrix(bD[1], bD[1], 0), Matrix(bD[1],
↪  bD[2], 0), Matrix(bD[1], bD[3], 0), Matrix(bD[1], bD[4], 0),
↪  Matrix(bD[1], bD[5], 0), Matrix(bD[1], bD[6], 0), Matrix(bD[1],
↪  bD[7], 0), Matrix(bD[1], bD[8], 0)], [M[1, 3], Matrix(bD[2],
↪  bD[2], 0), Matrix(bD[2], bD[3], 0), Matrix(bD[2], bD[4], 0),
↪  Matrix(bD[2], bD[5], 0), Matrix(bD[2], bD[6], 0), Matrix(bD[2],
↪  bD[7], 0), Matrix(bD[2], bD[8], 0)], [M[2, 3], Matrix(bD[3],
↪  bD[2], 0), Matrix(bD[3], bD[3], 0), Matrix(bD[3], bD[4], 0),
↪  Matrix(bD[3], bD[5], 0), Matrix(bD[3], bD[6], 0), Matrix(bD[3],
↪  bD[7], 0), Matrix(bD[3], bD[8], 0)], [M[3, 3], Matrix(bD[4],
↪  bD[2], 0), Matrix(bD[4], bD[3], 0), Matrix(bD[4], bD[4], 0),
↪  Matrix(bD[4], bD[5], 0), Matrix(bD[4], bD[6], 0), Matrix(bD[4],
↪  bD[7], 0), Matrix(bD[4], bD[8], 0)], [Matrix(bD[5], bD[1], 0),
↪  M[1, 2], Matrix(bD[5], bD[3], 0), Matrix(bD[5], bD[4], 0),
↪  Matrix(bD[5], bD[5], 0), Matrix(bD[5], bD[6], 0), Matrix(bD[5],
↪  bD[7], 0), Matrix(bD[5], bD[8], 0)], [Matrix(bD[6], bD[1], 0),
↪  Matrix(bD[6], bD[2], 0), M[2, 2], Matrix(bD[6], bD[4], 0),
↪  Matrix(bD[6], bD[5], 0), Matrix(bD[6], bD[6], 0), Matrix(bD[6],
↪  bD[7], 0), Matrix(bD[6], bD[8], 0)], [Matrix(bD[7], bD[1], 0),
↪  Matrix(bD[7], bD[2], 0), Matrix(bD[7], bD[3], 0), M[3, 2],
↪  Matrix(bD[7], bD[5], 0), Matrix(bD[7], bD[6], 0), Matrix(bD[7],
↪  bD[7], 0), Matrix(bD[7], bD[8], 0)], [Matrix(bD[8], bD[1], 0),
↪  Matrix(bD[8], bD[2], 0), Matrix(bD[8], bD[3], 0), Matrix(bD[8],
↪  bD[4], 0), M[1, 1], M[2, 1], M[3, 1], Matrix(bD[8], bD[8], 0)]]):
ExportMatrix("homAdjMat.tsv",adjacencyMatrix):
bD;
```

`outputExcRegModule.tpl`

```
read "exceptional.src":
with(exceptional):
initialize(3, 3, 3):
M := getMod([n,t,s,h]):
bD := [ColumnDimension(M[1, 3]), ColumnDimension(M[1, 2]),
↪  ColumnDimension(M[2, 2]), ColumnDimension(M[3, 2]),
↪  ColumnDimension(M[1, 1]), ColumnDimension(M[2, 1]),
↪  ColumnDimension(M[3, 1]), RowDimension(M[1, 1])]:
```

```
adjacencyMatrix := Matrix([[Matrix(bD[1], bD[1], 0), Matrix(bD[1],
↪  bD[2], 0), Matrix(bD[1], bD[3], 0), Matrix(bD[1], bD[4], 0),
↪  Matrix(bD[1], bD[5], 0), Matrix(bD[1], bD[6], 0), Matrix(bD[1],
↪  bD[7], 0), Matrix(bD[1], bD[8], 0)], [M[1, 3], Matrix(bD[2],
↪  bD[2], 0), Matrix(bD[2], bD[3], 0), Matrix(bD[2], bD[4], 0),
↪  Matrix(bD[2], bD[5], 0), Matrix(bD[2], bD[6], 0), Matrix(bD[2],
↪  bD[7], 0), Matrix(bD[2], bD[8], 0)], [M[2, 3], Matrix(bD[3],
↪  bD[2], 0), Matrix(bD[3], bD[3], 0), Matrix(bD[3], bD[4], 0),
↪  Matrix(bD[3], bD[5], 0), Matrix(bD[3], bD[6], 0), Matrix(bD[3],
↪  bD[7], 0), Matrix(bD[3], bD[8], 0)], [M[3, 3], Matrix(bD[4],
↪  bD[2], 0), Matrix(bD[4], bD[3], 0), Matrix(bD[4], bD[4], 0),
↪  Matrix(bD[4], bD[5], 0), Matrix(bD[4], bD[6], 0), Matrix(bD[4],
↪  bD[7], 0), Matrix(bD[4], bD[8], 0)], [Matrix(bD[5], bD[1], 0),
↪  M[1, 2], Matrix(bD[5], bD[3], 0), Matrix(bD[5], bD[4], 0),
↪  Matrix(bD[5], bD[5], 0), Matrix(bD[5], bD[6], 0), Matrix(bD[5],
↪  bD[7], 0), Matrix(bD[5], bD[8], 0)], [Matrix(bD[6], bD[1], 0),
↪  Matrix(bD[6], bD[2], 0), M[2, 2], Matrix(bD[6], bD[4], 0),
↪  Matrix(bD[6], bD[5], 0), Matrix(bD[6], bD[6], 0), Matrix(bD[6],
↪  bD[7], 0), Matrix(bD[6], bD[8], 0)], [Matrix(bD[7], bD[1], 0),
↪  Matrix(bD[7], bD[2], 0), Matrix(bD[7], bD[3], 0), M[3, 2],
↪  Matrix(bD[7], bD[5], 0), Matrix(bD[7], bD[6], 0), Matrix(bD[7],
↪  bD[7], 0), Matrix(bD[7], bD[8], 0)], [Matrix(bD[8], bD[1], 0),
↪  Matrix(bD[8], bD[2], 0), Matrix(bD[8], bD[3], 0), Matrix(bD[8],
↪  bD[4], 0), M[1, 1], M[2, 1], M[3, 1], Matrix(bD[8], bD[8], 0)]]):
ExportMatrix("excAdjMat.tsv",adjacencyMatrix):
bD;
```

These two files are then used to generate the Maple files to be run by Maple, producing a file containing the arrow matrices and returning a dimension vector.

```
mapleDir := GAPInfo.SystemEnvironment.PWD;;
path := Directory(mapleDir);;

importAdjMatFromMapleFile := function(mplFile, matFile)
## function requires the maple mpl file to give as output a dimension
↪  vector and write the matrix into the file matFile
        local maple, dimVector, i, adjMatrix, j, record, name, str,
         ↪  out;
        maple := Filename(DirectoriesSystemPrograms(),"maple");;
        str := "";
        out := OutputTextString(str, false);;
        Process( path, maple, InputTextNone(), out, ["-q",mplFile] );;
        CloseStream(out);
        NormalizeWhitespace(str);;
```

```
        dimVector := EvalString(str);
        adjMatrix := [];;
        i:=1;;
        for record in ReadCSV(matFile, true, "\t") do
                adjMatrix[i] := [];;
                j := 1;;
                for name in RecNames(record) do
                        adjMatrix[i][j] := Int(record.(name));
                        j := j+1;;
                od;
                i := i+1;;
        od;
        return [adjMatrix, dimVector];
end;

constructModuleFromAdjMatrix := function(adjMat, dimVect)
        local M, mats, startBasisOfVertex, totalDimension,
         ↪  sortedDimVect;
        startBasisOfVertex := [];;
        startBasisOfVertex[1] := 1;;
        for i in [2..8] do
                startBasisOfVertex[i] := startBasisOfVertex[i-1] +
                 ↪  dimVect[i-1];
        od;
        totalDimension := Sum(dimVect);
        mats := [];
        mats[1] := ["a1", TransposedMat(adjMat{[startBasisOfVertex[2].⌋
         ↪  .startBasisOfVertex[3]-1]}{[startBasisOfVertex[1]..startBa⌋
         ↪  sisOfVertex[2]-1]})];
        mats[2] := ["a3", TransposedMat(adjMat{[startBasisOfVertex[3].⌋
         ↪  .startBasisOfVertex[4]-1]}{[startBasisOfVertex[1]..startBa⌋
         ↪  sisOfVertex[2]-1]})];
        mats[3] := ["a2", TransposedMat(adjMat{[startBasisOfVertex[4].⌋
         ↪  .startBasisOfVertex[5]-1]}{[startBasisOfVertex[1]..startBa⌋
         ↪  sisOfVertex[2]-1]})];
        mats[4] := ["a12", TransposedMat(adjMat{[startBasisOfVertex[5]⌋
         ↪  ..startBasisOfVertex[6]-1]}{[startBasisOfVertex[2]..startB⌋
         ↪  asisOfVertex[3]-1]})];
        mats[5] := ["a32", TransposedMat(adjMat{[startBasisOfVertex[6]⌋
         ↪  ..startBasisOfVertex[7]-1]}{[startBasisOfVertex[3]..startB⌋
         ↪  asisOfVertex[4]-1]})];
```

```
        mats[6] := ["a22", TransposedMat(adjMat{[startBasisOfVertex[7]⌋
         ↪  ..startBasisOfVertex[8]-1]}{[startBasisOfVertex[4]..startB⌋
         ↪  asisOfVertex[5]-1]})];
        mats[7] := ["b1",
         ↪  TransposedMat(adjMat{[startBasisOfVertex[8]..totalDimensio⌋
         ↪  n]}{[startBasisOfVertex[5]..startBasisOfVertex[6]-1]})];
        mats[8] := ["b3",
         ↪  TransposedMat(adjMat{[startBasisOfVertex[8]..totalDimensio⌋
         ↪  n]}{[startBasisOfVertex[6]..startBasisOfVertex[7]-1]})];
        mats[9] := ["b2",
         ↪  TransposedMat(adjMat{[startBasisOfVertex[8]..totalDimensio⌋
         ↪  n]}{[startBasisOfVertex[7]..startBasisOfVertex[8]-1]})];
        for i  in [ 1 .. 9 ]  do
          if IsZero( mats[i][2] )  then
            Unbind( mats[i] );
          fi;
        od;

        sortedDimVect := [];
        sortedDimVect[1] := dimVect[1];
        sortedDimVect[2] := dimVect[2];
        sortedDimVect[3] := dimVect[5];
        sortedDimVect[4] := dimVect[4];
        sortedDimVect[5] := dimVect[7];
        sortedDimVect[6] := dimVect[3];
        sortedDimVect[7] := dimVect[6];
        sortedDimVect[8] := dimVect[8];
        M := RightModuleOverPathAlgebra(B, sortedDimVect,
         ↪  Compacted(mats));
        return M;
end;

constructTransModuleFromAdjMatrix := function(adjMat, dimVect)
        local M, mats, startBasisOfVertex, totalDimension,
         ↪  sortedDimVect;
        startBasisOfVertex := [];;
        startBasisOfVertex[1] := 1;;
        for i in [2..8] do
                startBasisOfVertex[i] := startBasisOfVertex[i-1] +
                 ↪  dimVect[i-1];
        od;
        totalDimension := Sum(dimVect);
        mats := [];
```

```
        mats[1] := ["b1",
         ↪  adjMat{[startBasisOfVertex[2]..startBasisOfVertex[3]-1]}{[⌋
         ↪  startBasisOfVertex[1]..startBasisOfVertex[2]-1]}];
        mats[2] := ["b3",
         ↪  adjMat{[startBasisOfVertex[3]..startBasisOfVertex[4]-1]}{[⌋
         ↪  startBasisOfVertex[1]..startBasisOfVertex[2]-1]}];
        mats[3] := ["b2",
         ↪  adjMat{[startBasisOfVertex[4]..startBasisOfVertex[5]-1]}{[⌋
         ↪  startBasisOfVertex[1]..startBasisOfVertex[2]-1]}];
        mats[4] := ["a12",
         ↪  adjMat{[startBasisOfVertex[5]..startBasisOfVertex[6]-1]}{[⌋
         ↪  startBasisOfVertex[2]..startBasisOfVertex[3]-1]}];
        mats[5] := ["a32",
         ↪  adjMat{[startBasisOfVertex[6]..startBasisOfVertex[7]-1]}{[⌋
         ↪  startBasisOfVertex[3]..startBasisOfVertex[4]-1]}];
        mats[6] := ["a22",
         ↪  adjMat{[startBasisOfVertex[7]..startBasisOfVertex[8]-1]}{[⌋
         ↪  startBasisOfVertex[4]..startBasisOfVertex[5]-1]}];
        mats[7] := ["a1", adjMat{[startBasisOfVertex[8]..totalDimensio⌋
         ↪  n]}{[startBasisOfVertex[5]..startBasisOfVertex[6]-1]}];
        mats[8] := ["a3", adjMat{[startBasisOfVertex[8]..totalDimensio⌋
         ↪  n]}{[startBasisOfVertex[6]..startBasisOfVertex[7]-1]}];
        mats[9] := ["a2", adjMat{[startBasisOfVertex[8]..totalDimensio⌋
         ↪  n]}{[startBasisOfVertex[7]..startBasisOfVertex[8]-1]}];
        for i  in [ 1 .. 9 ]  do
          if IsZero( mats[i][2] )  then
            Unbind( mats[i] );
          fi;
        od;
        sortedDimVect := [];
        sortedDimVect[8] := dimVect[1];
        sortedDimVect[3] := dimVect[2];
        sortedDimVect[2] := dimVect[5];
        sortedDimVect[5] := dimVect[4];
        sortedDimVect[4] := dimVect[7];
        sortedDimVect[7] := dimVect[3];
        sortedDimVect[6] := dimVect[6];
        sortedDimVect[1] := dimVect[8];
        M := RightModuleOverPathAlgebra(B, sortedDimVect,
         ↪  Compacted(mats));
        return M;
end;
```

```
constructIntSlopeModuleViaMaple := function(n, h, xi)
  local mapleFile, matrixData, foo;
  foo := false;
        mapleFile := Filename(path,"outputIntSlopeModule.mpl");;
  if (n <= 0) then
    n := p-n;;
    foo := true;;
  fi;
        PrintTo(mapleFile, "n:=", n, ":h:=", h, ":xi:=", xi, ":\n");;
        Exec("cat 'outputIntSlopeModule.tpl' >>
         ↪  'outputIntSlopeModule.mpl'");;
        matrixData :=
         ↪  importAdjMatFromMapleFile("outputIntSlopeModule.mpl",
         ↪  "homAdjMat.tsv");
  if foo then
    return constructTransModuleFromAdjMatrix(matrixData[1],
     ↪  matrixData[2]);
  else
    return constructModuleFromAdjMatrix(matrixData[1], matrixData[2]);
  fi;
end;

constructExcRegModuleViaMaple := function(n, t, s, h)
  local mapleFile, matrixData, foo;
  foo := false;;
        mapleFile := Filename(path,"outputExcRegModule.mpl");;
  if (n <= 0) then
    n := p-n;;
```

Note that the homogeneous modules for $\xi = \infty$ are constructed from those for $\xi = 1$ by replacing suitable arrow matrices.

```
  fi;
        PrintTo(mapleFile, "n:=", n ,":t:=", t, ":s:=", s, ":h:=", h,
         ↪  ":\n");;
        Exec("cat 'outputExcRegModule.tpl' >>
         ↪  'outputExcRegModule.mpl'");;
        matrixData :=
         ↪  importAdjMatFromMapleFile("outputExcRegModule.mpl",
         ↪  "excAdjMat.tsv");
  if foo then
    return constructTransModuleFromAdjMatrix(matrixData[1],
     ↪  matrixData[2]);
  else
```

```
    return constructModuleFromAdjMatrix(matrixData[1], matrixData[2]);
  fi;
end;

replaceArrowMatrix := function(arrow, matrix, module)
        local B, mats, newmats, i;
        B := ActingAlgebra(module);
        mats := MatricesOfPathAlgebraModule(module);
        newmats := mats;
        for i in [1..Size(mats)] do
                if i = arrow then
                        newmats[i] := matrix;
                else
                        newmats[i] := mats[i];
                fi;
        od;
        return RightModuleOverPathAlgebra(B, newmats);
end;

constructIntSlopeModuleAtInfinity := function(n, h)
  local tempModule, m, newMat;
  tempModule := constructIntSlopeModuleViaMaple(n, h, 1);
  if (n>=3) then
    m := QuoInt(n, p)-1;
    newMat :=
    ↪  MutableCopyMat(MatricesOfPathAlgebraModule(tempModule)[3]);
    newMat{[h+1..2*h]}{[h*m+1..h*(m+1)]} := IdentityMat(h, k);
    newMat{[h+1..2*h]}{[h*(2*m+1)+1..h*(2*m+2)]} := NullMat(h, h, k);
    for i in [2..h] do
      newMat[h+i][h*(2*m+1)+i-1] := 1;
    od;

    if RemInt(n, p) = 2 then
      newMat{[h+1..2*h]}{[h*(2*m+2)+1..h*(2*m+3)]} := NullMat(h, h, k);
      for i in [2..h] do
        newMat[h+i][h*(2*m+2)+i-1] := 1;
      od;
    fi;
    return replaceArrowMatrix(3, newMat, tempModule);
  elif (n < 0) then
    m := QuoInt(-n, p);
          newMat := MutableCopyMat(MatricesOfPathAlgebraModule(tempMod⌋
          ↪  ule)[1]);
```

```
          newMat{[h*m+1..h*(m+1)]}{[h+1..2*h]} := IdentityMat(h, k);
          newMat{[h*(2*m+1)+1..h*(2*m+2)]}{[h+1..2*h]} := NullMat(h,
           ↪  h, k);
          for i in [2..h] do
      newMat[h*(2*m+1)+i-1][h+i] := 1;
          od;

          if RemInt(-n, p) = 2 then
      newMat{[h*(2*m+2)+1..h*(2*m+3)]}{[h+1..2*h]} := NullMat(h, h, k);
      for i in [2..h] do
        newMat[h*(2*m+2)+i-1][h+i] := 1;
      od;
    fi;
    return replaceArrowMatrix(1, newMat, tempModule);
  else
    return false;
  fi;
end;
```

Finally, we construct the submodules exhibiting hyperfiniteness by the removal of certain basis elements and applications of τ.

```
offset := function(i, module)
        return Sum(DimensionVector(module){[1..i-1]});
end;

removeBasisElementsEmbedding := function(list, module)
        local M,j,basis;
        basis := ShallowCopy(BasisVectors(Basis(module)));
        for j in list do
                Unbind\[\](basis,j);
        od;
        M := SubRepresentationInclusion(module, basis);
        return M;
end;

keepBasisElementsEmbedding := function(list, module)
        local M,j,basis;
        basis := ShallowCopy(BasisVectors(Basis(module)));
        for j in DifferenceLists([1..Size(basis)],list) do
                Unbind\[\](basis,j);
        od;
        M := SubRepresentationInclusion(module,basis);
        return M;
```

```
end;

niceRankOneEmbeddingForHomModule := function(n, h, xi)
        local M,nbar,offset,list;
        if xi = "oo" then
                M := constructIntSlopeModuleAtInfinity(n, h);
        else
                M := constructIntSlopeModuleViaMaple(n, h, xi);
        fi;
        offset := function(i)
                return Sum(DimensionVector(M){[1..i-1]});
        end;
        if (n >= 3) then
                nbar := QuoInt(n, p);
                list := Flat([[offset(2)+1..offset(2)+h],
                 ↪  [offset(3)+1..offset(3)+2*h],[offset(7)+h*nbar+1..⌟
                 ↪  offset(7)+h*nbar+h]]);
                return removeBasisElementsEmbedding(list, M);
        elif (n < 0) then
                nbar := QuoInt(-n, p);
                if (RemInt(-n,p) = 0) then
                        list := Flat([[1..nbar*h],
                         ↪  [(nbar+1)*h+h+1..(2*nbar+1)*h],
                         ↪  [(2*nbar+2)*h+1..(3*nbar+2)*h]]);
                else
                        list := Flat([[1..nbar*h],
                         ↪  [(nbar+1)*h+h+1..(2*nbar+1)*h],
                         ↪  [offset(2)-(nbar+2)*h..offset(2)-2*h]]);
                fi;
                return keepBasisElementsEmbedding(list, M);
        else
                return false;
        fi;
end;

niceRankOneEmbeddingForExcModule := function(n, t, s, h)
        local hbar, baseEmbedding, embedding, homs, irreducible;
        if (n < p) then
                return false;
        fi;
        hbar := QuoInt(h, p);
        if h mod p = 0 then
                if t = 1 then
```

```
        baseEmbedding :=
        ↪ niceRankOneEmbeddingForHomModule(n, hbar,
        ↪ 1);
        if (n > 0 and s = 1) or (n < 0 and s = 2) then
            embedding := TransposeOfDualOfModuleHo⌟
            ↪ momorphism(baseEmbedding);
        elif (n > 0 and s = 2) or (n < 0 and s = 1)
        ↪ then
            embedding := DualOfTransposeOfModuleHo⌟
            ↪ momorphism(baseEmbedding);
        elif s = 3 then
            embedding := baseEmbedding;
        fi;
    elif t = 2 then
        baseEmbedding :=
        ↪ niceRankOneEmbeddingForHomModule(n, hbar,
        ↪ "oo");
        if s = 1 then
            embedding := baseEmbedding;
        elif s = 2 then
            embedding := TransposeOfDualOfModuleHo⌟
            ↪ momorphism(baseEmbedding);
        elif s = 3 then
            embedding := DualOfTransposeOfModuleHo⌟
            ↪ momorphism(baseEmbedding);
        fi;
    elif t = 3 then
        baseEmbedding :=
        ↪ niceRankOneEmbeddingForHomModule(n, hbar,
        ↪ 0);
        if s = 1 then
            embedding := baseEmbedding;
        elif s = 2 then
            embedding := TransposeOfDualOfModuleHo⌟
            ↪ momorphism(baseEmbedding);
        elif s = 3 then
            embedding := DualOfTransposeOfModuleHo⌟
            ↪ momorphism(baseEmbedding);
        fi;
    fi;
else
    baseEmbedding := niceRankOneEmbeddingForExcModule(n,
    ↪ t, s, (hbar+1)*p);
```

```
                if IsomorphicModules(Range(baseEmbedding),constructExc⌋
                ↪  RegModuleViaMaple(n, t, s, (hbar+1)*p)) = false
                ↪  then
                        return "Error: the constructed codomain of the
                        ↪  base embedding is not the desired module
                        ↪  in the exceptional tube!";
                fi;
                homs :=
                ↪  HomOverAlgebra(constructExcRegModuleViaMaple(n, t,
                ↪  s, h), Range(baseEmbedding));
                for irreducible in homs do
                        if IsInjective(irreducible) then
                                break;
                        fi;
                od;
                if IsInjective(baseEmbedding) and
                ↪  IsInjective(irreducible) then
                        embedding :=
                        ↪  IntersectionOfSubmodules(irreducible,
                        ↪  baseEmbedding)[2];
                elif IsInjective(baseEmbedding) = false then
                        return "Error: the embedding of the nice
                        ↪  submodule in the multiple-of-p case is not
                        ↪  injective!";
                elif IsInjective(irreducible) = false then
                        return "Error: the embedding of the smaller
                        ↪  quasi-length module into the base case
                        ↪  module is not injective!";
                fi;
        fi;
        return embedding;
end;

niceRankZeroOrOneEmbeddingForExcModule := function(n, t, s, h)
        local hbar, hprime, baseEmbedding, embedding, homs,
        ↪  irreducible;
        if (n > 0) then
                return false;
        fi;
        hbar := QuoInt(h, p);
        if h mod p = 0 then
                if t = 1 then
```

```
                baseEmbedding :=
                ↪  niceRankOneEmbeddingForHomModule(n, hbar,
                ↪  1);
                if s mod pp[t] = 1 then
                        embedding := DualOfTransposeOfModuleHo⌋
                        ↪  momorphism(baseEmbedding);
                elif s mod pp[t] = 2 then
                        embedding := TransposeOfDualOfModuleHo⌋
                        ↪  momorphism(baseEmbedding);
                elif s mod pp[t] = 0 then
                        embedding := baseEmbedding;
                fi;
        elif t = 2 then
                baseEmbedding :=
                ↪  niceRankOneEmbeddingForHomModule(n, hbar,
                ↪  "oo");
                if s mod pp[t] = 1 then
                        embedding := baseEmbedding;
                elif s mod pp[t] = 2 then
                        embedding := DualOfTransposeOfModuleHo⌋
                        ↪  momorphism(baseEmbedding);
                elif s mod pp[t] = 0 then
                        embedding := TransposeOfDualOfModuleHo⌋
                        ↪  momorphism(baseEmbedding);
                fi;
        elif t = 3 then
                baseEmbedding :=
                ↪  niceRankOneEmbeddingForHomModule(n, hbar,
                ↪  0);
                if s mod pp[t] = 1 then
                        embedding := baseEmbedding;
                elif s mod pp[t] = 2 then
                        embedding := DualOfTransposeOfModuleHo⌋
                        ↪  momorphism(baseEmbedding);
                elif s mod pp[t] = 0 then
                        embedding := TransposeOfDualOfModuleHo⌋
                        ↪  momorphism(baseEmbedding);
                fi;
        fi;
else
        hprime := RemInt(h,p);
```

```
                baseEmbedding :=
                ↪  niceRankZeroOrOneEmbeddingForExcModule(n, t,
                ↪  s+hprime, (hbar+1)*p);
                if IsomorphicModules(Range(baseEmbedding),
                ↪  constructExcRegModuleViaMaple(n, t, s+hprime,
                ↪  (hbar+1)*p)) = false then
                        return "Error: the constructed codomain of the
                        ↪  base embedding is not the desired module
                        ↪  in the exceptional tube!";
                fi;
                homs :=
                ↪  HomOverAlgebra(constructExcRegModuleViaMaple(n, t,
                ↪  s, h),Range(baseEmbedding));
                for irreducible in homs do
                        if IsInjective(irreducible) then
                                break;
                        fi;
                od;
                if IsInjective(baseEmbedding) and
                ↪  IsInjective(irreducible) then
                        embedding :=
                        ↪  IntersectionOfSubmodules(irreducible,
                        ↪  baseEmbedding)[2];
                elif IsInjective(baseEmbedding) = false then
                        return "Error: the embedding of the nice
                        ↪  submodule in the multiple-of-p case is not
                        ↪  injective!";
                elif IsInjective(irreducible) = false then
                        return "Error: the embedding of the smaller
                        ↪  quasi-length module into the base case
                        ↪  module is not injective!";
                fi;
        fi;
        return embedding;
end;

niceSubmoduleEmbeddingForExcModule := function(n, t, s, h)
        if n >= p then
                return niceRankOneEmbeddingForExcModule(n, t, s, h);
        elif n < 0 then
                return niceRankZeroOrOneEmbeddingForExcModule(n, t, s,
                ↪  h);
        else
```

```
                return false;
        fi;
end;
```

We can now check the codimensions of the modules constructed in such a way.

```
for n in [5..10] do
        Print(n, ":\n");
        for h in [1..8] do
        Print("h=", h, "\n");
                for t in [1..3] do
                Print("t=", t);
                        for s in [1..p] do
                                Print(" ", s, ": ");
                                emb := niceRankOneEmbeddingForExcModul⌋
                                ↪  e(n, t, s,
                                ↪  h);
                                Print(DimensionVector(CoKernel(emb)));
                        od;
                Print("\n");
                od;
        od;
od;
```