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Exceptional Sequences of Representations of Quivers

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ABSTRACT. We show that the braid group acts transitively on the set of exceptional sequences of representations of a quiver.

At the 1992 Canadian Mathematical Society Annual Seminar, A. N. Rudakov lectured on exceptional sequences of vector bundles for P^2, and more generally for Del Pezzo surfaces. This led us to consider the corresponding theory for representations of quivers. There is the notion of a complete exceptional sequence of representations of a quiver, and there is an action of the braid group on the set of such sequences. We show that this action is transitive. The proof uses a theorem of A. Schofield.

Exceptional sequences and the action of the braid group were discovered in the Moscow school of vector bundles, see [1,2,3]. The natural setting is in the context of triangulated categories, and this is described in [1].

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Occasionally there is an action of the braid group on the set of exceptional sequences for an abelian category. For example the argument of [2, §3.3] shows that this holds if the spaces $\mathrm{Ext}^i(X,Y)$ are naturally f.d. vector spaces for $i\geq 0$, $\mathrm{Ext}^i(X,Y)=0$ for $i\geq 3$, and whenever $\mathrm{Ext}^2(X,Y)\neq 0$ and $\mathrm{Ext}^1(X,X)=\mathrm{Ext}^1(Y,Y)=0$ we have $\dim \mathrm{Ext}^2(X,Y)<\dim \mathrm{Hom}(Y,X)$. Of course this applies to the category of representations of a quiver, since in this case $\mathrm{Ext}^i(X,Y)=0$ for $i\geq 2$. We do not need to quote this fact, however, since for quiver representations, the existence of a braid group action follows quite easily from properties of perpendicular categories.

Let k be an algebraically closed field, let Q be a quiver with no oriented cycles, and let kQ be the path algebra. By an <u>exceptional representation</u> we mean a finite dimensional left kQ-module X with $\mathrm{End}(X)=k$ and $\mathrm{Ext}(X,X)=0$. By an <u>exceptional sequence</u> $E=(X_1,\ldots,X_r)$ <u>of length</u> r we mean a sequence of exceptional representations satisfying $\mathrm{Hom}(X_j,X_i)=\mathrm{Ext}(X_j,X_i)=0$ for $1\leq i<j\leq r$. We say that an exceptional sequence is a <u>complete sequence</u> if it has length equal to the number of vertices of Q.

If C is a collection of representations, recall that the <u>perpendicular categories</u> are defined by

$$^{\perp}C = \{M \in kQ\text{-mod} \mid \mathrm{Hom}(M,X)=\mathrm{Ext}(M,X)=0 \text{ for all } X\in C\}$$
$$C^{\perp} = \{M \in kQ\text{-mod} \mid \mathrm{Hom}(X,M)=\mathrm{Ext}(X,M)=0 \text{ for all } X\in C\}.$$

In particular we can use this notion for an exceptional sequence $E=(X_1,\ldots,X_r)$. Now $E^{\perp}$ and $^{\perp}E$ may be calculated by induction on the length of E, and hence one can show that if Q has n vertices then $E^{\perp}$ and $^{\perp}E$ are equivalent to the categories of representations of quivers $Q(E^{\perp})$ and $Q(^{\perp}E)$ with n-r vertices and no oriented cycles, see for example

[4, Theorem 2.3]. Moreover the functors from $kQ(E^{\perp})$-mod and $kQ({}^{\perp}E)$-mod to kQ-mod are exact and induce isomorphisms on both Hom and Ext. Thus we can talk about simple objects of $E^{\perp}$, exceptional sequences for $E^{\perp}$, complete sequences for $E^{\perp}$, etc.

LEMMA 1. *Any exceptional sequence* $(X_1,..,X_a,Z_1,..,Z_c)$ *can be enlarged to a complete sequence* $(X_1,..,X_a,Y_1,..,Y_b,Z_1,..,Z_c)$.

Proof. It suffices to find a complete sequence $(Y_1,..,Y_b,Z_1,..,Z_c)$ for ${}^{\perp}(X_1,..,X_a)$, so we may assume that $a=0$. Next it suffices to find a complete sequence $(Y_1,..,Y_b)$ for $(Z_1,..,Z_c)^{\perp}$ so we may assume that $c=0$. Now the indecomposable projective, injective, or simple representations all give complete sequences when suitably ordered.

LEMMA 2. *If* $E=(X_1,..,X_n)$ *and* $F=(Y_1,..,Y_n)$ *are complete sequences which differ in at most one place, say* $X_j \cong Y_j$ *for* $j \neq i$, *then also* $X_i \cong Y_i$.

Proof. Passing to ${}^{\perp}(X_1,..,X_{i-1})$ we may suppose that $i=1$. Now passing to $(X_{i+1},..,X_n)^{\perp}$ we may suppose that Q has only one vertex. But then it has only one exceptional representation.

If E is an exceptional sequence, let $C(E)$ be the smallest full subcategory of kQ-mod which contains E and is closed under extensions, kernels of epis, and cokernels of monos.

LEMMA 3. *If* $E=(X_1,..,X_n)$ *is a complete sequence, then* $C(E) = kQ\text{-mod}$.

Proof. By induction on n, the number of vertices of Q. If n=0 there is nothing to prove, so suppose that n>0. Let $X=X_n$. Now $X_1,..,X_{n-1}$ is a complete sequence for $X^{\perp}$, so by the induction we have $C(X_1,..,X_{n-1}) = X^{\perp}$, and hence C(E) contains X and $X^{\perp}$.

Suppose that X is non-projective. The Bongartz completion of X is a tilting module $T=X\oplus Y$ with $Y\in X^{\perp}$. For each projective module P there is an exact sequence $0\longrightarrow P\longrightarrow T'\longrightarrow T''\longrightarrow 0$ with $T',T''\in\text{add}(T)$, so C(E) contains the projectives. Since any representation has a projective resolution it follows that C(E)=kQ-mod.

Suppose that X is projective, say corresponding to vertex i, so that $X^{\perp}$ consists of the representations which are zero at i. Thus C(E) contains the simples S_j corresponding to vertices $j\neq i$. There is also an exact sequence $0\longrightarrow \text{rad } X\longrightarrow X\longrightarrow S_i\longrightarrow 0$ with $\text{rad } X\in X^{\perp}$ so that $S_i\in C(E)$. Thus C(E)=kQ-mod.

LEMMA 4. *Let* $E=(X_1,..,X_r)$ *be an exceptional sequence.*

(1) $C(E) = {}^{\perp}(E^{\perp}) = {}^{\perp}F$ *where* F *is any complete sequence for* $E^{\perp}$.

(2) $C(E) = ({}^{\perp}E)^{\perp} = G^{\perp}$ *where* G *is any complete sequence for* ${}^{\perp}E$.

Proof. We have $C(E) \subseteq {}^{\perp}(E^{\perp}) \subseteq {}^{\perp}F$ since ${}^{\perp}(E^{\perp})$ contains E and is closed under extensions, kernels of epis and cokernels of monos. Now E is a complete sequence for ${}^{\perp}F$, so we have $C(E)={}^{\perp}F$ by Lemma 3. Part (2) is the same.

LEMMA 5. *If* E *is an exceptional sequence of length* r *then* C(E) *is equivalent to the category of representations of a quiver* Q(E) *with* r *vertices and no oriented cycles. Moreover the functor* kQ(E)-mod⟶kQ-mod *is exact and induces isomorphisms on both* Hom *and* Ext.

LEMMA 6. *If* (X,Y) *is an exceptional sequence then there are unique representations* $R_Y X$ *and* $L_X Y$ *with the property that* $(Y, R_Y X)$ *and* $(L_X Y, X)$ *are exceptional sequences in* C(X,Y).

The next result is due to Schofield [5].

LEMMA 7. *If* X *is exceptional and not simple then there is an exceptional sequence* (X,Y) *such that* X *is not a simple object of* C(X,Y).

Proof. We may suppose that X is sincere - otherwise we can pass to the support of X. We work by induction on the number n of vertices of Q. Now n≥2 since X is not simple. If n=2 there is an exceptional sequence (X,Y), we have C(X,Y)=kQ-mod and by assumption X is not simple. If n>2, let Y be a simple object of $^{\perp}X$. Now X is sincere as an object of $Y^{\perp}$ by [4, Lemma 4.2], and since $Y^{\perp}$ is equivalent to the representations of a quiver with n-1 vertices, the induction applies.

LEMMA 8. *Let* $E=(X_1,..,X_r)$ *be an exceptional sequence and let* $1 \le i < r$.

(1) $(X_1,X_2,..,X_{i-1},X_{i+1},Y,X_{i+2},..,X_r)$ *is an exceptional sequence in* C(E) *if and only if* $Y \cong R_{X_{i+1}} X_i$.

(2) $(X_1,X_2,..,X_{i-1},Z,X_i,X_{i+2},..,X_r)$ *is an exceptional*

sequence in C(E) *if and only if* $Z \cong L_{X_i} X_{i+1}$.

Proof. The stated sequences are exceptional since

$$R_{X_{i+1}} X_i, L_{X_i} X_{i+1} \in C(X_i, X_{i+1}) = {}^{\perp}[(X_i, X_{i+1})^{\perp}] = [{}^{\perp}(X_i, X_{i+1})]^{\perp}$$

and $X_1, \ldots, X_{i-1} \in {}^{\perp}(X_i, X_{i+1})$, $X_{i+2}, \ldots, X_r \in (X_i, X_{i+1})^{\perp}$. The uniqueness follows from Lemmas 2 and 5.

Let B_r be the braid group on r strings, so with generators $\sigma_1, \ldots, \sigma_{r-1}$ where σ_i moves the i-th over the (i+1)-th string, and with relations $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $i \neq j \pm 1$ and $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ for $1 \leq i \leq r-2$. Let $\mathcal{E}_r$ be the set of exceptional sequences of length r (up to isomorphism).

LEMMA 9. *The assignments*

$$\sigma_i\ (X_1, \ldots, X_r) = (X_1, X_2, \ldots, X_{i-1}, X_{i+1}, R_{X_{i+1}} X_i, X_{i+2}, \ldots, X_r)$$
$$\sigma_i^{-1} (X_1, \ldots, X_r) = (X_1, X_2, \ldots, X_{i-1}, L_{X_i} X_{i+1}, X_i, X_{i+2}, \ldots, X_r)$$

define an action of B_r *on* $\mathcal{E}_r$.

Proof. We have

$$\sigma_i^{-1} \sigma_i\ (X_1, \ldots, X_r) = (X_1, X_2, \ldots, X_{i-1}, Y, X_{i+1}, X_{i+2}, \ldots, X_r)$$

for some $Y \in C(X_1, \ldots, X_r)$, so $Y \cong X_i$ by uniqueness. Thus the stated actions are inverse. To verify the relation $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ note that $\sigma_i \sigma_{i+1} \sigma_i (X_1, \ldots, X_r)$ and $\sigma_{i+1} \sigma_i \sigma_{i+1} (X_1, \ldots, X_r)$ both have the form

$$(X_1, \ldots, X_{i-1}, X_{i+2}, R_{X_{i+2}} X_{i+1}, Y, X_{i+3}, \ldots, X_r)$$

with $Y \in C(X_1, \ldots, X_r)$, so they must be equal.

THEOREM. *If Q has n vertices then the action of* B_n *on the set* $\mathcal{E}_n$ *of complete sequences is transitive.*

Proof. We prove this by induction on n. If n=1 there is nothing to prove, while if n=2 it can be checked since every exceptional representation is preprojective or preinjective. Thus suppose n>2. Let $\mathcal{O}$ be an orbit for the action of B_n.

Let d be the minimum dimension of any representation in any complete sequence in $\mathcal{O}$. We show that d=1, so for a contradiction suppose otherwise. Let E be a complete sequence in $\mathcal{O}$ containing a representation X of dimension d. Applying $\sigma_1\sigma_2..\sigma_{i-1}$ if necessary we may assume that $E=(X,X_2,..,X_n)$. Let (X,Y) be the exceptional sequence given by Lemma 7. It extends to a complete sequence $F=(X,Y,Y_3,..,Y_n)$. Now $(X_2,..,X_n)$ and $(Y,Y_3,..,Y_n)$ are complete sequences for ${}^{\perp}X$, so by the induction they are in the same orbit under B_{n-1}, and hence E and F are in the same orbit under B_n. Now C(X,Y) contains a complete sequence (S,T) with S and T the simple objects of C(X,Y), and $\sigma_1^k F=(S,T,Y_3,..,Y_n)$ for some $k\in\mathbb{Z}$. By assumption X is not simple as an object of C(X,Y), so it involves both S and T, and hence dim S < dim X = d. But $\sigma_1^k F \in \mathcal{O}$, contradicting the minimality of d.

We have shown that $\mathcal{O}$ contains a sequence E which involves a simple S, and indeed we may assume that $E=(S,Z_2,..,Z_n)$. Let $P=(P_1,..,P_n)$ be the complete sequence of projectives, with P_j being the projective cover of S. As above, passing to ${}^{\perp}S$ and using the induction, we see that $\mathcal{O}$ contains the sequence $F=(S,P_1,..,P_{j-1},P_{j+1},..,P_n)$. Now $\sigma_{j-1}..\sigma_2\sigma_1 F = P$ since the two sides differ in at most one place. Thus $P\in\mathcal{O}$, and it follows that the action of B_n

on $\mathcal{E}_n$ is transitive.

COROLLARY. *Exceptional sequences* $E,F\in\mathcal{E}_r$ *are in the same orbit under* B_r *if and only if* $C(E)=C(F)$.

Finally note that if (X,Y) is an exceptional sequence,

$$\underline{\dim}\, R_Y X = \pm\, (\underline{\dim}\, X - \langle\underline{\dim}\, X, \underline{\dim}\, Y\rangle\, \underline{\dim}\, Y)$$
$$\underline{\dim}\, L_X Y = \pm\, (\underline{\dim}\, Y - \langle\underline{\dim}\, X, \underline{\dim}\, Y\rangle\, \underline{\dim}\, X),$$

so the theorem gives a convenient method for producing the real Schur roots for Q.

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