ON THE EXCEPTIONAL FIBRES OF KLEINIAN SINGULARITIES

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ABSTRACT. We give a new proof, avoiding case-by-case analysis, of a theorem of Y. Ito and I. Nakamura which provides a module-theoretic interpretation of the bijection between the irreducible components of the exceptional fibre for a Kleinian singularity, and the non-trivial simple modules for the corresponding finite subgroup of $\mathrm{SL}(2,\mathbb{C})$. Our proof uses a classification of certain cyclic modules for preprojective algebras.

INTRODUCTION

Let Γ be a finite subgroup of $\mathrm{SL}(2,\mathbb{C})$, let $X = \mathbb{C}^2/\Gamma$ be the corresponding Kleinian singularity and let $\pi : \tilde{X} \to X$ be its minimal resolution of singularities. The *exceptional fibre* E, the fibre of π over the singular point of X, is known to be a union of projective lines meeting transversally, and the graph whose vertices correspond to the irreducible components of E, with two vertices joined if and only if the components intersect, is a Dynkin diagram (of one of the types A_n, D_n, E_6, E_7, E_8).

If $N_0, N_1, \dots, N_n$ are a complete set of simple $\mathbb{C}\Gamma$-modules, with N_0 the trivial module, then the *McKay graph* of Γ has vertex set $\{0, 1, \dots, n\}$ and the number of edges between i and j is the multiplicity $[V \otimes N_i : N_j]$ where V is the natural 2-dimensional $\mathbb{C}\Gamma$-module. According to the McKay correspondence [12], this is an extended Dynkin diagram with extending vertex 0.

These two diagrams were related by Gonzalez-Sprinberg and Verdier [5], who showed that there is a natural bijection between the irreducible components of the exceptional fibre and the non-trivial irreducible representations of Γ. Recently Ito and Nakamura [6, 7] found a beautiful new interpretation of this bijection, and their work has already been used by Kapranov and Vasserot [9] in their proof that the derived category of $\tilde{X}$ is equivalent to the derived category of Γ-equivariant sheaves on $\mathbb{C}^2$. Unfortunately, both the work of Gonzalez-Sprinberg and Verdier, and of Ito and Nakamura, requires extensive case-by-case analysis for the different Dynkin diagrams. In this article we give a new proof of the theorem of Ito and Nakamura, which avoids such case-by-case analysis.

The theorem of Ito and Nakamura is as follows. Since Γ acts on $\mathbb{C}^2$, it also acts on the coordinate ring $R = \mathbb{C}[x, y]$, and on the Hilbert scheme $\mathrm{Hilb}^d(\mathbb{C}^2)$ of ideals of codimension d in R (as vector spaces). Ito and Nakamura observe that $\tilde{X}$ is isomorphic to

$$\mathrm{Hilb}^{\Gamma}(\mathbb{C}^2) = \{J \in \mathrm{Hilb}^{|\Gamma|}(\mathbb{C}^2) \mid J \text{ is } \Gamma\text{-invariant and } R/J \cong \mathbb{C}\Gamma \text{ as a } \mathbb{C}\Gamma\text{-module}\}.$$

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If $\mathfrak{m}$ is the ideal in R generated by x and y, then the exceptional fibre E corresponds to the $\mathfrak{m}$-primary ideals in $\mathrm{Hilb}^\Gamma(\mathbb{C}^2)$. It follows that any $J \in E$ contains the ideal $\mathfrak{n} = R(\mathfrak{m} \cap R^\Gamma)$, and that $V(J) = J/(\mathfrak{m}J + \mathfrak{n})$ is a $\mathbb{C}\Gamma$-module with $[V(J) : N_0] = 0$.

Theorem 1. *If $J \in E$ then $V(J)$ is a sum of one or two simple $\mathbb{C}\Gamma$-modules, and if two, they are non-isomorphic. If $i \neq 0$ then*

$$E(i) = \{J \in E \mid [V(J) : N_i] \neq 0\}$$

is a closed subset of E isomorphic to $\mathbb{P}^1$. Moreover $E(i)$ meets $E(j)$ if and only if i and j are adjacent in the McKay graph, and in this case $|E(i) \cap E(j)| = 1$.

In fact, the Hilbert scheme construction of $\tilde{X}$ is known to be equivalent to a moduli space construction of $\tilde{X}$ due to Kronheimer [11], reformulated using geometric invariant theory by Cassens and Slodowy [2]. We describe the corresponding reformulation of Theorem 1 in Section 4. We then prove this in Section 5, using a result about cyclic modules for preprojective algebras which is proved in Sections 2 and 3. This result, Lemma 2 should be of independent interest.

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1. Preprojective algebras and a homological formula

Let Q be a quiver with vertex set I and let K be a field. The *preprojective algebra* is

$$\Pi(Q) = K\overline{Q}/(\sum_{a \in Q}[a, a^*]),$$

where $\overline{Q}$ is the *double* of Q, obtained by adjoining an arrow $a^* : j \to i$ for each arrow $a : i \to j$ in Q, and $K\overline{Q}$ is the path algebra of $\overline{Q}$. See for example [4]. (For the definition of the path algebra, see for example [1].) Let e_i be the trivial path at vertex i. Any finite dimensional module M for $\Pi(Q)$ or $K\overline{Q}$ has a *dimension vector* $\underline{\dim}\, M \in \mathbb{N}^I$ whose ith component is $\dim e_i M$. Let $(-,-)$ be the symmetric bilinear form on $\mathbb{Z}^I$ defined by

$$(\alpha, \beta) = \sum_{i \in I} 2\alpha_i\beta_i - \sum_{\substack{a \in \overline{Q} \\ a:i \to j}} \alpha_i\beta_j.$$

Lemma 1. *If M and N are finite dimensional $\Pi(Q)$-modules, then*

$$\dim \mathrm{Ext}^1(M, N) = \dim \mathrm{Hom}(M, N) + \dim \mathrm{Hom}(N, M) - (\underline{\dim}\, M, \underline{\dim}\, N).$$

Proof. For simplicity we write Π for $\Pi(Q)$. It is easy to see that M has a projective resolution which starts

$$\cdots \to \bigoplus_{i \in I} \Pi e_i \otimes e_i M \xrightarrow{f} \bigoplus_{\substack{a \in \overline{Q} \\ a:i \to j}} \Pi e_j \otimes e_i M \xrightarrow{g} \bigoplus_{i \in I} \Pi e_i \otimes e_i M \xrightarrow{h} M \to 0,$$

where f is defined by

$$f(\sum_i p_i \otimes m_i) = \sum_{\substack{a \in Q \\ a:i \to j}} (p_i a^* \otimes m_i - p_j \otimes a^* m_j)_a - (p_j a \otimes m_j - p_i \otimes a m_i)_{a^*}$$

for $p_i \in \Pi e_i$ and $m_i \in e_i M$; g is defined on the summand corresponding to an arrow $a : i \to j$ in $\overline{Q}$ by $g(p \otimes m) = (pa \otimes m)_i - (p \otimes am)_j$ for p in Πe_j and m in $e_i M$; and h is multiplication. Computing the homomorphisms to N, and identifying $\mathrm{Hom}(\Pi e_j \otimes e_i M, N)$ with $\mathrm{Hom}_K(e_i M, e_j N)$, gives a complex

$$0 \to \bigoplus_{i \in I} \mathrm{Hom}_K(e_i M, e_i N) \to \bigoplus_{\substack{a \in \overline{Q} \\ a:i \to j}} \mathrm{Hom}_K(e_i M, e_j N) \to \bigoplus_{i \in I} \mathrm{Hom}_K(e_i M, e_i N)$$

in which the left hand cohomology is $\mathrm{Hom}(M, N)$ and the middle cohomology is $\mathrm{Ext}^1(M, N)$. Moreover, the alternating sum of the dimensions of the terms is $(\underline{\dim}\, M, \underline{\dim}\, N)$. It remains to prove that the cokernel of the right hand map has the same dimension as $\mathrm{Hom}(N, M)$. But using the trace map to identify $\mathrm{Hom}_K(U, V)^*$ with $\mathrm{Hom}_K(V, U)$, the dual of this complex is

$$\bigoplus_{i \in I} \mathrm{Hom}_K(e_i N, e_i M) \to \bigoplus_{\substack{a \in \overline{Q} \\ a:i \to j}} \mathrm{Hom}_K(e_i N, e_j M) \to \bigoplus_{i \in I} \mathrm{Hom}_K(e_i N, e_i M) \to 0,$$

and, up to changing the sign of components in the second direct sum corresponding to arrows which are not in Q, this is the same as the complex arising with M and N interchanged. The result follows. □

2. Classification of v-generated modules

Let Q be a quiver with vertex set I and let K be a field. Recall that, according to Kac's Theorem, the dimension vectors of indecomposable representations of Q are exactly the positive roots for a suitable root system in $\mathbb{Z}^I$.

If i is a vertex, we denote by S_i the simple $\Pi(Q)$-module whose dimension vector is the ith coordinate vector ϵ_i, and on which all arrows act as zero. A $\Pi(Q)$-module is said to be *nilpotent* if its only composition factors are the S_i.

If v is a vertex, we say that a $\Pi(Q)$-module M is *v-generated* if it is cyclic, generated by an element in $e_v M$. We have the following result, which should be of independent interest.

Lemma 2. *Let $\alpha \in \mathbb{N}^I$ and let v be a vertex with $\alpha_v = 1$.*

(1) *If there is a v-generated $\Pi(Q)$-module of dimension α, then α is a root.*
(2) *If α is a real root, then there is a unique v-generated module of dimension α.*
(2′) *The modules in (2) are nilpotent.*
(3) *If α is an imaginary root, and K is algebraically closed of characteristic zero, then there are infinitely many v-generated modules of dimension α.*

We give two entirely separate proofs. The first one proves (1), (2) and (2′). The second one, valid only when K is algebraically closed of characteristic zero, deduces (1), (2) and (3) rather easily from the fact that a certain moduli space can be described in two different ways. In our application later we have $K = \mathbb{C}$ and only need (1) and (2), so either proof would have sufficed.

If M is a module and i a vertex, then elements $\xi_1, \dots, \xi_d \in \mathrm{Ext}^1(M, S_i)$ define an extension

$$0 \to S_i^d \to E \to M \to 0.$$

The *universal extension* of M by S_i is the module E obtained by taking $\xi_1, \dots, \xi_d$ to be a basis of $\mathrm{Ext}^1(M, S_i)$. It is unique up to isomorphism. Note that E is

v-generated with $v \neq i$ if and only if M is v-generated and the ξ_j are linearly independent.

Proof of Lemma 2 (1), (2) *and* (2′). (1) We prove this by induction on $\sum_i \alpha_i$. Suppose that there is a v-generated module M of dimension α. If $(\alpha, \epsilon_i) > 0$ for some vertex $i \neq v$ then i must be loopfree. Since M is v-generated, we have $\mathrm{Hom}(M, S_i) = 0$, so $d = \dim \mathrm{Hom}(S_i, M) \geq (\alpha, \epsilon_i) > 0$ by the homological formula. Thus M has a v-generated quotient of dimension $\beta = \alpha - d\epsilon_i$. By induction β is a root, and so also is its reflection

$$s_i(\beta) = \beta - (\beta, \epsilon_i)\epsilon_i$$

which is equal to $\alpha + (d - (\alpha, \epsilon_i))\epsilon_i$. Thus α is a root by [8], §1 Condition (R2).

Thus suppose that $(\alpha, \epsilon_i) \leq 0$ for all $i \neq v$. If $(\alpha, \epsilon_v) \leq 0$, then since the existence of M clearly implies that α has connected support, it is in the fundamental region, hence a root. Now suppose that $(\alpha, \epsilon_v) > 0$. Leaving out the trivial case $\alpha = \epsilon_v$, this implies that v is connected in $\overline{Q}$ to only one vertex i with $\alpha_i > 0$, this vertex has $\alpha_i = 1$, and $(\alpha, \epsilon_i) = 1$. But now there is a unique arrow in $\overline{Q}$ from v to i, say a, and a unique reverse arrow, a^*. In any representation of dimension α, these arrows are represented by 1×1-matrices with product zero, so one of them must be zero. Now since M is v-generated, it must be a^* which is zero. Thus M has an i-generated submodule of dimension $s_v(\alpha) = \alpha - \epsilon_v$. By induction this is a root, hence so is α.

(2) Again we prove this by induction on $\sum_i \alpha_i$. Suppose that $(\alpha, \epsilon_i) > 0$ for some vertex $i \neq v$. Then i must be loopfree. Now $s_i(\alpha) = \alpha - (\alpha, \epsilon_i)\epsilon_i$ is a real root, so by induction there is a unique v-generated module N of this dimension. Now $s_i(\alpha) - \epsilon_i$ is not a root, since

$$(s_i(\alpha) - \epsilon_i, s_i(\alpha) - \epsilon_i) = 4 + 2(\alpha, \epsilon_i),$$

so by (1) we must have $\mathrm{Hom}(S_i, N) = 0$. Also $\mathrm{Hom}(N, S_i) = 0$ since N is v-generated. Thus $\dim \mathrm{Ext}^1(N, S_i) = (\alpha, \epsilon_i)$, and the universal extension

$$0 \to S_i^{(\alpha, \epsilon_i)} \to M \to N \to 0$$

is a v-generated module of dimension α. Moreover this module is unique, since any v-generated module M of dimension α has $\mathrm{Ext}^1(M, S_i) = 0$ (as a non-split extension gives a v-generated module of dimension $\alpha + \epsilon_i$, but this is not a root), so it has $\dim \mathrm{Hom}(S_i, M) = (\alpha, \epsilon_i)$, and by the uniqueness of N it fits into an exact sequence as above.

Thus suppose that $(\alpha, \epsilon_i) \leq 0$ for all $i \neq v$. Since α is a real root, it follows that $(\alpha, \epsilon_v) > 0$, and apart from the trivial case $\alpha = \epsilon_v$, we are in the situation as in (1) of arrows $a : v \to i$, $a^* : i \to v$, with $\alpha_i = 1$ and $(\alpha, \epsilon_i) = 1$. Now $\alpha - \epsilon_v = s_v(\alpha)$ is a real root, so there is a unique i-generated module of this dimension, and now taking a to be a non-zero 1×1 matrix and a^* to be zero, we clearly get a unique v-generated module of dimension α.

Finally (2′) follows by inspection. □

3. Moduli spaces

Let Q be a quiver with vertex set I and let K be an algebraically closed field of characteristic zero. If $\alpha \in \mathbb{N}^I$, then $K\overline{Q}$-modules of dimension vector α are given by elements of the variety

$$\operatorname{Rep}(\overline{Q},\alpha) = \prod_{\substack{a\in\overline{Q} \\ a:i\to j}} \operatorname{Mat}(\alpha_j \times \alpha_i, K).$$

We denote by $\operatorname{Rep}(\Pi(Q),\alpha)$ the closed subspace of $\operatorname{Rep}(\overline{Q},\alpha)$ corresponding to modules for $\Pi(Q)$. The group

$$\operatorname{GL}(\alpha) = \prod_{i\in I} \operatorname{GL}(\alpha_i, K)$$

acts on both of these spaces, and the orbits correspond to isomorphism classes.

Let θ be a homomorphism $\mathbb{Z}^I \to \mathbb{Z}$. A $K\overline{Q}$-module M is said to be *θ-semistable* (respectively *θ-stable*) if $\theta(\underline{\dim}\, M) = 0$, but $\theta(\underline{\dim}\, M') \geq 0$ for every submodule $M' \subseteq M$ (respectively $\theta(\underline{\dim}\, M') > 0$ for all non-zero proper submodules M' of M). King [10] has constructed a moduli space of θ-semistable $K\overline{Q}$-modules of dimension vector α. As a closed subset of this, there is a moduli space $\mathcal{M}_\theta(\Pi(Q),\alpha)$ of θ-semistable $\Pi(Q)$-modules of dimension α. Recall that two modules determine the same element in the moduli space if they have filtrations by θ-stable modules having the same associated graded modules. In particular, if

$$(\ddagger) \qquad \theta(\beta) \neq 0 \text{ for all } \beta \text{ strictly between } 0 \text{ and } \alpha$$

then all θ-semistable modules of dimension α are θ-stable, so the points of the moduli space correspond to isomorphism classes of θ-semistable modules.

If $\lambda \in K^I$ then the *deformed preprojective algebra* of weight λ is defined by

$$\Pi^\lambda(Q) = K\overline{Q}/(\sum_{a\in Q}[a,a^*] - \sum_{i\in I}\lambda_i e_i).$$

We denote by $\operatorname{Rep}(\Pi^\lambda(Q),\alpha)$ the closed subset of $\operatorname{Rep}(\overline{Q},\alpha)$ corresponding to $\Pi^\lambda(Q)$-modules, and by $\operatorname{Rep}(\Pi^\lambda(Q),\alpha) /\!\!/ \operatorname{GL}(\alpha)$ the affine quotient variety.

Lemma 3. *If $K = \mathbb{C}$, $\lambda \in \mathbb{Z}^I$ and θ is defined by $\theta(\beta) = \sum_i \lambda_i\beta_i$, then there is a set-theoretic bijection between $\mathcal{M}_\theta(\Pi(Q),\alpha)$ and $\operatorname{Rep}(\Pi^\lambda(Q),\alpha) /\!\!/ \operatorname{GL}(\alpha)$.*

Proof. This arises because both spaces occur as hyper-Kähler quotients. The proof is already familiar to specialists, see for example Section 8 of [14]. We make $\operatorname{Rep}(\overline{Q},\alpha)$ into a left module for the quaternions $\mathbb{H} = \mathbb{R}\oplus\mathbb{R}\mathbf{i}\oplus\mathbb{R}\mathbf{j}\oplus\mathbb{R}\mathbf{k}$ with the action of $\mathbf{i}$ given by the existing complex structure, and the action of $\mathbf{j}$ given by

$$\mathbf{j}(x_a, x_{a^*})_{a\in Q} = (-x_{a^*}^\dagger, x_a^\dagger)_{a\in Q},$$

where $\dagger$ is the conjugate transpose. Now the product of unitary groups

$$U(\alpha) = \prod_{i\in I} U(\alpha_i)$$

acts, and there are moment maps defined for $x \in \operatorname{Rep}(\overline{Q},\alpha)$ by

$$\mu_{\mathbb{C}}(x) = \sum_{a\in Q}[x_a, x_{a^*}] \in \prod_{i\in I}\operatorname{Mat}(\alpha_i,\mathbb{C}) \text{ and } \mu_{\mathbb{R}}(x) = \frac{i}{2}\sum_{a\in\overline{Q}}[x_a, x_a^\dagger] \in \prod_{i\in I}\operatorname{Lie} U(\alpha_i).$$

Letting $h = (\mathbf{i}-\mathbf{k})/\sqrt{2}$, direct calculation shows that

$$\mu_{\mathbb{C}}(hx) = \frac{1}{2}\big(\mu_{\mathbb{C}}(x)^\dagger - \mu_{\mathbb{C}}(x)\big) - i\mu_{\mathbb{R}}(x) \text{ and } \mu_{\mathbb{R}}(hx) = \frac{i}{2}\big(\mu_{\mathbb{C}}(x)^\dagger + \mu_{\mathbb{C}}(x)\big).$$

Since $\lambda \in \mathbb{Z}^I$, multiplication by h induces a bijection

$$\big(\mu_{\mathbb{C}}^{-1}(\lambda) \cap \mu_{\mathbb{R}}^{-1}(0)\big)/U(\alpha) \to \big(\mu_{\mathbb{C}}^{-1}(0) \cap \mu_{\mathbb{R}}^{-1}(i\lambda/2)\big)/U(\alpha).$$

Now [10], Proposition 6.5 (applied to the quiver $\overline{Q}$) implies that the orbit space $\big(\mu_{\mathbb{C}}^{-1}(\lambda) \cap \mu_{\mathbb{R}}^{-1}(0)\big)/U(\alpha)$ is bijective to the quotient $\mathrm{Rep}(\Pi^\lambda(Q),\alpha) /\!\!/ \mathrm{GL}(\alpha)$, and that $\big(\mu_{\mathbb{C}}^{-1}(0) \cap \mu_{\mathbb{R}}^{-1}(i\lambda/2)\big)/U(\alpha)$ is bijective to the moduli space $\mathcal{M}_\theta(\Pi(Q),\alpha)$. □

Proof of Lemma 2 (1), (2) *and* (3). (K algebraically closed of characteristic zero.) Let $\lambda \in \mathbb{Z}^I$ be a vector with $\lambda_i > 0$ for $i \neq v$ and $\sum_i \lambda_i \alpha_i = 0$, and let θ be as in Lemma 3. Clearly θ satisfies the condition (‡), and the moduli space $\mathcal{M}_\theta(\Pi(Q),\alpha)$ classifies the isomorphism classes of v-generated $\Pi(Q)$-modules of dimension α.

This moduli space is a variety which is defined over the algebraic closure of $\mathbb{Q}$, so to determine the number of points it contains, finite or infinite, we may assume that $K = \mathbb{C}$. By Lemma 3, there is a bijection between $\mathcal{M}_\theta(\Pi(Q),\alpha)$ and $\mathrm{Rep}(\Pi^\lambda(Q),\alpha) /\!\!/ \mathrm{GL}(\alpha)$. Now $\mathrm{Rep}(\Pi^\lambda(Q),\alpha) /\!\!/ \mathrm{GL}(\alpha)$ classifies the isomorphism classes of semisimple $\Pi^\lambda(Q)$-modules of dimension α. By the choice of λ, there is no dimension vector β strictly between 0 and α with $\sum_i \lambda_i\beta_i = 0$, and hence any $\Pi^\lambda(Q)$-module of dimension α must be simple ([4], Lemma 4.1).

Thus to prove (1), (2) and (3) it suffices to show that $\Pi^\lambda(Q)$ has no simple module of dimension α, a unique simple, or infinitely many simples, according to whether α is a non-root, a real root, or an imaginary root. This follows from the main theorem of [3]. Actually, for such special λ one doesn't need the full strength of that theorem. For example [3], Theorem 2.3 immediately implies that there is a simple module for $\Pi^\lambda(Q)$ of dimension α if and only if there is an indecomposable representation of Q of dimension α, so if and only if α is a root, by Kac's Theorem. □

4. Reformulation of the theorem

Let Γ be a finite subgroup of $\mathrm{SL}(2,\mathbb{C})$. We keep the notation of the introduction. Let Q be the quiver with vertex set $I = \{0,1,\dots,n\}$ obtained by choosing any orientation of the McKay graph. It is an extended Dynkin quiver with minimal positive imaginary root $\delta \in \mathbb{N}^I$ given by $\delta_i = \dim N_i$. We consider the preprojective algebra $\Pi(Q)$ with base field $K = \mathbb{C}$. (Note that a different orientation of Q would lead to an isomorphic preprojective algebra, see [4, Lemma 2.2]. However, some choice does have to be made in order to define the preprojective algebra.) Choose $\theta : \mathbb{Z}^I \to \mathbb{Z}$ with $\theta(\delta) = 0$ and $\theta(\epsilon_i) > 0$ for all $i \neq 0$. Since $\delta_0 = 1$, a module M of dimension δ is θ-semistable if and only if it is 0-generated. Moreover, since the condition (‡) of Section 3 holds, the points in the moduli space $\mathcal{M}_\theta(\Pi(Q),\delta)$ correspond to isomorphism classes of such modules. Now there is a projective morphism

$$\mathcal{M}_\theta(\Pi(Q),\delta) \to \mathcal{M}_0(\Pi(Q),\delta),$$

and by Cassens and Slodowy [2] this is the minimal resolution of the Kleinian singularity.

Let us explain why this is essentially the same as the Hilbert scheme construction of $\tilde{X}$ (see also [13]). Using the action of Γ on R, one can form the skew group algebra $R * \Gamma$, and letting

$$e = \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\gamma,$$

one can identify R together with the given action of Γ with the left $R*\Gamma$-module $(R*\Gamma)e$. Now any element $J \in \mathrm{Hilb}^{\Gamma}(\mathbb{C}^2)$ defines an $R*\Gamma$-module $N = R/J$ with the two properties

(1) $N \cong \mathbb{C}\Gamma$ as a $\mathbb{C}\Gamma$-module
(2) N is generated by an element in eN.

Moreover, (1) implies that eN is 1-dimensional, so that the generator in (2) is unique, up to a scalar, and one recovers J as the annihilator of the generating element. Now according to a calculation of Reiten and van den Bergh (see [4, Theorem 0.1]), the algebra $R*\Gamma$ is Morita equivalent to $\Pi(Q)$, with the module $(R*\Gamma)e$ corresponding to $\Pi(Q)e_0$, and $R*\Gamma$-modules whose underlying $\mathbb{C}\Gamma$-module is isomorphic to $\mathbb{C}\Gamma$ corresponding to $\Pi(Q)$-modules of dimension vector δ. (For the notion of Morita equivalence see [1].) It follows that $\mathcal{M}_\theta(\Pi(Q),\delta)$ can be considered as a moduli space of $R*\Gamma$-modules N satisfying (1) and (2), up to isomorphism. Thus $\mathcal{M}_\theta(\Pi(Q),\delta) \cong \mathrm{Hilb}^{\Gamma}(\mathbb{C}^2)$.

Under this isomorphism, the exceptional fibre in $\mathcal{M}_\theta(\Pi(Q),\delta)$ is given by the nilpotent modules. In fact an arbitrary $\Pi(Q)$-module of dimension δ is either nilpotent or simple, so its socle, $\operatorname{soc} M$, is either the whole of M and simple, or it is a sum of simples S_i. Clearly also, if M is 0-generated and of dimension δ, since $\delta_0 = 1$ we must have $[\operatorname{soc} M : S_0] = 0$.

Lemma 4. *Let $J \in \mathrm{Hilb}^{\Gamma}(\mathbb{C}^2)$ be in the exceptional fibre, and let M be the corresponding $\Pi(Q)$-module of dimension δ. If $i \neq 0$, then*

$$[V(J) : N_i] = \dim \operatorname{Ext}^1(M, S_i) = \dim \operatorname{Hom}(S_i, M) = [\operatorname{soc} M : S_i].$$

Proof. The right hand equality is obvious, and since M is generated by an element in $e_0 M$, we have $\operatorname{Hom}(M, S_i) = 0$, so the middle equality follows from the homological formula. Let N_i' be the simple $R*\Gamma$-module whose underlying $\mathbb{C}\Gamma$-module is equal to N_i, and on which x and y act as zero. Since $\mathfrak{n} = Re\mathfrak{n}$ and $eN_i' = 0$, we have $\operatorname{Hom}_{R*\Gamma}(\mathfrak{n}, N_i') = 0$, and hence by dimension shifting, since R is projective as a $R*\Gamma$-module, $\operatorname{Ext}^1_{R*\Gamma}(\mathfrak{n}, N_i') = 0$. Now since the first and last terms in the exact sequence

$$\operatorname{Hom}_{R*\Gamma}(R/\mathfrak{n}, N_i') \to \operatorname{Hom}_{R*\Gamma}(J/\mathfrak{n}, N_i') \to \operatorname{Ext}^1_{R*\Gamma}(R/J, N_i') \to \operatorname{Ext}^1_{R*\Gamma}(R/\mathfrak{n}, N_i')$$

are zero, the two middle terms are isomorphic. But $\operatorname{Hom}_{R*\Gamma}(J/\mathfrak{n}, N_i')$ has dimension $[V(J) : N_i]$, and $\operatorname{Ext}^1_{R*\Gamma}(R/J, N_i') \cong \operatorname{Ext}^1(M, S_i)$ by the Morita equivalence. □

Theorem 1 can thus be reformulated as follows.

Theorem 2. *The socle of any module in $\mathcal{M}_\theta(\Pi(Q),\delta)$ has at most two simple summands, and if two, they are non-isomorphic. If $i \neq 0$, then*

$$E(i) = \{M \mid [\operatorname{soc} M : S_i] \neq 0\}$$

is a closed subset of $\mathcal{M}_\theta(\Pi(Q),\delta)$ isomorphic to $\mathbb{P}^1$. Moreover $E(i)$ meets $E(j)$ if and only if i and j are adjacent in Q, and in this case $|E(i) \cap E(j)| = 1$.

Remark. The quotient $\mathfrak{m}/\mathfrak{n}$ considered by Ito and Nakamura corresponds, under the Morita equivalence between $R*\Gamma$ and $\Pi(Q)$, to the $\Pi(Q)$-module

$$P = \bigoplus_{\substack{a \in \overline{Q} \\ a:0 \to i}} \Pi(Q)e_i / \Pi(Q)e_0\Pi(Q)e_i \cong \bigoplus_{\substack{a \in \overline{Q} \\ a:0 \to i}} \Pi(Q')e_i,$$

where Q' is the Dynkin quiver obtained by deleting the vertex 0 from Q. Now preprojective algebras of Dynkin quivers are known to be finite-dimensional self-injective algebras, and it is easy to see that P is a projective-injective module whose top is isomorphic to its socle. Moreover, the decomposition of $\Pi(Q')$ as the direct sum of one copy of each indecomposable right $\mathbb{C}Q'$-module gives, on tensoring with P, a vector space decomposition of P whose summands correspond to the spaces $S_m(\mathfrak{m}/\mathfrak{n})[\rho]$ of Ito and Nakamura. Auslander-Reiten theory for Q' can be used to compute the dimensions of these summands. This gives another approach to the "duality theorems" of Ito and Nakamura [7].

5. Proof of Theorem 2

We keep the notation of Section 4. Recall that for an extended Dynkin quiver Q, the set of real roots is invariant under translation by δ.

Lemma 5. *There is no module in $\mathcal{M}_\theta(\Pi(Q),\delta)$ whose socle involves two copies of a simple S_i or simples S_i and S_j where i and j are not adjacent in Q.*

Proof. If there is such a module, then the quotient by the relevant length-two submodule is a 0-generated module of dimension $\delta - 2\epsilon_i$ or $\delta - \epsilon_i - \epsilon_j$, but neither of these are roots. ☐

Lemma 6. *If $i \neq 0$ and $j \neq 0$ are adjacent in Q, then there is a unique module in $\mathcal{M}_\theta(\Pi(Q),\delta)$ with socle $S_i \oplus S_j$.*

Proof. First existence. Since $\delta - \epsilon_i - \epsilon_j$ is a real root, there is a unique 0-generated module N of this dimension. Now

$$\dim \operatorname{Ext}^1(N, S_i) = \dim \operatorname{Hom}(S_i, N) + 1$$

and $\operatorname{Hom}(S_i, N) = 0$, for otherwise N has a quotient of dimension $\delta - 2\epsilon_i - \epsilon_j$, but this is not a root. Thus $\dim \operatorname{Ext}^1(N, S_i) = 1$, and similarly $\dim \operatorname{Ext}^1(N, S_j) = 1$. Now there is a "simultaneous universal extension"

$$0 \to S_i \oplus S_j \to M \to N \to 0.$$

Clearly M has dimension δ, and its socle contains $S_i \oplus S_j$. By Lemma 5 its socle can be no larger than this.

For uniqueness, note that any module M in $\mathcal{M}_\theta(\Pi(Q),\delta)$ with socle $S_i \oplus S_j$ fits in an exact sequence of the same form, and since N is unique up to isomorphism, and $\dim \operatorname{Ext}^1(N, S_i) = \dim \operatorname{Ext}^1(N, S_j) = 1$, the uniqueness of M follows. ☐

Lemma 7. *If $i \neq 0$ then $E(i)$ is closed in $\mathcal{M}_\theta(\Pi(Q),\delta)$ and $E(i) \cong \mathbb{P}^1$.*

Proof. Since $\delta + \epsilon_i$ is a real root, there is a unique 0-generated module L of this dimension. By the homological formula $\dim \operatorname{Hom}(S_i, L) \geq 2$, and we have equality, for otherwise L has a quotient of dimension $\delta - 2\epsilon_i$, but this is not a root.

Any module M in $E(i)$ has $\dim \operatorname{Ext}^1(M, S_i) = 1$, and the middle term of the non-split exact sequence must be isomorphic to L. Thus M is isomorphic to a quotient of L by a submodule isomorphic to S_i. Thus, taking cokernels defines a map $c : \mathbb{P}(\operatorname{Hom}(S_i, L)) \to E(i)$, which is onto, and clearly also 1-1 since L has trivial endomorphism ring. Since $\mathbb{P}(\operatorname{Hom}(S_i, L)) \cong \mathbb{P}^1$, this gives the result, except that we need to prove that c and its inverse are morphisms of varieties, and for this we need to go into the details of moduli spaces.

Let $\mathrm{Rep}(\Pi(Q),\delta)_\theta$ be the open set of θ-semistable elements of $\mathrm{Rep}(\Pi(Q),\delta)$. Since θ-semistable modules for $\Pi(Q)$ of dimension δ are automatically θ-stable, $\mathcal{M}_\theta(\Pi(Q),\delta)$ is the geometric quotient of $\mathrm{Rep}(\Pi(Q),\delta)_\theta$ by $\mathrm{GL}(\delta)$.

If a, b is a basis for $\mathrm{Hom}(S_i, L)$, then its coordinate ring is $\mathbb{C}[x,y]$ where x, y is the dual basis to a, b. We consider the map $S_i \otimes R \to L \otimes R$ sending $s \otimes r$ to $a(s) \otimes xr + b(s) \otimes yr$. The cokernel of this map is a $\Pi(Q)$-$\mathbb{C}[x,y]$-bimodule B. Tensoring B with any simple $\mathbb{C}[x,y]$-module, except for the one on which x and y act as zero, gives a θ-semistable $\Pi(Q)$-module of dimension δ. Thus B defines a morphism from $\mathrm{Hom}(S_i, L) \setminus \{0\}$ to $\mathrm{Rep}(\Pi(Q),\delta)_\theta$. This descends to a morphism from $\mathbb{P}(\mathrm{Hom}(S_i, L))$ to $\mathcal{M}_\theta(\Pi(Q),\delta)$, which is clearly equal to c, so c is a morphism. Note that this implies that $E(i)$ is a closed subset.

Now $E(i)$ is the image in $\mathcal{M}_\theta(\Pi(Q),\delta)$ of

$$F(i) = \{M \in \mathrm{Rep}(\Pi(Q),\delta)_\theta \mid \mathrm{Hom}(S_i, M) \neq 0\}.$$

If $M \in F(i)$ then $\dim \mathrm{Hom}(L, M) = 1$, so if U is the variety consisiting of pairs (M, f) where $M \in F(i)$ and $0 \neq f \in \mathrm{Hom}(L, M)$, then the map $U \to F(i)$ is a geometric quotient for the action of $\mathbb{C}^*$ which rescales f. Also, if V is the variety of triples (M, f, g) where $(M, f) \in U$ and $0 \neq g \in \mathrm{Hom}(S_i, L)$ is a map with $fg = 0$, then $V \to U$ is a geometric quotient for the action of $\mathbb{C}^*$ which rescales g. Now the natural map $V \to \mathbb{P}(\mathrm{Hom}(S_i, L))$ is invariant under all of the group actions, so it descends to a morphism from U, then to a morphism from $F(i)$, and finally to a morphism $E(i) \to \mathbb{P}(\mathrm{Hom}(S_i, L))$. Thus c^{-1} is a morphism. □

Theorem 2 now follows from Lemmas 5, 6 and 7. Note that the theorem holds, with this proof, for an arbitrary algebraically closed base field of characteristic zero.

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