# Classification of the indecomposable finite dimensional modules of clannish algebras

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## Abstract

In this thesis we classify all indecomposable finite dimensional modules of clannish algebras with idempotent relations on the special loops. To this end, we start with the introduction of the notion of asymmetric and symmetric strings and bands in terms of words. The classification will be given in terms of those. We first examine directions on special letters in these words of a clannish algebra. Then we reduce the case to skewed-gentle algebras and construct a bundle of semichains for such an algebra. Thus, we are able to reduce the classification problem for skewed-gentle algebras to the matrix problem of bundles of semichains studied by Bondarenko. From this problem, we extract one classification of the indecomposable finite dimensional modules of a skewed-gentle algebra. From this classification, we can deduce a classification for clannish algebras. Finally, we adjust this classification to obtain one similar to that obtained by Crawley-Boevey, in which the symmetric band modules are indexed by a vector space equipped with a pair of idempotent endomorphisms. In contrast to Crawley-Boevey's classification, however, ours gives representations with better bases, not requiring the introduction of a fixed non-zero non-identity element of the field, and so working over the field with two elements. Applied to the algebra generated by an idempotent and a square zero element, it confirms a conjecture of Crawley-Boevey in 1988.

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# Contents

# 1 Introduction

Interesting problems from representation theory can be described by quivers with certain relations. One type of such problems are *clannish* algebras. They were introduced in 1989 by Crawley-Boevey [CB89] and are the main subject of this thesis. Crawley-Boevey classified the modules of clannish algebras in terms of strings and bands if the underlying field has at least three elements. This is a direct conclusion of the classification of the indecomposable representations of *clans* in [CB89], whose notion was motivated in particular in order to solve the Gel'fand problem [Gel71]. The method used to obtain the classification for clans is the so called *functorial filtration*. It was developed by Gel'fand and Ponomarev [GP68] and set by Gabriel into the functorial setting. Ringel also applied this method for the classification of the indecomposable representations of the Dihedral 2-Groups [Rin75]. This classification is also given in terms of strings and bands. Crawley-Boevey used the same approach for another of his papers prior to [CB89]: in [CB88], Crawley-Boevey gives a classification of the indecomposable modules of the clannish algebra $\mathrm{k}\langle \varepsilon, a \mid \varepsilon^2 = \varepsilon, a^2 = 0\rangle$. Similar to the result in [CB89], this classification does not include algebras which have a base field with less than three elements. This is due to the introduction of $t$, a non-linear combination of the letter $\varepsilon$ and 1. However, Crawley-Boevey conjectured in the introduction of [CB88] (near the end of page 386) the existence of an analogous classification for arbitrary fields given in the original alphabet (replacing the letter $t$ by $\varepsilon$) of the algebra.
The aim of this thesis is to give a classification of the indecomposable finite dimensional modules of a clannish algebra in terms of *asymmetric and symmetric strings and bands* independent of the cardinality of the base field. The algebra considered in [CB88] will serve as one of our standard examples. In order to fill the gap in the existing classification, we will use a different approach than the functorial filtration method. To this end, we will at first consider *skewed-gentle* algebras, give a classification for them and then deduce from that one the classification for clannish algebras. We can proceed in this way since the skewed-gentle algebras belong by definition to the class of (quasi-)clannish algebras. Skewed-gentle algebras were introduced by Geiß and de la Peña as a specification of *quasi-clannish* algebras [GdlPn99] which are a generalisation of clannish algebras.
Similar to [BMM03], we will exploit the connection between skewed-gentle algebras and a matrix problem. Here, we reduce the classification problem of indecomposable modules of a skewed-gentle algebra to the matrix problem in terms of bundles of semichains introduced by Bondarenko in 1988 [Bon88, Bon91]. This will enable us to obtain a first formulation of the indecomposale finite dimensional modules of a skewed-gentle algebra in terms of strings and bands and deduce a respective first formulation for clannish algebras. However, this first formulation does not confirm Crawley-Boevey's

conjecture yet. In order to do so, we will refine the first classification result by applying results from [Bre74].
The same matrix problem which was studied by Bondarenko, was also studied by Deng in the context of bushes [Den00]. He found that his results can be applied to clannish algebras including base fields of cardinality two. However, the results by Deng cannot directly confirm the conjecture imposed by Crawley-Boevey in [CB88].
In this thesis, we draw clear parallels between chains and cycles as considered by Bondarenko and string and band modules of skewed-gentle algebras. We also include a detailed discussion of the relevant results of Bondarenko since they are pertinent to our approach. We include additional examples to shed some light on the results which are available in [Bon88] but not in the english translation [Bon91]. Additionally, they will help the reader to gain a better understanding of the technical construction given by Bondarenko.

## 1.1 Outline

This thesis is structured as follows:
In Chapter 2 we collect the preliminaries which are used in the later chapters. Those include the basics of representation theory as well as the notion of *clannish algebras*. Based on its definition, we define *words*. Here, we distinguish in particular between ordinary and special letters. This property is deduced from the definition of a clannish algebra. Furthermore, we introduce the properties *coadmissibility* and *minimality* for words and describe associated modules. For those, so called periodic words are of particular interest.
We proceed in Chapter 3 by defining *asymmetric* and *symmetric strings* and *bands* which are given by so called undirected words. The classification (Theorem 6.9, Theorem 6.10) will be given in terms of those. To any directed word, an undirected one can be associated. We introduce two types of directed words (*weakly consistent* and *consistent* words) and compare those types for directed words which have as associated undirected words asymmetric and symmetric strings and bands. The definition of the weakly consistent and consistent words depends on a linear ordering on certain words which we introduce in the previous chapter.
We start Chapter 4 by presenting the results of [Bon88, Bon91] and by giving detailed examples which we found to be missing in the existing literature. In doing so, we fix notation for this type of matrix problem. After introducing the basics on this topic, we give an explicit construction in order to show that for any skewed-gentle algebra $\Lambda$ there exists a bundle of semichains $\overline{\mathfrak{X}}_\Lambda$ (Theorem 4.70). From this construction on, we reduce ourselves to skewed-gentle algebras. The construction allows us to give an explicit description on how to obtain $\mathfrak{L}$–graphs from undirected words. We show in Theorem 4.113 that any asymmetric and symmetric string and band results in an $\mathfrak{L}$–graph which leads to a canonical $\overline{\mathfrak{X}}_\Lambda$–representation. We even find by Corollary

4.117 that there exists a 1-1-correspondence between their equivalence classes and the isomorphism classes of the $\mathfrak{L}$–chains which give canonical representations. We obtain similar results with respect to asymmetric and symmetric bands and simple $\mathfrak{L}$–cycles (Theorem 4.130, Corollary 4.142). We prove in Section 4.7 that the directions added on the constructed $\mathfrak{L}$–graphs coincide with those on letters of so called finite index in weakly consistent and consistent words.
The main results which are required in order to obtain the classification, are to be found in Chapter 5. We show that the category $\mathrm{mod}(\Lambda)$ of finite dimensional modules of a skewed-gentle algebra $\Lambda$ and the category $\mathrm{Rep}(\overline{\mathfrak{X}}_\Lambda)$ of representations of the bundle of semichains $\overline{\mathfrak{X}}_\Lambda$ are equivalent (Theorem 5.6). We present a classification of the finite dimensional modules of the skewed-gentle algebra $\Lambda$ in terms of strings and bands (Theorem 5.49). From this, we deduce a respective classification for clannish algebras $\overline{\Lambda}$ (Theorem 5.50).
In the final chapter, Chapter 6, we examine the symmetry axes of the symmetric bands more closely in the context of the four subspace problem. In order to do so, we apply results of [Bre74]. Finally, these results allow us to refine the classification result as formulated in Chapter 4 such that we will be able to confirm the conjecture stated by Crawley-Boevey in [CB88] (Theorem 6.10).

## 1.2 Main Theorem

### 1.2.1 Main Theorem for the algebra $\mathrm{k}\langle\varepsilon, a \mid \varepsilon^2 = \varepsilon, a^2 = 0\rangle$

We start this Subsection by first giving the respective result from [CB88].
To this end, let k be a field with at least three elements and let $\Lambda = \mathrm{k}\langle\varepsilon, a \mid \varepsilon^2 = \varepsilon, a^2 = 0\rangle$. We consider the alphabet $\Gamma = \{a, a^{-1}, t^*\}$, where $t = \lambda\varepsilon - \mu 1$ with $0 \neq \lambda, \mu \in \mathrm{k}$ and $\lambda \neq \mu$. We call $t^*$ a *special* letter with formal inverse given by itself. We build words by considering sequences in the letters of $\Gamma$, in which each $t^*$ is either followed by $a$ or $a^{-1}$, and each $a$ or $a^{-1}$ is followed by $t^*$.
For the classification, we want to consider only certain words. We distinguish between asymmetric and symmetric strings and bands. Strings are finite words in $a$, $a^{-1}$ and $t^*$, whose first and last letters are given by $t^*$. A string is symmetric if it is equal to its inverse which is obtained by reversing the order of the letters, and exchanging $a$ and $a^{-1}$. An example for an asymmetric string is the following: $t^*at^*$ since it is unequal to its inverse $t^*a^{-1}t^*$. An example for a symmetric string is given by $t^*at^*a^{-1}t^*$. Important is that we have that the sequence left of the middle $t^*$ is equal to the inverse of the sequence right of it. Thus, this letter $t^*$ in the middle of the word gives a symmetry axis.
Bands are defined by infinite periodic words. To specify a band it suffices to give one period. An example for an asymmetric band can be described

by its periodic part, the finite subsequence $t^*a$. The whole band is then of the form $\dots t^*at^*at^*a\dots$. Important is that when considering the whole word, it is not equal to its inverse shifted by any position. For a symmetric band, we want exactly the opposite. We want that it is equal to its inverse shifted by some position. An example for a symmetric band can be described by the finite subsequence $t^*at^*at^*a^{-1}t^*a^{-1}$ which results in the word $\dots t^*at^*at^*a^{-1}t^*a^{-1}t^*at^*at^*a^{-1}t^*a^{-1}\dots$. We see - by comparing the left subsequence to the inverse of the right subsequence - that there are two symmetry axes per periodic part in the band, given by the first $t^*$ and the one in fifth position.
We want to consider modules given by exactly those four types of words. For this we need to replace each letter $t^*$ by either $t$ or $t^{-1}$. We can consider $t^*$ as a placeholder for one of the other two letters. The question is which of the two letters to use to replace $t^*$. Crawley-Boevey gives an answer by giving an ordering on the words. We consider in a word the inverse of the subsequence left of a letter $t^*$ and compare it to the subsequence right of $t^*$. Comparing those two subsequences defines by which letter $t^*$ is replaced. Here, we replace $t^*$ by $t$ if the inverse of the left subsequence is bigger than the right subsequence. Otherwise, we replace $t^*$ by $t^{-1}$. There are some letters - those which we have called symmetry axes above - which do not obtain a unique replacement this way since the inverse of their left subsequence is equal to their right subsequence. But we will see in the presentation of the modules that we can omit this discussion.
Considering the words with $t^*$ replaced as described above, they describe modules. Each string results in a string module, and each band in a band module. For the bands, their repeating structure allows to only consider one of their periodic parts. We will give examples for the words above. To this end, we display each letter of the form $x$ by an arrow from the right to the left, and each letter of the form $x^{-1}$ by an arrow from the left to the right. Now setting vector spaces from a certain module category at its vertices describes the respective module, consisting of a vector space given by the direct sum of the vector spaces at its vertices and the action of the algebra described by the displayed arrows. Note that all considered vector spaces are finite dimensional.
For the asymmetric string $t^*at^*$ we replace each $t^*$ such that we consider $tat$ for the module. Let $V$ be a $\mathrm{k}$-module. The $\Lambda$-module $M(tat, V)$ with the $V_i's$ being disjoint copies of $V$ is described by

$$V_0 \xleftarrow{t} V_1 \xleftarrow{a} V_2 \xleftarrow{t} V_3\ . \tag{1}$$

This module gives an example in the image of the functor in [CB88, §1, p. 388], applied to $\mathrm{mod}\,\mathrm{k}$.
For the symmetric string $t^*at^*a^{-1}t^*$ we consider $tat^*a^{-1}t^{-1}$. Here, the letter $t^*$ in the middle has not been replaced since it is a symmetry axis, but is

determined by the category of the vector space. Let $V$ be in $\operatorname{mod} \mathrm{k}[s]/(q(s))$, where $q$ describes the quadratic polynomial with $q(t) = 0$. The $\Lambda$–module $M(tat^*a^{-1}t^{-1}, V)$ with the $V_i's$ being disjoint copies of $V$ is given by:

$$\begin{tikzcd} V_0 & V_1 \arrow[l, "t"'] & V_2 \arrow[l, "a"'] \arrow[loop right, "t=s"] \end{tikzcd} . \tag{2}$$

Since $t$ is a linear combination of $\varepsilon$ and 1, the module can also be displayed with $V$ being a $\mathrm{k}[f \mid f^2 = f]$–module and with $\varepsilon = f$ on the loop. This module then gives an example in the image of the functor in [CB88, §1, p. 388], applied to $\operatorname{mod} \mathrm{k}[f \mid f^2 = f]$.
For the asymmetric band given by $\ldots t^*at^*at^*a\ldots$ we consider the subsequence $ta$. We consider $V$ in $\operatorname{mod} \mathrm{k}[T, T^{-1}]$ where $T$ acts as the shift between the repetitions. The $\Lambda$–module $M(\ldots tata\ldots, V)$ with the $V_i's$ being disjoint copies of $V$ is given by

$$\begin{tikzcd} V_0 \arrow[r, bend right, "a"'] & V_1 \arrow[l, "t"'] \end{tikzcd} . \tag{3}$$

This module gives an example in the image of the functor in [CB88, §1, p. 388], applied to $\operatorname{mod} \mathrm{k}[T, T^{-1}]$.
We consider for the symmetric band $\ldots t^*at^*at^*a^{-1}t^*a^{-1}\ldots$ the subsequence $t^*at^*at^*a^{-1}t^{-1}a^{-1}$. Similar to the symmetric string case, the letters giving the symmetry axes are defined by the category of the vector space. Let $V$ be in $\operatorname{mod} \mathrm{k}\langle f, g\rangle/(q(f), q(g))$ with $q$ the quadratic polynomial as above. The $\Lambda$–module $M(\ldots t^*atat^*a^{-1}t^{-1}a^{-1}\ldots, V)$ with the $V_i$'s being disjoint copies of $V$ is given by

$$\begin{tikzcd} V_0 \arrow[loop left, "t=f"] & V_1 \arrow[l, "a"'] & V_2 \arrow[l, "t"'] & V_3 \arrow[l, "a"'] \arrow[loop right, "t=g"] \end{tikzcd} . \tag{4}$$

Similar to the module of a symmetric string, we can express this module in terms of $\varepsilon$ on the loops with $V$ being a $\mathrm{k}\langle f, g \mid f^2 = f, g^2 = g\rangle$–module. It gives an example in the image of the functor in [CB88, §1, p. 388], applied to $\operatorname{mod} \mathrm{k}\langle f, g \mid f^2 = f, g^2 = g\rangle$.
Crawley-Boevey states in [CB88] that the modules of this form give indecomposable finite dimensional modules of the algebra:

**Theorem 1.1.** *[CB88, Main Theorem] Let* k *be a field with at least three elements, let* $\Lambda = \mathrm{k}\langle \varepsilon, a \mid \varepsilon^2 = \varepsilon, a^2 = 0\rangle$. *Let* $\mathcal{M}$ *be a list of modules obtained from all asymmetric and symmetric strings and bands as described above in examples (1)-(4), with the modules* $V$ *running through a complete list of non-isomorphic indecomposable modules for each module category. Then* $\mathcal{M}$ *gives a complete list of non-isomorphic indecomposable* $\Lambda$*–modules.*

As we have seen, this result does not operate on the original alphabet of the algebra but introduces the letter $t^*$ $(t, t^{-1})$. Due to this introduction,

the field k is required to have at least three elements.
The main result of this thesis confirms Crawley-Boevey's conjecture from the same paper [CB88]. Here (near to the end of page 386), he conjectures that there is an analogous classification to the one given above in which the letter $t^*$ $(t, t^{-1})$ is replaced by the letter $\varepsilon^*$ $(\varepsilon, \varepsilon^{-1})$. This classification would hold for arbitrary fields.
In our setup, by replacing $t^*$ by $\varepsilon^*$, the above given examples for strings and bands correspond to the following:
The word $\varepsilon^* a \varepsilon^*$ is an asymmetric string and $\varepsilon^* a \varepsilon^* a^{-1} \varepsilon^*$ describes a symmetric string. Repeating the sequence $\varepsilon^* a$ gives an asymmetric band, while repeating $\varepsilon^* a \varepsilon^* a \varepsilon^* a^{-1} \varepsilon^* a^{-1}$ describes a symmetric band. We have replaced each $t^*$ by $\varepsilon^*$. It follows that we need to replace each of these $\varepsilon^*$ by either $\varepsilon$ or $\varepsilon^{-1}$. In order to do so, we proceed analogously as above with the $t^*$'s. Also for building the modules from those words, we proceed analogously as above. This leaves us with the following modules: For the asymmetric string we obtain from $\varepsilon a \varepsilon$ the $\Lambda$–module $M(\varepsilon a \varepsilon, V)$ with $V$ a k–module, and the $V_i$'s disjoint copies of $V$:

$$V_0 \xleftarrow{\varepsilon} V_1 \xleftarrow{a} V_2 \xleftarrow{\varepsilon} V_3 \,. \tag{5}$$

For the symmetric string we consider $\varepsilon a \varepsilon^* a^{-1} \varepsilon^{-1}$. The $\Lambda$–module $M(\varepsilon a \varepsilon^* a^{-1} \varepsilon^{-1}, V)$ with $V$ in $\operatorname{mod} \mathrm{k}[f \mid f^2 = f]$ and the $V_i$'s disjoint copies of $V$, is given by:

$$\begin{tikzcd} V_0 & V_1 \arrow[l, "\varepsilon"'] & V_2 \arrow[l, "a"'] \arrow[loop right, "\varepsilon = f"] \end{tikzcd} \,. \tag{6}$$

For the asymmetric band we consider $\varepsilon a$. The $\Lambda$–module $M(\dots \varepsilon a \varepsilon a \dots, V)$ with $V$ in $\operatorname{mod} k[T, T^{-1}]$ and the $V_i$'s being disjoint copies of $V$, is given by

$$\begin{tikzcd} V_0 \arrow[r, bend right, "a"'] & V_1 \arrow[l, "\varepsilon"'] \end{tikzcd} \,. \tag{7}$$

Finally, we consider the symmetric band. Here, we obtain the $\Lambda$–module $M(\dots \varepsilon^* a \varepsilon a \varepsilon^* a^{-1} \varepsilon^{-1} a^{-1} \dots, V)$ with $V$ in $\operatorname{mod} \mathrm{k}\langle f, g \mid f^2 = f, g^2 = g \rangle$, and all $V_i$'s being disjoint copies of $V$:

$$\begin{tikzcd} V_0 \arrow[loop left, "\varepsilon = f"] & V_1 \arrow[l, "a"'] & V_2 \arrow[l, "\varepsilon"'] & V_3 \arrow[l, "a"'] \arrow[loop right, "\varepsilon = g"] \end{tikzcd} \,. \tag{8}$$

Here, a depiction of the form $V_0 \xleftarrow{\varepsilon} V_1$ means that we have for $\varepsilon(v_1) = v_0$ that $\varepsilon(v_0) = v_0$, where $v_0 \in V_0$, $v_1 \in V_1$. This is due to the idempotent relation on $\varepsilon$.
Our main theorem applied to $\mathrm{k}\langle \varepsilon, a \mid \varepsilon^2 = \varepsilon, a^2 = 0 \rangle$ reads:
**Main Theorem.** *($\mathrm{k}\langle \varepsilon, a \mid \varepsilon^2 = \varepsilon, a^2 = 0 \rangle$) Let* k *be an arbitrary field, let $\Lambda = \mathrm{k}\langle \varepsilon, a \mid \varepsilon^2 = \varepsilon, a^2 = 0 \rangle$. Let $\mathcal{M}$ be a list of modules obtained from all asymmetric and symmetric strings and bands as described in examples (5)-(8), with the modules $V$ running through a complete list of non-isomorphic indecomposable modules for each module category. Then $\mathcal{M}$ gives a complete list of non-isomorphic indecomposable $\Lambda$–modules.*

### 1.2.2 General formulation of the Main Theorem

Let k be a field. Let $w$ be an asymmetric string and let $V$ be a k−module. Then we denote by $\mathcal{M}_1(w,V)$ the following module:

$$\begin{tikzcd} V_0 & V_1 \arrow[l, "w_1^{\kappa_1}"'] & V_2 \arrow[l, "w_2^{\kappa_2}"'] & \cdots \arrow[l, "w_3^{\kappa_3}"'] & V_n \arrow[l, "w_n^{\kappa_n}"'] \end{tikzcd}$$

where $\kappa_i \in \{+1,-1\}$ and $w = w_1 \dots w_n$, which we will discuss in detail in Section 2.4.
Let $w = u\varepsilon^* u^{-1}$ be a symmetric string (with $m$ being the length of $u$) and let $V$ be a $\mathrm{k}[f \mid f^2 = f]$−module. Then we denote by $\mathcal{M}_2(w,V)$ the following module:

$$\begin{tikzcd} V_0 & V_1 \arrow[l, "w_1^{\kappa_1}"'] & V_2 \arrow[l, "w_2^{\kappa_2}"'] & \cdots \arrow[l, "w_3^{\kappa_3}"'] & V_m \arrow[l, "w_m^{\kappa_m}"'] \arrow[loop right, "\varepsilon=f"] \end{tikzcd} .$$

Let $w_{\mathbb{Z}}$ be an asymmetric band of period $p$, an let $V$ be a $\mathrm{k}[T,T^{-1}]$−module. We denote by $\mathcal{M}_3(v,V)$ the module

$$\begin{tikzcd} V_0 \arrow[rrrr, bend right, "w_p^{\kappa_p}"'] & V_1 \arrow[l, "w_1^{\kappa_1}"'] & V_2 \arrow[l, "w_2^{\kappa_2}"'] & \cdots \arrow[l, "w_3^{\kappa_3}"'] & V_{p-1} \arrow[l, "w_{p-1}^{\kappa_{p-1}}"'] \end{tikzcd} .$$

Let $w_{\mathbb{Z}}$ be a symmetric band of period $p$ with periodic part $\varepsilon^* u \eta^* u^{-1}$, $|u| = m$, and let $V$ be a $\mathrm{k}\langle f,g \mid f^2 = f, g^2 = g\rangle$−module. We denote by $\mathcal{M}_4(v,V)$ the module

$$\begin{tikzcd} V_0 \arrow[loop left, "\varepsilon=f"] & V_1 \arrow[l, "w_2^{\kappa_2}"'] & V_2 \arrow[l, "w_3^{\kappa_3}"'] & \cdots \arrow[l, "w_4^{\kappa_4}"'] & V_m \arrow[l, "w_{m+1}^{\kappa_{m+1}}"'] \arrow[loop right, "\eta=g"] \end{tikzcd} .$$

The $V_j$'s in the modules are disjoint copies of the given $V$. The $\kappa_i$'s are directed according to the linear ordering (cf. Definition 2.41) on the words for special letters $w_i$ (cf. Section 2.3). Otherwise, $\kappa_i$ is given by the ordinary letters (cf. Section 2.3) in $w$ or $w_{\mathbb{Z}}$, respectively.
Our final classification result reads as follows:

**Main Theorem.** *Let $\Lambda$ be a clannish algebra. The modules of the form $\mathcal{M}_i(w,V)$, $i = 1,2,3,4$, with $w$ running through the sets of asymmetric and symmetric strings and bands, respectively, and $V$ running through a complete list of non-isomorphic indecomposable modules for each module category, give a complete list of pairwise non-isomorphic indecomposable modules of $\Lambda$.*

# 2 Preliminaries

We start this chapter by revising the basics of representation theory in Section 2.1. Moreover, we introduce clannish and skewed-gentle algebras in Section 2.2 following [CB89] and [GdlPn99]. Section 2.3 deals with directed and undirected alphabets obtained from a clannish algebra and words built from any of the alphabets. In the next chapter, we define the notion of asymmetric and symmetric strings and bands which are given by equivalence classes of certain words. We examine in Subsection 2.3.1 the so called $\mathbb{Z}$–words more closely which lead to the notion of bands. In particular, we determine certain properties of words in Subsections 2.3.2 and 2.3.3. These properties are required in order to describe the words which lead to $\mathfrak{L}$–graphs giving canonical $\overline{\mathfrak{X}}_\Lambda$–representations (compare Sections 4.5 and 4.6). We close the chapter with Section 2.4 which explains how directed words describe $\Lambda$–modules.

Throughout this thesis let k be a field. Note that k is not necessarily algebraically closed.
Moreover, we denote by $\mathbb{N}$ the natural numbers including 0.

## 2.1 Quivers and their representations

We follow [ARS97]. For k algebraically closed, see also [ASS06].

A *quiver* $Q$ is given by a quadruple $(Q_0, Q_1, s, t)$ consisting of a finite set of *vertices* $Q_0$, a finite set of *arrows* $Q_1$ and two maps $s, t : Q_1 \longrightarrow Q_0$, assigning to each arrow $x \in Q_1$ its *source* $s(x)$ and its *target* $t(x)$, giving $x : s(x) \longrightarrow t(x)$. A *loop at vertex* $i$ is an arrow $x \in Q_1$ with $s(x) = t(x) = i$.
A *path* $p$ in Q is given by a sequence $p = p_n \dots p_1$ of arrows $p_i \in \mathrm{Q}_1$, $1 \le i \le n$, such that $s(p_{i+1}) = t(p_i)$. The *length* of such a path $p$ is $n$. We denote by $s(p) \coloneqq s(p_1)$ the source of $p$ and its target by $t(p) \coloneqq t(p_n)$. For each vertex $i \in Q_0$ we have the *trivial path* $e_i$ of length 0 with $s(e_i) = t(e_i) = i$.
The *concatenation* $p \circ q$ of two paths $p = p_n \dots p_1$ and $q = q_m \dots q_1$ is given by

$$p \circ q = \begin{cases} p_n \dots p_1 q_m \dots q_1 & \text{if } s(p_1) = t(q_m), \\ p_n \dots p_1 & \text{if } q = e_{s(p_1)}, \\ q_m \dots q_1 & \text{if } p = e_{t(q_m)}, \\ 0 & \text{otherwise.} \end{cases}$$

In the following, we also write $pq$ instead of $p \circ q$ for short.
A k–linear *representation* of $Q$ is given by a tuple $V = (V_i, V_a)_{i \in Q_0, a \in Q_1}$ where $V_i$ is a vector space for each $i \in Q_0$ and $V_a : V_i \to V_j$ is a linear map for each $a : i \to j \in Q_1$. The representation $V$ is called *finite dimensional* if $V_i$ is finite dimensional for all $i \in Q_0$.
Let $V = (V_i, V_a)_{i,a}$ and $W = (W_i, W_a)_{i,a}$ be two representations of a given

quiver $Q$. A *morphism* $f : V \longrightarrow W$ is given by a family $f = (f_i)_{i\in Q_0}$ of $\mathrm{k}$-linear maps $f_i : V_i \longrightarrow W_i$ such that $f_j V_a = W_a f_i$ for $(a : i \to j) \in Q_1$, i.e. the following diagram commutes:

$$\begin{tikzcd} V_i \arrow[r, "f_i"] \arrow[d, "V_a"'] & W_i \arrow[d, "W_a"] \\ V_j \arrow[r, "f_j"] & W_j \end{tikzcd}$$

The composition of two morphisms of representations is given in the obvious way. The direct sum $V \oplus W$ of $V$ and $W$ is given by

$$\left( V_i \oplus W_i, V_a \oplus W_a = \begin{pmatrix} V_a & 0 \\ 0 & W_a \end{pmatrix} \right)_{i,a}.$$

A representation $V = (V_i, V_a)_{i,a}$ is called *indecomposable* if it cannot be written as a direct sum $V = W \oplus U$ of two non-zero representations $W$, $U$.
We denote the category of representations of a quiver $Q$ by $\mathrm{Rep}(Q)$. The full subcategory of finite dimensional representations is denoted by $\mathrm{rep}(Q)$. If $Q$ is finite, then both categories are abelian.
Moreover, one can associate to any quiver $Q$ the *path algebra* $\mathrm{k}Q$. This is the $\mathrm{k}$-algebra with underlying $\mathrm{k}$-vector space with basis given by the paths in $Q$. The product of two basis elements is given by the above concatenation. The path algebra is associative. Furthermore, $\mathrm{k}Q$ is finite dimensional if and only if $Q$ does not have oriented cycles, see [ARS97, §III.1, Proposition 1.1]. It is unital with $1 = \sum_{v\in Q_0} e_v$.
A *relation* $r$ on a quiver $Q$ is a $\mathrm{k}$-linear combination of paths $p_i$ which have lengths at least two. For $\mathrm{R}$ a set of relations on $Q$, the pair $(Q, \mathrm{R})$ is called a *quiver with relations*. Its associated path algebra $\mathrm{k}(Q, \mathrm{R})$ is given by $\mathrm{k}Q/(\mathrm{R})$. Generally, we are going to consider for any algebra $\Lambda$ left $\Lambda$-modules and denote by $\mathrm{Mod}(\Lambda)$ the category of all those modules. We denote the full subcategory of finite dimensional modules by $\mathrm{mod}(\Lambda)$.
It is a well-known result, e.g. [ARS97, §III.1, Theorem 1.5], that the categories $\mathrm{rep}(Q)$ and $\mathrm{mod}(\mathrm{k}Q)$ are equivalent. This induces an equivalence between $\mathrm{rep}(Q, \mathrm{R})$ and $\mathrm{mod}(\mathrm{k}Q/(\mathrm{R}))$ [ARS97, §III.1, Proposition 1.7].
Furthermore, the following Krull-Remak-Schmidt-Theorem is well-known:

**Theorem 2.1.** *[ARS97, Theorem 2.2 (b)] Let $\Lambda$ be a $\mathrm{k}$-algebra and let $\{V_i\}_{i\in I}$ and $\{W_j\}_{j\in J}$ be two finite families of finitely generated indecomposable $\Lambda$-modules. If*

$$\coprod_{i\in I} V_i \cong \coprod_{j\in J} W_j,$$

*then there exists a permutation $\pi : I \to J$ such that $V_i \cong W_{\pi(i)}$ for all $i \in I$.*

## 2.2 Clannish algebras

We consider clannish algebras in the sense of [CB18], with one minor restriction: we ask the relations on the special arrows to be idempotent relations.

Let Q be a quiver, Sp a subset of the loops of Q. We call any $\varepsilon \in \mathrm{Sp}$ a *special arrow*, and in contrast to that any $a \in \mathrm{Q}_1 \setminus \mathrm{Sp}$ an *ordinary arrow*. We denote the set of ordinary arrows by $\mathrm{Q}_1^{\mathrm{ord}}$. Let $\mathrm{R}^{\mathrm{Sp}} \coloneqq \{\varepsilon^2 - \varepsilon \mid \varepsilon \in \mathrm{Sp}\}$ describe the idempotent relations on the special arrows, and let R be a set of monomial relations on Q which do not start or end in a special loop, nor involve the square of one.

**Definition 2.2.** *Let $\Lambda = \mathrm{k}\,\mathrm{Q}/(\mathrm{R} \cup \mathrm{R}^{\mathrm{Sp}})$. We call the algebra $\Lambda$* clannish *if the following conditions hold:*

*(i)* *at most two arrows start at any vertex: $|\{a \in \mathrm{Q}_1 \mid s(a) = v\}| \leq 2$ for all $v \in \mathrm{Q}_0$,*

*(i)** *at most two arrows terminate at any vertex: $|\{a \in \mathrm{Q}_1 \mid t(a) = v\}| \leq 2$ for all $v \in \mathrm{Q}_0$,*

*(ii)* *for any $a \in \mathrm{Q}_1^{\mathrm{ord}}$, there is at most one $c \in \mathrm{Q}_1$ such that $ca \notin \mathrm{R}$,*

*(ii)** *for any $a \in \mathrm{Q}_1^{\mathrm{ord}}$, there is at most one $b \in \mathrm{Q}_1$ such that $ab \notin \mathrm{R}$.*

**Example 2.3.** 1. *The algebra $\Lambda = \mathrm{k}Q/(\mathrm{R} \cup \mathrm{R}^{\mathrm{Sp}})$ with quiver $Q$*

$$\begin{tikzcd} \bullet \arrow[loop left, "\varepsilon"] \arrow[loop right, "a"] \end{tikzcd}$$

$\mathrm{Sp} = \{\varepsilon\}$ *and* $\mathrm{R} = \{a^2\}$ *is a clannish algebra.*

2. *Let $Q$ be given by*

$$\begin{tikzcd} \bullet \arrow[loop left, "\varepsilon"] \arrow[r, "x"] & \bullet \arrow[loop above, "\kappa"] \arrow[r, "y"] & \bullet \arrow[loop right, "\eta"] \end{tikzcd}$$

*with* $\mathrm{Sp} = \{\varepsilon, \eta, \kappa\}$ *and* $\mathrm{R} = \{yx\}$. *Then* $\mathrm{k}Q/(\mathrm{R} \cup \mathrm{R}^{\mathrm{Sp}})$ *is a clannish algebra.*

3. *Consider the quiver $Q$*

$$\begin{tikzcd} 1 \arrow[r, "x"] & 2 \arrow[loop above, "\kappa"] \arrow[r, "y"] & 3 \\ 4 \arrow[ur, "z"] & & \end{tikzcd}$$

*with* $\mathrm{Sp} = \{\kappa\}$ *and* $\mathrm{R} = \varnothing$. *The path algebra* $\mathrm{k}Q/(\mathrm{R} \cup \mathrm{R}^{\mathrm{Sp}})$ *is not clannish since* $(i)^*$ *does not hold for vertex 2.*

4. *The algebra* $\Lambda = \mathrm{k}\,Q/(\mathrm{R} \cup \mathrm{R}^{\mathrm{Sp}})$ *with quiver* $Q$

$$\begin{tikzcd} 1 \arrow[r, "x"] & 2 \arrow[r, "y"] & 3 \\ 4 \arrow[ur, "z"] & & \end{tikzcd}$$

*with* $\mathrm{Sp} = \varnothing$*,* $\mathrm{R} = \{yx, yz\}$ *is clannish.*

Similar to *clannish* is – as the name already suggests – the notion of *quasi-clannish* which is a generalization of the former. To this end, let $Q$, Sp, $Q_1^{\mathrm{ord}}$ and $\mathrm{R}^{\mathrm{Sp}}$ be as above. Denote by R a set of relations, by $(\bar{\mathrm{R}})$ the ideal in $\mathrm{k}\,Q/(\mathrm{R}^{\mathrm{Sp}})$ generated by the classes of elements in R. Let $J$ be the ideal in $\mathrm{k}\,Q/(\mathrm{R}^{\mathrm{Sp}})$ generated by the ordinary arrows.
We denote by $g$ the following automorphism of $\mathrm{k}\,Q/(\mathrm{R}^{\mathrm{Sp}})$ of order 2:

$$\begin{aligned} g(1_v) &= 1_v && \text{for all } v \in Q_0, \\ g(a) &= \begin{cases} a & \text{if } t(a) \in Q_0^{\mathrm{ord}}, \\ -a & \text{if } t(a) \in Q_0^{\mathrm{Sp}}, \end{cases} && \text{for all } a \in Q_1^{\mathrm{ord}}, \\ g(\varepsilon) &= 1_{s(\varepsilon)} - \varepsilon && \text{for all } \varepsilon \in \mathrm{Sp}, \end{aligned}$$

where $Q_0^{\mathrm{Sp}} = \{s(\varepsilon) \mid \varepsilon \in \mathrm{Sp}\}$, $Q_0^{\mathrm{ord}} = Q_1 \backslash Q_0^{\mathrm{Sp}}$.

**Definition 2.4.** *[GdlPn99, Definition (4.2)] Let* $\Lambda = \mathrm{k}\,Q/(\mathrm{R} \cup \mathrm{R}^{\mathrm{Sp}})$*. Then* $\Lambda$ *is called* quasi-clannish *if the following conditions hold:*

*(i)* $(\bar{\mathrm{R}}) \subseteq J^2$ *is a* $\langle g \rangle$*–ideal in* $\mathrm{k}\,Q/(\mathrm{R}^{\mathrm{Sp}})$*,*

*(ii)* *at most two arrows start at any vertex:* $|\{a \in Q_1 \mid s(a) = v\}| \le 2$ *for all* $v \in Q_0$*,*

*(ii)** *at most two arrwos terminate at any vertex:* $|\{a \in Q_1 \mid t(a) = v\}| \le 2$ *for all* $v \in Q_0$*,*

*(iii)* *for any* $a \in Q_1^{\mathrm{ord}}$ *there is at most one* $b \in Q_1$ *with* $ab \notin \mathrm{R}$*,*

*(iii)** *for any* $a \in Q_1^{\mathrm{ord}}$ *there is at most one* $c \in Q_1$ *with* $ca \notin \mathrm{R}$*.*

**Lemma 2.5.** *[GdlPn99] Any clannish algebra is quasi-clannish.*

**Example 2.6.** *By Lemma 2.5, Example 2.3.1., 2. and 4. also are quasi-clannish algebras.*

The converse of Lemma 2.5 does not hold in general. But one can restrict the notion of quasi-clannish as follows:

**Definition 2.7.** *Let* $\Lambda = \mathrm{k}\,Q/(\mathrm{R} \cup \mathrm{R}^{\mathrm{Sp}})$ *be quasi-clannish. We call* $\Lambda$ string-quasi-clannish *if* R *consists of monomial relations only and no relation contains the square of a special loop.*

Then we obtain the following result:

**Lemma 2.8.** *Any string-quasi-clannish algebra is clannish.*

*Proof.* It is enough to check that no relation starts or ends in a special loop. We give a proof by contradiction.
Let $\varepsilon \in \mathrm{Sp}$ and let $p$ and $q$ be paths in $Q$ such that $\varepsilon p$ and $q\varepsilon$ are again paths. Assume for simplicity that $p$ and $q$ only consist of ordinary arrows that do not end in vertices incident to special loops. By Definition 2.7, $(\bar{\mathrm{R}})$ is $\langle g \rangle$– invariant in $\mathrm{k}\,Q/(\mathrm{R}^{\mathrm{Sp}})$. The action of $g$ on $\varepsilon p$ and $q\varepsilon$ is given by the following:

$$g(\varepsilon p) = (1_{s(\varepsilon)-\varepsilon})p = p - \varepsilon p, \tag{9}$$

$$g(q\varepsilon) = q(1_{s(\varepsilon)} - \varepsilon) = q - q\varepsilon. \tag{10}$$

Now (9) and (10) do not lie in $(\bar{\mathrm{R}})$ since $\mathrm{R}$ only consists of monomial relations. Thus, they give a contradiction. □

**Definition 2.9.** *[GdlPn99, Definition (4.2)] Let* $\Lambda = \mathrm{k}\,Q/(\mathrm{R} \cup \mathrm{R}^{\mathrm{Sp}})$ *be quasi-clannish. It is called* skewed-gentle *if it additionally satisfies the following conditions:*

*(iv)* $\mathrm{R}$ *consists of monomial relations of length* $2$*,*

*(v)* *for any* $a \in Q_1^{\mathrm{ord}}$ *there is at most one* $b \in Q_1$ *with* $ab \in \mathrm{R}$*,*

*(v)** *for any* $a \in Q_1^{\mathrm{ord}}$ *there is at most one* $c \in Q_1$ *with* $ca \in \mathrm{R}$*.*

**Example 2.10.** *1. The algebras in Example 2.3.1. and Example 2.3.2. are skewed-gentle.*

*2. The algebras in Example 2.3.3. and 2.3.4. are not skewed-gentle.*

*3. The algebra given by the following data is not skewed-gentle:*

$$Q: \quad 1 \xrightarrow{a} 2 \xrightarrow{b} 3 \xrightarrow{c} 4\,,$$

$\mathrm{Sp} = \varnothing$, $\mathrm{R} = \{cba\}$.

**Lemma 2.11.** *Any skewed-gentle algebra is clannish.*

*Proof.* Since any relation is of length two, it does not contain the square of a special loop. Lemma 2.8 yields the result. □

**Remark 2.12.** *Any algebra which is Morita equivalent to a clannish (quasi-clannish, skewed-gentle, respectively) algebra is also called clannish (quasi-clannsih, skewed-gentle, respectively).*

## 2.3 Words

In order to give a description for words of the clannish algebra $\Lambda$, we consider the latter in terms of the quiver Q with relations $\mathrm{R} \cup \mathrm{R}^{\mathrm{Sp}}$. We follow [CB18] for most definitions in this section.
Let $\Lambda = \mathrm{k}Q/(\mathrm{R} \cup \mathrm{R}^{\mathrm{Sp}})$ be a clannish algebra.

A *letter* is an arrow $x \in \mathrm{Q}_1$, its formal inverse $x^{-1}$, or a symbol $\varepsilon^*$ for any $\varepsilon \in \mathrm{Sp}$. The formal inverse of a symbol $\varepsilon^*$ is given by itself.
We call a letter $a^{\pm 1}$ for $a \in \mathrm{Q}_1^{\mathrm{ord}}$ of *ordinary type* or simply *ordinary*, and a letter $\varepsilon^{\pm 1}, \varepsilon^*$ for $\varepsilon \in \mathrm{Sp}$ of *special type* or simply *special*. We distinguish *direct letters*, which are of the form $x$ for some $x \in \mathrm{Q}_1$, from *inverse letters*, which are of the form $x^{-1}$ for some $x \in \mathrm{Q}_1$.
In the next step, we want to build words from certain sets of letters, so called *alphabets*. We are going to consider two types of words, coming from two types of alphabets.
We denote by

$$\Gamma_{\mathrm{d}}(\Lambda) \coloneqq \{a, a^{-1} \mid a \in \mathrm{Q}_1^{\mathrm{ord}}\} \cup \{\varepsilon, \varepsilon^{-1} \mid \varepsilon \in \mathrm{Sp}\}$$

the *directed alphabet* of $\Lambda$, and by

$$\Gamma_{\mathrm{ud}}(\Lambda) \coloneqq \{a, a^{-1} \mid a \in \mathrm{Q}_1^{\mathrm{ord}}\} \cup \{\varepsilon^* \mid \varepsilon \in \mathrm{Sp}\}$$

the *undirected alphabet* of $\Lambda$.
There exists the following forgetful map:

$$\begin{aligned} \phi_{\mathrm{ud}}^{\mathrm{d}} \colon \quad \Gamma_{\mathrm{d}}(\Lambda) &\longrightarrow \Gamma_{\mathrm{ud}}(\Lambda) \\ x^{\kappa} &\longmapsto \begin{cases} x^{\kappa} & \text{if } x \in \mathrm{Q}_1^{\mathrm{ord}}, \\ x^* & \text{if } x \in \mathrm{Sp}, \end{cases} \end{aligned} \tag{11}$$

for $\kappa \in \{+1, -1\}$.

**Example 2.13.** *Let $\Lambda$ be as in Example 2.2.1. Then its directed and undirected alphabet are given by*

$$\begin{aligned} \Gamma_{\mathrm{ud}}(\Lambda) &= \{a, a^{-1}, \varepsilon^*\}, \\ \Gamma_{\mathrm{d}}(\Lambda) &= \{a, a^{-1}, \varepsilon, \varepsilon^{-1}\}. \end{aligned}$$

**Example 2.14.** *Let $\Lambda$ be given by the quiver*

$$\begin{tikzcd} Q: & 1 \arrow[loop left, "\varepsilon"] \arrow[dr, "a"] & & 5 \arrow[loop right, "\kappa"] \arrow[dd, "e"] \\ & & 2 \arrow[ur, "c"] \arrow[dr, "d"] & \\ & 3 \arrow[loop left, "\eta"] \arrow[ur, "b"] & & 4 \end{tikzcd}$$

*with* $\mathrm{Sp} = \{\eta, \kappa, \varepsilon\}$, $\mathrm{R} = \{ca, db, ec\}$. *Its undirected and directed alphabets are*

$$\Gamma_{\mathrm{ud}}(\Lambda) = \{a, b, c, d, e, a^{-1}, b^{-1}, c^{-1}, d^{-1}, e^{-1}, \varepsilon^*, \kappa^*, \eta^*\},$$
$$\Gamma_{\mathrm{d}}(\Lambda) = \{a, b, c, d, e, a^{-1}, b^{-1}, c^{-1}, d^{-1}, e^{-1}, \varepsilon, \kappa, \eta, \varepsilon^{-1}, \kappa^{-1}, \eta^{-1}\}.$$

In the following, we might simply write $\Gamma_{\mathrm{ud}}$, $\Gamma_{\mathrm{d}}$ if the given algebra is clear from the context.
Let $\Gamma$ be either a directed or undirected alphabet for a clannish algebra $\Lambda$. Then a $\Gamma$−I−*word* $w_{\mathrm{I}}$ is given by a sequence of letters from $\Gamma$ of the following form:

$$w_{\mathrm{I}} = \begin{cases} w_1 \dots w_n & \text{if } \mathrm{I} = \{0, \dots, n\}, (n > 0), \\ w_1 w_2 \dots & \text{if } \mathrm{I} = \mathbb{N}, \\ \dots w_{-1} w_0 & \text{if } \mathrm{I} = -\mathbb{N}, \\ \dots w_{-2} w_{-1} w_0 \mid w_1 w_2 \dots & \text{if } \mathrm{I} = \mathbb{Z}, \end{cases} \tag{12}$$

such that

(i) for two consecutive letters $w_i$, $w_{i+1}$: $s(w_i) = t(w_{i+1})$ in Q,

(ii) for two consecutive letters $w_i$, $w_{i+1}$: $w_i^{-1} \neq w_{i+1}$,

(iii) if $r = r_1 \dots r_k \in \mathrm{R}$, then neither $r$ nor its inverse $r^{-1} = r_k^{-1} \dots r_1^{-1}$ occur as a consecutive subsequence of $w_{\mathrm{I}}$.

(iv) for $\varepsilon \in \mathrm{Sp}$, $\varepsilon^* \varepsilon^*$ does not occur as a consecutive subsequence of $w_{\mathrm{I}}$ in $\Gamma_{\mathrm{ud}}$, nor do $\varepsilon\varepsilon$ and $\varepsilon^{-1}\varepsilon^{-1}$ occur as a consecutive subsequence of $w_{\mathrm{I}}$ in $\Gamma_{\mathrm{d}}$.

Note that the "|" in the definition of a $\mathbb{Z}$−word is necessary to indicate the position of the letter $w_0$ and $w_1$ within the word.

**Example 2.15.** *Consider the algebra* $\Lambda$ *from Example 2.2.1.*
*Then for* $I = \{0, 1, 2, 3\}$,

$$w = \varepsilon^{-1} a \varepsilon$$

*is a* $\Gamma_{\mathrm{d}} - \mathrm{I}-$*word. For* $\mathrm{I} = \mathbb{Z}$,

$$w_{\mathbb{Z}} = \dots a\varepsilon^* a\varepsilon^* \mid a^{-1}\varepsilon^* a^{-1}\varepsilon^* \dots$$

*is a* $\Gamma_{\mathrm{ud}} - \mathbb{Z}-$*word, where* $w_{\mathbb{Z}}$ *consists of repetitions of the displayed subword to the left and to the right.*

**Example 2.16.** *Let* $\Lambda$ *be as in Example 2.14. Then*

$$w_{\mathbb{N}} = \varepsilon^* a^{-1} d^{-1} e\kappa^* cb\eta^* b^{-1} a\varepsilon^* a^{-1} d^{-1} e\kappa^* cb\eta^* b^{-1} a \dots$$

*is a* $\Gamma_{\mathrm{ud}} - \mathbb{N}-$*word and the following gives a* $\Gamma_{\mathrm{d}} - (-\mathbb{N})-$*word:*

$$w_{-\mathbb{N}} = \dots d^{-1} e\kappa e^{-1} da\varepsilon^{-1} a^{-1} d^{-1} e\kappa.$$

The length of a $\Gamma-\mathrm{I}-$word with $\mathrm{I}=\{0,\dots,n\}$ is given by $n$. For $I=\{0\}$, a $\Gamma-\mathrm{I}-$word is given by a *trivial word* $1_{v,\kappa}$ for some vertex $v\in Q_0$ and $\kappa\in\{+,-\}$. We call a $\Gamma-\mathrm{I}-$word $w_{\mathrm{I}}$ *directed* if $\Gamma=\Gamma_{\mathrm{d}}$ and *undirected* if $\Gamma=\Gamma_{\mathrm{ud}}$. With this adjective to describe the words, we might also drop the $\Gamma-$notation and say for instance "directed $\mathrm{I}-$word" instead of "$\Gamma_{\mathrm{d}}-\mathrm{I}-$word". We denote the set of all directed $\mathrm{I}-$words by $\mathcal{W}^{\mathrm{I}}(\Gamma_{\mathrm{d}}(\Lambda))$, and the set of all undirected $\mathrm{I}-$words by $\mathcal{W}^{\mathrm{I}}(\Gamma_{\mathrm{ud}}(\Lambda))$. If the given algebra is clear from the context, we write $\mathcal{W}^{\mathrm{I}}_{\star}$ instead of $\mathcal{W}^{\mathrm{I}}(\Gamma_{\star}(\Lambda))$ where $\star\in\{\mathrm{ud},\mathrm{d}\}$. We denote by $\mathcal{W}(\Gamma_{\star}(\Lambda))=\bigcup_{|\mathrm{I}|\in\mathbb{N}}\mathcal{W}^{\mathrm{I}}(\Gamma_{\star}(\Lambda))$ the set of all finite words of $\Gamma_{\star}(\Lambda)$, $\star\in\{\mathrm{ud},\mathrm{d}\}$. We write $\mathcal{W}_{\star}$ ($\star\in\{\mathrm{ud},\mathrm{d}\}$) for short. For any word in $\mathcal{W}_{\star}$, we drop the subscript in the notation when convenient and write $w\in\mathcal{W}_{\star}$ of length $n$ instead of $w_{\mathrm{I}}\in\mathcal{W}_{\star}$ with $I=\{0,\dots,n\}$.
Note that (11) induces the map

$$\Phi^{\mathrm{d}}_{\mathrm{ud}}\colon\mathcal{W}(\Gamma_{\mathrm{d}}(\Lambda))\longrightarrow\mathcal{W}(\Gamma_{\mathrm{ud}}(\Lambda)).\tag{13}$$

For a directed $\mathrm{I}-$word $v_{\mathrm{I}}$, we call an undirected $\mathrm{I}-$word $w_{\mathrm{I}}\in\Phi^{\mathrm{d}}_{\mathrm{ud}}(v_{\mathrm{I}})$ *an undirected version of* $v_{\mathrm{I}}$.
Vice versa, for some undirected $\mathrm{I}-$word $w_{\mathrm{I}}$, we call any directed $\mathrm{I}-$word $v_{\mathrm{I}}$ in the preimage $(\Phi^{\mathrm{d}}_{\mathrm{ud}})^{-1}(w_{\mathrm{I}})$ of $w_{\mathrm{I}}$ *a directed version of* $w_{\mathrm{I}}$.
For a given $\Gamma-\mathrm{I}-$word $w_{\mathrm{I}}$ there exists for every $i\in\mathrm{I}$ an associated vertex $v_i(w_{\mathrm{I}})$ in $\mathrm{Q}_0$, given by

$$\text{for } \mathrm{I}=\mathbb{Z}:\qquad v_i(w_{\mathbb{Z}})=s(w_i)=t(w_{i+1}),$$

$$\text{for } \mathrm{I}=\mathbb{N}:\qquad v_i(w_{\mathbb{N}})=\begin{cases} s(w_i)=t(w_{i+1}) & \text{if } i\geq 1,\\ t(w_1) & \text{if } i=0,\end{cases}$$

$$\text{for } \mathrm{I}=-\mathbb{N}:\qquad v_i(w_{-\mathbb{N}})=\begin{cases} s(w_i)=t(w_{i+1}) & \text{if } i\leq -1,\\ s(w_0) & \text{if } i=0,\end{cases}$$

$$\text{for } \mathrm{I}=\{0,\dots,n\}:\quad v_i(w)=\begin{cases}\begin{cases} s(w_i)=t(w_{i+1}) & \text{if } 0<i<n,\\ t(w_1) & \text{if } i=0,\\ s(w_n) & \text{if } i=n,\end{cases} & \text{if } w\neq 1_{v,\kappa} \text{ for all } v,\kappa,\\ \begin{cases} v \end{cases} & \text{if } w=1_{v,\kappa},\text{ some } v,\kappa.\end{cases}$$

**Example 2.17.** *We consider the word $w$ from Example 2.15. Then*

$$v_i(w)=s(w_i)=1 \textit{ for all } i\in\{0,\dots 3\}.$$

**Example 2.18.** *Let $\Lambda$ and $w_{\mathbb{N}}$ be given as in Example 2.16. Then*

$$\begin{aligned} v_5(w_{\mathbb{N}})&=s(w_5)=s(\kappa^*)=5=t(w_6)=t(c),\\ v_2(w_{\mathbb{N}})&=s(w_2)=s(a^{-1})=2.\end{aligned}$$

The *inverse* of a $\Gamma - \mathrm{I}$ –word $w_{\mathrm{I}}$ as in (12) is given by

$$w_{\mathrm{I}}^{-1} = \begin{cases} w_n^{-1} \dots w_1^{-1} & \text{if } I = \{0, \dots, n\}, (n > 0), \\ \dots w_2^{-1} w_1^{-1} & \text{if } I = \mathbb{N}, \\ w_0^{-1} w_{-1}^{-1} \dots & \text{if } I = -\mathbb{N}, \\ \dots w_2^{-1} w_1^{-1} w_0^{-1} \mid w_{-1}^{-1} w_{-2}^{-1} \dots & \text{if } I = \mathbb{Z}, \end{cases} \tag{14}$$

with $(x^{-1})^{-1} = x$ for any letter $x \in \Gamma$.
Note that the inverse of a $\Gamma - \mathbb{N}$ –word is given by a $\Gamma - (-\mathbb{N})$–word by definition, and, dually, the inverse of a $\Gamma - (-\mathbb{N})$–word is an $\Gamma - \mathbb{N}$ –word.
We define the inverse of a trivial word by $(1_{v,\kappa})^{-1} = 1_{v,-\kappa}$.

**Example 2.19.** *The inverse of $w$ from Example 2.15 is given by*

$$w^{-1} = \varepsilon^{-1} a^{-1} \varepsilon.$$

*For $w_{\mathbb{Z}}$ from the same example, we obtain the inverse*

$$w_{\mathbb{Z}}^{-1} = \dots a\varepsilon^* a\varepsilon^* \mid a^{-1} \dots.$$

**Example 2.20.** *The inverses of the words from Example 2.16 are given by*

$$(w_{\mathbb{N}})^{-1} = \dots a^{-1} b\eta^* b^{-1} c^{-1} \kappa^* e^{-1} da\varepsilon^*,$$
$$(w_{-\mathbb{N}})^{-1} = \kappa^{-1} e^{-1} da\varepsilon a^{-1} d^{-1} e\kappa^{-1} e^{-1} d \dots.$$

Now we choose for each letter $l \in \Gamma$ a *sign* $\mathrm{sgn}(l) \in \{+,-\}$ such that two distinct letters $l$ and $l'$ with the same starting vertex in Q have the same sign if and only if $\{l, l'\} = \{x^{-1}, y\}$ and either $xy \in \mathrm{R}$ or $x = y \in \mathrm{Sp}$. Thus, if $w_i$ and $w_{i+1}$ are two consecutive letters in a $\Gamma - \mathrm{I}$ –word $w_{\mathrm{I}}$, then $\mathrm{sgn}(w_i^{-1}) = -\mathrm{sgn}(w_{i+1})$.
The sign of a $\Gamma - \mathrm{I}$ –word $w_{\mathrm{I}}$ for $\mathrm{I} = \{0, \dots, n\}$ or $\mathrm{I} = \mathbb{N}$ is given by $\mathrm{sgn}(w_1)$, or, if $w_{\mathrm{I}} = 1_{v,\kappa}$, by $\kappa$. Simlarly, for $\mathrm{I} = -\mathbb{N}$, it is given by $\mathrm{sgn}(w_0)$. Additionally, we assume for a given algebra $\Lambda$ that the sign on a letter is compatible with both alphabets $\Gamma_{\mathrm{d}}$ and $\Gamma_{\mathrm{ud}}$, that is, for any two letters $x \in \Gamma_{\mathrm{d}}$, $y \in \Gamma_{\mathrm{ud}}$ with $\phi_{\mathrm{ud}}^{\mathrm{d}}(x) = y$ we have $\mathrm{sgn}(x) = \mathrm{sgn}(y)$.

**Example 2.21.** *Consider $\Lambda$ as in Example 2.3.1. and $\Gamma_{\mathrm{d}}(\Lambda)$, $\Gamma_{\mathrm{ud}}(\Lambda)$ from Example 2.13. It follows from the relations on $\Lambda$ that $\mathrm{sgn}(a) = \mathrm{sgn}(a^{-1})$ and $\mathrm{sgn}(\varepsilon) = \mathrm{sgn}(\varepsilon^{-1})$, but $\mathrm{sgn}(a) \neq \mathrm{sgn}(\varepsilon)$.*

**Example 2.22.** *Let $\Lambda$ be as in Example 2.14 and consider its undirected alphabet $\Gamma_{\mathrm{ud}}$. We obtain the following correspondences from the given relations:*

$$\begin{aligned} \mathrm{sgn}(c^{-1}) &= \mathrm{sgn}(a) \neq \mathrm{sgn}(d^{-1}), \\ \mathrm{sgn}(d^{-1}) &= \mathrm{sgn}(b) \neq \mathrm{sgn}(c^{-1}), \\ \mathrm{sgn}(e^{-1}) &= \mathrm{sgn}(c) \neq \mathrm{sgn}(\kappa^*), \\ \mathrm{sgn}(b^{-1}) &\neq \mathrm{sgn}(\eta^*), \\ \mathrm{sgn}(a^{-1}) &\neq \mathrm{sgn}(\varepsilon^*). \end{aligned}$$

*Thus, we can choose the signs as follows:*

$$\operatorname{sgn}(c^{-1}) = \operatorname{sgn}(d^{-1}) = \operatorname{sgn}(d) = \operatorname{sgn}(\kappa^*) = \operatorname{sgn}(\eta^*) = \operatorname{sgn}(\varepsilon^*) = +,$$
$$\operatorname{sgn}(e^{-1}) = \operatorname{sgn}(a^{-1}) = \operatorname{sgn}(b^{-1}) = \operatorname{sgn}(a) = \operatorname{sgn}(b) = \operatorname{sgn}(c) = \operatorname{sgn}(e) = -.$$

As for each letter $x \in \Gamma$, the source and target vertices $s(x)$ and $t(x)$ are defined via the quiver Q, we can extend this definition to some words. We define for $w_{\mathrm{I}}$ a $\Gamma-\mathrm{I}-$word with $I = \{0, \ldots, n\}$ or $I = \mathbb{N}$, the *source* to be given by $s(w_{\mathrm{I}}) = v_0(w_{\mathrm{I}})$. For $I = \{0, \ldots, n\}$, its *target* is given by $t(w_{\mathrm{I}}) = v_n(w_{\mathrm{I}})$, respectively for $I = -\mathbb{N}$ by $t(w_{\mathrm{I}}) = v_0(w_{\mathrm{I}})$.
The *composition* $vw$ of a $\Gamma - \mathrm{I}-$word $v$ and a $\Gamma - \mathrm{J}-$word $w$ is given by the concatenation of sequences of letters, provided $s(v) = t(w)$, $\operatorname{sgn}(v^{-1}) = -\operatorname{sgn}(w)$, and $vw$ is again a $\Gamma-\mathrm{I}'-$word for some $\mathrm{I}'$. Conventionally, we define $1_{v,\kappa}1_{v,\kappa} = 1_{v,\kappa}$. The composition of a $\Gamma - (-\mathbb{N})-$word $v_{-\mathbb{N}}$ and a $\Gamma - \mathbb{N}-$word $w_{\mathbb{N}}$ is indexed in a way such that

$$(vw)_{\mathbb{Z}} = \ldots v_{-1}v_0 \mid w_1 w_2 \ldots.$$

**Example 2.23.** *Let $\Lambda$ be as in Example 2.2.1.*

1. *Let $w = \varepsilon^{-1}a\varepsilon$, $v = a\varepsilon$. Then we know by Example 2.21 that* $\operatorname{sgn}(\varepsilon) = -\operatorname{sgn}(a)$ *and thus*
$$wv = \varepsilon^{-1}a\varepsilon a\varepsilon.$$
*Note that $vw$ is not a word, since the two words cannot be composed in this order.*

2. *Let $w_{-\mathbb{N}} = \ldots a\varepsilon a^{-1}$ be an $(-\mathbb{N})-$word and let $v_{\mathbb{N}} = \varepsilon a \ldots$ be an $\mathbb{N}-$word. Then the composition $(wv)_{\mathbb{Z}}$ is a $\mathbb{Z}-$word given by*
$$(wv)_{\mathbb{Z}} = \ldots a\varepsilon a^{-1} \mid \varepsilon a \ldots.$$

**Example 2.24.** *Consider $\Lambda$ as in Example 2.3.1. and $w_{-\mathbb{N}}$ and $w_{\mathbb{N}}$ from Example 2.16. Then we cannot compose them since* $\operatorname{sgn}(\kappa^*) = + = \operatorname{sgn}(\eta^*)$ *and also their concatenation does not result in a $\Gamma - \mathbb{Z}-$word.*

Any word which is bounded from below or above ($\mathrm{I} \in \{\mathbb{N}, -\mathbb{N}, \{0, \ldots, n\}\}$) can be composed with a suitable trivial word:

$$\begin{array}{llll}
1_{v,\kappa}w1_{\bar{v},\bar{\kappa}} & \text{if} & v = v_0(w), & \kappa = -\operatorname{sgn}(w_1), \\
& & \bar{v} = v_n(w), & \bar{\kappa} = -\operatorname{sgn}(w_n^{-1}), \\
1_{v,\kappa}w_{\mathbb{N}} & \text{if} & v = v_0(w_{\mathbb{N}}), & \kappa = -\operatorname{sgn}(w_1), \\
w_{-\mathbb{N}}1_{v,\kappa} & \text{if} & v = v_0(w_{-\mathbb{N}}), & \kappa = -\operatorname{sgn}(w_0^{-1}).
\end{array}$$

A *subword* of a $\Gamma - \mathrm{I}-$word $w_{\mathrm{I}}$ is a subsequence of consecutive letters of $w_{\mathrm{I}}$.

**Example 2.25.** *Any $\Gamma - \mathbb{Z}-$word has an $\mathbb{N}-$word and an $(-\mathbb{N})-$word as subwords.*

We fix some notation for certain subwords of a $\Gamma-\mathrm{I}-$word $w_{\mathrm{I}}$, where $i \in \mathrm{I}$:

$$\begin{aligned}
w_{\mathrm{I}}[< i] &\coloneqq \ldots w_{i-2}w_{i-1},\\
w_{\mathrm{I}}[\leq i] &\coloneqq \ldots w_{i-1}w_{i},\\
w_{\mathrm{I}}[> i] &\coloneqq w_{i+1}w_{i+2}\ldots,\\
w_{\mathrm{I}}[\geq i] &\coloneqq w_{i}w_{i+1}\ldots.
\end{aligned}$$

Thus, $w_{\mathrm{I}} = w_{\mathrm{I}}[\leq i]w_{\mathrm{I}}[> i] = w_{\mathrm{I}}[< i]w_{\mathrm{I}}[\geq i]$ for any $i \in \mathrm{I}$.
For a finite $\Gamma-$word $w$ of length $n$ it follows

$$\begin{aligned}
w[< 1] &= 1_{v,\kappa}, \text{ where } v = v_0(w), \kappa = \operatorname{sgn}(w_1),\\
w[\geq 1] &= w_{\mathrm{I}},\\
w[> n] &= 1_{v',\kappa'}, \text{ where } v' = v_n(w), \kappa' = -\operatorname{sgn}(w_n^{-1}),\\
w[\leq n] &= w_{\mathrm{I}}.
\end{aligned}$$

**Example 2.26.** *1. Let $\Lambda$ be as in Example 2.3.1. Consider $w$ as in Example 2.23.1. Then*

$$\begin{aligned}
w[< 3] &= \varepsilon^{-1}a,\\
w[< 2] &= \varepsilon^{-1},\\
w[< 1] &= 1_{1,\kappa}, \qquad \textit{where } \kappa = \operatorname{sgn}(\varepsilon^{-1}),\\
w[\leq 3] &= w,\\
w[\leq 2] &= w[< 3],\\
w[\leq 1] &= w[< 2],
\end{aligned}$$

*2. Let $z_{\mathbb{Z}} = (wv)_{\mathbb{Z}}$ from Example 2.23.2. Then*

$$\begin{aligned}
z[< 1] &= z[\leq 0] = w,\\
z[> 0] &= z[\geq 1] = v.
\end{aligned}$$

**Example 2.27.** *1. Consider $\Lambda$ from Example 2.3.1. and $w_{\mathbb{N}}$ as in Example 2.16. Then*

$$\begin{aligned}
w_{\mathbb{N}}[> 3] &= w_{\mathbb{N}}[\geq 4] = e\kappa^{*}cb\eta^{*}\ldots,\\
w_{\mathbb{N}}[< 1] &= 1_{v,\kappa}, \quad v = v_0(w_{\mathbb{N}}), \kappa = \operatorname{sgn}(w_1)
\end{aligned}$$

*2. For $w_{-\mathbb{N}}$ from Example 2.16 one has*

$$\begin{aligned}
w_{-\mathbb{N}}[\leq -2] &= \ldots a^{-1}d^{-1},\\
w_{-\mathbb{N}}[\geq -1] &= e\kappa.
\end{aligned}$$

There exists an equivalence relation on $\mathcal{W}_\star$, $\star \in \{\mathrm{d}, \mathrm{ud}\}$:

$$v \sim w \text{ if and only if } v = w^{-1} \text{ or } v = w. \tag{15}$$

**Example 2.28.** *1.) Let $\Lambda$ be as in Example 2.3.1. For $w = a\varepsilon^* a^{-1}\varepsilon^*$, $v = \varepsilon^* a\varepsilon a^{-1}$ we obtain $w \sim v$ since $v = w^{-1}$.*

*2.) Consider $\Lambda$ from Example 2.14. The words $v = da\varepsilon^*$ and $w = \varepsilon^* a^{-1}d^{-1}$ are equivalent.*

For $\Gamma - \mathbb{Z}$ –words we define the *shift* $w_\mathbb{Z}[m]$ of $w_\mathbb{Z}$ for some $m \in \mathbb{Z}$ by

$$w_\mathbb{Z}[m] \coloneqq \dots w_m \mid w_{m+1} \dots . \tag{16}$$

If there exists $p \in \mathbb{N}\backslash\{0\}$ minimal with the property $w_\mathbb{Z} = w_\mathbb{Z}[p]$, then we call $p$ the *period* of $w_\mathbb{Z}$ and can write $w_\mathbb{Z}$ in the following way:

$$w_\mathbb{Z} = \dots w_1 \dots w_p \mid w_1 \dots w_p \dots .$$

We call the finite subword $w_1 \dots w_p$ of $w_\mathbb{Z}$ *periodic part* and denote it by $\hat{w}_p$. We say that any letter of the form $w_{i+(k-1)p}$ for $k > 0$, $i \in \{1, \dots, p\}$ belongs to the *positive copy* $\hat{w}_p^{(k)}$ of $\hat{w}_p$, while any $w_{i+kp}$ for $k \leq 0$ and $i \in \{0, -1, \dots, -p+1\}$ belongs to the *negative copy* $\hat{w}_p^{(k)}$.

**Example 2.29.** *Consider $\Lambda$ as in Example 2.3.1. and $w_\mathbb{Z}$ with periodic part $\hat{w}_p = \varepsilon^* a$. Then*

$$\begin{array}{ll} w_1 w_2 = \hat{w}_p^{(1)} & w_3 w_4 = \hat{w}_p^{(2)}, \\ w_{-1} w_0 = \hat{w}_p^{(0)} & w_{-3} w_{-2} = \hat{w}_p^{(-1)} \end{array}$$

*since $p = 2$ and*

$$\begin{array}{llll} w_1 = w_{1+(1-1)2}, & w_2 = w_{2+(1-1)2}, & w_3 = w_{1+(2-1)2}, & w_4 = w_{2+(2-1)2}, \\ w_0 = w_{0+0\cdot 2}, & w_{-1} = w_{-1+0\cdot 2}, & w_{-2} = w_{0+(-1)\cdot 2}, & w_{-3} = w_{-1+(-1)\cdot 2} \end{array}$$

**Remark 2.30.** *Let $m \in \mathbb{Z}$ be positive. Then shifting by $-m$ means moving the letters of $w_\mathbb{Z}$ by $m$ positions to the right. Shifting by $m$ means moving the letters by $m$ positions to the left.*

**Lemma 2.31.** *The shift is additive:*

$$w_\mathbb{Z}[m+n] = (w_\mathbb{Z}[m])[n].$$

*Proof.* Consider the shifting map $\tau_m : \mathbb{Z} \to \mathbb{Z}$, defined by $\tau_m(i) = i + m$. With respect to the positions of the letters, the shift is exactly given by this map acting on their indices. One has $\tau_{m+n}(i) = i + m + n = \tau_n(\tau_m(i))$, giving the result. $\square$

We introduce an equivalence relation $\sim_{\mathbb{Z}}$ on $W_\star^{\mathbb{Z}}$ for $\star \in \{\mathrm{d}, \mathrm{ud}\}$ as follows:

$w_{\mathbb{Z}} \sim_{\mathbb{Z}} v_{\mathbb{Z}}$ if and only if $w_{\mathbb{Z}} = v_{\mathbb{Z}}[m]$ or $w_{\mathbb{Z}} = v_{\mathbb{Z}}^{-1}[m]$ for some $m \in \mathbb{Z}$.

**Example 2.32.** *1. Take $\Lambda$ as in Example 2.3.1. and consider the two following words of period 4:*

$$w_{\mathbb{Z}} = \dots \varepsilon a \varepsilon^{-1} a \mid \varepsilon a \varepsilon^{-1} a \dots,$$
$$v_{\mathbb{Z}} = \dots a^{-1} \varepsilon a^{-1} \varepsilon^{-1} \mid a^{-1} \varepsilon a^{-1} \varepsilon^{-1} \dots.$$

*The inverse of $w_{\mathbb{Z}}$ is given by*

$$w_{\mathbb{Z}}^{-1} = \dots \varepsilon a^{-1} \varepsilon^{-1} a^{-1} \mid \varepsilon a^{-1} \varepsilon^{-1} a^{-1}.$$

*It follows $v_{\mathbb{Z}} \sim_{\mathbb{Z}} w_{\mathbb{Z}}$ since $v_{\mathbb{Z}} = w_{\mathbb{Z}}^{-1}[-1]$.*

*2. Let $\Lambda$ be as in Example 2.28.2. and consider*

$$w_{\mathbb{Z}} = \dots \kappa^* c b \eta^* b^{-1} c^{-1} \mid \kappa^* c b \eta^* b^{-1} c^{-1} \dots,$$
$$v_{\mathbb{Z}} = \dots b^{-1} c^{-1} \kappa^* c b \eta^* b^{-1} \mid c^{-1} \kappa^* c b \eta^* b^{-1} \dots.$$

*Then $v_{\mathbb{Z}} \sim w_{\mathbb{Z}}$ since $v_{\mathbb{Z}} = w_{\mathbb{Z}}^{-1}[-2]$.*

**Lemma 2.33.** *The relations $\sim$ and $\sim_{\mathbb{Z}}$ are equivalence relations on $\mathcal{W}_\star$, $\mathcal{W}_\star^{\mathbb{Z}}$, respectively.*

The proof is given in the next subsection.
Since one can clearly distinguish between the equivalence relations $\sim$ and $\sim_{\mathbb{Z}}$, we drop the index of the second equivalence relation and use $\sim$ instead of $\sim_{\mathbb{Z}}$ for easier reading in the following.

### 2.3.1 Properties of $\mathbb{Z}$-words

In this subsection, we examine $\Gamma - \mathbb{Z}$-words more closely. We obtain nice properties with respect to their shifts and inverses.

**Lemma 2.34.** *Let $w_{\mathbb{Z}} \in \mathcal{W}_\star^{\mathbb{Z}}$, $\star \in \{\mathrm{d}, \mathrm{ud}\}$, and let $k \in \mathbb{Z}$. Then*

$$w_{\mathbb{Z}}^{-1}[-k] = (w_{\mathbb{Z}}[k])^{-1}.$$

*Proof.* The statement follows easily by comparing the two:
Write $w_{\mathbb{Z}} = \dots w_{-1} w_0 \mid w_1 w_2 \dots$. Then the shift by $k \in \mathbb{Z}$ is given by

$$w_{\mathbb{Z}}[k] = \dots w_{k-1} w_k \mid w_{k+1} w_{k+2} \dots$$

and its inverse by

$$(w_{\mathbb{Z}}[k])^{-1} = \dots w_{k+1}^{-1} w_k^{-1} \mid w_{k-1}^{-1} w_{k-2}^{-1} \dots.$$

On the other hand, the inverse of $w_{\mathbb{Z}}$ is

$$w_{\mathbb{Z}}^{-1} = \dots w_1^{-1} w_0^{-1} \mid w_{-1}^{-1} w_{-2}^{-1} \dots .$$

Shifting this by $-k$ gives

$$w_{\mathbb{Z}}^{-1}[-k] = \dots w_{k+1}^{-1} w_k^{-1} \mid w_{k-1}^{-1} w_{k-2}^{-1} \dots .$$

It follows that $w_{\mathbb{Z}}^{-1}[-k] = (w_{\mathbb{Z}}[k])^{-1}$. □

**Corollary 2.35.** *Let $w_{\mathbb{Z}} \in \mathcal{W}_\star^{\mathbb{Z}}$, $\star \in \{\mathrm{d}, \mathrm{ud}\}$, and let $k \in \mathbb{Z}$. Then*

$$w_{\mathbb{Z}}[k] = (w_{\mathbb{Z}}[k])^{-1} \text{ if and only if } w_{\mathbb{Z}} = (w_{\mathbb{Z}}[2k])^{-1}.$$

*Proof.* By Lemma 2.34, we have that $(w_{\mathbb{Z}}[k])^{-1} = w_{\mathbb{Z}}^{-1}[-k]$. Applying this to $w_{\mathbb{Z}}[k] = (w_{\mathbb{Z}}[k])^{-1}$ and then using a shift by $-k$ on both sides of the newly obtained equation gives

$$w_{\mathbb{Z}}^{-1}[-2k] = w_{\mathbb{Z}}.$$

Applying again Lemma 2.34 to the left hand term, it follows that

$$(w_{\mathbb{Z}}[2k])^{-1} = w_{\mathbb{Z}}.$$

□

**Example 2.36.** *Let $\Lambda$ be as in Example 2.3.1. Let*

$$w_{\mathbb{Z}} = \dots \varepsilon^* a^{-1} \varepsilon^* a \mid \varepsilon^* a^{-1} \varepsilon^* a \varepsilon^* a^{-1} \varepsilon^* a \dots .$$

*Note that $w_{\mathbb{Z}}$ is periodic with $p = 4$. Now consider the shift of $w_{\mathbb{Z}}$ by 2. Then*

$$\begin{aligned} w_{\mathbb{Z}}[2] &= \dots \varepsilon^* a^{-1} \varepsilon^* a \varepsilon^* a^{-1} \mid \varepsilon^* a \varepsilon^* a^{-1} \varepsilon^* a \dots, \\ (w_{\mathbb{Z}}[2])^{-1} &= \dots \varepsilon^* a^{-1} \varepsilon^* a \mid \varepsilon^* a^{-1} \varepsilon^* a \varepsilon^* a^{-1} \varepsilon^* a \dots \end{aligned}$$

*and thus $w_{\mathbb{Z}} = (w_{\mathbb{Z}}[2])^{-1}$. Similarly, we obtain that $w_{\mathbb{Z}}[1] = (w_{\mathbb{Z}}[1])^{-1}$.*

**Example 2.37.** *Consider Example 2.32.2. Then $v_{\mathbb{Z}} = w_{\mathbb{Z}}^{-1}[2]$. We have $v_{\mathbb{Z}} = (w_{\mathbb{Z}}[2])^{-1}$ as well.*

*Proof of Lemma 2.33.* We first show reflexivity, symmetry and transitivity for $\sim$. Then we show the same for $\sim_{\mathbb{Z}}$.

- Reflexivity of $\sim$ obviously holds since $w = w$ for any finite word $w$. Similarly, if $v = w^{-1}$, then also $w = v^{-1}$ and hence symmetry is given. For transitivity, we use the same arguments: let $v \sim w$, $w \sim u$. Without loss of generality, let $v = w^{-1}$. Then either $w = u$ and it follows $v = u^{-1}$, hence $v \sim u$; or $w = u^{-1}$ and $v = u$, hence $v \sim u$.

- For the relation $\sim_{\mathbb{Z}}$, reflexivity is given by $w_{\mathbb{Z}} = w_{\mathbb{Z}}[0]$. To show symmetry, assume $w_{\mathbb{Z}} \sim v_{\mathbb{Z}}$. Assume at first $w_{\mathbb{Z}} = v_{\mathbb{Z}}[m]$ for some $m \in \mathbb{Z}$. Then $v_{\mathbb{Z}} = w_{\mathbb{Z}}[-m]$ and thus $v_{\mathbb{Z}} \sim w_{\mathbb{Z}}$. If, on the other hand, $w_{\mathbb{Z}} = v_{\mathbb{Z}}^{-1}[m]$ for some $m \in \mathbb{Z}$, then it follows that $v_{\mathbb{Z}} = (w_{\mathbb{Z}}[-m])^{-1} = w_{\mathbb{Z}}^{-1}[m]$ (Lemma 2.34). Hence, $v_{\mathbb{Z}} \sim w_{\mathbb{Z}}$. To show transitivity, let $w_{\mathbb{Z}} \sim v_{\mathbb{Z}}$ and $v_{\mathbb{Z}} \sim u_{\mathbb{Z}}$. Assume at first $w_{\mathbb{Z}} = v_{\mathbb{Z}}[m]$ for some $m \in \mathbb{Z}$. If $v_{\mathbb{Z}} = u_{\mathbb{Z}}[k]$ for some $k \in \mathbb{Z}$, then $w_{\mathbb{Z}} = u_{\mathbb{Z}}[k][m] = u_{\mathbb{Z}}[k+m]$. Hence, $w_{\mathbb{Z}} \sim u_{\mathbb{Z}}$. Now assume that $v_{\mathbb{Z}} = u_{\mathbb{Z}}^{-1}[k]$ for some $k \in \mathbb{Z}$. Then we obtain that $w_{\mathbb{Z}} = u_{\mathbb{Z}}^{-1}[k+m]$ and thus $w_{\mathbb{Z}} \sim u_{\mathbb{Z}}$. Now assume that $w_{\mathbb{Z}} = v_{\mathbb{Z}}^{-1}[m]$ and $v_{\mathbb{Z}} = u_{\mathbb{Z}}^{-1}[k]$ for some $k \in \mathbb{Z}$. Applying Lemma 2.34 gives that $w_{\mathbb{Z}} = u_{\mathbb{Z}}[m-k]$ and thus $w_{\mathbb{Z}} \sim u_{\mathbb{Z}}$. If, on the other hand, $v_{\mathbb{Z}} = u_{\mathbb{Z}}[k]$, then, again by the same lemma, $w_{\mathbb{Z}} = u_{\mathbb{Z}}^{-1}[m-k]$. It follows that $w_{\mathbb{Z}} \sim u_{\mathbb{Z}}$. Hence transitivity also holds.

□

### 2.3.2 Coadmissible words

In this subsection we introduce the notion of a *coadmissible* word. It is useful with respect to the context of matrix problems for clannish algebras (cf. Chapter 4). The connection becomes clear in Sections 4.5 and 4.6.

**Definition 2.38.** *Let $w_{\mathrm{I}} \in \Gamma_\star$ for $\star \in \{\mathrm{d}, \mathrm{ud}\}$.*
*Then $w_{\mathrm{I}}$ is* left coadmissible *provided that either* I *is not bounded below, or* $\sup(\mathrm{I}) = n \in \mathbb{N}$ *and there does not exist a letter $l \in \Gamma_\star$ such that $lw_{\mathrm{I}}$ is again a word for $l = \varepsilon^{\pm 1}$ or $l = \varepsilon^*$ for some $\varepsilon \in \mathrm{Sp}$.*
*Similarly, $w_{\mathrm{I}}$ is* right coadmissible *provided that either* I *is not bounded above, or* $\sup(\mathrm{I}) = n \in \mathbb{N}$ *and there does not exist a letter $l \in \Gamma_\star$ such that $w_{\mathrm{I}}l$ is again a word for $l = \varepsilon^{\pm 1}$ or $l = \varepsilon^*$ for some $\varepsilon \in \mathrm{Sp}$.*
*We call $w_{\mathrm{I}}$* coadmissible *provided that $w_{\mathrm{I}}$ is both left and right coadmissible.*

**Example 2.39.** *Consider $\Lambda$ as in Example 2.3.1. Then*

1. *$w = \varepsilon a$ is left coadmissible, but not rightcoadmissible,*
2. *$w = a\varepsilon^*$ is right coadmissible, but not left coadmissible,*
3. *$w = \varepsilon a \varepsilon^{-1}$ is coadmissible.*

**Example 2.40.** *Let $\Lambda$ be as in Example 2.14.*

1. *The word $w_{\mathbb{N}}$ from Example 2.16 is left coadmissible.*
2. *The word $w_{-\mathbb{N}}$ from Example 2.16 is right coadmissible.*
3. *Let $w_{\mathbb{Z}}$ be as in Example 2.32.2. It is coadmissible.*
4. *The word $w = e\kappa^* c$ is also coadmissible.*

*5. The word $w = e$ is left coadmissible but not right coadmissible.*

Denote by

$$\widehat{\mathcal{W}}(\Gamma_\star(\Lambda)) \coloneqq \{w_{\mathrm{I}} \in \mathcal{W}(\Gamma_\star(\Lambda)) \mid w_{\mathrm{I}} \text{ right coadmissible } \}$$

the set of all right coadmissible words in $\Gamma_\star$, with $\star \in \{\mathrm{d}, \mathrm{ud}\}$. Now let

$$\mathcal{W}_\star^{\mathrm{pos}} \coloneqq \mathcal{W}_\star \cup \mathcal{W}_\star^{\mathbb{N}}$$

with $\star \in \{\mathrm{d}, \mathrm{ud}\}$ be the set of *positive* $\Gamma_\star$*−words*, and let

$$\mathcal{W}_{i,\kappa}^{\mathrm{pos}}(\Gamma_\star(\Lambda)) \coloneqq \{w \in \mathcal{W}_\star^{\mathrm{pos}} \mid t(w_1) = i, \mathrm{sgn}(w_1) = \kappa\}.$$

be a subset of $\mathcal{W}_\star^{\mathrm{pos}}$ consisting of words that have the same target vertex $i$ and the same sign $\kappa$. We also write $\mathcal{W}_{\star,i,\kappa}^{\mathrm{pos}}$ for short.
Define

$$\widehat{\mathcal{W}}_\star^{\mathrm{pos}} \coloneqq \mathcal{W}_\star^{\mathrm{pos}} \cap \widehat{\mathcal{W}}(\Gamma_\star(\Lambda))$$

to be the set of *positive right coadmissible* $\Gamma_\star$*−words.* Let

$$\widehat{\mathcal{W}}_{\star,i,\kappa}^{\mathrm{pos}} \coloneqq \mathcal{W}_{\star,i,\kappa}^{\mathrm{pos}} \cap \widehat{\mathcal{W}}(\Gamma_\star(\Lambda))$$

be the respective subset.

**Definition 2.41.** *We define for two words $v, w \in \widehat{\mathcal{W}}_{\star,i,\kappa}^{\mathrm{pos}}$, $\star \in \{\mathrm{d}, \mathrm{ud}\}$ that $v < w$ if*

*(1) $v = waz$ for some suitable word $z \in \widehat{\mathcal{W}}_\star^{\mathrm{pos}}$ and a direct letter $a$, or*

*(2) $w = vb^{-1}z$ for some suitable word $z \in \widehat{\mathcal{W}}_\star^{\mathrm{pos}}$ and an inverse letter $b^{-1}$, or*

*(3) $v = uaz$ and $w = ub^{-1}\tilde{z}$, for suitable words $z, \tilde{z} \in \widehat{\mathcal{W}}_\star^{\mathrm{pos}}$, $u \in \mathcal{W}_\star$, a direct letter $a$ and an inverse letter $b^{-1}$.*

**Theorem 2.42.** *The relation " $<$ " defines a lexicographical linear ordering on $\widehat{\mathcal{W}}_{\star,i,\kappa}^{\mathrm{pos}}$, for each $\star \in \{\mathrm{d}, \mathrm{ud}\}$, respectively.*

*Proof.* Let $v, w \in \widehat{\mathcal{W}}_{\star,i,\kappa}^{\mathrm{pos}}$ for $\star \in \{\mathrm{d}, \mathrm{ud}\}$, $v \neq w$. We show that either $v < w$ or $w < v$ holds. Let $n \in \mathbb{N}$ maximal such that $v_1 \dots v_n = w_1 \dots w_n$. Then there are three different possibilities:

(i) $v_{n+1} \neq w_{n+1}$,

(ii) $|v| = n$, $|w| > n$,

(iii) $|w| = n$, $|v| > n$.

Let us first consider the case $n = 0$ separately. We obtain in case (i) by definition of $\widehat{\mathcal{W}}^{\mathrm{pos}}_{\star,i,\kappa}$ that $\operatorname{sgn}(v_1) = \operatorname{sgn}(w_1)$. Thus, by definition of sign, $\{v_1, w_1\} = \{x, y^{-1}\}$ with either $yx = 0$ or $x = y \in \mathrm{Sp}$. Let us assume without loss of generality that $v_1 = y^{-1}$ and $w_1 = x$. Then we can write

$$\begin{aligned} v &= y^{-1}u, && \text{where } u = v[\geq 2],\\ w &= xz, && \text{where } z = w[\geq 2]. \end{aligned}$$

Thus, by (3) of Definition 2.41, we obtain that $v > w$. In case (ii) we have that $v = 1_{i,\kappa}$. We can write $w = 1_{i,\kappa}w$. Hence, $w = vw_1u$ for $u = w[\geq 2]$. If $w_1$ is an inverse letter, we obtain by (2) of Definition 2.41 that $v < w$, otherwise by (1) of Definition 2.41 that $w < v$. Case (iii) is analogous.
Let now $n > 0$. In case (i) we have $\operatorname{sgn}(v_n) = \operatorname{sgn}(w_n)$. It follows that $\operatorname{sgn}(v_{n+1}) = -\operatorname{sgn}(v_n^{-1}) = -\operatorname{sgn}(w_n^{-1}) = \operatorname{sgn}(w_{n+1})$. Thus, $\{v_{n+1}, w_{n+1}\} = \{x, y^{-1}\}$ with either $xy = 0$ or $x = y \in \mathrm{Sp}$. Without loss of generality assume that $v_{n+1} = y^{-1}$, $w_{n+1} = x$. Then we can write for $u = v_1 \dots v_n = w_1 \dots w_n$ that

$$\begin{aligned} w &= uxz, && \text{where } z = w[\geq n+2],\\ v &= uy^{-1}s, && \text{where } s = v[\geq n+2]. \end{aligned}$$

Condition (3) of Definition 2.41 gives that $w < v$. In case (ii), $\mu = \operatorname{sgn}(w_{n+1}) = -\operatorname{sgn}(w_n) = -\operatorname{sgn}(v_n)$. Hence we can write $v = v_1 \dots v_n 1_{s(w_n),\mu}$ and

$$w = vw_{n+1}u \quad \text{with} \quad u = w[\geq n+2].$$

If $w_{n+1}$ is now an inverse letter, it follows $w > v$ by (2) of Definition 2.41. Otherwise we obtain $w < v$ by (3). Case (iii) is again analogous to (ii). $\square$

**Remark 2.43.** *It follows from above that we have for $a, b \in \mathrm{Q}_1^{\mathrm{ord}}$ with $t(a) = s(b)$ in $Q$ and $\operatorname{sgn}(a) = \operatorname{sgn}(b^{-1}) = \kappa$ that*

$$a < 1_{t(a),\kappa} < b^{-1}.$$

We call two $\Gamma_\star$–words $v_{\mathrm{I}}$ and $w_{\mathrm{J}}$ *comparable* if $v_{\mathrm{I}}, w_{\mathrm{J}} \in \widehat{\mathcal{W}}^{\mathrm{pos}}_{\star,i,\kappa}$ for some $i \in \mathrm{I} \cap \mathrm{J}$, $\kappa \in \{+,-\}$, $\star \in \{\mathrm{d}, \mathrm{ud}\}$, and if $v_{\mathrm{I}} = w_{\mathrm{J}}$, $v_{\mathrm{I}} < w_{\mathrm{J}}$ or $w_{\mathrm{J}} < v_{\mathrm{I}}$. Otherwise we call them *incomparable*.

**Example 2.44.** *Let $\Lambda$ be as in Example 2.3.1. Then $w = a\varepsilon^*$ and $v = a^{-1}\varepsilon^*$ are two undirected finite right coadmissible words. One has $w = 1_{t(a),\kappa}az$, $v = 1_{t(a^{-1}),\kappa}a^{-1}z$ with $\kappa = -\operatorname{sgn}(a)$ and $z = u = \varepsilon^*$. By (3) of Definition 2.41 it follows that $w < v$.*

**Example 2.45.** *Consider $\Lambda$ as in Example 2.14.*

1. *Let $w_{\mathbb{N}} = \varepsilon^* a^{-1} d^{-1} e \kappa^* \mathbf{c} b \eta^* b^{-1} a \dots$ and $v_{\mathbb{N}} = \varepsilon^* a^{-1} d^{-1} e \kappa^* \mathbf{e^{-1}} d a \varepsilon^* \dots$ be two undirected right coadmissible $\mathbb{N}$–words. Then $w_{\mathbb{N}} < v_{\mathbb{N}}$ by (2) of Definition 2.41. We have used bold letters to indicate the positions of interest in $v_{\mathbb{N}}$ and $w_{\mathbb{N}}$.*

2. *Consider the following undirected finite right coadmissible words:*

$$w = e\kappa^* c, \quad v = e\kappa^* cb, \quad u = e\kappa^* cd^{-1}, \quad x = cb.$$

*Then $w < v$ by (1), $v < u$ by (2) of Definition 2.41, and $x$ is incomparable to $w, v, u$.*

Note that $\Phi^{\mathrm{d}}_{\mathrm{ud}}$ induces several other maps with the same assignment:

$$\begin{array}{rccc} \widehat{\Phi}^{\mathrm{d}}_{\mathrm{ud}}\colon & \widehat{\mathcal{W}}(\Gamma_{\mathrm{d}}(\Lambda)) & \longrightarrow & \widehat{\mathcal{W}}(\Gamma_{\mathrm{ud}}(\Lambda)), \\ (\Phi^{\mathrm{pos}})^{\mathrm{d}}_{\mathrm{ud}}\colon & \mathcal{W}^{\mathrm{pos}}_{\mathrm{d}} & \longrightarrow & \mathcal{W}^{\mathrm{pos}}_{\mathrm{ud}}, \\ \left(\widehat{\Phi}^{\mathrm{pos}}\right)^{\mathrm{d}}_{\mathrm{ud}}\colon & \widehat{\mathcal{W}}^{\mathrm{pos}}_{\mathrm{d}} & \longrightarrow & \widehat{\mathcal{W}}^{\mathrm{pos}}_{\mathrm{ud}}. \end{array}$$

We also can use the notions of directed and undirected versions with respect to the maps $\widehat{\Phi}^{\mathrm{d}}_{\mathrm{ud}}$, $(\Phi^{\mathrm{pos}})^{\mathrm{d}}_{\mathrm{ud}}$ and $\left(\widehat{\Phi}^{\mathrm{pos}}\right)^{\mathrm{d}}_{\mathrm{ud}}$.

**Example 2.46.** *Let $\Lambda$ be as in Example 2.3.1.*
*Let $v = \varepsilon a \varepsilon^{-1}$ be a directed word. Then $(\Phi)^{\mathrm{d}}_{\mathrm{ud}}(v) = w = \varepsilon^* a \varepsilon^*$.*
*Now consider the undirected word $w = \varepsilon^* a \varepsilon^*$. Then*

$$\left(\Phi^{\mathrm{d}}_{\mathrm{ud}}\right)^{-1}(w) = \{\varepsilon a \varepsilon, \varepsilon a \varepsilon^{-1}, \varepsilon^{-1} a \varepsilon^{-1}, \varepsilon^{-1} a \varepsilon\},$$

*i.e. it contains $v$ but there are also more directed versions of $w$.*

**Example 2.47.** *Consider $\Lambda$ from Example 2.3.1. and the words $w$, $s$ and $w_{\mathbb{N}}$ from Example 2.45 . Then*

$$\begin{aligned} \left(\Phi^{\mathrm{d}}_{\mathrm{ud}}\right)^{-1}(w) &= \{e\kappa^* c, e\kappa^{-1} c\}, \\ \left(\Phi^{\mathrm{d}}_{\mathrm{ud}}\right)^{-1}(s) &= \{cb\}, \\ \left(\Phi^{\mathrm{d}}_{\mathrm{ud}}\right)^{-1}(w_{\mathbb{N}}) &= \{\dots \varepsilon a^{-1} d^{-1} e\kappa cb\eta b^{-1} a, \dots \varepsilon^{-1} a^{-1} d^{-1} e\kappa cb\eta b^{-1} a, \\ &\qquad \dots \varepsilon^{-1} a^{-1} d^{-1} e\kappa^{-1} cb\eta b^{-1} a, \dots\}. \end{aligned}$$

### 2.3.3 Minimal words

Let I be finite throughout this subsection. Let $v$ be a $\Gamma_{\mathrm{ud}} - \mathrm{I}$ −word of length $m$ with $t(v_1) = s(\mu)$ for some $\mu \in \mathrm{Sp}$ and $s(v_m) = s(\eta)$, for some $\eta \in \mathrm{Sp}$.
Let $k \in \mathbb{N}$. Then we define

$$v^{[k]} \coloneqq v^{(1)} \kappa^{(1)} \dots \kappa^{(k-1)} v^{(k)}$$

with

$$v^{(i)} = \begin{cases} v & \text{if } i \text{ odd}, \\ v^{-1} & \text{if } i \text{ even}, \end{cases} \quad \text{and} \quad \kappa^{(i)} = \begin{cases} \eta^* & \text{if } i \text{ odd}, \\ \mu^* & \text{if } i \text{ even}. \end{cases}$$

Then $v^{[k]}$ is again undirected.

**Definition 2.48.** *Let $w$ be a $\Gamma_{\mathrm{ud}}-\mathrm{I}-$word of length $n$. We call $w$* composite *if $w$ is of the form $w = v^{[k]}$ for some word $v$ of length $m < n$ and some $k > 0$. Otherwise, we call $w$* minimal.

**Example 2.49.** *Let $\Lambda$ be as in Example 2.3.1. and consider its undirected alphabet. Let*

$$w = a\varepsilon^* a^{-1}\varepsilon^* a\varepsilon^* a^{-1}\varepsilon^* a.$$

*Then $w = v^{[5]}$ for $v = a$. Thus, $w$ is composite and $v$ is minimal.*

**Remark 2.50.** *We can also interpret the notion $t^{[p]}$ for some periodic $\Gamma_{\mathrm{d}}-\mathbb{Z}-$words $v_{\mathbb{Z}}$ as follows: Let $\hat{v}_p^{(i)} = \varepsilon^\kappa t\eta^\mu t^{-1}$ for all $i \in \mathbb{Z}$. Then $t^{[p]}$ describes the smallest subword of $v_{\mathbb{Z}}$ which contains $x \in \{t, t^{-1}\}$ $p$ times. Consider for instance $p = 3$. Then*

$$t^{[3]} = \varepsilon^\kappa t\eta^\mu t^{-1}\varepsilon^\kappa t.$$

**Lemma 2.51.** *Let $w$ be a finite $\Gamma_{\mathrm{ud}}-$word of length $n$ with $w = w^{-1}$. Then $w$ is composite.*

*Proof.* Writing the equality $w = w^{-1}$ in terms of letters gives

$$w_1 \dots w_n = w_n^{-1} \dots w_1^{-1}.$$

It follows that

$$w_i = w_{n+1-i}^{-1} \qquad \forall i \in \{1, \dots, n\}. \tag{17}$$

If $n$ is odd, then we can write $n = 2k + 1$ for some $k \in \mathbb{N}$. In particular, we obtain that $w_{k+1} = w_{k+1}^{-1}$. It follows that $w_{k+1} = \varepsilon^*$ for some $\varepsilon \in \mathrm{Sp}$. Moreover, we have that

$$w_1 \dots w_k = w_n^{-1} \dots w_{k+2}^{-1}.$$

Setting $u = w_1 \dots w_k$ we can write

$$w = u\varepsilon^* u^{-1}.$$

Hence, $w$ is composite.
If $n$ is even, then $n = 2k$ for some $k \in \mathbb{N}$. It follows by (17) that $w_k = w_{k+1}^{-1}$ which contradicts the definition of a word. □

The converse only holds conditionally:

**Lemma 2.52.** *Let $w$ be a composite finite $\Gamma_{\mathrm{ud}}-$word with $w = v^{[k]}$ for some minimal $v$ and some $k \in \mathbb{N}$. If $k$ is even, then $w = w^{-1}$.*

*Proof.* Since $k$ is even, $w$ is of the form

$$v\varepsilon^* \dots \varepsilon^* v^{-1}$$

for a suitable $\varepsilon \in \mathrm{Sp}$. Its inverse is given by

$$w^{-1} = v\varepsilon^* \dots \varepsilon^* v^{-1}.$$

It follows that $w = w^{-1}$. □

**Example 2.53.** *Let $\Lambda$ be given as in Example 2.3.1. Recall that $\Gamma_{\mathrm{ud}}(\Lambda) = \{a, a^{-1}, \varepsilon^*\}$.*
*Let $w = a\varepsilon^* a^{-1}\varepsilon^* a = v^{[3]}$ for $v = a$. Note that $k = 3$ is odd. The inverse of $w$ is given by*

$$w^{-1} = a^{-1}\varepsilon^* a\varepsilon^* a^{-1}.$$

*We see that $w \neq w^{-1}$.*
*Consider in contrast to that $x = v^{[4]} = a\varepsilon^* a^{-1}\varepsilon^* a\varepsilon^* a^{-1}$. Its inverse is*

$$x^{-1} = a\varepsilon^* a^{-1}\varepsilon^* a\varepsilon^* a^{-1}$$

*and thus $x = x^{-1}$.*

**Lemma 2.54.** *Let $w$ be a finite $\Gamma_{\mathrm{ud}}$−word with $w \neq w^{-1}$. If $w$ is coadmissible, then $w$ is not composite.*

*Proof.* Assume towards a contradiction that $w$ is coadmissible and composite. By Lemma 2.52 it follows that $w = v^{[k]}$ for some minimal $v$ with $k$ odd. In particular, $k \neq 2$ and there exist $\varepsilon, \eta \in \mathrm{Sp}$ such that $\varepsilon^* v\eta^*$ is again a word. It follows that $\varepsilon^* w\eta^*$ also is a word. Thus, $w$ is not coadmissible which gives a contradiction. □

**Example 2.55.** *Let $\Lambda$ be as in Example 2.14. Recall that*

$$\Gamma_{\mathrm{ud}}(\Lambda) = \{a, b, c, d, e, a^{-1}, b^{-1}, c^{-1}, d^{-1}, e^{-1}, \varepsilon^*, \kappa^*, \eta^*\}.$$

*Let $w = c^{-1}\kappa^* e^{-1}d$. Then $w \neq w^{-1}$ and $w$ is coadmissible. Futhermore, $w$ is not composite.*

## 2.4 Modules obtained from directed words

It is not possible to obtain modules from words in $\Gamma_{\mathrm{ud}}$ since the letters of special type $\varepsilon^*$ for $\varepsilon \in \mathrm{Sp}$ are not directed. In Section 3.1, techniques are introduced on how to give these letters a direction such that one obtains words in $\Gamma_{\mathrm{d}}$. From those we *can* directly obtain $\Lambda$–modules and hence it is enough to consider only directed words in this section. The general theory on this topic can be found in [Rin75]. Here, we mostly follow [CB88] and [CB18].
Let $w_{\mathrm{I}} \in \mathcal{W}_{\mathrm{d}}^{\mathrm{I}}$. The $\Lambda$–module $M(w_{\mathrm{I}})$ is given by a $\mathrm{k}$ –vector space with basis $b_i$, $i \in I$, and with action of $\Lambda$ in terms of the quiver $Q$ as follows:

$$e_v b_i = \begin{cases} b_i & \text{if } v_i(w_{\mathrm{I}}) = v, \\ 0 & \text{otherwise,} \end{cases}$$

for $e_v$ a trivial path in $\Lambda$ associated to the vertex $v$ in $Q$, and for $x \in \mathrm{Q}_1$:

$$xb_i = \begin{cases} b_{i-1} & \text{if } i-1 \in I, w_i = x, \\ b_{i+1} & \text{if } i+1 \in I, w_{i+1} = x^{-1}, \\ b_i & \text{if } w_i = x^{-1} \text{ or } w_{i+1} = x, \text{ and } x \in Q_1^{\mathrm{Sp}}, \\ 0 & \text{otherwise.} \end{cases}$$

**Example 2.56.** *Let $\Lambda = \mathrm{k}\langle \varepsilon, a\rangle / (\varepsilon^2 - \varepsilon, a^2)$, and thus $\Gamma_{\mathrm{d}} = \{a, a^{-1}, \varepsilon, \varepsilon^{-1}\}$. Let $w = a^{-1}\varepsilon a \varepsilon a \varepsilon^{-1} \in \mathcal{W}_{\mathrm{d}}$ be a word of length 6. Then the corresponding module $M(w)$ has as a $\mathrm{k}$-vector space basis $b_0, \dots, b_6$ and can be depicted as follows:*

$$\begin{tikzcd}
 & & & & & b_5 \arrow[dl, "a"'] \arrow[dr, "\varepsilon"] & \\
 & & & & b_4 \arrow[dl, "\varepsilon"'] & & b_6 \\
 & & & b_3 \arrow[dl, "a"'] & & & \\
b_0 \arrow[dr, "a"] & & b_2 \arrow[dl, "\varepsilon"'] & & & & \\
 & b_1 & & & & &
\end{tikzcd}$$

*or easier, as in the rest of this thesis:*

$$b_0 \xrightarrow{a} b_1 \xleftarrow{\varepsilon} b_2 \xleftarrow{a} b_3 \xleftarrow{\varepsilon} b_4 \xleftarrow{a} b_5 \xrightarrow{\varepsilon} b_6.$$

**Remark 2.57.** *It is important to keep the relations in mind when reading modules as depicted above. This applies in particular to special letters. Due to their idempotent relations, a depiction of the form*

$$b_{i-1} \xleftarrow{\varepsilon} b_i$$

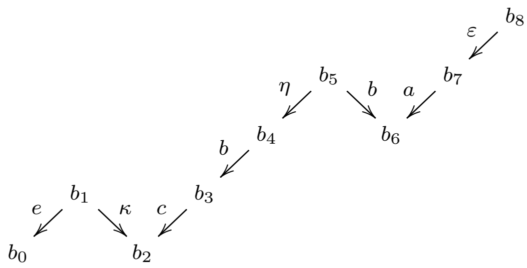
*for $\varepsilon \in \mathrm{Sp}$, is read as follows:*

$$\varepsilon(b_i) = b_{i-1},$$
$$\varepsilon(b_{i-1}) = b_{i-1}.$$

**Example 2.58.** *Let $\Lambda$ be given as in Example 2.14:*

$$Q: \quad \begin{tikzcd}
1 \arrow[loop left, "\varepsilon"] \arrow[dr, "a"] & & 5 \arrow[loop right, "\kappa"] \arrow[dd, "e"] \\
& 2 \arrow[ur, "c"] \arrow[dr, "d"] & \\
3 \arrow[loop left, "\eta"] \arrow[ur, "b"] & & 4
\end{tikzcd}$$

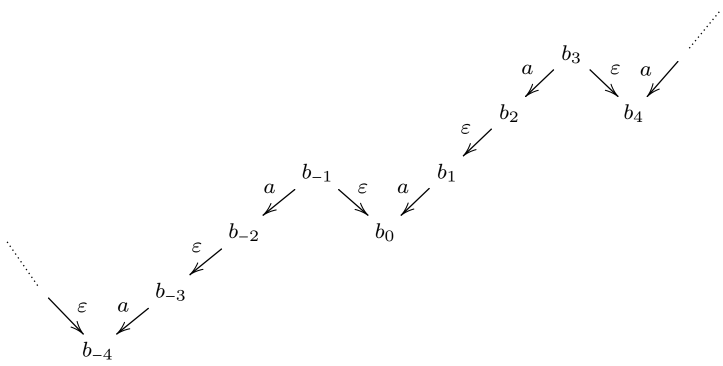
*with* $\mathrm{Sp} = \{\eta, \kappa, \varepsilon\}$, $\mathrm{R} = \{ca, db, ec\}$. *Consider* $w = e\kappa^{-1}cb\eta b^{-1}a\varepsilon$. *It is of length* 8. *The module* $M(w)$ *has basis* $b_0, \dots, b_8$ *as* $\mathrm{k}$*−vector space and is depicted as*



*or equivalently,*

$$b_0 \xleftarrow{e} b_1 \xrightarrow{\kappa} b_2 \xleftarrow{c} b_3 \xleftarrow{b} b_4 \xleftarrow{\eta} b_5 \xrightarrow{b} b_6 \xleftarrow{a} b_7 \xleftarrow{\varepsilon} b_8.$$

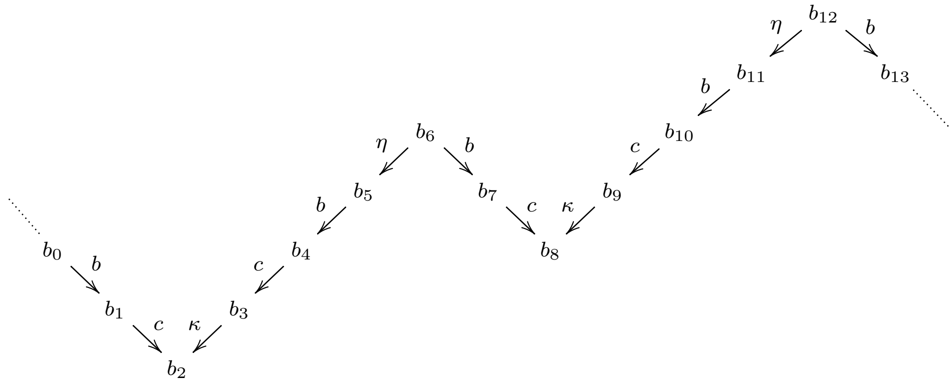
**Example 2.59.** *Let* $\Lambda = \mathrm{k}\langle \varepsilon, a\rangle/(\varepsilon^2 - \varepsilon, a^2)$ *as in Example 2.3.1., and thus* $\Gamma_{\mathrm{d}} = \{a, a^{-1}, \varepsilon, \varepsilon^{-1}\}$.
*Let* $w_{\mathbb{Z}} = \dots a\varepsilon a\varepsilon^{-1} \mid a\varepsilon a\varepsilon^{-1}a\varepsilon a\varepsilon^{-1} \dots \in \mathcal{W}_{\mathrm{d}}^{\mathbb{Z}}$ *of period* 4. *This gives a module* $M(w_{\mathbb{Z}})$ *that is infinite dimensional as a* $\mathrm{k}$*-vector space and can be depicted as*



*or as follows:*

$$\cdots \xrightarrow{\varepsilon} b_{-4} \xleftarrow{a} b_{-3} \xleftarrow{\varepsilon} b_{-2} \xleftarrow{a} b_{-1} \xrightarrow{\varepsilon} b_0 \xleftarrow{a} b_1 \xleftarrow{\varepsilon} b_2 \xleftarrow{a} b_3 \xrightarrow{\varepsilon} b_4 \xleftarrow{a} \cdots$$

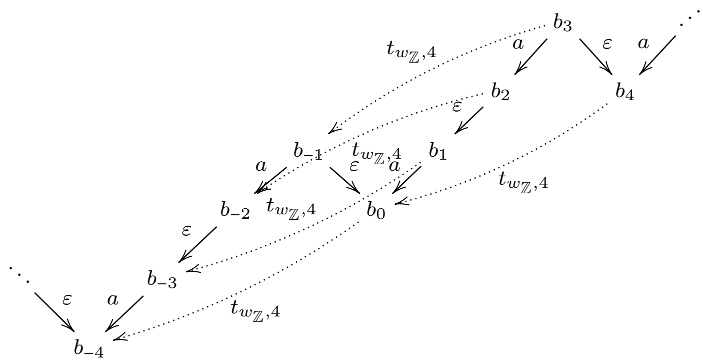
**Example 2.60.** *Let $\Lambda$ be as in Example 2.58 and let $w_{\mathbb{Z}}$ be a directed $\mathbb{Z}$–word in the respective alphabet with periodic part given by $\hat{w}_6 = b^{-1}c^{-1}\kappa c b \eta$. The underlying* k*–vector space of $M(w_{\mathbb{Z}})$ is infinite-dimensional. We depict the module as*




One can see in all examples, that the action of the formal inverses is depicted by arrows going from left to right, while the one of direct letters is depicted by arrows from right to left.

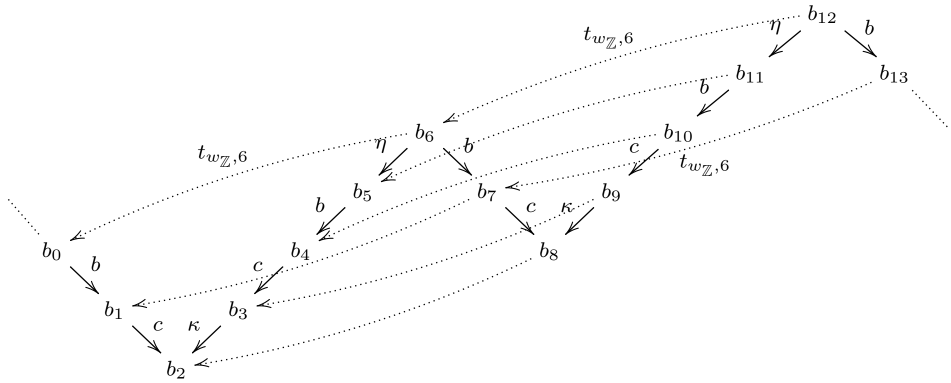
There are two important $\Lambda$–module isomorphisms. For $w_{\mathrm{I}}$ any directed I–word, the morphism $i_w : M(w) \longrightarrow M(w^{-1})$ is bijective by reversing the basis. For $w_{\mathbb{Z}} \in \mathcal{W}_{\mathrm{d}}^{\mathbb{Z}}$, there is a similar isomorphism $i_{w_{\mathbb{Z}}} : M(w_{\mathbb{Z}}) \longrightarrow M(w_{\mathbb{Z}}^{-1})$ by reversing the basis according to taking the inverse. For $w_{\mathbb{Z}}$ the map $t_{w_{\mathbb{Z}},k} : M(w_{\mathbb{Z}}) \longrightarrow M(w_{\mathbb{Z}}[k])$, $b_i \mapsto b_{i-k}$, $k \in \mathbb{Z}$ is also an isomorphism.

**Example 2.61.** *We consider Example 2.59. Then $p = 4$ and $t_{\mathbb{Z},4}$ acts as follows on the module $M(w_{\mathbb{Z}})$:*




**Example 2.62.** *Let $\Lambda$ and $w_{\mathbb{Z}}$ be as in Example 2.60. The period of $w_{\mathbb{Z}}$ is* 6

*and $t_{\mathbb{Z},6}$ acts on $M(w_{\mathbb{Z}})$ as depicted:*




Hence, words in the same equivalence class give rise to modules in the same isomorphism class. Note also that each map $t_{w_{\mathbb{Z}},p}$ gives rise to a periodic part and vice versa.

### 2.4.1 Modules from periodic words

Let $w_{\mathbb{Z}} \in \mathcal{W}_{\mathrm{d}}^{\mathbb{Z}}$ be of period $p$, i.e. $w_{\mathbb{Z}} = w_{\mathbb{Z}}[p]$. Then we can abbreviate the depiction of $M(w_{\mathbb{Z}})$ to the periodic part and obtain the picture of a classical band module (see [Rin75, CB88]).

**Example 2.63.** *We consider Example 2.59 from the previous subsection. Using periodicity of $w_{\mathbb{Z}}$, the depiction of $M(w_{\mathbb{Z}})$ can be reduced to*

$$\begin{tikzcd} b_0 & b_1 \arrow[l, "a"'] & b_2 \arrow[l, "\varepsilon"'] & b_3 \arrow[l, "a"'] \arrow[lll, bend left, "\varepsilon"] \end{tikzcd}$$

**Example 2.64.** *Consider again Example 2.62. Then we can depict $M(w_{\mathbb{Z}})$ in the following short form:*

$$\begin{tikzcd} b_0 \arrow[r, "b"] \arrow[rrrrr, bend right, "\eta"'] & b_1 \arrow[r, "c"] & b_2 & b_3 \arrow[l, "\kappa"'] & b_4 \arrow[l, "c"'] & b_5 \arrow[l, "b"'] \end{tikzcd} .$$

In this case, the $\Lambda$–module $M(w_{\mathbb{Z}})$ is free of rank $p$ over $\mathrm{k}[T, T^{-1}]$, for $T$ acting as $t_{w_{\mathbb{Z}},p}$, and hence becomes a $\Lambda - \mathrm{k}[T, T^{-1}]$–bimodule.
Thus, given a $\mathrm{k}[T, T^{-1}]$–module $V$, we can "extend" the module $M(w_{\mathbb{Z}})$ to the module $M(w_{\mathbb{Z}}, V) := M(w_{\mathbb{Z}}) \otimes_{\mathrm{k}[T,T^{-1}]} V \cong \bigoplus_{i \in I} V_i$ for some finite set I of cardinality $p$ and $V_i = V$ for all $i \in$ I: since $M(w_{\mathbb{Z}})$ is free of rank $p$ over $\mathrm{k}[T, T^{-1}]$, we have as k–vector spaces

$$M(w_{\mathbb{Z}}) \otimes_{\mathrm{k}[T,T^{-1}]} V \cong (\mathrm{k}[T, T^{-1}])^p \otimes_{\mathrm{k}[T,T^{-1}]} V \cong V^p.$$

The depiction of $M(w_{\mathbb{Z}}, V)$ is similar to the one of $M(w_{\mathbb{Z}})$, with disjoint copies of $V$ at the vertices instead of basis elements. Indexing the copies of

$V$ is helpful to read the action of $\Lambda$ which is given according to $w_{\mathbb{Z}}$.
To describe the action of $\Lambda$ on the vector space $M(w_{\mathbb{Z}}, V)$ more detailed, we introduce a more formal way of describing the module $M(w_{\mathbb{Z}}, V)$. To this end, let for $i \in \{0, \dots, p-1\}$

$$V_i = \{\bar{v} \mid \bar{v}_i = v \in V, \bar{v}_j = 0 \quad \forall j \neq i\},$$
$$\varphi_i : V \hookrightarrow V_i, \quad v \mapsto \bar{v} = (0, \dots, 0, v, 0 \dots, 0).$$

Then we have $\operatorname{im}(\varphi_i) = V_i \cong V$ for all $i \in \{0, \dots p-1\}$ and

$$M(w_{\mathbb{Z}}, V) = \bigoplus_{0 \le i \le p-1} V_i = \bigoplus_{0 \le i \le p-1} \operatorname{im}(\varphi_i) \tag{18}$$

as k$-$vector spaces. By the action of $\Lambda$ on $M(w_{\mathbb{Z}}, V)$, we have

$$w_i \varphi_{i+1} = \varphi_i, \tag{19}$$

that is, the following diagram commutes:

$$\begin{tikzcd}
\cdots & V_i & & V_{i+1} \arrow[ll, "w_i"'] & & V_{i+2} \arrow[ll, "w_{i+1}"'] & \cdots \\
& & & V \arrow[ull, "\varphi_i"] \arrow[u, "\varphi_{i+1}"'] \arrow[urr, "\varphi_{i+2}"'] & & &
\end{tikzcd}$$

Let $(v_0, \dots, v_{p-1}) \in \bigoplus \operatorname{im}(\varphi_i)$. Then

$$x(v_0, \dots, v_{p-1}) = \bigoplus_{i=0}^{p-1} x((0, \dots, 0, v_i, 0, \dots, 0)) \quad \text{for all } x \in Q_1,$$

where by (19) for all $x \in Q_1^{\text{ord}}$:

$$x((\delta_{ij} v)_i) = \begin{cases} (\delta_{ki} v)_i & \text{if } \exists \text{ arrow } x : V_j \to V_k, \\ 0 & \text{otherwise,} \end{cases}$$

and for all $\varepsilon \in \mathrm{Sp}$:

$$\varepsilon((\delta_{ij} v)_j) = \begin{cases} (\delta_{ij} v)_j & \text{if } \exists \text{ arrow } \varepsilon : V_l \to V_j, \\ (\delta_{ik} v)_k & \text{if } \exists \text{ arrow } \varepsilon : V_j \to V_k, \\ 0 & \text{otherwise.} \end{cases}$$

**Example 2.65.** *We consider again $M(w_{\mathbb{Z}})$ as in Example 2.63, and $V$ a* $\mathrm{k}[T, T^{-1}]$*-module. Then $M(w_{\mathbb{Z}}, V)$ is of the form*

$$\begin{tikzcd}
V_0 & V_1 \arrow[l, "a"'] & V_2 \arrow[l, "\varepsilon"'] & V_3 \arrow[l, "a"'] \arrow[lll, bend left, "\varepsilon"]
\end{tikzcd}$$

*where the $V_i$'s are disjoint copies of $V$. Also in this depiction, the special letters $\varepsilon$ and $\varepsilon^{-1}$ are acting with respect to the vector spaces as described in Remark 2.57. The action of $\Lambda$ on $M(w_{\mathbb{Z}}, V)$ is thus given as follows: let $(v_0, v_1, v_2, v_3) \in M(w_{\mathbb{Z}}, V)$. Then $a$ and $\varepsilon$ act according to the depiction:*

$$\begin{aligned} a(v_0, v_1, v_2, v_3) &= (v_1, 0, v_3, 0) \\ \varepsilon(v_0, v_1, v_2, v_3) &= (v_0 + v_3, v_1 + v_2, 0, 0) \end{aligned}$$

**Example 2.66.** *Consider again Example 2.62. Then we can depict for a given $\mathrm{k}[T, T^{-1}]$–module $V$, the module $M(w_{\mathbb{Z}}, V)$ in the following short form:*

$$\begin{tikzcd} V_0 \arrow[r, "b"] \arrow[rrrrr, bend right=10, "\eta"'] & V_1 \arrow[r, "c"] & V_2 & V_3 \arrow[l, "\kappa"'] & V_4 \arrow[l, "c"'] & V_5 \arrow[l, "b"'] \end{tikzcd}$$

*The action of $\Lambda$ on the module is given by*

$$\begin{aligned} (v_0, v_1, v_2, v_3, v_4, v_5) &\overset{b}{\longmapsto} (0, v_0, 0, 0, v_5, 0) \\ &\overset{c}{\longmapsto} (0, 0, v_1, v_4, 0, 0) \\ &\overset{\kappa}{\longmapsto} (0, 0, v_2 + v_3, 0, 0, 0) \\ &\overset{\eta}{\longmapsto} (0, 0, 0, 0, 0, v_0 + v_5). \end{aligned}$$

### 2.4.2 Modules from periodic words with $\hat{w}_p = \varepsilon^{\kappa} v \eta^{\mu} v^{-1}$

In this subsection we consider directed $\mathbb{Z}$–words $w_{\mathbb{Z}}$ of period $p = 2m+2$ with periodic part $w_p$ of the form $\varepsilon^{\kappa} v \eta^{\mu} v^{-1}$, where $\kappa, \mu \in \{+1, -1\}$, $v$ a subword of $w_{\mathbb{Z}}$ with $|v| = m$, and $\varepsilon, \eta \in \mathrm{Sp}$. We examine words of this form also more detailed in Subsection 3.2.4. They will play an important role in the classification of indecomposable $\Lambda$–modules (cf. Chapter 6).
For $V$ a $\mathrm{k}[T, T^{-1}]$–module, we have seen in the previous subsection how to depict the module $M(w_{\mathbb{Z}}, V)$. Let $\mu = \kappa = 1$, and let the $V_i$'s be disjoint copies of $V$. Then

$$\begin{tikzcd} M(w_{\mathbb{Z}}, V): V_1 \arrow[rrrrrrr, bend right=10, "\varepsilon = T"'] & V_2 \arrow[l, "v_1"'] & \cdots \arrow[l, "v_2"'] & V_m \arrow[l, "v_m"'] & V_{m+1} \arrow[l, "\eta"'] \arrow[r, "v_m"] & \cdots \arrow[r, "v_1"] & V_{2m+2} \end{tikzcd}$$

We can abbreviate this depiction using the symmetries in the two idempotents. To this end, let $W = V \oplus V$. Then $W$ is a $\mathrm{k}\langle f, e \mid f^2 = f, e^2 = e \rangle$–module and we can depict $M(w_{\mathbb{Z}}, W)$ as follows:

$$M(w_{\mathbb{Z}}, W): \quad \begin{tikzcd} W_0 \arrow[loop left, "\varepsilon = e"] & W_1 \arrow[l, "v_1"'] & \cdots \arrow[l, "v_2"'] & W_m \arrow[l, "v_m"'] \arrow[loop right, "\eta = f"] \end{tikzcd} \tag{20}$$

where the $W_i$'s are disjoint copies of $W$ and $M(w_{\mathbb{Z}}, W) = \bigoplus_{i=0}^{m} W_i$ as $\mathrm{k}$ –vector spaces. Note that one can consider the maps $\varphi_i$ from the previous subsection also with respect to $M(w_{\mathbb{Z}}, W)$. To this end, consider $\bar{\varphi}_i : W_i \longrightarrow W$, where $W_i = \{\bar{x} \mid \bar{x}_i = x \in W, \bar{x}_j = 0 \quad \forall j \neq i\}$, $0 \leq i \leq m$. One obtains as before $M(w_{\mathbb{Z}}, W) = \bigoplus_{i=0}^{m} W_i$ as vector spaces. The commutativity relations are now given by

$$\begin{aligned} v_i \bar{\varphi}_i &= \bar{\varphi}_{i-1}, \quad 1 \leq i \leq m \\ \bar{\varphi}_0 &= f \bar{\varphi}_0, \\ \bar{\varphi}_m &= e \bar{\varphi}_m. \end{aligned} \tag{21}$$

Thus, $\Lambda$ acts on $M(w_{\mathbb{Z}}, W)$ according to $w_{\mathbb{Z}}$, that is, with the notation from the previous subsection and $w \in W$:

$$x((\delta_{ij} w)_i) = \begin{cases} (\delta_{ik} w)_i & \text{if } \exists \text{ arrow } x : W_j \longrightarrow W_k, \\ 0 & \text{otherwise,} \end{cases}$$

for all $x \in \mathrm{Q}_1^{\mathrm{ord}}$, and for all $\varepsilon \in \mathrm{Sp}$

$$\varepsilon((\delta_{ij} w)_i) = \begin{cases} (\delta_{ij} w)_i & \text{if } \exists \text{ arrow } \varepsilon : W_l \longrightarrow W_j \text{ for some } l, \\ (\delta_{ik} w)_i & \text{if } \exists \text{ arrow } \varepsilon : W_j \longrightarrow W_k, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, we have that

$$\varepsilon((\delta_{i0} w)_i) = (\delta_{i0} f(w))_i, \text{ and} \tag{22}$$

$$\eta((\delta_{im} w)_i) = (\delta_{im} e(w))_i. \tag{23}$$

**Example 2.67.** *Let $W$ be a $\mathrm{k}\langle f, e \mid f^2 = f, e^2 = e\rangle$−module and let $\Lambda$ be as in Example 2.3.1. Consider the periodic word $w_{\mathbb{Z}}$ in $\Gamma_{\mathrm{d}}(\Lambda)$ with periodic part $\hat{w}_p = \varepsilon^{\kappa} a \varepsilon a \varepsilon^{-1} a \varepsilon^{\mu} a^{-1} \varepsilon a^{-1} \varepsilon^{-1} a^{-1}$, $\kappa, \mu \in \{+1, -1\}$. Then we depict $M(w_{\mathbb{Z}}, W)$ as*

$$\begin{tikzcd} W_0 \arrow[loop left, "\varepsilon=e"] & W_1 \arrow[l, "a"'] & W_2 \arrow[l, "\varepsilon"'] & W_3 \arrow[l, "a"'] \arrow[r, "\varepsilon"] & W_4 & W_5 \arrow[l, "a"'] \arrow[loop right, "\varepsilon=f"] \end{tikzcd}$$

*with the $W_i$'s disjoint copies of $W$ for all $i \in \{0, \dots 5\}$ and $\varepsilon$,$a$ acting according to the above descriptions:*

$$\begin{aligned} a(v_0, v_1, v_2, v_3, v_4, v_5) &= (v_1, 0, v_3, 0, v_5, 0), \\ \varepsilon(v_0, v_1, v_2, v_3, v_4, v_5) &= (f v_0, v_1 + v_2, 0, 0, v_3 + v_4, e v_5). \end{aligned}$$

**Example 2.68.** *Let $W$ be a $\mathrm{k}\langle f, e \mid f^2 = f, e^2 = e\rangle$−module and let $\Lambda$ be as in Example 2.58. Consider the periodic word $w_{\mathbb{Z}}$ with $\hat{w}_p = \eta^{\delta} b^{-1} c^{-1} \kappa^{\mu} bc$, $\delta, \mu \in \{+1, -1\}$. Then $M(w_{\mathbb{Z}}, W)$ is depicted as follows:*

$$\begin{tikzcd} W_0 \arrow[loop left, "\eta=e"] \arrow[r, "b"] & W_1 \arrow[r, "c"] & W_2 \arrow[loop right, "\kappa=f"] \end{tikzcd}$$

*where the $W_i$'s are disjoint copies of $W$, $0 \le i \le 2$, and $\Lambda$ acts as follows:*

$$\begin{aligned}(v_0, v_1, v_2) &\overset{b}{\longmapsto} (0, v_0, 0)\\ &\overset{c}{\longmapsto} (0, 0, v_1)\\ &\overset{\eta}{\longmapsto} (fv_0, 0, 0)\\ &\overset{\kappa}{\longmapsto} (0, 0, ev_2).\end{aligned}$$

# 3 Words of string and band type

As mentioned before (see Section 2.4) we would like to base the classification theorem on undirected words. Hence, we need to find a bridge from those to the $\Lambda$–modules in the classification statement.
To do so, we are going to introduce two different types of directed words in Section 3.1. One type is given by weakly consistent words for which the direction of the special letters depends on the directions of the ordinary arrows in the word. Here, the direction refers to the exponent of the letter and thus to its type (inverse, direct). In contrast to the weakly consistent words, we have the consistent words that include the special letters in the data set for the directions of the special letters. In particular, we introduce the $c^*$–index for letters of weakly consistent words and the $c$–index for letters of consistent words. Those indices measure the distance from a directed special letter to the letters which give conclusion on the type of directed word.
In Section 3.2, we introduce the notion of asymmetric and symmetric strings and bands. Our goal is to give the classification of the indecomposable finite dimensional modules in terms of those strings and bands. In order to do so, we will see in Chapter 4 that they result in $\mathfrak{L}$–graphs from which the canonical $\overline{\mathfrak{X}}_\Lambda$–representations are obtained. In preparation of these results, we take a closer look on the symmetries in symmetric bands in Subsection 3.2.5.
It is only natural to compare the weakly consistent and consistent directed versions of the strings and bands with the hope that one type of those will describe the finite dimensional indecomposable modules. This comparison is the content of Section 3.3. We will see that the directions coincide on letters of finite index, excluding one particular type of letter for symmetric strings (Theorems 3.53 and 3.61, Proposition 3.67).

## 3.1 Types of directed words

Througout this subsection let $\Gamma$ be either the directed or undirected alphabet of a clannish algebra $\Lambda$.

**Definition 3.1.** *Let $w_\mathrm{I}$ be a non-trivial $\Gamma-\mathrm{I}$–word. Then the* direction *of the letter $w_j$, $j \in \mathrm{I}$ is given by*

$$\mathrm{dir}(w_j) = \mathrm{dir}_j(w_\mathrm{I}) = \begin{cases} 1 & \text{if } w_j = x \text{ for some } x \in \mathrm{Q}_1, \\ -1 & \text{if } w_j = x^{-1} \text{ for some } x \in \mathrm{Q}_1, \\ 0 & \text{if } w_j = \varepsilon^* \text{ for some } \varepsilon \in \mathrm{Sp}\,. \end{cases}$$

Note that for $\Gamma = \Gamma_\mathrm{d}$ one has $\mathrm{dir}_j(w_\mathrm{I}) \neq 0$ for all $j \in \mathrm{I}$.
We can use the direction of a letter to visualize it, respectively the word it belongs to:

If $l \in \Gamma$ is a letter with $\mathrm{dir}(l) = 1$, then we depict it by an arrow going from right to left:

$$\xleftarrow{l}.$$

If, on the other hand, $h \in \Gamma$ is a letter with $\mathrm{dir}(h) = -1$, then it is of the form $h = l^{-1}$ for some $l \in \mathrm{Q}_1$ and we depict it by an arrow going in opposite direction:

$$\xrightarrow{l}.$$

A letter $l \in \Gamma$ of the form $l = \varepsilon^*$ (i.e. $l \in \Gamma_{\mathrm{ud}}$) is depicted by an edge:

$$\overset{l^*}{\text{---}}.$$

This visualization is very helpful to the reader: we are going to see later (in Subsections 3.1.1 and 3.1.2) that the directions on the special letters for the two mentioned types of directed words are given in an intuitive way with respect to this visualization.

**Example 3.2.** *Consider the finite directed word $v = \varepsilon^{-1}a\varepsilon$ of length 3 with alphabet given by $\Lambda$ as in Example 2.31. Then the directions of the single letters are given by*

$$\mathrm{dir}(v_1) = -1, \quad \mathrm{dir}(v_2) = 1, \quad \mathrm{dir}(v_3) = 1.$$

*If we consider the undirected word $w = a\varepsilon^* a^{-1}\varepsilon^*$ obtained from the undirected alphabet of the same $\Lambda$, then the directions are given by*

$$\mathrm{dir}(w_1) = 1, \quad \mathrm{dir}(w_2) = \mathrm{dir}(w_4) = 0, \quad \mathrm{dir}(w_3) = -1.$$

*According to the above description, we can depict the two words as follows:*

$$\begin{aligned} v &: \quad \xrightarrow{\varepsilon}\xleftarrow{a}\xleftarrow{\varepsilon}, \\ w &: \quad \xleftarrow{a}\overset{\varepsilon^*}{\text{---}}\xrightarrow{a}\overset{\varepsilon^*}{\text{---}}. \end{aligned}$$

### 3.1.1 Weakly consistent words

Throughout this section let $w_{\mathrm{I}}$ be a coadmissible undirected $\mathrm{I}$–word for either $\mathrm{I} = \{0, \ldots, n\}$ $(n > 0)$ or $\mathrm{I} = \mathbb{Z}$.
We are now going to define weakly consistent words.

**Definition 3.3.** *Let $v_{\mathrm{I}} \in (\Phi^{\mathrm{d}}_{\mathrm{ud}})^{-1}(w_{\mathrm{I}})$. We call $v_{\mathrm{I}}$* weakly consistent *provided that*

$$\mathrm{dir}(v_j) = \mathrm{dir}_j(v_{\mathrm{I}}) = \begin{cases} 1 & \textit{if } (w_{\mathrm{I}}[< j])^{-1} \geq w_{\mathrm{I}}[> j], \\ -1 & \textit{if } (w_{\mathrm{I}}[< j])^{-1} \leq w_{\mathrm{I}}[> j], \end{cases}$$

*for all $j \in \mathrm{I}$ with $w_j$ a special letter.*

In case of equality, the direction is not uniquely defined. This means we can either have $\mathrm{dir}_j(v_{\mathrm{I}}) = 1$ or $\mathrm{dir}_j(v_{\mathrm{I}}) = -1$ if $(w_{\mathrm{I}}[< j])^{-1} = w_{\mathrm{I}}[> j]$.
It is possible to construct a weakly consistent directed version from an undirected word by assigning exponents to the special letters in the undirected word as described in Definition 3.3.

**Remark 3.4.** *It is clear from the definition that there does not always exist a unique weakly consistent directed version $v_{\mathrm{I}}$ of $w_{\mathrm{I}}$, e.g. we will see in Section 3.2 that there is a unique weakly consistent directed version for words of so called asymmetric types, but not for symmetric ones.*

**Remark 3.5.** *Note that the directions on letters of ordinary type are the same in the undirected word $w_{\mathrm{I}}$ as in its weakly consistent directed version $v_{\mathrm{I}}$:*

$$\mathrm{dir}_j(w_{\mathrm{I}}) = \mathrm{dir}_j(v_{\mathrm{I}}) \text{ for all } j \in \mathrm{I} \text{ with } w_j \text{ of ordinary type.}$$

**Example 3.6.**

1. *Consider the algebra $\Lambda$ as in Example 2.3.1. Let $w = \varepsilon^* a \varepsilon^*$. Let $v \in (\Phi^{\mathrm{d}}_{\mathrm{ud}})^{-1}(w)$ be weakly consistent. We have $v = v_1 a v_3$ with $v_i \in \{\varepsilon, \varepsilon^{-1}\}$ for $i = 1, 3$. We have for the letter $v_1$ that $w[< 1]^{-1} = 1_{s(\varepsilon),\kappa}$, $\kappa = -\operatorname{sgn}(\varepsilon^*)$, $w[> 1] = a\varepsilon^*$. Hence, $w[> 1] = w[< 1]^{-1} a z$, for $z = \varepsilon^*$, and thus $w[< 1]^{-1} > w[> 1]$ and $\mathrm{dir}(v_1) = 1$. It follows similarly that $\mathrm{dir}(v_2) = 1$. Hence, $v$ is of the form $\varepsilon a \varepsilon$.*

2. *Let $\Lambda$ be as in Example 2.14. Consider $w = ad\varepsilon^* d^{-1} a^{-1}$ and let $v \in (\Phi^{\mathrm{d}}_{\mathrm{ud}})^{-1}(w)$. Then $v$ is given by $v = adv_3 d^{-1} a^{-1}$ with $v_3 \in \{\varepsilon, \varepsilon^{-1}\}$. We have*

$$(v[< 3])^{-1} = d^{-1} a^{-1} = v[> 3].$$

*Thus, $v$ is weakly consistent for any choice of $v_3$.*

To measure at which point of $w_{\mathrm{I}}$ the relevant information for the direction of a letter $v_j$ in a weakly consistent directed version $v_{\mathrm{I}}$ is found, we introduce an index for each $j \in \mathrm{I}$ for which $v_j$ is of special type:

**Definition 3.7.** *Let $v_{\mathrm{I}}$ be a weakly consistent directed version of the undirected $\mathrm{I}$−word $w_{\mathrm{I}}$. The* $c^*$−index *of a special letter $v_j$, $j \in \mathrm{I}$, is given by*

$$\mathrm{ind}^*_j(v_{\mathrm{I}}) \coloneqq \sup\{\mathrm{length}(z_{\mathrm{I}'}) \mid (w_{\mathrm{I}}[< j])^{-1} = z_{\mathrm{I}'} u_{\mathrm{J}}, w_{\mathrm{I}}[> j] = z_{\mathrm{I}'} x_{\mathrm{J}'}\}$$

*for some undirected subwords $z_{\mathrm{I}'}$, $u_{\mathrm{J}}$, $x_{\mathrm{J}'}$ of $w_{\mathrm{I}}$.*

Note that we can also define this index on the undirected version $w_{\mathrm{I}}$ since the direction of special letters in $v_{\mathrm{I}}$ only depends on $w_{\mathrm{I}}$. We define $\mathrm{ind}^*_j(w_{\mathrm{I}}) = \mathrm{ind}^*_j(v_{\mathrm{I}})$ for $v_{\mathrm{I}}$ a weakly consistent directed version of $w_{\mathrm{I}}$.

We denote by $\mathrm{J}^*$ the interval $[j - \mathrm{ind}^*_j(w_{\mathrm{I}}), j + \mathrm{ind}^*_j(w_{\mathrm{I}})]$ in $\mathbb{Z}$, and its left and right hand side subintervals by $\mathrm{J}^*_- = [j - \mathrm{ind}^*_j(w_{\mathrm{I}}), j]$, $\mathrm{J}^*_+ = [j, j + \mathrm{ind}^*_j(w_{\mathrm{I}})]$,

respectively.
The letters left (resp. right) of $\mathrm{J}^*$ determine the direction on $v_j$ for $v_{\mathrm{I}}$ weakly consistent. We denote them by

$$j^*_- := j - \mathrm{ind}^*_j(w_{\mathrm{I}}) - 1,$$
$$j^*_+ := j + \mathrm{ind}^*_j(w_{\mathrm{I}}) + 1.$$

For $I = \{0, \dots n\}$, we set conventionally $j^*_- = 0$ ($j^*_+ = n+1$) if $w_{j^*_-} = 1_{i,\kappa}$ ($w_{j^*_+} = 1_{i,\kappa}$) for $i = t(w_1)$, $\kappa = \mathrm{sgn}(w_1)$ ($i = s(w_n)$, $\kappa = \mathrm{sgn}(w_n)$, respectively).

**Remark 3.8.** *By definition, $w_{j^*_-}$, $w_{j^*_+}$ are of ordinary type. Hence,*

$$\mathrm{dir}_{j^*_-}(w_{\mathrm{I}}) = \mathrm{dir}_{j^*_+}(w_{\mathrm{I}}) = \mathrm{dir}_j(v_{\mathrm{I}}).$$

**Example 3.9.** *1. Consider $w$ ($v$, respectively) as in Example 3.6.1. Then*

$$\mathrm{ind}^*_1(w) = 0, \quad 1^* = [1,1], \quad 1^*_- = 0, \quad 1^*_+ = 2,$$
$$\mathrm{ind}^*_3(w) = 0, \quad 3^* = [3,3], \quad 3^*_- = 2, \quad 3^*_+ = 4.$$

*2. Let $\Lambda$ be as in Example 2.3.1 and let $w = \varepsilon^* a^{-1} \varepsilon^* a^{-1} \varepsilon^* a \varepsilon^* a \varepsilon^*$. Then*

$$\begin{aligned}
&\mathrm{ind}^*_1(w) = 0, \quad 1^* = [1,1], &&1^*_- = 0, \quad 1^*_+ = 2,\\
&\mathrm{ind}^*_3(w) = 0, \quad 3^* = [3,3], &&3^*_- = 2, \quad 3^*_+ = 4,\\
&\mathrm{ind}^*_5(w) = 4, \quad 5^* = [1,9], &&5^*_- = 0, \quad 5^*_+ = 10,\\
&\mathrm{ind}^*_7(w) = 0, \quad 7^* = [7,7], &&7^*_- = 6, \quad 7^*_+ = 8,\\
&\mathrm{ind}^*_9(w) = 0, \quad 9^* = [9,9], &&9^*_- = 8, \quad 9^*_+ = 10.
\end{aligned}$$

**Example 3.10.** *1. Consider $w$ and $v$ as in Example 3.6.1. Then*

$$v_{1^*_-} = 1_{t(v_1),\kappa} \text{ with } \kappa = \mathrm{sgn}(v_1),$$
$$v_{1^*_+} = v_2,$$
$$v_{3^*_-} = v_2,$$
$$v_{3^*_+} = 1_{s(v_3),\mu} \text{ with } \mu = \mathrm{sgn}(v_3),$$

*and thus*

$$\mathrm{dir}(w_2) = \mathrm{dir}(v_2),$$
$$\mathrm{dir}(v_1) = \mathrm{dir}(1_{t(v_1),\kappa}) = \mathrm{dir}(v_2),$$
$$\mathrm{dir}(v_3) = \mathrm{dir}(v_2) = \mathrm{dir}(1_{s(v_3),\mu}).$$

*2. Let $w$ and $v$ be as in Example 3.6.2. Then*

$$v_{3^*_-} = 1_{t(d),\kappa}, \quad \kappa = \mathrm{sgn}(d),$$
$$v_{3^*_+} = 1_{t(d),\kappa},$$

*i.e. $v_{3^*_-} = v_{3^*_+}$. It follows that $\mathrm{dir}(v_3) = 1$ and $\mathrm{dir}(v_3) = -1$ is possible.*

### 3.1.2 Consistent words

Throughout this subsection let $v_{\mathrm{I}}$ be a coadmissible directed $\mathrm{I}$–word for either $\mathrm{I} = \{0,\dots,n\}$ $(n > 0)$ or $\mathrm{I} = \mathbb{Z}$.

**Definition 3.11.** *We call* $v_{\mathrm{I}}$ consistent *provided that for any* $j \in \mathrm{I}$ *with* $v_j$ *special, we have that*

$$\operatorname{dir}_j(v_{\mathrm{I}}) = \begin{cases} 1 & \text{if } (v_{\mathrm{I}}[<j])^{-1} \geq v_{\mathrm{I}}[>j], \\ -1 & \text{if } (v_{\mathrm{I}}[<j])^{-1} \leq v_{\mathrm{I}}[>j]. \end{cases}$$

Again, in the case of equality, the direction is not uniquely defined, i.e. we can either have $\operatorname{dir}_j(v_{\mathrm{I}}) = 1$ or $\operatorname{dir}_j(v_{\mathrm{I}}) = -1$ if $(v_{\mathrm{I}}[<j])^{-1} = v_{\mathrm{I}}[>j]$.

**Example 3.12.** *Consider the coadmissible directed word* $v = \varepsilon a \varepsilon$ *from Example 3.6.1. Then we have that*

$$\begin{aligned} &v[<1]^{-1} = 1_{t(v_1),-\mu} \text{ with } \mu = \operatorname{sgn}(v_1), \quad v[>1] = a\varepsilon, \\ &v[<3]^{-1} = a^{-1}\varepsilon^{-1}, \quad v[>3] = 1_{s(v_3),\kappa} \text{ with } \kappa = \operatorname{sgn}(v_3). \end{aligned}$$

*Then clearly,* $v[<1]^{-1} > v[>1]$ *and* $v[<3]^{-1} > v[>3]$*, so* $v$ *is consistent. Thus, it is both weakly consistent and consistent.*

**Example 3.13.** *Let* $\Lambda$ *be as in Example 3.6 2. Consider the directed word*

$$v = \underset{v_1}{e}\,\underset{v_2}{\kappa}\,\underset{v_3}{c}\,\underset{v_4}{b}\,\underset{v_5}{\eta}\,\underset{v_6}{b^{-1}}\,\underset{v_7}{a}\,\underset{v_8}{\varepsilon^{-1}}\,\underset{v_9}{a^{-1}}\,\underset{v_{10}}{b}\,\underset{v_{11}}{\eta}\,\underset{v_{12}}{b^{-1}}\,\underset{v_{13}}{a}\,\underset{v_{14}}{\varepsilon^{-1}}\,\underset{v_{15}}{a^{-1}}\,\underset{v_{16}}{d^{-1}}\,\underset{v_{17}}{e^{-1}}\,\underset{v_{18}}{\kappa}.$$

*It is consistent with respect to the letters* $v_2$*,* $v_5$*,* $v_{11}$*,* $v_{14}$ *and* $v_{18}$*. The entire word is not consistent since it is not consistent with respect to* $v_8$*.*

For each $j \in \mathrm{I}$ with $v_j$ of special type, we define its *c-index* as follows:

**Definition 3.14.** *Let* $v_{\mathrm{I}}$ *be as above a directed version of some undirected word* $w_{\mathrm{I}}$*. The* $c$–index *for* $j \in \mathrm{I}$ *with* $v_j$ *special is given by*

$$\operatorname{ind}^c_j(v_{\mathrm{I}}) \coloneqq \sup\{\operatorname{length}(z_{\mathrm{I}'}) \mid (v_{\mathrm{I}}[<j])^{-1} = z_{\mathrm{I}'}u_{\mathrm{J}}, v_{\mathrm{I}}[>j] = z_{\mathrm{I}'}x_{\mathrm{J}'}\}$$

*for some subwords* $z_{\mathrm{I}'}$*,* $u_{\mathrm{J}}$*,* $x_{\mathrm{J}'}$ *of* $v_{\mathrm{I}}$*.*

Note that it is not as easy as in the case of weakly consistent words to construct a consistent directed version from an undirected word. However, it can be done by proceeding inductively on the $c$–index of the special letters. We introduce some notation with respect to the $c$–index. We denote by $\mathrm{J}^c$ the interval $[j - \operatorname{ind}^c_j(w_{\mathrm{I}}), j + \operatorname{ind}^c_j(w_{\mathrm{I}})]$ in $\mathbb{Z}$, and its left and right hand side subintervals by $\mathrm{J}^c_- = [j - \operatorname{ind}^c_j(w_{\mathrm{I}}), j]$, $\mathrm{J}^c_+ = [j, j + \operatorname{ind}^c_j(w_{\mathrm{I}})]$, respectively.

The letters left (resp. right) of $\mathrm{J}^c$ determine the direction on $v_j$ where $v_\mathrm{I}$ is consistent. We denote them by

$$j_-^c \coloneqq j - \mathrm{ind}_j^c(w_\mathrm{I}) - 1,$$
$$j_+^c \coloneqq j + \mathrm{ind}_j^c(w_\mathrm{I}) + 1.$$

For $I = \{0, \dots n\}$, we set conventionally $j_-^c = 0$ ($j_-^c = n+1$) if $w_{j_-^c} = 1_{i,\kappa}$ for $i = t(w_1)$, $\kappa = \mathrm{sgn}(w_1)$ ($i = s(w_n)$, $\kappa = \mathrm{sgn}(w_n)$). We proceed similarly with $j_+^c$.

**Example 3.15.** *Consider the consistent directed word $v$ in Example 3.12. Then, as mentioned above, we have $J^c = J^*$, $j_-^c = j_-^*$ and $j_+^c = j_+^*$, hence given as in Example 3.9.*

**Remark 3.16.** *As in the previous subsection, we do not have any conditions concerning the direction on letters of ordinary type, i.e. if $v_\mathrm{I}$ is a consistent directed version of an undirected word $w_\mathrm{I}$, then*

$$\mathrm{dir}_j(w_\mathrm{I}) = \mathrm{dir}_j(v_\mathrm{I}) \text{ for all } j \in \mathrm{I} \text{ with } w_j \text{ of ordinary type.}$$

*But in contrast to a weakly consistent orientation, we can now also have $v_{j_-^*}$, $v_{j_+^*}$ of special type. Still, we have by definition*

$$\mathrm{dir}_{j_-^c}(v_\mathrm{I}) = \mathrm{dir}_{j_+^c}(v_\mathrm{I}) = \mathrm{dir}_j(v_\mathrm{I}).$$

**Lemma 3.17.** *Let $w_\mathrm{I}$ be an undirected $\mathrm{I}$-word. Then*

$$\mathrm{ind}_j^*(w_\mathbb{Z}) \geq \mathrm{ind}_j^c(v_\mathbb{Z})$$

*for any $j \in \mathrm{I}$, any $v_\mathrm{I} \in (\Phi_{\mathrm{ud}}^{\mathrm{d}})^{-1}(w_\mathbb{Z})$.*

*Proof.* The inequality follows directly from Definition 3.7 and Definition 3.14. □

## 3.2 Strings and bands

In this section, we classify certain types of undirected words. Our underlying goal is to use these types to give a classification of the indecomposable $\Lambda$–modules.
Let $\Lambda$ be a clannish algebra throughout this section.

### 3.2.1 Asymmetric strings

**Definition 3.18.** *Let $w$ be a finite undirected word. It is said to be of* asymmetric string type *if $w$ is coadmissible and $w \neq w^{-1}$.*
*We denote by $\overline{\mathcal{W}}^a$ the set of all $w \in \mathcal{W}_{\mathrm{ud}}$ of asymmetric string type and by $\overline{\mathcal{W}}^a_\sim = \overline{\mathcal{W}}^a / \sim$ with the equivalence relation $\sim$ defined in Section 2.3. We call $w \in \overline{\mathcal{W}}^a_\sim$ an* asymmetric string.

**Example 3.19.** *Let $\Lambda$ be as in Example 2.3. The word $w = \varepsilon^* a \varepsilon^*$ is of asymmetric string type while $x = \varepsilon^* a \varepsilon^* a^{-1} \varepsilon^*$ is not since $x = x^{-1}$.*

Note that any asymmetric string $w$ is minimal by Lemma 2.54.
We obtain for $w$ of asymmetric string type, and for every directed version $v \in (\Phi^{\mathrm{d}}_{\mathrm{ud}})^{-1}(w)$, a module $M(v)$ that is depicted, for a $k$–basis $b_0, \dots, b_n$, as

$$b_0 \xleftarrow{v_1} b_1 \xleftarrow{v_2} \dots \xleftarrow{v_{n-1}} b_{n-1} \xleftarrow{v_n} b_n.$$

**Definition 3.20.** *Let $w$ be an asymmetric string and $v \in (\Phi^{\mathrm{d}}_{\mathrm{ud}})^{-1}(w)$. Then the module $M(v)$ is called* asymmetric string module.

**Proposition 3.21.** *Let $w$ be an asymmetric string. Then $v \in (\Phi^{\mathrm{d}}_{\mathrm{ud}})^{-1}(w)$ is weakly consistent if and only if the following holds for all $j \in \mathrm{I}$ with $v_j$ of special type:*

$$\mathrm{dir}(v_j) = \begin{cases} 1 & \text{if } (w[<j])^{-1} > w[>j] \\ -1 & \text{if } (w[<j])^{-1} < w[>j] \end{cases}. \tag{24}$$

*Proof.* It follows directly from Definition 3.3 that $v$ is weakly consistent if (24) holds.
For the other implication, it is enough to show that $(w[<j])^{-1} \neq w[>j]$ for all $j \in \mathrm{I}$ with $v_j$ of special type. Assume there exists $j \in \mathrm{I}$ with $(w[<j])^{-1} = w[>j]$. Let $u = (w[<j])^{-1} = w[>j]$. With $w_j = \varepsilon^*$, one can write $w$ as follows:

$$w = u^{-1} \varepsilon^* u.$$

Then $w = w^{-1}$, contradicting the definition of an asymmetric string. □

**Corollary 3.22.** *For each asymmetric string $w$ there exists a unique $v \in (\Phi^{\mathrm{d}}_{\mathrm{ud}})^{-1}(w)$ such that $v$ is weakly consistent.*

*Proof.* By Proposition 3.21, we know that any $\mathrm{dir}(v_j)$ with $v_j$ of special type is uniquely determined by $\mathrm{dir}(w_{j^*_-})$ and $\mathrm{dir}(w_{j^*_+})$. Since both $w_{j^*_-}$ and $w_{j^*_+}$ are of ordinary type, their directions are known. Thus, one can construct a weakly consistent directed version for any asymmetric string. Uniqueness follows by Proposition 3.21. □

### 3.2.2 Symmetric strings

**Definition 3.23.** *Let $w$ be a finite undirected word. It is said to be of* symmetric string type *if $w$ is coadmissible and can be written as $w = u\varepsilon^* u^{-1}$ for $u$ a minimal undirected word.*
*We denote by $\overline{\mathcal{W}}^s$ the set of all $w \in \mathcal{W}_{\mathrm{ud}}$ of symmetric string type and by $\overline{\mathcal{W}}^s_\sim = \overline{\mathcal{W}}^s / \sim$. We call $w \in \overline{\mathcal{W}}^s_\sim$ a* symmetric string.

By $w$ being of the form $w = u\varepsilon^* u^{-1}$, the condition $w = w^{-1}$ is implied (cf. Lemma 2.52). Note that $u$ is left coadmissible. Let $u$ be in the following of length $m$.

**Example 3.24.** *Let $\Lambda$ be as in Example 2.14. Then $w = \kappa^* c b \eta^* b^{-1} c^{-1} \kappa^*$ is of symmetric string type with $u = \kappa^* c b$.*

Similar to the case of asymmetric strings, we can obtain modules using the data of symmetric strings, or more precisely, using the data $u$.
Namely, for every $t \in (\Phi^{\mathrm{d}}_{\mathrm{ud}})^{-1}(u)$, there are directed words of the form $v^\pm = t\varepsilon^{\pm 1} t^{-1}$ with $v^\star \in (\Phi^{\mathrm{d}}_{\mathrm{ud}})^{-1}(w)$ and $\star \in \{+,-\}$. Now for each $v^\star$, $\star \in \{+,-\}$ arising in this way, we obtain two modules $M_i(v^\star)$, $i = 0,1$. Note that for any $i$, we have $M_i(v^+) \cong M_i(v^-)$. Hence, it is enough to consider either $v^+$ or $v^-$. We call the one word out of the two that we consider $v$. Taking into account that any idempotent $\varepsilon$ acting on a vector space $V$ gives the decomposition $V = \mathrm{im}(\varepsilon) \oplus \ker(\varepsilon)$, the module $M_i(v)$ is depicted as

$$b_0 \xleftarrow{t_1} b_1 \xleftarrow{t_2} \dots \xleftarrow{t_{m-1}} b_{m-1} \xleftarrow{t_m} b_m \circlearrowright_{\varepsilon = i}$$

with $\varepsilon$ acting as $i$.
Let $V$ be a $\mathrm{k}[f \mid f^2 = f]$–module. Then we can depict the above module also in the following way:

$$V_0 \xleftarrow{t_1} V_1 \xleftarrow{t_2} \dots \xleftarrow{t_{m-1}} V_{m-1} \xleftarrow{t_m} V_m \circlearrowright_{\varepsilon = f}$$

where the $V_i$'s are disjoint copies of $V$.

**Lemma 3.25.** *Let $w = u\varepsilon^* u^{-1}$ be a symmetric string. Then there exists a weakly consistent directed word $v \in (\Phi^{\mathrm{d}}_{\mathrm{ud}})^{-1}(w)$.*

*Proof.* Since $u$ is minimal, the direction of any $v_j$ of special type with $j \in \mathrm{I}$, $j \neq m+1$ is uniquely determined by $\mathrm{dir}(w_{j^*_-})$ and $\mathrm{dir}(w_{j^*_+})$.
For $j = m+1$ we have that $\mathrm{ind}^*_j(w) = m$. Thus, $(w[< j])^{-1} = w[> j] = u^{-1}$ and the direction of $v_{m+1}$ is not uniquely determined. Thus, choosing any orientation on $v_{m+1}$ gives a weakly consistent directed version. □

As one can see above, there are two options for a weakly consistent directed version $v$ of an asymmetric string $w = u\varepsilon^* u^{-1}$:

$$v^+ = t\varepsilon t^{-1}, \text{ i.e. } v_{m+1} = \varepsilon, \text{ and}$$
$$v^- = t\varepsilon^{-1}t^{-1}, \text{ i.e. } v_{m+1} = \varepsilon^{-1},$$

where $t \in (\Phi^{\mathrm{d}}_{\mathrm{ud}})^{-1}(u)$.

**Example 3.26.** *Consider $w$ from Example 3.24. Then*

$$v^+ = \kappa c b \eta b^{-1} c^{-1} \kappa^{-1}$$
$$v^- = \kappa c b \eta^{-1} b^{-1} c^{-1} \kappa^{-1}$$

*are both weakly consistent directed versions of $w$.*

It is not obvious whether there exists a unique consistent word $v$ with $(\Phi^{\mathrm{d}}_{\mathrm{ud}})(v) = w$ for $w$ a symmetric string.

### 3.2.3 Asymmetric bands

**Definition 3.27.** *Let $w_{\mathbb{Z}}$ be an undirected $\mathbb{Z}-$word. It is said to be of* asymmetric band type *if $w_{\mathbb{Z}} = w_{\mathbb{Z}}[p]$ for some $p > 0$, and $w_{\mathbb{Z}} \neq w_{\mathbb{Z}}^{-1}[k]$ for all $k \in \mathbb{Z}$. We denote by $\mathring{\mathcal{W}}^a$ the set of all $w_{\mathbb{Z}} \in \mathcal{W}^{\mathbb{Z}}_{\mathrm{ud}}$ of asymmetric band type and $\mathring{\mathcal{W}}^a_{\sim} = \mathring{\mathcal{W}}^a / \sim$ with the equivalence relation $\sim$ on $\mathbb{Z}-$words from Section 2.3. We call $w \in \mathring{\mathcal{W}}^a_{\sim}$ an* asymmetric band.

**Example 3.28.** 1. *Let $\Lambda$ be as in Example 2.3. Consider*

$$w_{\mathbb{Z}} = \ldots \varepsilon^* a \varepsilon^* \mid a \varepsilon^* a \ldots$$

*It is of asymmetric band type with $p = 2$ since*

$$w_{\mathbb{Z}}^{-1} = \ldots \varepsilon^* a^{-1} \varepsilon^* \mid a^{-1} \varepsilon^* a^{-1}$$

*yields that $w_{\mathbb{Z}}^{-1} \neq w_{\mathbb{Z}}$.*

2. *Consider $\Lambda$ from Example 2.14 and let $w_{\mathbb{Z}}$ with periodic part $\hat{w}_p = da\varepsilon^* a^{-1} b \eta^* b^{-1} c^{-1} \kappa^* e^{-1}$. It is of asymmetric band type with $p = 10$.*

We obtain for every directed version $v_{\mathbb{Z}} \in (\Phi^{\mathrm{d}}_{\mathrm{ud}})^{-1}(w_{\mathbb{Z}})$ of the same period as $w_{\mathbb{Z}}$, and for $V$ a $\mathrm{k}[T, T^{-1}]$–module, a module $M(v_{\mathbb{Z}}, V)$.
Similar as for $w_{\mathbb{Z}}$, we denote the periodic part of $v_{\mathbb{Z}}$ by $\hat{v}_p = v_1 \ldots v_p$. We depict $M(v_{\mathbb{Z}}, V)$ in the following way:

$$\begin{tikzcd} V_0 \arrow[rrr, bend right, "v_p"'] & V_1 \arrow[l, "v_1"'] & \ldots \arrow[l, "v_2"'] & V_{p-1} \arrow[l, "v_{p-1}"'] \end{tikzcd}$$

It is not as straightforward as in the string cases to make statements on the uniqueness of weakly consistent and consistent directed versions of asymmetric bands (compare Section 3.3).

**Lemma 3.29.** *Let $w_{\mathbb{Z}}$ be an asymmetric band of period $p$. Let $v_{\mathbb{Z}} \in (\Phi^{\mathrm{d}}_{\mathrm{ud}})^{-1}(w_{\mathbb{Z}})$ be of period $p$ and weakly consistent. Then*

$$\mathrm{dir}(v_j) = \begin{cases} 1 & \iff (w_{j-\frac{\tilde{p}}{2}} \dots w_{j-1})^{-1} > w_{j+1} \dots w_{j+\frac{\tilde{p}}{2}}, \\ -1 & \iff (w_{j-\frac{\tilde{p}}{2}} \dots w_{j-1})^{-1} < w_{j+1} \dots w_{j+\frac{\tilde{p}}{2}}, \end{cases}$$

*for all $j \in \mathbb{Z}$ with $v_j$ a special letter, where*

$$\tilde{p} = \begin{cases} p & \text{if } p \text{ even}, \\ p+1 & \text{if } p \text{ odd}. \end{cases}$$

*Proof.* Let $v_{\mathbb{Z}}$ be weakly consistent. Let $j \in \mathrm{I}$ with $v_j$ special. If

$$(w_{j-\frac{\tilde{p}}{2}} \dots w_{j-1})^{-1} > w_{j+1} \dots w_{j+\frac{\tilde{p}}{2}},$$

it follows that $(w_{\mathbb{Z}}[< j])^{-1} < w_{\mathbb{Z}}[> j]$. Then we have by Definition 3.3 that $\mathrm{dir}(v_j) = 1$. It follows similarly that $(w_{j-\frac{\tilde{p}}{2}} \dots w_{j-1})^{-1} < w_{j+1} \dots w_{j+\frac{\tilde{p}}{2}}$ implies $\mathrm{dir}(v_j) = -1$.
For the other implication, it is enough to show that

$$(w_{j-\frac{\tilde{p}}{2}} \dots w_{j-1})^{-1} \neq w_{j+1} \dots w_{j+\frac{\tilde{p}}{2}}. \tag{25}$$

Assume towards a contradiction that we have equality in (25). Consider $p$ to be even. Then we have that $\tilde{p} = p$. We have in particular that

$$w^{-1}_{j-\frac{p}{2}} = w_{j+\frac{p}{2}}. \tag{26}$$

The length of the subword $w_{j-\frac{p}{2}} \dots w_{j-1} w_j w_{j+1} \dots w_{j+\frac{p}{2}}$ is $p+1$. It follows by periodicity that $w_{j-\frac{p}{2}} = w_{j+\frac{p}{2}}$. Combining this equality with (26) results in $w_{j+\frac{p}{2}} = \varepsilon^*$ for some $\varepsilon \in \mathrm{Sp}$. Let $w_j$ be given by $\eta^*$, $\eta \in \mathrm{Sp}$ and denote by $u$ the subword $u = w_{j-\frac{p}{2}+1} \dots w_{j-1}$. We can thus write

$$w_{j-\frac{p}{2}+1} \dots w_{j-1} w_j w_{j+1} \dots w_{j+\frac{p}{2}} = u\eta^* u^{-1} \varepsilon^*. \tag{27}$$

Consider now $w_{\mathbb{Z}}[j - \frac{p}{2} + 1]$. It has periodic parts of form (27). Thus,

$$w_{\mathbb{Z}}[j - \frac{p}{2} + 1] = (w_{\mathbb{Z}}[j - \frac{p}{2} + 1])^{-1}.$$

It follows (Corollary 2.35 and Lemma 2.34) that

$$w_{\mathbb{Z}} = (w_{\mathbb{Z}}[2j - p])^{-1} = w_{\mathbb{Z}}^{-1}[-2j + p]. \tag{28}$$

This contradicts $w_{\mathbb{Z}}$ being an asymmetric band.
Let now $p$ be odd. It follows that $\tilde{p} = p + 1$. The length of the subword

$w_{j-\frac{p+1}{2}} w_{j-\frac{p+1}{2}+1} \dots w_j \dots w_{j+\frac{p+1}{2}-1} w_{j+\frac{p+1}{2}}$ is $p+2$. It follows by periodicity that

$$w_{j-\frac{p+1}{2}+1} = w_{j+\frac{p+1}{2}}. \tag{29}$$

We have by equality in (25) that

$$(w_{j-\frac{p+1}{2}+1})^{-1} = w_{j+\frac{p+1}{2}-1}. \tag{30}$$

Combining (29) and (30) gives that

$$w_{j+\frac{p+1}{2}} = \left(w_{j+\frac{p+1}{2}-1}\right)^{-1} \tag{31}$$

which contradicts the definition of a word. □

**Corollary 3.30.** *Let $w_{\mathbb{Z}}$ be an asymmetric band. Then there exists a unique directed version $v \in (\Phi^{\mathrm{d}}_{\mathrm{ud}})$ that is weakly consistent.*

*Proof.* By Lemma 3.29, one can always construct a unique weakly consistent directed version for an asymmetric band analogously to the asymmetric string case in Corollary 3.22. □

**Corollary 3.31.** *Let $w_{\mathbb{Z}}$ be an asymmetric band. Let $v_{\mathbb{Z}} \in (\Phi^{\mathrm{d}}_{\mathrm{ud}})^{-1}(w_{\mathbb{Z}})$ be a directed version. Then $v_{\mathbb{Z}}$ is consistent if and only if we have*

$$\mathrm{dir}(v_j) = \begin{cases} 1 & \text{if } (v_{j-\frac{\tilde{p}}{2}} \dots v_{j-1})^{-1} > v_{j+1} \dots v_{j+\frac{\tilde{p}}{2}}, \\ -1 & \text{if } (v_{j-\frac{\tilde{p}}{2}} \dots v_{j-1})^{-1} < v_{j+1} \dots v_{j+\frac{\tilde{p}}{2}}, \end{cases} \tag{32}$$

*for all $j \in \mathbb{Z}$ with $v_j$ of special type, where $\tilde{p}$ as in Lemma 3.29.*

*Proof.* Let $v_{\mathbb{Z}}$ be consistent. Then (32) holds by the same line of argument as in the proof of Lemma 3.29.
The converse implication follows from Lemma 3.29, since $\mathrm{ind}^c_j(v_{\mathbb{Z}}) \le \mathrm{ind}^*_j(v_{\mathbb{Z}})$ (see Lemma 3.17) for all $j \in \mathbb{Z}$. □

**Remark 3.32.** *It follows from Lemma 3.29 and Corollary 3.31 that every $j \in \mathbb{Z}$ giving a letter $w_j$ of special type in an asymmetric band $w_{\mathbb{Z}}$ has finite $c^*$−index. In particular, $\mathrm{ind}^*_j(w_{\mathbb{Z}}) < \frac{\tilde{p}}{2}$, and thus any $j \in \mathbb{Z}$ with $w_j$ of special type also has finite c−index.*

**Proposition 3.33.** *Let $w_{\mathbb{Z}}$ be an asymmetric band of period $p$. Then there exists a unique weakly consistent $\tilde{v}_{\mathbb{Z}} \in (\Phi^{\mathrm{d}}_{\mathrm{ud}})^{-1}(w_{\mathbb{Z}})$ of period $p$.
Furthermore, if there exists a consistent directed version $v_{\mathbb{Z}} \in (\Phi^{\mathrm{d}}_{\mathrm{ud}})^{-1}(w_{\mathbb{Z}})$, it is unique.*

*Proof.* Existence and uniqueness of a weakly consistent directed version is given by Corollary 3.30. Uniqueness of a consistent word follows from Corollary 3.31. □

### 3.2.4 Symmetric bands

**Definition 3.34.** *Let $w_{\mathbb{Z}}$ be an undirected $\mathbb{Z}-$word. It is said to be of* symmetric band type *if $w_{\mathbb{Z}} = w_{\mathbb{Z}}[p]$ for some $p > 0$ and $w_{\mathbb{Z}} = w_{\mathbb{Z}}^{-1}[k]$ for some $k \in \mathbb{Z}$.*
*We denote by $\mathring{\mathcal{W}}^s$ the set of all $w_{\mathbb{Z}} \in \mathcal{W}_{\mathbb{Z}}$ of symmetric band type and by $\mathring{\mathcal{W}}^s_{\sim} = \mathring{\mathcal{W}}^s / \sim$. We call $w \in \mathring{\mathcal{W}}^s_{\sim}$ a* symmetric band*.*

**Example 3.35.** *Let $\Lambda$ be as in Example 3.28.1. Then*

$$w_{\mathbb{Z}} = \dots \varepsilon^* a \varepsilon^* a^{-1} \varepsilon^* \mid a \varepsilon^* a^{-1} \dots$$

*is of symmetric band type with $\hat{w}_p = a\varepsilon^* a^{-1}\varepsilon^*$ since $w_{\mathbb{Z}}^{-1} = w_{\mathbb{Z}}$.*

**Lemma 3.36.** *Let $w_{\mathbb{Z}}$ be of symmetric band type. Then there exists $l \in \mathbb{Z}$ such that the periodic part of $w_{\mathbb{Z}}[l]$ is of the form $\varepsilon^* u \eta^* u^{-1}$ for a suitable undirected finite word $u$, and $\varepsilon, \eta \in \mathrm{Sp}$.*

*Proof.* First consider the case where $|\hat{w}_p| = 2$. By definition of clannish algebra, this is only possible for the algebra given by the quiver consisting of two special loops $\varepsilon, \eta$. Then it follows directly that $p = 2$ and $\hat{w}_p = \varepsilon^* \eta^*$ or $\hat{w}_p = \eta^* \varepsilon^*$. Hence the result follows with $u$ a trivial word.
Now let $|\hat{w}_p| > 2$. By definition of symmetric band type, there exists $k' > 0$ such that $w_{\mathbb{Z}} = w_{\mathbb{Z}}^{-1}[k']$.
If $k'$ is even, we can write $k' = 2k$ for some $k \in \mathbb{Z}$. Applying Corollary 2.35 gives

$$w_{\mathbb{Z}} = (w_{\mathbb{Z}}[-2k])^{-1} \text{ if and only if } w_{\mathbb{Z}}[-k] = (w_{\mathbb{Z}}[-k])^{-1}.$$

The second equation implies

$$\begin{aligned} w_{p-k} &= w_{p-k}^{-1}, \\ w_{p-(k-i)} &= w_{p-(k+i)}^{-1}, \text{ for all } i \in \mathbb{Z}. \end{aligned}$$

Hence, $w_{p-k}$ is a letter of special type.
We obtain two finite undirected words $v = w_{p-k+1} \dots w_p$ and $x = w_1 \dots w_{p-k-1}$ such that the periodic part is of the form $\hat{w}_p = x w_{p-k} v$. Without loss of generality assume $|x| < |v|$. Then there exists $1 \le j \le p$ such that $y = w_j \dots w_p$ gives

$$(yx)^{-1} = v. \tag{33}$$

Let $|v| = l$, $|x| = g$ and $|y| = h$ (i.e. $l = h + g$).
By (33), we have that

$$y_1 \dots y_h x_1 \dots x_g = v_l^{-1} \dots v_1^{-1},$$

so $y$ is a subword of $v^{-1}$, i.e., $y_1 \dots y_h = v_l^{-1} \dots v_{l-(h-1)}^{-1}$. But by periodicity, $y$ is also a subword of $v$, i.e., $y_1 \dots y_h = v_{l-(h-1)} \dots v_l$. Combining the last two equalities results in

$$v_l^{-1} \dots v_{l-(h-1)}^{-1} = y_1 \dots y_h = v_{l-(h-1)} \dots v_l.$$

Hence, $y$ is selfinverse, i.e., of the form $y = z\mu^* z^{-1}$, for some undirected (finite) word $z$ and some special letter $\mu$. Furthermore, $w_{\mathbb{Z}}$ is of the form

$$\dots z\mu^* z^{-1} \mid x w_{p-k} x^{-1} z \mu^* z^{-1} x w_{p-k} x^{-1} \dots.$$

Now the periodic part of $w_{\mathbb{Z}}$ is of the form $\hat{w}_p = x w_{p-k} x^{-1} z\mu^* z^{-1}$. For $m = |x|$ we obtain $w_{\mathbb{Z}}[m]$ with periodic part $w_{p-k} x^{-1} z \mu^* z^{-1} x$. Then for $u \coloneqq z^{-1}x$, $\varepsilon^* \coloneqq \mu^*$ and $\eta^* \coloneqq w_{p-k}$, we obtain $\hat{w}_p$ of the desired form.

Now consider $k'$ to be odd. Then we can write $k' = 2k + 1$ for some $k \in \mathbb{Z}$. We obtain, analogously to the previous part (compare structure of proof of Lemma 2.34) that

$$w_{\mathbb{Z}}[-(k+1)] = (w_{\mathbb{Z}}[-k])^{-1}.$$

This property results in

$$w_{p-k}^{-1} = w_{p-(k+1)},$$

a contradiction to the definition of a word. Hence, $k'$ cannot be odd.

If $k' = 0$, it follows that $w_p = w_p^{-1}$ and thus $w_p$ is of special type. Moreover, $w_{p-1}^{-1} \dots w_1^{-1} = w_1 \dots w_{p-1}$, i.e., $w_1 \dots w_{p-1} = u\varepsilon^* u^{-1}$ for some undirected finite word $u$ and some $\varepsilon \in \mathrm{Sp}$. With $\eta^* \coloneqq w_p$, it follows that $\hat{w}_p = u\varepsilon^* u^{-1}\eta^*$. Then $w_{\mathbb{Z}}[-1]$ has periodic part $\varepsilon^* u \eta^* u^{-1}$. □

**Corollary 3.37.** *We can assume for a symmetric band $w_{\mathbb{Z}}$ of period $p$ that $\hat{w}_p = \varepsilon^* u \eta^* u^{-1}$.*

*Proof.* By Lemma 3.36 there exists $k \in \mathbb{Z}$ such that $w_{\mathbb{Z}}[k]$ has periodic part of the form $\varepsilon^* u \eta^* u^{-1}$. Now $w_{\mathbb{Z}}[k] \sim w_{\mathbb{Z}}$, i.e., they lie in the same equivalence class in $\mathring{\mathcal{W}}^s_{\sim}$. Hence we can choose $w_{\mathbb{Z}}[k]$ as representative of $[w_{\mathbb{Z}}]$ and thus assume the periodic part to be of the form above. □

Assume from now on for $w_{\mathbb{Z}}$ a symmetric band of period $p$ that its periodic part is of the form $\hat{w}_p = \varepsilon^* u \eta^* u^{-1}$ with $|u| = m$, $\varepsilon, \eta \in \mathrm{Sp}$. Note that $u$ is minimal due to the minimality of the period $p$.
According to Subsection 2.4.2, we obtain for every $v_{\mathbb{Z}} \in (\Phi^{\mathrm{d}}_{\mathrm{ud}})^{-1}(w_{\mathbb{Z}})$ of period $p$, with $\hat{v}_p = \varepsilon^{\mu} t \eta^{\kappa} t^{-1}$, and for $V$ a $\mathrm{k}\langle e, f\rangle/(e^2 - e, f^2 - f)$–module, a $\Lambda$–module $M(v_{\mathbb{Z}}, V)$ (where $\mu, \kappa \in \{+1, -1\}$, $\varepsilon, \eta \in \mathrm{Sp}$, $t \in (\Phi^{\mathrm{d}}_{\mathrm{ud}})^{-1}(u)$). We depict $M(v_{\mathbb{Z}}, V)$ as described in Subsection 2.4.2 as

$$\begin{tikzcd} V_0 \arrow[loop left, "\varepsilon=e"] & V_1 \arrow[l, "t_1"'] & \dots \arrow[l, "t_2"'] & V_{m-1} \arrow[l, "t_{m-1}"'] & V_m \arrow[l, "t_m"'] \arrow[loop right, "\eta=f"] \end{tikzcd}$$

Assuming $\hat{w}_p = \varepsilon^* u\eta^* u^{-1}$, we can now determine the integer(s) $k$ for which we have $w_{\mathbb{Z}} = w_{\mathbb{Z}}^{-1}[k]$ (compare next subsection).

### 3.2.5 Symmetries in symmetric bands

We have seen in the previous subsection that the periodic part of a symmetric band is of a certain form. Now we want to show that those words admit exactly two types of reflection symmetries and one type of translation symmetry.
In order to do so, we consider the infinite dihedral group and its properties.

It is well known that the group of isometries on $\mathbb{Z}$, that is, the group of bijective maps $f : \mathbb{Z} \to \mathbb{Z}$ that are distance preserving with respect to the norm $d(x,y) = |x-y|$ on $\mathbb{Z}$, is given by the infinite dihedral group $\mathrm{D}_\infty$ ([Coh89, p. 20]).
Consider the following two kinds of isometries for $k \in \mathbb{Z}$:

$$\begin{aligned} r_k &: \mathbb{Z} \longrightarrow \mathbb{Z}, i \longmapsto 2k - i \\ \tau_k &: \mathbb{Z} \longrightarrow \mathbb{Z}, i \longmapsto i - k. \end{aligned}$$

**Definition 3.38.** *An isometry of the form $r_k$ is called* reflection (symmetry) *and one of the form $\tau_k$* translation (symmetry)*.*

Clearly, $r_k^2 = \mathrm{id}$ holds for every $k \in \mathbb{Z}$, i.e., $r_k$ is self-inverse, and $\tau_l^{-1} = \tau_{-l}$ for any $l \in \mathbb{Z}$. It follows for $i \in \mathbb{Z}$ that

$$r_k \tau_l r_k(i) = i + l = \tau_l^{-1}(i) \coloneqq \tau_{-l}(i).$$

In particular, this holds for $r \coloneqq r_0$ and $\tau \coloneqq \tau_1$ which generate $\mathrm{D}_\infty$ ([Coh89, p. 20]):

$$\mathrm{D}_\infty = \langle r, \tau \mid r^2 = \mathrm{id}, r\tau r = \tau^{-1} \rangle.$$

Note that reflections reverse the order. For $i < j$ in $\mathbb{Z}$ we have $r(j) < r(i)$.

**Definition 3.39.** *We define the action of $\mathrm{D}_\infty$ on $\mathcal{W}_{\mathbb{Z}}$ for $w_{\mathbb{Z}} = (w_i)_{i \in \mathbb{Z}}$ as follows:*

$$r(w_{\mathbb{Z}}) \coloneqq (w_{r(i)}^{-1})_{i \in \mathbb{Z}}, \tag{34a}$$

$$\tau(w_{\mathbb{Z}}) \coloneqq (w_{\tau(i)})_{i \in \mathbb{Z}}. \tag{34b}$$

**Lemma 3.40.** *The operations in (34a) and (34b) give a (left) group action of $\mathrm{D}_\infty$ on $\mathcal{W}_{\mathbb{Z}}$.*

*Proof.* Since we have defined the action on the generators of $\mathrm{D}_\infty$, it is enough to show well-definedness:

$$\begin{aligned} r^2(w_{\mathbb{Z}}) &= r(r(w_{\mathbb{Z}})) = r((w_{r(i)}^{-1})_{i \in \mathbb{Z}}) = (w_i)_{i \in \mathbb{Z}}, \\ r\tau r(w_{\mathbb{Z}}) &= r(\tau(r(w_{\mathbb{Z}}))) = r(\tau((w_{r(i)}^{-1})_{i \in \mathbb{Z}})) = r((w_{\tau r(i)}^{-1})_{i \in \mathbb{Z}}) = (w_{r\tau r(i)})_{i \in \mathbb{Z}} \\ &= (w_{\tau^{-1}(i)})_{i \in \mathbb{Z}} = \tau^{-1}(w_{\mathbb{Z}}). \end{aligned}$$

□

**Lemma 3.41.** *The following holds for any $k \in \mathbb{Z}$ and any $\mathbb{Z}$-word $w_{\mathbb{Z}}$:*

$$\tau_k(w_{\mathbb{Z}}) = w_{\mathbb{Z}}[-k].$$

*Proof.* The equation follows by definition of $\tau$ and the shift on words. □

Denote by $\mathrm{Sym}(\mathcal{W}_{\mathbb{Z}})$ the set of permutations on $\mathcal{W}_{\mathbb{Z}}$. Then by the action of $\mathrm{D}_\infty$ on $\mathcal{W}_{\mathbb{Z}}$, we obtain $r, \tau \in \mathrm{Sym}(\mathcal{W}_{\mathbb{Z}})$ and an injective map

$$\langle r, \tau \mid r^2 = \mathrm{id}, r\tau r = \tau^{-1} \rangle \hookrightarrow \mathrm{Sym}(\mathcal{W}_{\mathbb{Z}}).$$

Let from now on be $w_{\mathbb{Z}}$ a symmetric band of period $p$ with periodic part $\hat{w}_p = \varepsilon^* u \eta^* u^{-1}$.
We denote the *stabilizer group* of $w_{\mathbb{Z}}$ under $\mathrm{D}_\infty$ by $\mathrm{Stab}_{D_\infty}(w_{\mathbb{Z}})$.

**Lemma 3.42.** *For any $a \in \mathbb{Z}$, the composition $r\tau_a$ is a reflection of the form $r_{\frac{a}{2}}$ on $\mathcal{W}_{\mathbb{Z}}$.*

*Proof.* We have that

$$\begin{aligned} r\tau_a(w_{\mathbb{Z}}) &= (w^{-1}_{r\tau_a(i)})_{i\in\mathbb{Z}} = (w^{-1}_{r(i-a)})_{i\in\mathbb{Z}} = (w^{-1}_{a-i})_{i\in\mathbb{Z}}, \\ r_{\frac{a}{2}}(w_{\mathbb{Z}}) &= (w^{-1}_{r_{\frac{a}{2}}(i)})_{i\in\mathbb{Z}} = (w^{-1}_{2\frac{a}{2}-i})_{i\in\mathbb{Z}} = (w^{-1}_{a-i})_{i\in\mathbb{Z}}. \end{aligned}$$

Hence, they both act in the same way on $\mathcal{W}_{\mathbb{Z}}$. □

**Lemma 3.43.** *Let $\mathcal{S}_{n,a} \coloneqq \langle \tau_n, r\tau_a \rangle$ for $n, a \in \mathbb{Z}$. Then*

$$\mathcal{S}_{n,a} = \{\tau_{kn}, r\tau_{a+kn}\}_{k\in\mathbb{Z}}.$$

*Proof.* We first show that $\mathcal{S}_{n,a} \supseteq \{\tau_{kn}, r\tau_{a+kn}\}_{k\in\mathbb{Z}}$. Clearly, $\langle \tau_n \rangle = \{\tau_{kn}\}_{k\in\mathbb{Z}}$. Furthermore,

$$(r\tau_a)(\tau_{kn}) = r(\tau_a \tau_{kn}) = r\tau_{a+kn}$$

and the inclusion follows.
In order to show $\mathcal{S}_{n,a} \subseteq \{\tau_n, r\tau_{a+kn}\}_{k\in\mathbb{Z}}$, we assume for contradiction that this inclusion does not hold: Assume $r\tau_b \in \mathcal{S}_{n,a}$ with $b \neq a + kn$ for any $k \in \mathbb{Z}$. Then the following composition is also in $\mathcal{S}_{n,a}$:

$$(r\tau_a)(r\tau_b) = (r\tau_a r)\tau_b = \tau_{-a}\tau_b = \tau_{b-a}.$$

Here, we have used that $r\tau_l r = \tau_{-l}$ for any $l \in \mathbb{Z}$ which follows inductively from $r\tau r = \tau^{-1}$. Now $\tau_{b-a} \in \mathcal{S}_{n,a}$ if and only if $b - a = ln$ for some $l \in \mathbb{Z}$. This results in $b = a + ln$ which contradicts the assumption on $b$. □

**Proposition 3.44.** *Let $w_{\mathbb{Z}}$ be a symmetric band of period $p$, $\hat{w}_p = \varepsilon^* u \eta^* u^{-1}$. Then*

$$\mathrm{Stab}_{D_\infty}(w_{\mathbb{Z}}) = \mathcal{S}_{p,2}.$$

*Proof.* We first show the inclusion $\mathcal{S}_{p,2} \subseteq \mathrm{Stab}_{D_\infty}(w_\mathbb{Z})$. By Lemma 3.43 we have $\mathcal{S}_{p,2} = \{\tau_{kp}, r\tau_{2+kp}\}_{k\in\mathbb{Z}}$. Let us examine the elements of $S_{p,2}$.
By Lemma 3.41 and periodicity we know that $\tau_{kp}(w_\mathbb{Z}) = w_\mathbb{Z}[kp] = w_\mathbb{Z}$ for any $k \in \mathbb{Z}$. Thus, $\tau_{kp} \in \mathrm{Stab}_{D_\infty}(w_\mathbb{Z})$.
Let us consider now elements of the form $r\tau_{2+kp}$. If $k$ is even, we can write $k = 2l$ for some $l \in \mathbb{Z}$. Then $r\tau_{2+kp}$ is a reflection of the form $r_{1+lp}$ by Lemma 3.42. For $l \geq 0$ this describes the reflection in the first position of the positive copy $\hat{w}_p^{(l+1)}$. For $l < 0$, we rewrite $1 + lp$ as follows:

$$1 + lp = 1 + (l-1)p + p = -p + 1 + (l+1)p.$$

Thus, it describes for $l < 0$ the reflection in the first position of the negative copy $\hat{w}_p^{(l+1)}$.
Now consider $k$ to be odd. Then $k+1$ is even and $r\tau_{2+kp}$ is a reflection of the form $r_{1+\frac{kp}{2}}$. Using $p = 2m+2$, we can rewrite the subscript:

$$1 + \frac{kp}{2} = \frac{2 + (k+1)p - 2m - 2}{2} = \frac{k+1}{2}p - m. \tag{35}$$

For $\frac{k+1}{2} \leq 0$, $r_{1+\frac{kp}{2}}$ describes the reflection in position $m+2$ of the negative copy $\hat{w}_p^{(\frac{k+1}{2})}$. For $\frac{k+1}{2} > 0$, we rewrite (35) to

$$\frac{k+1}{2}p - m = \left(\frac{k+1}{2} - 1\right)p - m + 2m + 2 = \left(\frac{k+1}{2} - 1\right)p + m + 2.$$

Hence, $r_{1+\frac{kp}{2}}$ describes the reflection in position $m+2$ in the positive copy $\hat{w}_p^{(\frac{k+1}{2})}$.
It follows that $r\tau_{2+kp}(w_\mathbb{Z}) = w_\mathbb{Z}$ for any $k \in \mathbb{Z}$. Thus, any element of $\mathcal{S}_{p,2}$ stabilizes $w_\mathbb{Z}$, so $\mathcal{S}_{p,2} \subseteq Stab_{\mathrm{D}_\infty}(w_\mathbb{Z})$.
It remains to show equality. By definition, $Stab_{\mathrm{D}_\infty}(w_\mathbb{Z})$ is a subgroup of $\mathrm{D}_\infty$. The non-trivial subgroups of $\mathrm{D}_\infty$ are given by ([Spe56, §1.2.4, Example 2])

$$G_n = \langle r\tau_n \rangle, \text{ for some } n \in \mathbb{Z}, \tag{36}$$

$$G_{0,n} = \langle \tau_n \rangle, \text{ for some } n \geq 0, \tag{37}$$

$$G_{n,a} = \langle \tau_n, r\tau_a \rangle \text{ for some } n \geq 1, 0 \leq a < n. \tag{38}$$

Lemma 3.43 shows that $\mathcal{S}_{n,a} = G_{n,a}$ which is the largest subgroup of $\mathrm{D}_\infty$. Hence, it is enough to show that $\mathrm{Stab}_{D_\infty}(w_\mathbb{Z}) \neq \mathrm{D}_\infty$. To this end, assume towards a contradiction that we have equality. Then $\tau_l(w_\mathbb{Z}) = w_\mathbb{Z}$ for any $l \in \mathbb{Z}$. In particular, this holds for $l = p-1$. It follows by Lemma 3.41 that

$$w_\mathbb{Z}[p-1] = \tau_{p-1}(w_\mathbb{Z}) = w_\mathbb{Z},$$

contradicting minimality of the period $p$. Hence, $\mathrm{Stab}_{\mathrm{D}_\infty}(w_\mathbb{Z}) \neq \mathrm{D}_\infty$ and $\mathcal{S}_{p,2} = \mathrm{Stab}_{\mathrm{D}_\infty}(w_\mathbb{Z})$ follows. □

**Corollary 3.45.** *Let $w_{\mathbb{Z}}$ be a symmetric band of period $p > 0$ with periodic part $\hat{w}_p = \varepsilon^* u\eta^* u^{-1}$.*
*Then $w_{\mathbb{Z}} = w_{\mathbb{Z}}^{-1}[k]$ if and only if*

$$k \equiv -2 \pmod p, \quad k \equiv 2m \pmod p, \tag{39}$$
$$k \equiv -2(m+2) \pmod p \text{ or } k \equiv 2(2m+1) \pmod p. \tag{40}$$

*Proof.* Consider the equalities

$$\begin{aligned} w_{\mathbb{Z}}[1] &= (w_{\mathbb{Z}}[1])^{-1}, \\ w_{\mathbb{Z}}[m+2] &= (w_{\mathbb{Z}}[m+2])^{-1}, \\ w_{\mathbb{Z}}[-(2m+1)] &= (w_{\mathbb{Z}}[-(2m+1)])^{-1}, \\ w_{\mathbb{Z}}[-m] &= (w_{\mathbb{Z}}[-m])^{-1}, \end{aligned}$$

which result from the symmetries in the periodic parts. By Proposition 3.44 we know that these are all reflection symmetries of $w_{\mathbb{Z}}$. Applying Lemma 2.34 and Corollary 2.35 to the above equations yields the result. □

The previous Lemma and Proposition illustrate that the symmetry points are given by the letters with index $j \equiv 1, -m, -(2m+1), m+2 \pmod p$ which are exactly the symmetry points in the positive and negative copies of the periodic part. Hence, there does not exist a unique weakly consistent directed version for symmetric bands. The next lemma shows that we can use a simplified criterion to check on weak consistency for directed versions.

**Lemma 3.46.** *Let $w_{\mathbb{Z}}$ be a symmetric band with period $p$ and periodic part $\hat{w}_p = \varepsilon^* u\eta^* u^{-1}$. Let $v_{\mathbb{Z}} \in (\Phi^{\mathrm{d}}_{\mathrm{ud}})^{-1}(w_{\mathbb{Z}})$ be a directed version. Then $v_{\mathbb{Z}}$ is weakly consistent if and only if for all $j \in \mathbb{Z}$ with $j \not\equiv 1, -m, -(2m+1), m+2 \pmod p$ and $v_j$ of special type we have that*

$$\mathrm{dir}(v_j) = \begin{cases} 1 & \text{if } (w_{j-(m+1)} w_{j-m} \dots w_{j-1})^{-1} > w_{j+1} \dots w_{j+m} w_{j+m+1}, \\ -1 & \text{if } (w_{j-(m+1)} w_{j-m} \dots w_{j-1})^{-1} < w_{j+1} \dots w_{j+m} w_{j+m+1}. \end{cases}$$

*Proof.* By Corollary 3.45 we have for any $j \equiv 1, -m, -(2m+1), m+2 \pmod p$ that $(w_{\mathbb{Z}}[< j])^{-1} = w_{\mathbb{Z}}[> j]$. Hence, any letter $v_j$ indexed by such a $j$ can have direction 1 or −1 in a weakly consistent directed version.
Thus, it is enough to show that for $j \not\equiv 1, -m, -(2m+1), m+2 \pmod p$ we have that

$$(w_{j-(m+1)} w_{j-m} \dots w_{j-1})^{-1} \neq w_{j+1} \dots w_{j+m} w_{j+m+1}.$$

Assume towards a contradiction that the two terms are equal for the index

$j$. We write

$$\begin{aligned}
z_1 &= w_{j-(m+1)} w_{j-m} \dots w_{j-1}, \\
z_2 &= w_{j+1} \dots w_{j+m}, \\
z &= z_1 w_j z_2 = w_{j-(m+1)} w_{j-m} \dots w_{j-1} w_j w_{j+1} \dots w_{j+m}, \\
x_1 &= w_{j-m} \dots w_{j-1}, \\
x_2 &= w_{j+1} \dots w_{j+m} w_{j+m+1}, \\
x &= x_1 w_j x_2 = w_{j-m} \dots w_{j-1} w_j w_{j+1} \dots w_{j+m} w_{j+m+1}.
\end{aligned}$$

By assumption $z_1^{-1} = x_2$. Hence,

$$w_{j-i}^{-1} = w_{j+i} \quad \forall i \in \{1, \dots, m+1\}.$$

In particular, $w_{j-(m+1)}^{-1} = w_{j+(m+1)}$. Furthermore, $|z| = |x| = p$. It follows by periodicity that $w_{j-(m+1)} = w_{j+(m+1)}$. Then $w_{j-(m+1)} = w_{j-(m+1)}^{-1}$ is given by a special letter, say $\kappa^*$. Let $w_j = \mu^*$. Then we can write

$$w_{j-(m+1)} w_{j-m} \dots w_{j-1} w_j w_{j+1} \dots w_{j+m} = \kappa^* y \mu^* y^{-1}$$

where $y = w_{j-m} \dots w_{j-1}$. By periodicity, it follows that

$$w_{\mathbb{Z}}[j] = (w_{\mathbb{Z}}[j])^{-1}.$$

This can be rewritten to

$$w_{\mathbb{Z}} = w_{\mathbb{Z}}^{-1}[-2j]. \tag{41}$$

By choice of $j \not\equiv 1, -m, -(2m+1), m+2 \pmod p$, equation (41) gives a contradiction to Corollary 3.45. □

**Corollary 3.47.** *Let $w_{\mathbb{Z}}$ be a symmetric band of period $p$ and with periodic part $\hat{w}_p = \varepsilon^* u \eta^* u^{-1}$. Then we have for all $j \in \mathbb{Z}$ with $j \not\equiv 1, -m, -(2m+1), m+2 \pmod p$ that $\operatorname{ind}_j^*(w_{\mathbb{Z}}) < \infty$, in particular, $\operatorname{ind}_j^*(w_{\mathbb{Z}}) < \frac{p}{2}$.*

*Proof.* This result follows directly from Lemma 3.46. □

**Remark 3.48.** *Let $w_{\mathbb{Z}}$ be a symmetric band as in Corollary 3.47 and let $v_{\mathbb{Z}} \in (\Phi_{\mathrm{ud}}^{\mathrm{d}})^{-1}(w_{\mathbb{Z}})$. By Definition 3.7 and Definition 3.14, we have that $\operatorname{ind}_j^c(v_{\mathbb{Z}}) \leq \operatorname{ind}_j^*(v_{\mathbb{Z}})$ for all $j \in \mathbb{Z}$. Thus it follows by the above Corollary that $\operatorname{ind}_j^c(v_{\mathbb{Z}}) < \frac{p}{2} < \infty$ for all $j \in \mathbb{Z}$ with $j \not\equiv 1, -m, -(2m+1), m+2 \pmod p$.*

**Proposition 3.49.** *Let $w_{\mathbb{Z}}$ be a symmetric band of period $p$ with periodic part $\hat{w}_p = \varepsilon^* u \eta^* u^{-1}$. Then there exists a weakly consistent directed version $v_{\mathbb{Z}} \in (\Phi_{\mathrm{ud}}^{\mathrm{d}})^{-1}(w_{\mathbb{Z}})$.*

*Proof.* By Lemma 3.46 we know that for $j \not\equiv 1, -m, -(2m+1), m+2 \pmod p$ we have $\mathrm{ind}^*_j(w_\mathbb{Z}) < \frac{p}{2}$. Thus, we can simply set $\mathrm{dir}(v_j) = \mathrm{dir}(w_{j^*_-}) = \mathrm{dir}(w_{j^*_+})$. For any $j \equiv 1, -m, -(2m+1), m+2 \pmod p$, we know by Corollary 3.45 that $(w_\mathbb{Z}[< j])^{-1} = w_\mathbb{Z}[> j]$. Hence, setting $\mathrm{dir}(v_j)$ to be 1 or $-1$ gives a weakly consistent word. □

**Lemma 3.50.** *Let $w_\mathbb{Z}$ be a symmetric band with period $p > 0$ and periodic part $\hat{w}_p = \varepsilon^* u \eta^* u^{-1}$. Let $v_\mathbb{Z} \in (\Phi^{\mathrm{d}}_{\mathrm{ud}})^{-1}(w_\mathbb{Z})$ be a weakly consistent directed word as in Lemma 3.46. Let $j \in \mathbb{Z}$ with $w_j$ a special letter and $\mathrm{ind}^*_j(v_\mathbb{Z}) = d < \infty$. Then there do not exist $k, l \in \mathbb{Z}$ with $w_k, w_l$ special letters and $\mathrm{ind}^*_k(v_\mathbb{Z}) = \mathrm{ind}^*_l(v_\mathbb{Z}) = \infty$ and $|j-k| = |j-l| \le d$.*

*Proof.* Assume towards a contradiction that such indices $k, l \in \mathbb{Z}$ exist. Then, by Corollary 3.47, $w_k, w_l$ describe symmetry axes.
By assumption, $w_j$ lies exactly in the middle between $w_k$ and $w_l$. Let $x$ describe the subword between $w_k$ and $w_j$, hence $x^{-1}$ describes the subword between $w_j$ and $w_l$. By definition of $w_l, w_k$ as symmetry axes, we have

$$w_\mathbb{Z}[> l] = (w_\mathbb{Z}[< l])^{-1}$$

and

$$w_\mathbb{Z}[> k] = (w_\mathbb{Z}[< k])^{-1}.$$

By the symmetries in $j$,$k$ and $l$ we obtain

$$w_\mathbb{Z}[> l] = (w_\mathbb{Z}[< k])^{-1}.$$

Now we can write $w_\mathbb{Z}[< j] = x^{-1} w_l w_\mathbb{Z}[> l]$ and $(w_\mathbb{Z}[< j])^{-1} = x^{-1} w_k w_\mathbb{Z}[< k]^{-1}$. Thus $\mathrm{ind}^*_j(w_\mathbb{Z}) = \infty$, a contradiction to the assumption. □

We can also determine the stabilizer $\mathrm{Stab}_{\mathrm{D}_\infty}(w_\mathbb{Z})$ for $w_\mathbb{Z}$ an asymmetric band of period $p$:

**Proposition 3.51.** *Let $w_\mathbb{Z}$ be an asymmetric band of period $p$. Then*

$$\mathrm{Stab}_{\mathrm{D}_\infty}(w_\mathbb{Z}) = \{\tau_{kp}\}_{k \in \mathbb{Z}}.$$

*Proof.* We know by Lemma 3.41 that

$$\tau_{kp}(w_\mathbb{Z}) = w_\mathbb{Z}[-kp].$$

Thus, $\tau_{kp} \in \mathrm{Stab}_{\mathrm{D}_\infty}(w_\mathbb{Z})$ for any $k \in \mathbb{Z}$. Assume that $\{\tau_{kp}\}_{k\in\mathbb{Z}} \neq \mathrm{Stab}_{\mathrm{D}_\infty}(w_\mathbb{Z})$. We know that $\mathrm{Stab}_{\mathrm{D}_\infty}(w_\mathbb{Z})$ is a subgroup of $\mathrm{D}_\infty$. They are given by (36) - (38). Thus, $\mathrm{Stab}_{\mathrm{D}_\infty}$ contains a reflection of the form $r\tau_k = r_{\frac{k}{2}}$ with $k \in \{0, \ldots, p-1\}$. Then

$$r\tau_k(w_\mathbb{Z}) = (w^{-1}_{k-i})_{i\in\mathbb{Z}} = (w_i)_{i\in\mathbb{Z}}.$$

Denote by $\hat{u}_p$ the periodic part of $r\tau_k(w_{\mathbb{Z}})$. It is given by

$$\hat{u}_p = w_{k-1}^{-1} \dots w_1^{-1} w_p^{-1} \dots w_k^{-1}.$$

Since $r\tau_k \in \mathrm{Stab}_{\mathrm{D}_\infty}(w_{\mathbb{Z}})$ it follows that $\hat{w}_p = \hat{u}_p$:

$$w_1 \dots w_p = w_{k-1}^{-1} \dots w_1^{-1} w_p^{-1} \dots w_k^{-1}.$$

This equality yields that

$$w_1 \dots w_{k-1} = w_{k-1}^{-1} \dots w_1^{-1} \tag{42}$$

$$w_k \dots w_p = w_p^{-1} \dots w_k^{-1}. \tag{43}$$

We obtain that

$$w_{\mathbb{Z}} = w_{\mathbb{Z}}^{-1}[k-1]$$

which contradicts $w_{\mathbb{Z}}$ being asymmetric. $\square$

## 3.3 Comparison of weakly consistent and consistent words

In this section we examine correspondences of the two types of directed words introduced in Section 3.1. To this end, we can consider the asymmetric and symmetric strings and the asymmetric band as one case. The approach on symmetric bands is more complicated and thus is considered separately.

We see that for the asymmetric cases there exists a unique directed version that is both consistent and weakly consistent. In the case of symmetric string, there exist two possible directed versions which are also both consistent and weakly consistent.
The symmetric band is the most complicated case. Here, we know what the consistent directed versions look like with respect to the symmetry axes. Moreover, any consistent directed version is also weakly consistent. But the converse does not hold: only any weakly consistent directed version with symmetry axes oriented in the way of types 1) - 4) in Proposition 3.70 is also consistent.

Recall with the following example some notation from the Subsections 3.1.1 and 3.1.2:

**Example 3.52.** *Let $\Lambda$ be as in Example 2.14. Consider*

$$w = w_1 w_2 w_3 w_4 w_5 w_6 w_7 w_8 w_9 \tag{44}$$
$$= \varepsilon^* a^{-1} b \eta^* b^{-1} a \varepsilon^* a^{-1} d^{-1}. \tag{45}$$

*The word $v = \varepsilon^{-1} a^{-1} b \eta^{-1} b^{-1} a \varepsilon^{-1} a^{-1} d^{-1}$ is a weakly consistent and consistent directed version of $w$. The special letters in $v$ are indexed by $1, 4$ and $7$. For $j = 1$ and $j = 7$ we have $\mathrm{ind}_j^*(w) = \mathrm{ind}_j^c(w)$ and thus most of the following data coincides:*

$$\begin{aligned}
j = 1: \qquad & \mathrm{ind}_1^*(w) = \mathrm{ind}_1^c(v) = 0,\\
& 1^* = [1 - \mathrm{ind}_1^*(w), 1 + \mathrm{ind}_1^*(w)] = [1,1],\\
& 1^c = [1 - \mathrm{ind}_1^c(w), 1 + \mathrm{ind}_1^c(v)] = [1,1],\\
& w_{1_+^*} = 1 + \mathrm{ind}_1^*(w) - 1 = w_2,\\
& v_{1_+^c} = 1 + \mathrm{ind}_1^c(v) - 1 = w_2,\\
& w_{1_-^*} = 1 - \mathrm{ind}_1^*(w) + 1 = 1_{s(\varepsilon),\kappa},\\
& v_{1_-^c} = 1 - \mathrm{ind}_1^c(v) + 1 = 1_{s(\varepsilon),\kappa},\\
j = 7: \qquad & \mathrm{ind}_7^*(w) = \mathrm{ind}_7^c(v) = 1,\\
& 7^* = [6,8] = 7^c,\\
& w_{1_+^*} = v_{1_+^c} = w_9,\\
& w_{1_-^*} = v_{1_-^c} = w_5.
\end{aligned}$$

*For $j = 4$ we obtain different values:*

$$\begin{aligned} &\mathrm{ind}_4^*(w) = 3, && \mathrm{ind}_4^c(v) = 2, \\ &4^* = [1,7], && 4^c = [2,6], \\ &w_{4^*_-} = w_1, && v_{4^c_-} = w_1, \\ &w_{4^*_+} = w_8, && v_{4^c_+} = w_7. \end{aligned}$$

### 3.3.1 Directed words from strings and asymmetric bands

Let us first show that a weakly consistent orientation on asymmetric and symmetric strings and asymmetric bands is also consistent.

**Theorem 3.53.** *Let $w$ be a string, either asymmetric or symmetric, and let $w_{\mathbb{Z}}$ be an asymmetric band. Let $v \in (\Phi^{\mathrm{d}}_{\mathrm{ud}})^{-1}(w)$ ($v_{\mathbb{Z}} \in (\Phi^{\mathrm{d}}_{\mathrm{ud}})^{-1}(w_{\mathbb{Z}})$) be weakly consistent. If $w$ is a symmetric string of the form $u\varepsilon^* u^{-1}$, assume additionally that $v = t\varepsilon^\kappa t^{-1}$ for $t \in (\Phi^{\mathrm{d}}_{\mathrm{ud}})^{-1}(u)$ and $\kappa \in \{+1,-1\}$. Then $v$ ($v_{\mathbb{Z}}$, respectively) is consistent.*

**Remark 3.54.** *Recall that any weakly consistent directed version is uniquely given in both asymmetric cases. Only for the symmetric ones do we obtain more than one possible weakly consistent directed version (compare Section 3.2).*

*Proof of Theorem 3.53.* We show the statement for $w$ an asymmetric string. The other two cases are analogous.
Let $v \in (\Phi^{\mathrm{d}}_{\mathrm{ud}})^{-1}(w)$ be weakly consistent and assume towards a contradiction that $v$ is not consistent. Then there exists $1 \le j \le n$ with $w_j$ a special letter and $\mathrm{dir}(v_j) \neq \mathrm{dir}(v_{j^c_-})$ and $\mathrm{dir}(v_{j^c_-}) = \mathrm{dir}(v_{j^c_+})$. Thus, $\mathrm{ind}_j^*(w) > \mathrm{ind}_j^c(v)$ and $v_{j^c_-} = v_{j^c_+}$ are special letters. Let $x^{-1}$ denote the undirected subword between $w_{j^c_-}$ and $w_j$ in $w$; hence, $x$ is the undirected subword between $w_j$ and $w_{j^c_+}$. Set $y = w[< j^c_-]$, $z = w[> j^c_+]$. Then $w$ is of the form

$$y w_{j^c_-} x^{-1} w_j x w_{j^c_+} z.$$

Assume without loss of generality that $\mathrm{dir}(v_j) = 1$. Hence, $\mathrm{dir}(v_{j^c_-}) = \mathrm{dir}(v_{j^c_+}) = -1$. The weakly consistent orientation with respect to the positions $j^c_-, j^c_+$ and $j$ gives the following inequalities:

1) $y^{-1} < x^{-1} w_j x w_{j^c_+} z$,
2) $x w_{j^c_+} z < x w_{j^c_-} y^{-1}$, i.e. $z < y^{-1}$,
3) $x^{-1} w_j x w_{j^c_-} y^{-1} < z$.

Thus, we get

$$x^{-1} w_j x w_{j^c_-} y^{-1} < z < y^{-1} < x^{-1} w_j x w_{j^c_+} z.$$

Comparing the first and last word in this chain of inequalities gives $y^{-1} < z$, a contradiction to inequality 2). □

The next step is to show that a consistent word is also weakly consistent. This is more complicated to show than the previous statement. Let us therefor first introduce some auxiliary lemmas.

**Lemma 3.55.** *Let $w$ be a string, either asymmetric or symmetric, and let $w_{\mathbb{Z}}$ be an asymmetric band. Let $j \in I$ (for $I = \{0, \ldots, n\}$ or $I = \mathbb{Z}$, respectively) with $w_j$ of special type and $\operatorname{ind}_j^*(w) = d < \infty$ ($\operatorname{ind}_j^*(w_{\mathbb{Z}}) = d < \infty$, respectively). Then there do not exist $k, l \in I$ with $k < j < l$, $w_k$, $w_l$ both of special type, and $|j - k| = |j - l| \leq d$ such that $\operatorname{ind}_k^*(w) \geq d$ and $\operatorname{ind}_l^*(w) \geq d$ ($\operatorname{ind}_k^*(w_{\mathbb{Z}}) \geq d$ and $\operatorname{ind}_l^*(w_{\mathbb{Z}}) \geq d$, respectively).*

*Proof.* We first show the statement for $w$ an asymmetric string. For symmetric strings, the proof is analogous.
Assume towards a contradiction that such indices $k, l \in \mathbb{Z}$ exist with the above properties. Denote by $x$ the subword of $w$ between $w_j$ and $w_l$ (hence $x^{-1}$ gives the subword of $w$ between $w_k$ and $w_j$). Since $|j - k| = |j - l| \leq d$, we have $|x| < d$. Denote by $y$ the subword of $w$ between $w_{j_-^*}$ and $w_k$ (hence $y^{-1}$ describes the word between $w_l$ and $w_{j_+^*}$ and we have $|y| + |x| + 1 = d$). Also, $k_-^* < j_-^*$ and $j_+^* < l_+^*$, so let $z$ (respectively $u$) be the subword between $w_{k_-^*}$ and $w_{j_-^*}$ ($w_{j_+^*}$ and $w_{l_+^*}$, respectively). Thus, $w$ is of the form

$$\ldots w_{k_-^*} z w_{j_-^*} y w_k x^{-1} w_j x w_l y^{-1} w_{j_+^*} u w_{l_+^*} \ldots . \tag{46}$$

Assume without loss of generality that $|y| \leq |x|$. Then by symmetry in $w_k$ and $w_l$, $y$ is a finite subword of a word of the form

$$(\eta^* x^{-1} \varepsilon^* x)^h,$$

for some $h \in \mathbb{N}$, where $s^h = s \ldots s$ is a word given by $h$ copies of $s$ and $\eta^* = w_l$, $\varepsilon^* = w_j$. One has $|y| + |z| + 1 = \operatorname{ind}_k^*(w)$, $|y| + |u| + 1 = \operatorname{ind}_l^*(w)$.
For $x = x_1 \ldots x_f$ we know that $x_f$ is ordinary. By symmetry in position $l$ we obtain

$$x_f = (w_{j_+^*})^{-1}. \tag{47}$$

Similarly, symmetry in position $k$ results in

$$x_f^{-1} = (w_{j_-^*})^{-1}. \tag{48}$$

Combining (47) and (48) gives

$$w_{j_-^*} = (w_{j_+^*})^{-1}. \tag{49}$$

Thus, $\operatorname{ind}_j^*(w) \geq d + 1$, contradicting the assumption on its $c^*$–index.

Now let $w_{\mathbb{Z}}$ be an asymmetric band. Assume again towards a contradiction that the indices $k, l \in \mathbb{Z}$ as above exist. By Lemma 3.29, $\operatorname{ind}_j^*(w_{\mathbb{Z}})$ is finite and

it is enough to consider for $\hat{\jmath} \equiv j \pmod p$ the subword $t \coloneqq \hat{w}_p[> \hat{\jmath}]\hat{w}_p\hat{w}_p[< \hat{\jmath}]$ of $w_{\mathbb{Z}}$ to determine the weakly consistent orientation in position $j$. The letters $w_k$ and $w_l$ are by assumption in $t$. Also, it follows from $\mathrm{ind}_{k^*_-}$ and $\mathrm{ind}_{l^*_+}$ that both $w_{k^*_-}$ and $w_{l^*_+}$ are not contained in $t$. We extend $t$ to the left up to and including $w_{k^*_-}$, and to the right up to and including $w_{l^*_+}$. We denote the resulting word by $\hat{t}$.
Now $\hat{t}$ is a finite subword of $w_{\mathbb{Z}}$ which is of the same form as the subword in (46). Using the same arguments as in the asymmetric string case on $\hat{t}$, the proof for an asymmetric band follows analogously. □

**Lemma 3.56.** *Let $w$ be a string of length $n$, either asymmetric or symmetric, for some $n \in \mathbb{N}$, and let $w_{\mathbb{Z}}$ be an asymmetric band. Let $v \in (\Phi^{\mathrm{d}}_{\mathrm{ud}})^{-1}(w)$ ($v_{\mathbb{Z}} \in (\Phi^{\mathrm{d}}_{\mathrm{ud}})^{-1}(w_{\mathbb{Z}})$) be weakly consistent. If $w$ is a symmetric string of the form $u\varepsilon^* u^{-1}$, assume additionally that $v = t\varepsilon^{\kappa}t^{-1}$ for $t \in (\Phi^{\mathrm{d}}_{\mathrm{ud}})^{-1}(u)$ and $\kappa \in \{+1, -1\}$. Moreover, let $j \in I$ ($I = \{0, \dots n\}$ or $I = \mathbb{Z}$) with $w_j$ special and $\mathrm{ind}^*_j(v) = d$ ($\mathrm{ind}^*_j(v_{\mathbb{Z}}) = d$, respectively). Let $k, l \in I$ and $k < j < l$ with $|j - k| = |j - l| \le d$, $w_k$ and $w_l$ special letters, and $\mathrm{ind}^*_k(v) < d$, $\mathrm{ind}^*_l(v) < d$ ($\mathrm{ind}^*_k(v_{\mathbb{Z}}) < d$, $\mathrm{ind}^*_l(v_{\mathbb{Z}}) < d$, respectively).*
*Then either $\mathrm{dir}(v_k) = -\mathrm{dir}(v_l)$ or $\mathrm{dir}(v_k) = \mathrm{dir}(v_l) = \mathrm{dir}(v_j)$.*

*Proof.* We show the statement for $w$ an asymmetric string. For symmetric strings and asymmetric bands, the proof is analogous by the same arguments as in Lemma 3.55 (for symmetric bands first consider $j \neq m + 1$ which is analogous to the proof for asymmetric strings. For $j = m + 1$ it follows by symmetry of $w$ that $\mathrm{ind}^*_k(w) = \mathrm{ind}^*_l(w)$ and thus that $\mathrm{dir}(v_k) = -\mathrm{dir}(v_l)$).
Assume without loss of generality that $\mathrm{dir}(v_j) = -1$ and $\mathrm{ind}^*_l(w) \ge \mathrm{ind}^*_k(w)$. For $\mathrm{dir}(v_l) = 1$ we consider the following cases regarding the positions of $j^*_+$ and $l^*_+$:

- $l^*_+ < j^*_+$. Then $L^* \subset J^*$ and we need to distinguish
  1) $j < l^*_-$ and
  2) $l^*_- < j$.
- $j^*_+ < l^*_+$. Similarly to the above, we distinguish
  3) $j < l^*_-$ and
  4) $l^*_- < j$.
- $j^*_+ = l^*_+$. This case is given by 5) below and does not need further distinction.

We prove now for each of the five above cases that the statement holds.

1) $l^*_+ < j^*_+$ and $j < l^*_-$: Since $l^*_+ \in J^*$, we obtain by symmetry in position $j$ that $k^*_- \in J^*$, too. We obtain from the same symmetry that $\mathrm{ind}^*_k(w) = \mathrm{ind}^*_l(w)$. Then, again by symmetry in position $j$, $\mathrm{dir}(v_{l^*_+}) = -\mathrm{dir}(v_{k^*_-})$ and $\mathrm{dir}(v_{l^*_-}) = -\mathrm{dir}(v_{k^*_+})$. Hence, $\mathrm{dir}(v_l) = -\mathrm{dir}(v_k)$.

2) $l_+^+ < j_+^+$ and $l_-^* < j$: For two suitable undirected words $x_1$, $x_2$ with $\mathrm{ind}_l^*(w) = |x_1| + |x_2| + 1$, $w$ can be written within the interval $L^*$ as:

$$\dots w_{l_-^*} x_2^{-1} w_j x_1^{-1} w_l x_1 \eta^* x_2 w_{l_+^*} \dots \tag{50}$$

with $w_j = \eta^*$.
To determine position $k$, we use symmetry in $j$ and distinguish the following cases:

a) $|x_1| = |x_2| + 1$:
We have $x_1 = a x_2^{-1}$ for $a = w_{l_-^*} \in \mathrm{Q}_1^{\mathrm{ord}}$, and we can rewrite $w$ from (50) to:

$$\dots c^{-1} x_2^{-1} \eta^* x_2 a^{-1} w_k w_{l_-^*} x_2^{-1} w_j x_2 a^{-1} w_l a x_2^{-1} \eta^* x_2 w_{l_+^*} \dots$$

where $w_{l_+^*} = c$. Thus, $\mathrm{ind}_k^*(w) = 2|x_2| + 2 = \mathrm{ind}_l^*(w)$. Hence, by symmetry in $j$, $w_{k_-^*} = w_{l_+^*}^{-1}$ and thus $\mathrm{dir}(w_{k_-^*}) = -\mathrm{dir}(w_{l_+^*})$. It follows directly that $\mathrm{dir}(v_k) = -\mathrm{dir}(v_l)$.

b) $|x_1| > |x_2| + 1$:
We have $x_1 = x_3 a x_2^{-1}$ for some suitable undirected word $x_3$ and $a = w_{l_-^*}$. Using the symmetries in positions $l$ and $k$, we can deduce from (50) the following form of $w$:

$$\dots c^{-1} x_2^{-1} \eta^* x_2^{-1} a^{-1} x_3^{-1} w_k x_3 w_{l_-^*} x_2^{-1} w_j x_2 a^{-1} x_3^{-1} w_l x_3 a x_2^{-1} \eta^* x_2 w_{l_+^*} \dots$$

where $w_{l_+^*} = cb$, $\eta, \varepsilon \in \mathrm{Sp}$. Then $\mathrm{ind}_k^*(w) = 2|x_2| + |x_3| + 2 = \mathrm{ind}_l^*(w)$ and it follows by the same line of argument as in 2)a) that $\mathrm{dir}(v_k) = -\mathrm{dir}(v_l)$.

c) $|x_1| < |x_2| + 1$:
If $|x_1| = |x_2|$, then $w_l$ is of ordinary type which is a contradiction. Hence $|x_1| < |x_2|$ and we can write $x_2 = x_1^{-1} \varepsilon^* x_3$ for a suitable undirected word $x_3$ and $\varepsilon^* = w_l$ a special letter. For $|x_3| = 0$ one has $x_2 = x_1^{-1} \varepsilon^*$ and $w$ is given by

$$\dots w_{l_-^*} \varepsilon^* x_1 w_j x_1^{-1} w_l x_1 \eta^* x_1^{-1} \varepsilon^* w_{l_+^*} \dots .$$

It follows by symmetry in $j$ that $x_1 = a^{-1} \tilde{x}_1$ for $a = w_{l_-^*}$ and some suitable word $\tilde{x}_1$. Refining $w$ by this gives

$$\dots a^{-1} \varepsilon^* a^{-1} \tilde{x}_1 \eta^* \tilde{x}_1^{-1} a w_k a^{-1} \tilde{x}_1 w_j \tilde{x}_1^{-1} a w_l a^{-1} \tilde{x}_1 \eta^* \tilde{x}_1^{-1} a \varepsilon^* w_{l_+^*} \dots$$

with $w_j = \eta^*$, $w_k = \varepsilon^*$.
It follows from the above that $\mathrm{ind}_k^*(w) = 2|\tilde{x}_1| + 4 = \mathrm{ind}_l^*(w)$ and as above that $\mathrm{dir}(v_k) = -\mathrm{dir}(v_l)$.
Now let $|x_3| \geq 1$. Then $x_2 = x_1^{-1} \varepsilon^* x_3$. Assume for contradiction that $\mathrm{dir}(v_k) = \mathrm{dir}(v_l) = -\mathrm{dir}(v_j)$. Recall that we assume without

loss of generality that $\operatorname{dir}(v_j) = -1$. Hence, $\operatorname{dir}(v_k) = \operatorname{dir}(v_l) = 1$. We obtain from (50) by symmetry in position $l$ the following form for $w$:

$$\dots w_{l^*_-} x_3^{-1} w_k x_1 w_j x_1^{-1} w_l x_1 \eta^* x_1^{-1} \varepsilon^* x_3 w_{l^*_+} \dots$$

with $w_{l^*_-} = a$, $w_{l^*_+} = c$, $w_j = \eta^*$, $w_k = w_l = \varepsilon^*$. The weakly consistent orientation gives the following inequalities:

(i) from position $j$ (after eliminating $x_1^{-1}\varepsilon^*$ on both sides from the left):

$$x_3 a^{-1} \dots < x_1 \eta^* x_1^{-1} \varepsilon^* x_3 c \dots,$$

(ii) from position $k$: $x_1 \eta^* x_1^{-1} \varepsilon^* x_1 \eta^* x_1^{-1} \varepsilon^* x_3 c \dots < x_3 a^{-1} \dots$.

Extending (i) by $x_1 \eta^* x_1^{-1} \varepsilon^*$ from the left on both sides and applying (ii) and again (i) gives:

$$\begin{aligned} x_1 \eta^* x_1^{-1} \varepsilon^* x_3 a^{-1} \dots &< x_1 \eta^* x_1^{-1} \varepsilon^* x_1 \eta^* x_1^{-1} \varepsilon^* x_3 c \dots \\ &< x_3 a^{-1} \dots < x_1 \eta^* x_1^{-1} \varepsilon^* x_3 c \dots. \end{aligned}$$

Thus, comparing the first and last element of the inequality-chain and reducing by the same word as added before, we obtain

$$x_3 a^{-1} \dots < x_3 c \dots,$$

a contradiction to the definition of the linear order introduced in Section 2.3.

3) $j^*_+ < l^+_+$ and $j < l^*_-$: Let $x_1, x_3$ be two undirected suitable words with $|x_1| + |x_3| + 1 = \operatorname{ind}^*_l(w)$, such that (by symmetries in $j$ and $l$) $w$ is of the form:

$$\dots w_{j^*_-} x_1^{-1} w_k x_1 b^{-1} x_3 a^{-1} x_2^{-1} w_j x_2 w_{l^*_-} x_3^{-1} b x_1^{-1} w_l x_1 w_{j^*_+} x_3 w_{l^*_+} \dots$$

with $w_{j^*_+} = b^{-1}$ and $w_{l^*_-} = a$.
Since $\operatorname{dir}(w_{j^*_+}) = -1$ by assumption, it follows directly $\operatorname{ind}^*_k(w) = |x_1|$ and thus $w_{k^*_-} = w_{j^*_-}$. Hence, $\operatorname{dir}(v_k) = \operatorname{dir}(v_j) = -\operatorname{dir}(v_l)$.

4) $j^*_+ < l^*_+$ and $l^*_- < j$: Let $x_1, x_2$ and $x_3$ be three words such that $|x_1| + |x_2| + x_3| + 2 = \operatorname{ind}^*_l(w)$ and $w$ is of the form:

$$\dots w_{l^*_-} x_3^{-1} b x_2^{-1} w_j x_1^{-1} w_l x_1 \eta^* x_2 w_{j^*_+} x_3 w_{l^*_+} \dots \tag{51}$$

where $w_j = \eta^*$, $w_{j^*_+} = b^{-1}$.
Similar to case 2), we distinguish the following subcases:

a) $|x_1| = |x_2|$:
By symmetry in $j$, we obtain $w_l = b$ an ordinary letter. This contradicts the assumption on $w_l$.

b) $|x_1| = |x_2| + 1$:
We can write $x_1 = bx_2^{-1}$ and $w$ as follows:

$$\dots w_{l_-^*} x_3 b x_2^{-1} w_j x_2 b^{-1} w_l b x_2^{-1} \eta^* x_2 w_{j_+^*} x_3 w_{l_+^*} \dots .$$

We have $|x_3| > 0$ (otherwise it follows $w_l = w_{j_-^*}$, a contradiction to the different types of the two letters).
If $|x_3| = 1$, then $x_3 = \varepsilon^* = w_k$ and $w_{j_-^*} = w_{l_-^*}$. Thus, $\operatorname{dir}(v_l) = \operatorname{dir}(v_j)$, contradicting the assumption. Hence, $|x_3| > 1$.
We can write $x_3 = \varepsilon^* b x_4$. Assume towards a contradiction that $\operatorname{dir}(v_k) = \operatorname{dir}(v_l)(= -\operatorname{dir}(v_j))$. We consider $w$ of the following form:

$$\dots w_{l_-^*} x_4^{-1} b^{-1} w_k b x_2^{-1} w_j x_2 b^{-1} w_l b x_2^{-1} \eta^* x_2 w_{j_+^*} \varepsilon^* b x_4 w_{l_+^*} \dots$$

with $w_k = w_l = \varepsilon^*$, $w_j = \eta^*$, $w_{j_+^*} = b^{-1}$.
Since $v$ is weakly consistent, the following inequalities hold:

(i) for position $l$: $w_{l_+^*} \dots < w_{l_-^*}^{-1} \dots$,
(ii) for position $k$: $bx_4 w_{l_-^*}^{-1} \dots > bx_2^{-1}\eta^* x_2 b^{-1} \varepsilon^* b x_2^{-1} \eta^* x_2 b^{-1} \varepsilon^* b x_4 w_{l_+^*} \dots$,
(iii) for position $j$: $x_2 b^{-1} \varepsilon^* b x_4 w_{l_-^*}^{-1} \dots < x_2 b^{-1} \varepsilon^* b x_2^{-1} \eta^* x_2 b^{-1} \varepsilon^* b x_4 w_{l_+^*} \dots$.

We extend (iii) from the left by $x_2 b^{-1} \varepsilon^* b x_2^{-1} \eta^*$ and obtain the following chain of inequalities, using (iii), (ii) and then (iii) again.

$$\begin{aligned}
&x_2 b^{-1} \varepsilon^* b x_2^{-1} \eta^* x_2 b^{-1} \varepsilon^* b x_4 w_{l_-^*}^{-1} \dots \\
&< x_2 b^{-1} \varepsilon^* b x_2^{-1} \eta^* x_2 b^{-1} \varepsilon^* b x_2^{-1} \eta^* x_2 b^{-1} \varepsilon^* b x_4 w_{l_+^*} \dots \\
&< x_2 b^{-1} \varepsilon^* b x_4 w_{l_-^*}^{-1} \dots \\
&< x_2 b^{-1} \varepsilon^* b x_2^{-1} \eta^* x_2 b^{-1} \varepsilon^* b x_4 w_{l_+^*} \dots .
\end{aligned}$$

Comparing the first and last term gives

$$w_{l_-^*}^{-1} \dots < w_{l_+^*} \dots ,$$

a contradiction to (i).

c) $|x_1| > |x_2| + 1$:
We can write $x_1 = x_4 b x_2^{-1}$ for a suitable undirected word $x_4$ and obtain for $w$:

$$\dots w_{l_-^*} \dots x_4 b x_2^{-1} w_j x_2 b^{-1} x_4^{-1} w_l x_4 b x_2^{-1} \eta^* x_2 w_{j_+^*} x_3 w_{l_+^*} \dots$$

with $w_j = \eta^*$, $w_l = \varepsilon^*$ and $w_{j_+^*} = b^{-1}$.
Let us first consider the case $|x_4| = |x_3|$. It follows $x_4 = x_3^{-1}$ and thus by symmetry in position $j$ that $w_l = w_{l_-^*}^{-1}$. But $w_l$ is of special type while $w_{l_-^*}$ of ordinary type, giving a contradiction.

Now assume $|x_4| < |x_3|$. We obtain by symmetry in position $j$ that $x_3$ is of the form $x_3 = x_4^{-1}\varepsilon^* x_5$ for a suitable (possibly trivial) undirected word $x_5$. By symmetry in position $l$, we can write $w$ as follows:

$$\dots w_{l_-^*} x_5^{-1} w_k x_4 b x_2^{-1} w_j x_2 b^{-1} x_4^{-1} w_l x_4 b x_2^{-1} \eta^* x_2 b^{-1} x_4^{-1} \varepsilon^* x_5 w_{l_+^*} \dots$$

with $w_j = \eta^*$, $w_k = w_l = \varepsilon^*$.
Assume for contradiction that $\operatorname{dir}(v_l) = \operatorname{dir}(v_k) = -\operatorname{dir}(v_j)$. The directed version $v$ of $w$ is weakly consistent, giving:

(i) from position $k$:
$x_5 w_{l_-^*}^{-1} \dots > x_4 b x_2^{-1} \eta^* x_2 b^{-1} x_4^{-1} \varepsilon^* x_4 b x_2^{-1} \eta^* x_2 b^{-1} x_4^{-1} \varepsilon^* x_5 w_{l_+^*} \dots$,

(ii) from position $j$: $x_5 w_{l_-^*}^{-1} \dots < x_4 b x_2^{-1} \eta^* x_2 b^{-1} x_4^{-1} \varepsilon^* x_5 w_{l_+^*} \dots$,

(iii) from position $l$: $w_{l_-^*}^{-1} \dots > w_{l_+^*} \dots$

(Note that in (ii) we have reduced both sides from the left by $x_2 b^{-1} x_4^{-1} \varepsilon^*$).
Extending (ii) from the left by $v \coloneqq x_4 b x_2^{-1} \eta^* x_2 b^{-1} x_4^{-1} \varepsilon^*$ and applying (i) and (ii) afterwards gives

$$\begin{aligned} v x_5 w_{l_-^*}^{-1} \dots &< v x_4 b x_2^{-1} \eta^* x_2 b^{-1} x_4^{-1} \varepsilon^* x_5 w_{l_+^*} \dots \\ &< x_5 w_{l_-^*}^{-1} \dots < v x_5 w_{l_+^*} \dots \end{aligned}$$

Comparing the first and last term of this chain of inequalities gives

$$w_{l_-^*}^{-1} \dots < w_{l_+^*} \dots,$$

a contradiction to (iii).
Now consider the last case $|x_4| > |x_3|$. We can write $x_4 = x_5 w_{l_-^*} x_3^{-1}$ for some suitable $x_5$. We can deduce the following form for $w$:

$$\begin{aligned} \dots w_{j_-^*} x_2^{-1} \eta^* x_2 b^{-1} x_3 c^{-1} x_5^{-1} w_k x_5 w_{l_-^*} x_3^{-1} b x_2^{-1} w_j x_2 b^{-1} x_3 \\ c^{-1} x_5^{-1} w_l x_5 c x_3^{-1} b x_2^{-1} \eta^* x_2 w_{j_+^*} x_3 w_{l_+^*} \dots \end{aligned}$$

with $w_{l_+^*} = c$, $w_j = \eta^*$, $w_k = w_l = \varepsilon^*$ and $w_{j_+^*} = b^{-1}$.
Assume for contradiction $\operatorname{dir}(v_l) = \operatorname{dir}(v_k) = -\operatorname{dir}(v_j)$. We obtain from the weakly consistent orientation on $v$:

(i) from position $l$: $w_{l_-^*}^{-1} \dots > w_{l_+^*} \dots$,

(ii) from position $k$:

$$w_{j_-^*}^{-1} \dots > b^{-1} x_3 c^{-1} x_5^{-1} \varepsilon^* x_5 c x_3^{-1} b x_2^{-1} \eta^* x_2 b^{-1} x_3 w_{l_+^*} \dots,$$

(iii) from position $j$:

$$\begin{aligned} x_2 b^{-1} x_3 c^{-1} x_5^{-1} \varepsilon^* x_5 c x_3^{-1} b x_2^{-1} \eta^* x_2 b^{-1} x_3 w_{l_+^*} \dots \\ > x_2 b^{-1} x_3 w_{l_-^*}^{-1} x_5^{-1} \varepsilon^* x_5 c x_3^{-1} b x_2^{-1} \eta^* x_2 w_{j_-^*}^{-1} \dots \end{aligned}$$

(Note that we have reduced in case (ii) both words from the left by $x_5cx_3^{-1}bx_2^{-1}\eta^*x_2$.) Applying (iii), (ii) and again (iii) gives (after reducing in the last step by $x_2b^{-1}x_3c^{-1}x_5^{-1}\varepsilon^*x_5cx_3^{-1}bx_2^{-1}\eta^*$ from the left):

$$\begin{aligned}
&x_2b^{-1}x_3w_{l_-^*}^{-1}x_5^{-1}\varepsilon^*x_5cx_3^{-1}bx_2^{-1}\eta^*x_2w_{j_-^*}^{-1}\dots\\
&< x_2b^{-1}x_3c^{-1}x_5^{-1}\varepsilon^*x_5cx_3^{-1}bx_2^{-1}\eta^*x_2b^{-1}x_3w_{l_+^*}\dots\\
&< x_2w_{j_-^*}^{-1}\dots\\
&< x_2b^{-1}x_3w_{l_+^*}\dots
\end{aligned}$$

Comparing the first and last term gives

$$w_{l_-^*}^{-1}\dots < w_{l_+^*}\dots,$$

contradicting (i).

d) $|x_1| < |x_2|$:
In this case we can write $x_2 = x_1^{-1}\varepsilon^*x_4$ for some suitable word $x_4$ and consider $w$ in the following way:

$$\dots w_{l_-^*}x_3^{-1}bx_4^{-1}w_kx_1w_jx_1^{-1}w_lx_1\eta^*x_1^{-1}\varepsilon^*x_4w_{j_+^*}x_3w_{l_+^*}\dots$$

with $w_{j_+^*} = b^{-1}$, $w_j = \eta^*$, $w_k = w_l = \varepsilon^*$.
Assume for contradiction that $\mathrm{dir}(v_k) = \mathrm{dir}(v_l) = -\mathrm{dir}(v_j)$. The following inequalities hold:

(i) from position $k$: $x_4b^{-1}x_3w_{l_-^*}^{-1}\dots > x_1\eta^*x_1^{-1}\varepsilon^*x_1\eta^*x_1^{-1}\varepsilon^*x_4b^{-1}x_3w_{l_+^*}\dots$,

(ii) from position $j$: $x_1\eta^*x_1^{-1}\varepsilon^*x_4b^{-1}x_3w_{l_+^*}\dots > x_4b^{-1}x_3w_{l_-^*}^{-1}\dots$,

(iii) from position $l$: $w_{l_-^*}^{-1}\dots > w_{l_+^*}\dots$

We extend (ii) from the left by $x_1\eta^*x_1^{-1}\varepsilon^*$ and obtain the following chain of inequalities, applying (ii), (i) and (ii) again:

$$\begin{aligned}
&x_1\eta^*x_1^{-1}\varepsilon^*x_4b^{-1}x_3w_{l_-^*}^{-1}\dots\\
&< x_1\eta^*x_1^{-1}\varepsilon^*x_1\eta^*x_1^{-1}\varepsilon^*x_4b^{-1}x_3w_{l_+^*}\dots\\
&< x_4b^{-1}x_3w_{l_-^*}^{-1}\dots\\
&< x_1\eta^*x_1^{-1}\varepsilon^*x_4b^{-1}x_3w_{l_+^*}\dots.
\end{aligned}$$

Comparing the first and last term gives

$$w_{l_-^*}^{-1}\dots < w_{l_+^*}\dots,$$

a contradiction to (iii).

4*) In contrast to $w$ of the form as in (51), one can also have $w$ given by

$$\dots w_{l_-^*} z^{-1} w_j y^{-1} b x^{-1} w_l x w_{j_+^*} y \eta^* z w_{l_+^*} \dots \tag{52}$$

with $w_j = \eta^*$, $w_{j_+^*} = b^{-1}$. To analyse (52), we distinguish the following cases:

a*) $|z| = |y|$:
By symmetry in position $j$, we have $w_{l_-^*} = b^{-1}$. This contradicts $\operatorname{dir}(v_l) = 1$.

b*) $|z| < |y|$:
We can write $y = \bar{y} a z^{-1}$ for a suitable subword $\bar{y}$ and $w_{l_-^*} = a$. It follows that (52) can be refined to

$$\dots w_{j_-^*} x^{-1} w_k x b^{-1} \bar{y} w_{l_-^*} z^{-1} w_j z a^{-1} \bar{y}^{-1} b x^{-1} w_l x w_{j_+^*} \bar{y} a z^{-1} \eta^* z w_{l_+^*} \dots$$

Note that we have also used symmetry in $j$ to extend to the left. One has $\operatorname{ind}_k^*(w) = |x|$ and $w_{k_-^*} = w_{j_-^*}$. Hence, $\operatorname{dir}(v_k) = \operatorname{dir}(v_j) = -\operatorname{dir}(v_l)$.

c*) $|y| < |z|$:
We can write $z = y^{-1} b \bar{z}$ with $w_{j_+^*} = b^{-1}$ and $\bar{z}$ a suitable subword. Refining (52) results in

$$\dots w_{l_-^*} \bar{z}^{-1} b^{-1} y w_j y^{-1} b x^{-1} w_l x w_{j_+^*} y \eta^* y^{-1} b \bar{z} w_{l_+^*} \dots \tag{53}$$

Now $|\bar{z}| \neq |x|$ (otherwise $w_l = w_{l_-^*}$). Consider $|\bar{z}| > |x|$. Then $\bar{z} = x^{-1} \varepsilon^* \hat{z}$ with $w_l = \varepsilon^*$ and $\hat{z}$ a suitable subword. Thus, (53) can be refined to

$$\dots w_{l_-^*} \hat{z}^{-1} \varepsilon^* x b^{-1} y w_j y^{-1} b x^{-1} w_l x w_{j_+^*} y \eta^* y^{-1} b x^{-1} \varepsilon^* \hat{z} w_{l_+^*} \dots$$

If $|\hat{z}| = 0$, we can write $x = a^{-1} \tilde{x}$ with $w_{l_-^*} = a$ and $\tilde{x}$ a suitable subword. By symmetry in position $j$, we obtain $\operatorname{ind}_k^*(w) = |\tilde{x}| + 1$ and thus $w_{k_-^*} = w_{j_-^*}$. It follows that $\operatorname{dir}(v_k) = \operatorname{dir}(v_j) = -\operatorname{dir}(v_l)$. Now assume $|\hat{z}| > 0$ and for contradiction $\operatorname{dir}(v_k) = \operatorname{dir}(v_l)$. The following inequalities hold:

(i) position $k$: $\hat{z} a^{-1} \cdots > x b^{-1} y \eta^* y^{-1} b x^{-1} \varepsilon^* x b^{-1} y \eta^* y^{-1} b x^{-1} \varepsilon^* \hat{z} c \dots$

(ii) position $l$: $a^{-1} \cdots > c \dots$

(iii) position $j$: $y^{-1} b x^{-1} \varepsilon^* x b^{-1} y \eta^* y^{-1} b x^{-1} \varepsilon^* \hat{z} c \cdots > y^{-1} b x^{-1} \varepsilon^* \hat{z} a^{-1} \dots$

Reducing inequality (iii) from the left by $y^{-1} b x^{-1} \varepsilon^*$ and applying (i) afterwards, we obtain

$$\begin{aligned} x b^{-1} y \eta^* y^{-1} b x^{-1} \varepsilon^* \hat{z} c \dots &> \hat{z} a^{-1} \dots \\ &> x b^{-1} y \eta^* y^{-1} b x^{-1} \varepsilon^* x b^{-1} y \eta^* y^{-1} b x^{-1} \varepsilon^* \hat{z} c \dots \end{aligned}$$

We reduce the first and last term on the left by $xb^{-1}y\eta^* y^{-1}bx^{-1}\varepsilon^*$. Applying then (ii) results in

$$\hat{z}c\dots > xb^{-1}y\eta^* y^{-1}bx^{-1}\varepsilon^*\hat{z}\dots > \hat{z}a^{-1}\dots$$

Comparing the first and last term of this chain of inequalities contradicts the linear ordering on $\hat{\mathcal{W}}^{\mathrm{pos}}_{\star,i,\kappa}$.

Finally, consider $|\bar{z}| < |x|$. Then write $x = \tilde{x}a^{-1}z$ where $w_{l^*_-} = a$ and $\tilde{x}$ a suitable subword. The word $w$ as in (53) can be refined to

$$\dots w_{j^*_-}\bar{z}a^{-1}\tilde{x}^{-1}w_k\tilde{x}w_{l^*_-}\bar{z}^{-1}b^{-1}yw_jy^{-1}b\bar{z}a^{-1}\tilde{x}^{-1}w_lxw_{j^*_+}\dots$$

It follows that $\mathrm{ind}^*_k(w) = |x|$ and thus $w_{k^*_-} = w_{j^*_-}$. It follows that $\mathrm{dir}(v_k) = \mathrm{dir}(v_j) = -\mathrm{dir}(v_l)$.

5) $j^*_+ = l^*_+$:
It follows that $\mathrm{dir}(v_l) = \mathrm{dir}(w_{l^*_+}) = \mathrm{dir}(w_{j^*_+}) = \mathrm{dir}(v_j)$. By assumption $\mathrm{dir}(v_l) = -\mathrm{dir}(v_j)$ giving a contradiction.

For $\mathrm{dir}(v_j) = \mathrm{dir}(v_l)$, we automatically obtain the statement, since we either have $\mathrm{dir}(v_k) = -\mathrm{dir}(v_l)$ or $\mathrm{dir}(v_k) = \mathrm{dir}(v_l)$. $\square$

**Example 3.57.** *Consider $\Lambda$, $w$ and $v$ from Example 3.52. Recall that $v$ is weakly consistent. Note that $w$ is an asymmetric string. For $j = 4$, $k = 1$ and $l = 7$ we have*

$$\begin{aligned} |j-k| = |j-l| &= 3,\\ \mathrm{ind}^*_j(w) &= 3,\\ \mathrm{ind}^*_k(w) = 0 &< \mathrm{ind}^*_j(w),\\ \mathrm{ind}^*_l(w) = 1 &< \mathrm{ind}^*_j(w).\end{aligned}$$

*The assumptions of Lemma 3.56 are satisfied. We see that*

$$\mathrm{dir}(v_k) = \mathrm{dir}(v_l) = \mathrm{dir}(v_j).$$

**Example 3.58.** *Consider $\Lambda$ from Example 2.3.1. Let*

$$w: \quad \varepsilon^* a^{-1}\varepsilon^* a\varepsilon^* a\varepsilon^*$$

*be an asymmetric string. Then*

$$v: \quad \varepsilon^{-1}a^{-1}\varepsilon a\varepsilon a\varepsilon$$

*is a weakly consistent directed version of $w$. Take $j = 3$, $k = 2$, $l = 5$. Then we have*

$$\begin{aligned} |j-k| = |j-l| &= 2,\\ \mathrm{ind}^*_j(w) &= 2,\\ \mathrm{ind}^*_k(w) = 0 &< \mathrm{ind}^*_j(w),\\ \mathrm{ind}^*_l(w) = 0 &< \mathrm{ind}^*_j(w).\end{aligned}$$

*Furthermore, we see that*

$$\operatorname{dir}(v_k) = -\operatorname{dir}(v_l).$$

**Lemma 3.59.** *Let $w$ be a string of length $n$, either asymmetric or symmetric, for some $n \in \mathbb{N}$, and let $w_{\mathbb{Z}}$ be an asymmetric band. Let $v \in (\Phi^{\mathrm{d}}_{\mathrm{ud}})^{-1}(w)$ ($v_{\mathbb{Z}} \in (\Phi^{\mathrm{d}}_{\mathrm{ud}})^{-1}(w_{\mathbb{Z}})$, respectively) be weakly consistent. If $w$ is a symmetric string of the form $u\varepsilon^* u^{-1}$, assume additionally that $v = t\varepsilon^{\kappa} t^{-1}$ for $t \in (\Phi^{\mathrm{d}}_{\mathrm{ud}})^{-1}(u)$ and $\kappa \in \{+1, -1\}$. Let $j \in I$ ($I = \{1, \ldots, n\}$ or $I = \mathbb{Z}$) and $\operatorname{ind}^*_j(v) = d < \infty$ ($\operatorname{ind}^*_j(v_{\mathbb{Z}}) = d < \infty$, respectively). Let $k, l \in I$ with $k < j < l$, $|j - k| = |j - l| \le d$, $w_k$ and $w_l$ special letters, and $\operatorname{ind}^*_k(v) \ge d, \operatorname{ind}^*_l(v) < d$ ($\operatorname{ind}^*_k(v_{\mathbb{Z}}) \ge d, \operatorname{ind}^*_l(v_{\mathbb{Z}}) < d$, respectively). Then $\operatorname{dir}(v_j) = \operatorname{dir}(v_l)$.*

*Proof.* We show the statement for $w$ an asymmetric string. For symmetric strings and asymmetric bands, the proof is analogous by the same arguments as in Lemma 3.55.
Observe that $j + \operatorname{ind}^*_j(w) \le l + \operatorname{ind}^*_l(w)$ (otherwise $\operatorname{ind}^*_k(w) \le d$ by symmetry in position $j$).
Consider at first the case $j + \operatorname{ind}^*_j(w) = l + \operatorname{ind}^*_l(w)$. Then $j^*_+ = l^*_+$ and thus $\operatorname{dir}(w_{j^*_+}) = \operatorname{dir}(w_{l^*_+})$. It follows directly that $\operatorname{dir}(v_j) = -\operatorname{dir}(v_l)$.
Now let $j + \operatorname{ind}^*_j(w) < l + \operatorname{ind}^*_l(w)$. Then $j^*_+ < l^*_+$ also. Assume $j \notin L^*$. Denote by $x$ the subword between $w_l$ and $w_{j^*_+}$ and by $y$ the subword between $w_{j^*_+}$ and $w_{l^*_+}$. Assume for contradiction that $\operatorname{dir}(v_j) = \operatorname{dir}(v_l)$ and without loss of generality that $\operatorname{dir}(v_j) = -1$. By symmetry in the positions $l$ and $j$, $w$ is then of the form

$$\ldots w_{j^*_-} x^{-1} w_k x b^{-1} y f^{-1} \ldots w_j \ldots w_{l^*_-} \underbrace{y^{-1} b^{-1} x^{-1} w_l x w_{j^*_+} y}_{L^*} w_{l^*_+} \ldots,$$

where $w_{j^*_+} = b^{-1}, w_{j^*_-} = c^{-1}$, $w_{l^*_+} = a$ and $w_{l^*_-} = f$. It follows that $\operatorname{ind}^*_k(w) = |x| < d$ which is a contradiction.
Now consider $j \in L^*$. Denote by $y$ the subword of $w$ between the letters $w_l$ and $w_{j+^*}$, by $x$ the one between $w_j$ and $w_l$, and by $z$ the subword of $w$ between $w_{j^*_+}$ and $w_{l^*_+}$:

$$\ldots w_{j^*_-} y^{-1} w_k x^{-1} w_j x w_l y w_{j^*_+} z w_{l^*_+} \ldots \tag{54}$$

Then $\operatorname{ind}^*_j(w) = |x| + |y| + 1 > |y| + |z| + 1 = \operatorname{ind}^*_l(w)$ and thus $|x| > |z|$.
Assume towards a contradiction that $\operatorname{dir}(v_j) = -\operatorname{dir}(v_l)$ and without loss of generality that $\operatorname{dir}(v_j) = -1$. Let $w_{j^*_-} = c^{-1}$ and $w_{j^*_+} = b^{-1}$.
Assume at first $\operatorname{dir}(v_k) = 1$. We obtain the following inequalities from $v$ being weakly consistent:

(i) from position $k$: $y w^{-1}_{j^*_-} \ldots > x^{-1} w_j x w_l y w_{j^*_+} z w_{l^*_+} \ldots$,

(ii) from position $j$: $w_{j^*_+} z w_{l^*_+} \ldots > w^{-1}_{j^*_-} \ldots$,

(iii) from position $l$: $x^{-1}w_j x w_k y w_{j_-^*}^{-1} \ldots > y w_{j_+^*} z w_{l_+^*} \ldots$.

We extend (ii) from the left by the word $y$. Applying now the above inequalities in order starting with (i) gives

$$x^{-1}w_j x w_l y w_{j_+^*} z w_{l_+^*} \ldots < y w_{j_-^*}^{-1} \ldots < y w_{j_+^*} z w_{l_+^*} \ldots < x^{-1} w_j x w_k y w_{j_-^*}^{-1} \ldots$$

Since $w_k = w_l$, we can reduce the above chain to

$$w_{j_+^*} z w_{l_+^*} \ldots < w_{j_-^*}^{-1} \ldots \tag{55}$$

By assumption on $\mathrm{dir}(v_j)$, both letters $w_{j_-^*}$ and $w_{j_+^*}$ are inverse. Hence, (55) is contradicting the definition of the linear order "<" defined in Section 2.3. Let now $\mathrm{dir}(v_k) = -1$. Observe at first that $|x| \neq |y|$ in (54) (otherwise, by symmetry in $l$, $w_j = w_{j_+^*}$ giving a contradiction to them being, respectively, of special and ordinary type). We distinguish the different possibilities:

(i) Assume $|x| > |y|$. By symmetry in position $l$, we can write

$$x = \tilde{y} b y^{-1}.$$

Thus, (54) can be refined to

$$\ldots w_{j_-^*} y^{-1} w_k y b^{-1} \tilde{y}^{-1} w_j \tilde{y} b y^{-1} w_l y w_{j_+^*} z w_{l_+^*} \ldots$$

It follows that $w_{k_-^*} = w_{j_-^*}$. Thus, $\mathrm{ind}_j^*(w) = |y| < d$, giving a contradiction to the assumption on the $c^*$–index of $w_k$.

(ii) Assume $|x| < |y|$ in (54). By symmetry in position $l$, we can write $y = x^{-1}\eta^* \tilde{y}$ for a suitable subword $\tilde{y}$ and $w_j = \eta^*$. Refining (54) by this results in

$$\ldots w_{j_-^*} \tilde{y}^{-1} \eta^* x w_k x^{-1} w_j x w_l x^{-1} \eta^* \tilde{y} w_{j_+^*} z w_{l_+^*} \ldots \tag{56}$$

It follows that $|\tilde{y}| \neq |x|$ (otherwise $w_{j_-^*} = w_l$). Hence, let at first $|\tilde{y}| < |x|$. Then $x = \tilde{y} c \bar{x}$ for a suitable subword $\bar{x}$ and $w_{j_-^*} = c^{-1}$. The refined version of (56) is given by

$$\ldots w_{j_-^*} \tilde{y}^{-1} \eta^* \tilde{y} c \bar{x} w_k \bar{x}^{-1} c^{-1} \tilde{y}^{-1} w_j \tilde{y} c \bar{x} w_l \bar{x}^{-1} c^{-1} \tilde{y}^{-1} \eta^* \tilde{y} w_{j_+^*} z w_{l_+^*} \ldots$$

It follows that $w_{j_+^*} = w_{l_+^*}$ and thus $\mathrm{dir}(v_l) = -1$, a contradiction to the assumption on the direction of $v_l$.
Consider now the case $|\tilde{y}| > |x|$. Then $\tilde{y} = x \varepsilon^* \bar{y}$ for a suitable subword $\bar{y}$ and $w_l = \varepsilon^*$:

$$\ldots w_{j_-^*} \bar{y}^{-1} \varepsilon^* x^{-1} \eta^* x w_k x^{-1} w_j x w_l x^{-1} \eta^* x \varepsilon^* \bar{y} w_{j_+^*} z w_{l_+^*} \ldots \tag{57}$$

Refining $\bar{y}$ analogously as $y$ and $\tilde{y}$ above results in contradictions in the cases $|\bar{y}| \neq |x|$ and $|\bar{y}| < |x|$ (analogously to the above).

For $|\bar{y}| > |x|$ we can write $\bar{y} = x^{-1}\eta^* s$ for a suitable subword $s$. Thus, we consider now

$$\dots w_{j^*_-} s^{-1}\eta^* x\varepsilon^* x^{-1}\eta^* x w_k x^{-1} w_j x w_l x^{-1}\eta^* x\varepsilon^* x^{-1}\eta^* s w_{j^*_+} z w_{l^*_+}\dots \quad (58)$$

Now, comparing (56) - (58), we see that instead of refining further, we can consider $|s| \to 0$ instead. Assume without loss of generality that $|s| = 0$. This results in $x = c\tilde{x}$ for some suitable subword $\tilde{x}$. It follows that $w_{l^*_-} = w_{j^*_-}$ and thus $\operatorname{dir}(v_l) = -1$. This contradicts our assumption on the direction of $v_l$.

□

**Example 3.60.** *Let $\Lambda$ be as in Example 2.3.1. Let*

$$w: \quad \varepsilon^* a\varepsilon^* a\varepsilon^* a\varepsilon^* a^{-1}\varepsilon^* a\varepsilon^* a^{-1}\varepsilon^* a^{-1}\varepsilon^* a\varepsilon^*$$

*be an asymmetric string. The following word is a weakly consistent directed version of it:*

$$\begin{array}{rl} v: & \varepsilon\ a\ \varepsilon\ a\ \varepsilon\ a\ \varepsilon\ a^{-1}\ \varepsilon\ a\ \varepsilon^{-1}\ a^{-1}\ \varepsilon^{-1}\ a^{-1}\ \varepsilon^{-1}\ a\ \varepsilon \\ i \in \mathrm{I}: & 1\,2\,3\,4\,5\,6\,7\,8\ \ 9\,10\,11\ \ 12\ \ 13\ \ 14\ \ 15\ \ 16\,17 \end{array}$$

*Consider now $j = 11$, $k = 9$ and $l = 13$. We have*

$$\begin{aligned} |j-k| = |j-l| &= 2, \\ \operatorname{ind}^*_j(w) &= 2, \\ \operatorname{ind}^*_k(w) &= 6 > \operatorname{ind}^*_j(w), \\ \operatorname{ind}^*_l(w) &= 0 < \operatorname{ind}^*_j(w). \end{aligned}$$

*We see that*

$$\operatorname{dir}(v_j) = \operatorname{dir}(v_l).$$

With these auxiliary lemmas, we are able to prove the following theorem:

**Theorem 3.61.** *Let $w$ be an asymmetric or symmetric string and let $w_{\mathbb{Z}}$ be an asymmetric band. Let $v \in (\Phi^{\mathrm{d}}_{\mathrm{ud}})^{-1}(w)$ ($v_{\mathbb{Z}} \in (\Phi^{\mathrm{d}}_{\mathrm{ud}})^{-1}(w_{\mathbb{Z}})$, respectively) be a directed version of $w$ ($w_{\mathbb{Z}}$, respectively). Then $v$ ($v_{\mathbb{Z}}$, respectively) is weakly consistent if and only if it is consistent.*

*Proof.* We show the statement for $w$ an asymmetric string. For symmetric strings and asymmetric bands, the proof is analogous by the same arguments as in Lemma 3.55.
Let $w$ be an asymmetric string. Let $v \in (\Phi^{\mathrm{d}}_{\mathrm{ud}})^{-1}(w)$ be consistent and $z \in (\Phi^{\mathrm{d}}_{\mathrm{ud}})^{-1}(w)$ weakly consistent. We show the statement by induction on $\operatorname{ind}^*_j(w)$ for $j \in I = \{1, \dots, n\}$.
For $\operatorname{ind}^*_j(w) = 0$ it follows that $\operatorname{ind}^*_j(w) = \operatorname{ind}^c_j(w)$. Hence $w_{j^*_-} = w_{j^c_-}$ and

$w_{j_+^*} = w_{j_+^c}$ are of ordinary type. So $\operatorname{dir}(v_j) = \operatorname{dir}(z_j)$.
Now let $\operatorname{ind}_j^*(w) = d > 0$. Again, the statement is clear for $\operatorname{ind}_j^*(w) = \operatorname{ind}_j^c(w)$, so assume $\operatorname{ind}_j^*(w) > \operatorname{ind}_j^c(w)$. Let $k, l \in I$ such that $k = j_-^c$, $l = j_+^c$. Then

$$\operatorname{dir}(v_k) = \operatorname{dir}(v_l) = \operatorname{dir}(v_j) \tag{59}$$

and $|j-k| = |j-l| \le d$. Thus we can apply Lemma 3.55 and either have

a) $\operatorname{ind}_k^*(w) < d$ and $\operatorname{ind}_l^*(w) < d$, or

b) $\operatorname{ind}_k^*(w) \ge d$ and $\operatorname{ind}_l^*(w) < d$.

Let us first consider case *a*): by induction we obtain

$$\begin{aligned} \operatorname{dir}(v_k) &= \operatorname{dir}(z_k) \quad \text{and} \\ \operatorname{dir}(v_l) &= \operatorname{dir}(z_l). \end{aligned}$$

By Lemma 3.56 we either have $\operatorname{dir}(z_k) = \operatorname{dir}(z_l) = \operatorname{dir}(z_j)$ and we obtain by (59) the result. Alternatively, we have $-\operatorname{dir}(z_k) = \operatorname{dir}(z_l)$. Then it follows by induction that $\operatorname{dir}(v_k) = -\operatorname{dir}(v_l)$, a contradiction to (59), so this case does not occur.
In case *b*) it follows by induction that $\operatorname{dir}(v_l) = \operatorname{dir}(z_l)$. By Lemma 3.59 we obtain $\operatorname{dir}(z_j) = \operatorname{dir}(z_l)$ and using induction on $l$ and (59) results in $\operatorname{dir}(z_j) = \operatorname{dir}(v_j)$. □

**Corollary 3.62.** *Let $w = u\varepsilon^* u^{-1}$ be a symmetric string and $v \in (\Phi_{\mathrm{ud}}^{\mathrm{d}})^{-1}(w)$ be (weakly) consistent. Then $v = t\varepsilon^\kappa t^{-1}$ for $t \in (\Phi_{\mathrm{ud}}^{\mathrm{d}})^{-1}(u)$, $\kappa \in \{+1, -1\}$.*

**Corollary 3.63.** *For any asymmetric string (band, respectively), there exists a unique consistent directed version.*

*Proof.* By Corollary 3.22 we know that there exists a unique weakly consistent directed version for any asymmetric string. Corollary 3.30 gives the same for asymmetric bands. Theorem 3.61 states that any consistent directed version of an asymmetric string or band is also weakly consistent. Thus, uniqueness of a consistent directed version is inherited from the uniqueness of a weakly consistent directed version in both cases. □

### 3.3.2 Directed words from symmetric bands

For symmetric bands, the relation between consistent and weakly consistent orientations is not as intuitive as in the other cases. The two sets of types of directed versions are not equal, but there is merely an embedding of the consistent directed versions in the set of all weakly consistent directed versions.
As in the previous subsection, we will need some auxiliary lemmas in order to describe this embedding.

**Lemma 3.64.** *Let $w_{\mathbb{Z}}$ be a symmetric band of period $p > 0$, with periodic part $\hat{w}_p = \varepsilon^* u \eta^* u^{-1}$. Let $j \in \mathbb{Z}$ with $w_j$ a special letter and $\operatorname{ind}_j^*(w_{\mathbb{Z}}) = d < \infty$. Then there do not exists $k \neq l \in \mathbb{Z}$ with $w_k$, $w_l$ special letters, $|j-k| = |j-l| \leq d$ and $\operatorname{ind}_k^*(w_{\mathbb{Z}}) \geq d$, $\operatorname{ind}_l^*(w_{\mathbb{Z}}) \geq d$.*

*Proof.* Let us first make two observations: by assumption on $\operatorname{ind}_j(w_{\mathbb{Z}})$, we have that $j \neq 1, -m, -(2m+1), m+2 \pmod{p}$. Assume without loss of generality that $k < j < l$. It follows by Lemma 3.64 that $\operatorname{ind}_k^*(w_{\mathbb{Z}}) < \infty$ and $\operatorname{ind}_l^*(w_{\mathbb{Z}}) < \infty$.
Assume for contradiction that $k, l \in \mathbb{Z}$ as in the statement exist. By the second observation we know that neither $k$ nor $l$ gives the position of one of the symmetry axes. By Lemma 3.46 we only need to consider a finite subword of $w_{\mathbb{Z}}$ to determine the direction on $w_j$. Hence, the statement follows analogously to the proof of Lemma 3.55 in the previous subsection. □

**Lemma 3.65.** *Let $w_{\mathbb{Z}}$ be a symmetric band of period $p > 0$ with periodic part $\hat{w}_p = \varepsilon^* u \eta^* u^{-1}$. Let $v_{\mathbb{Z}} \in (\Phi_{\mathrm{ud}}^{\mathrm{d}})^{-1}(w_{\mathbb{Z}})$ be weakly consistent with periodic copies $\hat{v}_p^{(i)} = \varepsilon^{\mu_i} t \eta^{\kappa_i} t^{-1}$ for $t \in (\Phi_{\mathrm{ud}}^{\mathrm{d}})^{-1}(u)$ and $\mu_i, \kappa_i \in \{+1, -1\}$. Let $j \in \mathbb{Z}$ with $w_j$ a special letter and $\operatorname{ind}_j^*(w_{\mathbb{Z}}) = d < \infty$. Let $k, l \in \mathbb{Z}$ with $|j-k| = |j-l| \leq d$, $w_k, w_l$ both special letters and $\operatorname{ind}_k^*(w_{\mathbb{Z}}) < d$, $\operatorname{ind}_l^*(w_{\mathbb{Z}}) < d$.
Then either $\operatorname{dir}(v_j) = \operatorname{dir}(v_k) = \operatorname{dir}(v_l)$ or $\operatorname{dir}(v_k) = -\operatorname{dir}(v_l)$.*

*Proof.* By Lemma 3.46 we consider for $k, l, j$ each only finite subwords, so also in sum a finite subword of $w_{\mathbb{Z}}$. Thus, the proof is analogous to the proof of Lemma 3.56 of the previous subsection. □

**Lemma 3.66.** *Let $w_{\mathbb{Z}}$ be a symmetric band of period $p > 0$ with periodic part $\hat{w}_p = \varepsilon^* u \eta^* u^{-1}$. Let $v_{\mathbb{Z}} \in (\Phi_{\mathrm{ud}}^{\mathrm{d}})^{-1}(w_{\mathbb{Z}})$ be weakly consistent with periodic copies $\hat{v}_p^{(i)} = \varepsilon^{\mu_i} t \eta^{\kappa_i} t^{-1}$ for $t \in (\Phi_{\mathrm{ud}}^{\mathrm{d}})^{-1}(u)$ and $\mu_i, \kappa_i \in \{+1, -1\}$. Let $j \in \mathbb{Z}$ with $w_j$ a special letter and $\operatorname{ind}_j^*(w_{\mathbb{Z}}) = d < \infty$. Let $k, l \in \mathbb{Z}$ with $|j-k| = |j-l| \leq d$, $w_k, w_l$ both special letters and $\operatorname{ind}_k^*(w_{\mathbb{Z}}) \geq d$, $\operatorname{ind}_l^*(w_{\mathbb{Z}}) < d$.
Then $\operatorname{dir}(v_j) = \operatorname{dir}(v_l)$.*

*Proof.* By the same arguments as in the previous proofs, this proof is analogous to the proof of Lemma 3.59 of the previous subsection. □

Now we are able to relate consistent directed versions to weakly consistent ones of symmetric bands.

**Proposition 3.67.** *Let $w_{\mathbb{Z}}$ be a symmetric band of period $p$ with periodic part $\hat{w}_p = \varepsilon^* u \eta^* u^{-1}$. Let $x_{\mathbb{Z}} \in (\Phi_{\mathrm{ud}}^{\mathrm{d}})^{-1}(w_{\mathbb{Z}})$ be weakly consistent with periodic copies $\hat{x}_p^{(i)} = \varepsilon^{\sigma_i} s \eta^{\delta_i} s^{-1}$, where $s \in (\Phi_{\mathrm{ud}}^{\mathrm{d}})^{-1}(u)$ and $\sigma_i, \delta_i \in \{+1, -1\}$. Furthermore, let $v_{\mathbb{Z}} \in (\Phi_{\mathrm{ud}}^{\mathrm{d}})^{-1}(w_{\mathbb{Z}})$ be consistent with periodic copies $\hat{v}_p^{(i)} = \varepsilon^{\mu_i} t \eta^{\kappa_i} t^{-1}$ for $t \in (\Phi_{\mathrm{ud}}^{\mathrm{d}})^{-1}(u)$ and $\mu_i, \kappa_i \in \{+1, -1\}$.
Then one has for all $j \in \mathbb{Z}$ with $w_j$ a special letter and $\operatorname{ind}_j^*(w_{\mathbb{Z}}) < \infty$ that $\operatorname{dir}(v_j) = \operatorname{dir}(x_j)$.*

*Proof.* By Lemma 3.46, we only need to consider a finite subword for $j$ in order to check on consistency and weak consistency. Thus, the proof is analogous to the proof of Theorem 3.61, using Lemmas 3.64, 3.65 and 3.66. □

**Corollary 3.68.** *Let $w_{\mathbb{Z}}$ be a symmetric band of period $p$ with periodic part $\hat{w}_p = \varepsilon^* u \eta^* u^{-1}$. Then any consistent $v_{\mathbb{Z}} \in (\Phi^{\mathrm{d}}_{\mathrm{ud}})^{-1}(w_{\mathbb{Z}})$ preserves the symmetry axes of $w_{\mathbb{Z}}$ with respect to $u$.*

*Proof.* The statement follows directly from Proposition 3.67. □

**Example 3.69.** *Let $\Lambda$ be as in Example 2.3.1. and let $w_{\mathbb{Z}}$ be a symmetric band of period 8 with*

$$\begin{aligned} \hat{w}_p &= \varepsilon^* a \varepsilon^* a \varepsilon^* a^{-1} \varepsilon^* a^{-1}, \\ u &= a \varepsilon^* a. \end{aligned}$$

*Let $v_{\mathbb{Z}}$ be a consistent directed version of $w_{\mathbb{Z}}$. Then*

$$\begin{aligned} \hat{v}_p^{(i)} &= \varepsilon^{\mu_i} a \varepsilon a \varepsilon^{\kappa_i} a^{-1} \varepsilon^{-1} a^{-1}, \\ t &= a \varepsilon a, \end{aligned}$$

*for all $i \in \mathbb{Z}$, where $\mu_i, \kappa_i \in \{+1, -1\}$ and $t$ is a directed version of $u$.*

**Proposition 3.70.** *Let $w_{\mathbb{Z}}$ be a symmetric band of period $p$ with periodic part $\hat{w}_p = \varepsilon^* u \eta^* u^{-1}$. Let $v_{\mathbb{Z}} \in (\Phi^{\mathrm{d}}_{\mathrm{ud}})^{-1}(w_{\mathbb{Z}})$ be a directed version of $w_{\mathbb{Z}}$ with periodic copies $\hat{v}_p^{(i)} = \varepsilon^{\mu_i} t \eta^{\kappa_i} t^{-1}$ for $t \in (\Phi^{\mathrm{d}}_{\mathrm{ud}})^{-1}(u)$ and $\mu_i, \kappa_i \in \{+1, -1\}$. Then $v_{\mathbb{Z}}$ is consistent if and only if it is of one of the following four types:*

1) *$\mu_i = \kappa_i = -1$ for all $i \in \mathbb{Z}$,*

2) *$\mu_i = \kappa_i = 1$ for all $i \in \mathbb{Z}$,*

3) *(unique sink) there exists a unique $j \in \mathbb{Z}$ such that:*

$$\mu_i = \kappa_i = \begin{cases} -1 & \forall i \leq j, \\ 1 & \forall i > j \end{cases},$$

4) *(unique source) there exists a unique $j \in \mathbb{Z}$ such that:*

$$\mu_i = \kappa_i = \begin{cases} 1 & \forall i \leq j, \\ -1 & \forall i > j \end{cases}.$$

*Proof.* By Corollary 3.68 we know that the symmetry axes of $w_{\mathbb{Z}}$ are preserved by consistency, i.e. any consistent $x_{\mathbb{Z}} \in (\Phi^{\mathrm{d}}_{\mathrm{ud}})^{-1}(w_{\mathbb{Z}})$ has periodic copies of the form $\hat{x}_p^{(i)} = \varepsilon^{\mu_i} s \eta^{\kappa_i} s^{-1}$, for $\mu_i, \kappa_i \in \{+1, -1\}$, $s \in (\Phi^{\mathrm{d}}_{\mathrm{ud}})^{-1}(u)$. Hence, we can reduce each subword $u$ and $u^{-1}$ to a vertex and only consider the symmetry axes to show consistency. Thus, any periodic part $\hat{w}_p^{(i)} = \varepsilon^{\mu_i} u \eta^{\kappa_i} u^{-1}$ is being reduced to $\varepsilon^{\mu_i}\eta^{\kappa_i}$. To do so, we use a new indexing of the letters, only referring to those giving symmetry axes. We denote by $\check{v}_{\mathbb{Z}}$ the thus reduced version of $v_{\mathbb{Z}}$. Then, $v_{\mathbb{Z}}$ is consistent if and only if $\check{v}_{\mathbb{Z}}$ is consistent. We can picture $\check{v}_{\mathbb{Z}}$ with respect to 1) - 4) as follows:

1) $\ldots \longrightarrow\longrightarrow\longrightarrow\longrightarrow \ldots,$

2) $\ldots \longleftarrow\longleftarrow\longleftarrow\longleftarrow \ldots,$

3) $\ldots \longrightarrow\overset{j}{\longrightarrow}\longleftarrow\longleftarrow \ldots,$

4) $\ldots \longleftarrow\overset{j}{\longleftarrow}\longrightarrow\longrightarrow \ldots.$

Let us first show that $\check{v}_{\mathbb{Z}}$ given by 1) – 4) is consistent:

1) As we can see, we have $\mathrm{ind}_j^c(\check{v}_{\mathbb{Z}}) = 0$ for all $j \in \mathbb{Z}$. Hence, $\mathrm{dir}(\check{v}_j)$ is uniquely determined and we have

$$\check{v}_{j-1} = \check{v}_{j_-^{\mathrm{c}}} \quad \text{and} \quad \check{v}_{j+1} = \check{v}_{j_+^{\mathrm{c}}}$$

for any $j \in \mathbb{Z}$. Now $\check{v}_{j-1}^{-1} < \check{v}_{j+1}$ and thus

$$(\check{v}_{\mathbb{Z}}[< j])^{-1} < \check{v}_{\mathbb{Z}}[> j].$$

Thus, any $\check{v}_{\mathbb{Z}}$ of form 1) is consistent.

2) This case is analogous to 1) with reversed inequalities.

3) For all $j \in \mathbb{Z}$ with $\mathrm{ind}_j^c(\check{v}_{\mathbb{Z}}) = 0$, the result is clear by the same arguments as given in case 1).
Let us now consider $\check{v}_j$, $\check{v}_{j+1}$ for $j \in \mathbb{Z}$ with $\mathrm{ind}_j^c(\check{v}_{\mathbb{Z}}), \mathrm{ind}_{j+1}^c(\check{v}_{\mathbb{Z}}) > 0$. By the form of 3), there exists only one pair of indices with this property, namely the two neighbouring letters with opposite directions: $\mathrm{dir}(\check{v}_j) = -\mathrm{dir}(\check{v}_{j+1})$. We have

$$\begin{aligned} \text{for all } i \leq j: &\quad \mathrm{dir}(\check{v}_i) = \mathrm{dir}(\check{v}_j), \\ \text{for all } k \geq j+1: &\quad \mathrm{dir}(\check{v}_k) = \mathrm{dir}(\check{v}_{j+1}). \end{aligned}$$

It follows $\mathrm{dir}(\check{v}_i) = -\mathrm{dir}(\check{v}_k)$ for all $i \leq j$, $k \geq j+1$. Thus,

$$\begin{aligned} (\check{v}_{\mathbb{Z}}[< j])^{-1} &= \check{v}_{\mathbb{Z}}[> j], \text{ and} \\ (\check{v}_{\mathbb{Z}}[< (j-1)])^{-1} &= \check{v}_{\mathbb{Z}}[> (j+1)] \end{aligned}$$

giving $\mathrm{ind}_j^c(\check{v}_{\mathbb{Z}}) = \infty = \mathrm{ind}_{j+1}^c(\check{v}_{\mathbb{Z}})$. Hence, any $v_{\mathbb{Z}}$ of type 3) is consistent.

4) The proof of this case is analogous to case 3).

Conversely, assume that $\check{v}_{\mathbb{Z}}$ is consistent.
If all letters of $\check{v}_{\mathbb{Z}}$ have the same direction, i.e. $\mathrm{ind}_j^c(\check{v}_{\mathbb{Z}}) = 0$ for all $j \in \mathbb{Z}$, then $\check{v}_{\mathbb{Z}}$ is of type 1) or 2).
Assume now that there exists $j \in \mathbb{Z}$ with $\mathrm{ind}_j^c(\check{v}_{\mathbb{Z}}) > 0$, i.e., one has for its direct neighbours $\check{v}_{j-1}$, $\check{v}_{j+1}$ that $\mathrm{dir}(\check{v}_{j-1}) = -\,\mathrm{dir}(\check{v}_{j+1})$. Assume without loss of generality that $\mathrm{dir}(\check{v}_j) = \mathrm{dir}(\check{v}_{j-1})$. Then the vertex between $\check{v}_j$ and $\check{v}_{j+1}$ is either a source or a sink. Let $x = x_1 \dots x_k$ be a finite subword of $\check{v}_{\mathbb{Z}}$ with $x_1 = \check{v}_{j+1}$ and $\mathrm{dir}(x_i) = \mathrm{dir}(\check{v}_{j+1})$ for all $1 \le i \le k$. Similarly let $z = z_1 \dots z_l$ be a finite subword of $\check{v}_{\mathbb{Z}}$ with $z_l = \check{v}_j$ and $\mathrm{dir}(z_i) = \mathrm{dir}(\check{v}_j)$ for all $1 \le i \le l$. Assume without loss of generality that $|x| < |z|$. Denote by $y$ the $\mathbb{N}$-subword of $\check{v}_{\mathbb{Z}}$ starting on the right hand side of $x$ and assume $\mathrm{dir}(y_1) = -\,\mathrm{dir}(x_k)$. Thus, we assume that $\check{v}_{\mathbb{Z}}$ is none of the above types, in particular it is neither of type 3) nor of type 4). We obtain that

$$\begin{aligned} \mathrm{ind}_{j+1}^c(\check{v}) &= |x| - 1, \text{ and} \\ (j+1)_-^c &= (j+1) - (|x| - 1) - 1 = j + 1 - |x|, \\ (j+1)_+^c &= (j+1) + (|x| - 1) + 1 = j + 1 + |x|. \end{aligned}$$

It follows that $\mathrm{dir}(\check{v}_{j+1}) = \mathrm{dir}(\check{v}_j)$ giving a contradiction. □

**Corollary 3.71.** *Let $w_{\mathbb{Z}}$ be a symmetric band of period $p$ with periodic part $\hat{w}_p = \varepsilon^* u \eta^* u^{-1}$. Let $v_{\mathbb{Z}} \in (\Phi_{\mathrm{ud}}^{\mathrm{d}})^{-1}(w_{\mathbb{Z}})$ be weakly consistent, given by one of the types 1)–4) of Proposition 3.70 and with periodic copies $\hat{v}_p^{(i)} = \varepsilon^{\mu_i} t \eta^{\kappa_i} t^{-1}$ where $t \in (\Phi_{\mathrm{ud}}^{\mathrm{d}})^{-1}(u)$ and $\mu_i, \kappa_i \in \{+1, -1\}$. Then $v_{\mathbb{Z}}$ is consistent.*

*Proof.* The result follows from Proposition 3.67 and 3.70: by Proposition 3.67 we know that weakly consistent and consistent orientations coincide on those $j \in \mathbb{Z}$ with $\mathrm{ind}_j^*(w_{\mathbb{Z}}) < \infty$. Proposition 3.70 gives that letters $v_j$ with $\mathrm{ind}_j^*(w_{\mathbb{Z}}) = \infty$ are consistent for the types 1) - 4). □

## 3.4 Formulation of Main Theorem

We have now considered all components which are necessary to state the main theorem of this thesis. The proof of it will be given later (Chapter 6, Theorem 6.10).
For an easier formulation, we denote by $\mathcal{W}_i$ the following set of words, and by $\mathcal{C}_i$ the following category ($i = 1, 2, 3, 4$):

| $i$ | $\mathcal{W}_i$ | $\mathcal{C}_i$ |
|---|---|---|
| 1 | asymmetric strings | $\operatorname{mod} \mathrm{k}$ |
| 2 | symmetric strings | $\operatorname{mod} \mathrm{k}[f \mid f^2 = f]$ |
| 3 | asymmetric bands | $\operatorname{mod} \mathrm{k}[T, T^{-1}]$ |
| 4 | symmetric bands | $\operatorname{mod} \mathrm{k}\langle e, f \mid e^2 = e, f^2 = f\rangle$ |

Furthermore, we denote by $\mathcal{V}_i$ a complete set of all finite dimensional, pairwise non-isomorphic indecomposable modules of $\mathcal{C}_i$, $i = 1, 2, 3, 4$.
Let $w \in \mathcal{W}_1$, $V$ be a $\mathcal{C}_1$-module. Then we denote by $\mathcal{M}_1(w, V)$ the following module:

$$V_0 \xleftarrow{w_1^{\kappa_1}} V_1 \xleftarrow{w_2^{\kappa_2}} V_2 \xleftarrow{w_3^{\kappa_3}} \cdots \xleftarrow{w_n^{\kappa_n}} V_n \ ,$$

where

$$\kappa_i = \begin{cases} 1 & \text{if } (w[< i])^{-1} > w[> i], \\ -1 & \text{else,} \end{cases}$$

for all $i \in \mathrm{I}$ with $w_i$ a special letter, and where the $V_i$'s are disjoint copies of $V$.
Let $w = u\varepsilon^* u^{-1} \in \mathcal{W}_2$ and let $V$ be a $\mathcal{C}_2$–module. Then we denote by $\mathcal{M}_2(w, V)$ the following module:

$$\begin{tikzcd} V_0 & V_1 \arrow[l, "w_1^{\kappa_1}"'] & V_2 \arrow[l, "w_2^{\kappa_2}"'] & \cdots \arrow[l, "w_3^{\kappa_3}"'] & V_m \arrow[l, "w_m^{\kappa_m}"'] \arrow[loop right, "\varepsilon=f"] \end{tikzcd} ,$$

where

$$\kappa_i = \begin{cases} 1 & \text{if } (w[< i])^{-1} > w[> i], \\ -1 & \text{else,} \end{cases}$$

for all $1 \leq i \leq m$, with $w_i$ a special letter, and where the $V_i$'s are disjoint copies of $V$.
Let $w_{\mathbb{Z}} \in \mathcal{W}_3$ be of period $p$, an let $V$ be a $\mathcal{C}_3$–module. We denote by $\mathcal{M}_3(v, V)$ the module

$$\begin{tikzcd} V_0 \arrow[rrrr, bend right, "w_p^{\kappa_p}"'] & V_1 \arrow[l, "w_1^{\kappa_1}"'] & V_2 \arrow[l, "w_2^{\kappa_2}"'] & \cdots \arrow[l, "w_3^{\kappa_3}"'] & V_{p-1} \arrow[l, "w_{p-1}^{\kappa_{p-1}}"'] \end{tikzcd} ,$$

where

$$\kappa_i = \begin{cases} 1 & \text{if } (w_{\mathbb{Z}}[< i])^{-1} > w_{\mathbb{Z}}[> i], \\ -1 & \text{else,} \end{cases}$$

for $i \in \mathbb{Z}$ with $w_i$ a special letter, and where the $V_i$'s are disjoint copies of $V$.
Let $w_{\mathbb{Z}} \in \mathcal{W}_4$ be of period $p$ with periodic part $\hat{w}_p = \varepsilon^* u \eta^* u^{-1}$, and let $V$ be a $\mathcal{C}_4$–module. We denote by $\mathcal{M}_4(v, V)$ the module

$$\begin{tikzcd} V_0 \arrow[loop left, "\varepsilon=e"] & V_1 \arrow[l, "w_2^{\kappa_2}"'] & V_2 \arrow[l, "w_3^{\kappa_3}"'] & \cdots \arrow[l, "w_4^{\kappa_4}"'] & V_m \arrow[l, "w_{m+1}^{\kappa_{m+1}}"'] \arrow[loop right, "\eta=f"] \end{tikzcd},$$

where

$$\kappa_i = \begin{cases} 1 & \text{if } (w_{\mathbb{Z}}[< i])^{-1} > w_{\mathbb{Z}}[> i], \\ -1 & \text{else,} \end{cases}$$

for all $2 \le i \le m+1$ with $w_i$ a special letter, and where the $V_i$'s are disjoint copies of $V$.
The exponents $\kappa_i$ with $i \in \mathrm{I}$ such that $w_i$ is an ordinary letter, are given by the respective exponents in $w$ or $w_{\mathbb{Z}}$, respectively, in the above descriptions of $\mathcal{M}_i(w, V)$, $\mathcal{M}_i(w_{\mathbb{Z}}, V)$, respectively. Our final classification result reads as follows:

**Main Theorem:** Let $\Lambda$ be a clannish algebra. The modules of the form $M_i(w, V)$, $i = 1, 2, 3, 4$, with $w$ running through $\mathcal{W}_i$ and $V$ running through $\mathcal{V}_i$, give a complete list of finite dimensional, pairwise non-isomorphic indecomposable modules of $\Lambda$.

**Remark 3.72.** *We can see by the definition of $\mathcal{M}_1(w, V)$ and $\mathcal{M}_3(w, V)$ that we actually consider modules arising from weakly consistent directed versions of $w$ for the asymmetric cases. Recall that they coincide with consistent directed versions for those two cases.*
*Similarly, we know by Theorem 3.61 that the set of weakly consistent and consistent directed versions of symmetric strings coincide. We consider those versions here, too.*
*The case of a symmetric band is - as before - the most complicated one and requires further investigation. On first sight it might seem to behave similar to the other cases with respect to its directed versions, but we need to be careful here: the directions chosen on the symmetry axes are hidden in the action of $e$ and $f$ on the vector space.*

# 4 Formulation of a matrix problem

In this chapter, we reduce the setting to skewed-gentle algebras (see Section 4.3) and give an explicit construction of a bundle of semichains for a given skewed-gentle algebra. In order to do so, we first present in Section 4.1 the results and constructions on representations of bundles of semichains $\overline{\mathfrak{X}}$ from [Bon88, Bon91]. We introduce in particular the sets $\bar{\mathfrak{S}}(\mathfrak{L})$ of so-called simple, admissible $\mathfrak{L}$–chains, and the set of so-called simple $\mathfrak{L}$–cycles $\mathring{\mathfrak{S}}(\mathfrak{L})$. Starting from those sets, [Bon91] gives a nice construction of the canonical $\overline{\mathfrak{X}}$–representations which finally lead to a classification of those. By transforming our classification problem into the classification problem presented in those papers, we are able to use their classification results for our purposes. Before doing so, we give an explicit description of the category $\mathrm{Rep}(\overline{\mathfrak{X}})$ of representations of the bundle of semichains $\overline{\mathfrak{X}}$ (cf. Section 4.2).
We proceed in Section 4.4 with the above mentioned construction of a bundle of semichains $\overline{\mathfrak{X}}_\Lambda$ for a given skewed-gentle algebra $\Lambda$ and thus prove that such a bundle always exists for a skewed gentle algebra (Theorem 4.70). Furthermore, our method describes a unique way of constructing the bundle. In Section 4.5 and 4.6, we describe how to obtain $\mathfrak{L}$–graphs from words in $\Gamma_{\mathrm{ud}}(\Lambda)$. We find in particular that words given by asymmetric and symmetric strings result in simple, admissible $\mathfrak{L}$–chains (Theorem 4.113). Similarly, we can obtain such $\mathfrak{L}$–chains from symmetric bands (Theorem 4.113). We also see that asymmetric and symmetric bands result in simple $\mathfrak{L}$–cycles (Theorem 4.130). To this end, the notion of minimal and coadmissible words becomes important (cf. Chapter 2). Finally, we are able to show that there exists a 1-1-correspondence between strings and symmetric bands and the isomorphism classes of simple, admissible $\mathfrak{L}$–chains (Corollary 4.117). Similarly, we see that there exists a 1-1-correspondence between asymmetric and symmetric bands and the isomorphism classes of simple $\mathfrak{L}$–cycles (Corollary 4.142).
Another advantage of our construction is described in Section 4.7: we show here that the directions on $\mathfrak{L}$–graphs coincide with the directions on special letters of finite index due to the way we construct the bundle. Chapter 5 will show in which way this result helps us for the classification of the indecomposable modules.

## 4.1 Matrix problem for bundles of semichains

We introduce in this section the notion of a bundle of semichains and their representations after [Bon91]. Therefore, we stay close to the structure of [Bon91]. Furthermore, we outline the strategy for the classification of the indecomposable representations of bundles of semichains given in [Bon91] and proven in [Bon88]. We also cite useful properties with respect to this classification from [Bon88].

We start with some definitions and notation.

### 4.1.1 Bundles of semichains

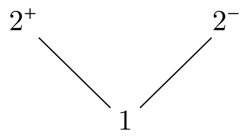
**Definition 4.1.** *Let* $\Pi$ *be a finite partially ordered set. Then* $\Pi$ *is a* semichain *if it can be written as a disjoint union*

$$\Pi = \bigsqcup_{i=1}^{k} \Pi_i,$$

*such that each* $\Pi_i$ *consists of either one element or two incomparable elements and such that for all* $x \in \Pi_i$ *and* $y \in \Pi_j$ *with* $i < j$ *one has* $x < y$.
*The sets* $\Pi_i$ *are called* links *of* $\Pi$.
*We denote the elements of a two-point link* $\Pi_i$ *by* $x^+$, $x^-$, *respectively, and write* $x^+ \lessgtr x^-$ *to denote their incomparability.*

**Remark 4.2.** *If each* $\Pi_i$ *consists of one point, then* $\Pi$ *is called a* chain.

**Example 4.3.** *The semichain* $\Pi = \Pi_1 \cup \Pi_2$ *with* $\Pi = \{1\}$, $\Pi_2 = \{2^+, 2^-\}$ *can be depicted in a Hasse diagram in the following way:*




Denote by $\mathfrak{X} = \{\mathfrak{C}_1, \dots, \mathfrak{C}_N, \mathfrak{R}_1, \dots, \mathfrak{R}_N\}$ a family of pairwise disjoint semichains, that is,

$$\begin{aligned} \mathfrak{C}_i \cap \mathfrak{C}_j &= \varnothing & \forall i \neq j, \\ \mathfrak{R}_i \cap \mathfrak{R}_j &= \varnothing & \forall i \neq j, \\ \mathfrak{C}_i \cap \mathfrak{R}_j &= \varnothing & \forall i, j. \end{aligned}$$

Moreover, denote by

$$\mathfrak{C} = \bigcup_{i=1}^{N} \mathfrak{C}_i, \quad \mathfrak{R} = \bigcup_{i=1}^{N} \mathfrak{R}_i$$

the union of the respective semichains.

**Definition 4.4.** *We call elements of* $\mathfrak{C}$ column labels *and those of* $\mathfrak{R}$ row labels. *Accordingly, any* $\mathfrak{C}_i$ *and* $\mathfrak{C}$ *itself, any* $\mathfrak{R}_i$, $\mathfrak{R}$ *itself is called a* column label set *and* row label set, *respectively.*

Denote by $\mathfrak{L}(\mathfrak{C}_i)$ or $\mathfrak{L}(\mathfrak{R}_i)$ the set of links of the semichain $\mathfrak{C}_i$ or $\mathfrak{R}_i$, respectively. The union of the sets $\mathfrak{L}(\mathfrak{C}_1), \dots, \mathfrak{L}(\mathfrak{C}_n)$ is denoted by $\mathfrak{L}(\mathfrak{C})$, and the union of $\mathfrak{L}(\mathfrak{R}_1), \dots, \mathfrak{L}(\mathfrak{R}_n)$ similarly by $\mathfrak{L}(\mathfrak{R})$. Moreover, we set $\mathfrak{L} = \mathfrak{L}(\mathfrak{X}) = \mathfrak{L}(\mathfrak{C}) \cup \mathfrak{L}(\mathfrak{R})$. Denote by $\mathfrak{X}_0 = \mathfrak{C} \cup \mathfrak{R}$ the set of elements of $\mathfrak{C}$ and $\mathfrak{R}$. Let $\sigma$ be an involution (that is, $\sigma^2 = \mathrm{id}$) on $\mathfrak{X}_0$ such that $\sigma(X) = X$ for all elements $X$ belonging to a two-point link.

**Definition 4.5.** *A* bundle of semichains *is a pair* $\overline{\mathfrak{X}} = (\mathfrak{X}, \sigma)$ *where* $\mathfrak{X} = \{\mathfrak{C}_1, \ldots, \mathfrak{C}_N, \mathfrak{R}_1, \ldots, \mathfrak{R}_N\}$ *is a family of pairwise disjoint semichains and* $\sigma$ *an involution on the set* $\mathfrak{X}_0$ *of elements of the semichains.*

**Example 4.6.** *Let* $N = 2$, $\mathfrak{X} = \{\mathfrak{C}_1, \mathfrak{C}_2, \mathfrak{R}_1, \mathfrak{R}_2\}$, *where* $\mathfrak{C}_1 = \{\mathfrak{C}_{11}^+ \lessgtr \mathfrak{C}_{11}^-\}$, $\mathfrak{C}_2 = \{\mathfrak{C}_{21}\}$, $\mathfrak{R}_1 = \{\mathfrak{R}_{11} > \mathfrak{R}_{12}\}$, $\mathfrak{R}_2 = \{\mathfrak{R}_{12}\}$. *Let* $\sigma$ *be given by*

$$\sigma : \mathfrak{R}_{11} \mapsto \mathfrak{C}_{21}, \quad \mathfrak{R}_{12} \mapsto \mathfrak{R}_{21}, \quad \mathfrak{C}_{11}^{\zeta} \mapsto \mathfrak{C}_{11}^{\zeta},\ \zeta \in \{+,-\}.$$

*Then* $\overline{\mathfrak{X}} = (\mathfrak{X}, \sigma)$ *is a bundle of semichains.*

**Example 4.7.** *Let* $N = 1$, $\mathfrak{C}_1 = \{\mathfrak{C}_{11}^+ \lessgtr \mathfrak{C}_{11}^-\}$, $\mathfrak{R}_1 = \{\mathfrak{R}_{11} > \mathfrak{R}_{12} > \mathfrak{R}_{13}\}$ *and* $\sigma$ *acts as follows:*

$$\begin{aligned} &\mathfrak{C}_{11}^{\zeta} \leftrightarrow \mathfrak{C}_{11}^{\zeta}, \quad \zeta \in \{+,-\}, \\ &\mathfrak{R}_{12} \leftrightarrow \mathfrak{R}_{12}, \\ &\mathfrak{R}_{11} \leftrightarrow \mathfrak{R}_{13}. \end{aligned}$$

*This gives for* $\mathfrak{X} = \{\mathfrak{C}_1, \mathfrak{R}_1\}$ *the bundle* $\overline{\mathfrak{X}} = \{\mathfrak{X}, \sigma\}$.

**Definition 4.8.** *A* representation *of the bundle* $\overline{\mathfrak{X}} = (\mathfrak{X}, \sigma)$ *over* k *is given by a collection of the form* $U = (U_X, U^i)_{X \in \mathfrak{X}_0, 1 \le i \le N}$ *such that*

- $U_X$ *is a finite dimensional* k*−vector space of the form* $\mathrm{k}^{n_X}$ *where* $n_X = \dim(U_X)$ *and such that* $\dim(U_X) = \dim(U_{\sigma(X)})$.
- $U^i : \bigoplus_{C \in \mathfrak{C}_i} U_C \longrightarrow \bigoplus_{R \in \mathfrak{R}_i} U_R$ *is a* k*−linear map for each* $1 \le i \le N$. *Equivalently,* $U^i$ *can be expressed as a finite matrix with band structure given as follows:*
  - · *horizontal bands are indexed by the elements* $R \in \mathfrak{R}_i$,
  - · *vertical bands are indexed by the elements* $C \in \mathfrak{C}_i$.

  *We denote by* $P(X)$ *the band indexed by* $X \in \mathfrak{X}_0$. *Then*

  - · *the band* $P(X)$ *has* $\dim(U_X)$ *rows (columns) if* $X \in \mathfrak{R}$ *(*$X \in \mathfrak{C}$*).*

**Remark 4.9.** *Note that for some elements* $X \in \mathfrak{X}_0$ *the corresponding band* $P(X)$ *may be empty.*

We are going to see examples for $\overline{\mathfrak{X}}$–representations in Subsection 4.1.4.

**Definition 4.10.** *Let* $U = (U_X, U^i)_{X,i}$ *and* $V = (V_X, V^i)_{X,i}$ *be two representations of* $\overline{\mathfrak{X}}$. *Then* $U$ *and* $V$ *are said to be* equivalent *if for any* $i \in \{1, \ldots, N\}$ *the matrices* $U^i$ *and* $V^i$ *can be obtained one from the other by a sequence of transformations of the following types:*

1. *Perform an arbitrary elementary transformation of the rows (respectively columns) within the band $P(X)$ in $U^i$, $X \in \mathfrak{R}_i$ ($X \in \mathfrak{C}_i$), for any $1 \le i \le N$, but*

   1a. *if $\sigma(X) = Y$ with $X \in \mathfrak{R}_i$, $Y \in \mathfrak{R}_j$ ($X \in \mathfrak{C}_i$, $Y \in \mathfrak{C}_j$), for some $1 \le j \le N$, then perform the same transformation on the band $P(Y)$ in $U^j$.*

   1b. *if $\sigma(X) = Y$ with $X \in \mathfrak{R}_i$, $Y \in \mathfrak{C}_j$ ($X \in \mathfrak{C}_i$, $Y \in \mathfrak{R}_j$), for some $1 \le j \le N$, then perform the inverse transformation on the band $P(Y)$ in $U^j$.*

2. *For $X < Y$ in $\mathfrak{R}_i$ ($\mathfrak{C}_i$) for some $1 \le i \le N$, add a multiple of the band $P(X)$ to the band $P(Y)$ within the matrix $U^i$.*

*Transformations of the types 1,1a,1b and 2 are called* admissible.

**Remark 4.11.** *We will make use of admissible transformations in particular in Section 4.2 in order to define the category of representations of a bundle of semichains.*

**Definition 4.12.** *Let $U$ and $W$ be two representations of the bundle $\overline{\mathfrak{X}}$. Then the* direct sum $U \oplus W$ *of the two representations is defined as follows:*

- *The vector space $(U \oplus W)_X$ for $X \in \mathfrak{X}_0$ is given by $U_X \oplus W_X$.*
- *The matrix $(U \oplus W)^i$ for $i \in \{1, \dots, N\}$ is given by*

$$U^i \oplus W^i : \left( \bigoplus_{C \in \mathfrak{C}_i} U_C \right) \oplus \left( \bigoplus_{C \in \mathfrak{C}_i} W_C \right) \xrightarrow{\begin{pmatrix} U^i & 0 \\ 0 & W^i \end{pmatrix}} \left( \bigoplus_{R \in \mathfrak{R}_i} U_R \right) \oplus \left( \bigoplus_{R \in \mathfrak{R}_i} W_R \right).$$

**Remark 4.13.** *Note that the intuitive definition of a direct sum of two representations would be to define $U^i \oplus W^i$ as maps*

$$\bigoplus_{C \in \mathfrak{C}_i} (U_C \oplus W_C) \longrightarrow \bigoplus_{R \in \mathfrak{R}_i} (U_R \oplus W_R).$$

*We use the natural isomorphism between these direct sums and the respective ones given in Definition 4.12 in order to define them as we did.*

**Definition 4.14.** *Let $U$ be a representation of the bundle $\overline{\mathfrak{X}}$. Then $U$ is* indecomposable *if it is not equivalent to the direct sum of two non-trivial representations of $\overline{\mathfrak{X}}$.*

### 4.1.2 $\mathfrak{L}$–graphs

The aim of this subsection is to define $\mathfrak{L}$– graphs. From those we construct in the next subsection certain representations of $\overline{\mathfrak{X}}$.

Recall that $\mathfrak{L} = \mathfrak{L}(\mathfrak{X})$ denotes the set of links arising from the semichains in $\mathfrak{X}$. In the following we identify links consisting of one point with the corresponding point itself. The number of points of a link $X$ is denoted by $r(X)$.
We are going to consider the following two symmetric binary relations on $\mathfrak{L}$:

**Definition 4.15.** *a) The relation $\alpha \subseteq \mathfrak{L} \times \mathfrak{L}$ is given by the tuples $(X, Y)$ such that*

$$\begin{array}{llll} \textit{either} & X \neq Y, & r(X) = r(Y) = 1, & \sigma(X) = Y, \\ \textit{or} & X = Y, & r(X) = 2. & \end{array}$$

*If $(X, Y) \in \alpha$, we write $X\alpha Y$.*

*b) The relation $\beta \subseteq \mathfrak{L} \times \mathfrak{L}$ is given by the tuples $(X, Y)$ such that*

$$\begin{array}{llll} \textit{either} & X \in \mathfrak{L}(\mathfrak{C}_i), & Y \in \mathfrak{L}(\mathfrak{R}_i), & \\ \textit{or} & X \in \mathfrak{L}(\mathfrak{R}_i), & Y \in \mathfrak{L}(\mathfrak{C}_i), & \textit{for some } 1 \leq i \leq N. \end{array}$$

For $\gamma \in \{\alpha, \beta\}$, we say that two links $X, Y \in \mathfrak{L}$ are in *$\gamma$–relation*, if $X\gamma Y$ holds. We write $X\bar{\gamma}Y$ if $X$ and $Y$ are not in $\gamma$–relation.

**Remark 4.16.** *Let $X \in \mathfrak{L}$. It follows by definition:*

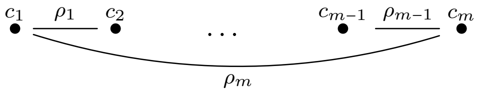
$$\begin{array}{l} X\alpha X \textit{ if and only if } X \textit{ is a two-point link}, \\ X\bar{\alpha}X \textit{ if and only if } X \textit{ is a one-point link.} \end{array}$$

**Example 4.17.** *Consider the bundle of semichains given in Example 4.7. Let $\mathfrak{C}_{11}$ denote the link containing the two points $\mathfrak{C}_{11}^{+}$ and $\mathfrak{C}_{11}^{-}$. We identify the other links only consisting of one point with the points themselves. Then we obtain the following relations:*

$$\begin{array}{ll} \mathfrak{R}_{1j}\beta\mathfrak{C}_{11} & \textit{for any } 1 \leq j \leq 3, \\ \mathfrak{C}_{11}\alpha\mathfrak{C}_{11}, & \\ \mathfrak{R}_{11}\alpha\mathfrak{R}_{13}, & \\ \mathfrak{R}_{12}\bar{\alpha}X & \textit{for any } X \in \mathfrak{C}_1 \cup \mathfrak{R}_1. \end{array}$$

Next, we define an $\mathfrak{L}$–graph. To this end, let $\Omega$ be the set of finite non-oriented graphs consisting of chains and cycles of the form

$$\overset{c_1}{\bullet} \xrightarrow{\ \rho_1\ } \overset{c_2}{\bullet} \qquad \ldots \qquad \overset{c_{m-1}}{\bullet} \xrightarrow{\ \rho_{m-1}\ } \overset{c_m}{\bullet} \qquad\qquad m \geq 1$$



$m \geq 2$

Denote in the following by

$$\tilde{m} = \begin{cases} m-1 & \text{if } C \text{ is a chain of length } m, \\ m & \text{if } C \text{ is a cycle of length } m. \end{cases}$$

the number of edges in the graph $C$. For cycles, we consider in the following indices $i > m$ modulo $m$.

**Definition 4.18.** *Let $C = (c_i, \rho_j)_{1 \leq i \leq m-1, 1 \leq j \leq \tilde{m}} \in \Omega$. An $\mathfrak{L}$–*graph *on $C$ is a function $g : C \to \mathfrak{L} \cup \{\alpha, \beta\}$ with*

$$\begin{aligned} g(c_i) &\in \mathfrak{L}, && \text{for all } 1 \leq i \leq m, \\ g(\rho_j) &\in \{\alpha, \beta\}, && \text{for all } 1 \leq j \leq \tilde{m}, \end{aligned}$$

*such that the following holds:*

*a) If $\rho_i$ connects the nodes $c_i$ and $c_{i+1}$ in $C$, then $g(c_i) g(\rho_i) g(c_{i+1})$.*

*b) For $\rho_i$ and $\rho_{i+1}$ neighbouring edges in $C$, then $g(\rho_i) \neq g(\rho_{i+1})$.*

*c) If $C$ is a cycle, and $g(c_m)$, $g(c_1)$ are one-point links, then $g(\rho_m) \neq \alpha$.*

*The* length *of an $\mathfrak{L}$–graph $g$ is denoted by $|g|$ and given by the number $m$ of nodes in $C$.*

**Remark 4.19.** *Note that we have added condition c) in Definition 4.18 in contrast to the original [Bon91]. This condition has to be given in order to guarantee a correct notion of equivalence.*

**Definition 4.20.** *We call an $\mathfrak{L}$–graph $g$ on a chain $C$ an $\mathfrak{L}$–*chain *and an $\mathfrak{L}$–graph $g$ on a cycle $C$ an $\mathfrak{L}$–*cycle*.*

We denote in the following $x_i = g(c_i)$ and $\lambda_{i,i+1} = g(\rho_i)$. Thus, we can describe an $\mathfrak{L}$–graph $g$ uniquely by the sequences $g_0 = \{x_0, \ldots, x_m\}$ and $g_1 = \{\lambda_{12}, \ldots, \lambda_{m-1,m}\}$ ($g_0 = \{\lambda_{12}, \ldots, \lambda_{m,1}\}$, respectively). Using for an $\alpha$–relation the symbol $\sim$, and for a $\beta$–relation the symbol $-$, we can depict an $\mathfrak{L}$–cycle as

$$x_1 \sim x_2 - \cdots - x_{m-1} \sim x_m \qquad \text{or} \qquad x_1 - x_2 \sim \cdots \sim x_{m-1} - x_m$$

depending on $g_1$. Recall that the second kind of depiction only is given if $x_1$ and $x_m$ are not one-point links. We depict an $\mathfrak{L}$–chain in a similar way.

**Example 4.21.** *Let $\overline{\mathfrak{X}}$ be given as in Example 4.7 with relations as in Example 4.17. Then*

1. $g : \mathfrak{R}_{12} - \mathfrak{C}_{11} \sim \mathfrak{C}_{11} - \mathfrak{R}_{13} \sim \mathfrak{R}_{11} - \mathfrak{C}_{11}$ *is an* $\mathfrak{L}$*–chain, and*

2. *the following is an* $\mathfrak{L}$*–cycle:*

$$g : \; \mathfrak{C}_{11} \text{ - } \mathfrak{R}_{13} \sim \mathfrak{R}_{11} \text{ - } \mathfrak{C}_{11} \sim \mathfrak{C}_{11} \text{ - } \mathfrak{R}_{11} \sim \mathfrak{R}_{13} \text{ - } \mathfrak{C}_{11}.$$

Using the description of an $\mathfrak{L}$–graph $g$ via the two sequences $g_0$ and $g_1$, it is easy to define the corresponding reversed graph $g^*$, or to compare $g$ to a second graph:

**Definition 4.22.** *Let $g$ be an $\mathfrak{L}$–graph with sequence of links $g_0 = \{x_1, \dots, x_m\}$ and sequence of relations $g_1 = \{\lambda_{12}, \dots, \lambda_{m-1,m}\}$ ($g_0 = \{\lambda_{12}, \dots, \lambda_{m,1}\}$, respectively). Its* reversed $\mathfrak{L}$–graph *$g^*$ is given by*

$$g_0^* = \{x_m, \dots, x_1\},$$
$$g_1^* = \begin{cases} \{\lambda_{m-1,m}, \dots, \lambda_{1,2}\} & \text{if } g \text{ an } \mathfrak{L} - \text{chain}, \\ \{\lambda_{m-1,m}, \dots, \lambda_{1,2}, \lambda_{m,1}\} & \text{if } g \text{ an } \mathfrak{L} - \text{cycle}. \end{cases}$$

**Definition 4.23.** *Let $g$ and $h$ be two $\mathfrak{L}$–graphs given by $g_0 = \{x_1, \dots, x_m\}$, $g_1 = \{\lambda_{12}, \dots, \lambda_{i,j}\}$ and $h_0 = \{y_1, \dots, y_{m'}\}$, $h_1 = \{\mu_{12}, \dots, \mu_{i',j'}\}$ where $(i,j) = (m-1, m)$ if $g$ is an $\mathfrak{L}$-chain and $(i,j) = (m,1)$ otherwise, $(i',j') = (m'-1, m')$ if $h$ is an $\mathfrak{L}$–chain and $(i',j') = (m',1)$ otherwise. Then $g = h$ if*

*(i)* $m = m'$,

*(ii)* $x_i = y_i$ *for all* $1 \le i \le m$,

*(iii)* $\lambda_{ij} = \mu_{ij}$ *for all* $1 \le i \le \tilde{m}$, $2 \le j \le m$ $(1 \le j \le m)$

**Example 4.24.** *Consider the $\mathfrak{L}$–graphs from Example 4.21. Then we obtain the following reversed graphs:*

1. $g^* = \mathfrak{C}_{11} - \mathfrak{R}_{11} \sim \mathfrak{R}_{13} - \mathfrak{C}_{11} \sim \mathfrak{C}_{11} - \mathfrak{R}_{12}$,

2. $g^* = g$.

We define next two properties of $\mathfrak{L}$–graphs. They are called *admissibility* and *simplicity*. Later, those properties will be useful in order to define so called canonical representations and in order to describe their constructions.

**Definition 4.25.** *An $\mathfrak{L}$–chain $g$ is called* admissible *if the following holds: if there exist $X, Y \in \mathfrak{L}$, $X \neq Y$ with $X \alpha Y$, and $c_i \in C$ with $g(c_i) = X$, then there exists an edge $\rho \in C$ containing $c_i$ (i.e., $\rho = \rho_i$ or $\rho = \rho_{i-1}$) with $g(\rho) = \alpha$.*

**Remark 4.26.** *This condition holds for any $\mathfrak{L}$–cycle.*

**Example 4.27.** *Both $\mathfrak{L}$–graphs from Example 4.21 are admissible. For the same bundle $\overline{\mathfrak{X}}$, the following $\mathfrak{L}$–chain is not admissible:*

$$g' : \mathfrak{R}_{12} - \mathfrak{C}_{11} \sim \mathfrak{C}_{11} - \mathfrak{R}_{13}.$$

*This is due to $x_4 = \mathfrak{R}_{13}$ being a one-point link which is in $\alpha$–relation with the one-point link $\mathfrak{R}_{11}$.*

**Remark 4.28.** *As we see in the above example, admissible $\mathfrak{L}$–chains are allowed to have two-point links as end points without any restriction (see Example 4.21.1).*
*The critical one-point links are those being in $\alpha$–relation with other one-point links. Hence, one-point links which are not in any $\alpha$–relation can also be end points in an admissible $\mathfrak{L}$–chain.*

The notion of simplicity is given in terms of isomorphisms.

**Definition 4.29.** *A* homomorphism $\tau : C \to C'$ of graphs *is given by a map on the nodes $c_i$ which preserves adjacency. That is, two neighbours $c_i$, $c_{i+1}$ in $C$ are sent to neighbours $c'_j = \tau(c_i)$, $c'_{j\pm1} = \tau(c_{i+1})$ in $C'$.*
*If the homomorphism $\tau$ is bijective, it is called* isomorphism.

**Definition 4.30.** *Let $\tau : C \to C'$ be an isomorphism of graphs and let $g$, $g'$ be two $\mathfrak{L}$–graphs defined on $C$, $C'$, respectively. Then $\tau$ is an* isomorphism of $\mathfrak{L}$–graphs *if $g = g'\tau$.*
*We denote the* group of automorphisms *on an $\mathfrak{L}$–graph $g$ by* $\mathrm{Aut}(g)$.
*A* rotation *for an $\mathfrak{L}$–cycle $g$ is given by an automorphism $\tau$ for which there exists $k \in \mathbb{Z}$ such that $\tau(c_i) = c_{i+k}$ for all $1 \leq i \leq m$. The group of rotations on $g$ is denoted by* $\mathrm{Rot}(g)$.

**Definition 4.31.** *We call an $\mathfrak{L}$–chain $g$* symmetric *if $\mathrm{Aut}(g)$ is not trivial. An $\mathfrak{L}$–cycle $g$ is called* symmetric *if the quotient group $\mathrm{Aut}(g)/\mathrm{Rot}(g)$ is not trivial.*

**Definition 4.32.** *We call an $\mathfrak{L}$–cycle $g$* simple *if* $\mathrm{Rot}(g) = \{\mathrm{id}\}$.

**Example 4.33.**
1. *Consider the $\mathfrak{L}$–cycle from Example 4.21.2. Then its underlying graph $C_g$ is of the form*

$$\underbrace{c_1 - c_2 - c_3 - c_4 - c_5 - c_6 - c_7 - c_8}.$$

*The automorphism group of $g$ is given by $\mathrm{Aut}(g) = \{\mathrm{id}, \tau\}$ where $\tau$ is an isomorphism on $C_g$, with the action*

$$c_4 \leftrightarrow c_5, \quad c_3 \leftrightarrow c_6, \quad c_2 \leftrightarrow c_7, \quad c_1 \leftrightarrow c_8.$$

*The morphism $\tau$ is clearly not a rotation, so $\mathrm{Rot}(g) = \{\mathrm{id}\}$. Thus, $g$ is symmetric and simple.*

2. *Consider for the same bundle as before the cycle*

$$g: \ \mathfrak{C}_{11} - \mathfrak{R}_{13} \sim \mathfrak{R}_{11} - \mathfrak{C}_{11} \sim \mathfrak{C}_{11} - \mathfrak{R}_{11} \sim \mathfrak{R}_{13} - \mathfrak{C}_{11}$$

   *It has the same underlying graph as given in 1. For* $\tau_{\pm 3} : c_i \mapsto c_{i\pm 3}$ *we have* $\operatorname{Aut}(g) = \operatorname{Rot}(g) = \{\tau_{\pm 3}, \mathrm{id}\}$. *Thus, $g$ is neither symmetric nor simple.*

3. *Consider again the same bundle as before. Let*

$$g = \mathfrak{C}_{11} - \mathfrak{R}_{13} \sim \mathfrak{R}_{11} - \mathfrak{C}_{11} \sim \mathfrak{C}_{11} - \mathfrak{R}_{13} \sim \mathfrak{R}_{11} - \mathfrak{C}_{11}$$

   *be an $\mathfrak{L}$−chain. It is symmetric since*

$$\tau : c_i \leftrightarrow c_{9-i}, \quad 1 \leq i \leq 4$$

   *gives a non-trivial automorphism on $g$ for the underlying graph*

$$C_g : c_1 - c_2 - c_3 - c_4 - c_5 - c_6 - c_7 - c_8.$$

**Remark 4.34.** *It is easy to see ([Bon88, §2]) that for a simple $\mathfrak{L}$−cycle $g$ one has* $|\operatorname{Aut}(g)| \leq 2$.
*If $g$ is additionally symmetric, then* $|\operatorname{Aut}(g)| = 2$.

The notion of simplicity for $\mathfrak{L}$−chains is given in different terms. In order to give this definition, we first introduce so called double ends for $\mathfrak{L}$−chains.

**Definition 4.35.** *Let $g$ be an $\mathfrak{L}$−chain of length $m > 1$. The left end $x_1$ of $g$ is called* double *if $\lambda_{12} = \beta$ and $x_1 \alpha x_1$. Analogously, the right end $x_m$ of $g$ is called* double *if $\lambda_{m-1,m} = \beta$ and $x_m \alpha x_m$. We denote the number of double ends of $g$ by* $\mathrm{d}(g)$.

**Remark 4.36.** *Note that in the case of $m = 1$, we define $\mathrm{d}(g) = 1$ if $x_1 \alpha x_1$.*

Let $h$ be an $\mathfrak{L}$−chain with $\mathrm{d}(h) = 2$. Then the $\mathfrak{L}$−chain $h^{[k]}$ for $k > 0$ is given by

$$h^{(1)} \sim h^{(2)} \sim \cdots \sim h^{(k)},$$

where

$$h^{(i)} = \begin{cases} h & \text{if } i \text{ odd}, \\ h^* & \text{if } i \text{ even}. \end{cases}$$

Note that we can also construct $h^{[2]}$ if only the right end of $h$ is double.

**Definition 4.37.** *Let $g$ be an $\mathfrak{L}$−chain. Then $g$ is called* composite *if there exists an $\mathfrak{L}$−chain $h$ and some $k > 1$ such that $g = h^{[k]}$.*
*Otherwise, $g$ is called* simple.

**Example 4.38.** *The $\mathfrak{L}$–chain $g$ from Exmaple 4.33.3. is composite with $k = 2$ and*

$$h = \mathfrak{C}_{11} - \mathfrak{R}_{13} \sim \mathfrak{R}_{11} - \mathfrak{C}_{11}.$$

*The $\mathfrak{L}$–chain $h$ is simple.*

**Lemma 4.39.** *An $\mathfrak{L}$–chain $g$ is symmetric if and only if it is composite of the form $g = h^{[k]}$ with $h$ simple and $k$ even.*

*Proof.* Let $g = h^{[k]}$ be composite with $h$ simple and $k$ even. Let $h$ be of length $n$ and its underlying graph $C_h$ be given by:

$$C_h : c_1 - c_2 - \cdots - c_n.$$

The $\mathfrak{L}$–chain $g$ is thus of length $n' = kn$. The map $\tau : C_g \to C_g$, $c_i \mapsto c_{n'+1-i}$ gives a non-trivial isomorphism of $g$. Hence, $g$ is symmetric.
Conversely, assume $g$ to be symmetric. Let $\tau \in \mathrm{Aut}(g)$ be non-trivial. By definition, the images of neighbouring nodes under $\tau$ are again neighbours. Thus, $\tau$ is either be given by a translation or a reflection. Since $g$ is not a cycle, $\tau$ cannot be given by a translation. Thus, it is given by a reflection. This implies $g$ being composite of form $g = h^{[k]}$, where $k$ is even for $h$ simple. □

**Lemma 4.40.** *An $\mathfrak{L}$–chain $g$ is simple (admissible) if and only if its reversed $\mathfrak{L}$–chain $g^*$ is simple (admissible).*

*Proof.* Follows from the definition of the reversed $\mathfrak{L}$–chain. □

**Definition 4.41.** *Let $C$ be an $\Omega$–graph and $g$ be an $\mathfrak{L}$–graph on $C$. A* subchain *of $g$ is given by restriction of $g$ to a connected subgraph of $C$.*

**Example 4.42.** *If $g$ is composite, say $g = h^{[k]}$ for some $h$, some $k$, then each $h^{(i)}$, $1 \le i \le k$, is a subchain of $g$.*

Finally we are able to denote the sets of $\mathfrak{L}$–chains and $\mathfrak{L}$–cycles which are used to construct the canonical representations of the bundle.

**Definition 4.43.** *We denote by $\bar{\mathfrak{S}}(\mathfrak{L})$ the set of simple admissible $\mathfrak{L}$–chains and by $\mathring{\mathfrak{S}}(\mathfrak{L})$ the set of simple $\mathfrak{L}$–cycles.*

### 4.1.3 Subchains via orientations

We set in this subsection the prerequisites for the construction of a representation of the bundle from each $\mathfrak{L}$–graph. To this end, we introduce four types of subchains which determine the maps of the representations. Subchains of these types will be called elementary subchains.

From now on consider an $\mathfrak{L}$–chain $g \in \bar{\mathfrak{S}}(\mathfrak{L})$ with $g_0 = \{x_1, \dots, x_m\}$. We embed $g$ into another $\mathfrak{L}$–chain denoted by $\tilde{g}$ as follows:

$$\tilde{g} = \begin{cases} g & \text{if } \mathrm{d}(g) = 0, \\ g^* \sim g & \text{if } \mathrm{d}(g) = 1, x_1 \text{ double end}, \\ g \sim g^* & \text{if } \mathrm{d}(g) = 1, x_m \text{ double end}, \\ g^* \sim g \sim g^* & \text{if } \mathrm{d}(g) = 2. \end{cases}$$

Consider in the next step $x_i = x_{i+1}$ for some $i \in \{1, \dots, m\}$. Then, clearly, $x_i \alpha x_{i+1}$, i.e. $\lambda_{i,i+1} = \alpha$. Starting from that relation in $\tilde{g}$, we construct a maximal symmetric subchain $\tilde{g}^{(i)} = w \sim w^*$, that is, the right end of $w$ is given by $x_i$ and the left end of $w^*$ by $x_{i+1}$. Now we consider the neighbours of that subchain in $\tilde{g}$: if $\tilde{g}^{(i)}$ does not contain the left (respectively, right) end of $\tilde{g}$, we denote by $y_i$ (respectively, $z_i$) the element in $\tilde{g}_0$ such that $y_i \beta w$ (respectively, $w^* \beta z_i$) extends $w$ (respectively, $w^*$) to another subchain of $\tilde{g}$. Otherwise, we set $y_i = \infty$ (respectively, $z_i = \infty$).

**Remark 4.44.** *Note that by construction of $\tilde{g}^{(i)}$, we have that $y_i \neq z_i$ (otherwise one obtains a contradiction to the simplicity of $g$) and either $y_i, z_i \in \mathfrak{L}(\mathfrak{C}_k) \cup \infty$, or $y_i, z_i \in \mathfrak{L}(\mathfrak{R}_k) \cup \infty$ for some $k \in \{1, \dots, N\}$. We assume $X < \infty$ for all $X \in \mathfrak{L}$, and thus obtain either $y_i < z_i$, or $z_i < y_i$.*

Some types of the elementary subchains depend on an orientation on the already mentioned subchain $x_i \sim x_{i+1}$ where $x_i = x_{i+1}$. We define this orientation.

**Definition 4.45.** *Let $x_i = x_{i+1} \in g_0$ as above.*
*We write $\overrightarrow{x_i \sim x_{i+1}}$ if one of the following conditions holds:*

*a) $y_i < z_i$ and $x_i \in \mathfrak{L}(\mathfrak{Y})$, $y_i \in \mathfrak{L}(\mathfrak{Y}) \cup \infty$, $\mathfrak{Y} \in \{\mathfrak{C}, \mathfrak{R}\}$,*

*b) $y_i > z_i$ and $x_i \in \mathfrak{L}(\mathfrak{Y})$, $y_i \in \mathfrak{L}(\overline{\mathfrak{Y}}) \cup \infty$, $\mathfrak{Y} \neq \overline{\mathfrak{Y}} \in \{\mathfrak{C}, \mathfrak{R}\}$.*

*We write $\overleftarrow{x_i \sim x_{i+1}}$ if one of the following conditions holds:*

*a) $y_i > z_i$ and $x_i \in \mathfrak{L}(\mathfrak{Y})$, $y_i \in \mathfrak{L}(\mathfrak{Y}) \cup \infty$, $\mathfrak{Y} \in \{\mathfrak{C}, \mathfrak{R}\}$,*

*b) $y_i < z_i$ and $x_i \in \mathfrak{L}(\mathfrak{Y})$, $y_i \in \mathfrak{L}(\overline{\mathfrak{Y}}) \cup \infty$, $\mathfrak{Y} \neq \overline{\mathfrak{Y}} \in \{\mathfrak{C}, \mathfrak{R}\}$.*

**Remark 4.46.** *Note that in case a) we always orient towards the larger link. In case b) we orient the other way around.*

We are now able to finally define the notion of an elementary subchain for $\mathfrak{L}$–chains.

**Definition 4.47.** *Let $g \in \bar{\mathfrak{S}}(\mathfrak{L})$. An* elementary subchain *of $g$ is given by any of its subchains that is of one of the following forms:*

*1)* $x_{i-1} - x_i$,

*2)* $\overrightarrow{x_{i-1} \sim x_i} - x_{i+1}$,

*3)* $x_{i-1} - \overleftarrow{x_i \sim x_{i+1}}$,

*4)* $\overrightarrow{x_{i-1} \sim x_i} - \overleftarrow{x_{i+1} \sim x_{i+2}}$.

Elementary subchains for $\mathfrak{L}$–cycles are defined similarly. Here, we set $\tilde{g} = g$ and obtain elementary subchains as defined above. But we need to take care in the case that $g$ is symmetric: if there exists an automorphism $\tau \neq \mathrm{id}$ with $\tau(c_i) = c_{i+1}$, then the elements $y_i$ and $z_i$ belong to $w \sim w^*$ and coincide. Hence, we cannot use the above construction. In this case, we assume

$$\overrightarrow{x_i \sim x_{i+1}} \quad \text{if } x_i \in \mathfrak{L}(\mathfrak{R}) \tag{60}$$

and

$$\overleftarrow{x_i \sim x_{i+1}} \quad \text{if } x_i \in \mathfrak{L}(\mathfrak{C}). \tag{61}$$

We denote by $e_{i,j}(g) = e_{j,i}(g)$ an elementary subchain of $g$ with ends given by $x_i$ and $x_j$. Note that we read the type of elementary subchain from left to right within the $\mathfrak{L}$–graphs, meaning that $i < j$ (in cylces considered modulo $m$ if necessary).

**Remark 4.48** ([Bon88, Lemma 2]). *An $\mathfrak{L}$–cycle $g$ always contains a maximal elementary subchain of length* 2. *That is a subchain $x_{i-1} - x_i$ which does not belong to any elementary subchain of greater length. We denote by $e_2(g)$ such a subchain with least $i \in \{1, \dots, m\}$.*

### 4.1.4 Construction of representations

In this subsection, we describe the canonical representations which we can construct from $\mathfrak{L}$–graphs in $\bar{\mathfrak{G}}(\mathfrak{L}) \cup \mathring{\mathfrak{G}}(\mathfrak{L})$. We restrict ourselves to the construction of the matrices of the respective representation, since we can conclude any information on the vector spaces from them.
The constructions require a lot of notation and description such that we first give an overview on the representations and a very rough idea on how we can obtain them, before going into a detail.

**Overview on canonical representations.**

where

$$\begin{aligned}
\varphi &= \varphi_0^n,\ \varphi_0 \text{ a monic, irreducible polynomial over } \mathrm{k}, \\
\delta_0(g) &= \delta(g)/2, \\
\delta(g) &= \#\{i \in \{1, \dots, m\} \mid x_i \neq x_{i+1}, x_i, x_{i+1} \in \mathfrak{L}(X), X \in \{\mathfrak{C}, \mathfrak{R}\}\}
\end{aligned}$$

| $\mathfrak{L}$–graph $g$ | | representation $U$ |
|---|---|---|
| | $\mathrm{d}(g) = 0$ | $U_1(g)$ |
| $\mathfrak{L}$–chain | $\mathrm{d}(g) = 1$ | $U_s(g), s = 1, 2$ |
| | $\mathrm{d}(g) = 2$ | $U_s(g, p), s = 1, 2, 3, 4, p \in \mathbb{N}$ |
| | symmetric, $\delta_0(g)$ even | $U(g, \varphi), \varphi_0 \neq t, t + 1$ |
| $\mathfrak{L}$–cycle | symmetric, $\delta_0(g)$ odd | $U(g, \varphi), \varphi_0 \neq t, t - 1$ |
| | non-symmetric | $U(g, \varphi), \varphi_0 \neq t$ |

**Overview on constructions.**
The general idea for constructing a matrix $U^i$, $1 \leq i \leq N$, of a representation $U$ for an $\mathfrak{L}$–graph $g \in \bar{\mathfrak{S}}(\mathfrak{L}) \cup \mathring{\mathfrak{S}}(\mathfrak{L})$ is the following:
The rows and columns of $U^i$ are divided into bands which are indexed by the elements of $\mathfrak{R}_i$ and $\mathfrak{C}_i$, respectively. Each $x_i \in g_0$ being equal to such an element indexes a row/column (or subband in case of a cycle) in the respective band. The entry in the intersection of a row (subband) $x_i$ and a column (subband) $x_j$ is 1 (identity block) if $x_i$ and $x_j$ are the ends of an elementary subchain. Otherwise, the entry is 0.
In detail, the construction is more complicated (e.g. in case of a cycle, there can also exist non-zero entries not given by an identity block) and requires a lot of descriptive notation.
First, we set some general notation:
Let $g \in \bar{\mathfrak{S}}(\mathfrak{L}) \cup \mathring{\mathfrak{S}}(\mathfrak{L})$, $g_0 = \{x_1, \dots, x_m\}$. Denote

$$g_0^\alpha = \{x_i \in g_0 \mid x_i \alpha x_i\}$$

and denote by $\Psi(g)$ the set of maps of the form

$$\psi : g_0^\alpha \to \{+1, -1\}$$

such that $\psi(x_i) = \pm 1$ whenever $x_i = x_{i \pm 1}$.

**Remark 4.49.** *[Bon88, §3.2.] Let $g \in \bar{\mathfrak{S}}(\mathfrak{L}) \cup \mathring{\mathfrak{S}}(\mathfrak{L})$ and let $\psi \in \Psi(g)$. Then for the reversed graph $g^*$ there exists a similar set $\Psi(g^*)$ with maps $\psi^*$ given by opposite sign:*

$$\psi^*(x_i) = -\psi(x_i), \quad x_i \in g_0.$$

Since $\psi \in \Psi(g)$ is not always uniquely defined (in particular, on double ends), we have the following options and conventionally assume the following for an $\mathfrak{L}$–chain $g$:

**Remark 4.50.** *Note that any $x_i \in g_0^\alpha$ is given by a two-point link $X \in \mathfrak{L}$. The elements $X^+$ and $X^-$ of $X$ each describe a band in the respective matrix. From the data given by $g$, we do not know to which band any $x_i \in g_0^\alpha$ with $x_i = X$ is assigned. This is fixed by the maps in $\Psi$.*

| d($g$) | $\lvert\Psi(g)\rvert$ | convention on $\psi_s \in \Psi(g)$, $s \in \{1,\dots,\lvert\Psi(g)\rvert\}$ |
|---|---|---|
| 0 | 1 | $\psi_1$ uniquely defined |
| 1 | 2 | $\psi_1(x_1) = -1 \quad \psi_2(x_1) = 1 \quad$ if $x_1$ is the double end<br>$\psi_1(x_m) = 1 \quad \psi_2(x_m) = -1 \quad$ if $x_m$ is the double end |
| 2 | 4 | $\psi_1(x_1) = -1 \quad \psi_2(x_1) = 1 \quad \psi_3(x_1) = -1 \quad \psi_4(x_1) = 1$<br>$\psi_1(x_m) = 1 \quad \psi_2(x_m) = 1 \quad \psi_3(x_m) = -1 \quad \psi_4(x_m) = -1$ |

Let $X, Y \in \mathfrak{L}$ with $X\bar{\alpha}X$, $Y\alpha Y$, $Z \in \mathfrak{X}_0$, $x_j \in g_0$ and $1 \le s \le 4$. We denote

$$\begin{aligned} g_0(X) &= \{x_i \in g_0 \mid x_i = X\}, \\ g_{0,s}^{\pm}(Y) &= \{x_i \in g_0 \mid x_i = Y, \psi_s(x_i) = \pm 1\}, \\ n(Z,g,s) &= \begin{cases} \lvert g_0(Z)\rvert & \text{if } Z = X \\ \lvert g_{0,s}^{\zeta}(Y)\rvert & \text{if } Z = Y^{\zeta} \end{cases}, \\ n(x_j) &= \#\{x_i \in g_0 \mid x_i = x_j, 0 \le i < j\}. \end{aligned}$$

**Remark 4.51.** *The sets $g_{0,s}^{\pm}(Y)$ are built in case of $Y$ being a two-point link. For one-point links $X$, we consider the set $g_0(X)$.*

To some extent, we can - with respect to the construction - group together representations of the types $U_1(g)$ and $U_s(g)$ for $s = 1, 2$. The representations $U(g, \varphi)$ and $U_s(g, p)$, $s = 1, 2, 3, 4$, $p \in \mathbb{N}$, are each similar to the construction of the first two, but will be treated separately. Thus, we start with the construction of $U(g)$ and deduce from it the constructions of the other representations.

**Construction of $U_1(g)$, $U_s(g)$.**
Let $g \in \bar{\mathfrak{S}}(\mathfrak{L})$ with $\mathrm{d}(g) = 1$. We start with the matrices $U^1, \dots, U^N$ of $U_1(g)$. Let $i \in \{1, \dots, N\}$ and let $X, Y, Z^+, Z^- \in \mathfrak{C}_i \cup \mathfrak{R}_i$. Then the structure of $U^i$ is given as follows:

- The bands of $U^i$:
  - The row bands are indexed by the elements of $\mathfrak{R}_i$.
  - The column bands are indexed by the elements of $\mathfrak{C}_i$.
- Order of bands:
  - The column/row band $P(X)$ is situated left/above of the column/row band $P(Y)$ if $X < Y$.
  - Additionally, if $Z^+$ and $Z^-$ are two incomparable elements, $P(Z^+)$ is situated left/above of $P(Z^-)$.
- Sizes of bands:
  - $\dim P(X) = n(X, g, 1)$,

· If $\sigma(X) = Y$, then $n(X, g, 1) = n(Y, g, 1)$.

- Within the bands:

  · The rows (columns) of $P(Z)$ are indexed by the elements in

  $$\begin{aligned} &g_0(Z), \text{ if } Z \text{ is an element of a one-point link},\\ &g_{0,1}^{\zeta}(Z), \text{ if } Z = Z^{\zeta} \text{ is an element of a two-point link }, \zeta \in \{+,-\}. \end{aligned}$$

  · The $j$–th row (column) within $P(Z)$ is indexed by the element $x_k \in g_0(Z)$ $(g_{0,1}^{\pm}(Z))$ such that $n(x_k) = j$.

  · If $x_k \sim x_{k+1}$ in $g$ with $x_k \neq x_{k+1}$, then $n(x_k) = n(x_{k+1})$.

To describe the entries of the matrix $U^i$, we denote by $x_k \cap x_j$ the entry in row $x_k$ and column $x_j$ of $U^i$. Then

$$x_k \cap x_j = \begin{cases} 1 & \text{if there exists } e_{k,j}(g),\\ 0 & \text{otherwise.} \end{cases}$$

The matrices $U_s(g)$ are constructed for each $s \in \{1,2\}$ as described above, using the respective map $\psi_s$ to determine the sets $g_{0,s}^{\pm}$, and using these sets to determine the elements indexing the respective bands. We obtain two representations from one $\mathfrak{L}$–chain.

**Example 4.52.** ***on*** *$U_1(g)$.*

1. *Let $N = 1$ and $\mathfrak{R}_1 = \{\mathfrak{R}_{11} > \mathfrak{R}_{12} > \mathfrak{R}_{13}\}$, $\mathfrak{C}_1 = \{\mathfrak{C}_{11}^{+} \approx \mathfrak{C}_{11}^{-}\}$ with $\sigma(\mathfrak{R}_{11}) = \mathfrak{R}_{13}$ and otherwise $\sigma$ acts as identity. Let*

$$\begin{array}{lcccccccc} g: & \mathfrak{R}_{12} - & \multicolumn{2}{c}{\overleftarrow{\mathfrak{C}_{11} \sim \mathfrak{C}_{11}}} & - \mathfrak{R}_{13} \sim & \mathfrak{R}_{11} - & \multicolumn{2}{c}{\overleftarrow{\mathfrak{C}_{11} \sim \mathfrak{C}_{11}}} & - \mathfrak{R}_{12} \\ & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\ \psi_1: & & +1 & -1 & & & & +1 & -1 \end{array}$$

*The elementary subchains of $g$ are:*

$$\begin{array}{llll} \textit{Type 1)} & e_{1,2}(g) = x_1 - x_2, & \textit{Type 3)} & e_{1,3}(g) = x_1 - \overleftarrow{x_2 \sim x_3},\\ & e_{3,4}(g) = x_3 - x_4, & & e_{5,7}(g) = x_5 - \overleftarrow{x_6 \sim x_7},\\ & e_{5,6}(g) = x_5 - x_6, & & \\ & e_{7,8}(g) = x_7 - x_8. & & \end{array}$$

*Following the above instructions, we thus obtain for the representation* $U_1(g) = (U_{\mathfrak{R}_{11}}, U_{\mathfrak{R}_{12}}, U_{\mathfrak{R}_{13}}, U_{\mathfrak{C}^+_{11}}, U_{\mathfrak{C}^-_{11}}, U^1)$ *that* $U^1$ *is of the form*

$$U^1 = \begin{array}{cc|cc|cc|} & & \mathfrak{C}^+_{11} & & \mathfrak{C}^-_{11} & \\ & & x_2 & x_6 & x_3 & x_7 \\ \hline \mathfrak{R}_{13} & x_4 & 0 & 0 & 1 & 0 \\ \hline \mathfrak{R}_{12} & x_1 & 1 & 0 & 1 & 0 \\ & x_8 & 0 & 0 & 0 & 1 \\ \hline \mathfrak{R}_{11} & x_5 & 0 & 1 & 0 & 1 \\ \hline \end{array}$$

*The vector spaces have the following dimensions:*

$$\dim(U_{\mathfrak{R}_{11}}) = \dim(U_{\mathfrak{R}_{13}}) = 1,$$
$$\dim(U_{\mathfrak{C}^-_{11}}) = \dim(U_{\mathfrak{C}^+_{11}}) = \dim(U_{\mathfrak{R}_{12}}) = 2.$$

2. *Let* $N = 5$ *and let the bundle* $\overline{\mathfrak{X}} = (\mathfrak{X}, \sigma)$ *be given by the semichains*

$$\begin{aligned} \mathfrak{C}_1 &= \{\mathfrak{C}^+_{11} \lessgtr \mathfrak{C}^-_{11}\}, & \mathfrak{R}_1 &= \{\mathfrak{R}_{11} > \mathfrak{R}_{12}\}, \\ \mathfrak{C}_2 &= \{\mathfrak{C}_{21} < \mathfrak{C}_{22} < \mathfrak{C}_{23}\}, & \mathfrak{R}_2 &= \{\mathfrak{R}_{21} > \mathfrak{R}_{22} > \mathfrak{R}_{23}\}, \\ \mathfrak{C}_3 &= \{\mathfrak{C}^+_{31} \lessgtr \mathfrak{C}^-_{31}\}, & \mathfrak{R}_3 &= \{\mathfrak{R}_{31} > \mathfrak{R}_{32}\}, \\ \mathfrak{C}_4 &= \{\mathfrak{C}_{41} < \mathfrak{C}_{41}\}, & \mathfrak{R}_4 &= \{\mathfrak{R}_{41} > \mathfrak{R}_{42}\}, \\ \mathfrak{C}_5 &= \{\mathfrak{C}^+_{51} \lessgtr \mathfrak{C}^-_{51}\}, & \mathfrak{R}_5 &= \{\mathfrak{R}_{51} > \mathfrak{R}_{52} > \mathfrak{R}_{53}\}, \end{aligned}$$

*and let* $\sigma$ *be acting as follows (and as identity otherwise):*

$$\begin{aligned} \mathfrak{C}_{21} &\leftrightarrow \mathfrak{R}_{12}, & \mathfrak{R}_{21} &\leftrightarrow \mathfrak{R}_{32}, \\ \mathfrak{C}_{23} &\leftrightarrow \mathfrak{R}_{51}, & \mathfrak{R}_{41} &\leftrightarrow \mathfrak{R}_{53}, \\ \mathfrak{C}_{41} &\leftrightarrow \mathfrak{R}_{23}. \end{aligned}$$

*Consider the* $\mathfrak{L}$*−chain*

$$\begin{array}{lcccccccc} g: & \mathfrak{R}_{42} - & \mathfrak{C}_{41} \sim & \mathfrak{R}_{23} - & \mathfrak{C}_{21} \sim & \mathfrak{R}_{12} - & \overleftarrow{\mathfrak{C}_{11} \sim} & \overleftarrow{\mathfrak{C}_{11}} - & \mathfrak{R}_{11} \\ & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\ \psi_1: & & & & & & & +1 & -1 \end{array}$$

*The elementary subchains of* $g$ *are given by*

*Type 1):* $x_i - x_{i+1} \quad \forall 1 \le i \le 8$ *odd,*
*Type 3):* $x_5 - \overleftarrow{x_6 \sim x_7}$.

*The representation $U_1(g)$ is given by the matrices*

$$U^1 = \begin{array}{cc|c|c|} & & \mathfrak{C}_{11}^+ & \mathfrak{C}_{11}^- \\ & & x_6 & x_7 \\ \hline \mathfrak{R}_{12} & x_5 & 0 & 1 \\ \hline \mathfrak{R}_{11} & x_8 & 1 & 1 \\ \hline \end{array}, \quad U^2 = \begin{array}{cc|c|} & & \mathfrak{C}_{21} \\ & & x_4 \\ \hline \mathfrak{R}_{23} & x_3 & 1 \\ \hline \end{array},$$

$$U^4 = \begin{array}{cc|c|} & & \mathfrak{C}_{41} \\ & & x_2 \\ \hline \mathfrak{R}_{42} & x_1 & 1 \\ \hline \end{array}$$

*and $U^3$ and $U^5$ are empty. The vector spaces of $U_1(g)$ have dimensions*

$$\dim(U_{\mathfrak{C}_{11}^{\zeta}}) = \dim(U_{\mathfrak{R}_{11}}) = \dim(U_{\mathfrak{R}_{12}}) = 1, \quad \zeta \in \{+,-\},$$
$$\dim(U_{\mathfrak{C}_{i1}}) = \dim(U_{\mathfrak{R}_{i1}}) = 1, \quad i \in \{2,4\}$$

*and all other vector spaces have dimension 0.*

**Example 4.53.** ***on*** $U_s(g)$***.***

1. *Consider the same setting as in Example 4.52.1. and let*

$$\begin{array}{lcccccc} g: & \mathfrak{R}_{12} - & \overleftarrow{\mathfrak{C}_{11} \sim \mathfrak{C}_{11}} & - \mathfrak{R}_{13} & \sim \mathfrak{R}_{11} & - \mathfrak{C}_{11} & \\ & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ \psi_1: & & +1 & -1 & & & +1 \\ \psi_2: & & +1 & -1 & & & -1 \end{array}$$

*The elementary subchains are*

*Type 1):* $e_{1,2}(g) = x_1 - x_2$, *Type 3):* $e_{1,3}(g) = x_1 - \overleftarrow{x_2 \sim x_3}$.
$e_{3,4}(g) = x_3 - x_4$,
$e_{5,6}(g) = x_5 - x_6$.

*We obtain the following two matrices $U_s^1$ for the representations $U_s(g)$, $s = 1,2$:*

$$U_1^1 = \begin{array}{cc|cc|c|} & & \mathfrak{C}_{11}^+ & & \mathfrak{C}_{11}^- \\ & & x_2 & x_6 & x_3 \\ \hline \mathfrak{R}_{13} & x_4 & 0 & 0 & 1 \\ \hline \mathfrak{R}_{12} & x_1 & 1 & 0 & 1 \\ \hline \mathfrak{R}_{11} & x_5 & 0 & 1 & 0 \\ \hline \end{array}, \quad U_2^1 = \begin{array}{cc|c|cc|} & & \mathfrak{C}_{11}^+ & \mathfrak{C}_{11}^- & \\ & & x_2 & x_3 & x_6 \\ \hline \mathfrak{R}_{13} & x_4 & 0 & 1 & 0 \\ \hline \mathfrak{R}_{12} & x_1 & 1 & 1 & 0 \\ \hline \mathfrak{R}_{11} & x_5 & 0 & 0 & 1 \\ \hline \end{array}$$

*The dimensions of the respective vector spaces are given by:*

$$\begin{array}{ll} s=1: & \dim(U_{\mathfrak{R}_{1i}}) = \dim(U_{\mathfrak{C}_{11}^-}) = 1, \quad 1 \le i \le 3 \\ & \dim(U_{\mathfrak{C}_{11}^+}) = 2, \\ s=2: & \dim(U_{\mathfrak{R}_{1i}}) = \dim(U_{\mathfrak{C}_{11}^+}) = 1, \quad 1 \le i \le 3, \\ & \dim(U_{\mathfrak{C}_{11}^-}) = 2. \end{array}$$

2. *Let $\overline{\mathfrak{X}} = (\mathfrak{X}, \sigma)$ be as in Example 4.52.2. Consider the following $\mathfrak{L}$–chain with two double ends:*

$$\begin{array}{lcccccccc}
g: & \mathfrak{R}_{52} - & \overleftarrow{\mathfrak{C}_{51} \sim \mathfrak{C}_{51}} & & - \mathfrak{R}_{51} \sim & \mathfrak{C}_{23} - & \mathfrak{R}_{21} \sim & \mathfrak{R}_{32} - & \mathfrak{C}_{31} \\
& x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\
\psi_1: & & +1 & -1 & & & & & +1 \\
\psi_2: & & +1 & -1 & & & & & -1
\end{array}$$

*The matrices $U_1^1$ and $U_1^4$ of the representation $U_1(g)$ are empty. The others are given by*

$$U_1^2 = \begin{array}{cc|c|} & & \mathfrak{C}_{23} \\ & & x_5 \\ \hline \mathfrak{R}_{21} & x_6 & 1 \\ \hline \end{array}, \quad U_1^3 = \begin{array}{cc|c|} & & \mathfrak{C}_{31}^{+} \\ & & x_8 \\ \hline \mathfrak{R}_{32} & x_7 & 1 \\ \hline \end{array},$$

$$U_1^5 = \begin{array}{cc|c|c|} & & \mathfrak{C}_{51}^{+} & \mathfrak{C}_{51}^{-} \\ & & x_2 & x_3 \\ \hline \mathfrak{R}_{52} & x_1 & 1 & 1 \\ \hline \mathfrak{R}_{51} & x_4 & 0 & 1 \\ \hline \end{array}$$

*The corresponding vector spaces $U_{\mathfrak{R}_{21}}$, $U_{\mathfrak{C}_{23}}$, $U_{\mathfrak{R}_{32}}$, $U_{\mathfrak{C}_{31}^{+}}$, $U_{\mathfrak{R}_{52}}$, $U_{\mathfrak{R}_{51}}$, $U_{\mathfrak{C}_{51}^{+}}$ and $U_{\mathfrak{C}_{51}^{-}}$ have dimension 1. The other vector spaces have dimension 0.*
*Let us now consider $s = 2$. The matrices $U_2^1$ and $U_2^4$ are also empty. The other matrices of the representation $U_2(g)$ are given by*

$$U_2^2 = U_1^2,$$
$$U_2^5 = U_1^5,$$
$$U_2^3 = \begin{array}{cc|c|} & & \mathfrak{C}_{31}^{-} \\ & & x_8 \\ \hline \mathfrak{R}_{32} & x_7 & 1 \\ \hline \end{array}$$

*The following vector spaces have dimension 1: $U_{\mathfrak{R}_{21}}$, $U_{\mathfrak{C}_{23}}$, $U_{\mathfrak{R}_{32}}$, $U_{\mathfrak{C}_{31}^{-}}$, $U_{\mathfrak{R}_{52}}$, $U_{\mathfrak{R}_{51}}$, $U_{\mathfrak{C}_{51}^{+}}$ and $U_{\mathfrak{C}_{51}^{-}}$. The rest of them has dimension 0.*

Note that we can obtain the entries of $U_1^2$ from those of $U_1^1$ in Example 4.53.1 by switching the columns. That is due to the following statement:

**Lemma 4.54** ([Bon88, §6, Statement 3.2.]). *Let $g \in \bar{\mathfrak{S}}(\mathfrak{L})$ with $\mathrm{d}(g) = 1$. Switching the positions of the bands indexed by two incomparable elements $X^{+} \not\lessgtr X^{-}$ gives an equivalent set of representations.*

**Construction of $U(g, \varphi)$.**
The case $U(g, \varphi)$ for $g \in \mathring{\mathfrak{S}}(\mathfrak{L})$ is similar to the previous one, but a bit more

complicated. Recall that $\varphi_0$ is a monic polynomial in $\mathrm{k}[t]$ which is irreducible over k. Note at first that the choice of $\varphi_0$ depends on the given $\mathfrak{L}$–cycle $g$: if $g$ is non-symmetric, then $\varphi_0 \neq t$. For $g$ symmetric, we distinguish between $\delta_0(g)$ being odd or even: in the first case, we assume that $\varphi_0 \neq t, t-1$, in the second case that $\varphi_0 \neq t, t+1$. Moreover, we consider – as in the case of $U_1(g)$ – the sets $g_0(Z)$ and $g^{\pm}_{0,1}(Z)$ as indices within the bands. But in contrast to the case of $\mathfrak{L}$–chains, the elements of the respective sets do not index rows and columns, but subbands within the bigger bands. The size of each such subband is given by $\ell = \deg(\varphi)$ where $\varphi = \varphi_0^n$. Let $x_k \cap x_j$ denote the intersection of the two respective subbands. Then for $g$ not being of length four and symmetric, the entries of $U^i$ are:

$$x_k \cap x_j = \begin{cases} 1_{\ell\times\ell} & \text{if there exists } e_{k,j}(g) \neq e_2(g) \\ F_\varphi & \text{if there exists } e_{k,j}(g) = e_2(g) \\ 0 & \text{otherwise} \end{cases},$$

where $F_\varphi$ denotes the Frobenius block of $\varphi$. Recall that for the polynomial $\varphi = t^n + a_{n-1}t^{n-1} \cdots + a_0$ its Frobenius block is given by the $(n \times n)$–matrix

$$F_\varphi = \begin{pmatrix} 0 & & & & -a_0 \\ 1 & \ddots & & & -a_1 \\ & \ddots & \ddots & & \vdots \\ & & \ddots & 0 & -a_{n-2} \\ & & & 1 & -a_{n-1} \end{pmatrix}$$

In the special case that $g$ is symmetric and $|g| = 4$, there exist two elements $x_l$, $x_{l+1}$ such that there are two elementary subchains $e^1_{l,l+1}(g)$ of length 2 and $e^2_{l,l+1}(g)$ of length 4. Then we combine the above instructions and set:

$$x_l \cap x_{l+1} = 1_{\ell\times\ell} + F_\varphi$$

**Remark 4.55.** *In the special case where $g$ is symmetric and $|g| = 4$, we have that $e_2(g) = e^1_{l,l+1}(g)$:*

$$\underbrace{\overset{e_2(g)}{\overleftrightarrow{x_1 \frown x_2 - x_3 \frown x_4}}}_{} \qquad \underbrace{\overrightarrow{x_1 \frown x_2} - \overleftarrow{x_3 \frown x_4}}_{e_2(g)}$$

**Remark 4.56** ([Bon88, §6]). *Admissible transformations of type 1 allow us to consider subchains of the form $\bar{e}_2(g) = x_{i-1} - x_i$ with $i$ not being the least in $\{1, \dots, m\}$ instead of $e_2(g)$. In this case, the block in $x_{i-1} \cap x_i$ is given by $F_\varphi$ or $F_\varphi^{-1}$.*

**Example 4.57.** ***on*** *$U(\varphi, g)$.*

1. *Consider the same setting as in Example 4.52, 1. and let*

$$\begin{array}{ccccccccc}
g_0: & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\
g: & \mathfrak{C}_{11} & - \ \mathfrak{R}_{13} \sim & \mathfrak{R}_{11} \ - & \mathfrak{C}_{11} & \sim \ \mathfrak{C}_{11} \ - & \mathfrak{R}_{11} \sim & \mathfrak{R}_{13} \ - & \mathfrak{C}_{11} \\
 & & & & & & & & \\
\psi_1: & -1 & & & +1 & -1 & & & +1 \\
\psi_2: & +1 & & & +1 & -1 & & & +1 \\
\psi_3: & -1 & & & +1 & -1 & & & -1 \\
\psi_4: & +1 & & & +1 & -1 & & & -1
\end{array}$$

*The $\mathfrak{L}$–cycle $g$ is symmetric, $\delta(g) = \#\{2,6\} = 2$ and $\delta_0(g) = 1$ is odd. Thus, $\varphi_0 \neq t, t-1$. We choose $\varphi_0 = t+1$, $\deg(\varphi) = 2$:*

$$\varphi = (t+1)^2 = t^2 + 2t + 1.$$

*The corresponding Frobenius matrix is given by*

$$F_\varphi = \begin{pmatrix} 0 & -1 \\ 1 & -2 \end{pmatrix}.$$

*We obtain the following matrices $U_s^1$ of $U_s(g,\varphi)$, $s = 1,2,3,4$ (recall that each $x_i \in g_0$ indexes a subband of size 2):*

$$U_1^1 = \begin{array}{cc|cccc|cccc|}
 & & \multicolumn{4}{c|}{\mathfrak{C}_{11}^{+}} & \multicolumn{4}{c|}{\mathfrak{C}_{11}^{-}} \\
 & & x_4 & & x_8 & & x_1 & & x_5 & \\
\hline
\mathfrak{R}_{13} & x_2 & 0 & 0 & 0 & 0 & \mathbf{0} & \mathbf{-1} & 0 & 0 \\
 & & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{-2} & 0 & 0 \\
 & x_7 & 0 & 0 & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & 0 & 0 \\
 & & 0 & 0 & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & 0 & 0 \\
\hline
\mathfrak{R}_{11} & x_3 & \mathbf{1} & \mathbf{0} & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{0} \\
 & & \mathbf{0} & \mathbf{1} & 0 & 0 & 0 & 0 & \mathbf{0} & \mathbf{1} \\
 & x_6 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{0} \\
 & & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{0} & \mathbf{1} \\
\hline
\end{array},$$

*with vector spaces of the following dimensions for $s = 1$:*

$$\dim(\mathfrak{R}_{11}) = \dim(\mathfrak{R}_{13}) = \dim(\mathfrak{C}_{11}^{+}) = \dim(\mathfrak{C}_{11}^{-}) = 4.$$

*In the following, we are going to neglect the subband structure within the matrices. In this example, it means that each entry denotes from now on a respective block matrix of size $2 \times 2$, that is, any 1 is going to denote a $2 \times 2$–identity matrix. In this version, the matrices $U_s^1$ are given by*

$$U_1^1 = \begin{array}{cc|cc|cc|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{\mathfrak{C}_{11}^+} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{\mathfrak{C}_{11}^-} & \multicolumn{1}{c}{} \\ \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{x_4} & \multicolumn{1}{c}{x_8} & \multicolumn{1}{c}{x_1} & \multicolumn{1}{c}{x_5} \\ \cline{3-6} \mathfrak{R}_{13} & x_2 & 0 & 0 & F_\varphi & 0 \\ & x_7 & 0 & 1 & 1 & 0 \\ \cline{3-6} \mathfrak{R}_{11} & x_3 & 1 & 0 & 0 & 1 \\ & x_6 & 0 & 0 & 0 & 1 \\ \cline{3-6} \end{array},$$

$$U_2^1 = \begin{array}{cc|ccc|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{\mathfrak{C}_{11}^+} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{\mathfrak{C}_{11}^-} \\ \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{x_1} & \multicolumn{1}{c}{x_4} & \multicolumn{1}{c}{x_8} & \multicolumn{1}{c}{x_5} \\ \cline{3-6} \mathfrak{R}_{13} & x_2 & F_\varphi & 0 & 0 & 0 \\ & x_7 & 1 & 0 & 1 & 0 \\ \cline{3-6} \mathfrak{R}_{11} & x_3 & 0 & 1 & 0 & 1 \\ & x_6 & 0 & 0 & 0 & 1 \\ \cline{3-6} \end{array},$$

$$U_3^1 = \begin{array}{cc|c|ccc|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{\mathfrak{C}_{11}^+} & \multicolumn{1}{c}{\mathfrak{C}_{11}^-} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{x_4} & \multicolumn{1}{c}{x_1} & \multicolumn{1}{c}{x_5} & \multicolumn{1}{c}{x_8} \\ \cline{3-6} \mathfrak{R}_{13} & x_2 & 0 & F_\varphi & 0 & 0 \\ & x_7 & 0 & 1 & 0 & 1 \\ \cline{3-6} \mathfrak{R}_{11} & x_3 & 1 & 0 & 1 & 0 \\ & x_6 & 0 & 0 & 1 & 0 \\ \cline{3-6} \end{array},$$

$$U_4^1 = \begin{array}{cc|cc|cc|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{\mathfrak{C}_{11}^+} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{\mathfrak{C}_{11}^-} & \multicolumn{1}{c}{} \\ \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{x_1} & \multicolumn{1}{c}{x_4} & \multicolumn{1}{c}{x_5} & \multicolumn{1}{c}{x_8} \\ \cline{3-6} \mathfrak{R}_{13} & x_2 & F_\varphi & 0 & 0 & 0 \\ & x_7 & 1 & 0 & 0 & 1 \\ \cline{3-6} \mathfrak{R}_{11} & x_3 & 0 & 1 & 1 & 0 \\ & x_6 & 0 & 0 & 1 & 0 \\ \cline{3-6} \end{array}$$

*The dimensions of the vector spaces are given by*

$$\dim(\mathfrak{R}_{11}) = \dim(\mathfrak{R}_{13}) = 4 \quad \textit{for all } s = 1, 2, 3, 4,$$

*and*

$$\begin{aligned} s = 1, 4 : \dim(\mathfrak{C}_{11}^+) &= 4, \quad \dim(\mathfrak{C}_{11}^-) = 4, \\ s = 2 : \dim(\mathfrak{C}_{11}^+) &= 6, \quad \dim(\mathfrak{C}_{11}^-) = 2, \\ s = 3 : \dim(\mathfrak{C}_{11}^+) &= 2, \quad \dim(\mathfrak{C}_{11}^-) = 6. \end{aligned}$$

2. *Consider the bundle of semichains as in Example 4.52.2. Consider the following $\mathfrak{L}$–cycle*

$$\begin{array}{lcccccccccccccccc}
g_0: & x_1 & & x_2 & & \overrightarrow{x_3} & & \overrightarrow{x_4} & & x_5 & & x_6 & & x_7 & & x_8 \\
g: & \mathfrak{C}_{23} & \sim & \mathfrak{R}_{51} & - & \mathfrak{C}_{51} & \sim & \mathfrak{C}_{51} & - & \mathfrak{R}_{53} & \sim & \mathfrak{R}_{41} & - & \mathfrak{C}_{41} & \sim & \mathfrak{R}_{23} \\
\psi_1: & & & & & +1 & & -1 & & & & & & & &
\end{array}$$

*It is non-symmetric, so we can choose* $\varphi = (t-1)^3 = t^3 - 3t^2 + 3t - 1$. *Its Frobenious block is thus given by*

$$F_\varphi = \begin{pmatrix} 0 & 0 & 3 \\ 1 & 0 & -3 \\ 0 & 1 & 1 \end{pmatrix}$$

*Its elementary subchains are:*

$$\begin{array}{ll}
\textit{Type 1):} & x_i - x_{i+1} \qquad \forall 1 \le i \le 8, \ \textit{even}, \\
& e_2(g) = x_8 - x_1, \\
\textit{Type 2):} & \overrightarrow{x_3 \sim x_4} - x_5.
\end{array}$$

*The representation* $U(g,\varphi)$ *consists of the two empty matrices* $U^1$, $U^3$ *and additionally the following ones:*

$$U^2 = \begin{array}{ccc} & & \mathfrak{C}_{23} \\ & & x_1 \\ \mathfrak{R}_{23} & x_8 & \boxed{F_\varphi} \end{array}, \quad U^4 = \begin{array}{ccc} & & \mathfrak{C}_{41} \\ & & x_7 \\ \mathfrak{R}_{41} & x_6 & \boxed{1} \end{array},$$

$$U^5 = \begin{array}{cc|c|c|} & & \mathfrak{C}_{51}^{+} & \mathfrak{C}_{51}^{-} \\ & & x_3 & x_4 \\ \cline{3-4} \mathfrak{R}_{53} & x_5 & 1 & 1 \\ \cline{3-4} \mathfrak{R}_{51} & x_2 & 1 & 0 \\ \cline{3-4} \end{array},$$

*The following vector spaces of the representation have dimension given by* $\deg(\varphi) = 3$*:* $U_{\mathfrak{R}_{23}}$, $U_{\mathfrak{C}_{23}}$, $U_{\mathfrak{R}_{41}}$, $U_{\mathfrak{C}_{41}}$, $U_{\mathfrak{R}_{51}}$, $U_{\mathfrak{R}_{53}}$, $U_{\mathfrak{C}_{51}^{\zeta}}$, *where* $\zeta \in \{+,-\}$. *All the other vector spaces of* $U(g,\varphi)$ *have dimension* $0$.

**Construction of $U_s(g,p)$.**

Recall that $g \in \bar{\mathfrak{S}}(\mathfrak{L})$, $s = 1,2,3,4$ and $p \in \mathbb{N}$ is fixed. To describe the matrices in $U_s(g,p)$, we need a slightly adjusted notation: Let $X, Y \in \mathfrak{L}$ with $X\bar{\alpha}X$,

$Y\alpha Y$ and let $Z \in \mathfrak{X}_0$. Denote

$$\begin{aligned}
g_0(p) &= \{(x_i, q) \mid x_i \in g_0, 1 \le q \le p\},\\
g_0(X,p) &= \{(x_i, q) \in g_0(p) \mid x_i = X\},\\
g_{0,s}^{\pm}(Y,p) &= \{(x_i, q) \in g_0(p) \mid x_i = Y, \psi_s(x_i) = \pm(-1)^{q+1}\},\\
n(Z,g,s,p) &= \begin{cases} |g_0(X,p)| & \text{if } Z = X\\ |g_{0,s}^{\zeta}(Y,p)| & \text{if } Z = Y^{\zeta}, \zeta \in \{+,-\}\end{cases},\\
n(x_j) &= p \cdot \#\{x_k \in g_0 \mid x_j = x_k, 1 \le k < j\},\\
n((x_j, q)) &= (n(x_j) - 1)p + q.
\end{aligned}$$

Now let $i \in \{1, \dots, N\}$, $X, Y, Z, Z^+, Z^- \in \mathfrak{C}_i \cup \mathfrak{R}_i$. Then the matrix $U^i$ is structured as follows:

- The bands of $U^i$:
  - The row and column bands are indexed by the elements of $\mathfrak{R}_i$, $\mathfrak{C}_i$, respectively.
- Order of bands:
  - The bands are ordered as in the previous cases.
- Size of bands:
  - $\dim P(X) = n(X, g, s, p)$,
  - If $\sigma(X) = Y$, then $n(X, g, s, p) = n(Y, g, s, p)$.
- Structure in bands:
  - The rows/columns of $P(Z)$ are indexed by elements in

    $$\begin{aligned} &g_0(Z,p) \text{ if } Z \text{ is an element in a one-point link,}\\ &g_{0,s}^{\zeta}(Z,p) \text{ if } Z = Z^{\zeta} \text{ is an element in a two-point link.}\end{aligned}$$

  - The row/column $(x_i, q_1)$ is situated above/left of the row/column $(x_j, q_2)$ if

    $$i < j \qquad \text{or} \qquad i = j,\ q_1 < q_2.$$

  - The $j$–th row (column) within $P(Z)$ is indexed by $(x_k, q) \in g_0(Z,p)$ $(g_{0,s}^{\pm}(Z,p))$ such that $n(x_k, q) = j$.
  - If $x_k \sim x_{k+1}$ in $g$ with $x_k \ne x_{k+1}$, then $n(x_k) = n(x_{k+1})$.

In order to describe the entries in $U^i$, we denote by $(x_k, q_1) \cap (x_j, q_2)$ the entry in row $(x_k, q_1)$ and column $(x_j, q_2)$. Then

$$(x_k, q_1) \cap (x_j, q_2) = \begin{cases} 1 & \text{if there exists } e_{k,j}(g) \text{ and (a), (b) or (c) holds,} \\ 0 & \text{otherwise} \end{cases} \tag{62}$$

where

(a) $q_1 = q_2$,

(b) $q_1 = q_2 - 1$, $q_2$ odd and either $k$ or $j$ is equal to 1,

(c) $q_1 = q_2 - 1$, $q_2$ even and either $k$ or $j$ is equal to $m$.

**Remark 4.58.** *The construction of $U_s(g,p)$ seems to be – in comparison to the other representations – chosen arbitrarily. The background is the following [Bon88, §3.2]:*
*When constructing $U_s(g,p)$, we actually consider $U_s(h,1)$ where $h = g^{[p]}$. For the latter, we consider*

$$\Psi(h) = \{\bar{\psi}_s \mid \psi_s \in \Psi(g)\}, \quad \bar{\psi}_s : h_0^\alpha \to \{+1,-1\},$$

*such that*

$$\bar{\psi}_s\big|_{g^{(j)}} = \begin{cases} \psi_s(g) & \text{if } g^{(j)} = g, \\ \psi_s^*(g) & \text{if } g^{(j)} = g^*. \end{cases}$$

*Note that $h_0^\alpha = \bigcup_{j=1}^p (g_0^{(j)})^\alpha$.*
*Denote in the following by $(x_k, q_j)$ the copy of $x_k$ in the subchain $g^{(q_j)}$ of $h$. The information on how to put an orientation in $h$ on its "joints" is vital to understand the connection to the earlier given construction:*

$$\overleftarrow{(x_i, q_k) \sim (x_i, q_{k+1})} \text{ if } i \in \{1, m\}, x_i \in \mathfrak{R}, \tag{63}$$

$$\overrightarrow{(x_i, q_k) \sim (x_i, q_{k+1})} \text{ if } i \in \{1, m\}, x_i \in \mathfrak{C}. \tag{64}$$

*Interpreting (a) - (c) in this context gives a clear idea on what is happening:*

*(a) We consider in each copy $g^{(q_j)}$ of $g$ the elementary subchains. If $e_{k,j}(g)$ exists, then (a) ensures that we take the respective elementary subchain $e_{k,j}(g^{(q_j)})$ in each copy $g^{(q_j)}$ into account, that is, we put 1 as entry in each copy $q_j$ of $x_k \cap x_j$.*

*(b) Here, we consider two neighbouring copies $g^{(q_1)} \sim g^{(q_2)}$ of $g$. Since $q_2$ is odd, $q_1$ is even and we know that they are of the form*

$$g^{(q_1)} = g^*, \qquad g^{(q_2)} = g.$$

*Thus, $g^{(q_1)}$ ends with the respective copy of $x_1 \in g_0$, and $g^{(q_2)}$ starts with the respective copy of $x_1 \in g_0$. The middle of $g^{(q_1)} \sim g^{(q_2)}$ is given by*

$$\cdots - x_1 \sim x_1 - \ldots \tag{65}$$

*with some direction given on $x_1 \sim x_1$. If $k = 1$ in (62) and (b) holds, then $(x_1, q_1) \in g^{(q_1)}$ is the end point of an elementary subchain, otherwise ($j = 1$), an end point is given by $(x_1, q_2) \in g^{(q_2)}$. It follows for (65):*

$$\begin{aligned} &\cdots - \overrightarrow{x_1 \sim x_1} - \ldots && \text{if } k = 1, \\ &\cdots - \overleftarrow{x_1 \sim x_1} - \ldots && \text{if } j = 1. \end{aligned}$$

*Hence, (b) ensures that we take all elementary subchains including the "joints" of the form $x_1 \sim x_1$ into account.*

(c) *It ensures – similar to (b) – that all elementary subchains including the "joints" of the form $x_m \sim x_m$ are taken into account.*

*We can conclude that we still follow the simple rule to put a 1 in the entry of a row and column if the elements indexing those are the ends of some elementary subchain in $g^{[p]}$, and we put 0 otherwise. The notation used in the construction is required since we consider different copies of $g$.*

In the following we work on $h = g^{[p]}$ to construct $U_s(g, p)$ but might still use the notation from the formal construction given above.

**Remark 4.59.** *[Bon88, §6, Statement 4.3.] Choosing the orientations at all joints in the opposite way than described in (63) and (64), gives an equivalent representation for each $s = 1, 2, 3, 4$.*

**Example 4.60.** ***on $U_s(g, p)$.***

1. *Consider the same setting as in Example 4.52.1. and let*

$$\begin{array}{ccccc} g: & \mathfrak{C}_{11} - & \mathfrak{R}_{13} \sim & \mathfrak{R}_{11} - & \mathfrak{C}_{11} \\ & x_1 & x_2 & x_3 & x_4 \\ \psi_1 & -1 & & & +1 \\ \psi_2 & +1 & & & +1 \\ \psi_3 & -1 & & & -1 \\ \psi_4 & +1 & & & -1 \end{array}$$

*We choose $p = 3$. Thus, we consider*

$$
\begin{aligned}
h =& g^{[p]} = g^{(1)} \sim g^{(2)} \sim g^{(3)} = g \sim g^{*} \sim g \\
=& \underbrace{\underset{x_1}{\mathfrak{C}_{11}} - \underset{x_2}{\mathfrak{R}_{13}} \sim \underset{x_3}{\mathfrak{R}_{11}} - \underset{x_4}{\overleftarrow{\mathfrak{C}_{11}}}}_{1} \sim \underbrace{\underset{x_4}{\mathfrak{C}_{11}} - \underset{x_3}{\mathfrak{R}_{11}} \sim \underset{x_2}{\mathfrak{R}_{13}} - \underset{x_1}{\overrightarrow{\mathfrak{C}_{11}}}}_{2} \sim \underbrace{\underset{x_1}{\mathfrak{C}_{11}} - \underset{x_2}{\mathfrak{R}_{13}} \sim \underset{x_3}{\mathfrak{R}_{11}} - \underset{x_4}{\mathfrak{C}_{11}}}_{3}
\end{aligned}
$$

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_4$ | $x_3$ | $x_2$ | $x_1$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\bar{\psi}_1$ | $-1$ | | | $+1$ | $+1$ | | | $-1$ | $-1$ | | | $+1$ |
| $\bar{\psi}_2$ | $+1$ | | | $+1$ | $+1$ | | | $+1$ | $+1$ | | | $+1$ |
| $\bar{\psi}_3$ | $-1$ | | | $-1$ | $-1$ | | | $-1$ | $-1$ | | | $-1$ |
| $\bar{\psi}_4$ | $+1$ | | | $-1$ | $-1$ | | | $+1$ | $+1$ | | | $-1$ |

*The elementary subchains of $h$ are:*

*Type 1)* $e_{(x_1,k),(x_2,k)}(h)$, $\quad k = 1, 2, 3,$

*Type 2)* $e_{(x_1,2),(x_2,3)}(h)$,

*Type 3)* $e_{(x_3,1),(x_4,2)}(h)$.

*The matrices $U_s^1$ in $U_s(g,3)$, $s = 1, 2, 3, 4$, are given by*

$$
U_1^1 = \begin{array}{cc|ccc|ccc|}
 & & & \mathfrak{C}_{11}^{+} & & & \mathfrak{C}_{11}^{-} & \\
 & & (x_4,1) & (x_4,2) & (x_4,3) & (x_1,1) & (x_1,2) & (x_1,3) \\
\hline
 & (x_2,1) & 0 & 0 & 0 & 1 & 0 & 0 \\
\mathfrak{R}_{13} & (x_2,2) & 0 & 0 & 0 & 0 & 1 & 0 \\
 & (x_2,3) & 0 & 0 & 0 & 0 & 1 & 1 \\
\hline
 & (x_3,1) & 1 & 1 & 0 & 0 & 0 & 0 \\
\mathfrak{R}_{11} & (x_3,2) & 0 & 1 & 0 & 0 & 0 & 0 \\
 & (x_3,3) & 0 & 0 & 1 & 0 & 0 & 0 \\
\hline
\end{array}
$$

$$
U_2^1 = \begin{array}{cc|cccccc|}
 & & \mathfrak{C}_{11}^{+} & & & & & \\
 & & (x_1,1) & (x_1,2) & (x_1,3) & (x_4,1) & (x_4,2) & (x_4,3) \\
\hline
 & (x_2,1) & 1 & 0 & 0 & 0 & 0 & 0 \\
\mathfrak{R}_{13} & (x_2,2) & 0 & 1 & 0 & 0 & 0 & 0 \\
 & (x_2,3) & 0 & 1 & 1 & 0 & 0 & 0 \\
\hline
 & (x_3,1) & 0 & 0 & 0 & 1 & 1 & 0 \\
\mathfrak{R}_{11} & (x_3,2) & 0 & 0 & 0 & 0 & 1 & 0 \\
 & (x_3,3) & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
\end{array}
$$

$$
U_3^1 = \begin{array}{cc|cccccc|}
 & & \mathfrak{C}_{11}^{-} & & & & & \\
 & & (x_1,1) & (x_1,2) & (x_1,3) & (x_4,1) & (x_4,2) & (x_4,3) \\
\hline
 & (x_2,1) & 1 & 0 & 0 & 0 & 0 & 0 \\
\mathfrak{R}_{13} & (x_2,2) & 0 & 1 & 0 & 0 & 0 & 0 \\
 & (x_2,3) & 0 & 1 & 1 & 0 & 0 & 0 \\
\hline
 & (x_3,1) & 0 & 0 & 0 & 1 & 1 & 0 \\
\mathfrak{R}_{11} & (x_3,2) & 0 & 0 & 0 & 0 & 1 & 0 \\
 & (x_3,3) & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
\end{array}
$$

$$
U_4^1 = \begin{array}{cc|ccc|ccc|}
 & & & \mathfrak{C}_{11}^{+} & & & \mathfrak{C}_{11}^{-} & \\
 & & (x_1,1) & (x_1,2) & (x_1,3) & (x_4,1) & (x_4,2) & (x_4,3) \\
\hline
 & (x_2,1) & 1 & 0 & 0 & 0 & 0 & 0 \\
\mathfrak{R}_{13} & (x_2,2) & 0 & 1 & 0 & 0 & 0 & 0 \\
 & (x_2,3) & 0 & 1 & 1 & 0 & 0 & 0 \\
\hline
 & (x_3,1) & 0 & 0 & 0 & 1 & 1 & 0 \\
\mathfrak{R}_{11} & (x_3,2) & 0 & 0 & 0 & 0 & 1 & 0 \\
 & (x_3,3) & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
\end{array}
$$

*The dimensions of the vector spaces are given by the respective band sizes, and* $\dim(\mathfrak{R}_{12}) = 0$*.*

2. *Consider the bundle of semichains from Example 4.52.2. and the following* $\mathfrak{L}$*–chain with two double ends:*

$$
\begin{array}{rcccccc}
g: & \mathfrak{C}_{11} - & \mathfrak{R}_{12} \sim & \mathfrak{C}_{21} - & \mathfrak{R}_{21} \sim & \mathfrak{R}_{32} - & \mathfrak{C}_{31} \\
 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\
\psi_1: & -1 & & & & & 1 \\
\psi_2: & 1 & & & & & 1 \\
\psi_3: & -1 & & & & & -1 \\
\psi_4: & 1 & & & & & -1
\end{array}
$$

*Let* $p = 2$*. Then we consider for the construction of the representations* $U_s(g, 2)$ *the* $\mathfrak{L}$*–chain* $h = g^{[2]}$*:*

$$
h: \quad \mathfrak{C}_{11} - \mathfrak{R}_{12} \sim \mathfrak{C}_{21} - \mathfrak{R}_{21} \sim \mathfrak{R}_{32} - \overleftarrow{\mathfrak{C}_{31} \sim \mathfrak{C}_{31}} - \mathfrak{R}_{32} \sim \mathfrak{R}_{21} - \mathfrak{C}_{21} \sim \mathfrak{R}_{12} - \mathfrak{C}_{11}
$$

$$
\begin{array}{r cccccc cccccc}
 & \multicolumn{6}{c}{\underbrace{x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6}_{1}} & \multicolumn{6}{c}{\underbrace{x_6 \quad x_5 \quad x_4 \quad x_3 \quad x_2 \quad x_1}_{2}} \\
\bar{\psi}_1: & -1 & & & & & 1 & -1 & & & & & 1 \\
\bar{\psi}_2: & 1 & & & & & 1 & -1 & & & & & -1 \\
\bar{\psi}_3: & -1 & & & & & -1 & 1 & & & & & 1 \\
\bar{\psi}_4: & 1 & & & & & -1 & 1 & & & & & -1
\end{array}
$$

*For any $s$, the matrices $U_s^4$ and $U_s^5$ are empty. Thus, the vector spaces $U_{X_{4j}}$ for $X \in \{\mathfrak{C},\mathfrak{R}\}$, $j \in \{1,2\}$, $U_{\mathfrak{C}_{51}^{\zeta}}$ for $\zeta \in \{+,-\}$ and $U_{\mathfrak{R}_{5j}}$ for $1 \le j \le 3$ have dimension $0$. Moreover, the vector spaces $U_{\mathfrak{R}_{11}}$, $U_{\mathfrak{C}_{23}}$, $U_{\mathfrak{C}_{22}}$, $U_{\mathfrak{R}_{22}}$, $U_{\mathfrak{R}_{23}}$ and $U_{\mathfrak{R}_{31}}$ also have dimension $0$. The dimensions of the other vector spaces are for any $s \in \{1,2,3,4\}$ given by*

$$\begin{aligned}
&\dim(U_{\mathfrak{R}_{12}}) = \dim(U_{\mathfrak{R}_{21}}) = \dim(U_{\mathfrak{R}_{32}}) = \dim(U_{\mathfrak{C}_{21}}) = 2,\\
&\dim(U_{\mathfrak{C}_{11}^{\zeta}}) = \dim(U_{\mathfrak{C}_{31}^{\zeta}}) = 1, \quad \zeta \in \{+,-\}.
\end{aligned}$$

*The other matrices are given as follows:*
$\mathbf{s = 1}$*:*

$$U_1^1 = \begin{array}{cc|c|c|} & & \mathfrak{C}_{11}^{+} & \mathfrak{C}_{11}^{-} \\ & & (x_1,2) & (x_1,1) \\ \cline{3-4} \mathfrak{R}_{12} & (x_2,1) & 0 & 1 \\ & (x_2,2) & 1 & 0 \\ \cline{3-4} \end{array}, \quad U_1^2 = \begin{array}{cc|cc|} & & \mathfrak{C}_{21} & \\ & & (x_3,1) & (x_3,2) \\ \cline{3-4} \mathfrak{R}_{21} & (x_4,1) & 1 & 0 \\ & (x_4,2) & 0 & 1 \\ \cline{3-4} \end{array},$$

$$U_1^3 = \begin{array}{cc|c|c|} & & \mathfrak{C}_{31}^{+} & \mathfrak{C}_{31}^{-} \\ & & (x_6,1) & (x_6,2) \\ \cline{3-4} \mathfrak{R}_{32} & (x_5,1) & 1 & 1 \\ & (x_5,2) & 0 & 1 \\ \cline{3-4} \end{array},$$

$\mathbf{s = 2}$*:*

$$U_2^1 = \begin{array}{cc|c|c|} & & \mathfrak{C}_{11}^{+} & \mathfrak{C}_{11}^{-} \\ & & (x_1,1) & (x_1,2) \\ \cline{3-4} \mathfrak{R}_{12} & (x_2,1) & 1 & 0 \\ & (x_2,2) & 0 & 1 \\ \cline{3-4} \end{array}, \quad U_2^2 = U_1^2, \quad U_2^3 = U_1^3,$$

$\mathbf{s = 3}$*:*

$$U_3^1 = U_1^1, \quad U_3^2 = U_2^2, \quad U_3^3 = \begin{array}{cc|c|c|} & & \mathfrak{C}_{31}^{+} & \mathfrak{C}_{31}^{-} \\ & & (x_6,2) & (x_6,1) \\ \cline{3-4} \mathfrak{R}_{32} & (x_5,1) & 1 & 1 \\ & (x_5,2) & 1 & 0 \\ \cline{3-4} \end{array},$$

$\mathbf{s = 4}$*:*

$$U_4^1 = U_1^1, \quad U_4^2 = U_3^2, \quad U_4^3 = U_3^3.$$

### 4.1.5 Classification Theorem

**Theorem 4.61.** *[Bon91, Main Theorem] Choosing one representative in each isomorphism class of $\mathfrak{L}$–chains and $\mathfrak{L}$–cycles of $\bar{\mathfrak{S}}(\mathfrak{L}) \cup \mathring{\mathfrak{S}}(\mathfrak{L})$ gives the following classification:*
*The set of representations of the form $U_s(g)$, $U_s(g,p)$ and $U(g,\varphi)$ associated to the representative $\mathfrak{L}$–graphs is a complete set of pairwise nonequivalent indecomposable representations of the bundle $\overline{\mathfrak{X}} = (\mathfrak{X}, \sigma)$.*

## 4.2 The category $\mathrm{Rep}(\overline{\mathfrak{X}})$

Let $\mathfrak{X} = (\mathfrak{C}_1, \dots, \mathfrak{C}_N, \mathfrak{R}_1, \dots, \mathfrak{R}_N)$ be as in Section 4.1, and let $\overline{\mathfrak{X}} = (\mathfrak{X}, \sigma)$ be a bundle of semichains with involution $\sigma$.
Let $U = (U_X, U^i)_{X \in \mathfrak{X}_0, 1 \le i \le N}$ be a representation of $\overline{\mathfrak{X}}$. For $S \subseteq \mathfrak{X}_0$ a subset, denote by $U_S$ the $\mathrm{k}$-vector space

$$U_S = \bigoplus_{Y \in S} U_Y.$$

Then $U^i : U_{\mathfrak{C}_i} \to U_{\mathfrak{R}_i}$ can be written in terms of an $(n^U_{\mathfrak{C}_i} \times n^U_{\mathfrak{R}_i})$–matrix, where $n^U_{\mathfrak{C}_i} = \dim(U_{\mathfrak{C}_i})$, $n^U_{\mathfrak{R}_i} = \dim(U_{\mathfrak{R}_i})$.
Similarly, we denote by $n^U_X$ the dimension of the vector space $U_X$, $X \in \mathfrak{X}_0$. We write $U^i_{XY} : U_X \to U_Y$ to denote the respective restriction (block matrix in) of $U^i$, $X \in \mathfrak{C}_i$, $Y \in \mathfrak{R}_i$.
Let $W = (W_X, W^i)_{X \in \mathfrak{X}_0, 1 \le i \le N}$ be a different representation of $\overline{\mathfrak{X}}$. Its map $W^i : W_{\mathfrak{C}_i} \to W_{\mathfrak{R}_i}$ is given by an $(n^W_{\mathfrak{C}_i} \times n^W_{\mathfrak{R}_i})$–matrix.

**Remark 4.62.** *Note that the bands in $U^i$ and $W^i$ are indexed by the same elements, but are of possibly different sizes.*

**Definition 4.63.** *Let $\overline{\mathfrak{X}} = (\mathfrak{X}, \sigma)$ be a bundle of semichains as described above. Then the category $\mathrm{Rep}(\overline{\mathfrak{X}})$ of representations of $\overline{\mathfrak{X}}$ is given by the following data:*

- *The **objects** of $\mathrm{Rep}(\overline{\mathfrak{X}})$ are given by representations of $\overline{\mathfrak{X}}$, that is, tuples of the form $U = (U_X, U^i)_{X \in \mathfrak{X}_0, 1 \le i \le N}$.*
- *A **morphism** $\theta : U \to W$ between two representations $U = (U_X, U^i)_{X,i}$ and $W = (W_X, W^i)_{X,i}$ is given by a tuple $\theta = (P, Q)$. Each entry of this tuple consists of $N$ $\mathrm{k}$-linear maps $P^1, \dots, P^N$, $Q^1, \dots, Q^N$, respectively, such that the conditions (i) - (iv) below are satisfied. As in the defintion of a $\overline{\mathfrak{X}}$–representation, one can also equivalently consider the $P^i$'s and $Q^i$'s as finite matrices with band structure given by the semichains of the bundle.*

  *(i)* $P^i : U_{\mathfrak{C}_i} \to W_{\mathfrak{C}_i}$ *and* $Q^i : U_{\mathfrak{R}_i} \to W_{\mathfrak{R}_i}$, $\forall 1 \le i \le N$,

  *(ii)* $Q^i U^i = W^i P^i$, $\forall 1 \le i \le N$,

  *(iii) for* $X, Y \in \mathfrak{X}_0$, $X \neq Y$ *and* $\sigma(X) = Y$*:*

  *(a) if* $X \in \mathfrak{R}_i$, $Y \in \mathfrak{R}_j$ *for some* $1 \le i, j \le N$, *then*

  $$Q^i_{XX} = Q^j_{YY},$$

  *(b) if* $X \in \mathfrak{C}_i$, $Y \in \mathfrak{C}_j$ *for some* $1 \le i, j \le N$, *then*

  $$P^i_{XX} = P^j_{YY},$$

(c) *if* $X \in \mathfrak{R}_i$ *and* $Y \in \mathfrak{C}_j$*, or* $X \in \mathfrak{C}_i$ *and* $Y \in \mathfrak{R}_j$*, for some* $1 \le i, j \le N$*, then*

$$Q^i_{XX} = P^j_{YY} \quad (P^i_{XX} = Q^j_{YY},\ \textit{respectively})$$

(iv) *for* $X, Y \in \mathfrak{X}_0$ *and* $\star$ *a block of the respective size with arbitrary (possibly zero) entries from* k*:*

(a) *if* $X, Y \in \mathfrak{C}_i$*, for some* $i \in \{1, \dots, N\}$*, then*

$$P^i_{XY} = \begin{cases} 0 & \textit{if } X < Y \textit{ or } X \bowtie Y \\ \star & \textit{if } X \ge Y \end{cases},$$

(b) *if* $X, Y \in \mathfrak{R}_i$ *for some* $i \in \{1, \dots, N\}$*, then*

$$Q^i_{XY} = \begin{cases} 0 & \textit{if } X > Y \textit{ or } X \bowtie Y \\ \star & \textit{if } X \le Y \end{cases}.$$

- *The* ***identity morphism*** *on a representation* $U$ *is given by* $\mathbb{1}_U = (P, Q)$ *where* $P^i = \mathbb{1}_{c_i \times c_i}$*,* $Q^i = \mathbb{1}_{r_i \times r_i}$*, where* $r_i = n^U_{\mathfrak{R}_i}$*,* $c_i = n^U_{\mathfrak{C}_i}$*.*
- *Let* $\theta = (P, Q) : U \to V$*,* $\varphi = (R, S) : V \to W$ *be two morphisms. Then their* ***composition*** *is given componentwise:* $\varphi \circ \theta = (RP, SQ)$ *such that*

$$S^i Q^i U^i = W^i R^i P^i \quad \forall 1 \le i \le N.$$

Note that $P^i$ and $Q^i$ inherit their band structures from $U^i$, $W^i$, respectively:

- $P^i_{XY}$ is of size $n^W_X \times n^U_Y$ with $X, Y \in \mathfrak{C}_i$.
- $Q^i_{XY}$ is of size $n^W_X \times n^U_Y$ with $X, Y \in \mathfrak{R}_i$.

It is a well-known fact that the Krull-Schmidt Theorem holds for $\mathrm{Rep}(\overline{\mathfrak{X}})$ (see [Bon91, KR77]) and that $\mathrm{Rep}(\overline{\mathfrak{X}})$ is additive.

**Example 4.64.** *Let* $N = 1$*,* $\mathfrak{X} = \{\mathfrak{C}_1, \mathfrak{R}_1\}$ *with*

$$\mathfrak{C}_1 = \{\mathfrak{C}^+_{11} \bowtie \mathfrak{C}^-_{11}\},$$
$$\mathfrak{R}_1 = \{\mathfrak{R}_{11} > \mathfrak{R}_{12} > \mathfrak{R}_{13}\}.$$

*Let* $\sigma$ *act as identity on the elements with the following exception:*

$$\sigma(\mathfrak{R}_{11}) = \mathfrak{R}_{13}.$$

*We know from the previous section that its representations of* $\overline{\mathfrak{X}} = (\mathfrak{X}, \sigma)$ *have the following band structure with* $n_{\mathfrak{R}_{11}} = n_{\mathfrak{R}_{13}}$*:*

$$\begin{array}{c|c|c|} & \mathfrak{C}^+_{11} & \mathfrak{C}^+_{11} \\ \cline{2-3} \mathfrak{R}_{13} & & \\ \cline{2-3} \mathfrak{R}_{12} & & \\ \cline{2-3} \mathfrak{R}_{11} & & \\ \cline{2-3} \end{array} \tag{66}$$

*Let $U$ and $W$ be two representations of $\overline{\mathfrak{X}}$ and $\theta = (P, Q) : U \to W$ be a morphism between them. Then the components of $\theta$ are of the following forms with respect to their bands:*

$$P^1 = \begin{array}{c|c|c|} & \mathfrak{C}_{11}^{+} & \mathfrak{C}_{11}^{-} \\ \hline \mathfrak{C}_{11}^{+} & \star & 0 \\ \hline \mathfrak{C}_{11}^{-} & 0 & \star \\ \hline \end{array}, \quad Q^1 = \begin{array}{c|c|c|c|} & \mathfrak{R}_{13} & \mathfrak{R}_{12} & \mathfrak{R}_{11} \\ \hline \mathfrak{R}_{13} & A & 0 & 0 \\ \hline \mathfrak{R}_{12} & \star & \star & 0 \\ \hline \mathfrak{R}_{11} & \star & \star & A \\ \hline \end{array}.$$

*Here, both $A$ and $\star$ denote a block of the respective size with arbitrary entries from $\mathrm{k}$. The two $A$–blocks are equal.*

**Example 4.65.** *Let $N = 1$ and $\mathfrak{X} = \{\mathfrak{C}_1, \mathfrak{R}_1\}$ with*

$$\mathfrak{C}_1 = \{\mathfrak{C}_{11} < \mathfrak{C}_{12} < \mathfrak{C}_{13}^{+} \approx \mathfrak{C}_{13}^{-} < \mathfrak{C}_{14}\}, \tag{67}$$

$$\mathfrak{R}_1 = \{\mathfrak{R}_{11} > \mathfrak{R}_{12}^{+} \approx \mathfrak{R}_{12}^{-} > \mathfrak{R}_{13}\}. \tag{68}$$

*The involution $\sigma$ acts as follows:*

$$\sigma : \quad \mathfrak{C}_{11} \mapsto \mathfrak{R}_{11}, \qquad \mathfrak{C}_{12} \mapsto \mathfrak{C}_{14},$$

*and as identity on the other elements. Thus, any representation of $\overline{\mathfrak{X}}$ is of the following form with $n_{\mathfrak{C}_{11}} = n_{\mathfrak{R}_{11}}$ and $n_{\mathfrak{C}_{12}} = n_{\mathfrak{C}_{14}}$:*

$$\begin{array}{c|c|c|c|c|c|} & \mathfrak{C}_{11} & \mathfrak{C}_{12} & \mathfrak{C}_{13}^{+} & \mathfrak{C}_{13}^{-} & \mathfrak{C}_{14} \\ \hline \mathfrak{R}_{13} & & & & & \\ \hline \mathfrak{R}_{12}^{+} & & & & & \\ \hline \mathfrak{R}_{12}^{-} & & & & & \\ \hline \mathfrak{R}_{11} & & & & & \\ \hline \end{array} \tag{69}$$

*Let $U$ and $W$ be two representations of $\overline{\mathfrak{X}}$ and $\theta = (P, Q) : U \to W$ a morphism. Its components are of the following forms with respect to their bands:*

$$P^1 = \begin{array}{c|c|c|c|c|c|} & \mathfrak{C}_{11} & \mathfrak{C}_{12} & \mathfrak{C}_{13}^{+} & \mathfrak{C}_{13}^{-} & \mathfrak{C}_{14} \\ \hline \mathfrak{C}_{11} & B & \star & \star & \star & \star \\ \hline \mathfrak{C}_{12} & 0 & A & \star & \star & \star \\ \hline \mathfrak{C}_{13}^{+} & 0 & 0 & \star & 0 & \star \\ \hline \mathfrak{C}_{13}^{-} & 0 & 0 & 0 & \star & \star \\ \hline \mathfrak{C}_{14} & 0 & 0 & 0 & 0 & A \\ \hline \end{array}, \quad Q^1 = \begin{array}{c|c|c|c|c|} & \mathfrak{R}_{13} & \mathfrak{R}_{12}^{+} & \mathfrak{R}_{12}^{-} & \mathfrak{R}_{11} \\ \hline \mathfrak{R}_{13} & \star & 0 & 0 & 0 \\ \hline \mathfrak{R}_{12}^{+} & \star & \star & 0 & 0 \\ \hline \mathfrak{R}_{12}^{-} & \star & 0 & \star & 0 \\ \hline \mathfrak{R}_{11} & \star & \star & \star & B \\ \hline \end{array}.$$

*Here, $A$ and $B$ denote blocks of the respective sizes with arbitrary entries from $\mathrm{k}$.*

We now prove the well-definedness of compositions in $\mathrm{Rep}(\overline{\mathfrak{X}})$.

**Lemma 4.66.** *Let $\theta = (P,Q) : U \to V$ and $\varphi = (R,S) : V \to W$ be morphisms in* $\operatorname{Rep}(\overline{\mathfrak{X}})$*. Then $\varphi \circ \theta \in \operatorname{Rep}(\overline{\mathfrak{X}})$.*

*Proof.* It is enough to check (iii) and (iv) of Definition 4.63. In order to do so, we denote in the following the ordering of the bands in the matrices by $<_{\pm}$. Let $X, Y \in \mathfrak{X}_0$. We write $X <_{\pm} Y$ if $X < Y$, or if $X = Z^+$ and $Y = Z^-$. In particular, it follows that

$$\begin{aligned} &P^i_{XY} = 0 \text{ and } R^i_{XY} = 0 && \text{for } X <_{\pm} Y, \\ &Q^i_{XY} = 0 \text{ and } S^i_{XY} = 0 && \text{for } X >_{\pm} Y. \end{aligned}$$

Since we use (iv) to prove (iii), we start with the former.

(iv) Let $X$ and $Y$ be in $\mathfrak{X}_0$. It is enough to consider the zero blocks in the composition.

(a) Let $X, Y \in \mathfrak{C}_i$ for some $i \in \{1, \dots, N\}$. Let $X < Y$. We obtain

$$\begin{aligned} (R^i P^i)_{XY} &= \sum_{Z \in \mathfrak{C}_i} R^i_{XZ} P^i_{ZY} \\ &= \sum_{Z <_{\pm} X} R^i_{XZ} P^i_{ZY} + \sum_{X <_{\pm} Z <_{\pm} Y} R^i_{XZ} P^i_{ZY} + \sum_{Y <_{\pm} Z} R^i_{XZ} P^i_{ZY} \\ &\quad + R^i_{XX} P^i_{XY} + R^i_{XY} P^i_{YY} \\ &= 0. \end{aligned} \tag{70}$$

Finally, let $X$ and $Y$ be incomparable, say $X <_{\pm} Y$. Then

$$\begin{aligned} (R^i P^i)_{XY} &= \sum_{Z <_{\pm} X} R^i_{XZ} P^i_{ZY} + \sum_{Y <_{\pm} Z} R^i_{XZ} P^i_{ZY} + R^i_{XX} P^i_{XY} \\ &\quad + R^i_{XY} P^i_{YY} \\ &= 0, \end{aligned} \tag{71}$$

and

$$\begin{aligned} (R^i P^i)_{YX} &= \sum_{Z <_{\pm} X} R^i_{YZ} P^i_{ZX} + \sum_{Y <_{\pm} Z} R^i_{YZ} P^i_{ZX} + R^i_{YX} P^i_{XX} \\ &\quad + R^i_{YY} P^i_{YX} \\ &= 0. \end{aligned} \tag{72}$$

Combining (70) - (72) gives

$$(R^i P^i)_{XY} = 0 \quad \text{if } X < Y \text{ or } X \lessgtr Y$$

(b) Proceeding analogously to (a) results in

$$(S^i Q^i)_{XY} = 0 \quad \text{if } X > Y \text{ or } X \lessgtr Y.$$

(iii) Let $X \neq Y \in \mathfrak{X}_0$ with $\sigma(X) = Y$. Note that $X$ and $Y$ are comparable.

(a) Let $X \in \mathfrak{R}_i$ and $Y \in \mathfrak{R}_j$ for some $i, j \in \{1, \dots, N\}$. We have that

$$Q^i_{XX} = Q^j_{YY} \qquad \text{and} \qquad S^i_{XX} = S^j_{YY}. \tag{73}$$

We obtain that

$$\begin{aligned}(S^iQ^i)_{XX} &= \sum_{Z <_\pm X} S^i_{XZ}Q^i_{ZX} + \sum_{X <_\pm Z} S^i_{XZ}Q^i_{ZX} + S^i_{XX}Q^i_{XX} \\ &= S^i_{XX}Q^i_{XX} \end{aligned} \tag{74}$$

and analogously, that

$$(S^jQ^j)_{YY} = S^j_{YY}Q^j_{YY} \tag{75}$$

Combining (73), (74) and (75) gives the desired result:

$$(S^iQ^i)_{XX} = (S^jQ^j)_{YY}.$$

(b) Consider $X \in \mathfrak{C}_i$ and $Y \in \mathfrak{C}_j$ for some $i \in \{1, \dots, N\}$. It follows analogously to (a) that

$$(R^iP^i)_{XX} = (R^jP^j)_{YY}.$$

(c) Let $X \in \mathfrak{C}_i$ and $Y \in \mathfrak{R}_j$ for some $i, j \in \{1, \dots, N\}$ ($X \in \mathfrak{R}_i$ and $Y \in \mathfrak{C}_j$). It follows analogously to (a) that

$$(R^iP^i)_{XX} = (S^jQ^j)_{YY} \quad ((R^jP^j)_{YY} = (S^iQ^i)_{XX}) \tag{76}$$

We know already from (a) and (b) that

$$\begin{aligned}(R^iP^i)_{XX} &= R^i_{XX}P^i_{XX}, \\ (S^jQ^j)_{YY} &= S^j_{YY}Q^j_{YY}.\end{aligned}$$

This results with (76) in

$$(S^jQ^j)_{YY} = S^j_{YY}Q^j_{YY} = R^i_{XX}P^i_{XX} = (R^iP^i)_{XX}$$

The case $X \in \mathfrak{R}_i$, $Y \in \mathfrak{C}_j$ for some $i, j \in \{1, \dots, N\}$ results analogously to the above in

$$(S^iQ^i)_{XX} = (R^jP^j)_{YY}.$$

□

**Remark 4.67.** *In terms of matrices, a morphism $\theta = (P, Q)$ is an isomorphism provided that any of its components $P^i$ and $Q^i$ has full rank.*
*Any admissible transformation has full rank and respects the conditions (i) - (iv) of a morphism in* $\mathrm{Rep}(\overline{\mathfrak{X}})$ *by definition. Thus, any admissible transformation gives an isomorphism in* $\mathrm{Rep}(\overline{\mathfrak{X}})$.

The converse of the previous remark is not trivial but we obtain the following:

**Lemma 4.68.** *Any isomorphism in* $\mathrm{Rep}(\overline{\mathfrak{X}})$ *is given by a finite product of admissible transformations.*

*Proof.* Let $\theta = (P, Q)$ be an isomorphism in $\mathrm{Rep}(\overline{\mathfrak{X}})$. Then, in terms of matrices, $P^i$ is an upper and $Q^i$ is a lower triangular matrix for any $1 \le i \le N$. We first consider $Q^i$.
We apply Gauss elimination to the block rows of $Q^i$. Note that for incomparable elements $X^+ \not\lessgtr X^-$ the block $Q^i_{X^+X^-}$ – which is situated below the diagonal block $Q^i_{X^+X^+}$ – is already 0 and does not need to be eliminated. Together with $Q^i$ having upper triangular form, Gauss elimination thus only requires admissible transformations of type 2. We denote the obtained matrix by $\tilde{Q}^i$:

$$\tilde{Q}^i = \left( \prod_{k,l,\lambda} G(k, l, \lambda) \right) Q^i, \tag{77}$$

where $G(k, l, \lambda)$ describes the operation on the row blocks $k$ and $l$ with $\lambda \in \mathrm{k}^\times$ (e.g. adding block row $k$ multiplied by $\lambda$ to block row $l$). Now $\tilde{Q}^i$ is of diagonal block form: $(X, Y \in \mathfrak{C})$

$$\tilde{Q}^i_{XY} = \begin{cases} 0, & \text{if } X \ne Y, \\ \star, & \text{if } X = Y. \end{cases}$$

Thus, we can write it as follows:

$$\tilde{Q}^i = \prod_{X \in \mathfrak{C}_i} D^i_X \tag{78}$$

where $D^i_X$ denotes the square matrix of same size as $\tilde{Q}^i$, with an arbitrary block $(D^i_X)_{XX}$, identity blocks in all other diagonal blocks $(D^i_X)_{YY}$, $Y \ne X$, and 0–entries in all off-diagonal blocks:

$$(D^i_X)_{ZY} = \begin{cases} A^i_X & \text{if } Z = Y = X \\ 1 & \text{if } Z = Y \ne X \\ 0 & \text{else} \end{cases}.$$

Any such $D_X^i$ is an admissible transformation of type 1a or 1b. Thus, combining (77) and (78), we can write $Q^i$ as a product of admissible transformations of type 2 and of type 1:

$$Q^i = \left(\prod_{k,l,\lambda} G(k,l,\lambda)\right)\left(\prod_{X\in\mathfrak{C}_i} D_X^i\right).$$

Now consider $P^i$. Proceeding analogously as for $Q^i$ with respect to $P^i$ being of upper triangular form and with respect to its block columns, we obtain

$$P^i = \left(\prod_{X\in\mathfrak{C}_i} \bar{D}_X^i\right)\left(\prod_{k,l,\lambda} \bar{G}^{-1}(k,l,\lambda)\right).$$

with similar notation as above. In particular, we denote by $\bar{A}_X^i$ its non-arbitrary block of $\bar{D}_X^i$ in position $XX$. Thus, $P^i$ can be written as a product of admissible transformations of type 1 and of type 2.
Note that the assumptions on certain matrices with respect to admissible transformations of type 1a and 1b are satisfied. This is due to $\tilde{Q}^i$ and $\tilde{P}^i$ being for any $1 \leq i \leq N$ of diagonal block form. Thus, for $X \neq Y \in \mathfrak{X}_0$ and $\sigma(X) = Y$, we have

- for $X \in \mathfrak{R}_i$, $Y \in \mathfrak{R}_j$ ($X \in \mathfrak{C}_i$, $Y \in \mathfrak{C}_j$) for some $i,j \in \{1,\dots,N\}$ that
$$A_X^i = A_Y^j \qquad (\bar{A}_X^i = \bar{A}_Y^j),$$
- for $X \in \mathfrak{R}_i$, $Y \in \mathfrak{C}_j$ ($X \in \mathfrak{C}_i$, $Y \in \mathfrak{R}_j$) for some $i,j \in \{1,\dots,N\}$ that
$$A_X^i = \bar{A}_Y^j \qquad (\bar{A}_X^i = A_Y^j).$$

Finiteness of the products follows in both cases from the finiteness of $Q^i$, $P^i$, respectively. □

**Remark 4.69.** *By Lemma 4.68, isomorphic representations in* $\mathrm{Rep}(\overline{\mathfrak{X}})$ *are given by equivalent ones (cf. Definition 4.10). In particular, the commutativity relation $Q^iU^i = W^iP^i$ implies for the isomorphism $\theta = (P,Q) : U \longrightarrow W$ that*

$$U^i = \left(Q^i\right)^{-1} W^i P^i.$$

*Thus, condition (iii) on morphisms ensures that any arbitrary elementary transformation on the band $P(X)$ implies the same or, respectively, inverse transformation on the band $P(Y)$, where $X$ and $Y$ are two links with $\sigma(X) = Y$.*

### 4.3 Reduction to skewed-gentle algebras

In the following subsections, we take the next step in order to obtain a classification of the indecomposable finite dimensional modules of a clannish algebra.
We can deduce from any clannish algebra $\bar{\Lambda}$ a skewed-gentle algebra $\Lambda$, by neglecting some of the zero relations of $\bar{\Lambda}$ such that all necessary conditions for $\Lambda$ are fulfilled (compare Definition 2.9, conditions (v) and (v)*).
In the following, we will restrict ourselves to skewed-gentle algebras $\Lambda$. Recall that any skewed-gentle algebra is clannish by Lemma 2.11. Hence, the previous results on clannish algebras also hold for skewed-gentle algebras.
Starting in the next subsection, we describe how to transform the setup of a skewed-gentle algebra $\Lambda$ into the setup of a bundle of semichains $\overline{\mathfrak{X}}_\Lambda$ as described in [Bon91]. The construction of $\overline{\mathfrak{X}}_\Lambda$ will be given in such a way that it is compatible with the directions on special letters of finite index for strings and bands (Proposition 4.145). Eventually, our construction will lead to an equivalence between the categories $\mathrm{Rep}(\overline{\mathfrak{X}}_\Lambda)$ and $\mathrm{mod}(\Lambda)$ (Theorem 5.6). Moreover, we obtain a classification of the indecomposable finite dimensional modules of a skewed-gentle algebra in terms of strings and bands (Theorem 5.49). This classification derives from the former mentioned equivalence. Finally, we will be able to give a reformulation of this classification which will lead to the classification for clannish algebras as conjectured in [CB88] (Theorem 6.10). Starting from the classification on skewed-gentle algebras, we will deduce a classification on clannish algebras as follows:
Let $\Lambda = \mathrm{k}Q/(\mathrm{R} \cup \mathrm{R}^{\mathrm{Sp}})$ be a skewed-gentle algebra and let $\bar{\Lambda} = \mathrm{k}Q/(\bar{\mathrm{R}} \cup \bar{\mathrm{R}}^{\mathrm{Sp}})$ be a clannish algebra. Assume that we obtain $\bar{\Lambda}$ from $\Lambda$ by adding the set of relations $\{r_1 \dots r_{k_i}\}_{i\in\mathrm{I}}$ to R. Denote by $(V_i, V_x)_{i\in Q_0, x\in Q_1}$ a $\Lambda$–representation. Take the list of indecomposable finite dimensional modules of $\Lambda$ given by its classification and dismiss all those modules $V$ which do not fulfill the relations $r \in \{r_1 \dots r_{k_i}\}_{i\in\mathrm{I}}$: $V_{r_1} \cdots V_{r_{k_i}} \neq 0$ for all $i \in \mathrm{I}$. The remaining ones give a classificaiton of the indecomposable finite dimensional modules of $\bar{\Lambda}$.

## 4.4 Construction of a bundle of semichains

In this section we want to give a description on how to transform the setup of a skewed-gentle algebra $\Lambda$ into the setup of a bundle of semichains $\overline{\mathfrak{X}}_\Lambda$ as described in [Bon91]. The construction of $\overline{\mathfrak{X}}_\Lambda$ will be given in such a way that it is compatible with the directions on special letters of finite index for strings and bands (Proposition 4.145). The construction given here coincides to some extend to the one of a bush given in [Den00] (see Remark 4.77). Eventually, our construction will lead to an equivalence between the categories $\mathrm{Rep}(\overline{\mathfrak{X}}_\Lambda)$ and $\mathrm{mod}(\Lambda)$ (Theorem 5.6).
For the rest of the chapter, let $\Lambda$ be a skewed-gentle algebra (unless stated otherwise) given by a quiver $Q$ with set of special loops given by Sp and a set of relations R. Let $V = (V_i, V_a)_{i\in Q_0, a\in Q_1}$ be an arbitrary representation of $\Lambda$.
Let $r = r_1 r_2 \in \mathrm{R}$ be a relation and let $a \in Q_1$ be an arrow. Then we write $a \in r$ if $r_1 = a$ or $r_2 = a$. Note that in this case we have that $a \in Q_1^{\mathrm{ord}}$.
The goal of this section is to prove the following statement:

**Theorem 4.70.** *Let $\Lambda = \mathrm{k}Q/(\mathrm{R}^{\mathrm{Sp}} \cup \mathrm{R})$ be a skewed-gentle algebra with* R *as described above. Then there exists for $\Lambda$ a bundle of semichains $\overline{\mathfrak{X}}_\Lambda$.*

We give the explicit construction of $\overline{\mathfrak{X}}_\Lambda$ in the Subsections 4.4.1 - 4.4.5 and the proof of the above theorem. Examples for the complete construction will be given in Subsection 4.4.7.

### 4.4.1 Filtrations from relations

In this subsection we describe how we obtain filtrations from the relations in $\mathrm{R} \cup \mathrm{R}^{\mathrm{Sp}}$. We need the filtrations in the following subsections in order to create semichains and assign them to the bundles.
We obtain five different types of filtrations. The types (1) - (3) are obtained from relations in R, while the filtrations of type (4) are obtained from $\mathrm{R}^{\mathrm{Sp}}$. Filtrations of type (5) will be called *standard filtration*. They do not arise from a relation.

First we describe the filtrations we obtain from R.
Let $i \in Q_0$ and $a \in Q_1^{\mathrm{ord}}$ with $s(a) = i$. If there does not exist $r \in \mathrm{R}$ with $a \in r$, then a filtration on $i$ is given by

$$(1) \quad F_i : 0 \subset \ker(a) \subset V_i,$$

where $\ker(a)$ describes the subspace generated by the kernel of $a$. We call any filtration of this form a *filtration of type (1).*
If there exists $r \in \mathrm{R}$ with $a \in r$, then there exists $b \in Q_1^{\mathrm{ord}}$ with $t(b) = i$ and $r = ab$. We consider

$$(2) \quad F_i : 0 \subset \mathrm{im}(b) \subset \ker(a) \subset V_i,$$

where $\mathrm{im}(b)$ denotes the subspace given by the image of $b$. This gives a *filtration of type (2)*.
Assume now that $t(a) = i$. We distinguish as before: if there does not exist $r \in \mathrm{R}$ with $a \in r$, then

$$(3) \quad F_i : 0 \subset \mathrm{im}(a) \subset V_i.$$

gives a *filtration of type (3)*.
If there exists a relation $r$ with $a \in r$, then we obtain again a filtration of type (2).

Let us now consider filtrations which we obtain from $\mathrm{R}^{\mathrm{Sp}}$.
Let $\varepsilon \in \mathrm{Sp}$ with $s(\varepsilon) = i$. Then we have the idempotent relation $\varepsilon^2 = \varepsilon$ on $\varepsilon$ and thus we can decompose $V_i$ into $V_i = \mathrm{im}(\varepsilon) \oplus \ker(\varepsilon)$. We obtain a filtration

$$(4) \qquad F_i : \quad \begin{tikzcd} & \mathrm{im}(\varepsilon) \arrow[dr] & \\ 0 \arrow[ur, hook] \arrow[dr, hook] & & V_i \\ & \ker(\varepsilon) \arrow[ur, hook] & \end{tikzcd} \tag{79}$$

which we call a *filtration of type (4)*. Instead of (79) we write:

$$0 \subset \mathrm{im}(\varepsilon) \oplus \ker(\varepsilon) = V_i$$

for this type of filtration.

At last, we consider a filtration that is not arising from a relation. We call this filtration *standard* or a *filtration of type (5)* and it is of the form

$$(5) \quad F_i : 0 \subset V_i.$$

This filtration arises at vertices $i \in Q_0$ which have at most one incident arrow.

Thus, if $i \in Q_0$ is not an isolated vertex (where isolated means that no arrows start and no arrows end in $i$), then we obtain two filtrations $F_i^{(1)}$, $F_i^{(2)}$ for $i$, of which at least one is of type (1) - (4), and the other is of type (1) - (5).
If, on the other hand, $i \in Q_0$ is an isolated vertex, then both $F_i^{(1)}$ and $F_i^{(2)}$ are filtrations of type (5).

### 4.4.2 Semichains from filtrations

In this subsection we describe how to construct several semichains from filtrations of type (1) - (5). Here, we are going to distinguish between the types

(1) - (3), (5) and type (4). This is due to the fact that (1) - (3),(5) have the form of a chain, while (4) is of diamond form. The goal is to obtain for each $F_i$ of the above types a semichain $S_i$ with elements corresponding in some way to the bases of the subspaces.

Let

$$F_i : 0 = V_{i0} \subset V_{i1} \subset \cdots \subset V_{in} = V_i$$

be a filtration of $V_i$ of type (1) - (3) or (5), $i \in Q_0$. Note that $n \leq 3$.
In the first step, we determine the bases of the respective subspaces and set

$$\begin{aligned}
&\mathcal{B}_{i0} = 0,\\
&\mathcal{B}_{i1} \text{ is a basis of } V_{i1},\\
&\mathcal{B}_{i2} \text{ is } \mathcal{B}_{i1} \text{ extended to a basis of } V_{i2},\\
&\quad\vdots
\end{aligned}$$

giving iteratively

$$\mathcal{B}_{ik} \text{ is } \mathcal{B}_{i,k-1} \text{ extended to a basis of } V_{ik}. \tag{80}$$

Additionally, we assume the following with respect to the bases $\{B_{ik}\}_k$:
Let $a : i \longrightarrow j \in Q_1$ and let

$$F_j : 0 = V_{j0} \subset V_{j1} \subset \cdots \subset V_{jm} = V_j$$

be a filtration on $V_j$. Note, that in case of $a$ being a loop, we have $i = j$ and thus $V_i = V_j$.
Let $k \in \{0, \dots, m\}$ such that $V_{jk}$ gives the subspace generated by the image of $a$, and let $l \in \{0, \dots, n\}$ such that $V_{il}$ is the subspace generated by the kernel of $a$, say

$$\begin{aligned}
V_{jk} &= \langle w_1, \dots, w_{\tilde{k}} \rangle,\\
V_{il} &= \langle v_1, \dots, v_{\tilde{l}} \rangle \quad \text{ and}\\
V_i / V_{il} &= \langle x_1 + V_{il}, \dots, x_{\tilde{n}} + V_{il} \rangle,
\end{aligned}$$

where the elements $w_1, \dots, w_{\tilde{k}}$ are linearly independent and same holds for $v_1, \dots, v_{\tilde{l}}$, and $x_1, \dots, x_{\tilde{n}}$. To simplify notation, in what follows, we will call $(x_1, \dots, x_{\tilde{n}})$ a *basis of* $V_i \ominus V_{il}$ and write $V_i \ominus V_{il}$ short for $\langle x_1, \dots, x_{\tilde{n}} \rangle$.
We assume that there exists for any $w_h \in \{w_1, \dots, w_{\tilde{k}}\}$ a unique $x_g \in \{x_1, \dots, x_{\tilde{n}}\}$ such that $a(x_g) = w_h$.

By definition of $\{\mathcal{B}_{ik}\}$ in (80), it follows that

$$\mathcal{B}_{ik} \supset \mathcal{B}_{il} \quad \forall l < k. \tag{81}$$

Hence,

$$\mathcal{B}^i_{k,k-1} \coloneqq \mathcal{B}_{i,k} \backslash \mathcal{B}_{i,k-1} \quad \forall k \in \{1, \dots, n\} \tag{82}$$

is well-defined. Using (82), we can now define the elements $S_{ik}$ of the semichain $S_i$ corresponding to $F_i$.
For $k \in \{1, \dots, n\}$, the element $S_{ik}$ corresponds to $\mathcal{B}^i_{k,k-1}$. We write

$$S_{ik} \mathrel{\hat{=}} \mathcal{B}^i_{k,k-1} \quad \forall k \in \{1, \dots, n\}. \tag{83}$$

Every element $S_{ik}$ belongs to a one-point link of the semichain. This one-point link will also be denoted by $S_{ik}$. It will be clear from the context, whether we refer to $S_{ik}$ as an element or as a link of the semichain $S_i$.
It remains to settle an ordering within the elements $\{S_{ik}\}_k$ of $S_i$. There are two possible ways to do that and we will use both in the following:

(i) We order the elements of $S_i$ with respect to the subspace inclusions in $F_i$, meaning

$$S_{i,k} > S_{i,k-1} \quad \forall k \in \{2, \dots, n\}. \tag{84}$$

We denote the resulting semichain by $S_i^{(c)}$ where the superscript stands for *compatible* with respect to the subspace inclusion:

$$S_i^{(c)} : \quad \{S_{i,1} < S_{i,2} < \dots < S_{i,n-1} < S_{i,n}\}. \tag{85}$$

(ii) We order the elements of $S_i$ in reversed order with respect to the subspace inclusions in $F_i$, giving

$$S_{i,k} < S_{i,k-1} \quad \forall k \in \{2, \dots, n\}. \tag{86}$$

We denote the resulting semichain by $S_i^{(r)}$, where the superscript stands for *reversed*:

$$S_i^{(r)} : \quad \{S_{i,n} < S_{i,n-1} < \dots < S_{i,2} < S_{i,1}.\} \tag{87}$$

**Remark 4.71.** *In case of $i$ being an isolated vertex, we obtain two standard filtrations. Each of them gives a semichain consisting of a single one-point link. We call this kind of semichain* standard.
*Since there is no ordering given on those semichains, we have*

$$S_i^{(c)} = S_i^{(r)} \tag{88}$$

*for any standard semichain $S_i$. Yet, for notational reasons, we will distinguish between those two copies of $S_i$ in the following.*

Thus, we have to choose $S_i$ from the set $\{S_i^{(c)}, S_i^{(r)}\}$. How to do so, we discuss in Subsection 4.4.4.

Let us now come to the semichains arising from filtrations of type (4). Hence, let

$$F_i : 0 = V_{i0} \subset (V_{i1} \oplus V_{i2}) = V_{i3} \tag{89}$$

with $V_{i3} = V_i$. Let $\varepsilon \in \mathrm{Sp}$ with $s(\varepsilon) = i$, and let $V_{i1} = \mathrm{im}(\varepsilon)$, $V_{i2} = \ker(\varepsilon)$.
Let $\mathcal{B}_{ik}$ be the basis of $V_{ik}$ for $k = 1, 2$. Then we have the following properties:

(i) $\mathcal{B}_{i1}$ is the basis of $\mathrm{im}(\varepsilon)$,

(ii) $\mathcal{B}_{i2}$ is the basis of $\ker(\varepsilon)$,

(iii) $\mathcal{B}_{i1} \cap \mathcal{B}_{i2} = 0$,

(iv) $V_i = \langle \mathcal{B}_{i1} \rangle \oplus \langle \mathcal{B}_{i2} \rangle$.

Again, we want to denote the elements of $S_i$ in terms of the bases of the respective subspaces of $F_i$. By (89) and (iv), there are only two bases to consider, namely $\mathcal{B}_{i1}$ and $\mathcal{B}_{i2}$.
Thus, the element $S_{ik}$ of $S_i$ corresponds to $\mathcal{B}_{ik}$, $k = 1, 2$. We write

$$S_{ik} \mathrel{\hat{=}} \mathcal{B}_{ik} \quad k = 1, 2.$$

By (iii), neither $\mathcal{B}_{i1} \subset \mathcal{B}_{i2}$, nor $\mathcal{B}_{i2} \subset \mathcal{B}_{i1}$. Hence, it is reasonable to consider $S_{i1}$ and $S_{i2}$ to be incomparable elements of $S_i$. We write

$$S_{i1} \lessgtr S_{i2}$$

to express the incomparability.
Together, these two elements form a link, denoted by $S_{i,\varepsilon}$, or simply by $S_\varepsilon$ (since $s(\varepsilon) = i$ by $\varepsilon$ being a loop), giving

$$S_i = \{S_{i1} \lessgtr S_{i2}\}. \tag{90}$$

We might by abuse of notation also denote the semichain $S_i$ by $S_\varepsilon$ if it is clear from the context, and refer to it as *special* semichain.

Hence, we obtain for each $i \in Q_0$ two semichains that we denote by $S_i^{(1)}$ and $S_i^{(2)}$. Each of them is of the form $S_i^{(r)}$, $S_i^{(c)}$ or $S_\varepsilon$.
Subsection 4.4.4 will give more information about the detailed choice of the semichains as one of the just mentioned forms.
First we want to extend the definiton of *signs* from letters to filtrations and semichains in the next section. Thus, the assignment of semichains to bundles in Subsection 4.4.4 can be given in a unique way.

### 4.4.3 Signs for filtrations and semichains

To describe the assignment of the semichains to the bundles $\mathfrak{R}$ and $\mathfrak{C}$, we will draw back on the notation of signs (compare Section 2.3).

Recall that we assign to each letter $x \in \Gamma_\star$, $\star \in \{\mathrm{ud}, \mathrm{d}\}$, a sign $\mathrm{sgn}(x) \in \{+,-\}$. Two letters $l, l'$ have the same sign if and only if $\{l, l'\} = \{x^{-1}, y\}$ for two arrows $x, y$, and either $xy \in \mathrm{R}$ or $x = y \in \mathrm{Sp}$.
Assume from now on that we have chosen the same sign for every $\varepsilon \in \mathrm{Sp}$:

$$\mathrm{sgn}(\varepsilon) = \kappa, \text{ for all } \varepsilon \in \mathrm{Sp}, \text{ some } \kappa \in \{+,-\}.$$

We want to use the signs of the letters appearing in a filtration to give their filtration a sign. To this end, let $F_i^{(j)}$, $j \in \{1,2\}$, be a filtration for some $i \in Q_0$. Depending on the type of $F_i^{(j)}$, we choose its sign as follows:

type (1): We choose the sign of $F_i^{(j)}$ with respect to the ordinary arrow determining it:

$$\mathrm{sgn}(F_i^{(j)}) = \mathrm{sgn}(a^{-1}). \tag{91}$$

type (2): We have for the two ordinary arrows determining $F_i^{(j)}$ that $\mathrm{sgn}(a^{-1}) = \mathrm{sgn}(b)$, and we choose the sign of the filtration according to this property:

$$\mathrm{sgn}(F_i^{(j)}) = \mathrm{sgn}(a^{-1}) = \mathrm{sgn}(b). \tag{92}$$

type (3): We proceed similar as in the case of type (1) which yields that

$$\mathrm{sgn}(F_i^{(j)}) = \mathrm{sgn}(a). \tag{93}$$

type (4): As in the cases of type (1) and (3), there is only one arrow determining subspaces within the filtration, and thus we set

$$\mathrm{sgn}(F_i^{(j)}) = \mathrm{sgn}(\varepsilon). \tag{94}$$

type (5): In this case we need to distinguish between $i$ being isolated or not. Let $F_i^{(\bar{j})}$ be the second filtration on the vector space $V_i$, hence, $j \neq \bar{j} \in \{1,2\}$. If $i$ is not isolated, then $F_i^{(\bar{j})}$ is not of type (5) and we can choose its sign according to the above description. Then we set

$$\mathrm{sgn}(F_i^{(j)}) = -\,\mathrm{sgn}(F_i^{(\bar{j})}). \tag{95}$$

Otherwise $i$ is isolated. Then we set conventionally

$$\mathrm{sgn}(F_i^{(1)}) = +, \quad \mathrm{sgn}(F_i^{(2)}) = -. \tag{96}$$

**Lemma 4.72.** *By choosing the signs as in (91) - (96), we have*

$$\operatorname{sgn}(F_i^{(1)}) \neq \operatorname{sgn}(F_i^{(2)}) \text{ for all } i \in Q_0.$$

*Proof.* Assume without loss of generality that $F_i^{(1)}$ is a filtration of type (5). It follows by construction above that $\operatorname{sgn}(F_i^{(1)}) \neq \operatorname{sgn}(F_i^{(2)})$ for $F_i^{(2)}$ a filtration of type (1)-(5).
Assume towards a contradiction that $\operatorname{sgn}(F_i^{(1)}) = \operatorname{sgn}(F_i^{(2)})$. Let $F_i^{(1)}$ be a filtration of type (2). We consider the different possibilities for $F_i^{(2)}$: If it is of type (1), then we have locally at vertex $i$ the situation

$$\begin{tikzcd} {} \arrow[r, "b"] & \bullet \arrow[r, "c"] \arrow[dr, "a"] & {} \\ & & {} \end{tikzcd}$$

with $ab = 0$. Hence, $\operatorname{sgn}(a^{-1}) = \operatorname{sgn}(b) = \operatorname{sgn}(F_i^{(1)})$ and $\operatorname{sgn}(F_i^{(2)}) = \operatorname{sgn}(c^{-1})$. Thus, by assumption $\operatorname{sgn}(c^{-1}) = \operatorname{sgn}(b)$. It follows $cb = 0$ as well, giving a contradiction to the assumption on the type of $F_i^{(2)}$ (since the relation $cb = 0$ would imply that $F_i^{(2)}$ is of type (2)). For $F_i^{(2)}$ of type (3) we obtain similarly a contradiction.
For $F_i^{(2)}$ of type (2), we locally have

$$\begin{tikzcd} {} \arrow[dr, "d"] & & \\ {} \arrow[r, "b"] & \bullet \arrow[r, "c"] \arrow[dr, "a"] & {} \\ & & {} \end{tikzcd}$$

with $ab = 0$ and $dc = 0$. Hence, since we assume the filtrations have the same sign. It follows that $\operatorname{sgn}(a^{-1}) = \operatorname{sgn}(b) = \operatorname{sgn}(d^{-1}) = \operatorname{sgn}(c)$. This implies that $ad = 0$ and $cb = 0$, a contradiction to $\Lambda$ being skewed-gentle.
Let $F_i^{(2)}$ now be of type (4). Then we have at vertex $i$:

$$\begin{tikzcd} {} \arrow[r, "b"] & \bullet \arrow[r, "a"] \arrow[loop above, "\varepsilon^*"] & {} \end{tikzcd}$$

with $ab = 0$, $\varepsilon \in \mathrm{Sp}$. For the filtrations to have the same sign, the relation $a\varepsilon = 0$ or the relation $\varepsilon b = 0$ must be satisfied. By definition of skewed-gentle, the relations may not start or end in a special loop. Hence, we obtain a contradiction.
It remains to consider cases where one of the filtrations is of type (4) and the other is not of type (2), not of type (5). Without loss of generality let $F_i^{(1)}$

be of type (1). If $F_i^{(2)}$ is of type (3), we consider at vertex $i$ the following situation:

$$\begin{tikzcd} {} \arrow[r, "b"] & \bullet \arrow[loop above, "\varepsilon^*"] \end{tikzcd}$$

Since $\operatorname{sgn}(F_i^{(1)}) = \operatorname{sgn}(F_i^{(2)})$, it follows that $\operatorname{sgn}(\varepsilon^*) = \operatorname{sgn}(b)$. This implies that $\varepsilon b = 0$ which gives a contradiction. We obtain a similar contradiction for $F_i^{(2)}$ being of type (1). □

Thus, we know how to obtain a sign for each of the two filtrations $F_i^{(1)}$, $F_i^{(2)}$ on a vertex $i \in Q_0$. By the previous lemma, it is clear that there does not exist $i \in Q_0$ such that its two filtrations have same sign.
We want to use this data in order to choose for each filtration $F_i^{(j)}$, $j = 1, 2$, a corresponding semichain $S_i^{(j)}$ in a unique way. In Subsection 4.4.2, we have already discussed what kind of possibilities we have for this choice.

For a filtration of type (4) there is no choice given since the corresponding semichain is given by (90).
Hence, let $F_i^{(j)}$ for $j \in \{1, 2\}$ be a filtration of type (1) - (3) or (5) with $\operatorname{sgn}(F_i^{(j)}) = \mu$ and $\mu \in \{+, -\}$.
The corresponding semichain $S_i^{(j)}$ can either be *compatible* with the subspace inclusions of $F_i^{(j)}$ as described in (85), or we use a *reversed* ordering in the semichain with respect to the subspace inclusions (cf. (86)). In the first case, we denote the semichain of the respective form by $S_i^{(j,c)}$, in the second case by $S_i^{(j,r)}$.
Recall that $\operatorname{sgn}(\varepsilon) = \kappa$ for all $\varepsilon \in \mathrm{Sp}$. We choose $S_i^{(j)}$ according to the sign of $F_i^{(j)}$ as follows:

$$\text{if } \mu = \kappa, \text{ set } S_i^{(j)} = S_i^{(j,c)}, \tag{97}$$

$$\text{otherwise, set } S_i^{(j)} = S_i^{(j,r)}, \tag{98}$$

and in any case set $\operatorname{sgn}(S_i^{(j)}) = \mu$.

This choice will help us in the next subsection to assign the semichains in a unique way to the different bundles. In addition, we obtain an "orientation" on the $\mathfrak{L}$–graphs which matches the directions on letters of finite index. Additionally, we will see that the letter $v_{m+1}$ which gives the symmetry axis in a symmetric string, is excluded from this result.

### 4.4.4 Assignment of semichains to bundles

When assigning the semichains to the bundles $\mathfrak{R}$ and $\mathfrak{C}$, we need to take the definition of admissible transformations into account. Here, the trans-

formations of type 2 are of importance since the allowed row and column operations in the matrices of the representations should coincide with those operations on the vector space which do not change the chosen (sub-)bases with respect to the filtrations. We have already done some of the work for this in the previous subsection.
As mentioned before, we would like to obtain a basis change matrix on each vector space $V_i$, $i \in Q_0$, in terms of the matrix problem. To this end, it is clear that the two filtrations at each vertex $i \in Q_0$ are assigned each to different bundles.
Keeping the goal in mind that the orientation Bondarenko gives on $\mathfrak{L}$–chains and $\mathfrak{L}$–cycles coincides with the one we have described on bands and strings (compare Chapter 3.3), we give the description of a certain assignment. We claim that the orientations then coincide with the directions on the respective letters of finite index, if the letter is not given by the symmetry axis of a symmetric string. (Proposition 4.145, Subsection 4.7).

Let $j \in \{1,2\}$ and let $\bar{\jmath}$ be its complement with respect to $\{1,2\}$. Then, as described in the previous section, we obtain for each $i \in Q_0$ two semichains $S_i^j$, $S_i^{\bar{\jmath}}$. Each of them has a sign, say for a fixed $i \in Q_0$:

$$\operatorname{sgn}(S_i^j) = \kappa, \quad \operatorname{sgn}(S_i^{\bar{\jmath}}) = -\kappa,$$

where $\kappa$ is still chosen in such a way that $\operatorname{sgn}(\varepsilon) = \kappa$ for all $\varepsilon \in \mathrm{Sp}$. We set

$$\mathfrak{R}_i = S_i^{\bar{\jmath}},$$
$$\mathfrak{C}_i = S_i^j.$$

Proceeding like this for any $i \in Q_0$, we obtain two bundles $\mathfrak{R} = \bigcup_{i=1}^n \mathfrak{R}_i$ and $\mathfrak{C} = \bigcup_{i=1}^n \mathfrak{C}_i$, where $|Q_0| = n$ and for any semichain $\mathfrak{R}_i \in \mathfrak{C}$ we have that $\operatorname{sgn}(\mathfrak{R}_i) = -\kappa$, for any $\mathfrak{C}_i \in \mathfrak{C}$ we have that $\operatorname{sgn}(\mathfrak{C}_i) = \kappa$.
Thus, all semichains in $\mathfrak{C}$ are of special or compatible type, while $\mathfrak{R}$ consists of semichains of reversed type.

**Remark 4.73.** *Note that there exists one exception for which the above construction does not work. For $\Lambda = \mathrm{k}Q/(\mathrm{R} \cup \mathrm{R}^{\mathrm{Sp}})$ given by*

$$Q: \quad \varepsilon \circlearrowright 1 \circlearrowleft \eta$$

*with $\mathrm{R} = \varnothing$ and $\mathrm{Sp} = \{\eta, \varepsilon\}$, we cannot use the construction given in this and the previous subsections. Instead, we consider $\Lambda' = \mathrm{k}Q'/(\mathrm{R}' \cup \mathrm{R}^{\mathrm{Sp}'})$ with*

$$Q: \quad \varepsilon \circlearrowright 1 \xrightarrow{a} 2 \circlearrowleft \eta\,,$$

*$\mathrm{R}' = \varnothing$ and $\mathrm{Sp}' = \{\eta, \varepsilon\}$. Considering representations $V'$ of $(Q', \mathrm{R}' \cup \mathrm{R}^{\mathrm{Sp}'})$ with $V'_a$ bijective is equivalent to considering representations $V$ of $(Q, \mathrm{R} \cup \mathrm{R}^{\mathrm{Sp}})$ with $V_1 \cong V'_1 \cong V'_2$ and $V_\varepsilon = V'_\varepsilon$, $V_\eta = V'_\eta$.*
*Hence, we replace $\Lambda$ in the following by $\Lambda'$ and use the mentioned identification of their respective representations.*

**Remark 4.74.** *Note that generally any assignment of the semichains to the row and column label sets gives a bundle of semichains. We can for example also assign semichains of the types $S_i^{j,c}$ to the row label set and similarly those of type $S_i^{j,r}$ to the column label set. We have chosen this particular assignment in order to obtain a compatibility with respect to directions (cf. Proposition 4.145).*

### 4.4.5 The involution $\sigma$

While the ordering within the semichains relates to admissible transformations of type 2, the involution $\sigma_\Lambda$ relates to the definition of admissible transformations of type 1 (compare Section 4.1 and [Bon91]). Hence, we need to take the admissible transformations of type 1 into account in order to define $\sigma_\Lambda$.
In Subsection 4.4.2, we have chosen some bases depending on each other; the basis of the image of an ordinary arrow is chosen with respect to its preimage without kernel. We choose $\sigma_\Lambda$ such that the admissible transformations reflect this correspondence.

By [Bon91], the involution is clearly defined on elements of two-point links. By construction, the two-point links are of the form $S_\varepsilon$ for $\varepsilon \in \mathrm{Sp}$. Recall that we denote its elements by $S_\varepsilon^+ \hat{=}$ basis of $\mathrm{im}(\varepsilon)$ and $S_\varepsilon^- \hat{=}$ basis of $\ker(\varepsilon)$. Then $\sigma_\Lambda$ acts as the identity on those two elements:

$$\sigma_\Lambda : \quad \begin{aligned} S_\varepsilon^+ &\mapsto S_\varepsilon^+ \\ S_\varepsilon^- &\mapsto S_\varepsilon^- . \end{aligned}$$

Any other link in our construction is given by a one-point link. We now describe how $\sigma_\Lambda$ acts on those. To this end, let $S_i^j$, $S_k^l$ be two elements of the form

$$S_i^j \quad \hat{=} \quad \text{basis of } V_i \ominus \ker(a), \tag{99}$$

$$S_k^l \quad \hat{=} \quad \text{basis of } \mathrm{im}(a) \tag{100}$$

for some $a : i \to k \in Q_1^{\mathrm{ord}}$. Recall that we have chosen the basis of $\mathrm{im}(a)$ depending on basis of $V_i \ominus \ker(a)$ in Subsection 4.4.2. Then $\sigma_\Lambda$ acts as follows:

$$S_i^j \overset{\sigma_\Lambda}{\longleftrightarrow} S_k^l .$$

On any other element $S_p^q$ which belongs to a one-point link, and which is not of the form (99)-(100), $\sigma$ acts as identity:

$$\sigma_\Lambda : \quad S_p^q \mapsto S_p^q .$$

**Definition 4.75.** *Let $\Lambda$ be a skewed-gentle algebra. We define its* associated bundle of semichains *by $\overline{\mathfrak{X}}_\Lambda = (\mathfrak{X}_\Lambda, \sigma_\Lambda)$, where $\mathfrak{X}_\Lambda$ and $\sigma_\Lambda$ are defined as in the construction above.*

**Remark 4.76.** *By the above definition of the involution $\sigma_\Lambda$, the* basis of $V_i \ominus \ker(a)$ *and* basis of $\operatorname{im}(a)$ *for any $a \in Q_1^{\mathrm{ord}}$ are connected in $\overline{\mathfrak{X}}_\Lambda$. By definition of admissible transformations of type 2, we need to change one of those bases whenever we change the other. Hence, $\overline{\mathfrak{X}}_\Lambda$ is compatible with this kind of dependences.*

**Remark 4.77.** *We find in [Den00, Example 3] a construction of a bush for a clannish algbra. This construction is similar to our construction for the following reasons. First, a bush is by definition a bundle of semichains. Second, Deng imposes additional conditions on the clannish algebra which coincide with our restriction to skewed-gentle algebras. Thus, Deng also considers skewed-gentle algebras. The semichains constructed in [Den00] correspond in the following way to the filtrations given in Subsection 4.4.1: the semichain given by $\{i^\varepsilon \lessgtr e\}$ corresponds to a filtration of type (4). The semichains $\{a^- < i^\varepsilon\}$ and $\{i^\varepsilon < b^+\}$ correspond to filtrations of type (1) and (3), respectively. A filtration of type (2) corresponds to the semichain $\{a^- < i^\varepsilon < b^+\}$.*
*But there are also some differences. First of all, Deng's construction does not determine a bush in a unique way. After having chosen signs on the letters, our construction is uniquely determined. Second, all semichains are chosen in a compatible order with respect to the subspace inclusions of the respective filtrations. In contrast to that, we include semichains with reversed order to obtain compatibility between the directions of special letters in words and relations of the form $x_i \alpha x_{i+1}$ in the $\mathfrak{L}$-graphs.*

### 4.4.6 The relations $\alpha$ and $\beta$

This subsection describes the $\alpha-$ and $\beta-$relations for our setup according to their definitions in Section 4.1. These relations follow naturally from our construction.

Note at first that these relations are defined between links of the semichains (in contrast to $\sigma_\Lambda$ which is defined between elements).

Let us first describe the $\beta-$relations. Recall that we assign for any vertex $i \in Q_0$ one of its semichains to $\mathfrak{R}$, and the other to $\mathfrak{C}$. Recall also that the vertices of $Q$ index the semichains in the bundles. We obtain

$$S_i^j \beta S_i^{\bar{j}} \quad \text{for any } i \in Q_0, j \in \{1,2\}, j \neq \bar{j} \in \{1,2\}.$$

Thus, $\beta$ indicates whether there is a switch between the two bases of $V_i$.

**Remark 4.78.** *Any link is in exactly one $\beta-$relation, up to symmetry.*

The $\alpha$–relations are formed in a more complicated manner since they depend on $\sigma$ and the number of elements in the links.
Our setup results in the following $\alpha$–relations:
Links of type $S_\varepsilon$, $\varepsilon \in \mathrm{Sp}$, are always in $\alpha$–relation with themselves, since any special loop gives a two-point link. Hence,

$$S_\varepsilon \alpha S_\varepsilon \quad \forall \varepsilon \in \mathrm{Sp}\,.$$

Now let $S_i^j$ describe a one-point link. Then, whenever $\sigma_\Lambda$ does not act as identity on such a link, we have

$$S_i^j \alpha(\sigma_\Lambda(S_i^j)).$$

**Remark 4.79.** *Any one-point link $S_p^q$ with $\sigma(S_p^q) = S_p^q$ is not in any $\alpha$–relation. Thus, those links do not have to be considered for admissibility. Links of the form $S_{\varepsilon^*}$ are also not of interest for admissibility since they are two-point links.*

**Remark 4.80.** *Note that only links of the form $S_{\varepsilon^*}$ for some $\varepsilon \in \mathrm{Sp}$ give candidates for double ends; they are the only links in $\alpha$–relation with themselves.*

We see later (compare Sections 4.5 and 4.6) that any $\alpha$–relation in an $\mathfrak{L}$–graph $g_w$ coming from a word $w$ represents a letter $w_i$ of $w$.
After giving the construction, we can formulate the proof of Theorem 4.70:

*Proof of Theorem 4.70.* The statement follows by the construction above. □

### 4.4.7 Examples

We give examples on how to construct a bundle of semichains for two skewed-gentle algebras.

**Example 4.81.** *Let $\Lambda$ be given as in Example 2.3.1. We choose the signs of the letters of $\Gamma_{\mathrm{ud}} = \{\varepsilon^*, a, a^{-1}\}$ as follows:*

$$\operatorname{sgn} \varepsilon^* = 1, \tag{101}$$
$$\operatorname{sgn} a = \operatorname{sgn}(a^{-1}) = -1. \tag{102}$$

*The filtrations are of type (4) and type (2), respectively:*

$$F_1^{(1)}: \quad 0 \subset \operatorname{im}(\varepsilon) \oplus \ker(\varepsilon) = V_1,$$
$$F_1^{(2)}: \quad 0 \subset \operatorname{im}(a) \subset \ker(a) \subset V_1$$

*and give the following semichains:*

$$S_1^{(1)} = \{S_{\varepsilon^*}^+ \lessgtr S_{\varepsilon^*}^-\},$$
$$S_1^{(2,r)} = \{S_{11} > S_{12} > S_{13}\},$$
$$S_1^{(2,c)} = \{S_{11} < S_{12} < S_{13}\}.$$

*The elements correspond in the following way to the bases of the subspaces:*

$$\begin{aligned}
S_{\varepsilon^*}^+ &\hat{=} \text{ basis of } \operatorname{im}(\varepsilon),\\
S_{\varepsilon^*}^- &\hat{=} \text{ basis of } \ker(\varepsilon),\\
S_{11} &\hat{=} \text{ basis of } \operatorname{im}(a),\\
S_{12} &\hat{=} \text{ basis of } \ker(a) \ominus \operatorname{im}(a),\\
S_{13} &\hat{=} \text{ basis of } V_1 \ominus \ker(a).
\end{aligned}$$

*The filtrations and semichais inherit their signs from (101) and (102) in the following way:*

$$\operatorname{sgn}(F_1^{(1)}) = 1, \qquad \operatorname{sgn}(S_1^{(1)}) = 1,$$
$$\operatorname{sgn}(F_1^{(2)}) = -1, \qquad \operatorname{sgn}(S_1^{(2,r)}) = \operatorname{sgn}(S_1^{(2,c)}) = -1.$$

*Hence, we choose* $S_1^{(2)} = S_1^{(2,r)}$ *and obtain*

$$\mathfrak{R}_1 = S_1^{(2)} \quad \textit{and}$$
$$\mathfrak{C}_1 = S_1^{(1)}.$$

*We set* $\mathfrak{C}_{\varepsilon^*}^+ = S_{\varepsilon^*}^+$, $\mathfrak{C}_{\varepsilon}^- = S_{\varepsilon^*}^-$, $\mathfrak{R}_{1i} = S_{1i}$, *where* $i = 1,2,3$. *The involution acts as identity on the elements, except for the pair* $\sigma_\Lambda(\mathfrak{R}_{11}) = \mathfrak{R}_{13}$. *Hence,* $\overline{\mathfrak{X}}_\Lambda = (\mathfrak{X}_\Lambda, \sigma_\Lambda)$ *is given with* $\sigma_\Lambda$ *as defined above, and with* $\mathfrak{X}_\Lambda = (\mathfrak{C}_1, \mathfrak{R}_1)$.
*We denote the two-point link containing* $\mathfrak{C}_{\varepsilon^*}^+$ *and* $\mathfrak{C}_{\varepsilon^*}^-$ *by* $\mathfrak{C}_{\varepsilon^*}$. *The following relations are given on the links:*

$$\begin{aligned}
\alpha - \textit{relations:}\quad & \mathfrak{R}_{11}\alpha\mathfrak{R}_{13}\\
& \mathfrak{C}_{\varepsilon^*}\alpha\mathfrak{C}_{\varepsilon^*}\\
\beta - \textit{relations:}\quad & \mathfrak{R}_{1i}\beta\mathfrak{C}_{\varepsilon^*}, \quad i = 1,2,3.
\end{aligned}$$

**Remark 4.82.** *The above algebra has motivated the Examples 4.7 and 4.17, where* $\mathfrak{C}_{11}$ *is to be identified with* $\mathfrak{C}_{\varepsilon^*}$.

**Example 4.83.** *Consider* $\Lambda$ *from Example 2.14. Recall that its undirected alphabet is given by*

$$\Gamma_{\mathrm{ud}} = \{a, a^{-1}, b, b^{-1}, c, c^{-1}, d, d^{-1}, e, e^{-1}, \varepsilon^*, \kappa^*, \eta^*\}$$

*and that the following zero relations hold:* $ca = 0$, $db = 0$, $ec = 0$.
*We choose the following signs according to those relations:*

$$\begin{aligned}
\operatorname{sgn}(\varepsilon^*) = \operatorname{sgn}(\eta^*) = \operatorname{sgn}(\kappa^*) &= -1, && (103)\\
\operatorname{sgn}(c^{-1}) = \operatorname{sgn}(a) &= -1, && (104)\\
\operatorname{sgn}(e^{-1}) = \operatorname{sgn}(c) &= \phantom{-}1, && (105)\\
\operatorname{sgn}(d^{-1}) = \operatorname{sgn}(b) &= \phantom{-}1, && (106)\\
\operatorname{sgn}(a^{-1}) &= \phantom{-}1, && (107)\\
\operatorname{sgn}(e) &= -1, && (108)\\
\operatorname{sgn}(b^{-1}) &= \phantom{-}1, && (109)\\
\operatorname{sgn}(d) &= \phantom{-}1. && (110)
\end{aligned}$$

*We consider the following filtrations:*

$$\begin{aligned}
&F_1^{(1)} : 0 \subset \operatorname{im}(\varepsilon) \oplus \ker(\varepsilon) = V_1, && F_1^{(2)} : 0 \subset \ker(a) \subset V_1,\\
&F_2^{(1)} : 0 \subset \operatorname{im}(a) \subset \ker(c) \subset V_2, && F_2^{(2)} : 0 \subset \operatorname{im}(b) \subset \ker(d) \subset V_2,\\
&F_3^{(1)} : 0 \subset \operatorname{im}(\eta) \oplus \ker(\eta) = V_3, && F_3^{(2)} : 0 \subset \ker(b) \subset V_3,\\
&F_4^{(1)} : 0 \subset \operatorname{im}(d) \subset V_4, && F_4^{(2)} : 0 \subset \operatorname{im}(e) \subset V_4,\\
&F_5^{(1)} : 0 \subset \operatorname{im}(\kappa) \oplus \ker(\kappa) = V_5, && F_5^{(2)} : 0 \subset \operatorname{im}(c) \subset \ker(e) \subset V_5.
\end{aligned}$$

*From those we derive these semichains:*

$$\begin{aligned}
S_1^{(1)} &: \{S_\varepsilon^+ \lessgtr S_\varepsilon^-\}, && \\
S_1^{(2,c)} &: \{S_{11} < S_{12}\}, && S_1^{(2,r)} : \{S_{11} > S_{12}\},\\
S_2^{(1,c)} &: \{S_{21}^{(1)} < S_{22}^{(1)} < S_{23}^{(1)}\}, && S_2^{(1,r)} : \{S_{21}^{(1)} > S_{22}^{(1)} > S_{23}^{(2)}\},\\
S_2^{(2,c)} &: \{S_{21}^{(2)} < S_{22}^{(2)} < S_{23}^{(2)}\}, && S_2^{(2,r)} : \{S_{21}^{(2)} > S_{22}^{(2)} > S_{23}^{(2)}\},\\
S_3^{(1)} &: \{S_\eta^+ \lessgtr S_\eta^-\}, && \qquad (111)\\
S_3^{(2,c)} &: \{S_{31} < S_{32}\}, && S_3^{(2,r)} : \{S_{31} > S_{32}\},\\
S_4^{(1,c)} &: \{S_{41}^{(1)} < S_{42}^{(1)}\}, && S_4^{(1,r)} : \{S_{41}^{(1)} > S_{42}^{(1)}\},\\
S_4^{(2,c)} &: \{S_{41}^{(2)} < S_{42}^{(2)}\}, && S_4^{(2,r)} : \{S_{41}^{(2)} > S_{42}^{(2)}\},\\
S_5^{(1)} &: \{S_\kappa^+ \lessgtr S_\kappa^-\}, && \\
S_5^{(2,c)} &: \{S_{51} < S_{52} < S_{53}\}, && S_5^{(2,r)} : \{S_{51} > S_{52} > S_{53}\}
\end{aligned}$$

*where the elements correspond to the following bases:*

$$\begin{aligned}
S^+_{\varepsilon^*} &\mathrel{\hat{=}} \text{basis of } \operatorname{im}(\varepsilon), & S^-_{\varepsilon^*} &\mathrel{\hat{=}} \text{basis of } \ker(\varepsilon),\\
S_{11} &\mathrel{\hat{=}} \text{basis of } \ker(a), & S_{12} &\mathrel{\hat{=}} \text{basis of } V_1 \ominus \ker(a),\\
S^{(1)}_{21} &\mathrel{\hat{=}} \text{basis of } \operatorname{im}(a), & S^{(1)}_{22} &\mathrel{\hat{=}} \text{basis of } \ker(c) \ominus \operatorname{im}(a),\\
S^{(1)}_{23} &\mathrel{\hat{=}} \text{basis of } V_2 \ominus \ker(c), & &\\
S^{(2)}_{21} &\mathrel{\hat{=}} \text{basis of } \operatorname{im}(b), & S^{(2)}_{22} &\mathrel{\hat{=}} \text{basis of } \ker(d) \ominus \operatorname{im}(b),\\
S^{(2)}_{23} &\mathrel{\hat{=}} \text{basis of } V_2 \ominus \ker(d), & &\\
S^+_{\eta^*} &\mathrel{\hat{=}} \text{basis of } \operatorname{im}(\eta), & S^-_{\eta^*} &\mathrel{\hat{=}} \text{basis of } \ker(\eta),\\
S_{31} &\mathrel{\hat{=}} \text{basis of } \ker(b), & S_{32} &\mathrel{\hat{=}} \text{basis of } V_3 \ominus \ker(b),\\
S^{(1)}_{41} &\mathrel{\hat{=}} \text{basis of } \operatorname{im}(d), & S^{(1)}_{42} &\mathrel{\hat{=}} \text{basis of } V_4 \ominus \operatorname{im}(d),\\
S^{(2)}_{41} &\mathrel{\hat{=}} \text{basis of } \operatorname{im}(e), & S^{(2)}_{42} &\mathrel{\hat{=}} \text{basis of } V_4 \ominus \operatorname{im}(e),\\
S^+_{\kappa^*} &\mathrel{\hat{=}} \text{basis of } \operatorname{im}(\kappa), & S^-_{\kappa^*} &\mathrel{\hat{=}} \text{basis of } \ker(\kappa),\\
S_{51} &\mathrel{\hat{=}} \text{basis of } \operatorname{im}(c), & S_{52} &\mathrel{\hat{=}} \text{basis of } \ker(e) \ominus \operatorname{im}(c),\\
S_{53} &\mathrel{\hat{=}} \text{basis of } V_5 \ominus \ker(e). & &
\end{aligned}$$

*The semichains in (111) inherit their signs from (103) - (110):*

$$\begin{aligned}
\operatorname{sgn}(S^{(1)}_1) &= -1, & \operatorname{sgn}(S^{(2,c)}_1) &= \operatorname{sgn}(S^{(2,r)}_1) = 1,\\
\operatorname{sgn}(S^{(1,c)}_2) &= \operatorname{sgn}(S^{(1,r)}_2) = -1, & \operatorname{sgn}(S^{(2,c)}_2) &= \operatorname{sgn}(S^{(2,r)}_2) = 1,\\
\operatorname{sgn}(S^{(1)}_3) &= -1, & \operatorname{sgn}(S^{(2,c)}_3) &= \operatorname{sgn}(S^{(2,r)}_3) = 1,\\
\operatorname{sgn}(S^{(1,c)}_4) &= \operatorname{sgn}(S^{(1,r)}_4) = 1, & \operatorname{sgn}(S^{(2,c)}_4) &= \operatorname{sgn}(S^{(2,r)}_4) = -1,\\
\operatorname{sgn}(S^{(1)}_5) &= -1, & \operatorname{sgn}(S^{(2,c)}_5) &= \operatorname{sgn}(S^{(2,r)}_5) = 1.
\end{aligned}$$

*Assigning the semichains to the row and column sets according to their signs results in:*

$$\begin{aligned}
\mathfrak{C}_1 &= S^{(1)}_1 & \mathfrak{R}_1 &= S^{(2,r)}_1\\
\mathfrak{C}_2 &= S^{(1,c)}_2 & \mathfrak{R}_2 &= S^{(2,r)}_2\\
\mathfrak{C}_3 &= S^{(1)}_3 & \mathfrak{R}_3 &= S^{(2,r)}_3\\
\mathfrak{C}_4 &= S^{(2,c)}_4 & \mathfrak{R}_4 &= S^{(1,r)}_4\\
\mathfrak{C}_5 &= S^{(1)}_5 & \mathfrak{R}_5 &= S^{(2,r)}_5
\end{aligned}$$

*Renaming the elements in the semichains regarded as elements of row or* *column sets gives* $\mathfrak{X}_\Lambda = \{\mathfrak{C}_1, \ldots, \mathfrak{C}_5, \mathfrak{R}_1, \ldots \mathfrak{R}_5\}$ *where*

$$\begin{array}{ll}
\mathfrak{C}_1 : \{\mathfrak{C}_\varepsilon^+ \lessgtr \mathfrak{C}_\varepsilon^-\} & \mathfrak{R}_1 : \{\mathfrak{R}_{11} > \mathfrak{R}_{12}\}, \\
\mathfrak{C}_2 : \{\mathfrak{C}_{21} < \mathfrak{C}_{22} < \mathfrak{C}_{23}\}, & \mathfrak{R}_2 : \{\mathfrak{R}_{21} > \mathfrak{R}_{22} > \mathfrak{R}_{23}\}, \\
\mathfrak{C}_3 : \{\mathfrak{C}_\eta^+ \lessgtr \mathfrak{C}_\eta^-\}, & \mathfrak{R}_3 : \{\mathfrak{R}_{31} > \mathfrak{R}_{32}\}, \\
\mathfrak{C}_4 : \{\mathfrak{C}_{41} < \mathfrak{C}_{42}\}, & \mathfrak{R}_4 : \{\mathfrak{R}_{41} > \mathfrak{R}_{42}\}, \\
\mathfrak{C}_5 : \{\mathfrak{C}_\kappa^+ \lessgtr \mathfrak{C}_\kappa^-\}, & \mathfrak{R}_5 : \{\mathfrak{R}_{51} > \mathfrak{R}_{52} > \mathfrak{R}_{53}\}.
\end{array} \tag{112}$$

*The involution $\sigma_\Lambda$ acts on the elements of the semichains in (112) in the following way:*

$$\begin{array}{ll}
\sigma_\Lambda : \mathfrak{R}_{12} \leftrightarrow \mathfrak{C}_{21} & \mathfrak{C}_{23} \leftrightarrow \mathfrak{R}_{51} \\
\quad \mathfrak{R}_{21} \leftrightarrow \mathfrak{R}_{32} & \mathfrak{R}_{23} \leftrightarrow \mathfrak{R}_{41} \\
\quad \mathfrak{C}_{41} \leftrightarrow \mathfrak{R}_{53}. &
\end{array}$$

*On any other element, $\sigma_\Lambda$ acts as identity.*
*We obtain the following $\alpha$–relations:*

$$\begin{array}{lr}
\mathfrak{C}_{21} \alpha \mathfrak{R}_{21} & \mathfrak{C}_{23} \alpha \mathfrak{R}_{51} \\
\mathfrak{R}_{21} \alpha \mathfrak{R}_{32} & \mathfrak{R}_{23} \alpha \mathfrak{R}_{41} \\
\mathfrak{C}_{41} \alpha \mathfrak{R}_{53} & \mathfrak{C}_\varepsilon \alpha \mathfrak{C}_\varepsilon \\
\mathfrak{C}_\eta \alpha \mathfrak{C}_\eta & \mathfrak{C}_\kappa \alpha \mathfrak{C}_\kappa
\end{array}$$

*and the following $\beta$–relations:*

$$\begin{array}{lr}
\mathfrak{C}_\varepsilon \beta \mathfrak{R}_{1i} & i = 1, 2 \\
\mathfrak{C}_{2i} \beta \mathfrak{R}_{2j} & i, j = 1, 2 \\
\mathfrak{C}_\eta \beta \mathfrak{R}_{3i} & i = 1, 2 \\
\mathfrak{C}_{4i} \beta \mathfrak{R}_{4j} & i, j = 1, 2 \\
\mathfrak{C}_\kappa \beta \mathfrak{R}_{5i} & i = 1, 2, 3.
\end{array}$$

**Remark 4.84.** *The above example gives algebra of Example 4.52.2.*

We have seen how to construct a bundle of semichains for a given skewed-gentle algebra. In the next step, we examine how to construct $\mathfrak{L}$–chains and $\mathfrak{L}$–cycles in $\overline{\mathfrak{X}}_\Lambda$ from words in $\Gamma_{\mathrm{ud}}(\Lambda)$.

### 4.5 $\mathfrak{L}$–chains from finite words

Recall that we consider a skewed-gentle algebra $\Lambda$. We consider finite $\Gamma_{\text{ud}}(\Lambda)$-words $w$. Starting from this data, we construct an $\mathfrak{L}$–chain $g_w$ in $\overline{\mathfrak{X}}_\Lambda$. The idea is to obtain for each letter $w_i$ of $w$ an $\alpha$–relation in $g_w$, and for each node between two letters we get two links in $g_w$ which are connected by a $\beta$–relation.
We see at the end of this section that asymmetric and symmetric strings, as well as the subword $u$ of the periodic part of a symmetric band give simple admissible $\mathfrak{L}$–chains (Theorem 4.113) and that any such $\mathfrak{L}$–chain can be constructed from those words (Theorem 4.116). These results yield the existence of a 1-1-correspondence between certain equivalence classes of words and certain isomorphism classes of $\mathfrak{L}$–chains (Corollary 4.117).
Before we get to this and related results, we give the construction of an $\mathfrak{L}$–chain $g_w$ for an arbitrary finite $\Gamma_{\text{ud}}(\Lambda)$–word $w$ and discuss its well-definedness and uniqueness.
Within the construction, we are going to refer to the start and target of a letter $w_i$ of $w$. When depicting $w$ in terms of arrows as described in Section 3.1, the start (target) of $w_i$ is always assigned to its right (left) hand side with respect to the depiction.

**Construction of $g_w$.** Let $w$ be an undirected finite word of length $n$. We construct the corresponding $\mathfrak{L}$–chain $g_w$ as follows:

1. Depict $w$ as $D_w : v_0 \overset{w_1}{\leftarrow} v_1 \overset{w_2}{\leftarrow} \dots \overset{w_n}{\leftarrow} v_n$.
2. Associate to each $v_i$ the values $v_i(s)$ and $v_i(t)$ – start and target of the letters $w_i$ and $w_{i+1}$, respectively.

Note that the vertices $v_i(s)$ and $v_i(t)$ describing the start and target of the letters in $w$ are to be distinguished from the associated vertices $v_i(Q) \coloneqq v_i(w)$ in the quiver (cf. Section 2.3).
Extending $w$ by trivial words to $1_{v_0(Q),\kappa} w 1_{v_n(Q),\mu}$, for appropiate $\kappa, \mu$, gives again a word. We consider $v_0(s)$ ($v_n(t)$) as start (target) of the respective trivial word.

3. We associate to each $v_i(s)$ a node $c_i$ in the graph $C_w$, and to each $v_i(t)$ a node $\bar{c}_i$. Thus, $C_{g_w}$ is a linear graph of form:
$$C_{g_w}: \quad c_0 - \bar{c}_0 - c_1 - \bar{c}_1 - \dots - c_n - \bar{c}_n.$$
4. Consider each letter $w_i$ as a map sending the element $v_i(s)$ to $v_{i-1}(t)$:
$$w_i : v_i(s) \mapsto v_{i-1}(t).$$
Assign to each of those a vector space $X, \bar{X}$, where $X, \bar{X}$ belong to one of the filtrations $F^{(1)}_{v_i(Q)}$, $F^{(2)}_{v_{i-1}(Q)}$, respectively. For some $i \in Q_0$ we have

that

$$\begin{aligned}
X \text{ in } F^{(j)}_{v_i(Q)} \text{ is assigned to } v_i(s) \text{ if } & \operatorname{sgn}(w_i^{-1}) = \operatorname{sgn}(F^{(j)}_{v_i(Q)}),\\
& v_i(s) \in X \text{ and } v_i(s) \notin V_{v_i(Q)} \ominus X,\\
\bar{X} \text{ in } F^{(\bar{j})}_{v_i(Q)} \text{ is assigned to } v_i(t) \text{ if } & \operatorname{sgn}(w_{i+1}) = \operatorname{sgn}(F^{(\bar{j})}_{v_i(Q)}),\\
& v_i(t) \in \bar{X} \text{ and } v_i(t) \notin V_{v_i(Q)} \ominus \bar{X},
\end{aligned}$$

where $j, \bar{j} \in \{1, 2\}$, $j \neq \bar{j}$.

Note that we treat $1_{v_0(Q),\kappa}$ as $w_0$, and $1_{v_n(Q),\mu}$ as $w_{n+1}$, in order to determine corresponding subspaces for $v_0(s)$ and $v_n(t)$, respectively.
In a filtration of linear form, there exists a maximal subspace as needed in step 4. In a filtration of diamond form, this maximal subspace is also defined since there only exist two proper subspaces which are complementary to each other.

5. Assign to each $v_i(s)$, $v_i(t)$ a link, using the associated subspaces according to Section 4.4.2:

$$\text{if } v_i(s) \text{ is assigned to } X, \text{ then the link } L_i \hat{=} \text{basis of } X \text{ is assigned to } v_i(s),$$
$$\text{if } v_i(t) \text{ is assigned to } \bar{X}, \text{ then the link } \bar{L}_i \hat{=} \text{basis of } \bar{X} \text{ is assigned to } v_i(t),$$

for $0 \leq i \leq n$, $L_i \neq \bar{L}_i \in \mathfrak{L}(\mathfrak{C}_{v_i(Q)} \cup \mathfrak{R}_{v_i(Q)})$.
Since $v_i(s)$ and $v_i(t)$ belong to uniquely determined subspaces $X, \bar{X}$ as described in step 4, the links $L_i$ and $\bar{L}_i$ are each uniquely given. Moreover, one of the two links belongs to the column label set $\mathfrak{L}(\mathfrak{C}_{v_i(Q)})$, and the other to the row label set $\mathfrak{L}(\mathfrak{R}_{v_i(Q)})$.

6. Order the links according to the ordering of their corresponding vertices in $w$:

$$\begin{array}{cccccc}
\overbrace{v_0(s) \quad v_0(t)}^{v_0} & & \overbrace{v_1(s) \quad v_1(t)}^{v_1} & \cdots & & \overbrace{v_n(s) \quad v_n(t)}^{v_n} \\
L_0 \qquad \bar{L}_0 & & L_1 \qquad \bar{L}_1 & \cdots & & L_n \qquad \bar{L}_n
\end{array}$$

7. Set $g_w : C_{g_w} \to \mathfrak{L} \cup \{\alpha, \beta\}$ with

$$\begin{cases} c_i \mapsto x_i \\ \bar{c}_i \mapsto \bar{x}_i \end{cases}, \quad \delta \mapsto \begin{cases} \lambda_{i,\bar{i}} & \text{if } c_i \overset{\delta}{-} \bar{c}_i \text{ in } C_{g_w} \\ \lambda_{\bar{i},i+1} & \text{if } \bar{c}_i \overset{\delta}{-} c_{i+1} \text{ in } C_{g_w} \end{cases},$$

where

$$\begin{aligned}
x_i &= L_i, & \bar{x}_i &= \bar{L}_i, & &\forall 0 \leq i \leq n,\\
\lambda_{i,\bar{i}} &= \beta, & & & &\forall 0 \leq i \leq n,\\
\lambda_{\bar{i},i+1} &= \alpha, & & & &\forall 0 \leq i < n.
\end{aligned}$$

This construction gives $g_w$ with

$$g_{w,0} = \{x_0, \bar{x}_0, \dots, x_n, \bar{x}_n\},$$
$$g_{w,1} = \{\lambda_{0,\bar{0}}, \lambda_{\bar{0},1}, \dots, \lambda_{n,\bar{n}}\}$$

which can be depicted as

$$g_w : x_0 - \bar{x}_0 \sim x_1 - \bar{x}_1 \sim \cdots \sim x_n - \bar{x}_n.$$

Recall that we work with undirected words, i.e., for any special letter $w_i = \varepsilon^*$ its direction is not defined. Thus, we are not able to determine for its start $v_i(s)$ and target $v_{i-1}(t)$ a subspace: they either belong to the kernel or the image of $\varepsilon$. But recall also that we only use the subspaces in order to determine a link for each start and target. The two-point link $\mathfrak{C}_{\varepsilon^*}$ is associated to both subspaces $\ker(\varepsilon)$ and $\mathrm{im}(\varepsilon)$. Thus, we can still uniquely assign the link $\mathfrak{C}_{\varepsilon^*}$ to $v_i(s)$ and $v_{i-1}(t)$ without knowing the direction of $w_i$. We write in abuse of notation $v_i(s) \in \varepsilon^*$ and basis of $\varepsilon^* \hat{=} \mathfrak{C}_{\varepsilon^*}$.
There are two other important properties of the construction worth mentioning:

**Remark 4.85.** *By step 7, we have that $L_i - \bar{L}_i$ and $\bar{L}_i \sim L_{i+1}$. This confirms that a $\beta$−relation indicates a change between the two bases of a vector space given by the respective filtrations, and that an $\alpha$−relation is given for each letter. Moreover, if we consider a subchain consisting of two links connected by an $\alpha$−relation, we can read from the arrangement of the links whether the corresponding letter is inverse or direct.*

**Remark 4.86.** *As mentioned above, the subspaces $X$ and $\bar{X}$ for $v_i(s)$ and $v_i(t)$, respectively, belong to two different filtrations. Thus, we obtain for $v_i(s)$ and $v_i(t)$ two links coming from different semichains. It follows that we obtain for each $i \in \{0, \dots, n\}$ one link from the column, and one link from the row label set.*

Before we start examining the well-definedness of the above construction, we give explicit examples:

**Example 4.87.** *Let $\Lambda$ be given as in Example 2.3.1. Recall that $Q$ is given by*

$$\begin{tikzcd} 1 \arrow[loop left, "\varepsilon"] \arrow[loop right, "a"] \end{tikzcd}$$

*with* $\mathrm{Sp} = \{\varepsilon\}$ *and* $\mathrm{R} = \{a^2\}$. *We choose the signs* $\mathrm{sgn}(\varepsilon^*) = \kappa$ *and* $\mathrm{sgn}(a) = \mathrm{sgn}(a^{-1}) = -\kappa$. *Recall from Example 4.81 that the semichains in $\overline{\mathfrak{X}}_\Lambda$ are of the following form:*

$$\mathfrak{C}_1 = \{\mathfrak{C}^+_{\varepsilon^*} \otimes \mathfrak{C}^-_{\varepsilon^*}\}$$
$$\mathfrak{R}_1 = \{\mathfrak{R}_{11} > \mathfrak{R}_{12} > \mathfrak{R}_{13}\},$$

*where*

$$\mathfrak{R}_{11} \mathrel{\hat{=}} \text{basis of } \operatorname{im}(a),$$
$$\mathfrak{R}_{12} \mathrel{\hat{=}} \text{basis of } \ker(a) \ominus \operatorname{im}(a),$$
$$\mathfrak{R}_{13} \mathrel{\hat{=}} \text{basis of } V_1 \ominus \ker(a).$$

*Recall that $\sigma_\Lambda$ acts as identity, apart from $\sigma_\Lambda(\mathfrak{R}_{11}) = \mathfrak{R}_{13}$. The signs of the semichains are given by*

$$\operatorname{sgn}(\mathfrak{C}_1) = \kappa,$$
$$\operatorname{sgn}(\mathfrak{R}_1) = -\kappa.$$

*Let $w = \varepsilon^* a \varepsilon^* a^{-1} \varepsilon^*$. Then $g_w$ is constructed by the following steps:*

*1.* $D_w : \quad v_0 \overset{\varepsilon^*}{—} v_1 \xleftarrow{a} v_2 \overset{\varepsilon^*}{—} v_3 \xrightarrow{a} v_4 \overset{\varepsilon^*}{—} v_5$

*2.*

$$\overbrace{v_0(s)\ \ v_0(t)}^{v_0} \overset{\varepsilon^*}{—} \overbrace{v_1(s)\ \ v_1(t)}^{v_1} \xleftarrow{a} \overbrace{v_2(s)\ \ v_2(t)}^{v_2} \overset{\varepsilon^*}{—} \overbrace{v_3(s)\ \ v_3(t)}^{v_3} \xrightarrow{a} \overbrace{v_4(s)\ \ v_4(t)}^{v_4} \overset{\varepsilon^*}{—} \overbrace{v_5(s)\ \ v_5(t)}^{v_5}$$

*3.*

$$C_w : \quad c_0 — \bar{c}_0 — c_1 — \bar{c}_1 — c_2 — \bar{c}_2 — c_3 — \bar{c}_3 — c_4 — \bar{c}_4 — c_5 — \bar{c}_5$$

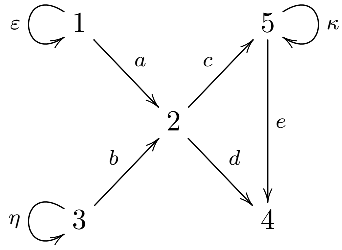
*4.+5.*

| $v_i$ | $v_i(s),\ v_i(t)$ | *sgn*$(w_i^{-1})$ / *sgn*$(w_{i+1})$ | *subspace* | *link* |
|---|---|---|---|---|
| $v_0$ | $v_0(s)$ | $-\kappa$ | $\ker(a)$ | $\mathfrak{R}_{12} = L_0$ |
| | $v_0(t)$ | $\kappa$ | $\varepsilon^*$ | $\mathfrak{C}_{\varepsilon^*} = \bar{L}_0$ |
| $v_1$ | $v_1(s)$ | $\kappa$ | $\varepsilon^*$ | $\mathfrak{C}_{\varepsilon^*} = L_1$ |
| | $v_1(t)$ | $-\kappa$ | $\operatorname{im}(a)$ | $\mathfrak{R}_{11} = \bar{L}_1$ |
| $v_2$ | $v_2(s)$ | $-\kappa$ | $V_1 \ominus \ker(a)$ | $\mathfrak{R}_{13} = L_2$ |
| | $v_2(t)$ | $\kappa$ | $\varepsilon^*$ | $\mathfrak{C}_{\varepsilon^*} = \bar{L}_2$ |
| $v_3$ | $v_3(s)$ | $\kappa$ | $\varepsilon^*$ | $\mathfrak{C}_{\varepsilon^*} = L_3$ |
| | $v_3(t)$ | $-\kappa$ | $V_1 \ominus \ker(a)$ | $\mathfrak{R}_{13} = \bar{L}_3$ |
| $v_4$ | $v_4(s)$ | $-\kappa$ | $\operatorname{im}(a)$ | $\mathfrak{R}_{11} = L_4$ |
| | $v_4(t)$ | $\kappa$ | $\varepsilon^*$ | $\mathfrak{C}_{\varepsilon^*} = \bar{L}_4$ |
| $v_5$ | $v_5(s)$ | $\kappa$ | $\varepsilon^*$ | $\mathfrak{C}_{\varepsilon^*} = L_5$ |
| | $v_5(t)$ | $-\kappa$ | $\ker(a)$ | $\mathfrak{R}_{12} = \bar{L}_5$ |

*6.+7.*

$$g_w : \quad L_0 - \bar{L}_0 \sim L_1 - \bar{L}_1 \sim L_2 - \bar{L}_2 \sim L_3 - \bar{L}_3 \sim L_4 - \bar{L}_4 \sim L_5 - \bar{L}_5 \tag{113}$$
$$\mathfrak{R}_{12} - \mathfrak{C}_{\varepsilon^*} \sim \mathfrak{C}_{\varepsilon^*} - \mathfrak{R}_{11} \sim \mathfrak{R}_{13} - \mathfrak{C}_{\varepsilon^*} \sim \mathfrak{C}_{\varepsilon^*} - \mathfrak{R}_{13} \sim \mathfrak{R}_{11} - \mathfrak{C}_{\varepsilon^*} \sim \mathfrak{C}_{\varepsilon^*} - \mathfrak{R}_{12} \tag{114}$$

**Example 4.88.** *Let $\Lambda$ be given as in Example 2.14. Recall that $Q$ is given by*




$\mathrm{Sp} = \{\eta, \varepsilon, \kappa\}$ *and* $\mathrm{R} = \{ca, db, ec\}$. *We choose the signs of the letters as in Example 4.83. We recall the semichains from (112):*

$$\begin{aligned}
&\mathfrak{C}_1 : \{\mathfrak{C}_\varepsilon^+ \lessgtr \mathfrak{C}_\varepsilon^-\} && \mathfrak{R}_1 : \{\mathfrak{R}_{11} > \mathfrak{R}_{12}\},\\
&\mathfrak{C}_2 : \{\mathfrak{C}_{21} < \mathfrak{C}_{22} < \mathfrak{C}_{23}\}, && \mathfrak{R}_2 : \{\mathfrak{R}_{21} > \mathfrak{R}_{22} > \mathfrak{R}_{23}\},\\
&\mathfrak{C}_3 : \{\mathfrak{C}_\eta^+ \lessgtr \mathfrak{C}_\eta^-\}, && \mathfrak{R}_3 : \{\mathfrak{R}_{31} > \mathfrak{R}_{32}\},\\
&\mathfrak{C}_4 : \{\mathfrak{C}_{41} < \mathfrak{C}_{42}\}, && \mathfrak{R}_4 : \{\mathfrak{R}_{41} > \mathfrak{R}_{42}\},\\
&\mathfrak{C}_5 : \{\mathfrak{C}_\kappa^+ \lessgtr \mathfrak{C}_\kappa^-\}, && \mathfrak{R}_5 : \{\mathfrak{R}_{51} > \mathfrak{R}_{52} > \mathfrak{R}_{53}\},
\end{aligned}$$

*where*

$$\begin{aligned}
&\mathfrak{R}_{11} \mathrel{\hat=} \text{basis of } \ker(a), && \mathfrak{R}_{12} \mathrel{\hat=} \text{basis of } V_1 \ominus \ker(a),\\
&\mathfrak{C}_{21} \mathrel{\hat=} \text{basis of } \operatorname{im}(a), && \mathfrak{C}_{22} \mathrel{\hat=} \text{basis of } \ker(c) \ominus \operatorname{im}(a),\\
&\mathfrak{C}_{23} \mathrel{\hat=} \text{basis of } V_2 \ominus \ker(c), && \\
&\mathfrak{R}_{21} \mathrel{\hat=} \text{basis of } \operatorname{im}(b), && \mathfrak{R}_{22} \mathrel{\hat=} \text{basis of } \ker(d) \ominus \operatorname{im}(b),\\
&\mathfrak{R}_{23} \mathrel{\hat=} \text{basis of } V_2 \ominus \ker(d), && \\
&\mathfrak{R}_{31} \mathrel{\hat=} \text{basis of } \ker(b), && \mathfrak{R}_{32} \mathrel{\hat=} \text{basis of } V_3 \ominus \ker(b),\\
&\mathfrak{C}_{41} \mathrel{\hat=} \text{basis of } \operatorname{im}(e), && \mathfrak{C}_{42} \mathrel{\hat=} \text{basis of } V_4 \ominus \operatorname{im}(e),\\
&\mathfrak{R}_{41} \mathrel{\hat=} \text{basis of } \operatorname{im}(d), && \mathfrak{R}_{42} \mathrel{\hat=} \text{basis of } V_4 \ominus \operatorname{im}(d),\\
&\mathfrak{R}_{51} \mathrel{\hat=} \text{basis of } \operatorname{im}(c), && \mathfrak{R}_{52} \mathrel{\hat=} \text{basis of } \ker(e) \ominus \operatorname{im}(c),\\
&\mathfrak{R}_{53} \mathrel{\hat=} \text{basis of } V_5 \ominus \ker(e).. &&
\end{aligned}$$

*Let $w = \eta^* b^{-1} c^{-1} \kappa^* e^{-1}$. Its corresponding $\mathfrak{L}$–chain $g_w$ is constructed as follows:*

1. $$D_w : \quad v_0 \overset{\eta^*}{—} v_1 \xrightarrow{b} v_2 \xrightarrow{c} v_3 \overset{\kappa^*}{—} v_4 \xrightarrow{e} v_5$$

2. $$\overbrace{v_0(s)\ \ v_0(t)}^{v_0} \overset{\eta^*}{—} \overbrace{v_1(s)\ \ v_1(t)}^{v_1} \xrightarrow{b} \overbrace{v_2(s)\ \ v_2(t)}^{v_2} \xrightarrow{c} \overbrace{v_3(s)\ \ v_3(t)}^{v_3} \overset{\kappa^*}{—} \overbrace{v_4(s)\ \ v_4(t)}^{v_4} \xrightarrow{e} \overbrace{v_5(s)\ \ v_5(t)}^{v_5}$$

3. $$C_w : \quad c_0 — \bar{c}_0 — c_1 — \bar{c}_1 — c_2 — \bar{c}_2 — c_3 — \bar{c}_3 — c_4 — \bar{c}_4 — c_5 — \bar{c}_5$$

4.+5.

| $v_i$ | $v_i(s)$, $v_i(t)$ | $sgn(w_i^{-1})$ / $sgn(w_{i+1})$ | *subspace* | *link* |
|---|---|---|---|---|
| $v_0$ | $v_0(s)$ | 1 | $\ker(b)$ | $\mathfrak{R}_{31} = L_0$ |
| | $v_0(t)$ | $-1$ | $\eta^*$ | $\mathfrak{C}_{\eta^*} = \bar{L}_0$ |
| $v_1$ | $v_1(s)$ | $-1$ | $\eta^*$ | $\mathfrak{C}_{\eta^*} = L_1$ |
| | $v_1(t)$ | 1 | $V_3 \ominus \ker(b)$ | $\mathfrak{R}_{32} = \bar{L}_1$ |
| $v_2$ | $v_2(s)$ | 1 | $\mathrm{im}(b)$ | $\mathfrak{R}_{21} = L_2$ |
| | $v_2(t)$ | $-1$ | $V_2 \ominus \ker(c)$ | $\mathfrak{C}_{23} = \bar{L}_2$ |
| $v_3$ | $v_3(s)$ | 1 | $\mathrm{im}(c)$ | $\mathfrak{R}_{51} = L_3$ |
| | $v_3(t)$ | $-1$ | $\kappa^*$ | $\mathfrak{C}_{\kappa^*} = \bar{L}_3$ |
| $v_4$ | $v_4(s)$ | $-1$ | $\kappa^*$ | $\mathfrak{C}_{\kappa^*} = L_4$ |
| | $v_4(t)$ | 1 | $V_5 \ominus \ker(e)$ | $\mathfrak{R}_{53} = \bar{L}_4$ |
| $v_5$ | $v_5(s)$ | $-1$ | $\mathrm{im}(e)$ | $\mathfrak{C}_{41} = L_5$ |
| | $v_5(t)$ | 1 | $V_4 \ominus \mathrm{im}(d)$ | $\mathfrak{R}_{42} = \bar{L}_5$ |

6.+7.

$$g_w : \quad \mathfrak{R}_{31}-\mathfrak{C}_{\eta^*}\sim\mathfrak{C}_{\eta^*}-\mathfrak{R}_{32}\sim\mathfrak{R}_{21}-\mathfrak{C}_{23}\sim\mathfrak{R}_{51}-\mathfrak{C}_{\kappa^*}\sim\mathfrak{C}_{\kappa^*}-\mathfrak{R}_{53}\sim\mathfrak{C}_{41}-\mathfrak{R}_{42}$$

Next we show the well-definedness of $g_w$. This includes several statements. In particular, we show that equivalent words result in isomorphic $\mathfrak{L}$–chains. We consider these results at first separately and then sum them up in the context of well-definedness.

**Lemma 4.89.** *The above construction results in an $\mathfrak{L}$–chain $g_w$ for any finite $\Gamma_{\mathrm{ud}}(\Lambda^*)$–word $w$. This $\mathfrak{L}$–chain is unique for any word $w$.*

*Proof.* Observe at first that $C_{g_w}$ is always given by a chain. Also, by step 7, condition (b) of the definition of an $\mathfrak{L}$–graph is satisfied. It remains to show condition (a) of an $\mathfrak{L}$–graph.
First we show that $L_i \beta \bar{L}_i$ holds for all $i \in \{0, \dots, n\}$. We have by step 4 and step 5 that $v_i(s) \in X$, $v_i(t) \in \bar{X}$ and basis of $X \mathrel{\hat{=}} L_i$, basis of $\bar{X} \mathrel{\hat{=}} \bar{L}_i$.
Since $\mathrm{sgn}(w_i^{-1}) \neq \mathrm{sgn}(w_{i+1})$ (cf. Section 2.3), $L_i$ and $\bar{L}_i$ do not belong to the same label set by construction. For $i$ fixed, assume without loss of generality that

$$L_i \in \mathfrak{L}(\mathfrak{C}),$$
$$\bar{L}_i \in \mathfrak{L}(\mathfrak{R}).$$

It follows by definition of the $\beta$–relation that $L_i \beta \bar{L}_i$ for any $i \in \{0, \dots, n\}$.
It remains to show that $\bar{L}_{i-1} \alpha L_i$ holds as well. Recall that for an ordinary arrow $(x : j \to k) \in Q_1^{\mathrm{ord}}$, the links corresponding to the bases of the subspaces $\mathrm{im}(x)$ and $V_j \ominus \ker(x)$ are connected by $\sigma_\Lambda$ and thus satisfy the $\alpha$–relation. For any special loop $\varepsilon \in \mathrm{Sp}$ we have $\mathfrak{C}_{\varepsilon^*} \alpha \mathfrak{C}_{\varepsilon^*}$.
We have for $w_i \in \{x, x^{-1}\}$ for some $(x : j \to k) \in Q_1^{\mathrm{ord}}$ the following depiction:

$$v_{i-1}(t) \xleftarrow{x} v_i(s) \qquad \text{if } w_i = x,$$
$$v_{i-1}(t) \xrightarrow{x} v_i(s) \qquad \text{if } w_i = x^{-1}.$$

Hence, $x^{-1} : v_i(s) \mapsto v_{i-1}(t)$ can be interpreted as $x : v_{i-1}(t) \mapsto v_i(s)$. Thus,

$$v_{i-1}(t) \in \begin{cases} \mathrm{im}(x) & \text{if } w_i = x, \\ V_j \ominus \ker(x) & \text{if } w_i = x^{-1}, \end{cases}$$
$$v_i(s) \in \begin{cases} V_j \ominus \ker(x) & \text{if } w_i = x, \\ \mathrm{im}(x) & \text{if } w_i = x^{-1}. \end{cases}$$

It follows that the corresponding links $\bar{L}_{i-1}$ and $L_i$ of $v_{i-1}(t)$ and $v_i(s)$ are one-point links and that $\sigma_\Lambda(\bar{L}_{i-1}) = L_i$ by the above. Hence, $\bar{L}_{i-1}\alpha L_i$ holds. If, on the other hand, $w_i = \varepsilon^*$ for some $\varepsilon \in \mathrm{Sp}$, then we depict the letter as

$$v_{i-1}(t) \overset{\varepsilon^*}{\text{---}} v_i(s).$$

As described above, we obtain as corresponding link both for $v_{i-1}(t)$ and $v_i(s)$ the two-point link $\mathfrak{C}_{\varepsilon^*} = \bar{L}_{i-1} = L_i$. It follows directly $\bar{L}_{i-1}\alpha L_i$.
Finally, we show that the resulting $\mathfrak{L}$−chain $g_w$ is unique for any $w$. The subspaces $X$ and $\bar{X}$ are uniquely given for each $v_i(s)$ with $w_i$, and each $v_i(t)$ with $w_{i+1}$ ordinary, respectively: each arrow $x \in Q_1^{\mathrm{ord}}$ gives rise to exactly one filtration at its starting and one at its terminating vertex in $Q$, due to the construction from Section 4.4. We assign $v_i(s)$ ($v_i(t)$) to a subspace $X$ ($\bar{X}$) according to the sign of the letter $w_i^{-1}$ ($w_{i+1}$). Hence, we consider for $v_i(s)$ ($v_i(t)$) a uniquely given filtration. It follows from $w_i \neq w_{i+1}$ that $X$ and $\bar{X}$ come from different filtrations. By the construction of semichains in Section 4.4.2, it follows that also the links assigned to $v_i(s)$ and $v_i(t)$ are uniquely determined for each of them.
In contrast to ordinary letters, special letters do not give rise to uniquely determined subspaces: since they are not oriented, it is not determined whether for $w_i = \varepsilon^*$ special its associated values $v_i(s)$ and $v_{i-1}(t)$ belong to $\ker(\varepsilon)$ or $\mathrm{im}(\varepsilon)$. But both subspaces correspond to one link $\mathfrak{C}_{\varepsilon^*}$. Thus – though the subspace assignment is not unique – the assignment to a link is unqiuely given. □

**Remark 4.90.** *Note that the $\mathfrak{L}$−chain construction is in particular unique for trivial words. To this end, consider $\Lambda$ as in Example 2.14 and let $w = 1_{2,+}$. We have that*

$$D_w : \quad v_0.$$

*We can extend $w$ to $w1_{2,+}$ and regard $w_0 = w_1 = 1_{2,+}$. Recall that the signs at vertex 2 are given by $\mathrm{sgn}(a) = \mathrm{sgn}(c^{-1}) = -$ and $\mathrm{sgn}(b) = \mathrm{sgn}(d^{-1}) = +$. We consider for the construction the signs*

$$\mathrm{sgn}(w_0^{-1}) = -,$$
$$\mathrm{sgn}(w_1) = +,$$

*and thus assign*

$$v_0(s) \in V_2 \ominus \ker(c),$$
$$v_0(t) \in V_2 \ominus \ker(d).$$

*Hence, we obtain two uniquely determined links to represent $v_0(s)$ and $v_0(t)$ in $g_w$. Since they are by signs in different label sets, they are in $\beta$–relation with each other:*

$$g_w : \quad \mathfrak{C}_{23} - \mathfrak{R}_{23}.$$

Before we come to well-definedness of the construction, we examine the following properties of isomorphisms between the constructed $\mathfrak{L}$–chains.

**Lemma 4.91.** *Let $v$ be a finite undirected word with $g_v = g_v^*$. Then $g_v$ is composite.*

*Proof.* Let $g_{v,0} = \{x_1, \dots, x_n\}$, $g_{v,0}^* = \{x_n, \dots, x_1\}$ and $g_{v,1} = \{\lambda_1, \dots, \lambda_{n-1}\}$, $g_{v,1}^* = \{\lambda_{n-1}, \dots, \lambda_1\}$. Then $x_{n-i} = x_{i+1}$ for all $i \in \{0, \dots, n-1\}$, and $\lambda_i = \lambda_{n-i}$ for all $i \in \{1, \dots, n-1\}$. Thus, we can write $g_v$ of the following form, knowing it is of even length by construction:

$$g_v : x_1\lambda_1 x_2 \dots x_{n/2}\lambda_{n/2}x_{n/2} \dots x_2\lambda_1 x_1.$$

By construction, $\lambda_{n/2}$ is given by an $\alpha$–relation, so with $h := x_1\lambda_1 x_2 \dots x_{n/2}$, we can write

$$g_v : h \sim h^*,$$

which shows that $g_v$ is composite. $\square$

**Example 4.92.** 1. *Let $\Lambda$ be given by*

$$Q : \quad \begin{tikzcd} \bullet \arrow[loop left, "\varepsilon"] \arrow[loop right, "a"] \end{tikzcd}$$

*with* $\mathrm{Sp} = \{\varepsilon\}$ *and* $\mathrm{R} = \{a^2\}$. *Let $v = \varepsilon^*$ and thus*

$$g_v : \quad \mathfrak{R}_{12} - \mathfrak{C}_{\varepsilon^*} \sim \mathfrak{C}_{\varepsilon^*} - \mathfrak{R}_{12} = g_v^* = h \sim h^*$$

*with $h = \mathfrak{R}_{12} - \mathfrak{C}_{\varepsilon^*}$.*

2. *Let $\Lambda$ be given by*

$$Q : \quad \begin{tikzcd} 1 \arrow[loop left, "\varepsilon"] \arrow[r, "a"] & 2 \arrow[loop right, "\eta"] \end{tikzcd}$$

*with* $\mathrm{Sp} = \{\varepsilon, \eta\}$ *and* $\mathrm{R} = \varnothing$. *The semichains of $\mathfrak{X}_\Lambda$ are given by*

$$\mathfrak{C}_1 = \{\mathfrak{C}_{\varepsilon^*}^+ \lessgtr \mathfrak{C}_{\varepsilon^*}^-\}, \qquad \mathfrak{R}_1 = \{\mathfrak{R}_{11} > \mathfrak{R}_{12}\},$$
$$\mathfrak{C}_2 = \{\mathfrak{C}_{\eta^*}^+ \lessgtr \mathfrak{C}_{\eta^*}^-\}, \qquad \mathfrak{R}_2 = \{\mathfrak{R}_{21} > \mathfrak{R}_{22}\},$$

*where*

$$\begin{aligned}\mathfrak{R}_{11} &\mathrel{\hat{=}} \text{basis of } \ker(a),\\ \mathfrak{R}_{12} &\mathrel{\hat{=}} \text{basis of } V_1 \ominus \ker(a),\\ \mathfrak{R}_{21} &\mathrel{\hat{=}} \text{basis of } \operatorname{im}(a),\\ \mathfrak{R}_{22} &\mathrel{\hat{=}} \text{basis of } V_2 \ominus \operatorname{im}(a).\end{aligned}$$

*The involution $\sigma_\Lambda$ sends $\mathfrak{R}_{12} \leftrightarrow \mathfrak{R}_{21}$ and acts as identity otherwise. Let $v = \varepsilon^* a^{-1} \eta^* a \varepsilon^*$ and*

$$h : \mathfrak{R}_{11} - \mathfrak{C}_{\varepsilon^*} \sim \mathfrak{C}_{\varepsilon^*} - \mathfrak{R}_{12} \sim \mathfrak{R}_{21} - \mathfrak{C}_{\eta^*}.$$

*Then $v^{-1} = v$ and*

$$(g_v : h \sim h^*) = g_v^* = g_{v^{-1}}.$$

**Lemma 4.93.** *Let $v$ and $w$ be two finite $\Gamma_{\mathrm{ud}}(\Lambda)$–words. Let $\tau : g_v \to g_w$ be an isomorphism. Then*

*a)* $\tau = \mathrm{id}$ *if $g_v$ and $g_w$ are simple,*

*b)* $\tau \in \{\mathrm{id}, \mathrm{rev}\}$ *if $g_v$ and $g_w$ are composite,*

*where* $\mathrm{rev}(c_i) = c_{n+1-i}$ *for any $i \in \{1, \dots, n\}$, and $n$ is the length of the underlying graph.*

*Proof.* a) Let the underlying graphs of $g_v$ and $g_w$ be given by

$$\begin{aligned} C_{g_v} &: c_1 - \cdots - c_n \quad \text{and}\\ C_{g_w} &: c'_1 - \cdots - c'_n,\end{aligned}$$

respectively. Recall that the image of two incident nodes under $\tau$ is again given by two incident nodes, i.e. if $c_i - c_{i+1}$, then $\tau(c_i) - \tau(c_{i+1})$. Let $g_v = g_w$. Then $\tau = \mathrm{id} \in \mathrm{Aut}(g_v)$. It remains to show, that $\tau$ cannot be of any other form. Assume towards a contradiction that there exists $\tau \in \mathrm{Aut}(g_v)$ with $\tau \neq \mathrm{id}$. Then there exist $i \neq j \in \{1, \dots, n\}$ with $\tau(c_i) = c'_j$. It follows that $\tau(c_{i+1}) \in \{c'_{j+1}, c'_{j-1}\}$, say without loss of generality $\tau(c_{i+1}) = c'_{j-1}$. Thus,

$$\begin{aligned}\tau(c_{i+k}) &= c'_{j-k} \quad \text{for } k \in \{i+1, \dots, \min\{n-i, j-1\}\},\\ \tau(c_{i-k}) &= c'_{j+k} \quad \text{for } k \in \{1, \dots, \min\{n-j, i-1\}\}.\end{aligned}$$

For reasons of well-definedness of the indices, it follows $i = j - 1 = \frac{n}{2}$ for $n$ even, and $i = j = \frac{n+1}{2}$ for $n$ odd. Hence, $\tau = \mathrm{rev}$ and $g_v = g_w^*$. By Lemma 4.91, $g_v$ is composite.

Let now $g_v \neq g_w$. We want to show that there does not exist an isomorphism between the two $\mathfrak{L}$–chains. Assume towards a contradiction otherwise. Then $\tau \neq \mathrm{id}$ and there exist $i \neq j \in \{1, \dots, n\}$ with $\tau(c_i) = c'_j$. Analogously to the above, we obtain that $g_v$ is composite which gives a contradiction.

b) Let $g_v = h^{[k]}$ and $g_w = m^{[l]}$ composite. If $g_v = g_w$, then $\tau = \mathrm{id} \in \mathrm{Aut}(g_v)$. If additionally $g_v = g_v^*$, then by same line of argument as in a), $\tau = \mathrm{rev} \in \mathrm{Aut}(g_v)$.
Consider now $g_v \neq g_w$. Then $\tau \neq \mathrm{id}$. By same line of argument as in a), it follows that $g_v = g_w^*$ and thus $\tau = \mathrm{rev}$.

□

**Remark 4.94.** *It follows that we have in Lemma 4.93, a) that $g_v = g_w$ and in Lemma 4.93, b) that $g_v = g_w$ or $g_v = g_w^*$.*

**Lemma 4.95.** *Let $v$ and $w$ be two finite $\Gamma_{\mathrm{ud}}(\Lambda)$–words with $v = w^{-1}$. Then $g_w^* = g_v$.*

*Proof.* Let $w = w_1 \dots w_n$ be an undirected finite word. Then we have that $v = w^{-1} = w_n^{-1} \dots w_1^{-1}$. We obtain the following correspondences on the associated values of the nodes $v_i$ of $w$ and $v'_i$ of $w^{-1}$, $0 \leq i \leq n$:

$$v'_i(t) \leftrightarrow v_{n-i}(s), \tag{115}$$
$$v'_i(s) \leftrightarrow v_{n-i}(t). \tag{116}$$

As maps, the letters act similar on the corresponding pairs. Thus, the links of corresponding associated values coincide for all $i \in \{0, \dots, n\}$ and we obtain by (115) and (116) that

$$\bar{L}_i = L_{n-i},$$
$$L_i = \bar{L}_{n-i}.$$

Using those equalities, we obtain as $\mathfrak{L}$–chains:

$$g_w : L_0 - \bar{L}_0 \sim L_1 - \dots \sim L_{n-1} - \bar{L}_{n-1} \sim L_n - \bar{L}_n$$
$$g_{w^{-1}} : \bar{L}_n - L_n \sim \bar{L}_{n-1} - \dots \sim \bar{L}_1 - L_1 \sim \bar{L}_0 - L_0.$$

It follows that $g_w^* = g_{w^{-1}}$. □

**Lemma 4.96.** *Let $v$ and $w$ be two finite $\Gamma_{\mathrm{ud}}(\Lambda)$– words. Then we have*

*(i)* $g_v = g_w$ *if and only if* $v = w$,

*(ii)* $g_v \cong g_w$ *if and only if* $v \sim w$,

*Proof.* (i) Let $v = w$. Denote by $v_i(-)$ the associated values of the nodes of $w$ and by $v_i'(-)$ those of $v$. Then we have for any $i \in \{0, \dots, n\}$ that $v_i(s) = v_i'(s)$ and $v_i(t) = v_i'(t)$ correspond to the same subspace, respectively. Hence

$$\begin{aligned} L_i^w &= L_i^v, && \text{and} \\ \bar{L}_i^w &= \bar{L}_i^v, && \text{for all } 0 \le i \le n. \end{aligned}$$

It follows $g_v = g_w$.
Conversely, let $g_v = g_w$. Then

$$\begin{aligned} x_i^v &= x_i^w, \quad \text{and} \quad \bar{x}_i^v = \bar{x}_i^w, && \text{for all } 0 \le i \le n, \\ \lambda_{i,\bar{i}}^v &= \lambda_{i,\bar{i}}^w, && \text{for all } 0 \le i \le n, \\ \lambda_{\bar{i},i+1}^v &= \lambda_{\bar{i},i+1}^w, && \text{for all } 0 \le i \le n-1. \end{aligned}$$

We have $v_i(s), v_i'(s) \in X_i$ and $v_i(t), v_i'(t) \in \bar{X}_i$ for all $0 \le i \le n$. It follows that $v_i = w_i$ for all $0 \le i \le n$ and thus $w = v$.

(ii) By definition, $v \sim w$ if and only if $v = w$ or $v = w^{-1}$. The case $v = w$ follows from $(i)$ with $\tau : g_v \to g_w$ given by the identity.
Let now $v = w^{-1}$ and let

$$\begin{aligned} g_{v,0} &= \{y_0, \bar{y}_0, \dots, y_n, \bar{y}_n\}, & g_{v,1} &= \{\lambda_{0\bar{0}}, \lambda_{\bar{0}1}, \dots, \lambda_{n\bar{n}}\}, \\ g_{w,0} &= \{x_0, \bar{x}_0, \dots, x_n, \bar{x}_n\}, & g_{w,1} &= \{\rho_{0\bar{0}}, \rho_{\bar{0}1}, \dots, \rho_{n\bar{n}}\}. \end{aligned}$$

We know by the previous lemma that $g_v = g_w^*$. Hence,

$$g_{v,0} = \{\bar{x}_n, x_n, \dots, \bar{x}_0, x_0\}, \quad g_{v,1} = \{\rho_{n\bar{n}}, \dots, \rho_{0\bar{0}}\}.$$

Denote by $C_{g_v}$, $C_{g_w}$ the respective underlying graphs of $g_v$ and $g_w$, consisting each of $2n$ nodes $c_{v,i}$, $c_{w,i}$, respectively ($n = |w| = |v|$). Then $\tau : C_{g_v} \to C_{g_w}$ gives an isomorphism between $g_v$ and $g_w$ by

$$\tau : c_{v,i} \mapsto c_{w,2n-i}.$$

This can be seen by renumbering $g_{v,0} = \{y_0, \dots, y_{2n}\}$ and the sets $g_{w,0}, g_{v,1}, g_{w,1}$ similarly. With these renumbered sets, we obtain that

$$y_i = x_{2n-i}, \quad \lambda_{ij} = \rho_{2n-i,2n-j}.$$

We conclude that $g_v \cong g_w$.
Conversely, let $g_v \cong g_w$ with $g_v \neq g_w$. Then the isomorphism $\tau : g_v \to g_w$ is given by the map rev (Lemma 4.93). Thus, $g_w = g_v^* = g_{v^{-1}}$. Uniqueness of the construction yields that $w = v^{-1}$ and thus $w \sim v$.

□

**Example 4.97.** *Let $\Lambda$ be as in Example 2.3.1. We have determined its corresponding bundle of semichains in Example 4.81.*
*Let $w = \varepsilon^* a \varepsilon^*$ and $v = \varepsilon^* a^{-1} \varepsilon^*$. Then $v = w^{-1}$ and it follows that $v \sim w$.*
*We obtain the following corresponding $\mathfrak{L}$–chains:*

$$\begin{aligned} g_v : \quad & \mathfrak{R}_{12} - \mathfrak{C}_{\varepsilon^*} \sim \mathfrak{C}_{\varepsilon^*} - \mathfrak{R}_{13} \sim \mathfrak{R}_{11} - \mathfrak{C}_{\varepsilon^*} \sim \mathfrak{C}_{\varepsilon^*} - \mathfrak{R}_{12}, \\ g_w : \quad & \mathfrak{R}_{12} - \mathfrak{C}_{\varepsilon^*} \sim \mathfrak{C}_{\varepsilon^*} - \mathfrak{R}_{11} \sim \mathfrak{R}_{13} - \mathfrak{C}_{\varepsilon^*} \sim \mathfrak{C}_{\varepsilon^*} - \mathfrak{R}_{12}. \end{aligned}$$

*Let the underlying graphs be given by*

$$\begin{aligned} C_{g_w} : \quad & c_1 - c_2 - c_3 - c_4 - c_5 - c_6 - c_7 - c_8, \\ C_{g_v} : \quad & c_1' - c_2' - c_3' - c_4' - c_5' - c_6' - c_7' - c_8'. \end{aligned}$$

*Then $\tau : C_{g_w} \longrightarrow C_{g_v}$ gives an isomorphism between $g_w$ and $g_v$ by*

$$c_i \mapsto c_{9-i} \qquad 1 \le i \le 8.$$

**Lemma 4.98.** *Let $w$ be a finite $\Gamma_{\mathrm{ud}}(\Lambda)$–word with corresponding $\mathfrak{L}$–chain $g_w$. Then $g_w$ is composite if and only if $w$ is composite. In particular, $g_w = g_v^{[k]}$ if and only if $w = v^{[k]}$.*

*Proof.* Let $g_w = h^{[k]}$ be composite for some $\mathfrak{L}$–chain $h$ and some $k \ge 2$. Assume without loss of generality that $d(h) = 2$. Let $h$ start in the link $\mathfrak{C}_{\varepsilon^*}$ and end in $\mathfrak{C}_{\eta^*}$ for $\eta, \varepsilon \in \mathrm{Sp}$. The key argument is that links of the form $\mathfrak{C}_{\mu^*}$ for $\mu \in \mathrm{Sp}$ correspond to both basis of $\mathrm{im}(\mu)$ and basis of $\ker(\mu)$. Thus, $h$ corresponds to some non-coadmissible word $v$ such that $\varepsilon^* v \eta^*$ is again a word. Set $g_v = h$. Any $\mathfrak{L}$–chain of the form

$$h \sim h^*$$

translates to $v\eta^* v^{-1}$ since $g_v^* = g_{v^{-1}}$. Similarly, any $\mathfrak{L}$–chain

$$h^* \sim h$$

translates to $v^{-1}\varepsilon^* v$. Since $g_w = g_v^{[k]}$, it follows by uniqueness of the construction that $w = v^{[k]} = v\eta^* v^{-1} \varepsilon^* \dots$.
Conversely, let $w = v^{[k]}$ for some $k \ge 2$ and with $v$ such that $\varepsilon^* v \eta^*$ is again a word. Thus, $g_v$ starts in $\mathfrak{C}_{\varepsilon^*}$ and ends in $\mathfrak{C}_{\eta^*}$. We know that any special letter $\mu^*$ is translated to

$$\mathfrak{C}_{\mu^*} \sim \mathfrak{C}_{\mu^*}$$

in a corresponding $\mathfrak{L}$–chain. Extending $v$ to $v\eta*$ yields the $\mathfrak{L}$–chain

$$g_v \sim \mathfrak{C}_{\eta*}.$$

Extending further to $v\eta^* v^{-1}$ yields

$$g_v \sim g_v^*.$$

Thus, the corresponding $\mathfrak{L}$–chain of $v^{[k]} = v\eta^* v^{-1}\varepsilon^* \dots$ is given by

$$g_v^{[k]} = g_v \sim g_v^* \sim \dots.$$

It follows by uniqueness of the construction that

$$g_w = g_v^{[k]}.$$

□

**Example 4.99.** *Let $\Lambda$ be as in Example 2.14 and $\overline{\mathfrak{X}}_\Lambda$ as in Example 4.83. Let $w = a^{-1}b\eta^* b^{-1}a$. Then*

$$g_w: \quad \mathfrak{C}_{\varepsilon^*} - \mathfrak{R}_{12} \sim \mathfrak{C}_{21} - \mathfrak{R}_{21} \sim \mathfrak{R}_{32} - \mathfrak{C}_{\eta^*} \sim \mathfrak{C}_{\eta^*} - \mathfrak{R}_{32} \sim \mathfrak{R}_{21} - \mathfrak{C}_{21} \sim \mathfrak{R}_{12} - \mathfrak{C}_{\varepsilon^*}$$

*and $g_w = h^{[2]}$ is composite for*

$$h: \quad \mathfrak{C}_{\varepsilon^*} - \mathfrak{R}_{12} \sim \mathfrak{C}_{21} - \mathfrak{R}_{21} \sim \mathfrak{R}_{32} - \mathfrak{C}_{\eta^*}.$$

*It follows that $h = g_v$ for the subword $v = a^{-1}b$ of $w$. Morover, we have that $w = v^{[2]}$.*

Statement (ii) Lemma 4.96 is of importance for our theory about strings and bands. It shows that equivalent words give isomorphic $\mathfrak{L}$–chains. We are especially interested in constructing $\mathfrak{L}$–graphs for symmetric and asymmetric strings and bands which are representatives of equivalence classes. Our hope is to get simple (admissible) $\mathfrak{L}$–graphs such that we obtain canonical representations from those.
To this end, we examine now the properties of $g_w$ for $w$ a (non-)coadmissible word. Write $\tilde{L}^i$ for a link given by $L_i$ or $\bar{L}_i$. We consider – with notation from the construction – the following sets:

$$\mathfrak{L}_w = \{L_i, \bar{L}_i\}_{i=1}^n, \tag{117}$$
$$\tilde{\mathfrak{L}}_w^\alpha = \{P = (\tilde{L}_i, \tilde{L}_j) \mid \tilde{L}_i \alpha \tilde{L}_j,\, j \in \{i-1, i+1\}\}. \tag{118}$$

Let $P, P'$ be two pairs from $\tilde{\mathfrak{L}}_w^\alpha$. We write $P \approx P'$ if $P = (\tilde{L}_i, \tilde{L}_j)$ and $P' = (\tilde{L}_j, \tilde{L}_i)$. We denote the respective set of pairs up to symmetry by

$$\mathfrak{L}_w^\alpha = \tilde{\mathfrak{L}}_w^\alpha / \approx, \text{ and furthermore,} \tag{119}$$
$$\mathfrak{L}_w^{\bar{\alpha}} = \{\tilde{L}_i \in L_w \mid \nexists P \in \mathfrak{L}_w^\alpha \text{ with } \tilde{L}_i \in P\}, \tag{120}$$
$$L_w^\beta = \{(L_i, \bar{L}_i)_{i=1}^n\}. \tag{121}$$

Properties of these sets will lead to properties of $g_w$ with respect to coadmissibility.

**Lemma 4.100.** *Let $w$ be a finite $\Gamma_{\mathrm{ud}}(\Lambda)-$ word which is not left coadmissible or not right coadmissible. Then $L_0 = \mathfrak{C}_{\varepsilon^*}$ or $\bar{L}_n = \mathfrak{C}_{\varepsilon^*}$, respectively, for $\varepsilon \in \mathrm{Sp}$ such that $\varepsilon^* w$ or $w\varepsilon^*$ is again a word.*

*Proof.* Assume $w$ is not left coadmissible. Then there exists $\varepsilon \in \mathrm{Sp}$ such that $\varepsilon^* w$ is again a word. Note that $\varepsilon$ is unique by Remark 4.73 and Definition 2.9. We have $v_0(s) \in \ker(\varepsilon)$ and hence $L_0 = \mathfrak{C}_{\varepsilon^*}$.
The statement follows similarly for $w$ not right coadmissible and $\bar{L}_n$. □

**Lemma 4.101.** *Let $w$ be a finite $\Gamma_{\mathrm{ud}}(\Lambda)-$ word. Assume that there exist $i \in \{0, \dots, n\}$ with $\tilde{L}_i = \mathfrak{C}_{\varepsilon^*}$ for some $\varepsilon \in \mathrm{Sp}$, and with $\tilde{L}_i \in \mathfrak{L}_w^{\bar{\alpha}}$. Then either $\tilde{L}_i = L_0$, or $\tilde{L}_i = \bar{L}_n$. In this case, $\tilde{L}_i$ gives a double end.*

*Proof.* Let $\tilde{L}_i = \mathfrak{C}_{\varepsilon^*} \in \mathfrak{L}_w^{\bar{\alpha}}$ for some $i$. Then $\tilde{L}_i \alpha \tilde{L}_i$ holds. By assumption, the pre- and successor links are not given by $\mathfrak{C}_{\varepsilon^*}$. Hence, $\tilde{L}_i$ can only be connected by $\beta-$relations to other links in $\mathfrak{L}_w$. By definition of an $\mathfrak{L}-$graph, $\alpha-$ and $\beta-$relations take turns in $g_{w,1}$, i.e., $\tilde{L}_i$ must be the first or last link of the $\mathfrak{L}-$chain $g_w$ (otherwise there would be two $\beta-$relations in a row). It follows $\tilde{L}_i = L_0$ or $\tilde{L}_i = \bar{L}_n$.
In any of these cases, $\tilde{L}_i \alpha \tilde{L}_i$ holds and $\tilde{L}_i$ is connected by a $\beta-$relation to the rest of $g_w$, hence it gives a double end. □

**Example 4.102.** *Let $\Lambda$ be as in Example 2.14 with $\overline{\mathfrak{X}}_\Lambda$ as in Example 4.83. Let $w = a^{-1} d^{-1}$. Its corresponding $\mathfrak{L}-$chain is given by*

$$g_w : \quad \mathfrak{C}_{\varepsilon^*} - \mathfrak{R}_{12} \sim \mathfrak{C}_{21} - \mathfrak{R}_{23} \sim \mathfrak{R}_{41} - \mathfrak{C}_{42}.$$

*Then $\mathfrak{L}_w^{\bar{\alpha}} = \{\mathfrak{C}_{\varepsilon^*}, \mathfrak{C}_{42}\}$. We have $L_0 = \mathfrak{C}_{\varepsilon^*}$ and $\mathfrak{C}_{\varepsilon^*} \alpha \mathfrak{C}_{\varepsilon^*}$. Hence, $\mathfrak{C}_{\varepsilon^*}$ gives a double end. The link $\mathfrak{C}_{42}$ is not of the form $\mathfrak{C}_{\mu^*}$ for any $\mu \in \mathrm{Sp}$. It does not give a double end since $\mathfrak{C}_{42} \bar{\alpha} \mathfrak{C}_{42}$.*

**Lemma 4.103.** *Let $w$ be a finite coadmissible $\Gamma_{\mathrm{ud}}(\Lambda)-$word. Then there does not exist $\tilde{L}_i \in \mathfrak{L}_w^{\bar{\alpha}}$ with $\tilde{L}_i = \mathfrak{C}_{\varepsilon^*}$ for any $\varepsilon \in \mathrm{Sp}$, $0 \leq i \leq n$.*

*Proof.* Assume towards a contradiction that there exist $i$ as above with $\tilde{L}_i = \mathfrak{C}_{\varepsilon^*} \in \mathfrak{L}_w^{\bar{\alpha}}$ for some $\varepsilon \in \mathrm{Sp}$. It follows by construction that $\tilde{L}_i = L_0$ or $\tilde{L}_i = \bar{L}_n$. Assume without loss of generality the first case. Then $v_0(s) \in \ker(\varepsilon)$ and thus basis of $\ker(\varepsilon) \mathrel{\hat{=}} \mathfrak{C}_{\varepsilon^*} = L_0$. Thus, $v_0(Q) = s(\varepsilon)$. It follows that $\varepsilon^* w$ is again a word. This contradicts $w$ being coadmissible. □

**Corollary 4.104.**
a) *Let $w$ be a left coadmissible finite $\Gamma_{\mathrm{ud}}(\Lambda)-$word. Then $L_0 \in \mathfrak{L}_w^{\bar{\alpha}}$ with $L_0 \neq \mathfrak{C}_{\varepsilon^*}$ for any $\varepsilon \in \mathrm{Sp}$. Moreover, $L_0$ corresponds to the basis of one of the following subspaces: $\ker(a)$ for some $a \in Q_1^{\mathrm{ord}}$, $V_{v_0(Q)} \ominus \mathrm{im}(b)$ for some $b \in Q_1^{\mathrm{ord}}$, or $V_{v_0(Q)}$.*

b) *Let $w$ be a right coadmissible finite $\Gamma_{\mathrm{ud}}(\Lambda)-$word. Then $\bar{L}_n \in \mathfrak{L}_w^{\bar{\alpha}}$ with $\bar{L}_n \neq \mathfrak{C}_{\varepsilon^*}$ for any $\varepsilon \in \mathrm{Sp}$. Moreover, $\bar{L}_n$ corresponds to the basis of one of the following subspaces: $\ker(a)$ for some $a \in Q_1^{\mathrm{ord}}$, $V_{v_0(Q)} \ominus \mathrm{im}(b)$ for some $b \in Q_1^{\mathrm{ord}}$, or $V_{v_0(Q)}$.*

*Proof.* a) By the same line of argument as in Lemma 4.103, it follows that $L_0 \neq \mathfrak{C}_{\varepsilon^*}$ for any special loop $\varepsilon$, and that $L_0 \in \mathfrak{L}_w^{\bar{\alpha}}$. Assume towards a contradiction that $L_0$ does not correspond to any of the bases of the subspaces. We already know that it also does not correspond to basis of $\ker(\varepsilon)$ nor to basis of $\operatorname{im}(\varepsilon)$ for any $\varepsilon \in \mathrm{Sp}$. Thus, by construction, $L_0$ either corresponds to basis of $\operatorname{im}(b)$ for some $b \in Q_1^{\mathrm{ord}}$, or to basis of $V_{v_0(Q)} \ominus \ker(a)$ for some $a \in Q_1^{\mathrm{ord}}$. The first case would require an additional letter $w_0 = b^{-1}$, the second an additional letter $w_0 = a$. Both cases give a contradiction to $w$ starting in $w_1$, i.e., to the length of $w$.

b) The result follows similar to a).

□

**Example 4.105.** *Let $\Lambda$ be as in Example 2.14 and let $w = \varepsilon^* a^{-1} d^{-1}$. Then $w$ is coadmissible and $g_w$ is given by*

$$g_w : \quad \mathfrak{R}_{11} - \mathfrak{C}_{\varepsilon^*} \sim \mathfrak{C}_{\varepsilon^*} - \mathfrak{R}_{12} \sim \mathfrak{C}_{21} - \mathfrak{R}_{23} \sim \mathfrak{R}_{41} - \mathfrak{C}_{42}.$$

*We have*

$$\mathfrak{L}_w^{\bar{\alpha}} = \{\mathfrak{R}_{11}, \mathfrak{C}_{42}\}.$$

*Both links are not of the form $\mathfrak{C}_{\mu^*}$ for any $\mu \in \mathrm{Sp}$.*

**Remark 4.106.** *The word $w$ from Example 4.102 is not left coadmissible and the link $\mathfrak{C}_{\varepsilon^*}$ belongs to the set $\mathfrak{L}_w^{\bar{\alpha}}$. Thus, we could assume that the converse of Lemma 4.103 holds. But that is not the case:*
*For instance, let $\Lambda$ be as in Example 2.3.1. and let $w = a$. Then $w$ is neither left coadmissible nor right coadmissible. Its corresponding $\mathfrak{L}$–chain is given by*

$$g_w : \quad \mathfrak{C}_{\varepsilon^*} - \mathfrak{R}_{11} \sim \mathfrak{R}_{13} - \mathfrak{C}_{\varepsilon^*}.$$

*We have $(\mathfrak{R}_{11}, \mathfrak{R}_{13}), (\mathfrak{C}_{\varepsilon^*}, \mathfrak{C}_{\varepsilon^*}) \in \tilde{\mathfrak{L}}_w^{\alpha}$. It follows that $\mathfrak{L}_w^{\bar{\alpha}} = \varnothing$. Hence, $\mathfrak{L}_w^{\bar{\alpha}}$ does not contain any link of the form $\mathfrak{C}_{\varepsilon^*}$, but $w$ is not coadmissible.*

The example given in the previous remark can be generalised:

**Lemma 4.107.** *Let $\Lambda$ be given by*

$$Q : \quad \varepsilon \circlearrowright \bullet \circlearrowleft a$$

*with $\mathrm{Sp} = \{\varepsilon\}$ and $\mathrm{R} = \{a^2\}$. Let $w$ be a finite $\Gamma_{\mathrm{ud}}(\Lambda)$–word with $w \neq 1_{i,\kappa}$ for any $i \in Q_0$, any $\kappa \in \{+,-\}$. Furthermore, let $w$ be neither left coadmissible nor right coadmissible. Then $\mathfrak{L}_w^{\bar{\alpha}} = \varnothing$.*

*Proof.* Since $w$ is neither left-coadmissible nor right-coadmissible, we know that $w_1 = x$ for $x \in \{a, a^{-1}\}$. Then $w_2 = \varepsilon^*$. Assume without loss of generality that $w_1 = a$. It follows that

$$\begin{aligned}
&v_0(s) \in \ker(\varepsilon), && \text{basis of } \ker(\varepsilon) \mathrel{\hat{=}} \mathfrak{C}_{\varepsilon^*} = L_0,\\
&v_0(t) \in \operatorname{im}(a), &&\\
&v_1(s) \in V_{v_1(Q)} \ominus \ker(a), &&\\
&v_1(t) \in \varepsilon^*, && \text{basis of } \varepsilon^* \mathrel{\hat{=}} \mathfrak{C}_{\varepsilon^*} = \bar{L}_1,\\
&v_2(s) \in \varepsilon^* && \text{basis of } \varepsilon^* \mathrel{\hat{=}} \mathfrak{C}_{\varepsilon^*} = L_2.
\end{aligned}$$

Hence, we have that

$$L_0 \alpha \bar{L}_1 \alpha L_2$$

and thus $L_0 \notin \mathfrak{L}_w^{\bar{\alpha}}$. Similarly, we obtain that $\bar{L}_n \notin \mathfrak{L}_w^{\bar{\alpha}}$. By construction, any other link in $g_{w,0}$ is in $\alpha$–relation with one of its neighbouring links and thus belongs to $\mathfrak{L}_w^{\alpha}$. □

If we exclude the algebra from Lemma 4.107, the converse of Lemma 4.103 and its Corollary 4.104 do hold:

**Lemma 4.108.** *Let $\Lambda$ be different from Lemma 4.107. Let $w$ be a finite $\Gamma_{\mathrm{ud}}(\Lambda)$–word which is not coadmissible. Then $\mathfrak{L}_w^{\bar{\alpha}} \neq \varnothing$.*

*Proof.* Since $w$ is not coadmissible, it is not left coadmissible or not right coadmissible or both. Assume without loss of generality that it is neither left coadmissible nor right coadmissible. Then there exist $\varepsilon, \mu \in \mathrm{Sp}$ such that $\varepsilon^* w \mu^*$ is again a word. Thus, when considering $g_w$, we have that

$$\begin{aligned}
v_0(s) &\in \ker(\varepsilon), \text{basis of } \varepsilon^* \mathrel{\hat{=}} \mathfrak{C}_{\varepsilon^*} = L_0,\\
v_n(t) &\in \ker(\mu), \text{basis of } \mu^* \mathrel{\hat{=}} \mathfrak{C}_{\mu^*} = \bar{L}_n.
\end{aligned}$$

By form of $\Lambda$, we know that $L_1$ and $\bar{L}_1$ are not in $\alpha$–relation with $L_0$. Similarly, we have that $\bar{L}_n \bar{\alpha} L_{n-1}$ and $\bar{L}_n \bar{\alpha} \bar{L}_{n-1}$. It follows that

$$L_0, L_n \in \mathfrak{L}_w^{\bar{\alpha}}.$$

□

**Corollary 4.109.** *Let $\Lambda$ be different from the algebra in Lemma 4.107.*

*a) If $w$ is a left coadmissible finite $\Gamma_{\mathrm{ud}}(\Lambda)$–word, then $\mathfrak{L}_w^{\bar{\alpha}} \neq \varnothing$.*

*b) If $w$ is a right coadmissible finite $\Gamma_{\mathrm{ud}}(\Lambda)$–word, then $\mathfrak{L}_w^{\bar{\alpha}} \neq \varnothing$.*

*Proof.* Both statements follow from Lemma 4.108. □

We have seen that the property of $\mathfrak{L}_w^{\bar{\alpha}}$ being empty or not does not give any hint about $w$ being coadmissible or not. But the types of links contained in $\mathfrak{L}_w^{\bar{\alpha}}$ do.

**Remark 4.110.** *The above statements give a hint about the connection between the terms of* admissibility *and* coadmissibility*. We have chosen the term* coadmissibility *for the respecting property since we do not only require admissibility on our $\mathfrak{L}$−chains. Recall, that admissibility states that*

*for $X, Y \in \mathfrak{L}$ with $X \neq Y$, $X\alpha Y$ and $g(c_i) = X$ for some $i$ , there exists an edge $\rho$ containing $c_i$ with $g(c_i) = \alpha$.*

*Thus, admissibility does not include two-point links. Coadmissibility on words gives admissibility on their corresponding $\mathfrak{L}$−graphs extended to two-point links:*

*for $X, Y \in \mathfrak{L}$ with $X = Y$ and $g(c_i) = X$ for some $i$ , there exists an edge $\rho$ containing $c_i$ with $g(c_i) = \alpha$.*

Finally, we see that the chains arising from asymmetric and symmetric strings or symmetric bands have certain properties:

**Proposition 4.111.** *a) Let $w$ be an asymmetric string. Then $d(g_w) = 0$.*

*b) Let $w = u\varepsilon^* u^{-1}$ be a symmetric string. Then $d(g_u) = 1$.*

*c) Let $w_{\mathbb{Z}}$ be a symmetric band with $\hat{w}_p = \varepsilon^* u \eta^* u^{-1}$. Then $d(g_u) = 2$.*

*Proof.* a) By definition, $w$ is coadmissible. Thus we have by Lemma 4.103 that $L_0 \neq \mathfrak{C}_{\varepsilon^*} \neq \bar{L}_n$ for any $\varepsilon \in \mathrm{Sp}$. By construction, only links of type $\mathfrak{C}_{\varepsilon^*}$ give double ends. Hence, it follows that $d(g_w) = 0$.

b) The word $u$ is left coadmissible, but not right coadmissible. Applying Lemma 4.108 yields that $\bar{L}_n = \mathfrak{C}_{\varepsilon^*} \in \mathfrak{L}_w^{\bar{\alpha}}$. Lemma 4.101 gives that $\bar{L}_n$ is a double end. Since $u$ is left coadmissible, we have that $L_0 \neq \mathfrak{C}_{\mu^*}$ for any $\mu \in \mathrm{Sp}$. Thus, $L_0$ does not give a double end. It follows that $d(g_u) = 1$.

c) Since $\varepsilon^* u \eta^*$ gives a word, $u$ is neiter left coadmissible nor right coadmissible. As in b), it follows that $L_0 = \mathfrak{C}_{\varepsilon^*}$ and $\bar{L}_n = \mathfrak{C}_{\eta^*}$. Hence, we obtain that $d(g_u) = 2$.

□

**Example 4.112.** *Let $\Lambda$ be as in Example 2.3.1. with $\overline{\mathfrak{X}}_\Lambda$ as in Example 4.81.*

*a) Let $w = \varepsilon^* a \varepsilon^*$ be an asymmetric string in $\Gamma_{\mathrm{ud}}(\Lambda)$. Then*

$$g_w: \quad \mathfrak{R}_{12} - \mathfrak{C}_{\varepsilon^*} \sim \mathfrak{C}_{\varepsilon^*} - \mathfrak{R}_{11} \sim \mathfrak{R}_{13} - \mathfrak{C}_{\varepsilon^*} \sim \mathfrak{C}_{\varepsilon^*} - \mathfrak{R}_{12}$$

*has no double end.*

*b) Let $w = \varepsilon^*$ be a symmetric string in $\Gamma_{\mathrm{ud}}(\Lambda)$ with $u = 1_{1,\mu}$, where $\mu = \mathrm{sgn}(\varepsilon^*)$. Then*

$$g_u: \quad \mathfrak{R}_{12} - \mathfrak{C}_{\varepsilon^*}$$

*and $d(g_u) = 1$, since its right end is double.*

*c) Let $w_{\mathbb{Z}}$ be a symmetric band in $\Gamma_{\mathrm{ud}}(\Lambda)$ with $\hat{w}_p = \varepsilon^* a \varepsilon^* a^{-1}$. It is of period 4 with $u = a$. We obtain*

$$g_u : \quad \mathfrak{C}_{\varepsilon^*} - \mathfrak{R}_{11} \sim \mathfrak{R}_{13} - \mathfrak{C}_{\varepsilon^*}$$

*with $d(g_u) = 2$.*

The final and most important result of this section ensures that we obtain for asymmetric and symmetric strings or symmetric bands $\mathfrak{L}$–chains which are simple and admissible. Thus, we make sure that a representative of an equivalence class of certain words gives a representative of a certain equivalence class of $\mathfrak{L}$–chains. Hence, the constructed $\mathfrak{L}$–chains give canonical $\overline{\mathfrak{X}}_\Lambda$–representations.

**Theorem 4.113.** *a) Let $w$ be an asymmetric string. Then $g_w \in \bar{\mathfrak{S}}(\mathfrak{L})$.*

*b) Let $w = u\varepsilon^* u^{-1}$ be a symmetric string. Then $g_u \in \bar{\mathfrak{S}}(\mathfrak{L})$.*

*c) Let $w_{\mathbb{Z}}$ be a symmetric band with $\hat{w}_p = \varepsilon^* u \eta^* u^{-1}$. Then $g_u \in \bar{\mathfrak{S}}(\mathfrak{L})$.*

*Proof.* We know by Lemma 4.89 that the construction gives an $\mathfrak{L}$–chain for the respective words in a) - c). It remains to prove simplicity and admissibility.

a) For $w$ an asymmetric string, we have that $d(g_w) = 0$, i.e., neither $L_0$ nor $\bar{L}_n$ are of the form $\mathfrak{C}_{\varepsilon^*}$ for any $\varepsilon \in \mathrm{Sp}$. By Lemma 4.104 we know that $L_0$ and $\bar{L}_n$ correspond each to a one-point link that is not in an $\alpha$–relation. Thus, admissibility at the beginning and end of $g_w$ is given (cf. Remark 4.79). By construction, we also have admissibility for the rest of the chain.
By Lemma 2.54 we know that $w$ is not composite. Lemma 4.98 yields that $g_w$ is also not composite.

b) The same line of argument as in a) gives admissibility for any link but the double end. Here, admissibility follows since the link is of the form $\mathfrak{C}_{\varepsilon^*}$ which is in $\alpha$–relation with itself and thus does not need to be considered for admissibility (cf. Remark 4.79).
Simplicity of $g_u$ follows as in a) by minimality of $u$.

c) Both admissibility and simplicity follow analogously to b) for both ends.

□

**Example 4.114.** *Consider again Example 4.112. Recall that by construction, we only have to check on admissibility at the start and end of the respective $\mathfrak{L}$–chain. We show that all three examples give $\mathfrak{L}$–chains in $\bar{\mathfrak{S}}(\mathfrak{L})$:*

a) *Both ends are given by* $\mathfrak{R}_{12}$. *Now* $\sigma_\Lambda$ *acts as identity on* $\mathfrak{R}_{12}$. *By Remark 4.79, this link is in no* $\alpha$*-relation and thus admissibilty at both ends is given. Furthermore, simplicity follows since* $\mathfrak{R}_{11} \neq \mathfrak{R}_{13}$.

b) *The link* $\mathfrak{R}_{12}$ *does not prevent admissibility by same line of argument as in a). The link* $\mathfrak{C}_{\varepsilon^*}$ *is a two-point link and does not need to be considered for admissibility. Hence,* $g_u$ *is admissible. It is simple since it does not contain any* $\alpha$*-relation.*

c) *Both ends are given by two-point links. Thus,* $g_u$ *is admissible. Since* $\mathfrak{R}_{11} \neq \mathfrak{R}_{13}$, *it is also simple.*

**Example 4.115.** *Let* $\Lambda$ *be as in Example 2.14.*

a) *Let* $w = da\varepsilon^*$ *be an asymmetric string. Then*

$$g_w: \quad \mathfrak{C}_{42} - \mathfrak{R}_{41} \sim \mathfrak{R}_{23} - \mathfrak{C}_{21} \sim \mathfrak{R}_{12} - \mathfrak{C}_{\varepsilon^*} \sim \mathfrak{C}_{\varepsilon^*} - \mathfrak{R}_{11}.$$

*The only links serving as double ends of a subchain are given by* $\mathfrak{C}_{\varepsilon^*}$. *It follows that* $d(g_w) = 0$. *The form of* $g_w$ *yields that it is simple. Furthermore, both* $\mathfrak{C}_{42}$ *and* $\mathfrak{R}_{11}$ *are in no* $\alpha$*-relation. Thus,* $g_w$ *is also admissible. Hence,* $g_w \in \bar{\mathfrak{S}}(\mathfrak{L})$.

b) *Let* $w = da\varepsilon^* a^{-1}d^{-1}$ *be a symmetric string. Its corresponding* $\mathfrak{L}$*-chain is given with* $u = da$ *by*

$$g_u: \mathfrak{C}_{42} - \mathfrak{R}_{41} \sim \mathfrak{R}_{23} - \mathfrak{C}_{21} \sim \mathfrak{R}_{12} - \mathfrak{C}_{\varepsilon^*}.$$

*The link* $\mathfrak{C}_{\varepsilon^*}$ *gives a double end. Analogoulsy to a),* $\mathfrak{C}_{42}$ *does not give a double end. Thus,* $d(g_u) = 1$. *Apart from the double end, we do not have another link in* $g_{u,0}$ *which is of the form* $\mathfrak{C}_{\mu^*}$ *for some* $\mu \in \mathrm{Sp}$. *Thus,* $g_u$ *is simple. Furthermore,* $g_u$ *is admissible since* $\mathfrak{C}_{42}$ *is not in any* $\alpha$*-relation. It follows that* $g_u \in \bar{\mathfrak{S}}(\mathfrak{L})$.

c) *Let* $w_{\mathbb{Z}}$ *be a symmetric band with* $\hat{w}_p = \varepsilon^* a^{-1} b \eta^* b^{-1} a$. *Then* $u = a^{-1}b$ *and*

$$g_u: \quad \mathfrak{C}_{\varepsilon^*} - \mathfrak{R}_{12} \sim \mathfrak{C}_{21} - \mathfrak{R}_{21} \sim \mathfrak{R}_{32} - \mathfrak{C}_{\eta^*}.$$

*Both* $\mathfrak{C}_{\varepsilon^*}$ *and* $\mathfrak{C}_{\eta^*}$ *give double ends:* $d(g_u) = 2$. *It follows also that* $g_u$ *is admissible. Moreover, there are no other links of the form* $\mathfrak{C}_{\mu^*}$ *for some* $\mu \in \mathrm{Sp}$ *contained in* $g_{u,0}$. *Hence,* $g_u$ *is simple and* $g_u \in \bar{\mathfrak{S}}(\mathfrak{L})$.

**Theorem 4.116.** *Let* $g \in \bar{\mathfrak{S}}(\mathfrak{L})$ *with*

a) $d(g) = 0$. *Then there exists a word* $w$ *of asymmetric string type with* $g_w = g$.

b) $d(g) = 1$. *Then there exists a word* $w$ *of symmetric string type with* $w = u\varepsilon^* u^{-1}$ *and* $g_u = g$.

*a)* $d(g) = 2$. *Then there exists a word* $w_{\mathbb{Z}}$ *of symmetric band type with periodic part* $\hat{w}_p = \varepsilon^* u \eta^* u^{-1}$ *for some* $\varepsilon, \eta \in \mathrm{Sp}$, *and* $g_u = g$.

*Proof.* Let $g_0 = \{x_0, \dots, x_m\}$.

a) Applying the construction backwards on $g$ results in an undirected word $w$ with $g_w = g$. It remains to show that $w$ is of asymmetric string type. We know that $g$ is simple which yields by Lemma 4.98 that $w$ is not composite. It follows that $w \neq w^{-1}$. Now assume towards a contradiction that $w$ is not left coadmissible. Then there exists $\varepsilon \in \mathrm{Sp}$ such that $\varepsilon^* w$ is again a word. It follows that $x_0 = \mathfrak{C}_{\varepsilon^*}$ which gives a double link.

b) By construction, we obtain a word $u$ with $g_u = g$ and with $u$ not left but right coadmissible if $x_0$ is a double end, and with $u$ not right but left coadmissible if $x_m$ is a double end. Assume without loss of generality that $x_m = \mathfrak{C}_{\varepsilon^*}$, for some $\varepsilon \in \mathrm{Sp}$, is a double end. Then $u\varepsilon^*$ is again a word and we can set $w = u\varepsilon^* u^{-1}$. It follows by definition of $u$ that $w$ is coadmissible. Analogously to a) we have that $u$ is simple. Hence, $w$ is of symmetric string type.

c) Similarly to b), we obtain by applying the construction backwards a word $u$ which is neither left nor right coadmissible with $g_u = g$. Since $g$ has two double ends, we know that $x_0 = \mathfrak{C}_{\varepsilon^*}$ for some $\varepsilon \in \mathrm{Sp}$ and that $x_m = \mathfrak{C}_{\eta^*}$ for some $\eta \in \mathrm{Sp}$. Thus, $\varepsilon^* u \eta^*$ is again a word. We set $\hat{w}_p = \varepsilon^* u \eta^* u^{-1}$ and consider $w_{\mathbb{Z}}$ with this periodic part. We know that $w_{\mathbb{Z}} = w_{\mathbb{Z}}^{-1}[-1]$ and that $w_{\mathbb{Z}} = w_{\mathbb{Z}}[p]$ by the form of $\hat{w}_p$. It remains to show that its period $p$ is given by $2|u| + 2$. We know by Lemma 4.98 that $u$ is simple. Thus, there does not exist any $p' < p$ with $w_{\mathbb{Z}} = w_{\mathbb{Z}}[p']$.

□

Thus, we clearly obtain by Theorem 4.113 and 4.116 a 1-1-correspondence between words of asymmetric string type and $\mathfrak{L}$–chains with no double ends, and a similar correspondence between words of symmetric string type and $\mathfrak{L}$–chains with one double end. We also obtain a 1-1-correspondence between words of symmetric band type and $\mathfrak{L}$–chains with two double ends. But we also get the following 1-1-correspondence:

**Corollary 4.117.** *There exists a 1-1-correspondence between the equivalence classes of asymmetric and symmetric strings, symmetric bands, and the isomorphism classes of* $\mathfrak{L}$*–chains in* $\bar{\mathfrak{S}}(\mathfrak{L})$.

*Proof.* Theorem 4.113 and Theorem 4.116 give a 1-1-correspondence between the set of words of asymmtric string, symmetric string and symmetric band type and the set of simple, admissible $\mathfrak{L}$–chains. By Lemma 4.96 we obtain a 1-1-correspondence between the equivalence classes on the words of asymmetric and symmetric string type and the isomorphism classes of the chains. We observe for words of symmetric band type that a corresponding $\mathfrak{L}$–chain constructed from its inverse, shift or inverse shift is also based on either $u$ or $u^{-1}$. Thus, we also obtain a 1-1-correspondence between the equivalence classes of words of symmetric band type and the isomorphism classes of $\mathfrak{L}$–chains with two double ends. $\square$

## 4.6 $\mathfrak{L}$–cycles from periodic words

Throughout this section, let $\Lambda$ be as Section 4.4. We consider $\mathbb{Z}$–words $w_{\mathbb{Z}}$ in $\Gamma_{\text{ud}}(\Lambda)$ of period $p$. For any such $w_{\mathbb{Z}}$ we want to obtain a corresponding $\mathfrak{L}$–cycle $g_{w_{\mathbb{Z}}}$. To this end, we give a construction. The idea is similar to the one of the previous section and we find similarities between the construction of $\mathfrak{L}$–cycles and the one of $\mathfrak{L}$–chains.
At the end of this section we see in Theorem 4.130 that asymmetric and symmetric bands result in simple $\mathfrak{L}$–cycles. Theorem 4.141 shows that there exists a correspondence between the set of words of band type and the set of simple $\mathfrak{L}$–cycles. In addition, we show that there exists a 1-1-correspondence between the equivalence classes of bands and the isomorphism classes of $\mathfrak{L}$–chains in $\mathring{\mathfrak{S}}\mathfrak{L}$ (Corollary 4.142).

We use the same notation for the construction as in Section 4.5 for the construction of $\mathfrak{L}$–chains.

**Construction of $g_{w_{\mathbb{Z}}}$.** Let $w_{\mathbb{Z}}$ be an undirected $\mathbb{Z}$–word of period $p$ and with $\hat{w}_p = w_1 \dots w_p$. We construct its corresponding $\mathfrak{L}$–cycle $g_{w_{\mathbb{Z}}}$ as follows:

1. Depict $\hat{w}_p$ as $D_{w_{\mathbb{Z}}}:\quad v_0 \overset{w_1}{\leftarrow} v_1 \overset{w_2}{\leftarrow} v_2 \dots \overset{w_p}{\leftarrow} v_p.$
2. Set $v_p = v_0$.
3. Associate to each $v_i$ the values $v_i(s)$ and $v_i(t)$, for all $0 \le i \le p-1$, which give the start and target of the letters $w_i$ and $w_{i+1}$, respectively.
4. Associate to each $v_i(s)$ a node $c_i$ in the graph $C_{g_{w_{\mathbb{Z}}}}$, and to each $v_i(t)$, $0 \le i \le p-1$ a node $\bar{c}_i$ in $C_{g_{w_{\mathbb{Z}}}}$, $0 \le i \le p-1$. The graph $C_{g_{w_{\mathbb{Z}}}}$ is cyclic and of the form
$$C_{g_{w_{\mathbb{Z}}}}:\quad \bar{c}_0 — c_1 — \bar{c}_1 — \cdots — c_{p-1} — \bar{c}_{p-1} — c_0$$
5. Consider each letter $w_i$ as a map sending $v_i(s)$ to $v_{i-1}(t)$, where $i \in \{1,\dots,p\}$. Assign to each of those a unique subspace $X, \bar{X}$ in one of the filtrations $F^{(j)}_{v_i(Q)}$, $F^{(k)}_{v_{i-1}(Q)}$, respectively, $j,k \in \{1,2\}$. For some $i \in Q_0$ we have that
$$\begin{aligned} X \text{ in } F^{(j)}_{v_i(Q)} \text{ is assigned to } v_i(s) \text{ if } & \operatorname{sgn}(w_i^{-1}) = \operatorname{sgn}(F^{(j)}_{v_i(Q)}),\\ & v_i(s) \in X \text{ and } v_i(s) \notin V_{v_i(Q)} \ominus X,\\ \bar{X} \text{ in } F^{(\bar{j})}_{v_i(Q)} \text{ is assigned to } v_i(t) \text{ if } & \operatorname{sgn}(w_{i+1}) = \operatorname{sgn}(F^{(\bar{j})}_{v_i(Q)}),\\ & v_i(t) \in \bar{X} \text{ and } v_i(t) \notin V_{v_i(Q)} \ominus \bar{X},\end{aligned}$$
for all $i \in \{0,\dots,p-1\}$, $\bar{j} \neq j$, $j,\bar{j} \in \{1,2\}$. Note that we identify $w_p$ with $w_0$ according to periodicity.

6. Using these subspaces, assign to each $v_i(s)$ and $v_i(t)$ the link corresponding to the basis of its subspace:

   if $v_i(s)$ is assigned to $X$, then the link $L_i \hat{=}$ basis of $X$ is assigned to $v_i(s)$,
   if $v_i(t)$ is assigned to $\bar{X}$, then the link $\bar{L}_i \hat{=}$ basis of $\bar{X}$ is assigned to $v_i(t)$,

   for $0 \le i \le p-1$, $L_i \neq \bar{L}_i \in \mathfrak{L}(\mathfrak{C} \cup \mathfrak{R})$.

7. Order the links according to their corresponding values $v_i(s)$ and $v_i(t)$:

$$\begin{array}{cccccccc} \overbrace{v_0(t)}^{v_0} & \overbrace{v_1(s) \quad v_1(t)}^{v_1} & \cdots & \overbrace{v_{p-1}(s) \quad v_{p-1}(t)}^{v_{p-1}} & \overbrace{v_0(s)}^{v_0} \\ \bar{L}_0 & L_1 \qquad \bar{L}_1 & \cdots & L_{p-1} \qquad \bar{L}_{p-1} & L_0 \end{array}$$

8. Set $g_{w_{\mathbb{Z}}} : C_{g_{w_{\mathbb{Z}}}} :\to \mathfrak{L} \cup \{\alpha, \beta\}$,

$$\begin{cases} c_i \mapsto x_i \\ \bar{c}_i \mapsto \bar{x}_i \end{cases}, \delta \mapsto \begin{cases} \lambda_{i,\bar{i}} & \text{if } c_i \overset{\delta}{-} \bar{c}_i \\ \lambda_{\bar{i},i+1} & \text{if } \bar{c}_{i+1} \overset{\delta}{-} c_i \end{cases},$$

   where

$$\begin{aligned} x_i &= L_i, \quad \bar{x}_i = \bar{L}_i, \\ \lambda_{i,\bar{i}} &= \beta, \\ \lambda_{\bar{i},i+1} &= \alpha, \end{aligned}$$

   for all $0 \le i \le p-1$, with the indices considered modulo $p$.

We obtain the $\mathfrak{L}$–cycle $g_{w_{\mathbb{Z}}}$ with

$$\begin{aligned} g_{w_{\mathbb{Z}},0} &= \{\bar{x}_0, x_1, \bar{x}_1, \ldots, x_{p-1}, \bar{x}_{p-1}, x_0\}, \\ g_{w_{\mathbb{Z}},1} &= \{\alpha, \beta, \alpha, \ldots, \alpha, \beta\}. \end{aligned}$$

**Remark 4.118.** *Note that one can also construct an $\mathfrak{L}$–cycle using $\hat{w}_{kp}$ given by $k$ copies of $\hat{w}_p$. These $\mathfrak{L}$–cycles are not simple. We denote in the following by $g_{w_{\mathbb{Z}}}$ the $\mathfrak{L}$–cycles constructed from $\hat{w}_p$.*

**Example 4.119.** *Let $\Lambda$ be as in Example 2.3.1. with $\overline{\mathfrak{X}}_\Lambda$ as in Example 4.81. Let $w_{\mathbb{Z}}$ be a periodic $\mathbb{Z}$–word with $\hat{w}_p = \varepsilon^* a$. Its $\mathfrak{L}$–cycle is constructed as follows:*

*1.* $\quad D_{w_{\mathbb{Z}}} : \quad v_0 \overset{\varepsilon^*}{—} v_1 \xleftarrow{a} v_2.$

*2.+3.* $\quad v_0(t) \xrightarrow{\varepsilon^*} v_1(s) \qquad v_1(t) \xleftarrow{a} v_0(s)$

*4.*

$$C_{g_{w_{\mathbb{Z}}}}: \quad \bar{c}_0 — c_1 — \bar{c}_1 — c_0$$

*5.+6.*

| $v_i$ | $v_i(s), v_i(t)$ | $\operatorname{sgn}(w_i^{-1}), \operatorname{sgn}(w_{i+1})$ | *subspace* | *link* |
|---|---|---|---|---|
| $v_0$ | $v_0(s)$ | $-1$ | $V_1 \ominus \ker(a)$ | $\mathfrak{R}_{13} = L_0$ |
| | $v_0(t)$ | $1$ | $\varepsilon^*$ | $\mathfrak{C}_{\varepsilon^*} = \bar{L}_0$ |
| $v_1$ | $v_1(s)$ | $1$ | $\varepsilon^*$ | $\mathfrak{C}_{\varepsilon^*} = L_1$ |
| | $v_1(t)$ | $-1$ | $\operatorname{im}(a)$ | $\mathfrak{R}_{11} = \bar{L}_1$ |

*7.+8.*

$$g_{w_{\mathbb{Z}}}: \quad \mathfrak{C}_{\varepsilon^*} \sim \mathfrak{C}_{\varepsilon^*} - \mathfrak{R}_{11} \sim \mathfrak{R}_{13}$$

**Example 4.120.** *Let $\Lambda$ be as in Example 2.14 and let $\overline{\mathfrak{X}}_\Lambda$ be as in Example 4.83. Let $w_{\mathbb{Z}}$ be a periodic $\mathbb{Z}$−word with $\hat{w}_p = \eta^* b^{-1} a \varepsilon^* a^{-1} b$. Its corresponding $\mathfrak{L}$−cycle $g_{w_{\mathbb{Z}}}$ is constructed by the following steps:*

*1.* $$D_{w_{\mathbb{Z}}}: \quad v_0 \overset{\eta^*}{—} v_1 \xrightarrow{b} v_2 \xleftarrow{a} v_3 \overset{\varepsilon^*}{—} v_4 \xrightarrow{a} v_5 \xleftarrow{b} v_6$$

*2.+3.* $v_0(t) \overset{\eta^*}{—} v_1(s)$ $\quad v_1(t) \xrightarrow{b} v_2(s)$ $\quad v_2(t) \xleftarrow{a} v_3(s)$ $\quad v_3(t) \overset{\varepsilon^*}{—} v_4(s)$ $\quad v_4(t) \xrightarrow{a} v_5(s)$ $\quad v_5(t) \xleftarrow{b} v_0(s)$

*4.*

$$C_{g_{w_{\mathbb{Z}}}}: \quad \bar{c}_0 — c_1 — \bar{c}_1 — c_2 — \bar{c}_2 — c_3 — \bar{c}_3 — c_4 — \bar{c}_4 — c_5 — \bar{c}_5 — c_0$$

*5.+6.*

| $v_i$ | $v_i(s), v_i(t)$ | $\operatorname{sgn}(w_i^{-1}), \operatorname{sgn}(w_{i+1})$ | *subspace* | *link* |
|---|---|---|---|---|
| $v_0$ | $v_0(s)$ | $1$ | $V_3 \ominus \ker(b)$ | $\mathfrak{R}_{32} = L_0$ |
| | $v_0(t)$ | $-1$ | $\eta^*$ | $\mathfrak{C}_{\eta^*} = \bar{L}_0$ |
| $v_1$ | $v_1(s)$ | $-1$ | $\eta^*$ | $\mathfrak{C}_{\eta^*} = L_1$ |
| | $v_1(t)$ | $1$ | $V_3 \ominus \ker(b)$ | $\mathfrak{R}_{32} = \bar{L}_1$ |
| $v_2$ | $v_2(s)$ | $1$ | $\operatorname{im}(b)$ | $\mathfrak{R}_{21} = L_2$ |
| | $v_2(t)$ | $-1$ | $\operatorname{im}(a)$ | $\mathfrak{C}_{21} = \bar{L}_2$ |
| $v_3$ | $v_3(s)$ | $1$ | $V_1 \ominus \ker(a)$ | $\mathfrak{R}_{12} = L_3$ |
| | $v_3(t)$ | $-1$ | $\varepsilon^*$ | $\mathfrak{C}_{\varepsilon^*} = \bar{L}_3$ |
| $v_4$ | $v_4(s)$ | $-1$ | $\varepsilon^*$ | $\mathfrak{C}_{\varepsilon^*} = L_4$ |
| | $v_4(t)$ | $1$ | $V_1 \ominus \ker(a)$ | $\mathfrak{R}_{12} = \bar{L}_4$ |
| $v_5$ | $v_5(s)$ | $-1$ | $\operatorname{im}(a)$ | $\mathfrak{C}_{21} = L_5$ |
| | $v_5(t)$ | $1$ | $\operatorname{im}(b)$ | $\mathfrak{R}_{21} = \bar{L}_5$ |

*7.+8.*

$g_{w_{\mathbb{Z}}}$ :

$$\mathfrak{C}_{\eta^*} \sim \mathfrak{C}_{\eta^*} - \mathfrak{R}_{32} \sim \mathfrak{R}_{21} - \mathfrak{C}_{21} \sim \mathfrak{R}_{12} - \mathfrak{C}_{\varepsilon^*} \sim \mathfrak{C}_{\varepsilon^*} - \mathfrak{R}_{12} \sim \mathfrak{C}_{21} - \mathfrak{R}_{21} \sim \mathfrak{R}_{32}$$

We denote the sets $\mathfrak{L}_{w_{\mathbb{Z}}}$ and $\mathfrak{L}^{\alpha}_{w_{\mathbb{Z}}}$ analogously to (117) and (119). Let $\tilde{L}_i \in \{L_i, \bar{L}_i\}$.

**Remark 4.121.** *For any $w_{\mathbb{Z}}$ of period $p$, we consider an $\mathfrak{L}$–cycle $g_{w_{\mathbb{Z}}}$ of even length such that*

$$\#\{\alpha \in g_{w_{\mathbb{Z}},1}\} = \#\{\beta \in g_{w_{\mathbb{Z}},1}\}.$$

*This means that each link $\tilde{L}_i \in \mathfrak{L}_{w_{\mathbb{Z}}}$ is in at least one $\alpha$–relation.*
*Note that in particular $\mathfrak{L}^{\bar{\alpha}}_{w_{\mathbb{Z}}} = \varnothing$, i.e., none of the links $\tilde{L}_i \in \mathfrak{L}_{w_{\mathbb{Z}}}$ corresponds to bases of subspaces of the types* $\ker(a)$ *(resp.* $\ker(a) \ominus \mathrm{im}(b)$ *if $ab = 0$), $V \ominus \mathrm{im}(a)$ or $V$, for any $a \in Q_1^{\mathrm{ord}}$.*

**Theorem 4.122.** *The above construction gives for any undirected periodic word $w_{\mathbb{Z}}$ a unique $\mathfrak{L}$–cycle $g_{w_{\mathbb{Z}}}$:*

$$w_{\mathbb{Z}} = u_{\mathbb{Z}} \quad \textit{if and only if} \quad g_{w_{\mathbb{Z}}} = g_{u_{\mathbb{Z}}}.$$

*Proof.* We first show that the construction results in an $\mathfrak{L}$–cycle: By the same line of argument as in Lemma 4.89, the construction gives an $\mathfrak{L}$–graph. Since the underlying graph is given by a cycle, the $\mathfrak{L}$–graph is an $\mathfrak{L}$–cycle.
Let $w_{\mathbb{Z}} = u_{\mathbb{Z}}$, i.e., $w_i = u_i$ for all $i \in \mathbb{Z}$. In particular, it follows $\hat{w}_p = \hat{u}_p$. By construction, we have $v_i^w(x) = v_i^u(x)$, for all $i \in \{0, \dots, p-1\}$ and $x \in \{s, t\}$. Here, we denote by $v_i^w$ the respective vertex in $D_{w_{\mathbb{Z}}}$, and by $v_i^u$ the respective vertex in $D_{u_{\mathbb{Z}}}$. We proceed similarly with the corresponding links and mark their correspondence by an appropiate superscript. The previous equalities yield that $L_i^w = L_i^u$ and $\bar{L}_i^w = \bar{L}_i^u$ for all $i \in \{0, \dots, p-1\}$. The order of the relations $\alpha$ and $\beta$ in $g_{w_{\mathbb{Z}},1}$ and $g_{u_{\mathbb{Z}},1}$ is fixed by construction. It follows that $g_{w_{\mathbb{Z}}} = g_{u_{\mathbb{Z}}}$.
Conversely, let $g_{w_{\mathbb{Z}}} = g_{u_{\mathbb{Z}}}$. Then $g_{w,0} = g_{u,0}$ and $g_{w,1} = g_{u,1}$. We have, in particular, that $L_i^w = L_i^u$ and $\bar{L}_i^w = \bar{L}_i^u$ for any $i \in \{0, \dots, p-1\}$. This implies that $v_i^w(x) = v_i^u(x)$ for any $i \in \{0, \dots, p-1\}$ and any $x \in \{s, t\}$. It follows that $w_i = u_i$ for all $i \in \{1, \dots, p\}$, i.e., $\hat{w}_p = \hat{u}_p$. Periodicity yields that $w_{\mathbb{Z}} = u_{\mathbb{Z}}$. □

**Proposition 4.123.** *Let $w_{\mathbb{Z}}$ and $u_{\mathbb{Z}}$ be two undirected $\mathbb{Z}$–words, both of period $p$ and with $u_{\mathbb{Z}} = w_{\mathbb{Z}}^{-1}$. Then $g_{u_{\mathbb{Z}}} = g^*_{w_{\mathbb{Z}}}$.*

*Proof.* Let $\hat{w}_p = w_1 \dots w_p$. Then we have that $\hat{u}_p = w_{p-1}^{-1} \dots w_1^{-1} w_p^{-1}$. We obtain that

$$D_{w_{\mathbb{Z}}}: \begin{tikzcd} v_0^w \arrow[rrr, bend right, "w_p"'] & v_1^w \arrow[l, "w_1"'] & \dots \arrow[l, "w_2"'] & v_{p-1}^w \arrow[l, "w_{p-1}"'] \end{tikzcd}$$

$$\begin{tikzcd} D_{u_\mathbb{Z}}: v_0^u & v_1^u \arrow[l, "w_{p-1}^{-1}"'] & \dots \arrow[l, "w_{p-2}^{-1}"'] & v_{p-1}^u \arrow[l, "w_1^{-1}"'] \arrow[from=1-1, to=1-4, bend right, "w_p^{-1}"'] \end{tikzcd}$$

We can rewrite $D_{u_\mathbb{Z}}$ in the following way:

$$\begin{tikzcd} D_{u_\mathbb{Z}}: v_0^u \arrow[r, "w_{p-1}"] & v_1^u \arrow[r, "w_{p-2}"] & \dots \arrow[r, "w_1"] & v_{p-1}^u \arrow[lll, bend left, "w_p"] \end{tikzcd}$$

Comparing $D_{w_\mathbb{Z}}$ with the rewritten $D_{u_\mathbb{Z}}$ yields that

$$\begin{aligned} v_i^u(s) &= v_{p-1-i}^w(t), \\ v_i^u(t) &= v_{p-1-i}^w(s) \end{aligned} \qquad \text{for all } i \in \{0, \dots, p-1\}.$$

This results in the following correspondence between the links:

$$\begin{aligned} L_i^u &= \bar{L}_{p-1-i}^w, \\ \bar{L}_i^u &= L_{p-1-i}^w \end{aligned} \qquad \text{for all } i \in \{0, \dots, p-1\}.$$

By construction, the order of the relations in any $\mathfrak{L}$–cycle $g_{x_\mathbb{Z}}$ is the same for any $\mathbb{Z}$–word $x_\mathbb{Z}$. Thus – by equality on the links – it follows that $g_{w_\mathbb{Z}}^* = g_{u_\mathbb{Z}}$. ☐

We can enhance the statement of Theorem 4.122:

**Theorem 4.124.** *Let $w_\mathbb{Z}$ and $u_\mathbb{Z}$ be two undirected $\mathbb{Z}$–words of period $p$. Then*

$$w_\mathbb{Z} \sim u_\mathbb{Z} \quad \textit{if and only if} \quad g_{w_\mathbb{Z}} \cong g_{u_\mathbb{Z}}.$$

*Proof.* By definition, $w_\mathbb{Z} \sim u_\mathbb{Z}$ if and only if $w_\mathbb{Z} = u_\mathbb{Z}[m]$ or $w_\mathbb{Z} = u_\mathbb{Z}^{-1}[m]$ for some $m \in \mathbb{Z}$.
Let $w_\mathbb{Z} = u_\mathbb{Z}[m]$ for some $m \in \mathbb{Z}$. It follows that

$$\hat{w}_p = u_m \dots u_p u_1 \dots u_{m-1}. \tag{122}$$

Denote the underlying graphs by $C_{w_\mathbb{Z}} = \{c_i', \delta_j'\}_{\substack{1 \le i \le 2p, \\ 1 \le j \le 2p}}$, $C_{u_\mathbb{Z}}\{c_i, \delta_j\}_{\substack{1 \le i \le 2p, \\ 1 \le j \le 2p}}$. We have that

$$\begin{aligned} g_{u_\mathbb{Z},0} &= \{x_0, \bar{x}_0, \dots, x_{p-1}, \bar{x}_{p-1}\}, \\ g_{w_\mathbb{Z},0} &= \{x_m, \bar{x}_m, \dots, x_{p-1}, \bar{x}_{p-1}, \dots, x_{m-1}, \bar{x}_{m-1}\}. \end{aligned}$$

It follows that $\tau : C_{u_\mathbb{Z}} \to C_{w_\mathbb{Z}}$ which acts as the shift by $-m$ ($c_i \mapsto c_{i-m}'$) gives an isomorphism between the two $\mathfrak{L}$–graphs $g_{u_\mathbb{Z}}$ and $g_{w_\mathbb{Z}}$.
Let $w_\mathbb{Z} = u_\mathbb{Z}^{-1}[m]$. We assume without loss of generality that $m \in \mathbb{Z}/p\mathbb{Z}$ and $m > 0$. Let $\hat{u}_p = u_1 \dots u_p$. It follows that

$$\hat{w}_p = u_{p-(m+1)}^{-1} \dots u_1^{-1} u_p^{-1} \dots u_{p-m}^{-1}. \tag{123}$$

Denote the underlying graphs of $g_{w_\mathbb{Z}}$ and $g_{u_\mathbb{Z}}$ by

$$C_{u_\mathbb{Z}} = \{c_i, \delta_j\}_{\substack{1\le i\le 2p,\\ 1\le j\le 2p}}, \text{ and}$$
$$C_{w_\mathbb{Z}} = \{c_i', \delta_j'\}_{\substack{1\le i\le 2p,\\ 1\le j\le 2p}}.$$

Futhermore, let

$$\begin{aligned} g_{u_\mathbb{Z},0} &= \{\bar{x}_0, \dots, x_{p-1}, \bar{x}_{p-1}, x_0\},\\ g_{w_\mathbb{Z},0} &= \{\bar{y}_0, \dots, y_{p-1}, \bar{y}_{p-1}, y_0\}. \end{aligned}$$

By (123), we can express $g_{w_\mathbb{Z},0}$ in terms of the links in $\mathfrak{L}_{u_\mathbb{Z}}$:

$$g_{w_\mathbb{Z},0} = \{x_{p-(m+1)}, \dots, \bar{x}_1, x_1, \bar{x}_0, \dots, \bar{x}_{p-m}, x_{p-m}, \bar{x}_{p-(m+1)}\}.$$

Consider the isomorphism $\pi : C_{g_{u_\mathbb{Z}}} \longrightarrow C_{g_{w_\mathbb{Z}}}$ given by $\tau_{2m-1} \circ \mathrm{rev}$. We want to show that $\pi$ gives an isomorphism between $g_{u_\mathbb{Z}}$ and $g_{w_\mathbb{Z}}$. To this end, we show that $g_{u_\mathbb{Z}} = g_{w_\mathbb{Z}} \circ \pi$. We renumber at first the sets $g_{u_\mathbb{Z},0}$ and $g_{w_\mathbb{Z},0}$ as follows:

$$\begin{aligned} g_{u_\mathbb{Z},0} &= \{x_0, x_1, \dots, x_{2p-3}, x_{2p-2}\},\\ g_{w_\mathbb{Z},0} &= \{y_0, y_1, \dots, y_{2p-3}, \bar{y}_{2p-2}\}\\ &= \{x_{2(p-(m+1))+1}, x_{2(p-(m+1))}, \dots, x_{2(p-m)+1}, x_{2(p-m)}\}\\ &= \{x_{2p-2m-1}, x_{2p-2m-2}, \dots, x_{2p-2m+1}, x_{2p-2m}\}. \end{aligned} \tag{124}$$

Note that we consider all indices modulo $p$. The map $\pi$ acts as follows on $C_{g_{u_\mathbb{Z}}}$:

$$\begin{aligned} &\{c_0, \dots, c_{2p-2}\}\\ \overset{\mathrm{rev}}{\longmapsto}&\{c_{2p-2}, c_0\}\\ \overset{\tau_{2m-1}}{\longmapsto}&\{c_{2p-2-2m+1}, c_{2p-2-2m}, \dots, c_{2p-2-2m+3}, c_{2p-2-2m+2}\}\\ =&\{c_{2p-2m-1}, c_{2p-2-2m}, \dots, c_{2p-2m+1}, c_{2p-2m}\} \end{aligned} \tag{125}$$

Comparing (124) and (125) yields that

$$g_{w_\mathbb{Z}}\pi = g_{u_\mathbb{Z}}.$$

It follows that $g_{w_\mathbb{Z}} \cong g_{u_\mathbb{Z}}$.
Conversely, let $g_{w_\mathbb{Z}} \cong g_{u_\mathbb{Z}}$ with $w_\mathbb{Z} \neq u_\mathbb{Z}$. We know by Lemma 4.126 that $\pi : C_{g_{w_\mathbb{Z}}} \longrightarrow C_{g_{u_\mathbb{Z}}}$ is given by a translation, a reflection or a composition of those. Any translation on $C_{g_{w_\mathbb{Z}}}$ by some $m \in \mathbb{Z}$ results in a shift by $2m$ on $w_\mathbb{Z}$. Similarly, any reflection on $C_{g_{w_\mathbb{Z}}}$ results in a shift composed with taking the inverse on $w_\mathbb{Z}$. Thus, $\pi$ implies that

$$w_\mathbb{Z} = u_\mathbb{Z}^{\mu}[m] \quad \text{for some } m \in \mathbb{Z} \text{ and } \mu \in \{+1, -1\}.$$

We obtain that $w_\mathbb{Z} = u_\mathbb{Z}[m]$ or $w_\mathbb{Z} = u_\mathbb{Z}^{-1}[m]$. In both cases, we obtain that $w_\mathbb{Z} \sim u_\mathbb{Z}$. □

**Example 4.125.** *1. Let $\Lambda$ be given as in Example 2.14. We consider $\overline{\mathfrak{X}}_\Lambda$ as in Example 4.83.*
*Let $w_\mathbb{Z}$ and $u_\mathbb{Z}$ be two undirected periodic $\mathbb{Z}-$words with $\hat{w}_p = \varepsilon^* a^{-1} b \eta^* b^{-1} a$ and $\hat{u}_p = \eta^* b^{-1} a \varepsilon^* a^{-1} b$. Then $w_\mathbb{Z} = u_\mathbb{Z}^{-1}[1]$. We obtain as corresponding $\mathfrak{L}-$cycles:*

$g_{w_\mathbb{Z}}$ :

$$\mathfrak{C}_{\varepsilon^*} \sim \mathfrak{C}_{\varepsilon^*} - \mathfrak{R}_{12} \sim \mathfrak{C}_{21} - \mathfrak{R}_{21} \sim \mathfrak{R}_{32} - \mathfrak{C}_{\eta^*} \sim \mathfrak{C}_{\eta^*} - \mathfrak{R}_{32} \sim \mathfrak{R}_{21} - \mathfrak{C}_{21} \sim \mathfrak{R}_{12}$$

*and*

$g_{u_\mathbb{Z}}$ :

$$\mathfrak{C}_{\eta^*} \sim \mathfrak{C}_{\eta^*} - \mathfrak{R}_{32} \sim \mathfrak{R}_{21} - \mathfrak{C}_{21} \sim \mathfrak{R}_{12} - \mathfrak{C}_{\varepsilon^*} \sim \mathfrak{C}_{\varepsilon^*} - \mathfrak{R}_{12} \sim \mathfrak{C}_{21} - \mathfrak{R}_{21} \sim \mathfrak{R}_{32}$$

*Denote the nodes of $C_{g_{w_\mathbb{Z}}}$ by $\{c_i\}_{1\leq i\leq 12}$ and those of $C_{g_{u_\mathbb{Z}}}$ by $\{c_i'\}_{1\leq i\leq 12}$. Then $\pi : C_{g_{u_\mathbb{Z}}} \longrightarrow C_{g_{w_\mathbb{Z}}}$ given by $\pi(c_i') = c_{i+6}$ yields that*

$$g_{w_\mathbb{Z}} \cong g_{u_\mathbb{Z}}.$$

*2. Let $\Lambda$ be as in Example 2.3.1. with $\overline{\mathfrak{X}}_\Lambda$ as in Example 4.81. Let $w_\mathbb{Z}$ and $u_\mathbb{Z}$ be periodic with $\hat{w}_p = a\varepsilon^* a\varepsilon^*$ and $\hat{u}_p = a^{-1}\varepsilon^* a\varepsilon^*$. Then*

$$g_{w_\mathbb{Z}} : \ \mathfrak{R}_{13} \sim \mathfrak{R}_{11} - \mathfrak{C}_{\varepsilon^*} \sim \mathfrak{C}_{\varepsilon^*} - \mathfrak{R}_{11} \sim \mathfrak{R}_{13} - \mathfrak{C}_{\varepsilon^*} \sim \mathfrak{C}_{\varepsilon^*}$$

*and*

$$g_{w_\mathbb{Z}} : \ \mathfrak{R}_{11} \sim \mathfrak{R}_{13} - \mathfrak{C}_{\varepsilon^*} \sim \mathfrak{C}_{\varepsilon^*} - \mathfrak{R}_{11} \sim \mathfrak{R}_{13} - \mathfrak{C}_{\varepsilon^*} \sim \mathfrak{C}_{\varepsilon^*}$$

*We see that $g_{w_\mathbb{Z}} \ncong g_{u_\mathbb{Z}}$ and that $u_\mathbb{Z}$ and $w_\mathbb{Z}$ are not equivalent as well.*

Theorem 4.124 shows that equivalence classes of words match by construction equivalence classes of $\mathfrak{L}-$cycles. We will enhance this theorem and show that symmetric and asymmetric bands match simple $\mathfrak{L}-$cycles (Theorem 4.130).
In order to do so, we examine the automorphism and rotation groups of $\mathfrak{L}-$cycles $g_{w_\mathbb{Z}}$ constructed from asymmetric or symmetric bands $w_\mathbb{Z}$ (Proposition 4.127).

We denote similarly as in $D_{w_\mathbb{Z}}$ (see step 1 of the construction) by $\{v_i\}_{i\in\mathbb{Z}}$ the vertices of $w_\mathbb{Z}$:

$$\dots \xleftarrow{w_i} v_i \xleftarrow{w_{i+1}} \dots$$

meaning that $v_i = t(w_{i+1}) = s(w_i)$. Note that we use $t(-)$ and $s(-)$ as start and target of the letters *within* $w_\mathbb{Z}$ – regardless of their use in Section 2.3. Note that each vertex $v_i$ gives two nodes in $C_{g_{w_\mathbb{Z}}} = \{c_i, \delta_i\}_{0\le i\le 2p-1}$:

$$v_i \mathrel{\hat{=}} \{c_{2i-1}, c_{2i}\} \tag{126}$$

meaning that $c_{2i-1} \mathrel{\hat{=}} v_i(s) = s(w_i)$ and $c_{2i} \mathrel{\hat{=}} v_i(t) = t(w_{i+1})$. It follows that $w_i \mathrel{\hat{=}} c_{2i-2} \sim c_{2i-1}$.

**Lemma 4.126.** *Let $C$ be a cyclic graph of even length and let $\pi$ be an automorphism on $C$. Then $\pi$ is given by a reflection of by a translation.*

*Proof.* This follows since the image of two neighbouring nodes under $\pi$ gives again two neighbours. □

**Proposition 4.127.** *We have that*

$$\mathrm{Aut}(g_{w_\mathbb{Z}}) = \begin{cases} \{\mathrm{id}\} & \text{if } w_\mathbb{Z} \text{ is an asymmetric band,} \\ \left\{\mathrm{id}, r_{\frac{1}{2}}\right\} & \text{if } w_\mathbb{Z} \text{ is a symmetric band.} \end{cases}$$

*Proof.* Let $w_\mathbb{Z}$ be an asymmetric band. Then $\mathrm{id} \in \mathrm{Aut}(g_{w_\mathbb{Z}})$. We know by Lemma 4.126 that any $\pi \in \mathrm{Aut}(g_{w_\mathbb{Z}})$ is given by a translation or by a reflection. Assume there exists some $\pi \neq \mathrm{id}$ in $\mathrm{Aut}(g_{w_\mathbb{Z}})$. If $\pi$ is given by a translation, then it is of the form $\pi_{2k}$ for some $k \in \{1, \dots, p - \frac{1}{2}\}$. This is due to taking the $\alpha-$ and $\beta-$relations into account. Since they take turns, we need to shift every node by a multiple of 2. By Definition, we have that $g_{w_\mathbb{Z}} = g_{w_\mathbb{Z}}\pi_{2k}$. This equality yields that

$$x_i = x_{\overline{i-2k}} \qquad \forall i \in \{0, \dots, 2p-1\} \tag{127}$$

where $g_{w_\mathbb{Z},0} = \{x_0, x_1, \dots, x_{2p-2}, x_{2p-1}\}$. We know that any $\alpha-$relation matches one letter in $\hat{w}_p$ as follows:

$$c_{2i-2} \sim c_{2i-1} \quad \leftrightarrow \quad v_{i-1}(t) \xleftarrow{w_i} v_i(s) \tag{128}$$

Thus, $\pi_{2k}$ matches the translation $\tau_k$ on $w_\mathbb{Z}$. We know by Lemma 3.41 that

$$\tau_l(w_\mathbb{Z}) = w_\mathbb{Z}[-l].$$

This yields together with (127) that

$$w_\mathbb{Z} = w_\mathbb{Z}[-l]$$

for $l < p$. This contradicts minimality of $p$.
Assume now that $\pi$ is given by a reflection. The construction gives that $x_{2i} \neq x_{2i-1}$ for all $i \in \{0, \dots, 2p-1\}$ since those links are connected by a $\beta$−relation. Hence, any reflection is given by a symmetry on an $\alpha$−relation. It follows that $\pi$ is of the form $\pi_{2k-\frac{3}{2}}$for some $k \in \{0, \dots, 2p-1\}$. Thus, its symmetry axis is given on the following $\alpha$−relation:

$$c_{2k-2} \sim c_{2k-1}.$$

It follows that

$$x_{2k-2} = x_{2k-1} = \mathfrak{C}_{\varepsilon^*} \quad \text{for some } \varepsilon \in \mathrm{Sp}\,.$$

Furthermore, we have by (128) that

$$w_k = \varepsilon^*$$

in $\hat{w}_p$. Using (128) on the image of $\pi_{2k-\frac{3}{2}}$ results in

$$w_{k-1} = w_{k+1}^{-1} \quad \forall i \in \mathbb{Z}\,.$$

It follows that $\pi_{2k-\frac{3}{2}}$ matches the reflection $r_k$ on $w_{\mathbb{Z}}$. We obtain that

$$w_{\mathbb{Z}} = w_{\mathbb{Z}}^{-1}[k]$$

which contradicts $w_{\mathbb{Z}}$ being asymmetric.

Let now $w_{\mathbb{Z}}$ be symmetric. Then $\mathrm{id} \in \mathrm{Aut}(g_{w_{\mathbb{Z}}})$. We know that $\hat{w}_p = \varepsilon^* u \eta^* u^{-1}$ for some $\eta, \varepsilon \in \mathrm{Sp}$. It follows by construction that

$$\begin{aligned}
g_{w_{\mathbb{Z}}}(c_0) &= \mathfrak{C}_{\varepsilon^*}\,,\\
g_{w_{\mathbb{Z}}}(c_1) &= \mathfrak{C}_{\varepsilon^*}\,,\\
g_{w_{\mathbb{Z}}}(c_{2m+2}) &= \mathfrak{C}_{\eta^*}\,,\\
g_{w_{\mathbb{Z}}}(c_{2m+3}) &= \mathfrak{C}_{\eta^*}\,,\\
g_{u,0} &= \{g_{w_{\mathbb{Z}}}(c_1), \dots, g_{w_{\mathbb{Z}}}(c_{2m+2})\},\\
g_{u^{-1},0} &= \{g_{w_{\mathbb{Z}}}(c_{2m+3}), \dots, g_{w_{\mathbb{Z}}}(c_{4m+4})\}.
\end{aligned}$$

In order to have $r_{\frac{1}{2}} \in \mathrm{Aut}(g_{w_{\mathbb{Z}}})$, we need to show that it acts as follows:

$$\begin{aligned}
c_0 &\leftrightarrow c_1\\
c_{2m+2} &\leftrightarrow c_{2m+3}\\
c_{2+i} &\leftrightarrow c_{4m+3-i} && \forall i \in \{0, \dots, 2m-1\}.
\end{aligned}$$

We obtain the following:

$$\begin{aligned}
r_{\frac{1}{2}}(c_0) &= c_{1-0} = c_1\\
r_{\frac{1}{2}}(c_1) &= c_{1-1} = c_0\\
r_{\frac{1}{2}}(c_{2m+3}) &= c_{1-2m-3} = c_{-2m-2} = c_{2m+2}\\
r_{\frac{1}{2}}(c_{2m+2}) &= c_{1-2m-2} = c_{-2m-1} = c_{2m+3}\\
r_{\frac{1}{2}}(c_{2+i}) &= c_{1-2-i} = c_{-1-i} = c_{4m+3-i},\\
r_{\frac{1}{2}}(c_{4m+3-i}) &= c_{1-4m-3+i} = c_{-4m-2+i} = c_{2+i}.
\end{aligned}$$

Note that we have used that $-i \equiv 4m+4-i \mod (4m+4)$ for any $i \in \{0,\dots,4m+3\}$. Hence, $r_{\frac{1}{2}} \in \mathrm{Aut}(g_{w_{\mathbb{Z}}})$. Note that the reflection $r_{2m+2+\frac{1}{2}}$ coincides with $r_{\frac{1}{2}}$ (see also Remark 4.128).
Assume now that $\pi \in \mathrm{Aut}(g_{w_{\mathbb{Z}}})$ with $\pi \notin \{\mathrm{id}, r_{\frac{1}{2}}\}$. Assume $\pi$ is given by a translation. Then $\pi = \pi_{2k-\frac{3}{2}}$ as described in the case of an asymmetric band (with $k \neq 1, m+2$). We obtain analogously to that case a contradiction. Assume that $\pi$ is given by a reflection. As explained above, $\pi$ is then of the form $r_{2k-\frac{1}{2}}$ for $k \in \{0,\dots,2p-1\}$ with $k \neq 1, 2m+2$. Recall that it matches the reflection $r_k$ on $w_{\mathbb{Z}}$. It follows that $w_k = \varepsilon^*$ gives a symmetry point in $w_{\mathbb{Z}}$. This gives a contradiction to Corollary 3.45. □

**Remark 4.128.** *Note that the maps $r_{\frac{1}{2}}$ and $r_{2m+2+\frac{1}{2}}$ coincide on $C_{g_{w_{\mathbb{Z}}}}$ for any symmetric band $w_{\mathbb{Z}}$:*

$$\begin{aligned}
r_{\frac{1}{2}}((c_i)_{i\in I}) &= (c_{\overline{1-i}})_{i\in I},\\
r_{2m+2+\frac{1}{2}}((c_i)_{i\in I}) &= (c_{\overline{4m+4+1-i}})_{i\in I} = (c_{\overline{1-i}})_{i\in I}.
\end{aligned}$$

*This follows from $2p = 4m+4$ and*

$$4m+6+1-i \equiv 2+1-i \equiv 3-i \mod 2p.$$

**Corollary 4.129.** *a) Let $w_{\mathbb{Z}}$ be an asymmetric band. Then*

$$|\mathrm{Aut}(g_{w_{\mathbb{Z}}})| = |\mathrm{Rot}(g_{w_{\mathbb{Z}}})| = 1.$$

*b) Let $w_{\mathbb{Z}}$ be a symmetric band. Then*

$$\begin{aligned}
|\mathrm{Aut}(g_{w_{\mathbb{Z}}})| &= 2\\
|\mathrm{Rot}(g_{w_{\mathbb{Z}}})| &= 1.
\end{aligned}$$

*Proof.* The statements follow from Proposition 4.127. □

The following theorem shows that asymmetric and symmetric bands induce canonical representations of $\overline{\mathfrak{X}}_{\Lambda}$.

**Theorem 4.130.** *a) Let $w_{\mathbb{Z}}$ be an asymmetric band. Then $g_{w_{\mathbb{Z}}} \in \mathring{\mathfrak{S}}(\mathfrak{L})$ and $g_{w_{\mathbb{Z}}}$ is non-symmetric.*

*b) Let $w_{\mathbb{Z}}$ be a symmetric band. Then $g_{w_{\mathbb{Z}}} \in \mathring{\mathfrak{S}}(\mathfrak{L})$ and $g_{w_{\mathbb{Z}}}$ is symmetric.*

*Proof.* a) Let $w_{\mathbb{Z}}$ be an asymmetric band. By Corollary 4.129, we know that $|\mathrm{Rot}(g_{w_{\mathbb{Z}}})| = 1$. Hence, $\mathrm{Rot}(g_{w_{\mathbb{Z}}})$ is trivial and $g_{w_{\mathbb{Z}}}$ is simple. By Theorem 4.122, $g_{w_{\mathbb{Z}}}$ is an $\mathfrak{L}$–cycle. It follows that $g_{w_{\mathbb{Z}}} \in \mathring{\mathfrak{S}}(\mathfrak{L})$.
By Corollary 4.129, it follows that $\mathrm{Aut}(g_{w_{\mathbb{Z}}})$ is trivial. Thus, $g_{w_{\mathbb{Z}}}$ is non-symmetric.

b) Let $w_{\mathbb{Z}}$ be a symmetric band. By Corollary 4.129, we know that $\mathrm{Rot}(g_{w_{\mathbb{Z}}})$ is trivial and thus $g_{w_{\mathbb{Z}}}$ is simple. It follows by Theorem 4.122 that $g_{w_{\mathbb{Z}}} \in \mathring{\mathfrak{S}}(\mathfrak{L})$.
Corollary 4.129 yields that $|\mathrm{Aut}(g_{w_{\mathbb{Z}}})| = 2$. Thus, $\mathrm{Aut}(g_{w_{\mathbb{Z}}})$ is not trivial and $g_{w_{\mathbb{Z}}}$ is symmetric.

□

**Example 4.131.** *1. Let $\Lambda$ be given as in Example 2.3.1. with $\overline{\mathfrak{X}}_\Lambda$ as constructed in Example 4.81.
Let $w_{\mathbb{Z}}$ be a symmetric band with $\hat{w}_p = \varepsilon^* a \varepsilon^* a^{-1}$. Then $u = a$ and $|u| = 1$. The corresponding $\mathfrak{L}$–cycle is given by*

$$g_{w_{\mathbb{Z}}}: \quad \begin{array}{ccccccccccccccc} x_0 & & x_1 & & x_2 & & x_3 & & x_4 & & x_5 & & x_6 & & x_7 \\ \mathfrak{C}_{\varepsilon^*} & \sim & \mathfrak{C}_{\varepsilon^*} & - & \mathfrak{R}_{11} & \sim & \mathfrak{R}_{13} & - & \mathfrak{C}_{\varepsilon^*} & \sim & \mathfrak{C}_{\varepsilon^*} & - & \mathfrak{R}_{13} & \sim & \mathfrak{R}_{11} \end{array}$$

*We have*

$$\mathrm{Aut}(g_{w_{\mathbb{Z}}}) = \{\mathrm{id}, r_{\frac{1}{2}} = r_{\frac{9}{2}}\}.$$

*Note that especially reflections can be deduced from the form of the $\mathfrak{L}$–graph. It follows that $g_{w_{\mathbb{Z}}}$ is simple and symmetric.*

*2. Let $\Lambda$ be as in Example 2.14 with $\overline{\mathfrak{X}}_\Lambda$ as in Example 4.83. Let $w_{\mathbb{Z}}$ be an asymmetric band with $\hat{w}_p = \kappa^* c d^{-1} e$. Then*

$$g_{w_{\mathbb{Z}}}: \quad \begin{array}{ccccccccccccccc} x_0 & & x_1 & & x_2 & & x_3 & & x_4 & & x_5 & & x_6 & & x_7 \\ \mathfrak{C}_{\kappa^*} & \sim & \mathfrak{C}_{\kappa^*} & - & \mathfrak{R}_{51} & \sim & \mathfrak{C}_{22} & - & \mathfrak{R}_{23} & \sim & \mathfrak{R}_{41} & - & \mathfrak{C}_{41} & \sim & \mathfrak{R}_{53} \end{array}$$

*It follows that $\mathrm{Aut}(g_{w_{\mathbb{Z}}}) = \{\mathrm{id}\}$. The only possible reflection would send $x_1 \leftrightarrow x_0$. It would follow $x_2 = x_7$ which does not hold. It follows that $g_{w_{\mathbb{Z}}}$ is simple and non-symmetric.*

**Remark 4.132.** *Note that simplicity of $g_{w_{\mathbb{Z}}}$ and minimality of the period $p$ are linked. The following example illustrates this.*

**Example 4.133.** *Let $\Lambda$ be as in Example 2.3.1. with $\overline{\mathfrak{X}}_\Lambda$ as in Example 4.81. Let $w_\mathbb{Z}$ be an asymmetric band of period $p$ with $\hat{w}_p = \varepsilon^* a$. Construct an $\mathfrak{L}$–cycle not from $\hat{w}_p$, but from $u = \varepsilon^* a \varepsilon^* a$ by applying the above construction to it. We obtain*

$$\begin{array}{lccccccccccccccc} & x_0 & & x_1 & & x_2 & & x_3 & & x_4 & & x_5 & & x_6 & & x_7 \\ g_u: & \mathfrak{C}_{\varepsilon^*} & \sim & \mathfrak{C}_{\varepsilon^*} & - & \mathfrak{R}_{11} & \sim & \mathfrak{R}_{13} & - & \mathfrak{C}_{\varepsilon^*} & \sim & \mathfrak{C}_{\varepsilon^*} & - & \mathfrak{R}_{11} & \sim & \mathfrak{R}_{13} \end{array}$$

*Then* $\operatorname{Aut}(g_u) = \{\mathrm{id}, \tau_{-4}\} = \operatorname{Rot}(g_u)$. *It follows that $g_u$ is not simple.*

We want to formalise the correspondence between translations and reflections on a word $w_\mathbb{Z}$ and those on the underlying graph of $g_{w_\mathbb{Z}}$ as used in the proof of Proposition 4.127.
This enables us to examine the relation between the stabilizer group $\operatorname{Stab}_{\mathrm{D}_\infty}(w_\mathbb{Z})$ of $w_\mathbb{Z}$ and the automorphism group $\operatorname{Aut}(g_{w_\mathbb{Z}})$ of its corresponding $\mathfrak{L}$–cycle $g_{w_\mathbb{Z}}$. We will see that there exists a 1-1-correspondence (Theorem 4.140).
Let $\varphi \in \mathrm{D}_\infty$. Recall that $\varphi$ acts as follows on an undirected $\mathbb{Z}$–word $w_\mathbb{Z}$:

$$\varphi(w_\mathbb{Z}) = \begin{cases} (w^{-1}_{\varphi(i)})_{i\in\mathbb{Z}}, & \text{if } \varphi = r_k \text{ for some } k \in \mathbb{Z}, \\ (w_{\varphi(i)})_{i\in\mathbb{Z}}, & \text{if } \varphi = \tau_k \text{ for some } k \in \mathbb{Z}, \end{cases}$$

where the translation $\tau_k$ acts as $i \mapsto i - k$ and the reflection $r_k$ as $i \mapsto 2k - i$.
Any $\varphi \in \mathrm{D}_\infty$ induces a map $\varphi_v$ on the vertices $v_i$ of $w_\mathbb{Z}$. To this end, we consider the vertices as tuple $(v_i)_{i\in\mathbb{Z}}$. We define $\varphi_v$ as follows:

$$\varphi_v((v_i)_{i\in\mathbb{Z}}) = \begin{cases} (v_{\varphi(i)})_{i\in\mathbb{Z}}, & \text{if } \varphi = \tau_k \text{ for some } k \in \mathbb{Z}, \\ (v_{\varphi(i)-1})_{i\in\mathbb{Z}}, & \text{if } \varphi = r_k \text{ for some } k \in \mathbb{Z}. \end{cases}$$

**Lemma 4.134.** *The induced map $\varphi_v$ is well-defined.*

*Proof.* We need to show that

$$v_{\varphi(i)} = s(w^{\mu_\varphi}_{\varphi(i)}) = t(w^{\mu_\varphi}_{\varphi(i+1)}) \tag{129}$$

with

$$\mu_\varphi = \begin{cases} -1 & \text{if } \varphi \text{ is a reflection}, \\ 1 & \text{if } \varphi \text{ is a translation}. \end{cases}$$

We show at first that (129) holds for any $\varphi = \tau_k \in \mathrm{D}_\infty$ for some $k \in \mathbb{Z}$. We have that

$$\begin{aligned} \varphi_v((v_i)_{i\in\mathbb{Z}}) &= (v_{i-k})_{i\in\mathbb{Z}}, \\ \varphi(w_\mathbb{Z}) &= (w_{i-k})_{i\in\mathbb{Z}}. \end{aligned}$$

For an arbitrary $i \in \mathbb{Z}$, we have that $v_{i-k}$ is in position $i-k$ in $\varphi_v((v_i)_{i\in\mathbb{Z}})$. In $\varphi(w_{\mathbb{Z}})$ we have $w_{i-k}$ in position $i-k$ and $w_{i+1-k}$ in position $i+1-k$. We obtain that

$$\begin{aligned} v_{i-k} &= s(w_{i-k}) = s(w_{\varphi(i)}) \\ &= t(w_{i+1-k}) = t(w_{\varphi(i+1)}). \end{aligned}$$

Let now $\varphi = r_k$ for some $k \in \mathbb{Z}$. Note that the order of indices reverses in a reflected word: if we have $v_i = s(w_i) = t(w_{i+1})$ in $w_{\mathbb{Z}}$, then we obtain after reflecting in $i$ that $v_i = s(w_{i+1}^{-1}) = t(w_i^{-1})$. Any start (target) of an inverse letter $x^{-1}$ is given by the target (start) of $x$. We obtain that

$$\begin{aligned} \varphi_v((v_i)_{i\in\mathbb{Z}}) &= (v_{2k-i-1})_{i\in\mathbb{Z}}, \\ \varphi(w_{\mathbb{Z}}) &= (w_{2k-i}^{-1})_{i\in\mathbb{Z}}. \end{aligned}$$

We know that

$$v_{2k-i-1} = s(w_{2k-i-1}) = t(w_{2k-i})$$

in $w_{\mathbb{Z}}$. We obtain that

$$\begin{aligned} w_{\varphi(i+1)}^{-1} &= w_{2k-i-1}, \\ w_{\varphi(i)}^{-1} &= w_{2k-i}. \end{aligned}$$

It follows that

$$v_{2k-i-1} = t(w_{\varphi(i+1)}^{-1}) = s(w_{\varphi(i)}^{-1}).$$

□

Any $\varphi \in \mathrm{D}_\infty$ induces also a map $\varphi_c$ on the nodes of the underlying graph $C_{g_{w_{\mathbb{Z}}}}$ of the corresponding $\mathfrak{L}$–cycle $g_{w_{\mathbb{Z}}}$. To this end, we denote also the nodes of the underlying graph as tuple $(c_i)_{i\in I}$ with $I = \{0, \dots, 2p-1\}$:

$$\varphi_c((c_i)_{i\in I}) = \begin{cases} \left(c_{\overline{2\varphi(i)-i}}\right)_{i\in I} & \text{if } \varphi = \tau_k \text{ for some } k \in \mathbb{Z}, \\ \left(c_{\overline{2\varphi(i)+i-3}}\right)_{i\in I} & \text{if } \varphi = r_k \text{ for some } k \in \mathbb{Z}, \end{cases}$$

where $\bar{x} \equiv x \mod (2p)$.

**Lemma 4.135.** *The induced map $\varphi_c$ is well-defined.*

*Proof.* First, we need to show that the images of neighbouring nodes are again neighbours. To this end, let $i \in \{0, \dots, 2p-1\}$. Consider the neighbours $c_{i-1}$, $c_i$ and $c_{i+1}$ in $C_{g_{w_{\mathbb{Z}}}}$. Let $\varphi_c = \tau_k$ for some $k \in \mathbb{Z}$. In $\varphi_c(C_{g_{w_{\mathbb{Z}}}})$ we obtain

$$\begin{aligned} &\text{in position } i-1: & c_{2(i-1-k)-(i-1)} &= c_{i-1-2k}, \\ &\text{in position } i: & c_{2(i-k)-i} &= c_{i-2k}, \\ &\text{in position } i+1: & c_{2(+1i-k)-(i+1)} &= c_{i+1-2k}. \end{aligned}$$

We see from the indices that those are indeed neighbouring nodes.
Consider now $\varphi_c = r_k$ for some $k \in \mathbb{Z}$. We obtain

$$\begin{aligned} &\text{in position } i-1: && c_{2(2k-(i-1))+(i-1)-3} = c_{4k-i-2},\\ &\text{in position } i: && c_{2(2k-i)+i-3)} = c_{4k-i-3}\\ &\text{in position } i+1: && c_{2(2k-(i+1))+(i+1)-3} = c_{4k-i-4}.\end{aligned}$$

and see that those are also neighbours.
Secondly, we show that the correspondence between the nodes and the associated values of each $v_i$ is preserved under $\varphi_v$ and $\varphi_c$. To this end, we neglect to consider the indices modulo $2p$. We have that

$$v_i \mathrel{\hat{=}} \{c_{2i-1}, c_{2i}\} \tag{130}$$

by construction (recall that $v_i(s) \mathrel{\hat{=}} c_{2i-1}$ and $v_i(t) \mathrel{\hat{=}} c_{2i}$). Let $\varphi_c = \tau_k$. We show that

$$v_{\varphi(i)} \mathrel{\hat{=}} \{c_{2\varphi(2i-1)-(2i-1)}, c_{2\varphi(2i)-2i}\}. \tag{131}$$

We have that $v_{\varphi(i)} = v_{i-k}$. This vertex corresponds to the nodes $\{c_{2i-2k-1}, c_{2i-2k}\}$. We have that

$$\begin{aligned} c_{2\varphi(2i)-2i} &= c_{2(2i-k)-2i} = c_{2i-2k},\\ c_{2\varphi(2i-1)-(2i-1)} &= c_{2(2i-1-k)-(2i-1)} = c_{2i-1-2k}\end{aligned}$$

and (131) follows.
Let $\varphi = r_k$ for some $k \in \mathbb{Z}$. We want to show that

$$v_{\varphi(i)-1} \mathrel{\hat{=}} \{c_{2\varphi(2i-1)+(2i-1)-3}, c_{2\varphi(2i)+2i-3}\}. \tag{132}$$

We have $v_{\varphi(i)-1} = v_{2k-i-1}$. This corresponds by (130) to $\{c_{4k-2i-3}, c_{4k-2i-2}\}$. We have

$$c_{2\varphi(2i-1)+(2i-1)-3} = c_{2(2k-(2i-1))+(2i-1)-3} = c_{4k-2i-2}, \tag{133}$$

$$c_{2\varphi(2i)+2i-3} = c_{2(2k-2i)+2i-3} = c_{4k-2i-3}. \tag{134}$$

We see that the set in (132) coincides with the values in (133) and (134). $\square$

Conversely, any $\pi$ acting on $C_{g_{w_{\mathbb{Z}}}}$ induces a map $\pi_w \in \mathrm{D}_\infty$. To define the induced map, we use again the correspondence

$$v_i \mathrel{\hat{=}} \{c_{2i-1}, c_{2i}\}$$

by$v_i(s) \mathrel{\hat{=}} c_{2i-1}$ and $v_i(t) \mathrel{\hat{=}} c_{2i}$ for all $i \in \{0, \dots, 2p-1\}$. We define $\pi_w$ as follows:

$$\pi_w(w_{\mathbb{Z}}) = \begin{cases} \left(w_{\frac{\pi(2i)}{2}}\right)_{i\in\mathbb{Z}} & \text{if } \pi = \tau_{2k} \text{ for some } k \in \mathbb{Z}/p\mathbb{Z},\\ \left(w^{-1}_{\frac{\pi(i)-i+3}{2}}\right)_{i\in\mathbb{Z}} & \text{if } \pi = r_{2k-\frac{3}{2}} \text{ for some } k \in \{0, \dots, 2p-1\}.\end{cases}$$

Recall that any translation is given by an even shift, and that any reflection is given by a symmetry on an $\alpha$–relation.

**Lemma 4.136.** *The induced map $\pi_w$ is well-defined.*

*Proof.* Each vertex $v_i$ of $\hat{w}_p$ corresponds to two nodes $\{c_{2i-1}, c_{2i}\}$ with $v_i(s) = c_{2i-1}$ and $v_i(t) = c_{2i}$. Each letter $w_i$ starts in $v_i(s)$ and ends in $v_{i-1}(t)$. Thus, each $w_i$ corresponds to the nodes $\{c_{2i-2}, c_{2i-1}\}$. We need to show that

$$(\pi_w(w_{\mathbb{Z}}))_i \mathrel{\hat{=}} \{c_{\pi(2i-2)}, c_{\pi(2i-1)}\}. \tag{135}$$

Let $\pi = \tau_{2k}$ for some $k \in \mathbb{Z}$. We have that

$$\pi_w(w_{\mathbb{Z}}) = \left(w_{\frac{\pi(2i)}{2}}\right)_{i\in\mathbb{Z}} = (w_{i-k})_{i\in\mathbb{Z}},$$

$$c_{\pi(2i-2)} = c_{2i-2-2k}, \tag{136}$$

$$c_{\pi(2i-1)} = c_{2i-1-2k}. \tag{137}$$

We know that $(\pi_w(w_{\mathbb{Z}}))_i = w_{i-k}$ starts in $v_{i-k}(s)$ and ends in $v_{i-k-1}(t)$. Thus, $w_{i-k}$ corresponds to the nodes

$$\{c_{2(i-k)-2}, c_{2(i-k)-1}\} = \{c_{2i-2k-2}, c_{2i-2k-1}\}.$$

The correspondence in (135) follows together with (136) and (137).
Let $\pi = r_{2k-\frac{3}{2}}$ for some $k \in \mathbb{Z}$. Then we have

$$c_{\pi(2i-2)} = c_{4k-3-2i+2} = c_{4k-2i-1}, \tag{138}$$

$$c_{\pi(2i-1)} = c_{4k-3-2i+1} = c_{4k-2i-2} \tag{139}$$

and

$$\pi_w(w_{\mathbb{Z}}) = \left(w_{\frac{\pi(i)-i+3}{2}}\right)_{i\in\mathbb{Z}} = (w^{-1}_{2k-i})_{i\in\mathbb{Z}}.$$

The letter $(\pi_w(w_{\mathbb{Z}}))_i = w^{-1}_{2k-i}$ starts in $v_{2k-i}(s)$ and ends in $v_{2k-i-1}(t)$ (note that the order is reversed due to the reflection). It follows with (138) and (139) that

$$w^{-1}_{2k-i} \mathrel{\hat{=}} \{c_{4k-2i-2}, c_{4k-2i-1}\} = \{c_{\pi(2i-1)}, c_{\pi(2i-2)}\}.$$

□

**Lemma 4.137.** *Let $w_{\mathbb{Z}}$ be an undirected $\mathbb{Z}$−word of period $p$. Let $\pi$ be a morphism on $C_{g_{w_{\mathbb{Z}}}}$. Then*

$$(\pi_w)_c = \pi.$$

*Proof.* Let $\pi = \tau_{2k}$ be a morphism on $C_{g_{w_{\mathbb{Z}}}}$ which is given by a translation. Then

$$\pi_w(w_{\mathbb{Z}}) = \left(w_{\frac{\pi(2i)}{2}}\right)_{i\in\mathbb{Z}} = (w_{i-k})_{i\in\mathbb{Z}} = \tau_k(w_{\mathbb{Z}}),$$

$$(\pi_w)_c((c_i)_{i\in I}) = \left(c_{\overline{2\tau_k(i)-i}}\right)_{i\in I} = \left(c_{\overline{i-2k}}\right)_{i\in I} = \pi((c_i)_{i\in I}).$$

Consider now $\pi = r_{2k-\frac{3}{2}}$. Then

$$\pi_w(w_{\mathbb{Z}}) = \left(w^{-1}_{\frac{\pi(i)-i+3}{2}}\right)_{i\in\mathbb{Z}} = \left(w^{-1}_{2k-i}\right)_{i\in\mathbb{Z}} = r_k(w_{\mathbb{Z}}),$$
$$(\pi_w)_c((c_i)_{i\in I}) = \left(c_{\overline{2r_k(i)+i-3}}\right)_{i\in I} = \left(c_{\overline{4k-i-3}}\right)_{i\in I} = \pi((c_i)_{i\in I}).$$

□

**Remark 4.138.** *Note that $(\varphi_c)_w = \varphi$ does not hold generally for $\varphi \in \mathrm{D}_\infty$. Let $w_{\mathbb{Z}}$ be a symmetric band of period $p$. Consider $r_1, r_{m+2} \in \mathrm{D}_\infty$. It follows that*

$$(r_1)_c((c_i)_{i\in I}) = \left(c_{\overline{2(2-i)+i-3}}\right)_{i\in I} = \left(c_{\overline{1-i}}\right)_{i\in I} = r_{\frac{1}{2}}((c_i)_{i\in I})$$
$$((r_1)_c)_w(w_{\mathbb{Z}}) = \left(w^{-1}_{\frac{r_{\frac{1}{2}}(i)-i+3}{2}}\right) = \left(w^{-1}_{\frac{4-2i}{2}}\right) = \left(w^{-1}_{2-i}\right)_{i\in\mathbb{Z}} = r_1(w_{\mathbb{Z}}).$$

*Hence, it holds that $((r_1)_c)_w = r_1$. But we obtain for $r_{m+2}$ the following:*

$$(r_{m+2})_c((c_i)_{i\in I}) = \left(c_{\overline{2(2m+4-i)+i-3}}\right)_{i\in I} = \left(c_{\overline{4m+8-2i+i-3}}\right)_{i\in I} = \left(c_{\overline{1-i}}\right)_{i\in I} \tag{140}$$
$$= r_{\frac{1}{2}}((c_i)_{i\in I}) = (r_1)_c((c_i)_{i\in I}). \tag{141}$$

*It follows that $((r_1)_c)_w = ((r_{m+2})_c)_w$ and thus, $((r_{m+2})_c)_w = r_1$.*

Let $w_{\mathbb{Z}}$ be in the following an asymmetric or symmetric band of period $p$. We define

$$\mathrm{Stab}^p_{\mathrm{D}_\infty}(w_{\mathbb{Z}}) := \mathrm{Stab}_{\mathrm{D}_\infty}(w_{\mathbb{Z}})/\approx$$

where $\varphi_k \approx \psi_l$ if and only if both are either of translation or reflection type, and $k - l \equiv 0 \pmod{p}$.

**Lemma 4.139.** *We have that*

$$\mathrm{Stab}^p_{\mathrm{D}_\infty}(w_{\mathbb{Z}}) = \begin{cases} \{[\mathrm{id}]\} & \text{if } w_{\mathbb{Z}} \text{ is an asymmetric band,} \\ \{[\mathrm{id}], [r], [r_{m+2}]\} & \text{if } w_{\mathbb{Z}} \text{ is a symmetric band,} \end{cases}$$

*with $r = r_0$.*

*Proof.* Recall that $\tau_0 = \mathrm{id}$. The statement follows from Proposition 3.51 for $w_{\mathbb{Z}}$ an asymmetric band: for any $k, l \in \mathbb{Z}$ we have that

$$\tau_{kp} \approx \tau_{lp}$$

since $(k-l)p \equiv 0 \mod p$.
For $w_{\mathbb{Z}}$ a symmetric band, recall that

$$\mathrm{Stab}_{\mathrm{D}_\infty}(w_{\mathbb{Z}}) = \{\tau_{kp}, r\tau_{2+kp}\}_{k\in\mathbb{Z}}$$

by Proposition 3.44 and Lemma 3.43. It follows similar to the case of an asymmetric band that $\tau_{kp} \approx \tau_{lp}$ for all $l, k \in \mathbb{Z}$. Let now $k, l \in \mathbb{Z}$. We examine when $r\tau_{2+kp}$ and $r\tau_{2+lp}$ are equivalent. We know that $r\tau_{2+ip} = r_{1+\frac{ip}{2}}$ for any $i \in \mathbb{Z}$. Thus, we know that both $r\tau_{2+kp}$ and $r\tau_{2+lp}$ are reflections. We calculate

$$1 + \frac{kp}{2} - \left(1 + \frac{lp}{2}\right) = \left(\frac{k-l}{2}\right)p. \tag{142}$$

We want (142) to be divisible by $p$. This is the case for $\frac{k-l}{2} \in \mathbb{N}$. Hence, either both $k$ and $l$ are even, or they are both odd.
Assume they are both even. Then we can write $k = 2k'$ and $l = 2l'$ for some $k', l' \in \mathbb{Z}$. Thus, (142) turns into

$$\left(\frac{2k' - 2l'}{2}\right)p = (k' - l')p \equiv 0 \mod p.$$

It follows that $r\tau_{2+kp} \approx r\tau_{2+lp}$ for any $k, l \in \mathbb{Z}$ which are even. We claim that $r\tau_{2+kp} \approx r_1$ for any $k \in \mathbb{Z}$ even: we can write

$$r\tau_{2+kp} = r\tau_{2+2k'p} = r_{1+k'p}.$$

We obtain that

$$1 - (1 + k'p) = -k'p \equiv 0 \mod p$$

which proves the claim.
Let now $k$ and $l$ in $\mathbb{Z}$ be odd. Then we can write $k = 2k' + 1$ and $l = 2l' + 1$ for some $k', l' \in \mathbb{Z}$. It follows with $p = 2m + 2$ that

$$r\tau_{2+kp} = r\tau_{2+(2k'+1)p} = r_{1+(k'+\frac{1}{2})p} = r_{1+(m+1)+k'p} = r_{m+2+k'p}.$$

Similarly, we obtain that $r\tau_{2+lp} = r_{m+2+l'p}$. Hence, we have that $r\tau_{2+kp} \approx r\tau_{2+lp}$ if $r_{m+2+k'p} \approx r_{m+2+l'p}$. The latter equivalence is given since

$$m + 2 + k'p - (m + 2 + l'p) = (k' - l')p \equiv 0 \mod p.$$

Moreover, we claim that $r_{m+2} \approx r\tau_{2+kp}$ for any $k \in \mathbb{Z}$ odd. This follows from

$$m + 2 - (m + 2 + k'p) = k'p \equiv 0 \mod p.$$

□

**Theorem 4.140.** *The map*

$$\theta : \mathrm{Stab}^p_{\mathrm{D}_\infty}(w_\mathbb{Z}) \longrightarrow \mathrm{Aut}(g_{w_\mathbb{Z}})$$
$$\varphi \longmapsto \varphi_c$$

*is surjective. For $w_\mathbb{Z}$ an asymmetric band, it is also injective. For $w_\mathbb{Z}$ a symmetric band, it is not injective.*

*Proof.* Let $w_{\mathbb{Z}}$ be a symmetric band.
Let us first show that $\theta$ is well-defined. This means that any two equivalent maps in $\mathrm{Stab}_{\mathrm{D}_\infty}(w_{\mathbb{Z}})$ are sent to the same element in $\mathrm{Aut}(g_{w_{\mathbb{Z}}})$.
To this end, let $k \neq l \in \mathbb{Z}$. Consider $\tau_{kp}, \tau_{lp} \in [\mathrm{id}]$. We have that

$$(\tau_{kp})_c((c_i)_{i\in I}) = \left(c_{\overline{2\tau_{kp}(i)-i}}\right)_{i\in I} = \left(c_{\overline{i-2kp}}\right)_{i\in I} = (c_{\overline{i}})_{i\in I},$$

and similarly $(\tau_{lp})_c((c_i)_{i\in I}) = (c_{\overline{i}})_{i\in I}$.
We consider next $r_{1+kp}, r_{1+lp} \in [r]$. We obtain that

$$(r_{1+kp})_c((c_i)_{i\in I}) = \left(c_{\overline{2r_{1+kp}(i)+i-3}}\right)_{i\in I} = \left(c_{\overline{1-i+4kp}}\right)_{i\in I} = \left(c_{\overline{1-i}}\right)_{i\in I},$$

and similarly that $(r_{1+lp})_c((c_i)_{i\in I}) = (c_{\overline{1-i}})_{i\in I}$.
Let $r_{m+2+kp}$ and $r_{m+2+lp}$ be in $[r_{m+2}]$. We obtain that

$$(r_{m+2+kp})_c((c_i)_{i\in I}) = \left(c_{\overline{2r_{m+2+kp}(i)+i-3}}\right)_{i\in I} = \left(c_{\overline{1-i+(4k+1)p}}\right)_{i\in I} = \left(c_{\overline{1-i}}\right)_{i\in I},$$

and similarly that $(r_{m+2+lp})_c((c_i)_{i\in I}) = (c_{\overline{1-i}})_{i\in I}$.
We show that $\theta$ is surjective.
Let $\pi = \mathrm{id} \in \mathrm{Aut}(g_{w_{\mathbb{Z}}})$. Then $\theta^{-1}(\pi) = \pi_w$ by Lemma 4.137. It is given by

$$\pi_w(w_{\mathbb{Z}}) = \left(w_{\frac{\mathrm{id}(2i)}{2}}\right)_{i\in\mathbb{Z}} = (w_i)_{i\in\mathbb{Z}} = \mathrm{id}(w_{\mathbb{Z}}).$$

Consider now $\pi = r_{\frac{1}{2}} \in \mathrm{Aut}(g_{w_{\mathbb{Z}}})$. Then $\theta^{-1}(\pi) = \pi_w$ which is given by

$$\pi_w(w_{\mathbb{Z}}) = \left(w^{-1}_{\frac{r_{\frac{1}{2}}(i)-i+1}{2}}\right)_{i\in\mathbb{Z}} = \left(w^{-1}_{\frac{4-2i}{2}}\right)_{i\in\mathbb{Z}} = \left(w^{-1}_{2-i}\right)_{i\in\mathbb{Z}} = r(w_{\mathbb{Z}}).$$

The map $\theta$ is not injective by Remark 4.128.
Let $w_{\mathbb{Z}}$ be an asymmetric band. Apart from injectivity, the statement follows analogously to the case of a symmetric band. Injectivity follows from $\theta\colon [\mathrm{id}] \mapsto \mathrm{id}$. □

Theorem 4.140 shows that the symmetries in $w_{\mathbb{Z}}$ and $g_{w_{\mathbb{Z}}}$ match each other. We can use this knowledge to show the following:

**Theorem 4.141.** *Let $g \in \mathring{\mathfrak{S}}(\mathfrak{L})$.*

*a) If $g$ is non-symmetric, then there exists a word $w_{\mathbb{Z}}$ of asymmetric band type with $g_{w_{\mathbb{Z}}} = g$.*

*a) If $g$ is symmetric, then there exists a word $w_{\mathbb{Z}}$ of symmetric band type with $g_{w_{\mathbb{Z}}} = g$.*

*Proof.* a) Applying the construction backwards yields the existence of a $\mathbb{Z}$–word $w_\mathbb{Z}$ with $w_\mathbb{Z} = w_\mathbb{Z}[p]$ for $p = |g|/2$ and with $g_{w_\mathbb{Z}} = g$. It remains to show that $w_\mathbb{Z} \neq w_\mathbb{Z}^{-1}[m]$ for all $m \in \mathbb{Z}$. We know that $\mathrm{Aut}(g) = \{\mathrm{id}\}$ since $g$ is non-symmetric. Theorem 4.140 and Theorem 4.139 yield that $\mathrm{Stab}^p_{\mathrm{D}_\infty}(w_\mathbb{Z}) = \{[\mathrm{id}]\}$. It follows that $w_\mathbb{Z} \neq w_\mathbb{Z}^{-1}[m]$ for all $m \in \mathbb{Z}$ and that $w_\mathbb{Z}$ is of asymmetric band type.

b) Applying the construction backwards yields the existence of a $\mathbb{Z}$–word $w_\mathbb{Z}$ with $w_\mathbb{Z} = w_\mathbb{Z}[p]$ for $p = |g|/2$ and with $g_{w_\mathbb{Z}} = g$. It remains to show that $w_\mathbb{Z} = w_\mathbb{Z}^{-1}[m]$ for some $m \in \mathbb{Z}$. We know by Proposition 4.127 that $\mathrm{Aut}(g) = \{\mathrm{id}, r_{\frac{1}{2}}\}$. We know by Theorem 4.139 that $\mathrm{Stab}^p_{\mathrm{D}_\infty}(w_\mathbb{Z}) = \{[\mathrm{id}], [r], [r_{m+2}]\}$. Thus, it follows by Theorem 4.140 that $\theta(r) = r_{\frac{1}{2}}$ with $\theta$ the map from Theorem 4.140. We have that $r(w_\mathbb{Z}) = w_\mathbb{Z}$ which corresponds to saying that $w_\mathbb{Z} = (w_\mathbb{Z}[1])^{-1}$. We can rewrite this equality by Lemma 2.34 to $w_\mathbb{Z} = w_\mathbb{Z}^{-1}[-1]$. This shows that $w_\mathbb{Z}$ is of symmetric band type.

□

Theorem 4.130 and Theorem 4.141 show that there is a 1-1-correspondence between words of band type and simple $\mathfrak{L}$–cycles. Similar as for $\mathfrak{L}$–chains (Corollary 4.117), we find another correspondence:

**Corollary 4.142.** *There exists a 1-1-correspondence between the equivalence classes of symmetric and asymmetric bands, and the isomorphism classes of $\mathfrak{L}$–cycles in $\mathring{\mathfrak{S}}(\mathfrak{L})$.*

*Proof.* The proof follows from Theorem 4.130, Theorem 4.141 and Theorem 4.124. □

### 4.7 Directions in the $\mathfrak{L}$–graphs

Recall that we assigned the constructed semichains to the bundles in a certain way in Section 4.4: any semichain in a row label set has a reversed ordering with respect to the subspace inclusions of the filtration it is built from. The orderings of the semichains in the column label set, on the other hand, preserve the subspace inclusions of the respective filtrations.
We have claimed in Section 4.4 that the way we cosntruct $\overline{\mathfrak{X}}_\Lambda$ ensures that the $\mathfrak{L}$–graphs $g_{w_{\mathrm{I}}}$ are compatible with (weakly) consistent versions $v_{\mathrm{I}}$ of $w_{\mathrm{I}}$. This means that the directions assigned to links in $g_{w_{\mathrm{I}}}$ of the forms

$$\overleftarrow{x_i \sim x_{i+1}} \qquad \text{and} \qquad \overrightarrow{x_i \sim x_{i+1}} \tag{143}$$

match the directions on the corresponding special letters in any (weakly) consistent version $v_{\mathrm{I}}$. We show this correspondence in Proposition 4.145 for asymmetric and symmetric strings and bands. Note that in particular the reversed ordering of the links in the row label sets contributes to this result. Futhermore, this correspondence does not depend on the signs of the letters chosen beforehand (as long as the conditions of the construction of $\overline{\mathfrak{X}}_\Lambda$ are respected). Finally, we should mention that the correspondence is only given for those letters of finite index, or – in case of a symmetric string – for letters with index unequal to $\frac{|w|+1}{2}$.
Let $\Lambda$ be as in Section 4.4 with corresponding bundle of semichains $\overline{\mathfrak{X}}_\Lambda$. Let $w_{\mathrm{I}}$ be a $\Gamma_{\mathrm{ud}}(\Lambda) - \mathrm{I}-$ word with $\mathrm{I} = \{0, \dots, n\}$ $(n > 0)$ or $\mathrm{I} = \mathbb{Z}$. For $\mathrm{I} = \mathbb{Z}$, assume additionally that $w_{\mathrm{I}}$ is of period $p$. Denote its corresponding $\mathfrak{L}$–graph by $g_{w_{\mathrm{I}}}$. Let $g_{w_{\mathrm{I}},0} = \{x_0, \dots, x_{n'}\}$ where $n'$ is given by $2n+1$ if $g_{w_{\mathrm{I}}}$ is an $\mathfrak{L}$–chain, and by $2p-1$ if $g_{w_{\mathrm{I}}}$ is an $\mathfrak{L}$–cycle.

**Definition 4.143.** *Let $w_{\mathrm{I}}$ be a $\Gamma_{\mathrm{ud}}(\Lambda) - \mathrm{I}-$word as above with $\mathfrak{L}$–graph $g_{w_{\mathrm{I}}}$. For any $x_i \in g_{w_{\mathrm{I}},0}$ with $x_i = x_{i+1}$ and $\lambda_{i,i+1} = \alpha$, we define the* direction *of $\lambda_{i,i+1}$ by*

$$\mathrm{dir}_{i,i+1}(g_{w_{\mathrm{I}}}) = \begin{cases} 1 & \text{if } \overleftarrow{x_i \sim x_{i+1}} \\ -1 & \text{if } \overrightarrow{x_i \sim x_{i+1}} \end{cases}.$$

**Remark 4.144.** *Note that $\mathrm{dir}_{i,i+1}(g_{w_{\mathrm{I}}})$ gives the negative of the orientation function $\varepsilon_0$ defined in [Bon88]:*

$$\begin{aligned} \varepsilon_0(x_i, x_{i+1}) &= 1 \text{ if } \overrightarrow{x_i \sim x_{i+1}}, \\ \varepsilon_0(x_i, x_{i+1}) &= -1 \text{ if } \overleftarrow{x_i \sim x_{i+1}}. \end{aligned}$$

The following proposition shows the compatibility between the directions of special letters $w_i$ of finite index in strings and bands (or index unequal to $\frac{|w|+1}{2}$ if $w$ is a symmetric string) and the direction on the respective subchain $x_k \sim x_{k+1}$ of the corresponding $\mathfrak{L}$–graph.

**Proposition 4.145.** *(i) Let $w$ be an asymmetric string of length $n$. Let $v \in (\Phi^{\mathrm{d}}_{\mathrm{ud}})^{-1}(w)$ be (weakly) consistent. Then*

$$\mathrm{dir}_{2j-1,2j}(g_w) = \mathrm{dir}_j(v)$$

*for all $j \in \{1,\dots,n\}$ with $w_j$ special.*

*(ii) Let $w = u\varepsilon^* u^{-1}$ be a symmetric string with $|u| = m$. Let $v \in (\Phi^{\mathrm{d}}_{\mathrm{ud}})^{-1}(w)$ be (weakly) consistent ($v = t\varepsilon^{\kappa}t^{-1}$ with $t \in (\Phi^{\mathrm{d}}_{\mathrm{ud}})^{-1}(u)$ and $\kappa = \{+1,-1\}$). Then*

$$\mathrm{dir}_{2j-1,2j}(g_u) = \mathrm{dir}_j(v) \tag{144}$$

*for all $j \in \{1,\dots,m\}$ with $w_j$ special.*

*(iii) Let $w_{\mathbb{Z}}$ be an asymmetric band of period $p$. Let $v_{\mathbb{Z}} \in (\Phi^{\mathrm{d}}_{\mathrm{ud}})^{-1}(w_{\mathbb{Z}})$ be (weakly) consistent. Then*

$$\mathrm{dir}_{2j-2,2j-1}(g_{w_{\mathbb{Z}}}) = \mathrm{dir}_{j+kp}(v_{\mathbb{Z}})$$

*for all $j \in \{1,\dots,p\}$ with $w_j$ special, and for all $k \in \mathbb{Z}$.*

*(iv) Let $w_{\mathbb{Z}}$ be a symmetric band of period $p$ with $\hat{w}_p = \varepsilon^* u\eta^* u^{-1}$, $|u| = m$. Let $v_{\mathbb{Z}} \in (\Phi^{\mathrm{d}}_{\mathrm{ud}})^{-1}(w_{\mathbb{Z}})$ be (weakly) consistent with periodic parts of the form $\hat{v}_p^{(i)} = \varepsilon^{\mu_i} t\eta^{\kappa_i} t^{-1}$, where $t \in (\Phi^{\mathrm{d}}_{\mathrm{ud}})^{-1}(u)$ and $\mu_i, \kappa_i \in \{+1,-1\}$. Then*

*(iv.1)* $\mathrm{dir}_{2j-2,2j-1}(g_{w_{\mathbb{Z}}}) = \mathrm{dir}_{j+kp}(v_{\mathbb{Z}})$ *for all $j \in \{1,\dots,p\}$ with $w_j$ special, $\mathrm{ind}^*_j(w_{\mathbb{Z}}) < \infty$, and for all $k \in \mathbb{Z}$.*

*(iv.2)* $\mathrm{dir}_{2j-1,2j}(g_u) = \mathrm{dir}_{j+1+kp}(v_{\mathbb{Z}}) = -\mathrm{dir}_{j+m+2+kp}(v_{\mathbb{Z}})$, *for all $j \in \{1,\dots,m\}$ with $w_j$ special, and for all $k \in \mathbb{Z}$.*

*Proof.* Note that we consider letters $w_j$ of special type with $\mathrm{ind}^*_j(w) \neq \frac{|w|+1}{2}$ for $w$ a symmetric string, or with $\mathrm{ind}^*_j(w_{\mathbb{Z}}) < \infty$ for $w_{\mathbb{Z}}$ a band. We distinguish for the proof between the cases a) and b) of Definition 4.45. Recall that we put in case a) the direction towards the bigger link of $y_i$ and $z_i$. In case b), we direct towards the smaller link. We have by construction that $x_i \in \mathfrak{L}(\mathfrak{C})$. Thus, we have $y_i, z_i \in \mathfrak{L}(\mathfrak{C}) \cup \{\infty\}$ in case a), and $y_i, z_i \in \mathfrak{L}(\mathfrak{R}) \cup \{\infty\}$ in case b).

Recall that a letter $w_i$ matches in the $\mathfrak{L}$–chain $g_w$ the subchain $x_{2i-1} \sim x_{2i}$. In an $\mathfrak{L}$–cycle, the letter $w_k \in \hat{w}_p$ matches the subchain $x_{2i-2} \sim x_{2i-1}$. We first consider the cases $(i), (ii)$ and $(iv.2.)$ of the proposition since they concern $\mathfrak{L}$–chains. Afterwards we deal with $(iii)$ and $(iv.1)$ which refer to $\mathfrak{L}$–cycles.

(i) We obtain for $w$ an asymmetric string the $\mathfrak{L}$–chain $g_w$ which is simple, admissible and has no double ends. We consider $\tilde{g}_w = g_w$ in order to determine the directions. Let $i \in \{1,\dots,n\}$, where $n = |w|$. Set $k = 2i-1$. Then $x_k \sim x_{k+1}$ matches $w_i$. Let $g_{l_k,0} = \{y_{k+1},\dots,x_{k-1}\}$ and

$g_{r_k,0} = \{x_{k+1} \dots, z_{k-1}\}$ be the subchains between $y_k$ and $x_k$, $x_{k+1}$ and $z_k$, respectively. We have that $g_{l_k} = g^*_{r_k}$.
We show at first that $y_k \neq \infty$ and $z_k \neq \infty$. We know by $d(g_w) = 0$ that $x_0 \mathrel{\hat{=}}$ basis of $\ker(a)$( basis of $\ker(a) \ominus \operatorname{im}(b)$) for some $a \in Q_1^{\text{ord}}$ ($b \in Q_1^{\text{ord}}, ab = 0$), and that $x_{2n+1} \mathrel{\hat{=}}$ basis of $\ker(c)$( basis of $\ker(c) \ominus \operatorname{im}(c)$) for some $c \in Q_1^{\text{ord}}$ ($d \in Q_1^{\text{ord}}, cd = 0$). Assume that $y_k = \infty$, $z_k \neq \infty$. It follows that $y_{k+1} = x_0 \mathrel{\hat{=}}$ basis of $\ker(a)$. We obtain by the symmetry in $x_k \sim x_{k+1}$ that $z_{k-1} \mathrel{\hat{=}}$ basis of $\ker(a)$. Since $z_k, z_{k-1} \in g_{w,0}$, it follows that $g_w$ terminates in $z_{k-1}$ which contradicts the length of $g_w$. The case $y_k \neq \infty$, $z_k = \infty$ gives analogously a contradiction. It remains to consider $y_k = \infty$, $z_k = \infty$. Then $y_{k+1} = x_0$ and $z_{k-1} = x_{2n+1}$. It follows that $g_{l_k} = x_0 - \dots \sim x_{k-1}$, $g_{r_k} = x_{k+1} \sim \dots - x_{2n+1}$. Thus, we obtain that $g_w = g_{l_k} - x_k \sim x_{k+1} - g^*_{l_k}$ is composite which gives a contradiction.
Let us now examine the directions given in Definition 4.45.

a) Assume without loss of generality that $y_k, z_k \in \mathfrak{L}(\mathfrak{C}_j)$ for some $j \in \{1, \dots, |Q_0|\}$, where

$$\mathfrak{C}_j = \{\mathfrak{C}_{j1} < \mathfrak{C}_{j2} < \mathfrak{C}_{j3}\}$$

with

$$\begin{aligned}\mathfrak{C}_{j1} &\mathrel{\hat{=}} \text{basis of } \operatorname{im}(a),\\ \mathfrak{C}_{j2} &\mathrel{\hat{=}} \text{basis of } \ker(b) \ominus \operatorname{im}(a),\\ \mathfrak{C}_{j3} &\mathrel{\hat{=}} \text{basis of } V_j \ominus \ker(b)\end{aligned}$$

for $a, b \in Q_1^{\text{ord}}$, $ba = 0$. Note that we do not have $z_k = \mathfrak{C}_{\varepsilon^*}$ and $y_k = \mathfrak{C}_{\eta^*}$ for $\eta, \varepsilon \in \text{Sp}$ by Definition 2.2. It follows by the previous discussion that $\mathfrak{C}_j$ is not given by two incomparable elements.
Let $y_k = \mathfrak{C}_{j2}$ and $z_k = \mathfrak{C}_{j1}$. We have $\operatorname{dir}_{k,k+1}(g_w) = 1$. Moreover, it follows that $y_k = x_0$ and that

$$w_2 = a : t(a) \longleftarrow s(a) \mathrel{\hat{=}} z_k \sim z_{k+1}.$$

For $g_{l_k} : x_1 \sim \dots \sim x_{k-1}$ we obtain that $x_0 - g_{l_k} - x_k$ corresponds to the subword $l_k$ of $w$. Similarly for $g_{r_k} : x_{k+1} \sim \dots \sim z_{k-1}$, the subword $r_k$ of $w$ corresponds to $x_{k+1} - g_{r_k} - x$ with $x = \mathfrak{C}_{j2}$. It follows that $l_k = w_1 \dots w_{i-1}$, $r_k = w_{i+1} \dots w_{z-1}$ and $r_k = l_k^{-1}$. Denote $z = w_{z+1} \dots w_n$. We obtain that

$$\begin{aligned}(w[< i])^{-1} &= l_k^{-1},\\ w[> i] &= r_k w_z z = l_k^{-1} a z.\end{aligned}$$

This yields $(w[< i])^{-1} > w[> i]$ and it follows that $\operatorname{dir}_i(w) = 1$.
Let now $z_k = \mathfrak{C}_{j3}$ and $y_k$ as above. Then $\operatorname{dir}_{k,k+1}(g_w) = -1$. Pro-

ceeding as above, we obtain $w_z = b^{-1}$ and

$$\begin{aligned}(w[<i])^{-1} &= l_k^{-1},\\ w[>i] &= r_k w_z z = l_k^{-1} b^{-1} z.\end{aligned}$$

Thus, $(w[<i])^{-1} < w[>i]$ and it follows that $\mathrm{dir}_i(w) = -1$.
Let now $y_k = \mathfrak{C}_{j1}$ and $z_k = \mathfrak{C}_{j2}$. Then $\mathrm{dir}_{k,k+1}(g_w) = -1$ and $z_k = x_{2n+1}$. We obtain that

$$w_y = a^{-1} : s(a) \longrightarrow t(a) \hat{=} y_{k-1} \sim y_k.$$

Let $l_k$ and $r_k$ be given as above. Let $y = w_1, \dots, w_{y-1}$. We obtain

$$\begin{aligned}w[>i] &= r_k = l_k^{-1},\\ (w[>i])^{-1} &= (y w_y l_k)^{-1} = l_k^{-1} a y^{-1}.\end{aligned}$$

It follows that $w[>i] > (w[<i])^{-1}$. Thus, we have $\mathrm{dir}_i(w) = -1$.
Let now $y_k = \mathfrak{C}_{j3}$ and keep $z_k = \mathfrak{C}_{j2}$. We have that $\mathrm{dir}_{k,k+1}(g_w) = 1$. We obtain for $w_y$ as above that $w_y = b$. It follows with $l_k$, $r_k$ and $y$ as above that

$$\begin{aligned}w[>i] &= r_k = l_k^{-1},\\ (w[>i])^{-1} &= (y w_y l_k)^{-1} = l_k^{-1} b^{-1} y^{-1}.\end{aligned}$$

It follows that $w[>i] < (w[<i])^{-1}$. Thus, we have $\mathrm{dir}_i(w) = 1$.
Consider now $y_k = \mathfrak{C}_{j1}$ and $z_k = \mathfrak{C}_{j3}$. We know that $y_k \neq x_0$ and $z_k \neq x_{2n+1}$. It follows that $\mathrm{dir}_{k,k+1}(g_w) = -1$. We obtain that

$$\begin{aligned}w_z &= b^{-1} : s(b) \longrightarrow t(b) \hat{=} z_k \sim z_{k+1},\\ w_y &= a^{-1} : s(a) \longrightarrow t(a) \hat{=} y_{k-1} \sim y_k.\end{aligned}$$

Let the subwords $y, z, l_k$ and $r_k$ of $w$ be given as follows:

$$\begin{aligned}y &= w_1 \dots w_{y-1},\\ z &= w_{z+1} \dots w_n,\\ l_k &= w_{y+1} \dots w_{i-1},\\ r_k &= w_{i+1} \dots w_{z-1}.\end{aligned}$$

Recall that $l_k$ and $r_k$ correspond up to the first and last link to $g_{l_k}$ and $g_{r_k}$. Moreover, we have that $r_k = l_k^{-1}$. We obtain that

$$\begin{aligned}(w[<i])^{-1} &= (y w_y l_k)^{-1} = l_k^{-1} a y^{-1},\\ w[>i] &= r_k w_z z = l_k^{-1} b^{-1} z.\end{aligned}$$

It follows that $(w[<i])^{-1} < w[>i]$. Thus, we have $\mathrm{dir}_i(w) = -1$.
Let now $y_k = \mathfrak{C}_{j3}$ and $z_k = \mathfrak{C}_{j1}$. This means that we have switched

the roles of $y_k$ and $z_k$ in the previous part. Thus, we obtain that $w_z = a$ and $w_y = b$. Furthermore, we get

$$\begin{aligned}(w[< i])^{-1} &= (yw_yl_k)^{-1} = l_k^{-1}b^{-1}y^{-1},\\ w[> i] &= r_kw_zz = l_k^{-1}az,\end{aligned}$$

where $y$, $z$, $l_k$ and $r_k$ as given as above. Hence, we have that $w[> i] < (w[< i])^{-1}$ and it follows that $\mathrm{dir}_i(w) = 1$.

b) Assume without loss of generality that $y_k, z_k \in \mathfrak{L}(\mathfrak{R}_j)$ for some $1 \le j \le |Q_0|$, where

$$\mathfrak{R}_j = \{\mathfrak{R}_{j1} > \mathfrak{R}_{j2} > \mathfrak{R}_{j3}\}.$$

We have that $\mathfrak{R}_{j1} \mathrel{\hat{=}}$ basis of $\mathrm{im}(a)$, $\mathfrak{R}_{j2} \mathrel{\hat{=}}$ basis of $\ker(b) \ominus \mathrm{im}(a)$, $\mathfrak{R}_{j3} \mathrel{\hat{=}}$ basis of $V_j \ominus \ker(a)$ for some $a, b \in Q_1^{\mathrm{ord}}$ with $ba = 0$.
Let $y_k = \mathfrak{R}_{j1}$ and $z_k = \mathfrak{R}_{j2}$. Then $\mathrm{dir}_{k,k+1}(g_w) = -1$. We obtain that

$$\begin{aligned}w_n &\mathrel{\hat{=}} z_{k-2} \sim z_{k-1},\\ w_y = a^{-1} : s(a) \longrightarrow t(a) &\mathrel{\hat{=}} y_{k-1} \sim y_k.\end{aligned}$$

Note that $x{-}g_{l_k}{-}x_k$ with $x = \mathfrak{R}_{12}$ matches a subword $l_k$ of $w$ which is given between $w_y$ and $w_i$. Similarly, denote by $r_k$ the subword $w_{i+1} \dots w_n$. Note that $r_k = l_k^{-1}$ since $g_{l_k} = g_{r_k}^*$. Let $y = w_1 \dots w_{y-1}$. It follows that

$$\begin{aligned}(w[< i])^{-1} &= (ya^{-1}l_k)^{-1} = l_k^{-1}ay^{-1} = r_kay^{-1},\\ w[> i] &= r_k.\end{aligned}$$

We obtain that $w[> i] > (w[< i])^{-1}$. It follows that $\mathrm{dir}_i(w) = -1$.
For $y_k = \mathfrak{R}_{j2}$ and $z_k = \mathfrak{R}_{j1}$ we have that $\mathrm{dir}_{k,k+1}(g_w) = 1$. We have

$$\begin{aligned}w_1 &\mathrel{\hat{=}} y_{k+1} \sim y_{k+2},\\ w_z = a : t(a) \longleftarrow s(a) &\mathrel{\hat{=}} z_k \sim z_{k+1}.\end{aligned}$$

It follows with $l_k = w_1 \dots w_{i-1}$, $l_k^{-1} = r_k = w_{i+1} \dots w_{z-1}$ and $z = w_{z+1} \dots w_n$ that

$$\begin{aligned}(w[< i])^{-1} &= l_k^{-1},\\ w[> i] &= l_k^{-1}az.\end{aligned}$$

Note that $l_k$ corresponds as before to the chain $y_j - g_{l_k} - x_k$ where $x_k \mathrel{\hat{=}} \ker(\varepsilon)$. We obtain that $\mathrm{dir}_i(w) = 1$.

Consider $y_k = \mathfrak{R}_{j3}$ and $z_k = \mathfrak{R}_{j2}$. Then $\mathrm{dir}_{k,k+1}(g_w) = 1$. We obtain

$$\begin{aligned} w_n &\mathrel{\hat=} z_{k-2} \sim z_{k-1}, \\ w_y = b : t(b) \longleftarrow s(b) &\mathrel{\hat=} y_{k-1} \sim y_k. \end{aligned}$$

Proceeding similar as above, we obtain that $\mathrm{dir}_i(w) = 1$.
Let $y_k = \mathfrak{R}_{j2}$ and $z_k = \mathfrak{R}_{j3}$. Then we have that $\mathrm{dir}_{k,k+1}(g_w) = -1$. We obtain the correspondences

$$\begin{aligned} w_z = b^{-1} : s(b) \longrightarrow t(b) &\mathrel{\hat=} z_k \sim z_{k+1}, \\ w_1 &\mathrel{\hat=} y_{k+1} \sim y_{k+1}. \end{aligned}$$

It follows as before that $\mathrm{dir}_i(w) = -1$.
Consider $y_k = \mathfrak{R}_{j1}$ and $z_k = \mathfrak{R}_{j3}$ resulting in $\mathrm{dir}_{k,k+1}(g_w) = -1$. We have

$$\begin{aligned} w_z = b^{-1} : s(b) \longrightarrow t(b) &\mathrel{\hat=} z_k \sim z_{k+1}, \\ w_y = a^{-1} : s(a) \longrightarrow t(a) &\mathrel{\hat=} y_{k-1} \sim y_k. \end{aligned}$$

We obtain as before two subwords $l_k$ and $r_k$ with $r_k = l_k^{-1}$:

$$\begin{aligned} l_k &= w_{y+1} \dots w_i, \\ r_k &= w_{i+1} \dots w_{z-1}. \end{aligned}$$

Denote by $y = w_1 \dots w_{y-1}$, $z = w_{z+1} \dots w_n$. We obtain that

$$\begin{aligned} (w[< i])^{-1} &= (y w_y l_k)^{-1} = l_k^{-1} a y^{-1}, \\ w[> i] &= r_k w_z z = l_k^{-1} b^{-1} z. \end{aligned}$$

It follows that $\mathrm{dir}_i(w) = -1$.
Finally, let $y_k = \mathfrak{R}_{j3}$ and $z_k = \mathfrak{R}_{j1}$ which give that $\mathrm{dir}_{k,k+1}(g_w) = 1$. We obtain with the notation from the previous case that $w_y = b$ and $w_z = a$. Thus, we obtain that $(w[< i])^{-1} > w[> i]$. It follows that $\mathrm{dir}_i(w) = 1$.

(ii) We construct for $w = u\varepsilon^* u^{-1}$ the simple admissible $\mathfrak{L}$–chain $g_u$ which has one double end given by $x_{2m+1}$. In order to determine the directions, we consider $\tilde{g}_u = g_u \sim g_u^*$. Let $i \in \{1, \dots, m\}$ and set $k = 2i-1$. We have that $x_k \sim x_{k+1}$ matches $w_i$. Let as before $g_{l_k,0} = \{y_{k+1}, \dots, x_{k-1}\}$ and $g_{r_k,0} = \{x_{k+2}, \dots, z_{k-1}\}$. We show that $y_k \neq \infty$ and $z_k \neq \infty$. We know that $x_0 \mathrel{\hat=}$ basis of $\ker(b) \ominus \mathrm{im}(a)$ and that $x_{2m+1} = \mathfrak{C}_{\varepsilon^*}$. Assume that $y_k \neq \infty$ and $z_k = \infty$. It follows by the form of $\tilde{g}_u$ that $|g_{r_k}| > |g_u^*|$. Since $g_{l_k} = g_{r_k}^*$, we obtain that

$$|g_{r_k}| + |g_{l_k}| > 2|g_u^*| = |\tilde{g}_u|.$$

This gives a contradiction since $g_{l_k}$ and $g_{r_k}$ are two non-intersecting subchains of $\tilde{g}_u$.
Let $y_k = \infty$ and $z_k \neq \infty$. Then we have that $y_{k+1} = x_0 \mathrel{\hat{=}}$ basis of $\ker(b) \ominus \operatorname{im}(a)$. Symmetry in $x_k \sim x_{k+1}$ yields that $z_{k-1} \mathrel{\hat{=}}$ basis of $\ker(b) \ominus \operatorname{im}(a)$. Thus, $z_{k-1} = x_r$ with $r \neq 0, 2m+1$ denotes the last link in $g_{u,0}$. It follows that $|g_u| < 2m+1$ which gives a contradiction.
Finally, assume that $y_k = \infty$ and $z_k = \infty$. Then $x_k = x_{2m+1}$. But there does not exist a link $x_{k+1}$ in $g_{u,0}$. Hence, $x_k = x_{2m+1}$ is not considered for directions.
We continue with the examination of the directions.

a) Assume without loss of generality that $y_k, z_k \in \mathfrak{L}(\mathfrak{C}_j)$ for some $j \in \{1, \dots, |Q_0|\}$. Note that $\mathfrak{C}_j$ is by the previous discussion not given by two incomparable elements. Assume that

$$\mathfrak{C}_j = \{\mathfrak{C}_{j1} < \mathfrak{C}_{j2} < \mathfrak{C}_{j3}\},$$

where we have $\mathfrak{C}_{j1} \mathrel{\hat{=}}$ basis of $\operatorname{im}(a)$,$\mathfrak{C}_{j2} \mathrel{\hat{=}}$ basis of $\ker(b) \ominus \operatorname{im}(a)$ and $\mathfrak{C}_{j3} \mathrel{\hat{=}}$ basis of $V_j \ominus \ker(a)$ for $a, b \in Q_1^{\text{ord}}$ with $ba = 0$.
Note that $x_0 \mathrel{\hat{=}}$ basis of $\ker(x)($ basis of $\ker(x) \ominus \operatorname{im}(y))$ for some $x \in Q_1^{\text{ord}}$ $(y \in Q_1^{\text{ord}}$ with $xy = 0)$, and $x_{2m+1} = \mathfrak{C}_{\varepsilon^*}$. Thus, we have that $z_k \neq \mathfrak{C}_{j2}$. If $y_k = \mathfrak{C}_{j2}$, then $y_k = x_0$. We consider the cases $y_k = \mathfrak{C}_{j2}$ and $z_k = \mathfrak{C}_{j1}$, $y_k = \mathfrak{C}_{j2}$ and $z_k = \mathfrak{C}_{j3}$, $y_k = \mathfrak{C}_{j1}$ and $z_k = \mathfrak{C}_{j3}$, $y_k = \mathfrak{C}_{j3}$ and $z_k = \mathfrak{C}_{j1}$. They follow analogously to the respective cases in (i), a).

b) Assume without loss of generality that $y_k, z_k \in \mathfrak{L}(\mathfrak{R}_j)$ for some $1 \leq j \leq |Q_0|$, where

$$\mathfrak{R}_j = \{\mathfrak{R}_{j1} > \mathfrak{R}_{j2} > \mathfrak{R}_{j3}\}.$$

We have that $\mathfrak{R}_{j1} \mathrel{\hat{=}}$ basis of $\operatorname{im}(a)$, $\mathfrak{R}_{j2} \mathrel{\hat{=}}$ basis of $\ker(b) \ominus \operatorname{im}(a)$, $\mathfrak{R}_{j3} \mathrel{\hat{=}}$ basis of $V_j \ominus \ker(a)$ for some $a, b \in Q_1^{\text{ord}}$ with $ba = 0$.
The proof follows analogously to (i), b) with the following restriction: $z_k \neq \mathfrak{R}_{j2}$. Thus, we only consider the cases $y_k = \mathfrak{R}_{j2}$ and $z_k = \mathfrak{R}_{j1}$, $y_k = \mathfrak{R}_{j2}$ and $z_k = \mathfrak{R}_{j3}$, $y_k = \mathfrak{R}_{j3}$ and $z_k = \mathfrak{R}_{j1}$, $y_k = \mathfrak{R}_{j1}$ and $z_k = \mathfrak{R}_{j3}$. Note that it follows for $y_k = \mathfrak{R}_{j2}$ that $y_k = x_0$.

(iv.2) We obtain the simple admissible $\mathfrak{L}$–chain $g_u$ with $d(g_u) = 0$, $x_0 = \mathfrak{C}_{\varepsilon^*}$ and $x_{2m+1} = \mathfrak{C}_{\eta^*}$. We consider $\tilde{g}_u = g_u^* \sim g_u \sim g_u^*$. Let $i \in \{1, \dots m\}$ and set $k = 2i - 1$. Then $w_i \mathrel{\hat{=}} x_k \sim x_{k+1}$. Let $g_{l_k}$ and $g_{r_k}$ as in the previous parts. We show that $y_k \neq \infty$ and $z_k \neq \infty$. To this end, assume that $z_k = \infty$ and $y_k = \infty$. Then we have that $g_{l_k} = \{x_{2m+1}, \dots, x_0, x_0, \dots, x_{k-1}\}$ and $g_{r_k} = \{x_{k+2}, \dots, x_{2m+1}, x_{2m+1}, \dots, x_0\}$. In particular, it follows that $x_0 - \cdots - x_k = x_{k+1} - \cdots - x_{2m+1}$. Thus, we can write $g_u = h \sim h^*$ for $h = x_0 - \cdots - x_k$. Hence, $g_u$ is composite which gives a contradiction.

Assume now that $y_k = \infty$ and $z_k \neq \infty$. This means that $z_k = \mathfrak{C}_{\eta^*}$. Note that $g^* \sim g \sim g^*$ matches the subword $u^{-1}\varepsilon^* u\eta^* u^{-1} =: s$ of $w_{\mathbb{Z}}$. Recall that we have for $g_{l_k} - x_k$ with $g_{l_k,0} = \{x_{2m+1}, \ldots, x_0, x_0, \ldots, x_{k-1}\}$ a matching subword $l_k$ of $s$. Similarly, we have for $x_{k+1} - g_{r_k}$ with $g_{r_k,0} = \{x_{k+2}, \ldots, x_{2m+1}, x_{2m+1}, \ldots, x_0\}$ the matching subword $r_k$ with $r_k = l_k^{-1}$. Note that

$$|l_k| = \frac{1}{2}(|g_{l_k}| + 1) > \frac{1}{2}|g_u^*| = m + 1.$$

It follows that $\operatorname{ind}_i^*(w_{\mathbb{Z}}) > m+1$. This gives a contradiction to Corollary 3.47 since $i \in \{1, \ldots, m\}$.

Assume now $y_k \neq \infty$ and $z_k = \infty$. We obtain analogously as in the previous case a contradiction.

a) Let $y_k$ and $z_k$ be without loss of generality in $\mathfrak{C}_j$ for some $j \in \{1, \ldots, |Q_0|\}$. Note that $\mathfrak{C}_j \neq \{\mathfrak{C}^+_{\mu^*} \otimes \mathfrak{C}^-_{\mu^*}\}$ for any $\mu \in \mathrm{Sp}$ by Definition 2.2 and since both $y_k$ and $z_k$ are unequal to $\infty$. Thus, let $\mathfrak{C}_j = \{\mathfrak{C}_{j1} < \mathfrak{C}_{j2} < \mathfrak{C}_{j3}\}$ as in (i),a). Note that $y_k \neq \mathfrak{C}_{j2}$ and $z_k \neq \mathfrak{C}_{j2}$ since $u$ is a subword of a periodic word. Thus, we consider the cases $y_k = \mathfrak{C}_{j1}$ and $z_k = \mathfrak{C}_{j3}$, $y_k = \mathfrak{C}_{j3}$ and $z_k = \mathfrak{C}_{j1}$. They follow analogously to the respective cases in (i),a).

b) Let $y_k$ and $z_k$ be without loss of generality in $\mathfrak{R}_j$ for some $j \in \{1, \ldots, |Q_0|\}$ where $\mathfrak{R}_j = \{\mathfrak{R}_{j1} > \mathfrak{R}_{j2} > \mathfrak{R}_{j3}\}$ as in (i),b). Note that $y_k \neq \mathfrak{R}_{j2}$ and $z_k \neq \mathfrak{R}_{j2}$ similar as in a). Thus, we consider the cases $y_k = \mathfrak{R}_{j1}$ and $z_k = \mathfrak{R}_{j3}$, $y_k = \mathfrak{R}_{j3}$ and $z_k = \mathfrak{R}_{j1}$. They follow analogously to the respective cases in (i),b).

We have dealt with the $\mathfrak{L}$–chains and consider next the $\mathfrak{L}$–cycles in the proposition.

(iii) We obtain the non-symmetric simple $\mathfrak{L}$–cycle $g_{w_{\mathbb{Z}}}$ of length $2p$. Let $i \in \{1, \ldots, p\}$ and set $k = 2i - 2$. Then $x_k \sim x_{k+1}$ matches $w_i$, and, in particular, $w_{i+kp}$ for all $k \in \mathbb{Z}$. Denote again by $g_{l_k,0} = \{y_{k+1}, \ldots, x_{k-1}\}$ the subchain between $y_k$ and $x_k$, and by $g_{r_k,0} = \{x_{k+2}, \ldots, z_{k-1}\}$ the subchain between $x_{k+1}$ and $z_k$. Recall that we can have $y_k = z_k$ in a symmetric $\mathfrak{L}$–cycle (see Subsection 4.1.3). Since $g_{w_{\mathbb{Z}}}$ is non-symmetric, this case does not occur.

a) Let $y_k$ and $z_k$ be without loss of generality in $\mathfrak{C}_j$ for some $j \in \{1, \ldots, |Q_0|\}$. We have $\mathfrak{C}_j \neq \{\mathfrak{C}^+_{\mu^*} \otimes \mathfrak{C}^-_{\mu^*}\}$ for all $\mu \in \mathrm{Sp}$ by $y_k \neq z_k$ and Definition 2.2. Thus, Let $\mathfrak{C}_j = \{\mathfrak{C}_{j1} < \mathfrak{C}_{j2} < \mathfrak{C}_{j3}\}$ be as in (i),a). Since $w_{\mathbb{Z}}$ is periodic, we have that $y_k \neq \mathfrak{C}_{j2}$ and $z_k \neq \mathfrak{C}_{j2}$. It remains to consider the cases $y_k = \mathfrak{C}_{j1}$ and $z_k = \mathfrak{C}_{j3}$, $y_k = \mathfrak{C}_{j3}$ and $z_k = \mathfrak{C}_{j1}$. They follow analogously as in (i),a).

b) Let $y_k, z_k \in \mathfrak{R}_j$ without loss of generality, $j \in \{1, \dots, |Q_0|\}$, where $\mathfrak{R}_j = \{\mathfrak{R}_{j1} > \mathfrak{R}_{j2} > \mathfrak{R}_{j3}\}$ as in (i),b). We have similar to a) that $y_k \neq \mathfrak{R}_{j2}$ and $z_k \neq \mathfrak{R}_{j2}$. The cases $y_k = \mathfrak{R}_{j1}, z_k = \mathfrak{R}_{j3}$ and $y_k = \mathfrak{R}_{j3}, z_k = \mathfrak{R}_{j1}$ follow as in (i),b).

(iv.1) We obtain a symmetric, simple $\mathfrak{L}$–cycle $g_{w_{\mathbb{Z}}}$ of length $2p = 4m + 4$. Let $i \in \{1, \dots, 2m+2\}$ with $\mathrm{ind}_i^*(w_{\mathbb{Z}}) < \infty$. Recall that $i \neq 1, m+2$ in this case. Set $k = 2i - 2$. Then $w_i$ matches $x_k \sim x_{k+1}$. Note that it follows by choice of $i$ that $k \neq 0, 2m+2$. Since $g_{w_{\mathbb{Z}}}$ is symmetric, we might have $z_k = y_k$. We show that this is not the case. To this end, assume otherwise: $z_k = y_k$. Then there exists and automorphism $\pi \in \mathrm{Aut}(g_{w_{\mathbb{Z}}})$ with $\pi \neq \mathrm{id}$ and $\pi(c_k) = c_{k+1}$. Since $g_{w_{\mathbb{Z}}}$ is simple, $\pi$ is not given by a translation, but by the reflection $r_{k+\frac{1}{2}}$. Recall that $\left(r_{k+\frac{1}{2}}\right)_w$ describes the induced map on $w_{\mathbb{Z}}$. We obtain with $k = 2i - 2$ that

$$(r_{k+\frac{1}{2}})_w(w_{\mathbb{Z}}) = \left(w_{\frac{2k+1-j-j+3}{2}}\right)_{j\in\mathbb{Z}} = (w_{k-j+2})_{j\in\mathbb{Z}} = (w_{2i-j})_{j\in\mathbb{Z}} = r_i(w_{\mathbb{Z}}).$$

We know by Lemma 4.136 that $(r_i)_c = r_{k+\frac{1}{2}}$. It follows by Theorem 4.140 that $r_i \in \mathrm{Stab}^p_{\mathrm{D}_\infty}(w_{\mathbb{Z}})$. Lemma 4.139 yields that $r_i = r_1$ or $r_i = r_{m+2}$. This contradicts the choice of $i \neq 1, m+2$.

a) Follows analogously to (iii), a).

b) Follows analogously to (iii), b).

□

**Remark 4.146.** *Let $w_j$ be a special letter as considered in Proposition 4.145 (i)-(iv). Recall that $w_j \hat{=} x_k \sim x_{k+1}$, where $k = 2j-1$ if we consider an $\mathfrak{L}$–chain, and $k = 2j-2$ if we consider an $\mathfrak{L}$–cycle. The proof of the proposition implies the following correspondences with the notation from the proof:*

$$\begin{aligned} z_k \sim z_{k+1} &\hat{=} w_{j_+^*} = w_{j_+^c}, \\ y_{k-1} \sim y_k &\hat{=} w_{j_-^*} = w_{j_-^c}. \end{aligned}$$

**Remark 4.147.** *We obtain by the symmetries in the respective words the following:*

*(i) In the same setting as in Proposition 4.145 (ii), we obtain that*

$$\mathrm{dir}_{2j-1,2j}(g_u) = -\,\mathrm{dir}_{|v|-j+1}(v) \tag{145}$$

*for all $j \in \{1, \dots, m\}$ with $w_j$ special.*

*(ii) In the same setting as in Proposition 4.145 (iv.2), we obtain that*

$$\mathrm{dir}_{2j-1,2j}(g_u) = -\,\mathrm{dir}_{-j+1+(k+1)p}(v_{\mathbb{Z}})$$

*for all $j \in \{1, \dots, m\}$ with $w_j$ special.*

**Example 4.148.** *Let $\Lambda$ be as in Example 2.3.1, meaning it is given by the quiver*

$$Q:\ \varepsilon \circlearrowright \bullet \circlearrowleft a$$

*and relations* $\mathrm{R} = \{a^2\}$, $\mathrm{R}^{\mathrm{Sp}} = \{\varepsilon^2 - \varepsilon\}$, *with* $\overline{\mathfrak{X}}_\Lambda$ *as in Example 4.83. Recall that its semichains are given by*

$$\begin{aligned}\mathfrak{C}_1 &= \{\mathfrak{C}^+_{\varepsilon^*} \lessgtr \mathfrak{C}^-_{\varepsilon^*}\},\\ \mathfrak{R}_1 &= \{\mathfrak{R}_{11} > \mathfrak{R}_{12} > \mathfrak{R}_{13}\}\end{aligned}$$

*with* $\sigma_\Lambda(\mathfrak{R}_{11}) = \mathfrak{R}_{13}$ *and otherwise* $\sigma_\Lambda$ *acts as identity.*

1. *Let* $w = \varepsilon^* a \varepsilon^* a \varepsilon^*$ *be an asymmetric string. Let* $v \in (\Phi^{\mathrm{d}}_{\mathrm{ud}})^{-1}(w)$ *be (weakly) consistent. It is uniquely given by* $v = \varepsilon a \varepsilon a \varepsilon$, *i.e.,* $\mathrm{dir}(v_1) = \mathrm{dir}(v_3) = \mathrm{dir}(v_5) = 1$ *(recall that* $\mathrm{dir}_j(v) = \mathrm{dir}(v_j)$*).*
   *The* $\mathfrak{L}$*−chain* $g_w$ *is given by:*

$$\begin{array}{cccccccccccc} \mathfrak{R}_{12} - & \multicolumn{2}{c}{\overleftarrow{\mathfrak{C}_{\varepsilon^*} \sim \mathfrak{C}_{\varepsilon^*}}} & - \mathfrak{R}_{13} \sim & \mathfrak{R}_{11} - & \multicolumn{2}{c}{\overleftarrow{\mathfrak{C}_{\varepsilon^*} \sim \mathfrak{C}_{\varepsilon^*}}} & - \mathfrak{R}_{13} \sim & \mathfrak{R}_{11} - & \multicolumn{2}{c}{\overleftarrow{\mathfrak{C}_{\varepsilon^*} \sim \mathfrak{C}_{\varepsilon^*}}} & - \mathfrak{R}_{12} \\ x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} & x_{11} \end{array}$$

   *It follows that* $\mathrm{dir}_{1,2}(g_w) = \mathrm{dir}_{5,6}(g_w) = \mathrm{dir}_{9,10}(g_w) = 1$.

b) *Let* $w = \varepsilon^* a \varepsilon^* a \varepsilon^* a^{-1} \varepsilon^* a^{-1} \varepsilon^*$ *be a symmetric band of length* 9. *Then* $u = \varepsilon^* a \varepsilon^* a$, $u^{-1} = a^{-1}\varepsilon^* a^{-1} \varepsilon^*$. *Let* $v \in (\Phi^{\mathrm{d}}_{\mathrm{ud}})^{-1}(w)$ *be (weakly) consistent. Then*

$$\begin{aligned}\mathrm{dir}(v_1) = \mathrm{dir}(v_3) &= \phantom{-}1,\\ \mathrm{dir}(v_7) = \mathrm{dir}(v_9) &= -1,\end{aligned}$$

   *i.e.,* $v = \varepsilon a \varepsilon a \varepsilon^\kappa a^{-1} \varepsilon^{-1} a^{-1} \varepsilon^{-1}$ *for* $\kappa \in \{+1, -1\}$. *The* $\mathfrak{L}$*−chain* $g_u$ *is given by*

$$\begin{array}{ccccccccccc} g_u: & \mathfrak{R}_{12} - & \multicolumn{2}{c}{\overleftarrow{\mathfrak{C}_{\varepsilon^*} \sim \mathfrak{C}_{\varepsilon^*}}} & - \mathfrak{R}_{13} \sim & \mathfrak{R}_{11} - & \multicolumn{2}{c}{\overleftarrow{\mathfrak{C}_{\varepsilon^*} \sim \mathfrak{C}_{\varepsilon^*}}} & - \mathfrak{R}_{13} \sim & \mathfrak{R}_{11} - & \mathfrak{C}_{\varepsilon^*} \\ & x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9. \end{array}$$

   *Thus, we have that*

$$\mathrm{dir}_{1,2}(g_u) = \mathrm{dir}_{5,6}(g_u) = 1.$$

c) *Let* $w_{\mathbb{Z}}$ *be an asymmetric band with periodic part* $\hat{w}_p = \varepsilon^* a$. *Let* $v_{\mathbb{Z}} \in (\Phi^{\mathrm{d}}_{\mathrm{ud}})^{-1}(w_{\mathbb{Z}})$ *be (weakly) consistent. Then*

$$\mathrm{dir}(v_{1+kp}) = 1 \quad \forall k \in \mathbb{Z}.$$

   *The corresponding* $\mathfrak{L}$*−cycle* $g_{w_{\mathbb{Z}}}$ *is given by*

$$g_{w_{\mathbb{Z}}}: \quad \begin{array}{ccccccc} x_0 & & x_1 & & x_2 & & x_3 \\ \overleftarrow{\mathfrak{C}_{\varepsilon^*} & \sim & \mathfrak{C}_{\varepsilon^*}} & - & \mathfrak{R}_{11} & \sim & \mathfrak{R}_{13} \end{array}$$

*It follows that* $\mathrm{dir}_{0,1} = 1$.

*d) Let* $w_{\mathbb{Z}}$ *be a symmetric band with* $\hat{w}_p = \varepsilon^* a \varepsilon^* a \varepsilon^* a^{-1} \varepsilon^* a^{-1}$, $u = a\varepsilon^* a$. *Let* $v_{\mathbb{Z}} \in \left(\Phi^{\mathrm{d}}_{\mathrm{ud}}\right)^{-1}(w_{\mathbb{Z}})$ *be (weakly) consistent (*$\hat{v}_p^{(i)} = \varepsilon^{\kappa_i} t \eta^* t^{-1}$ *with* $t \in \left(\Phi^{\mathrm{d}}_{\mathrm{ud}}\right)^{-1}(u)$ *and* $\kappa_i, \mu_i \in \{+1, -1\}$*). We have*

$$\begin{aligned} \mathrm{dir}(v_{3+kp}) &= \ \ 1, \\ \mathrm{dir}(v_{6+kp}) &= -1 \end{aligned}$$

*for all* $k \in \mathbb{Z}$*. We consider for* $w_{\mathbb{Z}}$

*(i) the* $\mathfrak{L}$*-chain* $g_u$*:*

$$g_u: \quad \begin{array}{ccccccccccccccc} \mathfrak{C}_{\varepsilon^*} & - & \mathfrak{R}_{13} & \sim & \mathfrak{R}_{11} & - & \overleftarrow{\mathfrak{C}_{\varepsilon^*} \sim \mathfrak{C}_{\varepsilon^*}} & - & \mathfrak{R}_{13} & \sim & \mathfrak{R}_{11} & - & \mathfrak{C}_{\varepsilon^*} \\ x_0 & & x_1 & & x_2 & & x_3 \quad\quad x_4 & & x_5 & & x_6 & & x_7 \end{array}$$

*Then we have that* $\mathrm{dir}_{3,4}(g_u) = 1$.

*(ii) the* $\mathfrak{L}$*-cycle* $g_{w_{\mathbb{Z}}}$*:*

$$\begin{array}{ccccccccccccccccccccccc} & x_3 & & x_4 & & x_5 & & x_6 & & x_7 & & x_8 & & x_9 & & x_{10} & & x_{11} & & x_{12} & \\ & \overleftarrow{\mathfrak{R}_{13} - \mathfrak{C}_{\varepsilon^*}} & \sim & \mathfrak{C}_{\varepsilon^*} & - & \mathfrak{R}_{11} & \sim & \mathfrak{R}_{13} & - & \mathfrak{C}_{\varepsilon^*} & \sim & \mathfrak{C}_{\varepsilon^*} & - & \mathfrak{R}_{13} & \sim & \mathfrak{R}_{11} & - & \mathfrak{C}_{\varepsilon^*} & \downarrow & \\ & \wr & & & & & & & & & & & & & & & & \wr & & \\ x_2 & \mathfrak{R}_{11} & & & & & & & & & & & & & & & & \mathfrak{C}_{\varepsilon^*} & & x_{13} \\ & | & & & & & & & & & & & & & & & & | & & \\ x_1 & \mathfrak{C}_{\varepsilon^*} & & & & & & & & & & & & & & & & \mathfrak{R}_{13} & & x_{14} \\ & \wr & & & & & & & & & & & & & & & & \wr & & \\ x_0 & \mathfrak{C}_{\varepsilon^*} & & & & & & & & & & & & & & & & \mathfrak{R}_{11} & & x_{15} \end{array}$$

*It follows that*

$$\begin{aligned} \mathrm{dir}_{4,5}(g_{w_{\mathbb{Z}}}) &= \ \ 1, \\ \mathrm{dir}_{12,13}(g_{w_{\mathbb{Z}}}) &= -1. \end{aligned}$$

# 5 The categories $\mathrm{mod}(\Lambda)$ and $\mathrm{Rep}(\overline{\mathfrak{X}}_\Lambda)$

We present the most important results in order to prove the main theorem in this chapter.
We start by describing the functor $F : \mathrm{Rep}(\overline{\mathfrak{X}}_\Lambda) \longrightarrow \mathrm{mod}(\Lambda)$ where $\Lambda$ is a skewed-gentle algebra. We show by the Propositions 5.3, 5.4 and 5.5 that the functor $F$ gives an equivalence of categories. Due to its length, the proof of Proposition 5.5 is moved to Section 5.2.
The nature of the functor allows us to analyse its image explicitely in terms of strings and bands. We find that we need to slightly adjust the directed alphabet of $\Lambda$ chosen in Chapter 2.3. For instance, we need to replace the letter $\varepsilon^{-1}$ by the letter $\bar{\varepsilon}^{-1}$ in order to describe the modules in the image of $F$ properly. This description is found in Section 5.4. However, we would like to be able to give a description of the indecomposable modules in terms of the original alphabet $\Gamma_{\mathrm{d}}(\Lambda)$. We examine this issue in Section 5.5 and summarise the results in Theorem 5.36. Finally, we are able to give a classification of the finite dimensional indecomposable modules of a skewed-gentle algebra in terms of strings and bands (Theorem 5.49). From this result, we can, according to Section 4.3, deduce a respective classification result for clannish algebras (Theorem 5.50). We use those results to state the classification in such terms that it proves the conjecture made by Crawley-Boevey in [CB88] (see Chapter 6).

## 5.1 Equivalence of categories

Let $\Lambda$ be as in Section 4.4 with bundle of semichains $\overline{\mathfrak{X}}_\Lambda$. We show in this section that there exists an equivalence between the category of representations of the bundle of semichains $\overline{\mathfrak{X}}_\Lambda$ and the category of finite dimensional $\Lambda$–modules. To this end, we first describe the action of the functor and prove then that it is faithful, full and dense (see [ASS06, Theorem 2.5., Appendix]).

Let us first recall the two categories $\mathrm{Rep}(\overline{\mathfrak{X}}_\Lambda)$ and $\mathrm{mod}(\Lambda)$. The objects of $\mathrm{Rep}(\overline{\mathfrak{X}}_\Lambda)$ are given by representations of the form $U = (U_X, U^i)_{X \in \mathfrak{X}_0, 1 \leq i \leq n}$, where $U_X$ is a $\mathrm{k}$–vector space of dimension $\dim P(X)$, $U^i : U_{\mathfrak{C}_i} \longrightarrow U_{\mathfrak{R}_i}$ is a $\mathrm{k}$–linear map and $n = |Q_0|$. Recall that any matrix $U^i$ is invertible by construction. A morphism $\theta : U \longrightarrow W$ between two representations is given by a tuple $(P, Q)$ where each entry consists of $n$ maps giving $\theta = (P^1, \dots, P^n, Q^1, \dots, Q^n)$. Each of the maps $P^i$ and $Q^i$ is given by a square matrix. For any $i \in \{1, \dots, n\}$ we have that $W^i P^i = Q^i U^i$.
Recall that $\Lambda = \mathrm{k}Q/(\mathrm{R} \cup \mathrm{R}^{\mathrm{Sp}})$ and that the objects of $\mathrm{mod}(\Lambda)$ can be regarded as representations of the form $V = (V_i, V_a)_{i \in Q_0, a \in Q_1}$. Here, $V_i$ denotes a $\mathrm{k}$–vector space and $V_a : V_i \longrightarrow V_j$ is a $\mathrm{k}$–linear map for the arrow $a : i \longrightarrow j$. A morphism $f : V \longrightarrow X$ between two representations is given by a tuple $f = (f_i)_{i \in Q_0}$ such that $f_i : V_i \longrightarrow X_i$ is a $\mathrm{k}$–linear map for any $i \in Q_0$ and

such that $W_a f_i = f_j V_a$ for any $a : i \longrightarrow j \in Q_1$.
Recall also that we have chosen $\operatorname{sgn}(\varepsilon) = \kappa$ for all $\varepsilon \in \mathrm{Sp}$.

Let

$$F : \operatorname{Rep}(\overline{\mathfrak{X}}_\Lambda) \longrightarrow \operatorname{mod}(\Lambda) \tag{146}$$

be given by the following:

(I) Let $U = (U_X, U^i)_{X,i} \in \operatorname{Ob}(\operatorname{Rep}(\overline{\mathfrak{X}}_\Lambda))$. Then $F(U) = V$ with $V = (V_i, V_a)_{i,a}$, $V_i = \bigoplus_{X \in \mathfrak{R}_i} U_X =: U_{\mathfrak{R}_i}$ and $V_a : V_i \longrightarrow V_j$ is given as follows:

(i) Let $a = \varepsilon \in \mathrm{Sp}$. We obtain for $V_\varepsilon$:

$$V_\varepsilon = U^i \tilde{V}_\varepsilon \left(U^i\right)^{-1},$$

where

$$\tilde{V}_\varepsilon = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & 0 \end{array}\right), \tag{147}$$

$i = s(\varepsilon)$, and the block $(\tilde{V}_\varepsilon)_{k,l}$ has size

$$\begin{cases} \dim(\operatorname{im}(\varepsilon)) \times \dim(\operatorname{im}(\varepsilon)) & \text{if } k = l = 1 \\ \dim(\operatorname{im}(\varepsilon)) \times \dim(\ker(\varepsilon)) & \text{if } k = 1, l = 2 \\ \dim(\ker(\varepsilon)) \times \dim(\operatorname{im}(\varepsilon)) & \text{if } k = 2, l = 1 \\ \dim(\ker(\varepsilon)) \times \dim(\ker(\varepsilon)) & \text{if } k = l = 2 \end{cases}.$$

(ii) Let $a \in Q_1^{\mathrm{ord}}$, $\operatorname{sgn}(a^{-1}) = -\kappa$, $\operatorname{sgn}(a) = -\kappa$. The basis of $V_i \ominus \ker(a)$ and the basis of $\operatorname{im}(a)$ correspond to links in the row label sets $\mathfrak{R}_i$ and $\mathfrak{R}_j$, respectively. Denote by $|\mathfrak{R}_i|$ the number of links in the semichain $\mathfrak{R}_i$. We obtain for $V_a : U_{\mathfrak{R}_i} \longrightarrow U_{\mathfrak{R}_j}$ a block matrix with rows corresponding to the links of $\mathfrak{R}_j$ and columns corresponding to the links of $\mathfrak{R}_i$:

$$(V_a)_{k,l} = \begin{cases} 1, & \text{if } k = |\mathfrak{R}_j|, l = 1, \\ 0, & \text{otherwise,} \end{cases}$$

where $1 \le k \le |\mathfrak{R}_j|, 1 \le l \le |\mathfrak{R}_i|$, and the block $(V_a)_{k,l}$ is of size $\dim P(\mathfrak{R}_{j,|\mathfrak{R}_j|+1-k}) \times \dim P(\mathfrak{R}_{i,|\mathfrak{R}_i|+1-l})$.

(iii) Let $a \in Q_1^{\mathrm{ord}}$, $\operatorname{sgn}(a^{-1}) = \kappa$, $\operatorname{sgn}(a) = -\kappa$. The basis of $V_i \ominus \ker(a)$ corresponds to a link in $\mathfrak{C}_i$, and the basis of $\operatorname{im}(a)$ corresponds to a link in $\mathfrak{R}_j$. We obtain

$$V_a = \tilde{V}_a \left(U^i\right)^{-1}$$

where

$$(\tilde{V_a})_{k,l} = \begin{cases} 1, & \text{if } k = |\mathfrak{R}_j|, l = 3, \\ 0, & \text{otherwise,} \end{cases}$$

$1 \le k \le |\mathfrak{R}_j|$, $1 \le l \le |\mathfrak{C}_i|$, and $(\tilde{V_a})_{k,l}$ is of size $\dim P(\mathfrak{R}_{j,|\mathfrak{R}_j|+1-k}) \times \dim P(\mathfrak{C}_{i,|\mathfrak{C}_i|+1-l})$.

(iv) Let $a \in Q_1^{\text{ord}}$ with $\text{sgn}(a^{-1}) = -\kappa$ and $\text{sgn}(a) = \kappa$. The basis of $V_i \ominus \ker(a)$ corresponds to a link in $\mathfrak{R}_i$, and the basis of $\text{im}(a)$ to a link in $\mathfrak{C}_j$. The map $V_a$ is given by

$$V_a = U^j \tilde{V_a}$$

where

$$(\tilde{V_a})_{k,l} = \begin{cases} 1 & \text{if } k = 1, l = 1 \\ 0 & \text{otherwise} \end{cases}$$

$1 \le k \le |\mathfrak{C}_j|$, $1 \le l \le |\mathfrak{R}_i|$, and $(\tilde{V_a})_{k,l}$ is of size $\dim P(\mathfrak{C}_{j,|\mathfrak{C}_j|+1-k}) \times \dim P(\mathfrak{R}_{i,|\mathfrak{R}_i|+1-l})$.

(v) Let $a \in Q_1^{\text{ord}}$ with $\text{sgn}(a^{-1}) = \kappa$ and $\text{sgn}(a) = \kappa$. We obtain that the basis of $V_i \ominus \ker(a)$ corresponds to a link in $\mathfrak{C}_i$. The basis of $\text{im}(a)$ corresponds to a link in $\mathfrak{C}_j$. It follows that $V_a$ is given by

$$V_a = U^j \tilde{V_a} \left(U^i\right)^{-1}$$

where

$$(\tilde{V_a})_{k,l} = \begin{cases} 1 & \text{if } k = 1, l = |\mathfrak{C}_i| \\ 0 & \text{otherwise} \end{cases},$$

$1 \le k \le |\mathfrak{C}_j|$, $1 \le l \le |\mathfrak{C}_i|$, and $(\tilde{V_a})_{k,l}$ is of size $\dim P(\mathfrak{C}_{j,|\mathfrak{C}_j|+1-k}) \times \dim P(\mathfrak{C}_{i,|\mathfrak{C}_i|+1-l})$.

(II) Let $\theta = (P, Q)$ be a morphism in $\text{Rep}(\overline{\mathfrak{X}}_\Lambda)$ with components $P^1, \dots, P^n$ and $Q^1, \dots, Q^n$. Then

$$F(\theta) = (Q^i)_{1 \le i \le n}.$$

Recall that the arrow $a : i \longrightarrow j \in Q_1^{\text{ord}}$ gives rise to one filtration in $V_i$ and to one in $V_j$. The matrix $\tilde{V_a}$ describes the action of $a$ with respect to those two filtrations. The filtration of $V_i$ can be of the form

$$0 \subset \ker(a) \subset V_i \tag{148}$$

$$\text{or} \quad 0 \subset \text{im}(c) \subset \ker(a) \subset V_i \tag{149}$$

if there exists $c \in Q_1^{\text{ord}}$ with $ac = 0$. Similarly, the filtration of $V_j$ can be given by

$$0 \subset \text{im}(a) \subset V_j \tag{150}$$
$$\text{or} \quad 0 \subset \text{im}(a) \subset \ker(x) \subset V_j \tag{151}$$

if there exists $x \in Q_1^{\text{ord}}$ with $xa = 0$.
The corresponding semichains consist of two or three links, respectively. Thus, $\tilde{V}_a$ has one of the following sizes with respect to its bands:

$$(3 \times 3), (3 \times 2), (2 \times 3), (2 \times 2).$$

The way we have constructed the semichains in Section 4.4.2 allows us to describe $\tilde{V}_a$ more detailed with respect to its block form. The Tables 1 and 2 give an overview on the details of (I) (ii) - (iv).

**Example 5.1.** *1. Let $\Lambda$ be given as in Example 2.3.1. Consider the $\overline{\mathfrak{X}}_\Lambda$-representation $U = (U^1, U_{\mathfrak{C}_{11}^+}, U_{\mathfrak{C}_{11}^-}, U_{\mathfrak{R}_{11}}, U_{\mathfrak{C}_{12}}, U_{\mathfrak{C}_{13}})$ with*

$$U^1 = \left(\begin{array}{cc|cc} 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 \end{array}\right).$$

*The respective vector spaces have the following dimensions:*

$$\begin{aligned} &\dim(U_{\mathfrak{C}_{11}^+}) = 2, && \dim(U_{\mathfrak{C}_{11}^-}) = 2, \\ &\dim(U_{\mathfrak{R}_{11}}) = 1, && \dim(U_{\mathfrak{R}_{12}}) = 2, \\ &\dim(U_{\mathfrak{R}_{13}}) = 1. \end{aligned}$$

*We have by construction that*

$$\tilde{V}_a = \left(\begin{array}{c|cc|c} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \end{array}\right), \quad \tilde{V}_\varepsilon = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

*Since $\tilde{V}_a : U_{\mathfrak{R}_1} \longrightarrow U_{\mathfrak{R}_1}$, we obtain that $V_a = \tilde{V}_a$. Moreover, we get that*

$$V_\varepsilon = U^1 \tilde{V}_\varepsilon \left(U^1\right)^{-1} = \left(\begin{array}{c|cc|c} 1 & 0 & -1 & 0 \\ \hline 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right).$$

*Thus, $F(U)$ is given by $V = (U_{\mathfrak{R}_1}, V_a, V_\varepsilon)$.*

| (I) | $\tilde{V_a}$ | semichains | $V_i \ominus \ker(a)$ | $\mathrm{im}(a)$ | $\tilde{V_a}$ as block matrix |
|---|---|---|---|---|---|
| (ii) | $U_{\mathfrak{R}_i} \to U_{\mathfrak{R}_j}$ | $\mathfrak{R}_i$={$\mathfrak{R}_{i1}$>$\mathfrak{R}_{i2}$>$\mathfrak{R}_{i3}$}<br>$\mathfrak{R}_j$={$\mathfrak{R}_{j1}$>$\mathfrak{R}_{j2}$>$\mathfrak{R}_{j3}$} | $\mathfrak{R}_{i3}$ | $\mathfrak{R}_{j1}$ | $\begin{array}{cc} & \begin{matrix}\mathfrak{R}_{i3} & \mathfrak{R}_{i2} & \mathfrak{R}_{i1}\end{matrix} \\ \begin{matrix}\mathfrak{R}_{j3}\\ \mathfrak{R}_{j2}\\ \mathfrak{R}_{j1}\end{matrix} & \begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix}\end{array}$ |
| | | $\mathfrak{R}_i$={$\mathfrak{R}_{i1}$>$\mathfrak{R}_{i2}$>$\mathfrak{R}_{i3}$}<br>$\mathfrak{R}_j$={$\mathfrak{R}_{j1}$>$\mathfrak{R}_{j2}$} | $\mathfrak{R}_{i3}$ | $\mathfrak{R}_{j1}$ | $\begin{array}{cc} & \begin{matrix}\mathfrak{R}_{i3} & \mathfrak{R}_{i2} & \mathfrak{R}_{i1}\end{matrix} \\ \begin{matrix}\mathfrak{R}_{j2}\\ \mathfrak{R}_{j1}\end{matrix} & \begin{pmatrix}0&0&0\\1&0&0\end{pmatrix}\end{array}$ |
| | | $\mathfrak{R}_i$={$\mathfrak{R}_{i1}$>$\mathfrak{R}_{i2}$}<br>$\mathfrak{R}_j$={$\mathfrak{R}_{j1}$>$\mathfrak{R}_{j2}$>$\mathfrak{R}_{j3}$} | $\mathfrak{R}_{i2}$ | $\mathfrak{R}_{j1}$ | $\begin{array}{cc} & \begin{matrix}\mathfrak{R}_{i2} & \mathfrak{R}_{i1}\end{matrix} \\ \begin{matrix}\mathfrak{R}_{j3}\\ \mathfrak{R}_{j2}\\ \mathfrak{R}_{j1}\end{matrix} & \begin{pmatrix}0&0\\0&0\\1&0\end{pmatrix}\end{array}$ |
| | | $\mathfrak{R}_i$={$\mathfrak{R}_{i1}$>$\mathfrak{R}_{i2}$}<br>$\mathfrak{R}_j$={$\mathfrak{R}_{j1}$>$\mathfrak{R}_{j2}$} | $\mathfrak{R}_{i2}$ | $\mathfrak{R}_{j1}$ | $\begin{array}{cc} & \begin{matrix}\mathfrak{R}_{i2} & \mathfrak{R}_{i1}\end{matrix} \\ \begin{matrix}\mathfrak{R}_{j2}\\ \mathfrak{R}_{j1}\end{matrix} & \begin{pmatrix}0&0\\1&0\end{pmatrix}\end{array}$ |
| (iii) | $U_{\mathfrak{C}_i} \to U_{\mathfrak{R}_j}$ | $\mathfrak{C}_i$={$\mathfrak{C}_{i1}$<$\mathfrak{C}_{i2}$<$\mathfrak{C}_{i3}$}<br>$\mathfrak{R}_j$={$\mathfrak{R}_{j1}$>$\mathfrak{R}_{j2}$>$\mathfrak{R}_{j3}$} | $\mathfrak{C}_{i3}$ | $\mathfrak{R}_{j1}$ | $\begin{array}{cc} & \begin{matrix}\mathfrak{C}_{i1} & \mathfrak{C}_{i2} & \mathfrak{C}_{i3}\end{matrix} \\ \begin{matrix}\mathfrak{R}_{j3}\\ \mathfrak{R}_{j2}\\ \mathfrak{R}_{j1}\end{matrix} & \begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}\end{array}$ |
| | | $\mathfrak{C}_i$={$\mathfrak{C}_{i1}$<$\mathfrak{C}_{i2}$<$\mathfrak{C}_{i3}$}<br>$\mathfrak{R}_j$={$\mathfrak{R}_{j1}$>$\mathfrak{R}_{j2}$} | $\mathfrak{C}_{i3}$ | $\mathfrak{R}_{j1}$ | $\begin{array}{cc} & \begin{matrix}\mathfrak{C}_{i1} & \mathfrak{C}_{i2} & \mathfrak{C}_{i3}\end{matrix} \\ \begin{matrix}\mathfrak{R}_{j2}\\ \mathfrak{R}_{j1}\end{matrix} & \begin{pmatrix}0&0&0\\0&0&1\end{pmatrix}\end{array}$ |
| | | $\mathfrak{C}_i$={$\mathfrak{C}_{i1}$<$\mathfrak{C}_{i2}$}<br>$\mathfrak{R}_j$={$\mathfrak{R}_{j1}$>$\mathfrak{R}_{j2}$>$\mathfrak{R}_{j3}$} | $\mathfrak{C}_{i2}$ | $\mathfrak{R}_{j1}$ | $\begin{array}{cc} & \begin{matrix}\mathfrak{C}_{i1} & \mathfrak{C}_{i2}\end{matrix} \\ \begin{matrix}\mathfrak{R}_{j3}\\ \mathfrak{R}_{j2}\\ \mathfrak{R}_{j1}\end{matrix} & \begin{pmatrix}0&0\\0&0\\0&1\end{pmatrix}\end{array}$ |
| | | $\mathfrak{C}_i$={$\mathfrak{C}_{i1}$<$\mathfrak{C}_{i2}$}<br>$\mathfrak{R}_j$={$\mathfrak{R}_{j1}$>$\mathfrak{R}_{j2}$} | $\mathfrak{C}_{i2}$ | $\mathfrak{R}_{j1}$ | $\begin{array}{cc} & \begin{matrix}\mathfrak{C}_{i1} & \mathfrak{C}_{i2}\end{matrix} \\ \begin{matrix}\mathfrak{R}_{j2}\\ \mathfrak{R}_{j1}\end{matrix} & \begin{pmatrix}0&0\\0&1\end{pmatrix}\end{array}$ |

Table 1: Details on $\tilde{V_a}$ I

| | | | | | |
|---|---|---|---|---|---|
| (iv) | $U_{\mathfrak{R}_i} \to U_{\mathfrak{C}_j}$ | $\mathfrak{R}_i$={$\mathfrak{R}_{i1}$>$\mathfrak{R}_{i2}$>$\mathfrak{R}_{i3}$} $\mathfrak{C}_j$={$\mathfrak{C}_{j1}$<$\mathfrak{C}_{j2}$<$\mathfrak{C}_{j3}$} | $\mathfrak{R}_{i3}$ | $\mathfrak{C}_{j1}$ | $\begin{array}{cc} & \begin{array}{ccc}\mathfrak{R}_{i3} & \mathfrak{R}_{i2} & \mathfrak{R}_{i1}\end{array} \\ \begin{array}{c}\mathfrak{C}_{j1}\\ \mathfrak{C}_{j2}\\ \mathfrak{C}_{j3}\end{array} & \begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}\end{array}$ |
| | | $\mathfrak{R}_i$={$\mathfrak{R}_{i1}$>$\mathfrak{R}_{i2}$>$\mathfrak{R}_{i3}$} $\mathfrak{C}_j$={$\mathfrak{C}_{j1}$<$\mathfrak{C}_{j2}$} | $\mathfrak{R}_{i3}$ | $\mathfrak{C}_{j1}$ | $\begin{array}{cc} & \begin{array}{ccc}\mathfrak{R}_{i3} & \mathfrak{R}_{i2} & \mathfrak{R}_{i1}\end{array} \\ \begin{array}{c}\mathfrak{C}_{j1}\\ \mathfrak{C}_{j2}\end{array} & \begin{pmatrix}1&0&0\\0&0&0\end{pmatrix}\end{array}$ |
| | | $\mathfrak{R}_i$={$\mathfrak{R}_{i1}$>$\mathfrak{R}_{i2}$} $\mathfrak{C}_j$={$\mathfrak{C}_{j1}$<$\mathfrak{C}_{j2}$<$\mathfrak{C}_{j3}$} | $\mathfrak{R}_{i2}$ | $\mathfrak{C}_{j1}$ | $\begin{array}{cc} & \begin{array}{cc}\mathfrak{R}_{i2} & \mathfrak{R}_{i1}\end{array} \\ \begin{array}{c}\mathfrak{C}_{j1}\\ \mathfrak{C}_{j2}\\ \mathfrak{C}_{j3}\end{array} & \begin{pmatrix}1&0\\0&0\\0&0\end{pmatrix}\end{array}$ |
| | | $\mathfrak{R}_i$={$\mathfrak{R}_{i1}$>$\mathfrak{R}_{i2}$} $\mathfrak{C}_j$={$\mathfrak{C}_{j1}$<$\mathfrak{C}_{j2}$} | $\mathfrak{R}_{i2}$ | $\mathfrak{C}_{j1}$ | $\begin{array}{cc} & \begin{array}{cc}\mathfrak{R}_{i2} & \mathfrak{R}_{i1}\end{array} \\ \begin{array}{c}\mathfrak{C}_{j1}\\ \mathfrak{C}_{j2}\end{array} & \begin{pmatrix}1&0\\0&0\end{pmatrix}\end{array}$ |
| (v) | $U_{\mathfrak{C}_i} \to U_{\mathfrak{C}_j}$ | $\mathfrak{C}_i$={$\mathfrak{C}_{i1}$<$\mathfrak{C}_{i2}$<$\mathfrak{C}_{i3}$} $\mathfrak{C}_j$={$\mathfrak{C}_{j1}$<$\mathfrak{C}_{j2}$<$\mathfrak{C}_{j3}$} | $\mathfrak{C}_{i3}$ | $\mathfrak{C}_{j1}$ | $\begin{array}{cc} & \begin{array}{ccc}\mathfrak{C}_{i1} & \mathfrak{C}_{i2} & \mathfrak{C}_{i3}\end{array} \\ \begin{array}{c}\mathfrak{C}_{j1}\\ \mathfrak{C}_{j2}\\ \mathfrak{C}_{j3}\end{array} & \begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix}\end{array}$ |
| | | $\mathfrak{C}_i$={$\mathfrak{C}_{i1}$<$\mathfrak{C}_{i2}$<$\mathfrak{C}_{i3}$} $\mathfrak{C}_j$={$\mathfrak{C}_{j1}$<$\mathfrak{C}_{j2}$} | $\mathfrak{C}_{i3}$ | $\mathfrak{C}_{j1}$ | $\begin{array}{cc} & \begin{array}{ccc}\mathfrak{C}_{i1} & \mathfrak{C}_{i2} & \mathfrak{C}_{i3}\end{array} \\ \begin{array}{c}\mathfrak{C}_{j1}\\ \mathfrak{C}_{j2}\end{array} & \begin{pmatrix}0&0&1\\0&0&0\end{pmatrix}\end{array}$ |
| | | $\mathfrak{C}_i$={$\mathfrak{C}_{i1}$<$\mathfrak{C}_{i2}$} $\mathfrak{C}_j$={$\mathfrak{C}_{j1}$<$\mathfrak{C}_{j2}$<$\mathfrak{C}_{j3}$} | $\mathfrak{C}_{i2}$ | $\mathfrak{C}_{j1}$ | $\begin{array}{cc} & \begin{array}{cc}\mathfrak{C}_{i1} & \mathfrak{C}_{i2}\end{array} \\ \begin{array}{c}\mathfrak{C}_{j1}\\ \mathfrak{C}_{j2}\\ \mathfrak{C}_{j3}\end{array} & \begin{pmatrix}0&1\\0&0\\0&0\end{pmatrix}\end{array}$ |
| | | $\mathfrak{C}_i$={$\mathfrak{C}_{i1}$<$\mathfrak{C}_{i2}$} $\mathfrak{C}_j$={$\mathfrak{C}_{j1}$<$\mathfrak{C}_{j2}$} | $\mathfrak{C}_{i2}$ | $\mathfrak{C}_{j1}$ | $\begin{array}{cc} & \begin{array}{cc}\mathfrak{C}_{i1} & \mathfrak{C}_{i2}\end{array} \\ \begin{array}{c}\mathfrak{C}_{j1}\\ \mathfrak{C}_{j2}\end{array} & \begin{pmatrix}0&1\\0&0\end{pmatrix}\end{array}$ |

Table 2: Details on $\tilde{V}_a$ II

*2. Let $\Lambda$ be as in Example 2.14. Let $U$ be a $\overline{\mathfrak{X}}_\Lambda$–representation given by*

$$U^1 = \left(\begin{array}{c|c} 1 & 1 \\ \hline 1 & 0 \end{array}\right), \qquad U^2 = \left( 1 \right), \qquad U^3 = 0$$
$$U^4 = \left( 1 \right), \qquad U^5 = 0$$

*and the non-trivial vector spaces of dimension*

$$\begin{array}{ll} \dim(U_{\mathfrak{C}_{11}^+}) = 1, & \dim(U_{\mathfrak{C}_{11}^-}) = 1, \\ \dim(U_{\mathfrak{R}_{11}}) = 1, & \dim(U_{\mathfrak{R}_{12}}) = 1, \\ \dim(U_{\mathfrak{R}_{23}}) = 1, & \dim(U_{\mathfrak{C}_{21}}) = 1, \\ \dim(U_{\mathfrak{R}_{41}}) = 1, & \dim(U_{\mathfrak{C}_{42}}) = 1. \end{array}$$

*We consider the following maps:*

$$\begin{array}{ll} \tilde{V}_a : U_{\mathfrak{R}_1} \longrightarrow U_{\mathfrak{C}_2}, & \tilde{V}_b : U_{\mathfrak{R}_3} \longrightarrow U_{\mathfrak{R}_2}, \\ \tilde{V}_c : U_{\mathfrak{C}_2} \longrightarrow U_{\mathfrak{R}_5}, & \tilde{V}_d : U_{\mathfrak{R}_2} \longrightarrow U_{\mathfrak{R}_4}, \\ \tilde{V}_e : U_{\mathfrak{R}_5} \longrightarrow U_{\mathfrak{C}_4}, & \tilde{V}_\varepsilon : U_{\mathfrak{C}_1} \longrightarrow U_{\mathfrak{C}_1}, \\ \tilde{V}_\eta : U_{\mathfrak{C}_3} \longrightarrow U_{\mathfrak{C}_3}, & \tilde{V}_\kappa : U_{\mathfrak{C}_5} \longrightarrow U_{\mathfrak{C}_5}. \end{array}$$

*Note that $\tilde{V}_b$, $\tilde{V}_c$, $\tilde{V}_e$, $\tilde{V}_\eta$ and $\tilde{V}_\kappa$ are zero maps. Thus, $V_b$, $V_c$, $V_e$, $V_\eta$ and $V_\kappa$ are also zero maps according to (I),(i)-(v). We obtain that*

$$\begin{aligned} V_a &= U^2 \tilde{V}_a = \left( \begin{array}{c|c} 1 & 0 \end{array} \right), \\ V_d &= \tilde{V}_d = \left( 1 \right), \\ V_\varepsilon &= U^1 \tilde{V}_\varepsilon \left(U^1\right)^{-1} = \left(\begin{array}{c|c} 0 & 1 \\ \hline 0 & 1 \end{array}\right). \end{aligned}$$

*Setting $V_i = U_{\mathfrak{R}_i}$ yields $F(U) = V$.*

**Proposition 5.2.** *The map $F$ as defined above is a functor.*

*Proof.* Let $\psi,\ \theta \in \mathrm{Morph}(\mathrm{Rep}(\overline{\mathfrak{X}}_\Lambda))$ with $\psi = (R,S) : V \longrightarrow W$ and $\theta = (P,Q) : U \longrightarrow V$. We show first that

$$F(\psi\theta) = F(\psi)F(\theta). \tag{152}$$

Recall that the composition $\psi\theta$ is given by $(RP, SQ)$. We have by definition of $F$ that

$$\begin{aligned} F(\psi) &= (S^i)_{1\leq i\leq n}, \\ F(\theta) &= (Q^i)_{1\leq i\leq n}, \text{ and} \\ F(\psi\theta) &= (S^i Q^i)_{1\leq i\leq n}. \end{aligned}$$

Thus, (152) holds.
Let $U \in \mathrm{Ob}(\mathrm{Rep}(\overline{\mathfrak{X}}_\Lambda))$ be an arbitrary representation. Then $1_U$ is given by the tuple $(P,Q)$ with $P^i = 1_{c_i \times c_i}$ and $Q^i = 1_{r_i \times r_i}$ for all $1 \le i \le n$, where $c_i = \dim(U_{\mathfrak{C}_i})$ and $r_i = \dim(U_{\mathfrak{R}_i})$. We obtain that

$$F(1_U) = (1_{r_i \times r_i})_{1 \le i \le n} = (1_{U_{\mathfrak{R}_i}})_{1 \le i \le n}.$$

Moreover, we have that $F(U) = V = (V_i, V_a)_{i \in Q_0, a \in Q_1} \in \mathrm{Ob}(\mathrm{mod}(\Lambda))$ where $V_i = U_{\mathfrak{R}_i}$. By definition, $1_V = (1_{V_i})_{1 \le i \le n} = (1_{U_{\mathfrak{R}_i}})_{1 \le i \le n}$. It follows that

$$F(1_U) = 1_{F(U)}.$$

Finally, we show that $F$ is well-defined. Let $T$ and $U$ be two arbitrary $\overline{\mathfrak{X}}_\Lambda$–representations. Let $V = F(U)$ and $W = F(T)$. By definition, $V$ and $W$ are $(Q, \mathrm{R} \cup \mathrm{R}^{\mathrm{Sp}})$–representations. Let $\theta = (P,Q) : U \longrightarrow T \in \mathrm{Morph}(\mathrm{Rep}(\overline{\mathfrak{X}}_\Lambda))$. It remains to show that

$$Q^j V_a = W_a Q^i \tag{153}$$

for all $a : i \longrightarrow j \in Q_1$. To this end, recall that the commutativity relation

$$Q^i U^i = W^i P^i \tag{154}$$

is given for all $1 \le i \le n$. Furthermore, we know that $Q^i$ is of lower triangular block form, and that $P^i$ is of upper triangular block form. Let us examine (153) for $a : i \longrightarrow j \in Q_1^{\mathrm{ord}}$. Then $V_a$ and $W_a$ are given by one of (I) (ii) - (v). Depending on the relations, $\tilde{V}_a$ and $\tilde{W}_a$ can be of block size $(3 \times 3)$, $(2 \times 3)$, $(3 \times 2)$ or $(2 \times 2)$. Note that $\tilde{V}_a$ and $\tilde{W}_a$ are of the same block form, though the sizes of the blocks can differ.

(ii) Let $V_a = \tilde{V}_a$, $W_a = \tilde{W}_a$. Then (153) is equivalent to $Q^j \tilde{V}_a = \tilde{W}_a Q^i$. Consider the $(3 \times 3)$–block form

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

for $\tilde{V}_a$ and $\tilde{W}_a$. Both $Q^i$ and $Q^j$ are also of $(3 \times 3)$–block form. Denote their entries by $q^i_{kl}$, $q^j_{kl}$, respectively, $1 \le k, l \le 3$. It follows that (153) is given if the following conditions hold:

$$q^i_{12} = q^i_{13} = 0, \tag{155}$$
$$q^j_{13} = q^j_{23} = 0, \tag{156}$$
$$q^j_{33} = q^i_{11}. \tag{157}$$

The conditions (155) and (156) follow from the triangular block forms of $Q^i$ and $Q^j$. We have by construction that

$$\mathfrak{R}_{j1} \hat{=} \text{basis of } \operatorname{im}(a), \tag{158}$$

$$\mathfrak{R}_{i3} \hat{=} \text{basis of } V_i \ominus \ker(a). \tag{159}$$

It follows that

$$Q^i_{\mathfrak{R}_{i3},\mathfrak{R}_{i3}} = Q^j_{\mathfrak{R}_{j1},\mathfrak{R}_{j1}}. \tag{160}$$

Recall the order of the links in $\mathfrak{R}_i$ and $\mathfrak{R}_j$: $\mathfrak{R}_{i1} > \mathfrak{R}_{i2} > \mathfrak{R}_{i3}$ and $\mathfrak{R}_{j1} > \mathfrak{R}_{j2} > \mathfrak{R}_{j3}$. We obtain that $q^j_{33} = Q^j_{\mathfrak{R}_{j1},\mathfrak{R}_{j1}}$ and that $q^i_{11} = Q^i_{\mathfrak{R}_{i3},\mathfrak{R}_{i3}}$. It follows that (157) is equivalent to (160).
Consider the $(2 \times 3)$−block form

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The block matrix $Q^j$ is of size $(2 \times 2)$; $Q^i$ is given by a $(3 \times 3)$−block matrix. We denote their entries again by $q^i_{kl}$ and $q^j_{kl}$. We obtain that (153) holds if

$$q^i_{12} = q^i_{13} = 0, \tag{161}$$

$$q^j_{12} = 0, \tag{162}$$

$$q^j_{22} = q^i_{11}. \tag{163}$$

By the triangular block forms of $Q^i$ and $Q^j$ we know that (161) and (162) hold. The semichains $\mathfrak{R}_i$ and $\mathfrak{R}_j$ have length 3 and 2, respectively. Condition (163) follows as above from $\mathfrak{R}_{i3} \hat{=}$ basis of $V_i \ominus \ker(a)$ and $\mathfrak{R}_{j1} \hat{=}$ basis of $\operatorname{im}(a)$.
Consider the $(3 \times 2)$−block form

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then $Q^i$ is given by a $(2 \times 2)$−block matrix, and $Q^j$ by a $(3 \times 3)$−block matrix. If the following conditions are satisfied, then (153) holds:

$$q^j_{13} = q^j_{23} = 0, \tag{164}$$

$$q^i_{12} = 0, \tag{165}$$

$$q^i_{11} = q^j_{33}. \tag{166}$$

We have $\mathfrak{R}_i = \{\mathfrak{R}_{i1} > \mathfrak{R}_{i2}\}$ and $\mathfrak{R}_j = \{\mathfrak{R}_{j1} > \mathfrak{R}_{j2} > \mathfrak{R}_{j3}\}$, where $\mathfrak{R}_{i2} \hat{=}$ basis of $V_i \ominus \ker(a)$ and $\mathfrak{R}_{j1} \hat{=}$ basis of $\operatorname{im}(a)$. Thus, (166) follows from $Q^i_{\mathfrak{R}_{i2},\mathfrak{R}_{i2}} = q^i_{11}$ and $Q^j_{\mathfrak{R}_{j1},\mathfrak{R}_{j1}} = q^j_{33}$. The lower triangular block

form of $Q^i$ and $Q^j$ implies (164) and (165).
Consider the $(2 \times 2)$–block matrix

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

for $\tilde{V}_a$ and $\tilde{W}_a$. Both $Q^i$ and $Q^j$ are of block form $(2 \times 2)$. We obtain that (153) is given under the following conditions:

$$q^j_{12} = 0, \tag{167}$$

$$q^i_{12} = 0, \tag{168}$$

$$q^i_{11} = q^j_{22}. \tag{169}$$

Similarly to the previous cases, (167) and (168) follow from the triangular block forms of $Q^i$ and $Q^j$. Both $\mathfrak{R}_i$ and $\mathfrak{R}_j$ consist each of two links. In particular, we have that $\mathfrak{R}_{i2} \mathrel{\hat{=}}$ basis of $V_i \ominus \ker(a)$ and $\mathfrak{R}_{j1} \mathrel{\hat{=}}$ basis of $\operatorname{im}(a)$. This implies that $Q^i_{\mathfrak{R}_{i2},\mathfrak{R}_{i2}} = Q^j_{\mathfrak{R}_{j1},\mathfrak{R}_{j1}}$. By $Q^i_{\mathfrak{R}_{i2},\mathfrak{R}_{i2}} = q^i_{11}$ and $Q^j_{\mathfrak{R}_{j1},\mathfrak{R}_{j1}} = q^j_{22}$, (169) follows.

(iii) Let $V_a = \tilde{V}_a \left(U^i\right)^{-1}$ and $W_a = \tilde{W}_a \left(T^i\right)^{-1}$. Applying (154) to (153), the latter is equivalent to

$$Q^j \tilde{V}_a = \tilde{W}_a P^i.$$

This equality implies similar conditions to the ones in (ii). These conditions are satisfied due to the block forms of $Q^j$ and $P^i$ and the block correspondences given by $\sigma_\Lambda(\mathfrak{R}_{j1}) = \mathfrak{C}_{i3}$ ($\sigma_\Lambda(\mathfrak{R}_{j1}) = \mathfrak{C}_{i2}$, respectively).

(iv) Let $V_a = U^j \tilde{V}_a$ and $W_a = T^j \tilde{W}_a$. Due to (154), (153) is equivalent to

$$P^j \tilde{V}_a = \tilde{W}_a Q^i.$$

It follows analgously to the previous cases that (153) holds under similar conditions as in (iii). They are implied by the block form of $Q^i$ and $P^j$ and the block correspondence given by the connection of $\mathfrak{C}_{j1}$ and $\mathfrak{R}_{i3}$ ($\mathfrak{R}_{i2}$, respectively) under $\sigma_\Lambda$.

(v) Let $V_a = U^j \tilde{V}_a U^i$ and $W_a = T^j \tilde{W}_a T^i$. We obtain that (153) is equivalent to

$$P^j \tilde{V}_a = \tilde{W}_a P^i$$

by (154). We obtain that (153) holds if similar conditions as in (ii) with respect to $P^i$ and $P^j$ are satisfied. Due to the triangular block forms of $P^i$ and $P^j$, we know that certain entries are zero. Furthermore, we know that two blocks coincide since $\mathfrak{C}_{j1}$ and $\mathfrak{C}_{i3}$ ($\mathfrak{C}_{i2}$ respectively) are connected by $\sigma_\Lambda$. This data confirms that the conditions in doubt are fulfilled.

□

**Proposition 5.3.** *The functor $F$ is dense.*

*Proof.* Let $V = (V_i, V_a)_{i\in Q_0, a\in Q_1} \in \mathrm{Ob}(\mathrm{mod}(\Lambda))$. By construction of $\overline{\mathfrak{X}}_\Lambda$, we have a correspondence between filtrations on $V_i$ arising from the arrows $a \in Q_1$ and semichains in $\overline{\mathfrak{X}}_\Lambda$. Thus, we can deduce from $V$ a naturally partitioned matrix

$$U^i : V_i^c \longrightarrow V_i^r$$

for each $i \in Q_0$, where $V_i^c \cong V_i^r \cong V_i$. In particular, $V_i^c$ is given in the basis determined by the filtration $F_i^{(j)}$ that corresponds to $S_i^{(c)}$. Similarly, $V_i^r$ is given in the basis determined by the filtration $F_i^{(\bar{\jmath})}$ which corresponds to $S_i^{(r)}$, where $j \neq \bar{\jmath}$.
It follows that $U^i$ is a basis change matrix for all $i \in Q_0$, partitioned according to the filtrations $F_i^{(j)}$ and $F_i^{(\bar{\jmath})}$ with bands indexed by the elements of the corresponding semichains.
Set $U_{\mathfrak{R}_i} = V_i^r$ and $U_{\mathfrak{C}_i} = V_i^c$ for all $i \in Q_0$. Furthermore, take $U_X$ to be for any $X \in \mathfrak{X}_0$ the subspace given by the basis corresponding to $X$ by the filtration associated to $\mathfrak{R}_i$ ($\mathfrak{C}_i$).
By construction, we obtain for $U = (U^i, U_X)_{i,X}$ that $F(U) \cong V$. □

**Proposition 5.4.** *The functor $F$ is faithful.*

*Proof.* Let $\theta = (P, Q) : U \longrightarrow T \in \mathrm{Morph}(\mathrm{Rep}(\overline{\mathfrak{X}}_\Lambda))$ with $F(\theta) = 0$. We have by definition of $F$ that $F(\theta) = (Q^i)_{1\leq i\leq n}$. It follows that the $Q$−component of $\theta$ is zero. Recall that $Q^i U^i = W^i P^i$ for all $1 \leq i \leq n$, and that every $U^i$ and every $W^i$ is invertible by construction. It follows that

$$P^i = \left(W^i\right)^{-1} Q^i U^i = 0$$

for all $1 \leq i \leq n$. Thus, the $P$−component of $\theta$ is zero. We obtain that $\theta = 0$. □

**Proposition 5.5.** *The functor $F$ is full.*

*Proof.* Due to its technicality, we refer to Subsection 5.2 for the proof. □

**Theorem 5.6.** *The functor $F : \mathrm{Rep}(\overline{\mathfrak{X}}_\Lambda) \longrightarrow \mathrm{mod}(\Lambda)$ gives and equivalence of categories.*

*Proof.* The proof follows from the Propositions 5.4, 5.3 and 5.5. □

## 5.2 Proof of Proposition 5.5

Let $V$ and $W$ be two $(Q, \mathrm{R} \cup \mathrm{R}^{\mathrm{Sp}})$–representations and let $f : V \longrightarrow W$ be a morphism. We want to show that there exists $\theta \in \mathrm{Morph}(\mathrm{Rep}(\overline{\mathfrak{X}}_{\Lambda}))$ such that $F(\theta) = f$.
By Proposition 5.3, we know that there exist two $\overline{\mathfrak{X}}_{\Lambda}$–representations $U$ and $X$ with $F(U) = V$ and $F(W) = X$. Set

$$Q^i = f_i, \tag{170}$$

$$P^i = \left(X^i\right)^{-1} Q^i U^i, \tag{171}$$

for all $1 \leq i \leq n$. Recall that the matrices in $U$ and $X$ are invertible by construction. We claim that $F(\theta) = f$ for $\theta = (P, Q)$ with components as described in (170) and (171).
Let $i \in \{1, \ldots, n\}$. Assume without loss of generality that $i \in Q_0$ is not isolated. It follows by (171) that $X^i P^i = Q^i U^i$. We show that the respective other conditions of Definition 4.63 are satisfied for $Q^i$ and $P^i$. We have the following possibilities in $Q$ on the vertex $i$:

(1) $\begin{tikzcd} i \arrow[loop left, "\varepsilon"] \end{tikzcd}$

(2) $\begin{tikzcd} i \arrow[r, "a"] & j \end{tikzcd}$

(3) $\begin{tikzcd} k \arrow[r, "b"] & i \end{tikzcd}$

(4) $\begin{tikzcd} i \arrow[loop left, "\varepsilon"] \arrow[r, "a"] & j \end{tikzcd}$

(5) $\begin{tikzcd} k \arrow[r, "b"] & i \arrow[loop right, "\varepsilon"] \end{tikzcd}$

(6) $\begin{tikzcd} k \arrow[r, "b"] & i \arrow[r, "a"] & j \end{tikzcd}$ with $ab \neq 0$

(7) $\begin{tikzcd} k \arrow[r, "b"] & i \arrow[r, "a"] & j \end{tikzcd}$ with $ab = 0$

(8) $\begin{tikzcd} k \arrow[r, "b"] & i \arrow[loop above, "\varepsilon"] \arrow[r, "a"] & j \end{tikzcd}$ with $ab = 0$

(9) $\begin{tikzcd} l \arrow[dr, "c"] & & \\ k \arrow[r, "b"] & i \arrow[r, "a"] & j \end{tikzcd}$ with $ab \neq 0,\ ca = 0$

(10) $\begin{tikzcd} k \arrow[r, "b"] & i \arrow[r, "a"] \arrow[dr, "d"'] & j \\ & & p \end{tikzcd}$ with $ab \neq 0,\ db = 0$

(11) $$\begin{tikzcd} l \arrow[dr, "c"] & & \\ k \arrow[r, "b"] & i \arrow[r, "a"] \arrow[dr, "d"'] & j \\ & & p \end{tikzcd} \qquad \text{with } ab \neq 0,\ dc \neq 0,\ db = 0,\ ac = 0.$$

Thus, we regard commutativity relations with respect to the ordinary arrows $a$, $b$, $c$ and $d$, and with respect to the special arrow $\varepsilon$. We first compute those commutativity relations. In order to do so, we distinguish the different possible semichain assignments according to (I), (ii) - (v) in the definition of the functor $F$.

($\Sigma$) Commutativity relations on vertex $i$ with respect to $\varepsilon : i \longrightarrow i \in \mathrm{Sp}$. By definition of $f$, the commutativity relation $Q^i V_\varepsilon = W_\varepsilon Q^i$ holds. Furthermore, we have that $V_\varepsilon = U^i \tilde{V}_\varepsilon \left(U^i\right)^{-1}$ with

$$\tilde{V}_\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{172}$$

The maps $W_\varepsilon$ and $\tilde{W}_\varepsilon$ are of similar (block) form. Recall that the equality $X^i P^i = Q^i U^i$ holds for all $i$. Thus, we can rewrite the commutativity relation $Q^i V_\varepsilon = W_\varepsilon Q^i$ to

$$P^i \tilde{V}_\varepsilon = \tilde{W}_\varepsilon P^i. \tag{173}$$

Denote the entries of $P^i$ by $p^i_{kl}$ for $1 \leq k, l \leq 2$. Inserting (172) and the respective block form of $\tilde{W}_\varepsilon$ into (173) gives

$$\begin{pmatrix} p^i_{11} & 0 \\ p^i_{21} & 0 \end{pmatrix} = \begin{pmatrix} p^i_{11} & p^i_{12} \\ 0 & 0 \end{pmatrix}.$$

We obtain that $P^i$ is given by the following block form:

$$P^i = \begin{pmatrix} p^i_{11} & 0 \\ 0 & p^i_{22} \end{pmatrix}. \tag{174}$$

(A) Commutativity relations on vertex $i$ with respect to $a : i \longrightarrow j$. By definition of $f$ we have that

$$Q^j V_a = W_a Q^i. \tag{175}$$

Recall that $V_a$ ($W_a$) is given in terms of $\tilde{V}_a$ ($\tilde{W}_a$). We distinguish the following possibilities according to Table 1 and Table 2:

(ii) We consider $V_a = \tilde{V}_a : U_{\mathfrak{R}_i} \longrightarrow U_{\mathfrak{R}_j}$, $W_a = \tilde{W}_a : X_{\mathfrak{R}_i} \longrightarrow X_{\mathfrak{R}_j}$. Hence, we can rewrite (175) to

$$Q^j \tilde{V}_a = \tilde{W}_a Q^i. \tag{176}$$

Let $\tilde{V}_a$ and $\tilde{W}_a$ be given by $(3 \times 3)$–matrices. They are both of the block form

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and give maps between filtrations of the form (149) and (151). Denote the entries of $Q^j$ by $q^j_{kl}$ and those of $Q^i$ by $q^i_{kl}$, $1, \leq k, l \leq 3$. Equation (176) results in

$$\begin{pmatrix} q^j_{13} & 0 & 0 \\ q^j_{23} & 0 & 0 \\ q^j_{33} & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ q^i_{11} & q^i_{12} & q^i_{13} \end{pmatrix}.$$

Thus, the matrix $Q^i$ is of form

$$Q^i = \begin{pmatrix} q^i_{11} & 0 & 0 \\ q^i_{21} & q^i_{22} & q^i_{23} \\ q^i_{31} & q^i_{32} & q^i_{33} \end{pmatrix}. \tag{177}$$

Furthermore, we have that $q^i_{11} = q^j_{33}$.
Let $\tilde{V}_a$ and $\tilde{W}_a$ be given by $(2 \times 3)$–matrices. They describe maps between filtrations of the forms (149) and (150) and are of the following block form:

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The commutativity relation in (176) results in

$$\begin{pmatrix} q^j_{12} & 0 & 0 \\ q^j_{22} & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ q^i_{11} & q^i_{12} & q^i_{13} \end{pmatrix}.$$

We obtain that $Q^i$ is of form (177) and that $q^j_{22} = q^i_{11}$.
Let $\tilde{V}_a$ and $\tilde{W}_a$ be given by $(3 \times 2)$–matrices. Thus, they give maps between filtrations of the forms (148) and (151) and are of the block form

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

We obtain by (176) that

$$\begin{pmatrix} q^j_{13} & 0 \\ q^j_{23} & 0 \\ q^j_{33} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ q^i_{11} & q^i_{12} \end{pmatrix}.$$

The blocks $q^j_{33}$ and $q^i_{11}$ are equal to each other. It also follows that $Q^i$ is of the following block form:

$$Q^i = \begin{pmatrix} q^i_{11} & 0 \\ q^i_{21} & q^i_{22} \end{pmatrix}. \tag{178}$$

Finally, let $\tilde{V_a}$ and $\tilde{W_a}$ be given by $(2\times 2)$–matrices which describe maps between filtrations of the forms (148) and (150). Their block form is given by

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The commutativity relation (176) gives

$$\begin{pmatrix} q^j_{12} & 0 \\ q^j_{22} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ q^i_{11} & q^i_{12} \end{pmatrix}.$$

It follows that $q^j_{22} = q^i_{11}$. We obtain that $Q^i$ is of same block form as in (178).

(iii) We consider $V_a = \tilde{V_a}\left(U^i\right)^{-1}$, $W_a = \tilde{W_a}\left(X^i\right)^{-1}$. Thus, $\tilde{V_a}$ is a map between the vector spaces $U_{\mathfrak{C}_i}$ and $U_{\mathfrak{R}_j}$ and similarly $\tilde{W_a} : X_{\mathfrak{C}_i} \longrightarrow X_{\mathfrak{R}_j}$. We can rewrite (175) as follows:

$$Q^j\tilde{V_a}\left(U^i\right)^{-1} = \tilde{W_a}\left(X^i\right)^{-1} Q^i.$$

Applying the commutativity relation $Q^iU^i = P^iX^i$ gives

$$Q^j\tilde{V_a}\left(U^i\right)^{-1} = \tilde{W_a}P^i\left(U^i\right)^{-1}.$$

Hence, it is enough to consider the commutativity relation

$$Q^j\tilde{V_a} = \tilde{W_a}P^i. \tag{179}$$

Denote the entries of $P^i$ by $p^i_{kl}$ with $k, l$ in the respective range. We proceed similarly to case (ii). Let $\tilde{V_a}$ and $\tilde{W_a}$ be $(3\times 3)$–matrices. Their blockform is given by

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We obtain from (179) that

$$\begin{pmatrix} 0 & 0 & q^j_{13} \\ 0 & 0 & q^j_{23} \\ 0 & 0 & q^j_{33} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ p^i_{31} & p^i_{32} & p^i_{33} \end{pmatrix}.$$

It follows that $q^j_{33} = p^i_{33}$ and that $P^i$ is of form

$$P^i = \begin{pmatrix} p^i_{11} & p^i_{12} & p^i_{13} \\ p^i_{21} & p^i_{22} & p^i_{23} \\ 0 & 0 & p^i_{33} \end{pmatrix}. \tag{180}$$

Let $\tilde{V}_a$ and $\tilde{W}_a$ be given by $(2\times 3)$–matrices. They are of block form

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The equality (179) results in

$$\begin{pmatrix} 0 & 0 & q^j_{12} \\ 0 & 0 & q^j_{22} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ p^i_{31} & p^i_{32} & p^i_{33} \end{pmatrix}.$$

This equality yields that $q^j_{22} = p^i_{33}$ and that $P^i$ is of same block form as in (180).
Let $\tilde{V}_a$ and $\tilde{W}_a$ be given by $(3\times 2)$–matrices. Then they have the following block form:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

We obtain by (179) that

$$\begin{pmatrix} 0 & q^j_{13} \\ 0 & q^j_{23} \\ 0 & q^j_{33} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ p^i_{21} & p^i_{22} \end{pmatrix}.$$

It follows that $q^j_{33} = p^i_{22}$ and that $P^i$ is given by

$$P^i = \begin{pmatrix} p^i_{11} & p^i_{12} \\ 0 & p^i_{22} \end{pmatrix}. \tag{181}$$

Let $\tilde{V}_a$ and $\tilde{W}_a$ be given by a $(2\times 2)$–block matrix of the form

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

The equality in (179) results in

$$\begin{pmatrix} 0 & q^j_{12} \\ 0 & q^j_{22} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ p^i_{21} & p^i_{22} \end{pmatrix}.$$

Thus, we have that $q^j_{22} = p^i_{22}$ and $P^i$ has block form as in (181).

(iv) Let $V_a = U^j \tilde{V}_a$, $W_a = X^j \tilde{W}_a$. We have that $\tilde{V}_a : U_{\mathfrak{R}_i} \longrightarrow U_{\mathfrak{C}_j}$ and $\tilde{W}_a : X_{\mathfrak{R}_i} \longrightarrow X_{\mathfrak{C}_j}$. We rewrite (175) to $Q^j U^j \tilde{V}_a = X^j \tilde{W}_a Q^i$. Applying the relation $Q^j U^j = X^j P^j$ to the left hand side, yields that (175) is equivalent to

$$P^j \tilde{V}_a = \tilde{W}_a Q^i. \tag{182}$$

Let $\tilde{V}_a$ and $\tilde{W}_a$ be given by $(3 \times 3)$–block matrices. They are of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The commutativity relation (182) results in

$$\begin{pmatrix} p^j_{11} & 0 & 0 \\ p^j_{21} & 0 & 0 \\ p^j_{31} & 0 & 0 \end{pmatrix} = \begin{pmatrix} q^i_{11} & q^i_{12} & q^i_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We obtain that $q^i_{11} = p^j_{11}$. Moreover, $Q^i$ is of the block form

$$Q^i = \begin{pmatrix} q^i_{11} & 0 & 0 \\ q^i_{21} & q^i_{22} & q^i_{23} \\ q^i_{31} & q^i_{32} & q^i_{33} \end{pmatrix}. \tag{183}$$

Let now $\tilde{V}_a$ and $\tilde{W}_a$ be given by $(2 \times 3)$–block matrices. Their block form is given by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We obtain by (182) the equality

$$\begin{pmatrix} p^j_{11} & 0 & 0 \\ p^j_{21} & 0 & 0 \end{pmatrix} = \begin{pmatrix} q^i_{11} & q^i_{12} & q^i_{13} \\ 0 & 0 & 0 \end{pmatrix}.$$

It follows that $p^j_{11} = q^i_{11}$ and that $Q^i$ is of the same block form as in (183).
Let $\tilde{V}_a$ and $\tilde{W}_a$ be given by the $(3 \times 2)$–block matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

We obtain by the commutativity relation (182) that

$$\begin{pmatrix} p^j_{11} & 0 \\ p^j_{21} & 0 \\ p^j_{31} & 0 \end{pmatrix} = \begin{pmatrix} q^i_{11} & q^i_{12} \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus, the blocks $q^i_{11}$ and $p^j_{11}$ coincide. We obtain also that $Q^i$ has block form

$$Q^i = \begin{pmatrix} q^i_{11} & 0 \\ q^i_{21} & q^i_{22} \end{pmatrix}. \tag{184}$$

Finally, let $\tilde{V}_a$ and $\tilde{W}_a$ be given by $(2 \times 2)$–block matrices of the form

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then (182) results in

$$\begin{pmatrix} p^j_{11} & 0 \\ p^j_{21} & 0 \end{pmatrix} = \begin{pmatrix} q^i_{11} & q^i_{12} \\ 0 & 0 \end{pmatrix}.$$

We obtain that $q^i_{11} = p^j_{11}$ and that $Q^i$ has block form as in (184).

(v) Let $V_a = U^j \tilde{V}_a \left(U^i\right)^{-1}$, $W_a = X^j \tilde{W}_a \left(X^i\right)^{-1}$. We obtain that $\tilde{V}_a : U_{\mathfrak{C}_i} \longrightarrow U_{\mathfrak{C}_j}$ and $\tilde{W}_a : X_{\mathfrak{C}_i} \longrightarrow X_{\mathfrak{C}_j}$. Furthermore, (175) can be rewritten to $Q^j U^j \tilde{V}_a \left(U^i\right)^{-1} = X^j \tilde{W}_a \left(X^i\right)^{-1} Q^i$. Applying the commutativity relation $Q^j U^j = X^j P^j$ to the left hand side, and $Q^i U^i = X^i P^i$ to the right hand side, yields that (175) is equivalent to

$$P^j \tilde{V}_a = \tilde{W}_a P^i. \tag{185}$$

Let $\tilde{V}_a$ and $\tilde{W}_a$ be given by $(3 \times 3)$–block matrices. They are of the form

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Inserting this matrix in the respective positions in (185) gives

$$\begin{pmatrix} 0 & 0 & p^j_{11} \\ 0 & 0 & p^j_{21} \\ 0 & 0 & p^j_{31} \end{pmatrix} = \begin{pmatrix} p^i_{31} & p^i_{32} & p^i_{33} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We obtain that $p^j_{11} = p^i_{33}$ and that

$$P^i = \begin{pmatrix} p^i_{11} & p^i_{12} & p^i_{13} \\ p^i_{21} & p^i_{22} & p^i_{23} \\ 0 & 0 & p^i_{33} \end{pmatrix}. \tag{186}$$

Let $\tilde{V}_a$ and $\tilde{W}_a$ be given by $(2 \times 3)$–block matrices. They are of the form

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus, (185) results in

$$\begin{pmatrix} 0 & 0 & p^j_{11} \\ 0 & 0 & p^j_{21} \end{pmatrix} = \begin{pmatrix} p^i_{31} & p^i_{32} & p^i_{33} \\ 0 & 0 & 0 \end{pmatrix}.$$

We obtain that $p^j_{11} = p^i_{32}$ and that $P^i$ is given by a block form as in (186).
Let $\tilde{V}_a$ and $\tilde{W}_a$ be given by $(3 \times 2)$–block matrices:

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

We obtain by (185) the equality

$$\begin{pmatrix} 0 & p^j_{11} \\ 0 & p^j_{21} \\ 0 & p^j_{31} \end{pmatrix} = \begin{pmatrix} p^i_{21} & p^i_{22} \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

It follows that $p^j_{11} = p^i_{22}$ and that the block form of $P^i$ is given by

$$P^i = \begin{pmatrix} p^i_{11} & p^i_{12} \\ 0 & p^i_{22} \end{pmatrix}. \tag{187}$$

Let $\tilde{V}_a$ and $\tilde{W}_a$ be given by block matrices of block sizes $2 \times 2$:

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

The equality (185) results in

$$\begin{pmatrix} 0 & p^j_{11} \\ 0 & p^j_{21} \end{pmatrix} = \begin{pmatrix} p^i_{21} & p^i_{22} \\ 0 & 0 \end{pmatrix}.$$

We obtain that $p^j_{11} = p^i_{22}$ and that $P^i$ is of block form as in (187).

(B) Commutativity relations on vertex $i$ with respect to $b : k \longrightarrow i \in Q_1^{\text{ord}}$. This case is analogous to case (A). Note in particular that $\tilde{V}_b$ is given by the same block forms as $\tilde{V}_a$ in case (A). Due to the analogy, we only give the results of the calculations:

(ii) We consider $V_b = \tilde{V}_b$, $W_b = \tilde{W}_b$. We obtain for $\tilde{V}_b$ and $\tilde{W}_b$ both given by a $(3 \times 3)-$ or $(3 \times 2)-$block matrix, that $Q^i$ is of the following block form:

$$Q^i = \begin{pmatrix} q_{11}^i & q_{12}^i & 0 \\ q_{21}^i & q_{22}^i & 0 \\ q_{31}^i & q_{32}^i & q_{33}^i \end{pmatrix}.$$

Furthermore, we get in both cases that $q_{33}^i = q_{11}^k$. Let $\tilde{V}_b$ and $\tilde{W}_b$ be both given by a $(2 \times 3)-$ or $(2 \times 2)-$ block matrix. Then we obtain that

$$Q^i = \begin{pmatrix} q_{11}^i & 0 \\ q_{21}^i & q_{22}^i \end{pmatrix}$$

and the equality $q_{22}^i = q_{11}^k$.

(iii) Let $V_b = \tilde{V}_b \left(U^k\right)^{-1}$, $W_b = \tilde{W}_b \left(X^k\right)^{-1}$. Let $\tilde{V}_b$ and $\tilde{W}_b$ be both given by a $(3 \times 3)-$ or $(3 \times 2)-$block matrix. We obtain the following block form of $Q^i$:

$$Q^i = \begin{pmatrix} q_{11}^i & q_{12}^i & 0 \\ q_{21}^i & q_{22}^i & 0 \\ q_{31}^i & q_{32}^i & q_{33}^i \end{pmatrix}.$$

If $\tilde{V}_b$ and $\tilde{W}_b$ are both given by $(3 \times 3)-$block matrices, we have that $q_{33}^i = p_{33}^k$. If $\tilde{V}_b$ and $\tilde{W}_b$ are of size $3 \times 2$, we get that $q_{33}^i = p_{22}^k$. If $\tilde{V}_b$ and $\tilde{W}_b$ are both given by a $(2 \times 3)-$ or $(2 \times 2)-$block matrix, then $Q^i$ is of the following block form:

$$Q^i = \begin{pmatrix} q_{11}^i & 0 \\ q_{21}^i & q_{22}^i \end{pmatrix}.$$

Moreover, we obtain that $q_{22}^i = p_{33}^k$ if $\tilde{V}_b$ and $\tilde{W}_b$ are $(2 \times 3)-$block matrices, and that $q_{22}^i = p_{22}^k$ if $\tilde{V}_b$ and $\tilde{W}_b$ are $(2 \times 2)-$matrices.

(iv) We consider $V_b = U^i \tilde{V}_b$ and $W_b = X^i \tilde{W}_b$. Let $\tilde{V}_b$, $\tilde{W}_b$ be both given by a $(3 \times 3)-$ or $(3 \times 2)-$matrix. Then $P^i$ is given by

$$P^i = \begin{pmatrix} p_{11}^i & p_{12}^i & p_{13}^i \\ 0 & p_{22}^i & p_{23}^i \\ 0 & p_{32}^i & p_{33}^i \end{pmatrix}.$$

We obtain also that $p^i_{11} = q^k_{11}$ in both cases.
Let now $\tilde{V}_b$ and $\tilde{W}_b$ both be given by a $(2 \times 3)-$ or $(2 \times 2)-$block matrix. Then we obtain the following block form of $P^i$:

$$P^i = \begin{pmatrix} p^i_{11} & p^i_{12} \\ 0 & p^i_{22} \end{pmatrix}.$$

Furthermore, we get again the equality $p^i_{11} = q^k_{11}$ in both cases.

(v) Let $V_b = U^i \tilde{V}_b \left(U^k\right)^{-1}$, $W_b = X^i \tilde{W}_b \left(X^k\right)^{-1}$. We obtain for $\tilde{V}_b$, $\tilde{W}_b$ both given by $(3 \times 3)-$ or $(3 \times 2)-$block matrices that

$$P^i = \begin{pmatrix} p^i_{11} & p^i_{12} & p^i_{13} \\ 0 & p^i_{22} & p^i_{23} \\ 0 & p^i_{32} & p^i_{33} \end{pmatrix}.$$

Additionally, we obtain the equality $p^i_{11} = p^k_{33}$ in the $(3 \times 3)-$case, and $p^i_{11} = p^k_{22}$ in the $(3 \times 2)-$case.
Let $\tilde{V}_b$, $\tilde{W}_b$ be given by $(2 \times 3)-$ or $(2 \times 2)-$block matrices. Then the block form of $P^i$ is given by

$$P^i = \begin{pmatrix} p^i_{11} & p^i_{12} \\ 0 & p^i_{22} \end{pmatrix}.$$

Moreover, we obtain that $p^i_{11} = p^k_{33}$ if $\tilde{V}_b$, $\tilde{W}_b$ are of size $(2 \times 3)$, and $p^i_{11} = p^k_{22}$ if $\tilde{V}_b$, $\tilde{W}_b$ are of size $(2 \times 2)$.

(C) Commutativity relations on vertex $i$ with respect to $c : l \longrightarrow i \in Q^{\text{ord}}_1$. We only consider $V_c$, $W_c$ in the cases (7) and (9). Thus, we know that $\tilde{V}_c$ terminates in $V_i$ with basis given according to the filtration $0 \subset \text{im}(c) \subset \ker(a) \subset V_i$. Thus, the terminating vector space of $\tilde{V}_c$ coincides with the starting vector space of $\tilde{V}_a$. It follows that $\tilde{V}_c$ is given by a $(3 \times 3)-$ or $(3 \times 2)-$block matrix. The same follows analogously for $\tilde{W}_c$. Proceeding anlaogously to (A) results in the following forms of $Q^i$ and $P^i$ (for both choices of $\tilde{V}_c$ and $\tilde{W}_c$):

(ii) Let $V_c = \tilde{V}_c$, $W_c = \tilde{W}_c$. We obtain

$$Q^i = \begin{pmatrix} q^i_{11} & q^i_{12} & 0 \\ q^i_{21} & q^i_{22} & 0 \\ q^i_{31} & q^i_{32} & q^i_{33} \end{pmatrix} \tag{188}$$

and the equality $q^i_{33} = q^l_{11}$.

(iii) Let $V_c = \tilde{V}_c \left(U^l\right)^{-1}$, $W_c = \tilde{W}_c \left(X^l\right)^{-1}$. Then $Q^i$ is of the block form given in (188). Moreover, we obtain that $q^i_{33} = p^l_{33}$ for the $(3\times3)-$block matrix $\tilde{V}_b$, and $q^i_{33} = p^l_{22}$ for the $(3\times2)-$block matrix.

(iv) Let $V_c = U^i \tilde{V}_c$, $W_c = X^i \tilde{W}_c$. Then we have that

$$P^i = \begin{pmatrix} p^i_{11} & p^i_{12} & p^i_{13} \\ 0 & p^i_{22} & p^i_{23} \\ 0 & p^i_{32} & p^i_{33} \end{pmatrix}. \tag{189}$$

It also follows that $p^i_{11} = q^l_{11}$.

(v) Let $V_c = U^i \tilde{V}_c \left(U^l\right)^{-1}$, $W_c = X^i \tilde{W}_c \left(X^l\right)^{-1}$. We obtain that $P^i$ is of the block form (189). If $\tilde{V}_b$, $\tilde{W}_b$ are given by a $(3 \times 3)$–matrix, we have that $p^i_{11} = p^l_{33}$. In the other case we get that $p^i_{11} = p^l_{22}$.

(D) Commutativity relations on vertex $i$ with respect to $d : i \longrightarrow p \in Q_1^{\mathrm{ord}}$. We consider $V_d$ and $W_d$ in the cases (8) and (9). Thus, $\tilde{V}_d$ starts in $V_i$ with basis according to the filtration $0 \subset \mathrm{im}(b) \subset \ker(d) \subset V_i$. It follows that $\tilde{V}_b$ terminates in the same vector space in which $\tilde{V}_d$ starts. Because of this, $\tilde{V}_d$ is given by a $(2 \times 3)$– or $(3 \times 3)$–block matrix. We obtain similar results for $\tilde{W}_d$. Proceeding analogously to (A) results in the following block matrices for $Q^i$ and $P^i$ (for both choices of $\tilde{V}_d$, $\tilde{W}_d$ in each case):

(ii) We consider $V_d = \tilde{V}_d$, $W_d = \tilde{W}_d$. Then $Q^i$ is given by

$$Q^i = \begin{pmatrix} q^i_{11} & 0 & 0 \\ q^i_{21} & q^i_{22} & q^i_{23} \\ q^i_{31} & q^i_{32} & q^i_{33} \end{pmatrix}. \tag{190}$$

We obtain that $q^i_{11} = q^p_{33}$ if $\tilde{V}_b$, $\tilde{W}_b$ are of size $(3 \times 3)$. If $\tilde{V}_b$, $\tilde{W}_b$ are $(2 \times 3)$–matrices, the respective equality is given by $q^i_{11} = q^p_{22}$.

(iii) Let $V_d = \tilde{V}_d \left(U^i\right)^{-1}$, $W_d = \tilde{W}_d \left(X^i\right)^{-1}$. We obtain

$$P^i = \begin{pmatrix} q^i_{11} & q^i_{12} & q^i_{13} \\ q^i_{21} & q^i_{22} & q^i_{23} \\ 0 & 0 & q^i_{33} \end{pmatrix} \tag{191}$$

and the equality $p^i_{33} = q^p_{33}$ if $\tilde{V}_b$, $\tilde{W}_b$ are given by square matrices, and $p^i_{33} = q^p_{22}$ if $\tilde{V}_b$, $\tilde{W}_b$ are non-square.

(iv) We consider $V_d = U^p \tilde{V}_d$, $W_d = X^p \tilde{W}_d$. Then $Q^i$ has the same block form as in (190). Furthermore, it follows that $p^p_{11} = q^i_{11}$ in both cases.

(v) Let $V_d = U^p \tilde{V}_d \left(U^i\right)^{-1}$, $W_d = X^p \tilde{W}_d \left(X^i\right)^{-1}$. Then $P^i$ has block form as in (191) and $p^p_{11} = p^i_{33}$.

With the information gathered in $(\Sigma)$, (A) - (D), we are now able to analyse the shape and correspondences within blocks of $Q^i$ and $P^i$ in all ten cases. We want the following properties:

- $Q^iU^i = W^iP^i$
- $Q^i$ is of lower triangular block form
- $P^i$ is of upper triangular block form (diagonal form with respect to $\varepsilon$)
- Let $X$ denote the link corresponding to the basis of the image of $a$ ($b$, $c$, $d$, respectively) and let $Y$ denote the link corresponding to a basis of the starting vector space without the basis of the kernel of $a$ ($b$, $c$, $d$, respectively). We know that $\sigma_\Lambda(X) = Y$. Then the respective blocks in $Q^i$, $P^i$ and $Q^j$, $P^j$ ($Q^k$, $P^k$, $Q^l$, $P^l$, $Q^p$, $P^p$, respectivley) coincide.

The first property follows by the definition of $P^i$ in (171). The last property is given by the block equalities named in (A) - (D), (ii) - (v). We deduce the shapes of $Q^i$ and $P^i$ in each of the cases (1) - (10) from the information given by the computations (A) - (D) and ($\Sigma$):

(1) In this case, $\varepsilon$ is the only arrow incident to the vertex $i$. It follows from (174) in ($\Sigma$) that $P^i$ has the required diagonal block form. By construction, $\varepsilon$ determines the basis of $U_{\mathfrak{C}_i}$. The basis of $U_{\mathfrak{R}_i}$ is given by the standard basis which corresponds to the only link of the semichain $\mathfrak{R}_i$. Thus, $Q^i$ is given by a $(1)$–block matrix and we are done.

(2) In this case, we either have that $\tilde{V}_a$ and $\tilde{W}_a$ are both given by a $(2\times 2)$– or a $(3\times 2)$–matrix. It follows from (A) that we obtain for the cases (ii) and (iv) that $Q^i$ is of the form $Q^i = \begin{pmatrix} q^i_{11} & 0 \\ q^i_{21} & q^i_{22} \end{pmatrix}$. The second filtration on the vertex $i$ is the standard filtration. It follows that the $P^i$ is a $(1\times 1)$–matrix and thus fulfills the requirements.
Similarly, we have for the cases (iii) and (v) that $P^i = \begin{pmatrix} p^i_{11} & 0 \\ 0 & p^i_{22} \end{pmatrix}$, and that $Q^i$ is a $(1\times 1)$–matrix.

(3) This case is analogous to case (2).

(4) The starting and terminating vector space of $\tilde{V}_\varepsilon$ are given by $U_{\mathfrak{C}_i}$. The starting vector space of $\tilde{V}_a$ differs from it and is given by $U_{\mathfrak{R}_i}$. It follows that we only combine (A) (ii) and (iv) with ($\Sigma$):

| (A) | give | ($\Sigma$) | gives | results in |
|---|---|---|---|---|
| (ii)/(iv) | $q^i_{12} = 0$ | | $q^i_{12} = q^i_{21} = 0$ | $Q^i = \begin{pmatrix} q^i_{11} & 0 \\ q^i_{21} & q^i_{22} \end{pmatrix}$, $P^i = \begin{pmatrix} p^i_{11} & 0 \\ 0 & p^i_{22} \end{pmatrix}$. |

We see that $Q^i$ and $P^i$ satisfy the required conditions on their block form.

(5) The terminating vector space of $\tilde{V}_b$ is given by $U_{\mathfrak{R}_i}$, and the same property holds for $\tilde{W}_b$. Thus, we combine (B) (ii) and (iii) with $(\Sigma)$ and obtain analogously to (4) that $Q^i$ and $P^i$ are of the required from.

(6) Note that the terminating vector space of $\tilde{V}_b$ is given in a different basis than the starting vector space of $\tilde{V}_a$. Thus, we can have the following combinations of (A) and (B):

| | | | | |
|---|---|---|---|---|
| (A) | (ii) | with | (B) | (iv),(v) |
| (A) | (iii) | with | (B) | (ii),(iii) |
| (A) | (iv) | with | (B) | (iv),(v) |
| (A) | (v) | with | (B) | (ii),(iii). |

Recall that $\tilde{V}_a$ starts in $U_{\mathfrak{C}_i}$ in the cases (iii) and (v). It starts in $U_{\mathfrak{R}_i}$ in the cases (ii) and (iv). The terminating vector spaces are given for $\tilde{V}_b$ similarly: in the cases (iv) and (v) it is given by $U_{\mathfrak{C}_i}$ and in (ii) and (iii) by $U_{\mathfrak{R}_i}$. Summing up the results gives

| (A) | give(s) | (B) | give(s) | results in |
|---|---|---|---|---|
| (ii)/(iv) | $q^i_{12} = 0$ | (iv)/(v) | $p^i_{21} = 0$ | $Q^i = \begin{pmatrix} q^i_{11} & 0 \\ q^i_{21} & q^i_{22} \end{pmatrix}$, $P^i = \begin{pmatrix} p^i_{11} & p^i_{12} \\ 0 & p^i_{22} \end{pmatrix}$ |
| (iii)/(v) | $p^i_{21} = 0$ | (ii)/(iii) | $q^i_{12} = 0$ | $Q^i = \begin{pmatrix} q^i_{11} & 0 \\ q^i_{21} & q^i_{22} \end{pmatrix}$, $P^i = \begin{pmatrix} p^i_{11} & p^i_{12} \\ 0 & p^i_{22} \end{pmatrix}$ |

We can see in the table above that $Q^i$ and $P^i$ have in each combination the required form.

(7) We have in contrast to (6) that the terminating vector space of $\tilde{V}_b$ and the starting vector space of $\tilde{V}_a$ coincide. Note that $\tilde{V}_a$ is given by a $(2 \times 3)$– or $(3 \times 3)$–block matrix. The map $\tilde{V}_b$ is given by a $(3 \times 2)$– or $(3 \times 3)$–block matrix. This allows the following combinations of (A) and (B) and gives the respective information:

| (A) | give(s) | (B) | give(s) |
|---|---|---|---|
| (ii)/(iv) | $q^i_{12} = q^i_{13} = 0$ | (ii)/(iii) | $q^i_{13} = q^i_{23} = 0$ |
| (iii)/(v) | $p^i_{31} = p^i_{32} = 0$ | (iv)/(v) | $p^i_{21} = p^i_{31} = 0.$ |

The vector space whose basis is not determined by the filtration $0 \subset \mathrm{im}(b) \subset \ker(a) \subset V_i$, is given by the standard basis. The map inheriting its block structure from this vector space is given by a $(1 \times 1)$–matrix.

Summarising, we obtain the following block forms:

$$\text{(A) (ii) and (iv):} \quad Q^i = \begin{pmatrix} q^i_{11} & 0 & 0 \\ q^i_{21} & q^i_{22} & 0 \\ q^i_{31} & q^i_{32} & q^i_{33} \end{pmatrix}, \; P^i = \begin{pmatrix} p^i_{11} \end{pmatrix}$$

$$\text{(A) (iii) and (v):} \quad Q^i = \begin{pmatrix} q^i_{11} \end{pmatrix}, \; P^i = \begin{pmatrix} p^i_{11} & p^i_{12} & p^i_{13} \\ 0 & p^i_{22} & p^i_{23} \\ 0 & 0 & p^i_{33} \end{pmatrix}.$$

(8) This case is analogous to the cases (4) and (7) combined. This implies that we consider the combinations from (7) for (A) (ii) and (iv). We obtain for all those combinations that

$$Q^i = \begin{pmatrix} q^i_{11} & 0 & 0 \\ q^i_{21} & q^i_{22} & 0 \\ q^i_{31} & q^i_{32} & q^i_{33} \end{pmatrix}.$$

In contrast to (7), the matrix $P^i$ is not given by one block, but is determined by $\tilde{V}_\varepsilon$ as in (3):

$$P^i = \begin{pmatrix} p^i_{11} & 0 \\ 0 & p^i_{22} \end{pmatrix}.$$

(9) Note that we have the same combinations of (A) and (B) as in (6). The terminating vector space of $\tilde{V}_c$ coincides with the starting vector space of $\tilde{V}_a$. We obtain for (A) and (C) the following possible combinations:

| | | | | |
|---|---|---|---|---|
| (A) | (ii) | with | (C) | (ii),(iii) |
| (A) | (iii) | with | (C) | (iv), (v) |
| (A) | (iv) | with | (C) | (ii),(iii) |
| (A) | (v) | with | (C) | (iv), (v). |

We sum up the information in the following table:

| (A) | give(s) | (B) | give(s) | (C) | give(s) |
|---|---|---|---|---|---|
| (ii)/(iv) | $q^i_{12} = q^i_{13} = 0$ | (iv)/(iii) | $p^i_{21} = q^i_{23} = 0$ | (ii)/(iii) | $q^i_{13} = 0$ |
| (iii)/(v) | $p^i_{31} = p^i_{32} = 0$ | (ii)/(iii) | $q^i_{12} = 0$ | (iv)/(v) | $p^i_{21} = p^i_{31} = 0$ |

We obtain the block forms

$$\text{(A) (ii) and (iv):} \quad Q^i = \begin{pmatrix} q^i_{11} & 0 & 0 \\ q^i_{21} & q^i_{22} & 0 \\ q^i_{31} & q^i_{32} & q^i_{33} \end{pmatrix}, P^i = \begin{pmatrix} p^i_{11} & p^i_{12} \\ 0 & p^i_{22} \end{pmatrix} \qquad (192)$$

$$\text{(A) (iii) and (v):} \quad Q^i = \begin{pmatrix} q^i_{11} & 0 \\ q^i_{21} & q^i_{22} \end{pmatrix}, P^i = \begin{pmatrix} p^i_{11} & p^i_{12} & p^i_{13} \\ 0 & p^i_{22} & p^i_{23} \\ 0 & 0 & p^i_{33} \end{pmatrix} \qquad (193)$$

which are of the wanted form.

(10) We consider the same combinations of (A) and (B) as in (6). The following combinations are allowed for (B) and (D):

| | | | | |
|---|---|---|---|---|
| (B) | (ii) | with | (D) | (ii),(iv) |
| (B) | (iii) | with | (D) | (ii), (iv) |
| (B) | (iv) | with | (D) | (iii),(v) |
| (B) | (v) | with | (D) | (iii), (v). |

We sum up the information obtained from the commutativities of the respective possible combinations:

| (A) | give(s) | (B) | give(s) | (D) | give(s) |
|---|---|---|---|---|---|
| (ii)/(iv) | $q^i_{21} = 0$ | (iv)/(v) | $p^i_{21} = p^i_{31} = 0$ | (iii)/(v) | $p^i_{31} = p^i_{32} = 0$ |
| (iii)/(v) | $p^i_{21} = 0$ | (ii)/(iii) | $q^i_{13} = q^i_{23} = 0$ | (ii)/(iv) | $q^i_{12} = q^i_{13} = 0$ |

We obtain the block forms

$$\text{(A) (ii) and (iv):} \quad Q^i = \begin{pmatrix} q^i_{11} & 0 \\ q^i_{21} & q^i_{22} \end{pmatrix}, \; P^i = \begin{pmatrix} p^i_{11} & p^i_{12} & p^i_{13} \\ 0 & p^i_{22} & p^i_{23} \\ 0 & 0 & p^i_{33} \end{pmatrix}$$

$$\text{(A) (iii) and (v):} \quad Q^i = \begin{pmatrix} q^i_{11} & 0 & 0 \\ q^i_{21} & q^i_{22} & 0 \\ q^i_{31} & q^i_{32} & q^i_{33} \end{pmatrix}, \; P^i = \begin{pmatrix} p^i_{11} & p^i_{12} \\ 0 & p^i_{22} \end{pmatrix}.$$

The block matrix $Q^i$ is of lower triangular and $P^i$ of upper triangular form. Thus, they satisfy the condition.

(11) This case is a combination of the cases (9) (with respect to the arrows $c$ and $a$) and (10) (with respect to the arrows $b$ and $d$). Thus, we consider the same combinations of (A) and (C) ((B) and (D)) as in (9) ((10)). We sum up the information obtained from those combinations:

| (A) | give(s) | (B) | give(s) | (C) | give(s) | (D) | give(s) |
|---|---|---|---|---|---|---|---|
| (ii)/(iv) | $q^i_{12} = 0$ <br> $q^i_{13} = 0$ | (iv)/(v) | $p^i_{21} = 0$ <br> $p^i_{31} = 0$ | (ii)/(iii) | $q^i_{13} = 0$ <br> $q^i_{23} = 0$ | (iii)/(v) | $p^i_{31} = 0$ <br> $p^i_{32} = 0$ |
| (iii)/(v) | $p^i_{31} = 0$ <br> $p^i_{32} = 0$ | (ii)/(iii) | $q^i_{13} = 0$ <br> $q^i_{23} = 0$ | (iv)/(v) | $p^i_{21} = 0$ <br> $p^i_{31} = 0$ | (ii)/(iv) | $q^i_{12} = 0$ <br> $q^i_{13} = 0$ |

In combination, this data results for (A) (ii)-(v) in the following block forms of $P^i$ and $Q^i$:

$$Q^i = \begin{pmatrix} q^i_{11} & 0 & 0 \\ q^i_{21} & q^i_{22} & 0 \\ q^i_{31} & q^i_{32} & q^i_{33} \end{pmatrix}, \quad P^i \begin{pmatrix} p^i_{11} & p^i_{12} & p^i_{13} \\ 0 & p^i_{22} & p^i_{23} \\ 0 & 0 & q^i_{33} \end{pmatrix}.$$

We see that they are both of the required lower and respectively upper triangular block form.

## 5.3 On the computation of inverses

Let $U$ be a $\overline{\mathfrak{X}}_0$–representation and let $U^i$ be one of the linear maps of $U$, $i \in \{1, \ldots, n\}$. In this section, we describe how to obtain the inverse of $U^i$. Recall that $U^i$ is invertible by construction.

We observe at first that any row and any column band of $U^i$ consists of maximal two non-zero block entries and at least one non-zero block entry. Recall that we only obtain elementary subchains of length 2 or 3. Any link $x$ ($y$) which is not contained in an elementary subchain of length 3 thus gives a row (column) consisting of exactly one non-zero entry. Let $x$ and $y$ be contained in the elementary subchain $e_{x,y}(g)$ of length 2. We have that

$$U^i(x,z) = \begin{cases} 1 & \text{if } z = y \text{ and } e_{x,y}(g) \neq e_2(g), \\ F_\varphi & \text{if } z = y \text{ and } e_{x,y}(g) = e_2(g), \\ 0 & \text{else,} \end{cases}$$

$$U^i(z,y) = \begin{cases} 1 & \text{if } z = x \text{ and } e_{x,y}(g) \neq e_2(g), \\ F_\varphi & \text{if } z = x \text{ and } e_{x,y}(g) = e_2(g), \\ 0 & \text{else.} \end{cases}$$

For any elementary subchain of length 3, we consider several entries in $U^i$. Any elementary subchain of the form

$$x - \overleftarrow{y \sim z} - a$$

with $y, z \in \mathfrak{C}_i$, $x, a \in \mathfrak{R}_i$, results in the following entries in $U^i$:

$$\begin{array}{c|cc} & y & z \\ \hline x & 1 & 1 \\ a & 0 & 1 \end{array} \quad \text{or} \quad \begin{array}{c|cc} & y & z \\ \hline a & 0 & 1 \\ x & 1 & 1 \end{array} \tag{194}$$

An elementary subchain of the form

$$w - \overrightarrow{x \sim y} - z$$

with $x, y \in \mathfrak{C}_i$ and $w, z \in \mathfrak{R}_i$ gives the following entries in $U^i$:

$$\begin{array}{c|cc} & x & y \\ \hline w & 1 & 0 \\ z & 1 & 1 \end{array} \quad \text{or} \quad \begin{array}{c|cc} & x & y \\ \hline z & 1 & 1 \\ w & 1 & 0 \end{array} \tag{195}$$

In case of an $\mathfrak{L}$–cycle, the elementary subchain $e_2(g)$ can be involved. If $e_2(g) = z - a$, then $x - \overleftarrow{y \sim z} - a$ results in the following entries in $U^i$:

$$\begin{array}{c|cc} & y & z \\ \hline x & 1 & 1 \\ a & 0 & F_\varphi \end{array} \quad \text{or} \quad \begin{array}{c|cc} & y & z \\ \hline a & 0 & F_\varphi \\ x & 1 & 1 \end{array} \tag{196}$$

The elementary subchain $w - \overrightarrow{x \sim y} - z$ with $e_2(g) = w - x$ gives

$$\begin{array}{c|cc} & x & y \\ \hline w & F_\varphi & 0 \\ z & 1 & 1 \end{array} \quad \text{or} \quad \begin{array}{c|cc} & x & y \\ \hline z & 1 & 1 \\ w & F_\varphi & 0 \end{array} \tag{197}$$

Note that the entries described in (194) - (197) do not have to arise in one block as depicted here. The entries can be separated by several columns or rows. In order to compute the inverse of $U^i$, we can rearrange the columns and rows such that the non-zero entries are given by blocks as described above (i.e. the respective entries are not separated by zero entries). Those blocks lie on the diagonal. We denote the rearranged matrix by $\underline{U}^i$ and neglect any band structure for this matrix. In order to compute the inverse of $\underline{U}^i$, it is enough to compute the inverses of the respective blocks:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \tag{198}$$

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \tag{199}$$

$$\begin{pmatrix} 1 & 1 \\ 0 & F_\varphi \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -F_\varphi^{-1} \\ 0 & F_\varphi^{-1} \end{pmatrix}, \qquad \begin{pmatrix} 0 & F_\varphi \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -F_\varphi^{-1} & 1 \\ F_\varphi & 0 \end{pmatrix} \tag{200}$$

$$\begin{pmatrix} F_\varphi & 0 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} F_\varphi^{-1} & 0 \\ -F_\varphi^{-1} & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 \\ F_\varphi & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & F_\varphi^{-1} \\ 1 & -F_\varphi^{-1} \end{pmatrix}. \tag{201}$$

Rearranging the rows and columns of $\left(\underline{U}^i\right)^{-1}$ back to the order given in $U^i$, we obtain $\left(U^i\right)^{-1}$ with the band structure deduced from $U^i$.

**Example 5.7.** *Let $\Lambda$ be given as in Example 2.3.1. by the quiver*

$$\varepsilon \circlearrowleft 1 \circlearrowright a$$

*with* $\mathrm{Sp} = \{\varepsilon\}$, $\mathrm{R} = \{a^2\}$.

*(i) Let $w = \varepsilon^* a \varepsilon^*$ be an asymmetric string. Its corrsponding $\mathfrak{L}$−chain is given by*

$$\begin{array}{ccccccccccccccccc} g_w: & \mathfrak{R}_{12} & - & \mathfrak{C}_{\varepsilon*} & \sim & \mathfrak{C}_{\varepsilon*} & - & \mathfrak{R}_{11} & \sim & \mathfrak{R}_{13} & - & \mathfrak{C}_{\varepsilon^*} & \sim & \mathfrak{C}_{\varepsilon*} & - & \mathfrak{R}_{12} \\ & x_0 & & x_1 & & x_2 & & x_3 & & x_4 & & x_5 & & x_6 & & x_7 \end{array}$$

with $\overleftarrow{x_1 \sim x_2}$ and $\overleftarrow{x_5 \sim x_6}$.

*The $\overline{\mathfrak{X}}_0$−representation $U(g_w)$ has one matrix $U^1$. It is given by*

$$U^1 = \begin{array}{c|cc|cc|} & x_1 & x_5 & x_2 & x_6 \\ \hline x_4 & 0 & 1 & 0 & 1 \\ \hline x_0 & 1 & 0 & 1 & 0 \\ x_7 & 0 & 0 & 0 & 1 \\ \hline x_3 & 0 & 0 & 1 & 0 \\ \hline \end{array}$$

*We consider the following $\underline{U}^1$:*

$$\underline{U}^1 = \begin{array}{c|cccc|} & x_1 & x_2 & x_5 & x_6 \\ \hline x_0 & 1 & 1 & 0 & 0 \\ x_3 & 0 & 1 & 0 & 0 \\ x_7 & 0 & 0 & 0 & 1 \\ x_4 & 0 & 0 & 1 & 1 \\ \hline \end{array}$$

*Applying (198) to the respective blocks yields*

$$\left(\underline{U}^1\right)^{-1} = \begin{array}{c|cccc|} & x_0 & x_3 & x_7 & x_4 \\ \hline x_1 & 1 & -1 & 0 & 0 \\ x_2 & 0 & 1 & 0 & 0 \\ x_5 & 0 & 0 & -1 & 1 \\ x_4 & 0 & 0 & 1 & 0 \\ \hline \end{array}$$

*We obtain be reodering of the columns and rows:*

$$U^1 = \begin{array}{c|c|cc|c|} & x_4 & x_0 & x_7 & x_3 \\ \hline x_1 & 0 & 1 & 0 & -1 \\ x_5 & 1 & 0 & -1 & 0 \\ \hline x_2 & 0 & 0 & 0 & 1 \\ x_6 & 0 & 0 & 1 & 0 \\ \hline \end{array}$$

(ii) *Let $w_{\mathbb{Z}}$ be a symmetric band with $\hat{w}_p = \varepsilon^* a \varepsilon^* a^{-1}$. Its corresponding $\mathfrak{L}$–cycle is given by*

$$g_{w_{\mathbb{Z}}}: \overset{x_0}{\mathfrak{C}_{\varepsilon*}} \sim \overset{x_1}{\mathfrak{C}_{\varepsilon*}} - \overset{x_2}{\mathfrak{R}_{11}} \sim \overset{x_3}{\mathfrak{R}_{13}} - \overset{x_4}{\mathfrak{C}_{\varepsilon^*}} \sim \overset{x_5}{\mathfrak{C}_{\varepsilon*}} - \overset{x_6}{\mathfrak{R}_{13}} \sim \overset{x_7}{\mathfrak{R}_{11}}$$

*We obtain*

$$U^1 = \begin{array}{c|cc|cc|} & x_0 & x_4 & x_1 & x_5 \\ \hline x_3 & 0 & 1 & 0 & 1 \\ x_6 & 0 & 0 & 0 & 1 \\ \hline x_2 & 0 & 0 & F_\varphi & 0 \\ x_7 & 1 & 0 & 1 & 0 \\ \hline \end{array}$$

*We rearrange as follows:*

$$\underline{U}^1 = \begin{array}{c|cccc|} & x_5 & x_4 & x_1 & x_0 \\ \hline x_3 & 1 & 1 & 0 & 0 \\ x_6 & 1 & 0 & 0 & 0 \\ x_2 & 0 & 0 & F_\varphi & 0 \\ x_7 & 0 & 0 & 1 & 1 \\ \hline \end{array}$$

*and compute its inverse according to (199) and (201):*

$$\left(\underline{U}^1\right)^{-1} = \begin{array}{c|cccc|} & x_3 & x_6 & x_2 & x_7 \\ \hline x_5 & 0 & 1 & 0 & 0 \\ x_4 & 1 & -1 & 0 & 0 \\ x_1 & 0 & 0 & F_\varphi^{-1} & 0 \\ x_0 & 0 & 0 & -F_\varphi^{-1} & 1 \\ \hline \end{array}$$

*Arranging the rows and columns according to* $U^1$ *gives*

$$\left(U^1\right)^{-1} = \begin{array}{c|cc|cc|} & x_3 & x_6 & x_2 & x_7 \\ \hline x_0 & 0 & 0 & -F_\varphi^{-1} & 1 \\ x_4 & 1 & -1 & 0 & 0 \\ \hline x_1 & 0 & 0 & F_\varphi^{-1} & 0 \\ x_5 & 0 & 1 & 0 & 0 \\ \hline \end{array}$$

**Remark 5.8.** *Keep in mind that* $\underline{U}^i$ *must not always be unique. Consider again Example 5.7 (ii). In* $\underline{U}^1$*, we can also switch the columns* $x_5$ *and* $x_4$ *and the rows* $x_7$ *and* $x_2$*. Then we obtain the matrix*

$$\left(\underline{U}^1\right)' = \begin{array}{c|cccc|} & x_4 & x_5 & x_1 & x_0 \\ \hline x_3 & 1 & 1 & 0 & 0 \\ x_6 & 0 & 1 & 0 & 0 \\ x_7 & 0 & 0 & 1 & 1 \\ x_2 & 0 & 0 & F_\varphi & 0 \\ \hline \end{array}$$

*Its inverse is given by*

$$\left[\left(\underline{U}^1\right)'\right]^{-1} = \begin{array}{c|cccc|} & x_3 & x_6 & x_7 & x_2 \\ \hline x_4 & 1 & -1 & 0 & 0 \\ x_5 & 0 & 1 & 0 & 0 \\ x_1 & 0 & 0 & 0 & F_\varphi^{-1} \\ x_0 & 0 & 0 & 1 & -F_\varphi^{-1} \\ \hline \end{array}$$

*Rearranging its rows and columns back, we obtain the same* $\left(U^1\right)^{-1}$ *as in the example.*

## 5.4 The image of $F$ in terms of strings and bands

We analyse in this section the modules in the image of $F$ in terms of strings and bands as defined in Subsection 3.2. Recall that an $\alpha$–relation in an $\mathfrak{L}$–graph corresponds to a letter $w_i$. A $\beta$–relation indicates a change between the bases according to the two constructed filtrations of $V_i$. Thus, we are especially interested in the $\alpha$–relations of the $\mathfrak{L}$–graph. In particular, we want to examine the entries in the matrix which come from links incident to the $\alpha$–relations and see what happens to them under the action of $F$.
Keep in mind that we are going to compose the results in the notation of strings and bands and thus also change within the proofs to this notation at some point.

**Lemma 5.9.** *Let $g$ be an $\mathfrak{L}$–graph and denote by $U_s(g)$ ($U(g,V)$) a corresponding representation where $V$ is - in case of $g$ being a cycle - the vector space given according to some $\varphi$ in the construction, and some $s$ such that $\psi_s \in \Psi(g)$.*

*(i) Any subchain of the form $x_{i-1} - \overrightarrow{x_i \sim x_{i+1}} - x_{i+2}$ of $g$ that, in case of a cycle, does not contain $e_2(g)$, results in the following action in $F(U_s(g))$ ($F(U(g,V))$):*

$$x_{i-1} \xrightarrow{\bar{\varepsilon}=-1} x_{i+2} \text{ if } g \text{ is an } \mathfrak{L}-\text{chain with } \psi_s(x_i) = 1, \psi_s(x_{i+1}) = -1,$$
$$x_{i-1} \xrightarrow{\varepsilon=-1} x_{i+2} \text{ if } g \text{ is an } \mathfrak{L}-\text{chain with } \psi_s(x_i) = -1, \psi_s(x_{i+1}) = 1,$$
$$V_{i-1} \xrightarrow{\bar{\varepsilon}=-1} V_{i+2} \text{ if } g \text{ is an } \mathfrak{L}-\text{cycle, with } V_{i-1}, V_{i+2} \text{ disjoint copies of } V,$$

*(ii) Any subchain of the form $x_{i-1} - \overleftarrow{x_i \sim x_{i+1}} - x_{i+2}$ of $g$ that, in case of a cycle, does not contain $e_2(g)$, results in the following action in $F(U_s(g))$ ($F(U(g,V))$):*

$$x_{i-1} \xleftarrow{\varepsilon=-1} x_{i+2} \text{ if } g \text{ is an } \mathfrak{L}-\text{chain with } \psi_s(x_i) = 1, \psi_s(x_{i+1}) = -1,$$
$$x_{i-1} \xleftarrow{\bar{\varepsilon}=-1} x_{i+2} \text{ if } g \text{ is an } \mathfrak{L}-\text{chain with } \psi_s(x_i) = -1, \psi_s(x_{i+1}) = 1,$$
$$V_{i-1} \xleftarrow{\varepsilon=-1} V_{i+2} \text{ if } g \text{ is an } \mathfrak{L}-\text{cycle, with } V_{i-1}, V_{i+2} \text{ disjoint copies of } V$$

*Proof.* Recall that $V_\varepsilon = U^i \tilde{V}_\varepsilon \left(U^i\right)^{-1}$ where $i = s(\varepsilon)$ and $U^i$ belongs to $U_s(g)$ ($U(g,V)$). Denote by $M(x_i, x_j)$ the entry in a matrix $M$ in the row indexed by $x_i$ and in the column indexed by $x_j$. In case of an $\mathfrak{L}$–cycle, this entry is a block of size $\deg(\varphi) \times \deg(\varphi)$. We show the statement for $g$ an $\mathfrak{L}$–chain. The proof for $g$ an $\mathfrak{L}$–cycle is analaogous. Recall that any $\psi \in \Psi(g)$ is uniquely defined for $\mathfrak{L}$–cycles (Subsection 4.1.4).

(i) Let first $\psi_s(x_i) = 1$ and $\psi_s(x_{i+1}) = -1$. Note that there do not exist elementary subchains of type 4 for $\overline{\mathfrak{X}}_\Lambda$ (see Remark 4.73). Thus, we can determine the respective entries in $U^i$, $\left(U^i\right)^{-1}$ and $\tilde{V}_\varepsilon$ (see Section 5.3):

$$U^i(x_{i-1},x) = \begin{cases} 1 & \text{if } x = x_i, \\ 0 & \text{else}, \end{cases}$$
$$U^i(x_{i+2},x) = \begin{cases} 1 & \text{if } x \in \{x_i, x_{i+1}\}, \\ 0 & \text{else}, \end{cases}$$
$$U^i(x,x_i) = \begin{cases} 1 & \text{if } x \in \{x_{i-1}, x_{i+2}\}, \\ 0 & \text{else}, \end{cases}$$

$$U^i(x,x_{i+1}) = \begin{cases} 1 & \text{if } x = x_{i+2}, \\ 0 & \text{else}, \end{cases} \tag{202}$$
$$\tilde{V}_\varepsilon(x,y) = \begin{cases} 1 & \text{if } x = y \in P(\mathfrak{C}^+_{\varepsilon^*}), \\ 0 & \text{else}, \end{cases} \tag{203}$$
$$\left(U^i\right)^{-1}(x_i,x) = \begin{cases} 1 & \text{if } x = x_{i-1}, \\ 0 & \text{else}, \end{cases} \tag{204}$$
$$\left(U^i\right)^{-1}(x_{i+1},x) = \begin{cases} -1 & \text{if } x = x_{i-1}, \\ 1 & \text{if } x = x_{i+2}\}, \\ 0 & \text{else}, \end{cases} \tag{205}$$
$$\left(U^i\right)^{-1}(x,x_{i-1}) = \begin{cases} 1 & \text{if } x = x_i, \\ -1 & \text{if } x = x_{i-1}, \\ 0 & \text{else}, \end{cases} \tag{206}$$
$$\left(U^i\right)^{-1}(x,x_{i+2}) = \begin{cases} 1 & \text{if } x = x_{i+1}, \\ 0 & \text{else}. \end{cases} \tag{207}$$

We compute $\tilde{V}_\varepsilon \left(U^i\right)^{-1}$ and denote its entries by $(\cdot,\cdot)$:

$$(x_i, x) = \sum_l \tilde{V}_\varepsilon(x_i, l)\left(U^i\right)^{-1}(l, x) = \left(U^i\right)^{-1}(x_i, x) = \begin{cases} 1 & \text{if } x = x_{i-1}, \\ 0 & \text{else}, \end{cases} \tag{208}$$

$$(x_{i+1}, x_i) = \sum_l \tilde{V}_\varepsilon(x_{i+1}, l)\left(U^i\right)^{-1}(l, x) = 0, \tag{209}$$

$$(x, x_{i-1}) = \sum_l \tilde{V}_\varepsilon(x, l)\left(U^i\right)^{-1}(l, x_{i-1}) \tag{210}$$

$$= \tilde{V}_\varepsilon(x, x_i) - \tilde{V}_\varepsilon(x, x_{i+1}) = \begin{cases} 1 & \text{if } x = x_i \\ 0 & \text{else} \end{cases}, \tag{211}$$

$$(x, x_{i+2}) = \sum_l \tilde{V}_\varepsilon(x, l)\left(U^i\right)^{-1}(l, x_{i+2}) \tag{212}$$

$$= \tilde{V}_\varepsilon(x, x_{i+1}) = 0. \tag{213}$$

Applying (208)-(213), we compute $V_\varepsilon = U^i \tilde{V}_\varepsilon \left(U^i\right)^{-1}$:

$$V_\varepsilon(x_{i-1}, x) = \sum_l U^i(x_{i-1}, l)(l, x) = (x_i, x) = \begin{cases} 1 & \text{if } x = x_{i-1}, \\ 0 & \text{else}, \end{cases}$$

$$V_\varepsilon(x, x_{i-1}) = \sum_l U^i(x, l)(l, x_{i-1}) = \begin{cases} 1 & \text{if } x \in \{x_{i-1}, x_{i+2}\}, \\ 0 & \text{else}, \end{cases}$$

$$V_\varepsilon(x_{i+2}, x) = \sum_l U^i(x_{i+2}, l)(l, x)$$

$$= (x_i, x) + (x_{i+1}, x) = \begin{cases} 1 & \text{if } x = x_{i-1}, \\ 0 & \text{else}, \end{cases}$$

$$V_\varepsilon(x, x_{i+2}) = \sum_l U^i(x, l)(l, x_{i+2}) = 0.$$

We see that $\varepsilon$ acts as follows on the basis element $x_{i-1}$ of $V_i$:

$$x_{i-1} \xrightarrow{\varepsilon} x_{i-1} + x_{i+2}. \tag{214}$$

We can rewrite (214) to

$$x_{i-1} \xrightarrow{\bar{\varepsilon}} -x_{i+2}$$

which gives the statement.

Let now $\psi_((x_i) = -1$ and $\psi_((x_{i+1}) = 1$. Proceeding analogously to the first part, we obtain that

$$x_{i-1} \xrightarrow{\varepsilon} -x_{i+2}$$

(ii) The statement follows analogously to (i). For $\psi_s(x_i) = 1$ and $\psi_s(x_{i+1}) = -1$, the entries of interest in $U^i$ and $\left(U^i\right)^{-1}$ are given as follows:

$$U^i(x_{i-1}, x) = \begin{cases} 1 & \text{if } x \in \{x_i, x_{i+1}\}, \\ 0 & \text{else,} \end{cases}$$

$$U^i(x_{i+2}, x) = \begin{cases} 1 & \text{if } x = x_{i+1}, \\ 0 & \text{else,} \end{cases}$$

$$U^i(x, x_i) = \begin{cases} 1 & \text{if } x = x_{i-1}, \\ 0 & \text{else,} \end{cases}$$

$$U^i(x, x_{i+1}) = \begin{cases} 1 & \text{if } x \in \{x_{i-1}, x_{i+2}\}, \\ 0 & \text{else,} \end{cases}$$

$$\left(U^i\right)^{-1}(x_i, x) = \begin{cases} -1 & \text{if } x = x_{i+2}, \\ \;\;1 & \text{if } x = x_i, \\ 0 & \text{else,} \end{cases}$$

$$\left(U^i\right)^{-1}(x_{i+1}, x) = \begin{cases} 1 & \text{if } x = x_{i+2}\}, \\ 0 & \text{else,} \end{cases}$$

$$\left(U^i\right)^{-1}(x, x_{i-1}) = \begin{cases} 1 & \text{if } x = x_i, \\ 0 & \text{else,} \end{cases}$$

$$\left(U^i\right)^{-1}(x, x_{i+2}) = \begin{cases} -1 & \text{if } x = x_i, \\ \;\;1 & \text{if } x = x_{i+1}, \\ \;\;0 & \text{else.} \end{cases}$$

Computing $V_\varepsilon$ results in

$$x_{i+2} \xrightarrow{\varepsilon} -x_{i-1}.$$

Similarly, we obtain in the case $\psi_s(x_i) = -1$, $\psi_(x_{i+1}) = 1$ that

$$x_{i+2} \xrightarrow{\bar{\varepsilon}} -x_{i-1}.$$

□

**Remark 5.10.** *Note that the case $\psi_s(x_i) = -1$, $\psi_s(x_{i+1}) = 1$ only occurs for $\mathfrak{L}$–chains $g$ with two double ends when using the method described in Remark 4.58 in order to construct the $\overline{\mathfrak{X}}$–representation $U_s(g, p)$. We observe that we either have $\psi_s(x_i) = -1$, $\psi_s(x_{i+1}) = 1$ or $\psi_s(x_i) = 1$, $\psi_s(x_{i+1}) = -1$ for such $\mathfrak{L}$–chains. This is due to the definition of the maps $\psi_s^*(g)$ for the $\mathfrak{L}$–chain $g$: $\psi_s^*(x_i^*) = -\psi_s(x_{m+1-i})$ (see Remark 4.49) where $x_i^* \in g_0^*$, $x_{m+1-i} \in g_0$, and we have in particular that $x_i^* = x_{m+1-i}$ (see Definition 4.22).*

**Lemma 5.11.** *Let $g$ be an $\mathfrak{L}$–graph and denote by $U(g)$ ($U(g,V)$) a corresponding representation where $V$ describes - in case of $g$ being an $\mathfrak{L}$–cycle - the vector space corresponding to some $\varphi$ in the construction. Let $x_i \mathrel{\hat{=}}$ basis of $\mathrm{im}(a)$ and $x_{i+1} \mathrel{\hat{=}}$ basis of $V_i \ominus \ker(a)$ be two links in $g_0$, where $a : i \longrightarrow j \in Q_1^{\mathrm{ord}}$.*

*(i) Let $x_i, x_{i+1} \in \mathfrak{C}$. Then any subchain of the form $x_{i-1} - x_i \sim x_{i+1} - x_{i+2}$ of $g$ that, in case of a cycle, does not contain $e_2(g)$, results in the following action in $F(U(g))$ ($F(U(g,V))$):*

$$x_{i-1} \xleftarrow{a} x_{i+2} \text{ if } g \text{ is an } \mathfrak{L}-\text{chain}$$
$$V_{i-1} \xleftarrow{a} V_{i+2} \text{ if } g \text{ is an } \mathfrak{L}-\text{cycle}, V_{i-1}, V_{i+2} \text{ disjoint copies of } V$$

*(ii) Let $x_i, x_{i+1} \in \mathfrak{R}$. Any subchain of the form $x_{i-1} - \overleftarrow{x_i \sim x_{i+1}} - x_{i+2}$ of $g$ that, in case of a cycle, does not contain $e_2(g)$, results in the following action in $F(U(g))$ ($F(U(g,V))$):*

$$x_i \xleftarrow{a} x_{i+1} \text{ if } g \text{ is an } \mathfrak{L}-\text{chain}$$
$$V_i \xleftarrow{a} V_{i+1} \text{ if } g \text{ is an } \mathfrak{L}-\text{cycle}, V_i, V_{i+1} \text{ disjoint copies of } V$$

*(iii) Let $x_i \in \mathfrak{C}$, $x_{i+1} \in \mathfrak{R}$. Any subchain of the form $x_{i-1} - \overleftarrow{x_i \sim x_{i+1}} - x_{i+2}$ of $g$ that, in case of a cycle, does not contain $e_2(g)$, results in the following action in $F(U(g))$ ($F(U(g,V))$):*

$$x_{i-1} \xleftarrow{a} x_{i+1} \text{ if } g \text{ is an } \mathfrak{L}-\text{chain},$$
$$V_{i-1} \xleftarrow{a} V_{i+1} \text{ if } g \text{ is an } \mathfrak{L}-\text{chain}, V_{i-1}, V_{i+1} \text{ disjoint copies of } V$$

*(iv) Let $x_i \in \mathfrak{R}$, $x_{i+1} \in \mathfrak{C}$. Any subchain of the form $x_{i-1} - \overleftarrow{x_i \sim x_{i+1}} - x_{i+2}$ of $g$ that, in case of a cycle, does not contain $e_2(g)$, results in the following action in $F(U(g))$ ($F(U(g,V))$):*

$$x_i \xleftarrow{a} x_{i+2} \text{ if } g \text{ is an } \mathfrak{L}-\text{chain},$$
$$V_i \xleftarrow{a} V_{i+2} \text{ if } g \text{ is an } \mathfrak{L}-\text{cycle}, V_i, V_{i+2} \text{ disjoint copies of } V$$

*Proof.* We use the same notation as in the proof of Lemma 5.9. We show the statement for $g$ an $\mathfrak{L}$–chain. The proof for $g$ an $\mathfrak{L}$–cycle is analogous.

(i) Note that both links $x_{i-1}$ and $x_{i+2}$ belong to row label sets. Thus, $x_{i-1} \neq x_{i-2}$ and $x_{i+2} \neq x_{i+3}$. It follows that $x_i$ belongs to exactly one elementary subchain which is of length 2 ($x_{i-1} - x_i$) and $x_{i+1}$ also belongs to exactly one elementary subchain which is of length 2 ($x_{i+1} -$

$x_{i+2}$). Recall that $V_a = U^j \tilde{V}_a \left(U^i\right)^{-1}$. We obtain for the respective matrix components:

$$U^j(x, x_i) = \begin{cases} 1 & \text{if } x = x_{i-1}, \\ 0 & \text{else}, \end{cases} \qquad U^j(x_{i-1}, x) = \begin{cases} 1 & \text{if } x = x_i, \\ 0 & \text{else}, \end{cases} \tag{215}$$

$$U^i(x, x_{i+1}) = \begin{cases} 1 & \text{if } x = x_{i+2}, \\ 0 & \text{else}, \end{cases} \qquad U^i(x_{i+2}, x) = \begin{cases} 1 & \text{if } x = x_{i+1}, \\ 0 & \text{else}, \end{cases} \tag{216}$$

$$\tilde{V}_\varepsilon(x, x_{i+1}) = \begin{cases} 1 & \text{if } x = x_i, \\ 0 & \text{else}, \end{cases} \qquad \tilde{V}_\varepsilon(x_i, x) = \begin{cases} 1 & \text{if } x = x_{i+1}, \\ 0 & \text{else}. \end{cases} \tag{217}$$

We compute the relevant entries of the inverse of $U^i$:

$$\left(U^i\right)^{-1}(x_{i+1}, x) = \begin{cases} 1 & \text{if } x = x_{i+2}, \\ 0 & \text{else}, \end{cases} \tag{218}$$

$$\left(U^i\right)^{-1}(x, x_{i+2}) = \begin{cases} 1 & \text{if } x = x_{i+1}, \\ 0 & \text{else}. \end{cases} \tag{219}$$

Denote by $(\cdot,\cdot)$ the respective entries of the product $\tilde{V}_a \left(U^i\right)^{-1}$. We obtain the following from (215) - (219):

$$(x, x_{i+2}) = \sum_l \tilde{V}_a(x, l) \left(U^i\right)^{-1}(l, x_{i+2}) = \tilde{V}_a(x, x_{i+1}) = \begin{cases} 1 & \text{if } x = x_i, \\ 0 & \text{else}, \end{cases} \tag{220}$$

$$(x_i, x) = \sum_l \tilde{V}_a(x_i, l) \left(U^i\right)^{-1}(l, x) = \left(U^i\right)^{-1}(x_{i+1}, x) = \begin{cases} 1 & \text{if } x = x_{i+2}, \\ 0 & \text{else}. \end{cases} \tag{221}$$

We use the results from (220) and (221) to finish the computation of $V_a$:

$$V_a(x_{i-1}, x) = \sum_l U^j(x_{i-1}, l)(l, x) = (x_i, x) = \begin{cases} 1 & \text{if } x = x_{i+2}, \\ 0 & \text{else}, \end{cases}$$

$$V_a(x, x_{i+2}) = \sum_l U^j(x, l)(l, x_{i+2}) = U^j(x, x_i) = \begin{cases} 1 & \text{if } x = x_{i-1}, \\ 0 & \text{else}. \end{cases}$$

We see that $V_a$ acts as follows on the respective basis elements:

$$x_{i-1} \xleftarrow{a} x_{i+2} \tag{222}$$

(ii) Note that $x_{i-1}$ and $x_{i+2}$ belong to column label sets. Thus, we can have that $x_{i-1} = x_{i-2} = \mathfrak{C}_{\varepsilon^*}$ for some $\varepsilon \in \mathrm{Sp}$ with $\overrightarrow{x_{i-2} \sim x_{i-1}}$. Similarly, we can have that $x_{i+2} = x_{i+3} = \mathfrak{C}_{\eta^*}$ for some $\eta \in \mathrm{Sp}$ with $\overleftarrow{x_{i+2} \sim x_{i+3}}$. Note that we cannot have both cases at once by exluding $\mathrm{k}\langle \eta, \varepsilon \mid \eta^2 = \eta, \varepsilon^2 = \varepsilon \rangle$ (Remark 4.73). Depending on those cases, $x_i$ ($x_{i+1}$) can belong to one or two elementary subchains.
Recall that $V_a = \tilde{V}_a$. Thus, the number of elementary subchains which contain $x_i$ ($x_{i+1}$) does not affect $V_a$ and it follows for all cases:

$$\tilde{V}_a(x, x_{i+1}) = \begin{cases} 1 & \text{if } x = x_i, \\ 0 & \text{else,} \end{cases} \quad \tilde{V}_a(x_i, x) = \begin{cases} 1 & \text{if } x = x_{i+1}, \\ 0 & \text{else.} \end{cases}$$

We obtain that $V_a$ acts as follows on the respective basis elements:

$$x_i \xleftarrow{a} x_{i+1}.$$

(iii) Note that $x_{i-1}$ belongs to a row label set and that $x_{i+2}$ belongs to a column label set. We can have that $x_{i+2} = x_{i+3}$ with $\overleftarrow{x_{i+2} \sim x_{i+3}}$. In this case, $x_{i+1}$ belongs to two elementary subchains. They affect the entries of $U^i$. Recall that $V_a = U^j \tilde{V}_a$. We see that the number of elementary subchains containing $x_{i+1}$ does not affect $V_a$.
The relevant entries of the matrices are given by

$$U^j(x, x_i) = \begin{cases} 1 & \text{if } x = x_{i-1}, \\ 0 & \text{else,} \end{cases} \qquad U^j(x_{i-1}, x) = \begin{cases} 1 & \text{if } x = x_i, \\ 0 & \text{else,} \end{cases}$$
$$\tilde{V}_a(x, x_{i+1}) = \begin{cases} 1 & \text{if } x = x_i, \\ 0 & \text{else,} \end{cases} \qquad \tilde{V}_a(x_i, x) = \begin{cases} 1 & \text{if } x = x_{i+1}, \\ 0 & \text{else.} \end{cases}$$

We obtain for $V_a$:

$$V_a(x_{i-1}, x) = \sum_l U^j(x_{i-1}, l)\tilde{V}_a(l, x) = \tilde{V}_a(x_i, x) = \begin{cases} 1 & \text{if } x = x_{i+1}, \\ 0 & \text{else,} \end{cases}$$
$$V_a(x, x_{i+1}) = \sum_l U^j(x, l)\tilde{V}_a(l, x_{i+1}) = U^j(x, x_i) = \begin{cases} 1 & \text{if } x = x_{i-1}, \\ 0 & \text{else.} \end{cases}$$

Thus, $V_a$ acts as follows on the respective basis elements:

$$x_{i-1} \xleftarrow{a} x_{i+1}.$$

(iv) Note that $x_{i-1}$ belongs to a column label set, and that $x_{i+2}$ belongs to a row label set. We can have that $x_{i-1} = x_{i-2}$ with $\overrightarrow{x_{i-2} \sim x_{i-1}}$. In this case, $x_i$ belongs to two elementary subchains. They affect the entries of $U^j$. Recall that $V_a = \tilde{V}_a \left(U^i\right)^{-1}$. Thus, the number of elementary

subchains containing $x_i$ does not affect the entries of $V_a$.
The relevant entries of the matrices $U^i$ and $\tilde{V}_a$ are given as follows:

$$U^i(x_{i+2},x) = \begin{cases} 1 & \text{if } x = x_{i+1}, \\ 0 & \text{else,} \end{cases} \qquad U^i(x,x_{i+1}) = \begin{cases} 1 & \text{if } x = x_{i+2}, \\ 0 & \text{else,} \end{cases}$$

$$\tilde{V}_a(x,x_{i+1}) = \begin{cases} 1 & \text{if } x = x_i, \\ 0 & \text{else,} \end{cases} \qquad \tilde{V}_a(x_i,x) = \begin{cases} 1 & \text{if } x = x_{i+1}, \\ 0 & \text{else.} \end{cases}$$

Computing the inverse of $U^i$ according to Section 5.3 results in

$$\left(U^i\right)^{-1}(x_{i+1},x) = \begin{cases} 1 & \text{if } x = x_{i+2}, \\ 0 & \text{else,} \end{cases}$$
$$\left(U^i\right)^{-1}(x,x_{i+2}) = \begin{cases} 1 & \text{if } x = x_{i+1}, \\ 0 & \text{else.} \end{cases}$$

We compute the respective entries of $V_a$:

$$\begin{aligned} V_a(x,x_{i+2}) &= \sum_l \tilde{V}_a(x,l)\left(U^i\right)^{-1}(l,x_{i+2}) = \tilde{V}_a(x,x_{i+1}) = \begin{cases} 1 & \text{if } x = x_i, \\ 0 & \text{else,} \end{cases} \\ V_a(x_i,x) &= \sum_l \tilde{V}_a(x_i,l)\left(U^i\right)^{-1}(l,x) = \left(U^i\right)^{-1}(x_{i+1},x) \\ &= \begin{cases} 1 & \text{if } x = x_{i+2}, \\ 0 & \text{else.} \end{cases} \end{aligned}$$

It follows that $V_a$ acts as follows on the respective basis elements:

$$x_i \xleftarrow{a} x_{i+2}.$$

□

**Remark 5.12.** *We obtain an analogous result to Lemma 5.11 if we switch the roles of $x_i$ and $x_{i+1}$: in this case, we have that that $x_i \mathrel{\hat{=}}$ basis of $V_i \ominus \ker(a)$, $x_{i+1} \mathrel{\hat{=}}$ basis of $\operatorname{im}(a)$. The respective action in $F(U(g))$ ($F(U(g,V))$) is then described by $a^{-1}$.*

**Lemma 5.13.** *Let $g$ be an $\mathfrak{L}-$chain with one double end. Let $U_s(g)$ be a corresponding representation ($s \in \{1,2\}$).*

*(i) If $x_1 = \mathfrak{C}_{\varepsilon^*}$ is the double end for some $\varepsilon \in \mathrm{Sp}$, then the subchain $x_1 - x_2$ results in the following action on $x_2$ in $F(U_s(g))$:*

$$\varepsilon(x_2) = \begin{cases} 0 & \textit{if } s = 1, \\ x_2 & \textit{if } s = 2. \end{cases}$$

(ii) *If* $x_m = \mathfrak{C}_{\varepsilon^*}$ *is the double end for some* $\varepsilon \in \mathrm{Sp}$*, the subchain* $x_{m-1} - x_m$ *results in the following action on* $x_{m-1}$ *in* $F(U_s(g))$*:*

$$\varepsilon(x_{m-1}) = \begin{cases} 0 & \textit{if } s = 2, \\ x_{m-1} & \textit{if } s = 1. \end{cases}$$

*Proof.* (i) We proceed similar as in Lemma 5.9 and compute $V_\varepsilon$. We know that $x_1 \in \mathfrak{C}_i$ for some $i \in Q_0$ and $x_2 \in \mathfrak{R}_i$. It follows that $x_3 \neq x_2$ (otherwise $x_3 = x_2 \in \mathfrak{C}_i$). We obtain the following entries:

$$U^i(x, x_1) = \begin{cases} 1 & \text{if } x = x_2, \\ 0 & \text{else,} \end{cases}$$

$$U^i(x_2, x) = \begin{cases} 1 & \text{if } x = x_1, \\ 0 & \text{else,} \end{cases}$$

$$V_\varepsilon(x, y) = \begin{cases} 1 & \text{if } x = y \in \mathfrak{C}^+_{\varepsilon^*}, \\ 0 & \text{else.} \end{cases}$$

It follows that

$$\left(U^i\right)^{-1}(x_1, x) = \begin{cases} 1 & \text{if } x = x_2, \\ 0 & \text{else,} \end{cases}$$

$$\left(U^i\right)^{-1}(x, x_2) = \begin{cases} 1 & \text{if } x = x_1, \\ 0 & \text{else.} \end{cases}$$

Denote by $(\cdot,\cdot)$ the matrix given by the product $V_\varepsilon U^i$. Recall that $\psi_1(x_1) = -1$ and $\psi_2(x_1) = 1$. Thus, $x_1$ belongs to $\mathfrak{C}^-_{\varepsilon^*}$ for $s = 1$, and to $\mathfrak{C}^+_{\varepsilon^*}$ for $s = 2$. We obtain the following:

$\mathbf{s = 1}$:

$$(x_1, x) = \sum_l V_\varepsilon(x_1, l)\left(U^i\right)^{-1}(l, x) = 0, \tag{223}$$

$$(x, x_2) = \sum_l V_\varepsilon(x, l)\left(U^i\right)^{-1}(l, x_2) = V_\varepsilon(x, x_1) = 0 \text{ for all } x, \tag{224}$$

$\mathbf{s = 2}$:

$$(x_1, x) = \sum_l V_\varepsilon(x_1, l)\left(U^i\right)^{-1}(l, x) = \left(U^i\right)^{-1}(x_1, x) = \begin{cases} 1 & \text{if } x = x_2, \\ 0 & \text{else,} \end{cases} \tag{225}$$

$$(x, x_2) = \sum_l V_\varepsilon(x, l)\left(U^i\right)^{-1}(l, x_2) = V_\varepsilon(x, x_1) = \begin{cases} 1 & \text{if } x = x_1, \\ 0 & \text{else.} \end{cases} \tag{226}$$

We use (223) - (226) to compute $V_\varepsilon$. Its relevant entries are given as follows:

$\mathbf{s = 1}$ :

$$V_\varepsilon(x_2, x) = \sum_l U^i(x_2, l)(l, x) = (x_1, x) = 0 \text{ for all } x, \tag{227}$$

$$V_\varepsilon(x, x_2) = \sum_l U^i(x, l)(l, x_2) = 0 \text{ for all } x, \tag{228}$$

$\mathbf{s = 2}$ :

$$V_\varepsilon(x_2, x) = \sum_l U^i(x_2, l)(l, x) = (x_1, x) = \begin{cases} 1 & \text{if } x = x_2, \\ 0 & \text{else,} \end{cases} \tag{229}$$

$$V_\varepsilon(x, x_2) = \sum_l U^i(x, l)(l, x_2) = U^i(x, x_1) = \begin{cases} 1 & \text{if } x = x_2, \\ 0 & \text{else.} \end{cases} \tag{230}$$

We obtain from (227) - (228) that

$$\varepsilon(x_2) = 0 \text{ if } s = 1,$$

and

$$\varepsilon(x_2) = x_2 \text{ if } s = 2,$$

from (229) - (230).

(ii) Recall that $\psi_1(x_m) = 1$ and $\psi_2(x_m) = -1$. The proof is analogous to (i).

□

**Lemma 5.14.** *Let $g$ be an $\mathfrak{L}$−chain with two double ends. Denote by $U_s(g, p)$ a corresponding representation ($p \in \mathbb{N}\backslash\{0\}$, $s \in \{1, 2, 3, 4\}$).*

*(i) Let $x_1 = \mathfrak{C}_{\varepsilon^*}$ for some $\varepsilon \in \mathrm{Sp}$. The subchain $(x_1, 1) - (x_2, 1)$ results in the following action on $(x_2, 1)$ in $F(U_s(g, p))$:*

$$\varepsilon((x_2, 1)) = \begin{cases} 0 & \text{if } s \in \{1, 3\}, \\ 1 & \text{if } s \in \{2, 4\}. \end{cases}$$

*(ii) Let $x_m = \mathfrak{C}_{\eta^*}$ for some $\eta \in \mathrm{Sp}$. If $p$ is odd, the subchain $(x_{m-1}, p) - (x_m, p)$ results in the following action on $(x_{m-1}, p)$ in $F(U_s(g, p))$:*

$$\eta((x_{m-1}, p)) = \begin{cases} 0 & \text{if } s \in \{1, 2\}, \\ 1 & \text{if } s \in \{3, 4\}. \end{cases}$$

*If $p$ is even, the subchain $(x_1, p) - (x_2, p)$ results in the following action on $(x_2, p)$ in $F(U_s(g, p))$:*

$$\varepsilon((x_2, p)) = \begin{cases} 0 & \text{if } s \in \{2, 4\}, \\ 1 & \text{if } s \in \{1, 3\}. \end{cases}$$

*Proof.* (i) We know that $(x_2, 1) \neq (x_3, 1)$ since $x_2 \in \mathfrak{R}$. Thus, $(x_1, 1)$ is only contained in the elementary subchain $(x_1, 1) - (x_2, 1)$ of $g^{[p]}$. Recall that $\psi_1((x_1, 1)) = \psi_3((x_1, 1)) = 1$ and $\psi_2((x_1, 1)) = \psi_4((x_1, 1)) = -1$. Thus, the cases $s = 1, 3$ follow analogously to Lemma 5.13, (i) with $s = 1$. Similarly, the cases $s = 2, 4$ are analogous to the case $s = 2$ of Lemma 5.13, (i).

(ii) Let $p$ be odd. We have that $(x_m, p)$ is only contained in the elementary subchain $(x_{m-1}, p) - (x_m, p)$. Recall that $\psi_s^*((x_m, p)) = \psi_s(x_m)$. It follows that $\psi_1((x_m, p)) = \psi_2((x_m, p)) = 1$ and that $\psi_3((x_m, p)) = \psi_4((x_m, p)) = -1$. The cases $s = 1, 2$ follow analogously to Lemma 5.13 (ii) for $s = 1$. Similarly, the cases $s = 3, 4$ follow analogously to Lemma 5.13 (ii) for $s = 2$.
Consider now $p$ to be even. Recall that $\psi_s^*((x_1, p)) = -\psi_s(x_1)$. Hence, we obtain that $\psi_1((x_1, p)) = \psi_3((x_1, p)) = 1$ and that $\psi_2((x_1, p)) = \psi_4((x_1, p)) = 1$. The cases $s = 1, 3$ follow analogoulsy to Lemma 5.13 (ii) for $s = 2$. Similarly, the cases $s = 2, 4$ follow analogously to Lemma 5.13 (ii) for $s = 1$.

$\square$

**Lemma 5.15.** *Let $g$ be an $\mathfrak{L}$−cycle and let $U(g, \varphi)$ be a corresponding representation. Let $e_2(g) = x_{i-1} - x_i$.*

*(i) Let $x_{i-1} \in \mathfrak{R}$ and $x_i \in \mathfrak{C}$. Then $e_2(g)$ results in the following action in $F(U(g, \varphi))$:*

- *for $x_i \mathrel{\hat{=}}$ basis of $\operatorname{im}(b)$ and $x_{i+1} \mathrel{\hat{=}}$ basis of $V_k \ominus \ker(b)$ for some $b \in Q_1^{\mathrm{ord}}$, $s(b) = k$:*

$$\bar{V}_{i-1} \xleftarrow{b=F_\varphi} \bar{V}_j,$$

- *for $x_i \mathrel{\hat{=}}$ basis of $V_k \ominus \ker(b)$ and $x_{i+1} \mathrel{\hat{=}} \operatorname{im}(b)$ for some $b \in Q_1^{\mathrm{ord}}$, $s(b) = k$:*

$$\bar{V}_{i-1} \xrightarrow{b=F_\varphi^{-1}} \bar{V}_j,$$

- *for $x_i = x_{i+1} = \mathfrak{C}_{\varepsilon^*}$ for some $\varepsilon \in \mathrm{Sp}$, $s(\varepsilon) = k$:*

$$\bar{V}_{i-1} \xrightarrow{\bar{\varepsilon}=-F_\varphi^{-1}} \bar{V}_j,$$

*where*

$$j = \begin{cases} i+1 & \text{if } x_{i+1} \in \mathfrak{R}, \\ i+2 & \text{if } x_{i+1} \in \mathfrak{C}, \end{cases}$$

*and $\bar{V}$ a $\mathrm{k}$−vector space of dimension $\deg \varphi$ with the $\bar{V}_i$'s disjoint copies of $\bar{V}$.*

*(ii) Let $x_{i-1} \in \mathfrak{C}$ and $x_i \in \mathfrak{R}$. Then $e_2(g)$ results in the following action in $F(U(g,\varphi))$:*

- *for* $x_{i-2} \,\hat{=}\,$ basis of $\operatorname{im}(a)$ *and* $x_{i-1} \,\hat{=}\,$ basis of $V_k \ominus \ker(a)$ *for some* $a \in Q_1^{\mathrm{ord}}$*,* $s(a) = k$*:*

$$\bar{V}_j \xleftarrow{a=F_\varphi^{-1}} \bar{V}_i,$$

- *for* $x_{i-2} \,\hat{=}\,$ basis of $V_k \ominus \ker(a)$ *and* $x_{i-1} \,\hat{=}\,$ basis of $\operatorname{im}(a)$ *for some* $a \in Q_1^{\mathrm{ord}}$*,* $s(a) = k$*:*

$$\bar{V}_j \xrightarrow{a=F_\varphi} \bar{V}_i,$$

- *for* $x_{i-2} = x_{i-1} = \mathfrak{C}_{\varepsilon^*}$ *for some* $\varepsilon \in \mathrm{Sp}$*,* $s(\varepsilon) = k$*:*

$$\bar{V}_j \xleftarrow{\varepsilon=-F_\varphi^{-1}} \bar{V}_i,$$

*where*

$$j = \begin{cases} i-2 & \text{if } x_{i-2} \in \mathfrak{R}, \\ i-3 & \text{if } x_{i-2} \in \mathfrak{C}, \end{cases} \tag{231}$$

*and* $\bar{V}$ *a* $\mathrm{k}$*−vector space of dimension* $\deg\varphi$ *with the* $\bar{V}_i$*'s disjoint copies of* $\bar{V}$*.*

*Proof.* Keep in mind that we consider $\mathfrak{L}$−cycles and thus, that each link indicates a subband of size $\deg(\varphi)$. We keep the computations in the proof in terms of the links and will switch in the last step to the computations in terms of vector spaces.

(i) We consider the subchain $x_{i-2} \sim x_{i-1} - x_i \sim x_{i+1} - x_{i+2}$ with $x_{i-1} \in \mathfrak{R}$ and $x_i \in \mathfrak{C}$. Assume without loss of generality that $x_{i-1} \,\hat{=}\,$ basis of $V_l \ominus \ker(a)$ and $x_{i-2} \,\hat{=}\,$ basis of $\operatorname{im}(a)$ for $a : l \longrightarrow h \in Q_1^{\mathrm{ord}}$. Note that $e_2(g)$ determines the entries in the matrix $U^l$:

$$U^l(x_{i-1}, x_i) = F_\varphi. \tag{232}$$

Moreover, $V_a$ is given independent of $U^l$: we have that $V_a = U^h \tilde{V}_a$ for $x_{i-2} \in \mathfrak{C}$, and $V_a = \tilde{V}_a$ for $x_{i-2} \in \mathfrak{R}$, where $\operatorname{im}(a) \subset V_h$.
Assume that $k = l$, that is $x_i \,\hat{=}\,$ basis of $V_l \ominus \ker(b)$, $x_{i+1} \,\hat{=}\,$ basis of $\operatorname{im}(b)$ for some $b : l \longrightarrow m \in Q_1^{\mathrm{ord}}$, and assume that $x_{i+1} \in \mathfrak{C}$. We have that $V_b = U^m \tilde{V}_b \left(U^l\right)^{-1}$. Denote by $(\cdot,\cdot)$ the product $\tilde{V}_b \left(U^l\right)^{-1}$. We have that

$$\left(U^l\right)^{-1}(x, x_{i-1}) = \begin{cases} F_\varphi^{-1} & \text{if } x = x_i, \\ 0 & \text{else,} \end{cases}, \tag{233}$$

$$\left(U^l\right)^{-1}(x_i, x) = \begin{cases} F_\varphi^{-1} & \text{if } x = x_{i-1}, \\ 0 & \text{else.} \end{cases} \tag{234}$$

Similar to the previous proofs, we obtain that

$$\begin{aligned}
(x_{i+1},x) &= \sum_j \tilde{V}_b(x_{i+1},j)\left(U^l\right)^{-1}(j,x) = \left(U^l\right)^{-1}(x_i,x) \\
&= \begin{cases} F_\varphi^{-1} & \text{if } x = x_{i-1}, \\ 0 & \text{else,} \end{cases} \\
(x,x_{i-1}) &= \sum_j \tilde{V}_b(x,j)\left(U^l\right)^{-1}(j,x_{i-1}) = \tilde{V}_b(x,x_i)F_\varphi^{-1} \\
&= \begin{cases} F_\varphi^{-1} & \text{if } x = x_{i+1}, \\ 0 & \text{else.} \end{cases}
\end{aligned}$$

It follows that

$$\begin{aligned}
V_b(x,x_{i-1}) &= \sum_j U^m(x,j)(j,x_{i-1}) = U^m(x,x_{i+1})F_\varphi^{-1} \\
&= \begin{cases} F_\varphi^{-1} & \text{if } x = x_{i+2}, \\ 0 & \text{else,} \end{cases} \\
V_b(x_{i+2},x) &= \sum_j U^m(x_{i+2},j)(j,x) = (x_{i+1},x) \\
&= \begin{cases} F_\varphi^{-1} & \text{if } x = x_{i-1}, \\ 0 & \text{else.} \end{cases}
\end{aligned}$$

We obtain that $\bar{V}_{i-1} \xrightarrow{b=F_\varphi^{-1}} \bar{V}_{i+2}$.
Consider now $x_{i+1} \in \mathfrak{R}$. Then $V_b$ is given by $V_b = \tilde{V}_b\left(U^l\right)^{-1}$. The relevant entries of $U^l$ are given as in (233) and (234). We obtain for $V_b$ the following:

$$\begin{aligned}
V_b(x,x_{i-1}) &= \sum_j \tilde{V}_b(x,j)\left(U^l\right)^{-1}(j,x_{i-1}) = \tilde{V}_b(x,x_i)F_\varphi^{-1} \\
&= \begin{cases} F_\varphi^{-1} & \text{if } x = x_{i+1}, \\ 0 & \text{else,} \end{cases} \\
V_b(x_{i+1},x) &= \sum_j \tilde{V}_b(x_{i+1},j)\left(U^l\right)^{-1}(j,x) = \left(U^l\right)^{-1}(x_i,x) \\
&= \begin{cases} F_\varphi^{-1} & \text{if } x = x_{i-1}, \\ 0 & \text{else.} \end{cases}
\end{aligned}$$

We obtain that $\bar{V}_{i-1} \xrightarrow{b=F_\varphi^{-1}} \bar{V}_{i+1}$.

Assume that $k \neq l$, that is, $x_i \,\hat{=}$ basis of $\operatorname{im}(b)$, $x_{i+1} \,\hat{=}$ basis of $V_k \ominus \ker(b)$ where $b: k \longrightarrow l \in Q_1^{\text{ord}}$. Assume additionally that $x_{i+1} \in \mathfrak{C}$. It follows that $V_b = U^l \tilde{V}_b \left(U^k\right)^{-1}$ with

$$U^l(x_{i-1}, x) = \begin{cases} F_\varphi & \text{if } x = x_i, \\ 0 & \text{else,} \end{cases} \tag{235}$$

$$U^l(x, x_i) = \begin{cases} F_\varphi & \text{if } x = x_{i-1}, \\ 0 & \text{else.} \end{cases} \tag{236}$$

Similar to the previous cases, we obtain for $(\cdot,\cdot) = \tilde{V}_b \left(U^k\right)^{-1}$ that

$$\begin{aligned}
(x_i, x) &= \sum_j \tilde{V}_b(x_i, j) \left(U^k\right)^{-1}(j, x) = \left(U^k\right)^{-1}(x_{i+1}, x) \\
&= \begin{cases} 1 & \text{if } x = x_{i+2}, \\ 0 & \text{else,} \end{cases} \\
(x, x_{i+2}) &= \sum_j \tilde{V}_b(x, j) \left(U^k\right)^{-1}(j, x_{i+2}) = \tilde{V}_b(x, x_{i+1}) \\
&= \begin{cases} 1 & \text{if } x = x_i, \\ 0 & \text{else.} \end{cases}
\end{aligned}$$

It follows that

$$\begin{aligned}
V_b(x_{i-1}, x) &= \sum_j U^l(x_{i-1}, j)(j, x) = F_\varphi(x_i, x) \\
&= \begin{cases} F_\varphi & \text{if } x = x_{i+2}, \\ 0 & \text{else,} \end{cases} \\
V_b(x, x_{i+2}) &= \sum_j U^l(x, j)(j, x_{i+2}) = U^l(x, x_i) \\
&= \begin{cases} F_\varphi & \text{if } x = x_{i-1}, \\ 0 & \text{else.} \end{cases}
\end{aligned}$$

We obtain that $\bar{V}_{i-1} \xleftarrow{b=F_\varphi} \bar{V}_{i+2}$.
Assume that $x_{i+1} \in \mathfrak{R}$. Then $V_b = U^l \tilde{V}_b$. The entries of $U^l$ are given as described in (235) and (236). We obtain for $V_b$ the following:

$$\begin{aligned}
V_b(x, x_{i+1}) &= \sum_j U^l(x, j)\tilde{V}_b(j, x_{i+1}) = U^l(x, x_i) = \begin{cases} F_\varphi & \text{if } x = x_{i-1}, \\ 0 & \text{else,} \end{cases} \\
V_b(x_{i-1}, x) &= \sum_j U^l(x_{i-1}, j)\tilde{V}_b(j, x) = F_\varphi \tilde{V}_b(x_i, x) = \begin{cases} F_\varphi & \text{if } x = x_{i+1}, \\ 0 & \text{else.} \end{cases}
\end{aligned}$$

Thus, the computations result in $\bar{V}_{i-1} \xleftarrow{b=F_\varphi} \bar{V}_{i+1}$.

Assume that $x_i = x_{i+1} = \mathfrak{C}_{\varepsilon^*}$ for some $\varepsilon \in \mathrm{Sp}$ with $s(\varepsilon) = l$. Then we have that $x_{i+1} \in \mathfrak{C}$. By definition of $e_2(g)$, it follows that $\mathrm{dir}_{i,i+1}(g) = -1$. Note that $x_i \in \mathfrak{C}^+_{\varepsilon^*}$ and $x_{i+1} \in \mathfrak{C}^-_{\varepsilon^*}$. We want to compute $V_\varepsilon = U^l \tilde{V}_\varepsilon \left(U^l\right)^{-1}$. Here, the relevant entries of $U^l$ are given by

$$U^l(x, x_i) = \begin{cases} 1 & \text{if } x = x_{i+2}, \\ F_\varphi & \text{if } x = x_{i-1}, \\ 0 & \text{else}, \end{cases} \qquad U^l(x, x_{i+1}) = \begin{cases} 1 & \text{if } x = x_{i+2}, \\ 0 & \text{else}, \end{cases}$$

$$U^l(x_{i+2}, x) = \begin{cases} 1 & \text{if } x \in \{x_{i+1}, x_i\}, \\ 0 & \text{else}, \end{cases} \qquad U^l(x_{i-1}, x) = \begin{cases} F_\varphi & \text{if } x = x_i, \\ 0 & \text{else}. \end{cases}$$

Thus, we obtain according to Section 5.3 that the respective entries of its inverse are the following:

$$\left(U^l\right)^{-1}(x_i, x) = \begin{cases} F_\varphi^{-1} & \text{if } x = x_{i-1}, \\ 0 & \text{else}, \end{cases}$$

$$\left(U^l\right)^{-1}(x_{i+1}, x) = \begin{cases} -F_\varphi^{-1} & \text{if } x = x_{i-1}, \\ 1 & \text{if } x = x_{i+2}, \\ 0 & \text{else}, \end{cases}$$

$$\left(U^l\right)^{-1}(x, x_{i+2}) = \begin{cases} 1 & \text{if } x = x_{i+1}, \\ 0 & \text{else}, \end{cases}$$

$$\left(U^l\right)^{-1}(x, x_{i-1}) = \begin{cases} F_\varphi^{-1} & \text{if } x = x_i, \\ -F_\varphi^{-1} & \text{if } x = x_{i+1}, \\ 0 & \text{else}. \end{cases}$$

We denote by $(\cdot, \cdot)$ the entries of the product $\tilde{V}_\varepsilon \left(U^l\right)^{-1}$. We obtain the following:

$$\begin{aligned}
(x_i, x) &= \sum_j \tilde{V}_\varepsilon(x_i, j) \left(U^l\right)^{-1}(j, x) = \left(U^l\right)^{-1}(x_i, x) \\
&= \begin{cases} F_\varphi^{-1} & \text{if } x = x_{i-1}, \\ 0 & \text{else}, \end{cases} \\
(x_{i+1}, x) &= \sum_j \tilde{V}_\varepsilon(x_{i+1}, j) \left(U^l\right)^{-1}(j, x) = 0 \text{ for any } x \in \mathfrak{C}_l, \\
(x, x_{i-1}) &= \sum_j \tilde{V}_\varepsilon(x, j) \left(U^l\right)^{-1}(j, x_{i-1}) \\
&= \tilde{V}_\varepsilon(x, x_i) F_\varphi^{-1} + \tilde{V}_\varepsilon(x, x_{i+1}) \left(-F_\varphi^{-1}\right) \\
&= \tilde{V}_\varepsilon(x, x_i) F_\varphi^{-1} = \begin{cases} F_\varphi^{-1} & \text{if } x = x_i, \\ 0 & \text{else}, \end{cases} \\
(x, x_{i+2}) &= \sum_j \tilde{V}_\varepsilon(x, j) \left(U^l\right)^{-1}(j, x_{i+2}) = \tilde{V}_\varepsilon(x, x_{i+1}) = 0 \text{ for any } x \in \mathfrak{C}_l.
\end{aligned}$$

Multiplying the above from the left by $U^l$ results in

$$\begin{aligned}
V_\varepsilon(x, x_{i-1}) &= \sum_j U^l(x, j)(j, x_{i-1}) = U^l(x, x_i) F_\varphi^{-1} = \begin{cases} 1 & \text{if } x = x_{i-1}, \\ F_\varphi^{-1} & \text{if } x = x_{i+2}, \\ 0 & \text{else}, \end{cases} \\
V_\varepsilon(x_{i+2}, x) &= \sum_j U^l(x_{i+2}, j)(j, x) = (x_i, x) + (x_{i+1}, x) \\
&= (x_i, x) = \begin{cases} F_\varphi^{-1} & \text{if } x = x_{i-1}, \\ 0 & \text{else}. \end{cases}
\end{aligned}$$

Hence, we obtain that $\varepsilon$ acts as follows on $V_{i-1}$:

$$\begin{aligned}
\bar{V}_{i-1} &\xrightarrow{\varepsilon = F_\varphi^{-1}} \bar{V}_{i+2} \text{ and} \\
\bar{V}_{i-1} &\xrightarrow{\varepsilon} \bar{V}_{i-1}.
\end{aligned}$$

Considering $V_{\bar{\varepsilon}} = 1 - V_\varepsilon$ yields that

$$\bar{V}_{i-1} \xrightarrow{\bar{\varepsilon} = -F_\varphi^{-1}} \bar{V}_{i+2}.$$

(ii) The proof is analogous to (i). Note that we have for $x_{i-2} = x_{i-1} = \mathfrak{C}_{\varepsilon^*}$ for some $\varepsilon \in \mathrm{Sp}$, that $\mathrm{dir}_{i-2,i-1}(g) = 1$ due to the definition of $e_2(g)$.

□

**Remark 5.16.** *We see in the proof above that $e_2(g)$ influences the map corresponding to the incident link which belongs to a column label set.*

**Remark 5.17.** *In case of a symmetric band $w_{\mathbb{Z}}$, we assume that its periodic part is of the form $\hat{w}_p = \varepsilon^* u \eta^* u^{-1}$ (cf. Subsection 3.2.4). We know by construction that*

$$e_2(g_{w_{\mathbb{Z}}}) = x_1 - \bar{x}_1,$$

*where $\bar{x}_0 = x_1 = \mathfrak{C}_{\varepsilon^*}$ and $\mathrm{dir}_{0,1}(g_{w_{\mathbb{Z}}}) = 1$. Thus, $\bar{x}_1 \in \mathfrak{R}$ and it follows that*

$$\bar{V}_0 \xleftarrow{\varepsilon = -F_\varphi^{-1}} \bar{V}_1$$

*in $F(U(g_{w_{\mathbb{Z}}}, \varphi))$.*

The above results show that we have for strings $w$ that each link $x_i \in g_{w,0}$ with $x_i \in \mathfrak{R}$ corresponds to a basis element in $F(U_s(g_w))$. Similarly, we have for bands $w_{\mathbb{Z}}$ that each link $x_i \in g_{w_{\mathbb{Z}},0}$ with $x_i \in \mathfrak{R}$ corresponds to a copy $V_i$ of $V$ in $F(U(g_{w_{\mathbb{Z}}}, \varphi))$.

Recall from Chapter 4.7 that the direction on links of $\mathfrak{L}$–graphs coincides with the directions on the respective special letter if the letter has finite index. This information together with the Lemmas 5.9 - 5.15 enables us to directly read the image of $F$ from the original string or band and its corresponding $\mathfrak{L}$–graph. To this end, we introduce an alphabet fitting our needs with respect to the image of $F$.

**Definition 5.18.** *Let $\Lambda$ be a skewed-gentle algebra. Let*

$$\Sigma_{\mathrm{d}}(\Lambda) \coloneqq \{\varepsilon^{\pm 1}, \bar{\varepsilon}^{\pm 1}, x^{\pm 1} \mid \varepsilon \in \mathrm{Sp}, x \in Q_1^{\mathrm{ord}}\}$$

*be an extension of $\Gamma_{\mathrm{d}}(\Lambda)$. We denote the respective set of directed words by*

$$\mathcal{W}(\Sigma_{\mathrm{d}}(\Lambda)) \coloneqq \{w_{\mathrm{I}} \mid w_i \in \Sigma_{\mathrm{d}}(\Lambda) \text{ for all } i \in \mathrm{I}\}.$$

There exists the following forgetful map:

$$\begin{aligned} \psi_\Lambda^\Sigma : \quad \Sigma_{\mathrm{d}}(\Lambda) &\longrightarrow \Gamma_{\mathrm{d}}(\Lambda) \\ x^\kappa &\longmapsto \begin{cases} x^\kappa & \text{if } x \neq \bar{\varepsilon} \text{ for any } \varepsilon \in \mathrm{Sp}, \\ \bar{x}^\kappa & \text{else.} \end{cases} \end{aligned} \tag{237}$$

We consider the following subset of $\Sigma_{\mathrm{d}}(\Lambda)$:

$$\tilde{\Sigma}_{\mathrm{d}}(\Lambda) \coloneqq \{\varepsilon, \bar{\varepsilon}^{-1}, x^{\pm 1} \mid \varepsilon \in \mathrm{Sp}, x \in Q_1^{\mathrm{ord}}\} \tag{238}$$

and denote the respective set of words by $\mathcal{W}\big(\tilde{\Sigma}_{\mathrm{d}}(\Lambda)\big)$. We denote the restriction of $\psi_\Lambda^\Sigma$ to $\tilde{\Sigma}_{\mathrm{d}}(\Lambda)$ by $\tilde{\psi}_\Lambda^\Sigma$.
Similarly as (11) induces the map (13) in Section 2.3, the map (237) induces the following map:

$$\Psi_\Lambda^\Sigma : \mathcal{W}(\Sigma_{\mathrm{d}}(\Lambda)) \longrightarrow \mathcal{W}(\Gamma_{\mathrm{d}}(\Lambda))$$

We denote the respective restriction by $\tilde{\Psi}_\Lambda^\Sigma$.
Note that we can compose the map $\tilde{\Psi}_\Lambda^\Sigma$ with $\Phi_{\mathrm{ud}}^{\mathrm{d}}$. We denote this composition by $\tilde{\Psi}_{\mathrm{ud}}^\Sigma$.

**Remark 5.19.** *The notation above can be also be applied to clannish algebras.*

**Example 5.20.** *Let $\Lambda$ be given by the two-loop quiver in Example 2.3.1.*

*(i) Let $w = \varepsilon^* a^{-1}\varepsilon^*$ be an asymmetric string. Let $v \in (\Phi^{\mathrm{d}}_{\mathrm{ud}})^{-1}(w)$ be given by $v = \varepsilon^{-1}a^{-1}\varepsilon^{-1}$. Note that $v$ is (weakly) conistent. Let $t \in \left(\tilde{\Psi}^{\Sigma}_{\Lambda}\right)^{-1}(v)$ be given by $t = \bar{\varepsilon}^{-1}a^{-1}\bar{\varepsilon}^{-1}$. Then $t \in \left(\tilde{\Psi}^{\Sigma}_{\mathrm{ud}}\right)^{-1}(w)$.*

*(ii) Let $w_{\mathbb{Z}}$ be a symmetric band with $\hat{w}_p = \varepsilon^* u \varepsilon^* u^{-1}$, where $u = a\varepsilon^* a$. Let $v_{\mathbb{Z}} \in (\Phi^{\mathrm{d}}_{\mathrm{ud}})^{-1}(w_{\mathbb{Z}})$ with $\hat{v}_p = \varepsilon a \varepsilon a \varepsilon a^{-1}\varepsilon^{-1}a^{-1}$. Note that $v_{\mathbb{Z}}$ is (weakly) conistent. Let $t_{\mathbb{Z}} \in \left(\tilde{\Psi}^{\Sigma}_{\Lambda}\right)^{-1}(v_{\mathbb{Z}})$ with $\hat{t}_p = \varepsilon a \varepsilon a \varepsilon a^{-1}\bar{\varepsilon}^{-1}a^{-1}$. Then $t_{\mathbb{Z}} \in \left(\tilde{\Psi}^{\Sigma}_{\mathrm{ud}}\right)^{-1}(w_{\mathbb{Z}})$.*

Having introduced this notation, we are able to compose the corresponding statements.

**Theorem 5.21.** *(i) Let $w$ be an asymmetric string. Let $g_w$ be the $\mathfrak{L}$-chain corresponding to $w$ with $\overline{\mathfrak{X}}_0$-representation $U_1(g_w)$. Then there exists $v \in \left(\tilde{\Psi}^{\Sigma}_{\mathrm{ud}}\right)^{-1}(w)$ (weakly) consistent with*

$$F(U_1(g_w)) = M(v).$$

*(ii) Let $w = u\varepsilon^* u$ be a symmetric string. Let $g_u$ be the $\mathfrak{L}$-chain corresponding to $w$ with $\overline{\mathfrak{X}}_0$-representation $U_s(g_u)$. Then there exists $v \in \left(\tilde{\Psi}^{\Sigma}_{\mathrm{ud}}\right)^{-1}(w)$ (weakly) consistent with*

$$F(U_s(g_u)) = M_i(t)$$

*where $t \in (\Psi^{\Sigma}_{\mathrm{ud}})^{-1}(u)$ with $v = t\varepsilon^{\kappa}t^{-1}$, $\kappa \in \{+1,-1\}$, and*

$$i = \begin{cases} 1 & \text{if } s = 1, \\ 0 & \text{if } s = 2. \end{cases}$$

*(iii) Let $w_{\mathbb{Z}}$ be an asymmetric band. Let $g_{w_{\mathbb{Z}}}$ be the $\mathfrak{L}$-cycle corresponding to $w_{\mathbb{Z}}$ with $\overline{\mathfrak{X}}_0$-representation $U(g_{w_{\mathbb{Z}}},\varphi)$. Then there exists $v_{\mathbb{Z}} \in \left(\tilde{\Psi}^{\Sigma}_{\mathrm{ud}}\right)^{-1}(w_{\mathbb{Z}})$ (weakly) consistent with*

$$F(U(g_{w_{\mathbb{Z}}},\varphi)) = M(v_{\mathbb{Z}},V)$$

*where $V$ is a $\mathrm{k}[T,T^{-1}]$-vector space with $\dim(V) = \deg(\varphi)$.*

*(iv) Let $w_{\mathbb{Z}}$ be a symmetric band with periodic part $\hat{w}_p = \varepsilon^* u \eta^* u^{-1}$.*

*(iv.i)* *Let $g_{w_{\mathbb{Z}}}$ be the $\mathfrak{L}$–cycle corresponding to $w_{\mathbb{Z}}$ with $\overline{\mathfrak{X}}_0$–representation $U(g_{w_{\mathbb{Z}}},\varphi)$. Then there exists $v_{\mathbb{Z}} \in \left(\tilde{\Psi}^{\Sigma}_{\mathrm{ud}}\right)^{-1}(w_{\mathbb{Z}})$ (weakly) consistent with periodic parts $\hat{v}_p^{(i)} = \varepsilon t \eta t^{-1}$ for all $i \in \mathbb{Z}$ and with*

$$F(U(g_{w_{\mathbb{Z}}},\varphi)) = M(v_{\mathbb{Z}}, V)$$

*where $V$ is a $\mathrm{k}[T,T^{-1},(T+1)^{-1}]$–vector space if $\delta_0(g)$ is even, or $V$ is a $\mathrm{k}[T,T^{-1},(T-1)^{-1}]$–vector space if $\delta_0(g)$ is odd, with $\dim(V) = \deg(\varphi)$.*

*(iv.ii)* *Let $g_u$ be the $\mathfrak{L}$–chain corresponding to $w_{\mathbb{Z}}$ with $\overline{\mathfrak{X}}_0$–representations $U_s(g_u,p)$, $p \in \mathbb{N}$. Then there exist words $v_{\mathbb{Z}} \in (\Psi^{\Sigma}_{\mathrm{ud}})^{-1}(w_{\mathbb{Z}})$ (weakly) consistent with periodic parts $\hat{v}_p^{(i)} = xtyt^{-1}$ for all $i \in \mathbb{Z}$, $x, y \in \mathrm{Sp}$, $x \in \{\varepsilon, \bar{\varepsilon}\}$, $y \in \{\eta, \bar{\eta}\}$, with*

$$\{F(U_s(g_u,p))\}_{1 \le s \le 4} = \{M_{0,1,x,y}(t^{[p]})\}_{x \in \{\varepsilon,\bar{\varepsilon}\}, y \in \{\eta,\bar{\eta}\}}$$

*where $t^{[p]}$ is given as defined in Subsection 2.3.3, Remark 2.50,*

$$x = \begin{cases} \varepsilon & \text{if } s = 1,3, \\ \bar{\varepsilon} & \text{if } s = 2,4, \end{cases}$$

$$y = \begin{cases} \eta & \text{if } s = 1,2, \\ \bar{\eta} & s = 3,4, \end{cases}$$

*and with the module $M_{i,j,x,y}(t^{[p]})$ of the form*

$$x{=}i \quad \xleftarrow{t} \; \xleftarrow{y} \; \xrightarrow{t} \; \xleftarrow{x} \quad \cdots \quad x{=}j \quad \text{if } p \text{ is even},$$

$$x{=}i \quad \xleftarrow{t} \; \xleftarrow{y} \; \xrightarrow{t} \; \xleftarrow{x} \quad \cdots \quad y{=}j \quad \text{if } p \text{ is odd},$$

*with the k–th copy of $t^{\kappa}$ acting on a respective copy of the basis elements $(b_0, \ldots, b_m)$.*

*Proof.* The proof follows from the Lemmas 5.9 - 5.15. □

**Remark 5.22.** *We obtain in (iv.ii) of the above lemma a direct assignment between the two sets. It is of the following form:*

$$\begin{aligned} F(U_1(g_u,p)) &= M_{0,1,\varepsilon,\eta}(t^{[p]}) \\ F(U_2(g_u,p)) &= M_{0,1,\bar{\varepsilon},\eta}(t^{[p]}) \\ F(U_3(g_u,p)) &= M_{0,1,\varepsilon,\bar{\eta}}(t^{[p]}) \\ F(U_4(g_u,p)) &= M_{0,1,\bar{\varepsilon},\bar{\eta}}(t^{[p]}) \end{aligned}$$

*We consider $t^{[p]}$ as subword of $v_{\mathbb{Z}}$ with periodic parts $\hat{v}_p^{(i)} = \varepsilon t \eta t^{-1}$ for $s = 1$. Similarly, for $s = 2$, we consider it as a subword of $v_{\mathbb{Z}}$ with $\hat{v}_p^{(i)} = \bar{\varepsilon} t \eta t^{-1}$, and for $s = 3$ we consider $v_{\mathbb{Z}}$ with $\hat{v}_p^{(i)} = \varepsilon t \bar{\eta} t^{-1}$, for $s = 4$ we have $\hat{v}_p^{(i)} = \bar{\varepsilon} t \bar{\eta} t^{-1}$.*

Note that we explicitely know the form of the words $v$, $v_{\mathbb{Z}}$, respectively, in $\mathcal{W}(\Sigma_{\mathrm{d}}(\Lambda))$ in the above theorem, according to the respective lemmas. We examine in the next section whether we can restrict the statement to words from $\mathcal{W}(\Gamma_{\mathrm{d}}(\Lambda))$.

**Example 5.23.** *Let $\Lambda$ be as in Example 2.14. Recall that it is given by the quiver*

$$\begin{tikzcd} Q: & 1 \arrow[loop left, "\varepsilon"] \arrow[dr, "a"] & & 5 \arrow[loop right, "\kappa"] \arrow[dd, "e"] \\ & & 2 \arrow[ur, "c"] \arrow[dr, "d"] & \\ & 3 \arrow[loop left, "\eta"] \arrow[ur, "b"] & & 4 \end{tikzcd}$$

*with* $\mathrm{Sp} = \{\varepsilon, \eta, \kappa\}$ *and* $\mathrm{R} = ca, ec, db\}$.

*(i) Let $w = d^{-1}e\kappa^{*}c$ be an asymmetric string. Its corresponding $\mathfrak{L}$−chain is given by*

$$\begin{array}{ccccccccccccccccccccc} g_w: & \mathfrak{C}_{22} & - & \mathfrak{R}_{23} & \sim & \mathfrak{R}_{41} & - & \mathfrak{C}_{41} & \sim & \mathfrak{R}_{53} & - & \multicolumn{3}{c}{\overleftarrow{\mathfrak{C}_{\kappa^*} \sim \mathfrak{C}_{\kappa^*}}} & - & \mathfrak{R}_{51} & \sim & \mathfrak{C}_{23} & - & \mathfrak{R}_{22} \\ & x_0 & & x_1 & & x_2 & & x_3 & & x_4 & & x_5 & & x_6 & & x_7 & & x_8 & & x_9 \end{array}$$

*We obtain that $F(U_1(g))$ is given by*

$$x_1 \xrightarrow{d} x_2 \xleftarrow{e} x_4 \xleftarrow{\kappa=-1} x_7 \xleftarrow{c} x_9.$$

*(ii) Let $w = \varepsilon^{*}a^{-1}b\eta^{*}b^{-1}a\varepsilon^{*}$ be a symmetric string. Its corresponding $\mathfrak{L}$−chain is given by*

$$\begin{array}{cccccccccccccccc} g_u: & \mathfrak{R}_{11} & - & \multicolumn{3}{c}{\overrightarrow{\mathfrak{C}_{\varepsilon^*} \sim \mathfrak{C}_{\varepsilon^*}}} & — & \mathfrak{R}_{12} & \sim & \mathfrak{C}_{21} & - & \mathfrak{R}_{12} & \sim & \mathfrak{R}_{32} & - & \mathfrak{C}_{\eta^*} \\ & x_0 & & x_1 & & x_2 & & x_3 & & x_4 & & x_5 & & x_6 & & x_7 \end{array}$$

*where $u = \varepsilon^{*}a^{-1}b$. This gives the following for $F(U_s(g))$:*

$$\begin{tikzcd} x_0 \arrow[r, "\bar{\varepsilon}=-1"] & x_3 \arrow[r, "a"] & x_5 & x_6 \arrow[l, "b"'] \arrow[loop right, "\eta"] \end{tikzcd}$$

*with*

$$\eta = \begin{cases} 1 & \textit{if } s = 2, \\ 0 & \textit{if } s = 1. \end{cases}$$

**Example 5.24.** *Let $\Lambda$ be as in Example 2.3.1: $\Lambda = \mathrm{k}Q/(\mathrm{R} \cup \mathrm{R}^{\mathrm{Sp}})$ where*

$$\begin{tikzcd} 1 \arrow[loop left, "\varepsilon"] \arrow[loop right, "a"] \end{tikzcd}$$

*with* $\mathrm{R} = \{a^2\}$ *and* $\mathrm{Sp} = \{\varepsilon\}$.

*(i) Let* $w_{\mathbb{Z}}$ *be a symmetric band with* $\hat{w}_p = \varepsilon^* a \varepsilon^* a^{-1}$. *The corresponding* $\mathfrak{L}$*−cycle is given by*

$$g_{w_{\mathbb{Z}}}: \quad \overleftarrow{\mathfrak{C}_{\varepsilon^*} \sim \mathfrak{C}_{\varepsilon^*}} - \mathfrak{R}_{11} \sim \mathfrak{R}_{13} - \overleftarrow{\mathfrak{C}_{\varepsilon^*} \sim \mathfrak{C}_{\varepsilon^*}} - \mathfrak{R}_{13} \sim \mathfrak{R}_{11}$$

*Take* $\varphi_0 \neq t, t-1$ *(*$\delta_0(g_{w_{\mathbb{Z}}}) = 1$*) and let* $\bar{V}$ *be a* $\mathrm{k}$*−vector space of dimension* $\deg(\varphi)$. *Then* $F(U(g_{w_{\mathbb{Z}}}, \varphi))$ *is given by*

$$\begin{tikzcd} \bar{V}_0 \arrow[rrr, bend right, "\varepsilon=-F_{\varphi}^{-1}"'] & \bar{V}_1 \arrow[l, "a"'] & \bar{V}_2 \arrow[l, "\varepsilon=-1"'] \arrow[r, "a"] & \bar{V}_3 \end{tikzcd}$$

*where the* $\bar{V}_i$*'s are disjoint copies of* $\bar{V}$.

*(ii) Note that there is also a corresponding* $\mathfrak{L}$*−chain* $g_{w_{\mathbb{Z}}}$ *for the symmetric band given in (i). It is given by*

$$g_{w_u}: \quad \mathfrak{C}_{\varepsilon^*} — \mathfrak{R}_{11} \sim \mathfrak{R}_{13} — \mathfrak{C}_{\varepsilon^*}.$$

*In order to construct* $U_s(g_{w_{\mathbb{Z}}}, 2)$, *we consider*

$$\begin{array}{rcccccccccccccccc}
 & x_0 & & x_1 & & x_2 & & x_3 & & x_4 & & x_5 & & x_6 & & x_7 \\
h: & \mathfrak{C}_{\varepsilon^*} & - & \mathfrak{R}_1 & \sim & \mathfrak{R}_{13} & - & \mathfrak{C}_{\varepsilon^*} & \overleftarrow{\sim} & \mathfrak{C}_{\varepsilon^*} & - & \mathfrak{R}_{13} & \sim & \mathfrak{R}_{11} & - & \mathfrak{C}_{\varepsilon^*} \\
\psi_1^*: & -1 & & & & & & 1 & & -1 & & & & & & 1 \\
\psi_2^*: & 1 & & & & & & 1 & & -1 & & & & & & -1 \\
\psi_3^*: & -1 & & & & & & -1 & & 1 & & & & & & 1 \\
\psi_4^*: & 1 & & & & & & -1 & & 1 & & & & & & -1
\end{array}$$

*Applying Lemma 5.14 to h results in* $F(U_s(g_{w_{\mathbb{Z}}}, 2))$ *being given by*

$$s = 1: \quad \begin{tikzcd} b_0 \arrow[loop left, "\varepsilon=0"] & b_1 \arrow[l, "a"'] & b_1^{(1)} \arrow[l, "\varepsilon=-1"'] \arrow[r, "a"] & b_0^{(1)} \arrow[loop right, "\varepsilon=1"] \end{tikzcd}$$

$$s = 2: \quad \begin{tikzcd} b_0 \arrow[loop left, "\bar{\varepsilon}=0"] & b_1 \arrow[l, "a"'] & b_1^{(1)} \arrow[l, "\varepsilon=-1"'] \arrow[r, "a"] & b_0^{(1)} \arrow[loop right, "\bar{\varepsilon}=1"] \end{tikzcd}$$

$$s = 3: \quad \begin{tikzcd} b_0 \arrow[loop left, "\varepsilon=0"] & b_1 \arrow[l, "a"'] & b_1^{(1)} \arrow[l, "\bar{\varepsilon}=-1"'] \arrow[r, "a"] & b_0^{(1)} \arrow[loop right, "\varepsilon=1"] \end{tikzcd}$$

$$s = 4: \quad \begin{tikzcd} b_0 \arrow[loop left, "\bar{\varepsilon}=0"] & b_1 \arrow[l, "a"'] & b_1^{(1)} \arrow[l, "\bar{\varepsilon}=-1"'] \arrow[r, "a"] & b_0^{(1)} \arrow[loop right, "\bar{\varepsilon}=1"] \end{tikzcd}$$

*and* $b_i^{(1)}$ *a copy of* $b_i$. *We can also display the modules in a different way. For example, for* $s = 1$, *this reads:*

$$\begin{tikzcd} V_1 \arrow[loop left, "\varepsilon=\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}"] & V_2 \arrow[l, "a"'] \arrow[loop right, "\varepsilon=\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}"] \end{tikzcd}$$

*for* $V$ *a* $\mathrm{k}$*−vector space of dimension two, and with* $V_1$ *and* $V_2$ *being disjoint copies of* $V$.

*(iii) Let $w_{\mathbb{Z}}$ be an asymmetric band with $\hat{w}_p = a^{-1}\varepsilon^* a^{-1}\varepsilon^*$. Its corresponding $\mathfrak{L}$−cycle is given by*

$$g_{w_{\mathbb{Z}}}: \ \mathfrak{R}_{13} \sim \mathfrak{R}_{11} - \overrightarrow{\mathfrak{C}_{\varepsilon^*} \sim \mathfrak{C}_{\varepsilon^*}} - \mathfrak{R}_{13} \sim \mathfrak{R}_{11} - \overrightarrow{\mathfrak{C}_{\varepsilon^*} \sim \mathfrak{C}_{\varepsilon^*}}$$

*Take $\varphi_0 \neq t$. We obtain that $F(U(g_{w_{\mathbb{Z}}}, \varphi))$ is given by*

$$\begin{tikzcd} \bar{V}_0 \arrow[r, "\bar{\varepsilon}=-F_{\varphi}^{-1}"] & \bar{V}_1 \arrow[r, "a"] & \bar{V}_2 \arrow[r, "\bar{\varepsilon}=-1"] & \bar{V}_3 \arrow[lll, bend left, "a"] \end{tikzcd}$$

## 5.5 The image of $F$ in terms of $\mathcal{W}(\Gamma_{\mathrm{d}}(\Lambda))$

We have seen in the previous section that the image of the functor $F$ can be desribed in terms of words in the alphabet $\Sigma_{\mathrm{d}}(\Lambda)$. By definition, we have that $\mathcal{W}(\Gamma_{\mathrm{d}}(\Lambda)) \subset \mathcal{W}(\tilde{\Sigma}_{\mathrm{d}}(\Lambda)) \subset \mathcal{W}(\Sigma_{\mathrm{d}}(\Lambda))$. In this section, we examine whether we can also express the image of $F$ in terms of $\mathcal{W}(\Gamma_{\mathrm{d}}(\Lambda))$. We will see that the answer is affirmative for the image of canonical representations arising from $\mathfrak{L}$–cycles and from $\mathfrak{L}$–chains without two double ends. Recall that we consider $\Lambda$ to be a skewed-gentle algebra.

To this end, we show first that two words $v_{\mathrm{I}}$, $v'_{\mathrm{I}}$ in $\mathcal{W}(\tilde{\Sigma}_{\mathrm{d}}(\Lambda))$ which differ only in one special letter by $v_j = \bar{v}'_j$ for some $j \in \mathrm{I}$, give rise to isomorphic modules.
Let $\mathrm{I} = \{0, \dots, n\}$. We denote by $\mu_j$ the map

$$\begin{aligned} \mu_j : \quad & \mathcal{W}(\Sigma_{\mathrm{d}}(\Lambda)) && \longrightarrow \mathcal{W}(\Sigma_{\mathrm{d}}(\Lambda)) \\ & (\dots, v_{j-1}, v_j, v_{j+1}, \dots) && \longmapsto (\dots, v_{j-1}, \bar{v}_j, v_{j+1}, \dots) \end{aligned}$$

sending the special letter $v_j$ of a word $v_{\mathrm{I}}$ to $\bar{v}_j$, where $j \in \{1, \dots, n\}$.
Let $\mathrm{I} = \mathbb{Z}$ and let $v_{\mathrm{I}}$ be a $\mathbb{Z}$–word of period $p$. Assume that $\hat{v}_p^{(i)} = \hat{v}_p^{(k)}$ for all $i, k \in \mathbb{Z}$. We denote then by $\mu_j$ the map

$$\begin{aligned} \mu_j : \quad & \mathcal{W}(\Sigma_{\mathrm{d}}(\Lambda)) && \longrightarrow \mathcal{W}(\Sigma_{\mathrm{d}}(\Lambda)) \\ & (v_1, \dots, v_j, \dots, v_p)^{(i)} && \longmapsto (v_1, \dots, \bar{v}_j, \dots, v_p)^{(i)} \end{aligned}$$

for all $i \in \mathbb{Z}$, sending the special letter $v_j$ of any periodic part of the word $v_{\mathrm{I}}$ to $\bar{v}_j$, where $j \in \{1, \dots, p\}$.
For example, $\mu_j$ sends $v_{\mathrm{I}}$ to $v'_{\mathrm{I}}$ and vice versa, since $\mu_j$ is self-inverse.

**Remark 5.25.** *Due to the definition of $\mu_j(v_{\mathrm{I}})$, we have that*

$$\operatorname{dir}(v_i) = \operatorname{dir}((\mu_j(v_{\mathrm{I}}))_i) \ \forall i \in \mathrm{I} \ \textit{and for any } j \in \mathrm{I}.$$

*Furthermore, we have that*

$$\begin{aligned} v_i &= (\mu_j(v_{\mathrm{I}}))_i \ \forall i \neq j, \\ v_j &= \overline{(\mu_j(v_{\mathrm{I}}))_j} \end{aligned}$$

**Proposition 5.26.** *Let $w$ be an asymmetric string in $\Gamma_{\mathrm{ud}}(\Lambda)$. Let $v, v' \in \mathcal{W}(\Sigma_{\mathrm{d}}(\Lambda))$ such that $\mu_j(v) = v'$ for some $j \in \mathrm{I}$, $\Psi_\Lambda^\Sigma(v) = \Psi_\Lambda^\Sigma(v')$ is (weakly) consistent and $\Psi_{\mathrm{ud}}^\Sigma(v) = \Psi_{\mathrm{ud}}^\Sigma(v') = w$. Then $M(v) \cong M(v')$.*

*Proof.* We show the statement by induction on $\operatorname{ind}_j^*(v)$. Let $\mathrm{I} = \{0, \dots, n\}$. Assume that $\operatorname{dir}(v_j) = -1$ and $v_j = \varepsilon^{-1}$ for some $\varepsilon \in \mathrm{Sp}$ (the other cases follow analogously). Denote the basis of the k–vector space $M(v)$ by $(b_0, \dots, b_n)$ where $v_i(b_i) = b_{i-1}$ for all $1 \leq i \leq n$. Similarly, denote by $(c_0, \dots, c_n)$ the basis

of $M(v')$.
Let $\mathrm{ind}_j^*(w) = 0$. Let $f : M(v) \longrightarrow M(v')$ be given by

$$b_i \longmapsto \begin{cases} c_i & \forall i \le j-1, \\ -c_i & \forall i \ge j+1, \end{cases} \qquad \forall i \ne j,$$
$$b_j \longmapsto c_{j-1} - c_j.$$

Then $f$ is injective. It follows that $f$ is bijective. It remains to show that $f$ is a morphism of $\Lambda$–modules. To this end, note that $\mathrm{ind}_j^*(v') = \mathrm{ind}_j^*(v)$. Thus, we have that $\mathrm{dir}(v'_{j-1}) = \mathrm{dir}(v'_{j+1}) = \mathrm{dir}(v'_j) = -1$, and $v'_{j-1} = x^{-1}$, $v'_{j+1} = y^{-1}$ where $x$ and $y$ are ordinary letters. In particular, we have that $v_{j-1} = v'_{j-1}$ and $v_{j+1} = v'_{j+1}$. For $j \in \mathrm{I}$, we have that $v_j = \varepsilon^{-1}$ and $v'_j = \bar{\varepsilon}^{-1}$. We obtain by definition of $f$:

$$\varepsilon f(b_{j-1}) = \varepsilon(c_{j-1}) = (1_{s(\varepsilon)} - \varepsilon)(c_{j-1}) = c_{j-1} - c_j,$$
$$f\varepsilon(b_{j-1}) = f(b_j) = c_{j-1} - c_j,$$

where $s(\varepsilon)$ denotes the start vertex of $\varepsilon$ in the quiver. Similarly, we have that

$$fy(b_j) = f(b_{j+1}) = -c_{j+1},$$
$$yf(b_j) = y(c_{j-1} - c_j) = -c_{j+1}. \tag{239}$$

Recall that $yx = 0$ since $\Lambda$ is skewed-gentle. This implies that $y(c_{j-1}) = 0$ which gives (239).
For all other indices, we have that $b_i = c_i$ and $v_i = v'_i$. Thus, $f$ gives a module isomorphism between $M(v)$ and $M(v')$.

Let now $\mathrm{ind}_j^*(w) = d > 0$. Let $w_{j_-^*} = y$ and $w_{j_+^*} = x$. Let $k, l \in J^*$ such that $w_k$ and $w_l$ are of special type and $|j-k| = |j-l| \le d$. Assume for now that this is the only pair in $J^*$ with those properties. Let $w_k = w_l = \eta^*$ for some $\eta \in \mathrm{Sp}$. We either have $\mathrm{dir}(v_k) = -\mathrm{dir}(v_l)$ or $\mathrm{dir}(v_k) = \mathrm{dir}(v_l)$. In both cases, by Lemma 3.55, we either have that $\mathrm{ind}_k *(w) < d$ and $\mathrm{ind}_l^*(w) < d$, or that $\mathrm{ind}_k^*(w) < d$ and $\mathrm{ind}_l^*(w) \ge d$.
Let us first consider $\mathrm{dir}(v_k) = -\mathrm{dir}(v_l)$ with $\mathrm{ind}_k^*(w) < d$, $\mathrm{ind}_l^*(w) < d$. Assume without loss of generality that $\mathrm{dir}(v_k) = \mathrm{dir}(v_j) = -1$ and thus $\mathrm{dir}(v_l) = 1$. By induction, we have that

$$M(\mu_l(v)) \cong M(v) \cong M(\mu_k(v)). \tag{240}$$

It follows that $M(v) \cong M(\mu_l\mu_k(v))$. Thus, we can assume $v_k$ and $v_l$ to be such that $v_k = v_l^{-1}$. Recall that we also have by definition of $\mu_j$ that $v'_k = (v'_l)^{-1}$. Let $f : M(v) \longrightarrow M(v')$ be given by

$$b_i \longmapsto \begin{cases} c_i & \forall i \le j-1, \\ -c_i & \forall i \ge j_+^*, \end{cases} \quad \forall i \notin \{j, \dots, j_+^* - 1\},$$
$$b_{j+i} \longmapsto c_{j-i-1} - c_{j+i} \qquad \text{for all } i \in \mathrm{I} \text{ such that } j+i \in \{j, \dots, j_+^* - 1\}.$$

This map is injective and by $M(v)$ and $M(v')$ having same dimension, $f$ is bijective. It remains to show that $f$ is a morphism of $\Lambda$–modules.
Since $f : b_i \mapsto c_i$ and $v_i = v'_i$ for all $i \leq j-1$, it remains to check the commutativity relation for all $i \geq j$. Consider $i = j$. We have $v_j = \varepsilon^{-1}$ and $v'_j = \bar{\varepsilon}^{-1}$ and the commutativity relation to be checked is given by $f\varepsilon = \varepsilon f$. We have that $\bar{\varepsilon}(c_{j-1}) = c_j$ which yields that $\varepsilon(c_{j-1}) = c_{j-1} - c_j = f(b_j)$. Consider now $i \in \mathrm{I}$ such that $j + i \in \{j+1, \ldots, j_+^* - 1\}$. By symmetry in $v_j$ we have that $v_{j+i} = (v_{j-i})^{-1}$. Assume without loss of generality that $\mathrm{dir}(v_{j+i}) = 1$ and $\mathrm{dir}(v_{j-i}) = -1$. Furthermore, by definition of $\mu_j$, we have that $v_{j+i} = v'_{j+i}$ and that $v_{j-i} = v'_{j-i}$. Thus, it follows for $v_{j-i}(b_{j-i}) = b_{j-i-1}$ that $v_{j+i}(b_{j-i-1}) = b_{j-i}$. The commutativity relation thus follows from the following:

$$\begin{aligned}
f v_{j+i}(b_{j+i}) &= f(b_{j+i-1}) = c_{j-i} - c_{j+i-1},\\
v'_{j+i} f(b_{j+i}) &= v_{j+i}(c_{j-i-1} - c_{j+i}) = c_{j-i} - c_{j+i-1}.
\end{aligned}$$

The commutativity relation follows for all other indices $j+i \in \{j+2, \ldots, j_+^* - 1\}$ analogously, in particular, since we have that $v_k = (v_l)^{-1}$.
The next in line is to show that $fx = xf$ for the index $j_+^*$. We have that $x(b_{j_+^*-1}) = b_{j_+^*}$. Moreover, we have by symmetry in position $j$ that $x(b_{j_-^*}) = 0$: otherwise, we have that $v_{j_-^*+1} = x^{-1}$ and symmetry yields that $v_{j_+^*-1} = x$ which contradicts the definition of a word. Thus, we obtain that

$$\begin{aligned}
fx(b_{j_+^*-1}) &= f(b_{j_+^*}) = -c_{j_+^*},\\
xf(b_{j_+^*-1}) &= x(c_{j_-^*} - c_{j_+^*-1}) = -c_{j_+^*}.
\end{aligned}$$

For all $i \geq j_+^*$ we have that $f : b_i \mapsto c_i$ and $v_i = v'_i$. Hence, the commutativity follows for all $i \geq j_+^*$. The existence of the commutativity relations shows that $f : M(v) \longrightarrow M(v')$ is a $\Lambda$–module isomorphism.
The case $\mathrm{dir}(v_k) = -\mathrm{dir}(v_l)$ with $\mathrm{ind}_k^*(w) < d$ and $\mathrm{ind}_l *(w) \geq d$ follows analogously (by Lemma 3.59: $\mathrm{dir}(v_k) = \mathrm{dir}(v_j)$).
Consider now the case where $\mathrm{dir}(v_k) = \mathrm{dir}(v_l)$. It follows by Lemma 3.56 and 3.59 that $\mathrm{dir}(v_k) = \mathrm{dir}(v_l) = \mathrm{dir}(v_j)$ in both cases ($\mathrm{ind}_k^*(w) < d$, $\mathrm{ind}_l^*(w) < d$, or $\mathrm{ind}_k^*(w) < d$, $\mathrm{ind}_l^*(w) \geq d$). By induction, we can use again the bijections given in (240). Thus, we can assume $v_k$ and $v_l$ to be such that $v_k = \bar{v}_l$. Let$f : M(v) \longrightarrow M(v')$ be given by

$$\begin{aligned}
b_i &\longmapsto \begin{cases} c_i & \forall i \leq j-1, \\ -c_i & \forall i \geq l, \end{cases} && \forall i \notin \{j, \ldots, l-1\},\\
b_{j+1} &\longmapsto c_{j-i-1} - c_{j+i} && \text{for all } i \in \mathrm{I} \text{ such that } j+i \in \{j, \ldots, l-1\}.
\end{aligned}$$

It follows analogously to the previous case that $f$ is a module isomorphism. The key step in this case is given for the index $l$. Assume that $v_l = \eta^{-1}$ for some $\eta \in \mathrm{Sp}$. Then $v_l(b_l) = b_{l-1}$ yields that $\eta(b_{l-1}) = b_l$. It follows that

$v_k = \bar{\eta}^{-1}$ with $\bar{\eta}(b_{k-1}) = b_k$. In particular, it follows that $\eta(b_k) = 0$. We obtain for the index $l$ that

$$\eta f(b_{l-1}) = \eta(b_k - b_{l-1}) = -c_l,$$
$$f\eta(b_{l-1}) = f(b_l) = -c_l.$$

□

Finally, let $\{k_i\}_{1\le i\le n}$ and $\{l_i\}_{1\le i\le n}$ be in $J^*$ such that all $w_{k_i}$ and $w_{l_i}$ are of special type and $|j-k_i| = |j-l_i| \le d$ for all $i \in \{1,\dots,n\}$. Assume additionally that all pairs of this form in $J^*$ are described by the above sets. For all pairs $(k_i, l_i)$ with $\mathrm{dir}(w_{k_i}) = -\mathrm{dir}(w_{l_i})$ we proceed as described above, starting with the pair $(k_j, l_j)$ with smallest $|j-k_j|$. This is the pair $(w_{k_j}, w_{l_j})$ which is closest to $w_j$. From this one, we continue towards the margin of $J^*$. We proceed with all pairs of this form, until we either reach a pair $(k_p, l_p)$ with $\mathrm{dir}(w_{k_p}) = \mathrm{dir}(w_{l_p})$, or we reach the margin of $J^*$. In the first case, we also proceed with the pair $(k_p, l_p)$ as described above. The result follows.

**Example 5.27.** *Let $\Lambda$ be given by the two-loop quiver with relations as described in Example 2.3.1. Let $w = a\varepsilon^* a\varepsilon^* a^{-1}\varepsilon^* a$ be an asymmetric string in $\Gamma_{\mathrm{ud}}(\Lambda)$. Let $v = a\varepsilon a\varepsilon a^{-1}\varepsilon a$ be a directed version of $w$ and let $v' = \mu_4(v) = a\varepsilon a\bar{\varepsilon} a^{-1}\varepsilon a$.*
*In order to see that $M(v) \cong M(v')$, consider first $v'' = a\bar{\varepsilon} a\bar{\varepsilon} a^{-1}\varepsilon a$. Let the bases as $\mathrm{k}-$vector spaces be given by $(b_0,\dots,b_7)$ for $M(v)$, by $(c_0,\dots,c_7)$ for $M(v')$ and by $(d_0,\dots,d_7)$ for $M(v'')$. In the proof of Proposition 5.26, it is actually shown that $M(v'') \cong M(v)$ and that $M(v'') \cong M(v')$. In detail, the module isomorphism $f : M(v) \to M(v'')$ is given by*

$$\begin{aligned} b_i &\longmapsto -d_i && \forall i \le 1,\\ b_{3-i} &\longmapsto d_{3+i+1} - d_{3-i} && \forall i \in \{0,1\},\\ b_i &\longmapsto d_i && \forall i \ge 4. \end{aligned}$$

*The module isomorphism $g : M(v'') \to M(v')$ is given by*

$$\begin{aligned} d_i &\longmapsto c_i && \forall i \ge 2,\\ d_1 &\longmapsto c_2 - c_1, &&\\ d_0 &\longmapsto -c_0. && \end{aligned}$$

*Thus, the module isomorphism $M(v) \cong M(v')$ is given by $g \circ f$.*

**Proposition 5.28.** *Let $w = u\varepsilon^* u^{-1}$ be a symmetric string in $\Gamma_{\mathrm{ud}}(\Lambda)$ for $\Lambda$ a skewed-gentle algebra, $\varepsilon \in \mathrm{Sp}$, $|u| = m$. Let $v = t\varepsilon^{\kappa} t^{-1}, v' = t'\varepsilon^{\kappa'}(t')^{-1} \in \mathcal{W}(\Sigma_{\mathrm{d}}(\Lambda))$ with $\kappa, \kappa' \in \{+1,-1\}$, such that $\mu_j(t) = t'$ for some $j \in \{1,\dots,m\}$, $\Psi^{\Sigma}_{\Lambda}(v) = \Psi^{\Sigma}_{\Lambda}(v')$ is (weakly) consistent and $\Psi^{\Sigma}_{\mathrm{ud}}(v) = \Psi^{\Sigma}_{\mathrm{ud}}(v') = w$. Then $M_i(v) \cong M_i(v')$ for $i = 0,1$.*

*Proof.* Assume that $\mathrm{dir}(v_j) = -1$, $v_j = \eta^{-1}$. The other cases follow analogously. Recall that the $\Lambda$–module $M_i(v)$ can be written as follows:

$$b_0 \xleftarrow{v_1} b_1 \xleftarrow{v_2} \cdots \xleftarrow{v_{m-1}} b_{m-1} \xleftarrow{v_m} b_m \circlearrowright \varepsilon{=}i \tag{241}$$

Thus, the proof is analogous to the proof of Proposition 5.26, apart from the case $v_{m+1} \in J^*$ in the induction step.
Let $j \neq m+1$ such that $v_{m+1} \in J^*$. We have that $\mathrm{ind}^*_{m+1} \geq d$. Let $k \in J^*$ such that $|j-k| = |j-(m+1)|$. By Lemma 3.55 we know that $\mathrm{ind}^*_k(w) < d$. Lemma 3.59 yields that $\mathrm{dir}(v_k) = \mathrm{dir}(v_j)$. Hence, $v_k \in \{\varepsilon^{-1}, \bar{\varepsilon}^{-1}\}$ since $v_{m+1} = \varepsilon$. Let the basis of $M_i(v)$ be as described above be given by $(b_0, \ldots, b_{2m})$ and the one of $M_i(v')$ be given by $(c_0, \ldots, c_{2m})$. Let $f : M_i(v) \to M_i(v')$ with $i = 1$ be given by

$$\begin{aligned} b_q &\longmapsto c_i && \forall i \leq j-1, \\ b_{q+i} &\longmapsto c_{j-q-+1} - c_{j+q} && \forall q \in \mathrm{I} \text{ such that } j+q \in \{j, \ldots, m\}, \\ b_q &\longmapsto -c_i && \forall i \geq m+1. \end{aligned}$$

Then $f$ is an injective and thus bijective map between the vector spaces $M_i(v)$ and $M_i(v')$. In order to show that $f$ is a $\Lambda$–module morphism, we examine the commutativity relations of the form $f\lambda = \lambda f$, $\lambda \in \Lambda$. We have that $f : b_q \mapsto c_q$ and $v_q = v'_q$ for all $q \leq j-1$. Thus, the above relation directly follows for those indices. A similar argument yields the relations for any $q \geq m+1$. The cases $j \leq q \leq m$ follow analogously to the case of an asymmetric string using symmetry in $j$ and definition of $\mu_j$. The special case to consider here is $q = m$. Recall that $f : b_m \mapsto c_k - c_m$. By induction, we know that $M(\mu_k(v')) \cong M(v')$ and similarly, that $M(\mu_k(v)) \cong M(v)$. By induction, we can also assume that $v_k = \varepsilon^{-1}$. Thus, $c_k \in \mathrm{im}(\varepsilon)$. It follows that

$$\begin{aligned} f\varepsilon(b_m) &= f(b_m) = c_k - c_m, \\ \varepsilon f(b_m) &= \varepsilon(c_k - c_m) = c_k - c_m, \end{aligned}$$

yielding the commutativity relation. Note that we write here $\varepsilon(b_m) = b_m$ according to the depiction (241). Consider now the case $i = 0$. By induction, we can consider $v_k = \bar{\varepsilon}^{-1}$. Thus, $c_k \in \mathrm{im}(\bar{\varepsilon}) = \ker(\varepsilon)$. We obtain that $\varepsilon(c_k - c_m) = 0$. The commutativity relation follows. $\square$

**Example 5.29.** *Let $\Lambda$ be given by the two-loop quiver with relations as described in Example 2.3.1. Let $w = u\varepsilon^* u^{-1}$ be a symmetric string in $\Gamma_{\mathrm{ud}}(\Lambda)$ with $u = \varepsilon^* a^{-1} \varepsilon^* a$. Consider $v = \varepsilon^{-1} a^{-1} \varepsilon^{-1} a \varepsilon^{\kappa} a^{-1} \varepsilon a \varepsilon$ and $v' = \varepsilon^{-1} a^{-1} \bar{\varepsilon}^{-1} a \varepsilon^{\kappa} a^{-1} \bar{\varepsilon} a \varepsilon$ in $\tilde{\Sigma}_{\mathrm{d}}(\Gamma)$. Note that $\mu_3(\varepsilon^{-1} a^{-1} \varepsilon^{-1} a) = \varepsilon^{-1} a^{-1} \bar{\varepsilon}^{-1} a$. The word $v$ is in particular given in terms of the alphabet $\Gamma_{\mathrm{d}}(\Lambda)$ and (weakly) consistent. Consider additionally $v'' = \bar{\varepsilon}^{-1} a^{-1} \bar{\varepsilon}^{-1} a \varepsilon^{\kappa} a^{-1} \varepsilon a \bar{\varepsilon}$. We have that $\mu_1(\bar{\varepsilon}^{-1} a^{-1} \bar{\varepsilon}^{-1} a) = \varepsilon^{-1} a^{-1} \bar{\varepsilon}^{-1} a$. Denote by $(b_0, \ldots, b_4, b_5, \ldots, b_9)$ the basis of the $\mathrm{k}$–vector space*

*$M_i(v)$, by $(c_0,\dots,c_4,c_5,\dots,c_9)$ the basis of $M_i(v')$ and by $(d_0,\dots,d_4,d_5,\dots,d_9)$ the basis of $M_i(v'')$.*
*Let $i=1$. We obtain that $M_1(v)\cong M_1(v')$ as modules by sending*

$$\begin{aligned} b_i &\longmapsto c_i && \forall i\le 2,\\ b_3 &\longmapsto c_2-c_3,\\ b_4' &\longmapsto c_1-c_4,\\ b_i &\longmapsto -c_i && \forall i\ge 5. \end{aligned}$$

*Now let $i=0$ and consider the module homomorphism $f:M_0(v)\to M_0(v'')$ which is given by*

$$\begin{aligned} b_0 &\longmapsto d_0,\\ b_1 &\longmapsto d_0-d_1,\\ b_2 &\longmapsto d_2,\\ b_3 &\longmapsto -d_2-d_3,\\ b_4 &\longmapsto -d_1-d_4,\\ b_i &\longmapsto d_i && \forall i\ge 5. \end{aligned}$$

*It follows that $f$ is a module isomorphism. Consider next the module homomorphism $g:M_0(v')\to M_0(v'')$ given by*

$$\begin{aligned} c_0 &\longmapsto d_0,\\ c_1 &\longmapsto d_0-d_1,\\ c_i &\longmapsto -d_i && \forall 2\le i\le 4,\\ c_i &\longmapsto d_i && \forall i\ge 5. \end{aligned}$$

*It follows that $g$ is an isomorphism. The module isomorphism $M_0(v)\cong M_0(v')$ is given by $g^{-1}\circ f$.*

Let $v_{\mathbb{Z}}$ be a $\Gamma_{\mathrm{d}}-\mathbb{Z}$–word, $V$ be a $\mathrm{k}[T,T^{-1}]$–module with $T$ acting as $A\in\mathrm{End}(V)$. We encode this information in the following additionally and may write $M(v_{\mathbb{Z}},(V,A))$ instead of $M(v_{\mathbb{Z}},V)$, if convenient.

**Proposition 5.30.** *Let $w_{\mathbb{Z}}$ be an asymmetric band of period $p$ with periodic part $\hat{w}_p$. Let $v_{\mathbb{Z}}$, $v'_{\mathbb{Z}}\in\mathcal{W}(\Sigma_{\mathrm{d}}(\Lambda))$ with $\hat{v}_p^{(i)}=\hat{v}_p^{(k)}$, $\hat{v'}_p^{(i)}=\hat{v'}_p^{(k)}$ for all $i,k\in\mathbb{Z}$ and such that $\mu_j(v_{\mathbb{Z}})=v'_{\mathbb{Z}}$ for some $j\in\{1,\dots,p\}$, $\Psi_\Lambda^\Sigma(v_{\mathbb{Z}})=\Psi_\Lambda^\Sigma(v'_{\mathbb{Z}})$ is (weakly) consistent and $\Psi_{\mathrm{ud}}^\Sigma(v_{\mathbb{Z}})=\Psi_{\mathrm{ud}}^\Sigma(v'_{\mathbb{Z}})=w_{\mathbb{Z}}$. Then*

$$M(v_{\mathbb{Z}},(V,A)\cong M(v'_{\mathbb{Z}},(V,-A))$$

*for $V$ a $k[T,T^{-1}]$–module, $A\in\mathrm{End}(V)$ invertible.*

*Proof.* We show the statement by induction on the $c^*$–index of $w_j$. Recall that $\mathrm{ind}_j^*(w_{\mathbb{Z}}) < \frac{p}{2}$ (cf. Corollary 3.31, Remark 3.32)). Thus, it follows that $|J^*| \le p-2$ and we can assume for simplicity that $J^* \subseteq [2, p-1]$ (otherwise, the proof follows analogously with a respective shift on the indices).
Let $\mathrm{ind}_j^*(w_{\mathbb{Z}}) = 0$. We have that $v_{j-1} = v_{j_-^*}$ and that $v_{j+1} = v_{j_+^*}$, and similarly in $v'_{\mathbb{Z}}$. Thus, $\mathrm{dir}(v_{j-1}) = \mathrm{dir}(v_{j+1}) = \mathrm{dir}(v_j)$, and similarly in $v'_{\mathbb{Z}}$. Assume that $v_j = \varepsilon^{-1}$. The case $v_j = \varepsilon$ follows analogusly. We have that $v'_j = \bar{\varepsilon}^{-1}$. Let $v_{j-1} = x^{-1}$ and $v_{j+1} = y^{-1}$. Recall that $v_{j-1} = v'_{j-1}$ and $v_{j+1} = v'_{j+1}$. Let $v_i(b_i) = b_{i-1}$ for all $i \in \mathbb{Z}$.
Consider the ring isomorphism $g : \mathrm{k}[T, T^{-1}] \longrightarrow \mathrm{k}[T, T^{-1}]$, $T \mapsto -T$. As first step, we show that $h : {}_{\Lambda}M(v_{\mathbb{Z}})_{\mathrm{k}[T,T^{-1}]} \longrightarrow {}_{\Lambda}M(v'_{\mathbb{Z}})_g$ is an isomorphism of bimodules. Here, $M(v'_{\mathbb{Z}})_g$ denotes the right $\mathrm{k}[T, T^{-1}]$–module $M(v'_{\mathbb{Z}})$ restricted to $g$. Recall that $M(v_{\mathbb{Z}})$ ecomes a $\Lambda - \mathrm{k}[T, T^{-1}]$–modules by $T$ acting as $t_{v_{\mathbb{Z}},p}$. Hence, we have that $b_i T := t_{v_{\mathbb{Z}},p}(b_i)$ for any $b_i \in M(v_{\mathbb{Z}})$, with $t_{v_{\mathbb{Z}},p}$ the shift by $-p$ on $v_{\mathbb{Z}}$. We denote the respective operation of $M(v_{\mathbb{Z}})_g$ by $b_i \star T := b_i(-T)$.
Let $q \in \mathbb{Z}$ be the index of the periodic parts of $v_{\mathbb{Z}}$, $v'_{\mathbb{Z}}$, respectively. We choose for $q$ even:

$$\begin{aligned} h: \quad & b_{i+qp} && \longmapsto \begin{cases} c_{i+qp} & , 1 \le i \le j-1, \\ -c_{i+qp} & , j+1 \le i \le p, \end{cases} \\ & b_{j+qp} && \longmapsto c_{j-1+qp} - c_{j+qp}, \end{aligned}$$

and for $q$ odd:

$$\begin{aligned} h: \quad & b_{i+qp} && \longmapsto \begin{cases} -c_{i+qp} & , 1 \le i \le j-1, \\ c_{i+qp} & , j+1 \le i \le p, \end{cases} \\ & b_{j+qp} && \longmapsto -c_{j-1+qp} + c_{j+qp}. \end{aligned}$$

By definition, $h$ is a bijective map between k–vector spaces. Let us show next that it is a morphism of left $\Lambda$–modules. To this end, we observe that the key points are given at $v_j$ and $v_{j+1}$ in $\hat{v}_p^{(k)}$. Let $k$ be even. We obtain that

$$\begin{aligned} h\varepsilon(b_{j-1+qp}) &= h(b_{j+qp}) = c_{j-1+qp} - c_{j+qp}, \\ \varepsilon h(b_{j-1+qp}) &= \varepsilon(c_{j-1+qp}) = (1-\bar{\varepsilon})(c_{j-1+qp}) = c_{j-1+qp} - \bar{\varepsilon}(c_{j-1+qp}), \\ &= c_{j-1+qp} - c_{j+qp}, \end{aligned}$$

and that

$$\begin{aligned} hy(b_{j+qp}) &= h(b_{j+1+qp}) = -c_{j+1+qp}, \\ yh(b_{j+qp}) &= y(c_{j-1+qp} - c_{j+qp}) = y(-c_{j+qp}) = -c_{j+1+qp}. \end{aligned}$$

Here, we have used in the last equation that $yx = 0$ by definition of $\Lambda$ being skewed-gentle. This gives that $y(c_{j-1+qp}) = 0$.
For $q$ odd, we have that

$$
\begin{aligned}
h\varepsilon(b_{j-1+qp}) &= h(b_{j+qp}) = -c_{j-1+qp} + c_{j+qp},\\
\varepsilon h(b_{j-1+qp}) &= \varepsilon(-c_{j-1+qp}) = (1-\bar{\varepsilon})(-c_{j-1+qp}) = -c_{j-1+qp} - \bar{\varepsilon}(-c_{j-1+qp}),\\
&= -c_{j-1+qp} + c_{j+qp},
\end{aligned}
$$

and that

$$
\begin{aligned}
hy(b_{j+qp}) &= h(b_{j+1+qp}) = c_{j+1+qp},\\
yh(b_{j+qp}) &= y(-c_{j-1+qp} + c_{j+qp}) = y(c_{j+qp}) = c_{j+1+qp}.
\end{aligned}
$$

These equations give commutativity relations at the positions $j$ and $j+1$ in each periodic part. The other commutativity relations follow by definition of $h$. Hence, $h$ is a morphism of left $\Lambda$–modules.
Finally, we show that $h$ is a morphism between the modules $M(v_{\mathbb{Z}})_{\mathrm{k}[T,T^{-1}]}$ and $M(v'_{\mathbb{Z}})_g$. Let $q$ be even. Then we have that

$$
h(b_{i+qp}T) = h(b_{i+(q-1)p}) = \begin{cases} -c_{i+(q-1)p}, & 1 \le i \le j-1,\\ \phantom{-}c_{i+(q-1)p}, & j+1 \le i \le p,\\ -c_{j-1+(q-1)p} + c_{j+(q-1)p}, & i = j,\end{cases}
$$

$$
\begin{aligned}
(h(b_{i+qp})) \star T &= \begin{cases} \phantom{-}c_{i+qp} \star T & 1 \le i \le j-1,\\ -c_{i+qp} \star T & j+1 \le i \le p,\\ (c_{j-1+qp} - c_{j+qp}) \star T & i = j,\end{cases}\\
&= \begin{cases} -c_{i+(q-1)p} & 1 \le i \le j-1,\\ \phantom{-}c_{i+(q-1)p} & j+1 \le i \le p,\\ -c_{j-1+qp} + c_{j+qp} & i = j.\end{cases}
\end{aligned}
$$

The above yields that $h(b_{i+qp}T) = h(b_{i+qp}) \star T$ for $q$ even. Similarly, we obtain for $q$ odd the respective commutativity relations.
The above yields that

$$
M(v_{\mathbb{Z}}, (V, A)) = M(v_{\mathbb{Z}}) \otimes_{\mathrm{k}[T,T^{-1}]} (V, A) \cong M(v'_{\mathbb{Z}})_g \otimes_{\mathrm{k}[T,T^{-1}]} (V, A).
$$

Note that we can write

$$
M(v'_{\mathbb{Z}})_g \otimes_{\mathrm{k}[T,T^{-1}]} (V, A) \cong \left(M(v'_{\mathbb{Z}}) \otimes_{\mathrm{k}[T,T^{-1}]} \mathrm{k}[T, T^{-1}]_g\right) \otimes_{\mathrm{k}[T,T^{-1}]} (V, A).
$$

Recall that $T$ acts as $A$ on the vector space $V$. We consider for $v \in V$ the element $1 \otimes v \in \mathrm{k}[T, T^{-1}]_g \otimes_{\mathrm{k}[T,T^{-1}]} (V, A)$:

$$
T(1 \otimes v) = T \otimes v = (-1 \star T) \otimes v = -1 \otimes Tv = -1 \otimes Av.
$$

Thus, it follows that

$$M(v'_{\mathbb{Z}})_g \otimes_{\mathrm{k}[T,T^{-1}]} (V,A) \cong M(v'_{\mathbb{Z}}) \otimes_{\mathrm{k}[T,T^{-1}]} (V,-A).$$

Summarising, we obtain that

$$M(v_{\mathbb{Z}},(V,A)) \cong M(v'_{\mathbb{Z}},(V,-A)).$$

Let now $\mathrm{ind}^*_j(w_{\mathbb{Z}}) = d > 0$.
Assume that $\mathrm{dir}(v_j) = -1$ and that $v_j = \varepsilon^{-1}$ (the cases $v_j = \bar{\varepsilon}^{-1}$ and $\mathrm{dir}(v_j) = 1$ follow analogously). Let $k,l \in J^*$ such that $|k-j| = |l-j|$ and $w_k = w_l = \kappa^*$ for some $\kappa \in \mathrm{Sp}$. Assume without loss of generality that $k < j < l$ and that $(k,l)$ is the pair in $J^*$ with $|k-j| = |l-j|$ minimal. Let $w_{j^*_-} = x^{-1}$ and $w_{j^*_+} = y^{-1}$. Let $\mathrm{dir}(v_k) = \mathrm{dir}(v_l)$. We observe that we have in this case that their directions are equal to $\mathrm{dir}(v_j)$ (compare Lemmas 3.56, 3.59). We have by Lemma 3.55 that at least one of $\mathrm{ind}^*_k(w_{\mathbb{Z}})$ and $\mathrm{ind}^*_l(w_{\mathbb{Z}})$ is smaller than $d$. Assume without loss of generality that $\mathrm{ind}^*_k(w_{\mathbb{Z}}) < d$. We can assume by induction that $M(\mu_k(v'_{\mathbb{Z}}),(V,-A)) \cong M(v'_{\mathbb{Z}},(V,A))$. We can also assume by induction that $v_k = \bar{v}_l$, say, $v_k = \bar{\eta}^{-1}$ and $v_l = \eta^{-1}$. Similarly we have that $M(\mu_k(v_{\mathbb{Z}}),(V,-A)) \cong M(v_{\mathbb{Z}},(V,A))$. Thus, we can also assume that $v'_k = \bar{\eta}^{-1}$ and $v'_l = \eta^{-1}$. Let $h : {}_{\Lambda}M(v_{\mathbb{Z}})_{\mathrm{k}[T,T^{-1}]} \longrightarrow {}_{\Lambda}M(v_{\mathbb{Z}})_g$ be given as follows for $q$ even (with $q$ as in the induction basis):

$$\begin{aligned}
b_{i+qp} &\longmapsto c_{i+qp} && 1 \le i \le j-1,\\
b_{j+i+qp} &\longmapsto c_{j-i-1+qp} - c_{j+i+qp} && \forall i \text{ such that } j+i \in \{j,\dots,l-1\},\\
b_{i+qp} &\longmapsto -c_{i+qp} && l \le i \le p,
\end{aligned}$$

and for $q$ odd:

$$\begin{aligned}
b_{i+qp} &\longmapsto -c_{i+qp} && 1 \le i \le j-1,\\
b_{j+i+qp} &\longmapsto -c_{j-i-1+qp} + c_{j+i+qp} && \forall i \text{ such that } j+i \in \{j,\dots,l-1\},\\
b_{i+qp} &\longmapsto c_{i+qp} && l \le i \le p.
\end{aligned}$$

By definition, $h$ is a bijective map between the two vector spaces. We show next that $h$ is a morphism of $\Lambda$–modules. Here, the key points are given by the indices $j$ and $l$. Let $q$ be even. We obtain that

$$\begin{aligned}
&h\varepsilon(b_{j-1+qp}) = h(b_{j+qp}) = c_{j-1+qp} - c_{j+qp},\\
&\varepsilon h(b_{j-1+qp}) = \varepsilon(c_{j-1+qp}) = (1_{s(\varepsilon)} - \bar{\varepsilon})(c_{j-1+qp}) = c_{j-1+qp} - c_{j+qp},\\
&h\eta(b_{l-1+qp}) = h(b_{l+qp}) = -c_{l+qp},\\
&\eta h(b_{l-1+qp}) = \eta(c_{k+qp} - c_{l-1+qp}) = -c_{l+qp},
\end{aligned}$$

where $s(\varepsilon)$ denotes the starting vertex of $\varepsilon$ in the quiver. We have used in the last equation that $c_{k+qp} \in \mathrm{im}(\bar{\eta}) = \ker(\eta)$. Now let $q$ be odd:

$$\begin{aligned}
&h\varepsilon(b_{j-1+qp}) = h(b_{j+qp}) = -c_{j-1+qp} + c_{j+qp},\\
&\varepsilon h(b_{j-1+qp}) = \varepsilon(-c_{j-1+qp}) = (1_{s}(\varepsilon) - \bar{\varepsilon})(c_{j-1+qp}) = -c_{j-1+qp} + c_{j+qp},\\
&h\eta(b_{l-1+qp}) = h(b_{l+qp}) = c_{l+qp},\\
&\eta h(b_{l-1+qp}) = \eta(c_{k+qp} - c_{l-1+qp}) = c_{l+qp}.
\end{aligned}$$

It follows that $h$ is morphism between $\Lambda$–modules. Finally, we examine the interaction of $T$ and $h$. Again, let $q$ be even:

$$\begin{aligned}
h(b_{i+qp}) \star T &= \begin{cases} c_{i+kp}(-T) & 1 \le i \le j-1,\\ (c_{j-f-1+qp} - c_{j+f+qp})(-T) & i = j+f \in \{j,\dots,l-1\},\\ -c_{i+qp}(-T) & l \le i \le p, \end{cases}\\
&= \begin{cases} -c_{i+(q-1)p} & 1 \le i \le j-1,\\ -c_{j-f-1+(q-1)p} + c_{j+f+(q-1)p} & i = j+f \in \{j,\dots,l-1\},\\ c_{i+(q-1)p} & l \le i \le p, \end{cases}\\
h(b_{i+qp}T) &= h(b_{i+(q-1)p})\\
&= \begin{cases} -c_{i+(q-1)p} & 1 \le i \le j-1,\\ -c_{j-f-1+(q-1)p} + c_{j+f+(q-1)p} & i = j+f \in \{j,\dots,l-1\},\\ c_{i+(q-1)p} & l \le i \le p. \end{cases}
\end{aligned}$$

Proceeding analogously for $q$ being odd yields that $h(b_{i+qp}) \star T = h(b_{i+qp}T)$ for all $1 \le i \le p$, $q \in \mathbb{Z}$. It follows that $h$ is a bimodule morphism between the modules ${}_{\Lambda}M(\mu_k(v_{\mathbb{Z}}))_{\mathrm{k}[T,T^{-1}]}$ and ${}_{\Lambda}M(\mu_k(v'_{\mathbb{Z}}))_g$. By the same argumentation as in the induction basis, we obtain additionally that $M(\mu_k(v_{\mathbb{Z}}),(V,-A)) \cong M(\mu_k(v'_{\mathbb{Z}}),(V,A))$. It follows that

$$M(v_{\mathbb{Z}},(V,A)) \cong M(v'_{\mathbb{Z}},(V,-A)).$$

Let now $\mathrm{dir}(v_k) = -\mathrm{dir}(v_l)$. Similar to the previous case, we can assume by induction that $v_k = v_l^{-1}$, say $v_k = \eta$, $v_l = \eta^{-1}$: by induction we have that $M(\mu_k(v_{\mathbb{Z}}),(V,-A)) \cong M(v_{\mathbb{Z}},(V,A))$ and $M(\mu_k(v_{\mathbb{Z}}),(V,-A)) \cong M(v'_{\mathbb{Z}},(V,A))$. Consider the map $h : {}_{\Lambda}M(\mu_k(v_{\mathbb{Z}}))_{\mathrm{k}[T,T^{-1}]} \longrightarrow {}_{\Lambda}M(\mu_k(v'_{\mathbb{Z}}))_g$ given as follows: for $q$ even:

$$\begin{aligned}
b_{i+qp} &\longmapsto c_{i+qp} && 1 \le i \le j-1,\\
b_{j+i+qp} &\longmapsto c_{j-i-1+qp} - c_{j+i+qp} && \forall i \text{ such that } j+i \in \{j,\dots,j_+^*-1\},\\
b_{i+qp} &\longmapsto -c_{i+qp} && j_+^* \le i \le p,
\end{aligned}$$

and for $q$ odd:

$$\begin{aligned}
b_{i+qp} &\longmapsto -c_{i+qp} && 1 \le i \le j-1,\\
b_{j+i+qp} &\longmapsto c_{j-i-1+qp} + c_{j+i+qp} && \forall i \text{ such that } j+i \in \{j,\dots,j_+^*-1\},\\
b_{i+qp} &\longmapsto c_{i+qp} && j_+^* \le i \le p.
\end{aligned}$$

Proceeding analogously to the previous case, it follows that $h$ is a morphism of left $\Lambda$–modules. Here, the key points are given by the indices $j$ and $j_+^*$ and we use that $yx = 0$ by definition of $\Lambda$ being skewed-gentle. Hence, we obtain that $y(b_{j_-^*+qp}) = 0$ for any $q \in \mathbb{Z}$. We obtain analogously to the previous case that $h$ is an isomorphism between the modules ${}_\Lambda M(\mu_k(v_\mathbb{Z}))_{\mathrm{k}[T,T^{-1}]}$ and ${}_\Lambda M(\mu_k(v'_\mathbb{Z}))_g$. Thus, we have that $M(\mu_k(v_\mathbb{Z}),(V,A)) \cong M(\mu_k(v'_\mathbb{Z}),(V,-A))$. It follows that

$$M(v_\mathbb{Z},(V,A)) \cong M(v'_\mathbb{Z},(V,-A)).$$

□

**Example 5.31.** *Let $\Lambda$ be given as in Example 2.3.1. We consider the asymmetric band $w_\mathbb{Z}$ in $\mathcal{W}(\Gamma_{\mathrm{ud}}(\Lambda))$ with periodic part $\hat{w}_p = \varepsilon^* a \varepsilon^* a^{-1} \varepsilon^* a^{-1}$. Let $v_\mathbb{Z}$, $v'_\mathbb{Z} \in \mathcal{W}(\Sigma_{\mathrm{d}}(\Lambda))$ with periodic parts given by $\hat{v}_p = \varepsilon^{-1} a \varepsilon^{-1} a^{-1} \varepsilon^{-1} a^{-1}$ and $\hat{(v'_p)} = \varepsilon^{-1} a \bar{\varepsilon}^{-1} a^{-1} \varepsilon^{-1} a^{-1}$. We observe at first that $\mathrm{ind}_3^*(v_\mathbb{Z}) = 2$ and $\mathrm{ind}_5^*(v_\mathbb{Z}) = 0$.*
*Since $v_1 = \varepsilon^{-1}$, we use induction on $v_5$ which yields that $\mu_5(v_\mathbb{Z})$ has periodic part $(\widehat{\mu_5(v_\mathbb{Z})})_p = \varepsilon^{-1} a \varepsilon^{-1} a^{-1} \bar{\varepsilon}^{-1} a^{-1}$ and $\mu_5(v'_\mathbb{Z})$ has periodic part $(\widehat{\mu_5(v'_\mathbb{Z})})_p = \varepsilon^{-1} a \bar{\varepsilon}^{-1} a^{-1} \bar{\varepsilon}^{-1} a^{-1}$. Additionally, we have by induction that $M(v_\mathbb{Z},(V,A)) \cong M(\mu_5(v_\mathbb{Z}),(V,-A))$ and $M(\mu_5(v'_\mathbb{Z}),(V,A)) \cong M(v'_\mathbb{Z},(V,-A))$. The induction step yields that $M(\mu_5(v_\mathbb{Z}),(V,-A)) \cong M(\mu_5(v_\mathbb{Z}),(V,A))$. With this, we obtain the wanted isomorphism.*

**Proposition 5.32.** *Let $w_\mathbb{Z}$ be an asymmetric or symmetric band of period $p$. Let $v_\mathbb{Z} \in \mathcal{W}(\Sigma_{\mathrm{d}}(\Lambda))$ such that $\Phi_{\mathrm{ud}}^{\mathrm{d}}(v_\mathbb{Z}) = w_\mathbb{Z}$ and $\hat{v}_p^{(i)} = \hat{v}_p^{(k)}$ for all $i,k \in \mathbb{Z}$. Denote by $M(v_\mathbb{Z})$ the bimodule in which any special letter $v_j$ of $\hat{v}_p^{(i)}$ sends $b_j$ to $b_{j-1}$. Denote by $M'(v_\mathbb{Z})$ the bimodule in which $v_j$ sends $b_j$ to $-b_{j-1}$ for some $j \in \{1,\dots,p\}$ with $v_j$ special. Then*

$$M'(v_\mathbb{Z},(V,A)) \cong M(v_\mathbb{Z},(V,-A)).$$

*Proof.* The proof follows analogously to the induction step of the proof of Proposition 5.30. □

**Proposition 5.33.** *Let $w_\mathbb{Z}$ be a symmetric band of period $p$ with periodic part $\hat{w}_p = \varepsilon^* u \eta^* u^{-1}$ where $\varepsilon, \eta \in \mathrm{Sp}$, $|u| = m$. Let $v_\mathbb{Z}$, $v'_\mathbb{Z} \in \mathcal{W}(\Sigma_{\mathrm{d}}(\Lambda))$ such that*

- $\mathrm{dir}(v_{1+kp}) = \mathrm{dir}(v_{m+2+kp}) = \mathrm{dir}(v'_{1+kp}) = \mathrm{dir}(v'_{m+2+kp}) = 1$ *for all* $k \in \mathbb{Z}$,
- $\hat{v}_p^{(i)} = \hat{v}_p^{(k)}$ *and* $\hat{v'}_p^{(i)} = \hat{v'}_p^{(k)}$ *for all* $i,k \in \mathbb{Z}$,
- $\mu_j(v_\mathbb{Z}) = v'_\mathbb{Z}$ *for some* $j \in \{1,\dots,p\}$ *with* $j \neq 1, m+2$,
- $\Psi_{\mathrm{d}}^{\Sigma}(v_\mathbb{Z}) = \Psi_{\mathrm{d}}^{\Sigma}(v'_\mathbb{Z})$ *is (weakly) consistent ,*
- $\Psi_{\mathrm{ud}}^{\Sigma}(v_\mathbb{Z}) = \Psi_{\mathrm{ud}}^{\Sigma}(v'_\mathbb{Z}) = w_\mathbb{Z}$.

*Then*

$$M(v_{\mathbb{Z}},(V,A)) \cong M(v'_{\mathbb{Z}},(V,-A)) \tag{242}$$

*where $V$ is a $k[T,T^{-1},(T-1)^{-1}]$– ($k[T,T^{-1},(T+1)^{-1}]$–)module, $A \in \mathrm{End}(V)$ invertible with 1 (−1) not an eigenvalue.*

*Proof.* The periodic part of $w_{\mathbb{Z}}$ is of the form $\hat{w}_p = \varepsilon^* u\eta^* u^{-1}$. By assumption, we can assume without loss of generality that $\hat{v}_p = \varepsilon t\eta t^{-1}$ with $\Psi^{\Sigma}_{\mathrm{d}}(t) = u$. The proof follows analogously to the proof of Propositon 5.30. In order to apply the induction, we need to take into consideration that $\mathrm{ind}^*_{1+qp}(w_{\mathbb{Z}}) = \infty = \mathrm{ind}^*_{m+2+qp}(w_{\mathbb{Z}})$ for all $q \in \mathbb{Z}$. □

**Proposition 5.34.** *Let $w_{\mathbb{Z}}$ be a symmetric band of period $p$ with periodic part given by $\hat{w}_p = \varepsilon^* u\eta^* u^{-1}$, $\eta,\varepsilon \in \mathrm{Sp}$, $|u| = m$. Let $v_{\mathbb{Z}}$, $v'_{\mathbb{Z}} \in \mathcal{W}(\Sigma_{\mathrm{d}}(\Lambda))$ such that*

- $\hat{v}^{(i)}_p \varepsilon t\eta t^{-1}$, $\hat{v'}^{(i)}_p = \varepsilon t'\eta(t')^{-1}$ *for all $i \in \mathbb{Z}$,*
- $\mathrm{dir}(v_{1+kp}) = \mathrm{dir}(v_{m+2+kp}) = \mathrm{dir}(v'_{1+kp}) = \mathrm{dir}(v'_{m+2+kp}) = 1$ *for all $k \in \mathbb{Z}$*
- $\mu_j(t) = t'$ *for some $j \in \{1,\dots,m\}$,*
- $\Psi^{\Sigma}_{\mathrm{d}}(v_{\mathbb{Z}}) = \Psi^{\Sigma}_{\mathrm{d}}(v'_{\mathbb{Z}})$ *is (weakly) consistent ,*
- $\Psi^{\Sigma}_{\mathrm{ud}}(v_{\mathbb{Z}}) = \Psi^{\Sigma}_{\mathrm{ud}}(v'_{\mathbb{Z}}) = w_{\mathbb{Z}}$.

*Let $p \in \mathbb{N}$. Then*

$$M_{i,j,\varepsilon,\eta}(t^{[p]}) = M_{i,j,\varepsilon,\eta}((t')^{[p]}) \tag{243}$$

*where $i,j \in \{0,1\}$, $t^{[p]},(t')^{[p]}$ are defined as in Subsection 2.3.3 and with modules $M_{i,j,\varepsilon,\eta}(-)$ defined as in Theorem 5.21.*

*Proof.* We consider $t^{[p]}$ and $(t')^{[p]}$ as subwords of $v_{\mathbb{Z}}$, $v'_{\mathbb{Z}}$, respectively, and thus will use the $c^*$–indices of $v_{\mathbb{Z}}$, $v'_{\mathbb{Z}}$ for the induction. Recall that this is conform with the way of orientation in $\mathfrak{L}$–chains with two double ends. Note that any $\eta$ (any $\varepsilon$) in $t^{[p]}$ which is given between $t$ and $t^{-1}$ ($t^{-1}$ and $t$) has infinite $c^*$–index. The proof is analogous to the proof of Proposition 5.28. □

**Remark 5.35.** *The above statement also holds if the periodic parts are given by*

- $\hat{v}^{(i)}_p = \bar{\varepsilon}t\eta t^{-1}$ *and* $\hat{v}'^{(i)}_p = \bar{\varepsilon}t'\eta(t')^{-1}$, *or*
- $\hat{v}^{(i)}_p = \varepsilon t\bar{\eta} t^{-1}$ *and* $\hat{v}'^{(i)}_p = \varepsilon t'\bar{\eta}(t')^{-1}$, *or*
- $\hat{v}^{(i)}_p = \bar{\varepsilon}t\bar{\eta} t^{-1}$ *and* $\hat{v}'^{(i)}_p = \bar{\varepsilon}t'\bar{\eta}(t')^{-1}$,

*for all* $i \in \mathbb{Z}$.

We are able to refine Theorem 5.21 by the above results as follows with respect to the alphabets:

**Theorem 5.36.** *Let* $\Lambda$ *be a skewed-gentle algebra as above.*

*(i)* *Let* $w \in \Gamma_{\mathrm{ud}}(\Lambda)$ *be an asymmetric string. Let* $v \in \left(\Phi^{\mathrm{d}}_{\mathrm{ud}}\right)^{-1}(w)$ *be (weakly) consistent. Let* $v' \in \tilde{\Sigma}_{\mathrm{d}}(\Lambda)$ *such that* $F(U(g_w)) = M(v')$. *Then*
$$M(v') \cong M(v).$$

*(ii)* *Let* $w \in \Gamma_{\mathrm{ud}}(\Lambda)$ *be a symmetric string of the form* $w = u\varepsilon^* u^{-1}$. *Let* $v \in \left(\Phi^{\mathrm{d}}_{\mathrm{ud}}\right)^{-1}(w)$ *be (weakly) consistent. Let* $v' \in \tilde{\Sigma}_{\mathrm{d}}(\Lambda)$ *be such that* $F(U_s(g_u)) = M_i(v')$ *where*
$$i = \begin{cases} 1 & \text{if } s = 1, \\ 0 & \text{if } s = 2. \end{cases}$$
*Then* $M_i(v') \cong M_i(v)$ *for* $i \in \{0, 1\}$.

*(iii)* *Let* $w_{\mathbb{Z}} \in \Gamma_{\mathrm{ud}}(\Lambda)$ *be an asymmetric band. Let* $v_{\mathbb{Z}} \in \left(\Phi^{\mathrm{d}}_{\mathrm{ud}}\right)^{-1}(w_{\mathbb{Z}})$ *be (weakly) consistent with periodic parts* $\hat{v}_p^{(i)} = \hat{v}_p^{(k)}$ *for all* $i, k \in \mathbb{Z}$. *Let* $v'_{\mathbb{Z}} \in \tilde{\Sigma}_{\mathrm{d}}(\Lambda)$ *such that* $F(U(g_{w_{\mathbb{Z}}}, \varphi)) = M(v'_{\mathbb{Z}}, (V, A))$ *where* $A = \sigma_1 F_{\varphi}^{\sigma_2} \in \mathrm{End}(V)$, $\sigma_1, \sigma_2 \in \{+1, -1\}$ *and* $V$ *a* $\mathrm{k}[T, T^{-1}]$*–module of dimension* $\deg(\varphi)$. *Then*
$$M(v_{\mathbb{Z}}, (V, A)) \cong M(v'_{\mathbb{Z}}, (V, (-1)^{\omega} A))$$
*where* $\omega$ *denotes the number of all special letters which do not act as* $A$ *in* $\hat{v}_p^{(i)}$, *plus the number of inverse special letters in* $\hat{v}_p^{(i)}$.

*(iv)* *Let* $w_{\mathbb{Z}} \in \Gamma_{\mathrm{ud}}(\Lambda)$ *be a symmetric band with periodic part* $\hat{w}_p = \varepsilon^* u \eta^* u^{-1}$. *Let* $v_{\mathbb{Z}} \in \left(\Psi^{\Sigma}_{\mathrm{ud}}\right)^{-1}(w_{\mathbb{Z}})$ *be (weakly) consistent with periodic parts* $\hat{v}_p^{(i)} = \varepsilon t \eta t^{-1}$ *if* $s = 1$, $\hat{v}_p^{(i)} = \bar{\varepsilon} t \eta t^{-1}$ *if* $s = 2$, $\hat{v}_p^{(i)} = \varepsilon t \bar{\eta} t^{-1}$ *if* $s = 3$, $\hat{v}_p^{(i)} = \bar{\varepsilon} t \bar{\eta} t^{-1}$ *if* $s = 4$, *for all* $i \in \mathbb{Z}$, $t \in \left(\Phi^{\mathrm{d}}_{\mathrm{ud}}\right)^{-1}(u)$. *Let* $v'_{\mathbb{Z}} \in \Sigma_{\mathrm{d}}(\Lambda)$ *such that* $\{F(U_s(g_u, p))\}_s \cong \{M_{i,j,x,y}((t')^{[p]})\}_{x,y}$, *where* $\hat{v}_p'^{(i)} = \varepsilon t' \eta (t')^{-1}$ *for* $s = 1$, $\hat{v}_p'^{(i)} = \bar{\varepsilon} t' \eta (t')^{-1}$ *for* $s = 2$, $\hat{v}_p'^{(i)} = \varepsilon t' \bar{\eta} (t')^{-1}$ *for* $s = 3$, $\hat{v}_p'^{(i)} = \bar{\varepsilon} t' \bar{\eta} (t')^{-1}$ *for* $s = 4$, *for all* $i \in \mathbb{Z}$. *Then*
$$M_{0,1,x,y}(t^{[p]}) \cong M_{0,1,x,y}((t')^{[p]}),$$
*where* $x \in \{\varepsilon, \bar{\varepsilon}\}$, $y \in \{\eta, \bar{\eta}\}$.

*(v) Let $w_{\mathbb{Z}} \in \Gamma_{\mathrm{ud}}(\Lambda)$ be a symmetric band with periodic part $\hat{w}_p = \varepsilon^* u\eta^* u^{-1}$. Let $v_{\mathbb{Z}} \in \left(\Phi^{\mathrm{d}}_{\mathrm{ud}}\right)^{-1}(w_{\mathbb{Z}})$ be weakly consistent with periodic parts $\hat{v}_p^{(i)} = \varepsilon t\eta t^{-1}$ for all $i \in \mathbb{Z}$, $t \in \left(\Phi^{\mathrm{d}}_{\mathrm{ud}}\right)^{-1}(u)$. Let $v'_{\mathbb{Z}} \in \tilde{\Sigma}_{\mathrm{d}}(\Lambda)$ such that $F(U(g_{w_{\mathbb{Z}}}, \varphi)) = M(v'_{\mathbb{Z}}, (V, A))$ where $p \in \mathbb{N}$, $A = -F_\varphi^{-1} \in \mathrm{End}(V)$, $V$ a $\mathrm{k}[T, T^{-1}, (T+1)^{-1}]$–module ($\mathrm{k}[T, T^{-1}, (T-1)^{-1}]$–module) if $\delta_0(g_{w_{\mathbb{Z}}})$ is odd (even). Then*

$$M(v_{\mathbb{Z}}, (V, A)) \cong M(v'_{\mathbb{Z}}, (V, (-1)^\omega A))$$

*where $\omega$ denotes the number of inverse special letters which do not act as $A$ in $\hat{v}_p^{(i)}$, plus the number of all special letters in $\hat{v}_p^{(i)}$.*

*Proof.* We observe that $v_{(\mathbb{Z})} = \mu_{j_1} \circ \cdots \circ \mu_{j_k}(v'_{(\mathbb{Z})})$ where $v'_{j_1}, \ldots, v'_{j_k}$ denote the inverse special letters of $v'_{(\mathbb{Z})}$ with $j_1, \ldots, j_k \notin \{1, m+2\}$ in case (iv). The results follow by applying the following Propositions on the respective special letters:

(i) Proposition 5.26

(ii) Proposition 5.28

(iii) Proposition 5.30 and Proposition 5.32

(iv) Proposition 5.34

(v) Proposition 5.33 and Proposition 5.32.

□

**Remark 5.37.** *In case of part (v) of the above theorem, we can write*

$$\omega = 3\bar{\omega} + 1$$

*where $\bar{\omega}$ describes the number of special letters in $t$: by symmetry in $\hat{v}_p^{(i)}$ we have that the number of all special letters in this subword is equal to $2\bar{\omega} + 2$. Also, the number of inverse special letters in $\hat{v}_p^{(i)}$ is given by $\bar{\omega}$ by symmetry and since the symmetry axes $v_1$ and $v_{m+2}$ are given by direct letters. Hence, we consider $3\bar{\omega}+2$. We know by construction that $v_1$ acts as $-F_\varphi^{-1}$. It follows that $\omega = 3\bar{\omega} + 1$.*
*Thus, it is enough to consider $\bar{\omega}+1$ instead of $\omega$ in calculations for this case.*

We can examine the value of $\omega$ in part (v) of the above theorem more closely. To this end, we first revisit parts of the construction of $\mathfrak{L}$–graphs coming from words.

**Lemma 5.38.** *Let $w_{\mathrm{I}}$ be a $\Gamma_{\mathrm{ud}}(\Lambda)$–word and denote by $g_{w_{\mathrm{I}}}$ its corresponding $\mathfrak{L}$–graph. Let $x_j = \mathfrak{C}_{\delta^*}$ and $x_k = \mathfrak{C}_{\kappa^*}$ for some $\delta, \kappa \in \mathrm{Sp}$, $j, k \in \{1, \ldots, |g_{w_{\mathrm{I}}}|\}$*

*with $j < k$. Assume that there does not exist $l \in \{j, \dots, k\}$ with $x_l = \mathfrak{C}_{\zeta^*}$ for any $\zeta \in \mathrm{Sp}$. Then*

$$\#\{i \in \{j, \dots, k\} \mid x_i \neq x_{i+1},\ \textit{either}\ x_i, x_{i+1} \in \mathfrak{L}(\mathfrak{C})\ \textit{or}\ x_i, x_{i+1} \in \mathfrak{L}(\mathfrak{R})\}$$

*is odd.*

*Proof.* Subchains of the form $x_i$—$x_{i+1}$ do not contribute to the above set since we have by definition of the relation $\beta$ that $x_i \in \mathfrak{L}(\mathfrak{C})$, $x_{i+1} \in \mathfrak{L}(\mathfrak{R})$, or vice versa. We observe also that subchains of the form $x_i \sim x_{i+1}$ only contribute to the mentioned set if the links are not of the form $\mathfrak{C}_{\zeta^*}$ for any $\zeta \in \mathrm{Sp}$. By definition, we can neglect subchains of the form $x_i \sim x_{i+1}$ with $x_i$ and $x_{i+1}$ not belonging to the same set of links with respect to columns and rows. Hence, the statement follows by a combinatorial argument. To this end, we mark links of $\mathfrak{L}(\mathfrak{C})$ by a black bullet, and those of $\mathfrak{L}(\mathfrak{R})$ by a white bullet. With $x_j$ we start in a black one, and obtain the following picture:

$$\bullet\!-\!\!-\!\circ \sim \circ\!-\!\!-\!\bullet \sim \bullet\!-\!\!-\!\circ \sim \circ\!-\!\!-\!\cdots\!-\!\!-\!\bullet$$

where the final bullet corresponds to $x_k$. It follows that between $x_j$ and $x_k$ there can be a series of relations of the form $x_i \sim x_{i+1}$ of the wanted form. This series starts and ends in such a relation with $x_i, x_{i+1} \in \mathfrak{L}(\mathfrak{R})$. It follows by definition of $g_u$ that the above set has odd cardinality. □

**Lemma 5.39.** *Let $w_{\mathbb{Z}}$ be a symmetric band with $\hat{w}_p = \varepsilon^* u \eta^* u^{-1}$. Let $g_{w_{\mathbb{Z}}}$ be its corresponding $\mathfrak{L}$–cycle. Then*

*(i) $\delta_0(g_{w_{\mathbb{Z}}})$ is even if and only if the number of special letters in $u$ is odd,*

*(ii) $\delta_0(g_{w_{\mathbb{Z}}})$ is odd if and only if the number of special letters in $u$ is even.*

*Proof.* Recall that $\delta_0(g_{w_{\mathbb{Z}}}) = \delta(g_{w_{\mathbb{Z}}})/2$. Denote by $k_i$ the number of indices contributing to $\delta(g_u)$ between two links of the form $\mathfrak{C}_{\delta^*}$, $\mathfrak{C}_{\kappa^*}$ for some $\delta, \kappa \in \mathrm{Sp}$ (without any links of the form $\mathfrak{C}_{\zeta^*}$ lying between them). Keep in mind that $g_u$ is a subchain of $g_{w_{\mathbb{Z}}}$. Lemma 5.38 yields that each $k_i$ is odd. Moreover, we have that

$$\delta(g_{w_{\mathbb{Z}}}) = 2\left(\sum_{i=1}^{n} k_i\right),$$

and thus it follows

$$\delta_0(g_{w_{\mathbb{Z}}}) = \sum_{i=1}^{n} k_i.$$

We obtain that $\delta_0(g_{w_{\mathbb{Z}}})$ is even if and only if $n$ is even, and that $\delta_0(g_{w_{\mathbb{Z}}})$ is odd if and only if $n$ is odd.

The subchain $g_u$ starts in the link $\mathfrak{C}_{\varepsilon^*}$ and ends in $\mathfrak{C}_{\eta^*}$. It follows that $n$ is even if and only if the number of special letters in $u$ is odd. It is odd if and only if the number of special letters in $u$ is even. This observation yields the statement. □

**Lemma 5.40.** *We have*

$$(-1)^{\bar{\omega}+1} = (-1)^{\delta_0(g_{w_{\mathbb{Z}}})}$$

*in Theorem 5.36,(v) where $\bar{\omega}$ denotes the number of special letters in $t$.*

*Proof.* Assume that $\delta_0(g_{w_{\mathbb{Z}}})$ is odd. We know by Lemma 5.39 that the number of special letters in $t$ is even. Thus, $\bar{\omega}$ is even and it follows that $\bar{\omega}+1$ is odd. Analogously, it follows for $\delta_0(g_{w_{\mathbb{Z}}})$ even that $\bar{\omega}+1$ is odd. □

**Corollary 5.41.** *Let $w_{\mathbb{Z}}$ be a symmetric band with periodic part $\hat{w}_p = \varepsilon^* u \eta^* u^{-1}$ for some $\varepsilon, \eta \in \mathrm{Sp}$. Let $v_{\mathbb{Z}} \in \left(\Phi^{\mathrm{d}}_{\mathrm{ud}}\right)^{-1}(w_{\mathbb{Z}})$ be (weakly) consistent with $\hat{v}_p^{(i)} = \varepsilon t \eta t^{-1}$ for all $i \in \mathbb{Z}$, with $t \in \left(\Phi^{\mathrm{d}}_{\mathrm{ud}}\right)^{-1}(u)$. Let $V$ be a $\mathrm{k}[T, T^{-1}, (T-1)^{-1}]$-module. Then*

$$F(U(g_{w_{\mathbb{Z}}}, \varphi)) \cong M(v_{\mathbb{Z}}, (V, (-1)^{\delta_0(g_{w_{\mathbb{Z}}})+1} F_{\varphi}^{-1})) \tag{244}$$

*Proof.* The isomorphy above follows from Theorem 5.36, (v) and Lemma 5.40. Note that we have to additionally take the letter $v_{m+2} = \eta$ into account. It is sending $b_{m+2}$ to $-b_{m+1}$ in $F(U(g_{w_{\mathbb{Z}}}), \varphi)$. It remains to show that 1 is not an eigenvalue of $(-1)^{\delta_0(g_{w_{\mathbb{Z}}})+1} F_{\varphi}^{-1}$. Let $\delta_0(g_{w_{\mathbb{Z}}})$ be odd. We obtain that $(-1)^{\delta_0(g_{w_{\mathbb{Z}}})+1} = 1$. By construction, we have that $\varphi_0 \neq t, t-1$. It follows that 1 is not an eigenvalue of $F_{\varphi}^{-1}$.
Let $\delta_0(g_{w_{\mathbb{Z}}})$ be even. Then we have that $(-1)^{\delta_0(g_{w_{\mathbb{Z}}})+1} = -1$. By construction, $\varphi_0 \neq t, t+1$ which yields that $-1$ is not an eigenvalue of $F_{\varphi}^{-1}$. This implies that 1 is not an eigenvalue of $-F_{\varphi}^{-1}$. □

We want to give the classification Theorem 4.61 in terms of the image of $F$. In order to exclude isomorphic modules, we first recall some general results which have already been alluded to in Section 2.4.

**Lemma 5.42.** *Let $w$ be an asymmetric string and let $v \in \left(\Phi^{\mathrm{d}}_{\mathrm{ud}}\right)^{-1}(w)$ be weakly consistent. Then*

$$M(v) \cong M(v')$$

*for any $v' \in [v]$.*

*Proof.* We have for $v' \in [v]$ that $v' \sim v$. It follows that either $v' = v$ or $v' = v^{-1}$. The isomorphism between the modules is given by the identity in the first case, and by $i_w$ in the second case. Recall that $i_w$ reverses the basis (cf. Section 2.4). □

**Lemma 5.43.** *Let $w$ be a symmetric string and let $v \in \left(\Phi^{\mathrm{d}}_{\mathrm{ud}}\right)^{-1}(w)$ be weakly consistent. Then*

$$M_i(v) \cong M_i(v')$$

*for any $v' \in [v]$, $i \in \{0, 1\}$.*

*Proof.* Let $w = u\varepsilon^* u^{-1}$ and $v = t\varepsilon t^{-1}$. By definition, $v' = v$ or $v' = v^{-1} = t\varepsilon^{-1}t^{-1}$. The result follows due to the same symmetry in both words. $\square$

Recall that two $\mathbb{Z}$−words $v_{\mathbb{Z}}$ and $v'_{\mathbb{Z}}$ are said to be equivalent if $v'_{\mathbb{Z}} = v_{\mathbb{Z}}[m]$ or $v'_{\mathbb{Z}} = v_{\mathbb{Z}}^{-1}[m]$ for some $m \in \mathbb{Z}$. For bands, we will consider equivalences given by shift and by inverses separately.

**Lemma 5.44.** *Let $w_{\mathbb{Z}}$ be an asymmetric or symmetric band and let $v_{\mathbb{Z}} \in \left(\Phi^{\mathrm{d}}_{\mathrm{ud}}\right)^{-1}(w_{\mathbb{Z}})$ be weakly consistent with $\hat{v}_p^{(i)} = \hat{v}_p^{(k)}$ for all $i, k \in \mathbb{Z}$. Then*

$$M(v_{\mathbb{Z}}^{-1}) \cong M(v_{\mathbb{Z}})_g$$

*as $\mathrm{k}[T, T^{-1}]$−modules, where $g : \mathrm{k}[T, T^{-1}] \to \mathrm{k}[T, T^{-1}]$ sending $T \mapsto T^{-1}$. We have in particular that*

$$M(v_{\mathbb{Z}}, (V, A)) \cong M(v_{\mathbb{Z}}^{-1}, (V, A^{-1}))$$

*as $\Lambda - \mathrm{k}[T, T^{-1}]$−bimodules.*

*Proof.* The isomorphism $h : M(v_{\mathbb{Z}}) \to M(v_{\mathbb{Z}}^{-1})$ between the vector spaces is given by reversing the basis. Recall that $T$ acts as the shift $t_{w_{\mathbb{Z}},p}$ on $M(v_{\mathbb{Z}})$. It follows that $b_{i+qp}T = b_{i+(q-1)p}$ in $M(v_{\mathbb{Z}})$. We have in $M(v_{\mathbb{Z}}^{-1})$ that $b_{i+qp}T = b_{i+(q+1)p}$ by reverse of the basis. Hence, $b_{i+qp}g(T) = b_{i+qp}T^{-1} = b_{i+(q-1)p}$.
It follows from $h$ that

$$M(v_{\mathbb{Z}}, V) = M(v_{\mathbb{Z}}) \otimes_{\mathrm{k}[T,T^{-1}]} V \cong M(v_{\mathbb{Z}}^{-1})_g \otimes_{\mathrm{k}[T,T^{-1}]} V$$

as bimodules. We obtain for $m \in M(v_{\mathbb{Z}}^{-1})$, $v \in V$ that

$$mg(T) \otimes v = m \otimes g^{-1}(T)v.$$

This yields the following isomorphism of bimodules:

$$M(v_{\mathbb{Z}}^{-1})_g \otimes_{\mathrm{k}[T,T^{-1}]} V \cong M(v_{\mathbb{Z}}^{-1}) \otimes_{\mathrm{k}[T,T^{-1}]} {}_gV$$

which results in

$$M(v_{\mathbb{Z}}, (V, A)) \cong M(v_{\mathbb{Z}}^{-1}, (V, A^{-1})).$$

Note that T acts as $A^{-1}$ on the right hand side. $\square$

**Lemma 5.45.** *Let $w_{\mathbb{Z}}$ be an asymmetric or symmetric band and let $v_{\mathbb{Z}} \in \left(\Phi^{\mathrm{d}}_{\mathrm{ud}}\right)^{-1}(w_{\mathbb{Z}})$ be weakly consistent with $\hat{v}_p^{(i)} = \hat{v}_p^{(k)}$ for all $i, k \in \mathbb{Z}$. Then*

$$M(v_{\mathbb{Z}}[m]) \cong M(v_{\mathbb{Z}}) \qquad \textit{for any } m \in \mathbb{Z},$$

*as $\mathrm{k}[T, T^{-1}]$−modules. We have in particular that*

$$M(v_{\mathbb{Z}}[m], (V, A)) \cong M(v_{\mathbb{Z}}, (V, A))$$

*as $\Lambda - \mathrm{k}[T, T^{-1}]$−bimodules.*

*Proof.* The isomorphy between the $\mathrm{k}[T,T^{-1}]$–modules $M(v_{\mathbb{Z}}[m])$ and $M(v_{\mathbb{Z}})$ is given by the respective shift on the basis. Thus, the second isomorphy between the two $\Lambda - \mathrm{k}[T,T^{-1}]$–modules follows directly. □

**Remark 5.46.** *We see by Lemma 5.44 and Lemma 5.45 that equivalent words do not necessarily give isomorphic modules with the same $A \in \mathrm{End}(V)$. However, we can also see by the above results that running through the list of $A \in \mathrm{End}(V)$ acting as $T$ for one representative of an equivalence class will give a complete set of modules for this class.*

**Lemma 5.47.** *Let $w_{\mathbb{Z}}$ be a symmetric band with periodic part $\hat{w}_p = \varepsilon^* u \eta^* u^{-1}$. Let $v_{\mathbb{Z}} \in \left(\Psi^{\Sigma}_{\mathrm{ud}}\right)^{-1}(w_{\mathbb{Z}})$ be weakly consistent with periodic parts of one of the following forms:*

*(i)* $\hat{v}_p^{(i)} = \varepsilon t \eta t^{-1}$

*(ii)* $\hat{v}_p^{(i)} = \varepsilon t \bar{\eta} t^{-1}$

*(iii)* $\hat{v}_p^{(i)} = \bar{\varepsilon} t \eta t^{-1}$

*(iv)* $\hat{v}_p^{(i)} = \bar{\varepsilon} t \bar{\eta} t^{-1}$

*for all $i \in \mathbb{Z}$, $t \in \left(\Phi^{\mathrm{d}}_{\mathrm{ud}}\right)^{-1}(u)$, $p \in \mathbb{N}\backslash\{0\}$. Then*

$$\{M_{i,j,x,y}(t^{[p]})\}_{i,j,x,y} \cong \{M_{i',j',x',y'}(t'^{[p]})\}_{i',j',x',y'} \tag{245}$$

*as $\Lambda$–modules for any $v'_{\mathbb{Z}} \in [v_{\mathbb{Z}}]$ with $t'$ a subword of $v_{\mathbb{Z}}$, $t' \sim t$, and where $i \neq j \in \{0,1\}$, $i' \neq j' \in \{0,1\}$, $x \in \{\varepsilon, \bar{\varepsilon}\}$, $y \in \{\eta, \bar{\eta}\}$ and $x' \in \{\varepsilon, \bar{\varepsilon}\}$, $y' \in \{\eta, \bar{\eta}\}$, or $x' \in \{\varepsilon^{-1}, \bar{\varepsilon}^{-1}\}$, $y' \in \{\eta^{-1}, \bar{\eta}^{-1}\}$, or with the roles of $x'$ and $y'$ switched.*

*Proof.* Let $v'_{\mathbb{Z}} \in [v_{\mathbb{Z}}]$. Then $v'_{\mathbb{Z}} = v_{\mathbb{Z}}[m]$ or $v'_{\mathbb{Z}} = v_{\mathbb{Z}}^{-1}[m]$ for some $m \in \mathbb{Z}$. Let $p$ be even. Let at first $v'_{\mathbb{Z}} = v_{\mathbb{Z}}[m]$ with $t' = t$ (i.e. $m = kp$ for some $k \in \mathbb{Z}$). We have that $x' \in \{\varepsilon, \bar{\varepsilon}\}$ and $y' \in \{\eta, \bar{\eta}\}$. The isomorphism of sets (245) is given by the identity. Let $v'_{\mathbb{Z}} = v_{\mathbb{Z}}^{-1}[m]$ with $t' = t$. Then $x' \in \{\varepsilon^{-1}, \bar{\varepsilon}^{-1}\}$ and $y' \in \{\eta^{-1}, \bar{\eta}^{-1}\}$. By reversing the basis elements, we see that (245) is given by

$$M_{i,j,x,y}(t^{[p]}) \cong M_{j,i,x^{-1},y^{-1}}(t^{[p]}) \tag{246}$$

Similarly, we obtain an isomorphism of the form

$$M_{i,j,x,y}((t^{-1})^{[p]}) \cong M_{j,i,x^{-1},y^{-1}}((t^{-1})^{[p]}) \tag{247}$$

This refers to modules given by $v_{\mathbb{Z}}[m']$ with $t' = t^{-1}$ and $v_{\mathbb{Z}}^{-1}[m]$ with $t' = t^{-1}$. Let $v'_{\mathbb{Z}} = v_{\mathbb{Z}}[m]$ with $t' = t^{-1}$ (i.e. $m \neq kp$ for any $k \in \mathbb{Z}$). Exploiting the properties $\mathrm{im}(\varepsilon) = \ker(\bar{\varepsilon})$, $\ker(\varepsilon) = \mathrm{im}(\bar{\varepsilon})$, and similar properties for $\eta$, we find that (245) is given by

$$M_{i,j,x,y}(t^{[p]}) \cong M_{j,i,\bar{y}^{-1},\bar{x}^{-1}}((t^{-1})^{[p]}) \tag{248}$$

Combining (246)-(248) gives the isomorphism in (245) for any $v'_{\mathbb{Z}}$ as described above.
Now consider $p$ to be odd. Let $v'_{\mathbb{Z}} = v_{\mathbb{Z}}^{-1}[m]$. Then $x' \in \{\varepsilon^{-1}, \bar{\varepsilon}^{-1}\}$, $y' \in \{\eta^{-1}, \bar{\eta}^{-1}\}$. Let $t' = t^{-1}$. Reversing the basis elements yields that the isomorphism in (245) is given by

$$M_{i,j,x,y}(t^{[p]}) \cong M_{j,i,y^{-1},x^{-1}}((t^{-1})^{[p]}). \tag{249}$$

Now consider $t' = t$. We obtain a similar isomorphism as in (249) where the second module refers to $v_{\mathbb{Z}}[m]$.
Finally, we consider modules given by $v_{\mathbb{Z}}$ and $v'_{\mathbb{Z}} = v_{\mathbb{Z}}[m]$. For those modules we obtain that the isomorphism in (245) is given by

$$M_{i,j,x,y}(t^{[p]}) \cong M_{j,i,\bar{y},\bar{x}}((t^{-1})^{[p]}). \tag{250}$$

Here, we use the properties $\mathrm{im}(\varepsilon) = \ker(\bar{\varepsilon})$, $\ker(\varepsilon) = \mathrm{im}(\bar{\varepsilon})$, and same for $\eta$. Combining the isomorphisms (249) and (250)in both cases, yields the isomorphism in (245) for any $v'_{\mathbb{Z}}$ with $v'_{\mathbb{Z}} \sim v_{\mathbb{Z}}$. $\square$

**Example 5.48.** *Let $w_{\mathbb{Z}}$ be a symmetric band of period $p$ with periodic part $\hat{w}_p = \varepsilon^* u \eta^* u^{-1}$. Let $v_{\mathbb{Z}}$ be as in Lemma 5.47. We consider $p = 4$. Assume without loss of generality that $|t| = 1$. We obtain for the periodic parts of $v_{\mathbb{Z}}$ as in Lemma 5.47 (i) - (iv) the following modules:*

*(i)* $M_{0,1,\varepsilon,\eta}(t^{[p]})$: $\varepsilon{=}0 \circlearrowright b_1 \xleftarrow{t} b'_1 \xleftarrow{\eta} b_2 \xrightarrow{t} b'_2 \xleftarrow{\varepsilon} b_3 \xleftarrow{t} b'_3 \xleftarrow{\eta} b_4 \xrightarrow{t} b'_4 \circlearrowleft \varepsilon{=}1$

*(ii)* $M_{0,1,\varepsilon,\bar{\eta}}(t^{[p]})$: $\varepsilon{=}0 \circlearrowright b_1 \xleftarrow{t} b'_1 \xleftarrow{\bar{\eta}} b_2 \xrightarrow{t} b'_2 \xleftarrow{\varepsilon} b_3 \xleftarrow{t} b'_3 \xleftarrow{\bar{\eta}} b_4 \xrightarrow{t} b'_4 \circlearrowleft \varepsilon{=}1$

*(iii)* $M_{0,1,\bar{\varepsilon},\eta}(t^{[p]})$: $\bar{\varepsilon}{=}0 \circlearrowright b_1 \xleftarrow{t} b'_1 \xleftarrow{\eta} b_2 \xrightarrow{t} b'_2 \xleftarrow{\bar{\varepsilon}} b_3 \xleftarrow{t} b'_3 \xleftarrow{\eta} b_4 \xrightarrow{t} b'_4 \circlearrowleft \bar{\varepsilon}{=}1$

*(iv)* $M_{0,1,\bar{\varepsilon},\bar{\eta}}(t^{[p]})$: $\bar{\varepsilon}{=}0 \circlearrowright b_1 \xleftarrow{t} b'_1 \xleftarrow{\bar{\eta}} b_2 \xrightarrow{t} b'_2 \xleftarrow{\bar{\varepsilon}} b_3 \xleftarrow{t} b'_3 \xleftarrow{\bar{\eta}} b_4 \xrightarrow{t} b'_4 \circlearrowleft \bar{\varepsilon}{=}1$

*Consider in contrast to those the following modules arising from $v_{\mathbb{Z}}^{-1}[m]$:*

*(i)* $M_{1,0,\varepsilon^{-1},\eta^{-1}}(t^{[p]})$: $\varepsilon{=}1 \circlearrowright c_1 \xleftarrow{t} c'_1 \xrightarrow{\eta} c_2 \xrightarrow{t} c'_2 \xrightarrow{\varepsilon} c_3 \xleftarrow{t} c'_3 \xrightarrow{\eta} c_4 \xrightarrow{t} c'_4 \circlearrowleft \varepsilon{=}0$

*(ii)* $M_{1,0,\varepsilon^{-1},\bar{\eta}^{-1}}(t^{[p]})$: $\varepsilon{=}1 \circlearrowright c_1 \xleftarrow{t} c'_1 \xrightarrow{\bar{\eta}} c_2 \xrightarrow{t} c'_2 \xrightarrow{\varepsilon} c_3 \xleftarrow{t} c'_3 \xrightarrow{\bar{\eta}} c_4 \xrightarrow{t} c'_4 \circlearrowleft \varepsilon{=}0$

*(iii)* $M_{1,0,\bar{\varepsilon}^{-1},\eta^{-1}}(t^{[p]})$: $\bar{\varepsilon}{=}1 \circlearrowright c_1 \xleftarrow{t} c'_1 \xrightarrow{\eta} c_2 \xrightarrow{t} c'_2 \xrightarrow{\bar{\varepsilon}} c_3 \xleftarrow{t} c'_3 \xrightarrow{\eta} c_4 \xrightarrow{t} c'_4 \circlearrowleft \bar{\varepsilon}{=}0$

(iv) $M_{1,0,\bar{\varepsilon}^{-1},\bar{\eta}^{-1}}(t^{[p]})$: $\bar{\varepsilon}{=}1 \circlearrowright c_1 \xleftarrow{t} c_1' \xrightarrow{\bar{\eta}} c_2 \xrightarrow{t} c_2' \xrightarrow{\bar{\varepsilon}} c_3 \xleftarrow{t} c_3' \xrightarrow{\bar{\eta}} c_4 \xrightarrow{t} c_4' \circlearrowleft \bar{\varepsilon}{=}0$

*We obtain an isomorphism $M_{0,1,x,y}(t^{[p]}) \to M_{1,0,x^{-1},y^{-1}}(t^{[p]})$, with $x \in \{\varepsilon, \bar{\varepsilon}\}$, $y \in \{\eta, \bar{\eta}\}$, by sending $b_i \mapsto c_{4+1-i}$, $b_i' \mapsto c_{4+1-i}'$ for all $1 \le i \le 4$.*
*In addition, we consider the following modules arising from $v_{\mathbb{Z}}[m]$, $m \neq kp$ for any $k$:*

(i) $M_{0,1,\eta,\varepsilon}((t^{-1})^{[p]})$: $\eta{=}0 \circlearrowright d_1 \xrightarrow{t} d_1' \xleftarrow{\varepsilon} d_2 \xleftarrow{t} d_2' \xleftarrow{\eta} d_3 \xrightarrow{t} d_3' \xleftarrow{\varepsilon} d_4 \xleftarrow{t} d_4' \circlearrowleft \eta{=}1$

(ii) $M_{0,1,\eta,\bar{\varepsilon}}((t^{-1})^{[p]})$: $\eta{=}0 \circlearrowright d_1 \xrightarrow{t} d_1' \xleftarrow{\bar{\varepsilon}} d_2 \xleftarrow{t} d_2' \xleftarrow{\eta} d_3 \xrightarrow{t} d_3' \xleftarrow{\bar{\varepsilon}} d_4 \xleftarrow{t} d_4' \circlearrowleft \eta{=}1$

(iii) $M_{0,1,\bar{\eta},\varepsilon}((t^{-1})^{[p]})$: $\bar{\eta}{=}0 \circlearrowright d_1 \xrightarrow{t} d_1' \xleftarrow{\varepsilon} d_2 \xleftarrow{t} d_2' \xleftarrow{\bar{\eta}} d_3 \xrightarrow{t} d_3' \xleftarrow{\varepsilon} d_4 \xleftarrow{t} d_4' \circlearrowleft \bar{\eta}{=}1$

(iv) $M_{0,1,\bar{\eta},\bar{\varepsilon}}((t^{-1})^{[p]})$: $\bar{\eta}{=}0 \circlearrowright d_1 \xrightarrow{t} d_1' \xleftarrow{\bar{\varepsilon}} d_2 \xleftarrow{t} d_2' \xleftarrow{\bar{\eta}} d_3 \xrightarrow{t} d_3' \xleftarrow{\bar{\varepsilon}} d_4 \xleftarrow{t} d_4' \circlearrowleft \bar{\eta}{=}1$

*The isomorphism $g : M_{1,0,x,y}(t^{[p]}) \to M_{0,1,\bar{x}^{-1},\bar{y}^{-1}}((t^{-1})^{[p]})$, with $x \in \{\varepsilon^{-1}, \bar{\varepsilon}^{-1}\}$, $y \in \{\eta^{-1}, \bar{\eta}^{-1}\}$, is given as follows:*

$$\begin{aligned}
d_1 &\mapsto c_4,\\
d_1' &\mapsto c_4',\\
d_2 &\mapsto -c_3 + c_4',\\
d_2' &\mapsto -c_3' + c_4,\\
d_3 &\mapsto c_2 - c_3',\\
d_3' &\mapsto c_2' - c_3,\\
d_4 &\mapsto c_1 + c_2',\\
d_4' &\mapsto c_1' + c_2.
\end{aligned}$$

We are now able to reformulate Theorem 4.61 in terms of the image of $F$:

**Theorem 5.49.** *Choose for each asymmetric and symmetric band and string one representative in the equivalence class of its directed version which is given by $v$, $v_{\mathbb{Z}}$, respectively, as in Theorem 5.36. Then the set of representations of the form $M(v)$, $M_i(v)$, $M_{0,1,x,y}(t^{[p]})$, $M(v_{\mathbb{Z}}, (V, A))$ associated to the representatives gives a complete set of pairwise non-isomorphic indecomposable represenations of the skewed-gentle algebra $\Lambda$.*

*Proof.* Both indecomposability and completeness follow from Theorem 5.6 and Theorem 5.36. The Lemmas 5.42 - 5.47 yield that the modules are pairwise non-isomorphic. □

**Theorem 5.50.** *Choose for each asymmetric and symmetric string and band one representative as in Theorem 5.49. Then the set of representations of* *the form $M(v)$, $M_i(v)$, $M_{i,j}(t^{[p]})$, $M(v_{\mathbb{Z}}, (V, A))$ associated to the representatives gives a complete set of pairwise non-isomorphic indecomposable represenations of the clannish algebra $\Lambda$.*

*Proof.* The statement follows from Theorem 5.49 as explained in Section 4.3. $\square$

Theorem 5.50 gives a classification of the finite dimensional modules of clannish algebras. However, it cannot yet confirm the classification given by Crawley-Boevey given in [CB89] and his conjecture from [CB88]. In order to confirm both of them, we need to refine the above statement further. This is the goal of the next chapter.

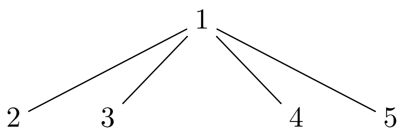
# 6 Symmetric bands in the context of the 4-subspace-problem

The directions put on the subchains corresponding to the symmetry axes in symmetric bands may seem to be arbitrary on first sight (cf. Chapter 4). The case of directions on the joints of the composite $\mathfrak{L}$–chain arising from one with two double ends behaves similarly. In order to gain a better understanding on why those directions are chosen in this way, we examine the symmetry axes and their roles in the modules more closely in this chapter. To this end, we reduce the case of a symmetric band module to the four-subspace problem. This allows us to apply results from [Bre74] and find answers here (cf. Theorem 6.5, Theorem 6.7). The results of this chapter explain the choices taken on the directions in [Bon88, Bon91]. Finally, they allow us to reformulate Theorem 5.49 and thus confirm the conjecture made by Crawley-Boevey for an arbitrary field in [CB88] (Theorem 6.10).

## 6.1 Reduction to the 4-subspace-problem

The 4-subspace-problem describes the problem of classifying all indecomposable modules of a quiver of type

$\tilde{D}_4$ :




where $V_2, V_3, V_4$ and $V_5$ are subspaces of $V_1$ for any representation $V$ (see [SS07, Chapter XIII.3.]).
Gelfand an Ponomarev gave a solution to the problem in 1970 for the base field being algebraically closed ([GP72]). It was followed by a classification by Brenner for an arbitrary skew field in 1974 ([Bre74]).
We want to use this classification in order to describe the modules arising from symmetric bands in more detail. This new knowledge allows us to analyze the indecomposables resulting from the matrix problem $\overline{\mathfrak{X}}_\Lambda$ in this setup.

Let $w_{\mathbb{Z}}$ be a symmetric band of period $p$ with $\hat{w}_p = \varepsilon^* u \eta^* u^{-1}$, $|u| = m$, $p = 2m+2$ and $\varepsilon, \eta \in \mathrm{Sp}$. According to its periodic part, we can also depict $w_{\mathbb{Z}}$ in the following form:

$$w_{\mathbb{Z}} : \qquad \varepsilon^* \circlearrowright \xleftarrow{\;u\;} \circlearrowleft \eta^* \; . \tag{251}$$

Let $v_{\mathbb{Z}} \in \left(\Phi_{\mathrm{ud}}^{\mathrm{d}}\right)^{-1}(w_{\mathbb{Z}})$ with $\hat{v}_p^{(i)} = \varepsilon^{\kappa} t \eta^{\mu} t^{-1}$, $t \in \left(\Phi_{\mathrm{ud}}^{\mathrm{d}}\right)^{-1}(u)$, $\kappa, \mu \in \{+1, -1\}$ for all $i \in \mathbb{Z}$. The band module $M(v_{\mathbb{Z}}, V)$ can be depicted as follows:

$$\begin{tikzcd}
V \arrow[d, "\varepsilon^{\kappa}"'] & V \arrow[l, "t"'] \\
V & V \arrow[l, "t"] \arrow[u, "\eta^{\mu}"]
\end{tikzcd}$$

Similar to (251), it can be rewritten to

$$\begin{tikzcd}
V \oplus V \arrow[loop left, "\varepsilon"] & V \oplus V \arrow[l, "t"'] \arrow[loop right, "\eta"]
\end{tikzcd}$$

Let $W = V \oplus V$. It follows that $W$ is a $\mathrm{k}\langle e, f \mid e^2 = e, f^2 = f\rangle$–module:

$$\begin{tikzcd}
W \arrow[loop left, "\varepsilon=f"] \arrow[loop right, "\eta=e"]
\end{tikzcd} \tag{252}$$

Consider next modules of the form $M_{i,j,x,y}(t^{[p]})$:

$$\begin{tikzcd}
b_0^{(p)} \arrow[d, "x"] & b_m^{(p)} \arrow[l, "t"'] \arrow[loop right, "y=j"] \\
b_0^{(p-1)} & b_m^{(p-1)} \arrow[l, "t"'] \\
b_0^{(2)} & b_m^{(2)} \arrow[l, "t"] \arrow[d, "y"] \\
b_0^{(1)} \arrow[loop left, "x=i"] & b_m^{(1)} \arrow[l, "t"']
\end{tikzcd} \tag{253}$$

Now let $W$ be a $\mathrm{k}$–vector space of dimension $p$. We can rewrite (253) with the respective choices of $x$ and $y$ as follows:

$$\begin{tikzcd}
W \arrow[loop left, "\varepsilon"] & W \arrow[l, "t"'] \arrow[loop right, "\eta"]
\end{tikzcd}$$

As above, we see that $W$ is a $\mathrm{k}\langle e, f \mid e^2 = e, f^2 = f\rangle$–module:

$$\begin{tikzcd}
W \arrow[loop left, "\varepsilon=f"] \arrow[loop right, "\eta=e"]
\end{tikzcd} \tag{254}$$

By this way of depicting, it is easy to see that $\varepsilon$ acts as $f$ and $\eta$ as $e$ on $W$. Since both are idempotents, they each give a vector space decomposition of $W$ of the form $\mathrm{im}(f) \oplus \ker(f) = W = \mathrm{im}(e) \oplus \ker(e)$. Hence, each of the two maps $e$ and $f$ is characterised by its decomposing property and one can equivalently to (254) consider

$$\begin{tikzcd}
 & & W & & \\
\mathrm{im}(f) \arrow[urr, hook] & \ker(f) \arrow[ur, hook] & & \mathrm{im}(e) \arrow[ul, hook] & \ker(e) \arrow[ull, hook]
\end{tikzcd} \tag{255}$$

It naturally follows that we can consider the classification of modules as in (254) in terms of a 4-subspace-problem.

In this section, we use the same notation as in [Bre74]. Hence, the 4-subspace-problem to be considered is given by $(U;\mathbf{K})$, $\mathbf{K} = (K_1, K_2, K_3, K_4)$:

$$\begin{tikzcd}
 & & U & & \\
K_1 \arrow[urr, hook] & K_2 \arrow[ur, hook] & & K_3 \arrow[ul, hook'] & K_4 \arrow[ull, hook']
\end{tikzcd}$$

Due to (255), we are interested in the modules of defect 0, that is, (cf. [Bre74, §1])

$$\rho(U;\mathbf{K}) = \sum_{i=1}^{4} \dim(K_i) - 2\dim(U) = 0.$$

Moreover, we want one type of those modules to satisfy

$$K_1 \oplus K_2 = U = K_3 \oplus K_4, \tag{256}$$

and have unique non-trivial intersection between two of the subspaces. The second type of them should satisfy any direct sum decomposition:

$$U = K_i \oplus K_j \quad \forall i \neq j. \tag{257}$$

Thus, the cases of interest from [Bre74] are case (i) (second type of modules) and cases (ii) and (iii) (first type of modules) of [Bre74, §5]. Note that case (i) corresponds to the homogeneous tubes of the AR-quiver of $\tilde{D}_4$, and the cases (ii) and (iii) correspond to two of the 2-tubes of the AR-quiver.
To examine the different cases, we recall the following result from [Bre74, §2]:

**Lemma 6.1** (Gelfand and Ponomarev)**.** *If $(U;\boldsymbol{K})$ is indecomposable, and $\rho(U,\boldsymbol{K}) = 0$, then either*

*(a) For all $i \neq j$, $1 \leq i,j \leq 4$, $K_i \oplus K_j = U$, or*

*(b) There exist $i', j' \in \{1,2,3,4\}$, $i' \neq j'$ such that $K_{i'} \cap K_{j'} \neq 0$ and, if $i \in \{i', j'\}$ and $j \notin \{i', j'\}$, then $K_i \oplus K_j = U$.*

### 6.1.1 Case (i)

We consider the following setting:

$$\begin{aligned}
&U = \zeta_1 Q \oplus \zeta_2 Q,\\
&K_1 = \zeta_1 Q, \quad K_2 = \zeta_2 Q, \quad K_3 = (\zeta_1 + \zeta_2)Q, \quad K_4 = (\zeta_1 + \zeta_2 A)Q,
\end{aligned}$$

where $\zeta_i \in \mathrm{Hom}_{\mathrm{k}}(Q, U)$ injective, and $A \in \mathrm{End}_{\mathrm{k}}(Q)$ invertible, indecomposable and for which 0 and 1 are not eigenvalues. Here, $A$ indecomposable means that if we consider $Q$ as $\mathrm{k}[x]$–module, $Q$ will be indecomposable. This results in

$$\begin{aligned}
&Q \cong \mathrm{k}^m,\\
&U \cong \mathrm{k}^{2m} = \{(p,q) \mid p, q \in \mathrm{k}^m\},\\
&K_1 \cong \{(p,0) \mid p \in \mathrm{k}^m\},\\
&K_2 \cong \{(0,q) \mid q \in \mathrm{k}^m\},\\
&K_3 \cong \{(p,p) \mid p \in \mathrm{k}^m\},\\
&K_4 \cong \{(p, A(p)) \mid p \in \mathrm{k}\}.
\end{aligned}$$

We examine the intersections of two subspaces: it is easy to see from the above notation of the subspaces that the following intersections are trivial:

$$K_1 \cap K_2 = 0, \quad K_1 \cap K_3 = 0, \quad K_2 \cap K_3 = 0.$$

We obtain the following for the other intersections:
Let $(p,q) \in K_1 \cap K_4$. Then $q = A(z) = 0$ for some $z \in \mathrm{k}^m$, and $p = z$. Since 0 is not an eigenvalue of $A$, it follows $z = 0$ and thus $p = 0 = q$.
Let $(p,q) \in K_2 \cap K_4$. Then $p = 0$ and $q = A(p) = A(0) = 0$ ($A$ non-singular).
Finally, let $(p,q) \in K_3 \cap K_4$. Then $p = q = A(z)$ for some $z \in \mathrm{k}^m$, and $p = z$. Thus, $A(z) = z$ holds. Now 1 is not an eigenvalue of $A$, so $z = 0$, giving also $p = q = 0$. Thus, we obtain

$$K_1 \cap K_4 = 0, \quad K_2 \cap K_4 = 0, \quad K_3 \cap K_4 = 0,$$

which shows that there do not exist $i \neq j \in \{1,2,3,4\}$ with $K_i \cap K_j \neq 0$. Lemma 6.1 gives

$$K_i \oplus K_j = U \quad \forall i \neq j, 1 \leq i, j \leq 4.$$

It follows that this case gives modules of the second type (cf. 257).

**Remark 6.2.** *Recall that for $A \in \mathrm{End}(Q)$, we can consider $Q$ as $\mathrm{k}[x]$–module by defining ([Jac85, §3.2])*

$$(a_0 + a_1 x + \cdots + a_m x^m)x = a_0 x + a_1(Ax) + \cdots + a_m(A^m x)$$

### 6.1.2 Case (ii)

We examine the case for the identity permutation $\iota$ and then conclude from this the respective results for the other permutations. Hence, we consider the following setting:

$$\begin{aligned}
&U = \zeta_1 Q \oplus \zeta_2 Q,\\
&K_1 = \zeta_1 Q, \quad K_2 = \zeta_2 Q, \quad K_3 = (\zeta_1 + \zeta_2)Q, \quad K_4 = (\zeta_1 + \zeta_2 J)Q
\end{aligned}$$

where $\zeta_i \in \mathrm{Hom}_{\mathrm{k}}(Q,U)$ injective, $i = 1,2$, and $J \in \mathrm{End}_{\mathrm{k}}(Q)$, nilpotent and indecomposable.
To make notation within the following computations easier, we use

$$\begin{aligned}
&Q \cong \mathrm{k}^m, \\
&U \cong \mathrm{k}^{2m} = \{(p,q) \mid p,q \in \mathrm{k}^m\}, \\
&K_1 \cong \{(p,0) \mid p \in \mathrm{k}^m\}, \\
&K_2 \cong \{(0,q) \mid q \in \mathrm{k}^m\}, \\
&K_3 \cong \{(p,p) \mid p \in \mathrm{k}^m\}, \\
&K_4 \cong \{(p,J(p)) \mid p \in \mathrm{k}^m\}.
\end{aligned}$$

We find that $K_1 \cap K_4 \neq 0$:
Let $(p,q) \in K_1 \cap K_4$. Then there exists $z \in K^m$ with $J(z) = q$ and $p = q$. We know that $\ker(J) \neq 0$ since $J$ is nilpotent. Thus, there exists $q' \in \ker(J)$ such that $(q',0) \in K_1 \cap K_4$.
Having found a non-trivial intersection, we can apply Lemma 6.1 which yields that

$$K_i \oplus K_j = U \quad \forall \{i,j\} \neq \{1,4\}, i \neq j,$$

and, in particular,

$$K_1 \oplus K_2 = U = K_3 \oplus K_4.$$

Thus, the following permutations of the subspaces (only denoting the indices) give a decomposition of the wanted form:

$$\begin{aligned}
&(1,2,3,4),(2,1,3,4),(1,2,4,3),(2,1,4,3), \\
&(3,4,1,2),(3,4,2,1),(4,3,1,2),(4,3,2,1), \\
&(1,3,2,4),(1,3,4,2),(3,1,2,4),(3,1,4,2), \\
&(2,4,1,3),(4,2,1,3),(2,4,3,1),(4,2,3,1).
\end{aligned}$$

Note that of the given permutations in [Bre74], the permutation $\iota$ corresponds to $(1,2,3,4)$, $(12)$ to $(2,1,3,4)$, $(13)$ to $(3,2,1,4)$, $(24)$ to $(1,4,3,2)$, $(34)$ to $(1,2,4,3)$ and $(12)(34)$ to $(2,1,4,3)$. Hence, it follows that the permutations $\iota$, $(12)$, $(34)$ and $(12)(34)$ give indecomposable modules of the wanted first type (cf. 256). The permutations $(13)$ and $(24)$ do not.

### 6.1.3 Case (iii)

We proceed similar to case (ii). The setting is the following:

$$\begin{aligned}
&U = \zeta_1 Q \oplus \zeta_2 Q \oplus \mu_3 X, \\
&K_1 = \zeta_1 Q \oplus \mu_3 X, \quad K_2 = \zeta_2 Q \oplus \mu_3 X, \quad K_3 = (\zeta_1 + \zeta_2)Q, \\
&K_4 = (\zeta_1 + \zeta_2(J+1) + \mu_3 b)Q
\end{aligned}$$

where $\zeta_i$ and $J$ are as in case (ii), and $\mu_3 \in \mathrm{Hom}_{\mathrm{k}}(X, U)$, $b \in \mathrm{Hom}_{\mathrm{k}}(Q, U)$ and satisfies $b(\ker(J)) = X$.
We simplify notation (as in case (ii)) in the following way:

$$\begin{aligned}
&Q \cong \mathrm{k}^m, \\
&X \cong \mathrm{k}, \\
&U \cong \mathrm{k}^{2m+1} = \{(p, q, x) \mid p, q \in \mathrm{k}^m, x \in \mathrm{k}\}, \\
&K_1 \cong \{(p, 0, x) \mid p \in \mathrm{k}^m, x \in \mathrm{k}\}, \\
&K_2 \cong \{(0, q, x) \mid q \in \mathrm{k}^m, x \in \mathrm{k}\}, \\
&K_3 \cong \{(p, p, 0) \mid p \in \mathrm{k}^m\}, \\
&K_4 \cong \{(p, (J + 1)(p), \mu_3 b(p)) \mid p \in \mathrm{k}^m\}
\end{aligned}$$

It is easy to see that $(0, 0, x) \in K_1 \cap K_2$ for any $x \in \mathrm{k}$, i.e. $K_1 \cap K_2 \neq 0$. By Lemma 6.1, it follows that

$$K_i \oplus K_j = U \quad \forall \{i, j\} \neq \{1, 2\}, i \neq j.$$

Hence, the following permutations of the subspaces give modules of the form (256):

$$\begin{gathered}
(1,3,2,4),(3,1,2,4),(1,3,4,2),(3,1,4,2), \\
(2,4,1,3),(2,4,3,1),(4,2,1,3),(4,2,3,1), \\
(1,4,2,3),(1,4,3,2),(4,1,2,3),(4,1,3,2), \\
(2,3,1,4),(3,2,1,4),(2,3,4,1),(3,2,4,1).
\end{gathered}$$

Comparing those to the list of permutations given in [Bre74], we observe that $\iota$ corresponds to $(1,2,3,4)$, (23) to $(1,3,2,4)$, (24) to $(1,4,3,2)$, (13) to $(3,2,1,4)$, (14) to $(4,2,3,1)$ and (13)(24) corresponds to $(3,4,1,2)$. It follows that (23), (24), (13) and (14) give indecomposable modules of the wanted first type. The permutations $\iota$ and (13)(24) do not.

## 6.2 Interpretation of the cases (ii) and (iii) in terms of strings

In this subsection, we use the results on the cases (ii) and (iii) of the previous subsection in order to find indecomposable $\mathrm{k}\langle e, f\rangle/(e^2 - e, f^2 - f)$–modules corresponding to the modules of the form $F(U(g_u, p))$ from Section 5.4.
Before we state the respective result, we consider the following auxiliary lemma:

**Lemma 6.3.** *Let $\Theta$ be a lower triangular matrix of size $n \times n$ with*

- $\Theta_{ii} = \Theta_{jj}$ *for all* $i, j$,
- $\Theta_{i,i-(i-1)} = \Theta_{i+1,i+1-(i-1)} = \Theta_{i+2,i+2-(i-1)} = \cdots = \Theta_{n,n-(i-1)}$ *for all* $i \geq 3$ *odd,*

- $\Theta_{i,i-(i-1)} = \Theta_{i+1,i+1-(i-1)} = \Theta_{i+2,i+2-(i-1)} = \cdots = \Theta_{n,n-(i-1)} = 0$ *for all* $i \geq 2$ *even,*

*or an upper triangular matrix with*

- $\Theta_{ii} = \Theta_{jj}$ *for all* $i, j$*,*
- $\Theta_{i-(i-1),i} = \Theta_{i+1-(i-1),i+1} = \Theta_{i+2-(i-1),i+2} = \cdots = \Theta_{n-(i-1),n}$ *for all* $i \geq 3$ *odd,*
- $\Theta_{i-(i-1),i} = \Theta_{i+1-(i-1),i+1} = \Theta_{i+2-(i-1),i+2} = \cdots = \Theta_{n-(i-1),n} = 0$ *for all* $i \geq 2$ *even,*

*and with* $\Theta^2 = \Theta$ *in any case. Then* $\Theta_{ij} = 0$ *for all* $i \neq j$*.*

*Proof.* Assume without loss of generality that $\Theta$ is lower triangular. Assume towards a contradiction that $\Theta_{ij} \neq 0$ for some $i \neq j$. Denote by $J$ the set of indices of non-zero minor diagonals of $\Theta$. Let $D$ be the diagonal given by the smallest index in $J$ with entries $0 \neq l \in \mathrm{k}$. Let the entries of the main diagonal be denoted by $a$. By the idempotent property on $\Theta$ we get that $a^2 = a$ and $l = al + la$. Thus, $a \in \{0, 1\}$. For $a = 0$ it follows that $l = 0$. If $a = 1$, we obtain $2l = l$ and thus, $l = 0$. Hence, $D$ is given by a zero diagonal and its index is not the smallest element in $J$. It follows inductively that $J$ does not contain a smallest element, yielding that $J = \varnothing$. $\square$

**Theorem 6.4.** *Let* $A = \mathrm{k}\langle e, f\rangle/(e^2 - e, f^2 - f)$*. Denote* $\bar{e} = 1 - e$*,* $\bar{f} = 1 - f$*. Then the following* $A$*–modules are pairwise non-isomorphic and indecomposable:*

*1) of dimension* $2n$*,* $n \geq 1$*:*

*a)* $$\begin{tikzcd} b_1 \arrow[loop left, "f=0"] & b_2 \arrow[l, "e"'] & \cdots \arrow[l, "f"'] & b_{2n-1} \arrow[l, "f"'] & b_{2n} \arrow[l, "e"'] \arrow[loop right, "f=1"] \end{tikzcd}$$

*b)* $$\begin{tikzcd} b_1 \arrow[loop left, "\bar{f}=0"] & b_2 \arrow[l, "e"'] & \cdots \arrow[l, "\bar{f}"'] & b_{2n-1} \arrow[l, "\bar{f}"'] & b_{2n} \arrow[l, "e"'] \arrow[loop right, "\bar{f}=1"] \end{tikzcd}$$

*c)* $$\begin{tikzcd} b_1 \arrow[loop left, "f=0"] & b_2 \arrow[l, "\bar{e}"'] & \cdots \arrow[l, "f"'] & b_{2n-1} \arrow[l, "f"'] & b_{2n} \arrow[l, "\bar{e}"'] \arrow[loop right, "f=1"] \end{tikzcd}$$

*d)* $$\begin{tikzcd} b_1 \arrow[loop left, "\bar{f}=0"] & b_2 \arrow[l, "\bar{e}"'] & \cdots \arrow[l, "\bar{f}"'] & b_{2n-1} \arrow[l, "\bar{f}"'] & b_{2n} \arrow[l, "\bar{e}"'] \arrow[loop right, "\bar{f}=1"] \end{tikzcd}$$

*2) of dimension* $2n + 1$*,* $n \geq 0$*:*

*a)* $$\begin{tikzcd} b_1 \arrow[loop left, "f=0"] & b_2 \arrow[l, "e"'] & \cdots \arrow[l, "f"'] & b_{2n-1} \arrow[l, "f"'] & b_{2n} \arrow[l, "e"'] & b_{2n+1} \arrow[l, "f"'] \arrow[loop right, "e=1"] \end{tikzcd}$$

*b)* $$\begin{tikzcd} b_1 \arrow[loop left, "\bar{f}=0"] & b_2 \arrow[l, "e"'] & \cdots \arrow[l, "\bar{f}"'] & b_{2n-1} \arrow[l, "\bar{f}"'] & b_{2n} \arrow[l, "e"'] & b_{2n+1} \arrow[l, "\bar{f}"'] \arrow[loop right, "e=1"] \end{tikzcd}$$

*c)* $$\begin{tikzcd} b_1 \arrow[loop left, "f=0"] & b_2 \arrow[l, "\bar{e}"'] & \cdots \arrow[l, "f"'] & b_{2n-1} \arrow[l, "f"'] & b_{2n} \arrow[l, "\bar{e}"'] & b_{2n+1} \arrow[l, "f"'] \arrow[loop right, "\bar{e}=1"] \end{tikzcd}$$

*d)* $$\begin{tikzcd} b_1 \arrow[loop left, "\bar{f}=0"] & b_2 \arrow[l, "\bar{e}"'] & \cdots \arrow[l, "\bar{f}"'] & b_{2n-1} \arrow[l, "\bar{f}"'] & b_{2n} \arrow[l, "\bar{e}"'] & b_{2n+1} \arrow[l, "\bar{f}"'] \arrow[loop right, "\bar{e}=1"] \end{tikzcd}$$

*Proof.* First, we show indecomposability. To this end, we compute the endomorphism ring of each module. If its only idempotents are given by 1 and 0, it is local and thus the module is indecomposable.

1) We show the statement for the case 1a). The cases 1b), c) and d) follow analogously.
Denote the respective module in 1a) by $M_{2n}$. The actions of $e$ and $f$ on the vector space $\mathrm{k}^{2n}$ in terms of matrices are the following:

$$f = \begin{pmatrix} \boxed{0} & & & & \\ & \boxed{\begin{matrix} 1 & 1 \\ 0 & 0 \end{matrix}} & & & \\ & & \ddots & & \\ & & & \boxed{\begin{matrix} 1 & 1 \\ 0 & 0 \end{matrix}} & \\ & & & & \boxed{1} \end{pmatrix}, \quad e = \begin{pmatrix} \boxed{\begin{matrix} 1 & 1 \\ 0 & 0 \end{matrix}} & & \\ & \ddots & \\ & & \boxed{\begin{matrix} 1 & 1 \\ 0 & 0 \end{matrix}} \end{pmatrix}$$

i.e.,

$$\begin{aligned} f_{i,j} &= \begin{cases} 1 & \text{if } i > 1 \text{ even and } j \in \{i, i+1\}, \text{ or } i = j = 2n, \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} 1 & \text{if } j \neq 1, j \text{ even and } i = j, \text{ or } j \text{ odd and } i = j-1, \\ 0 & \text{otherwise} \end{cases} \\ e_{i,j} &= \begin{cases} 1 & \text{if } i \text{ odd and } j \in \{i, i+1\}, \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} 1 & \text{if } j \text{ odd and } i = j, \text{ or } j \text{ even and } i = j-1, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Let $\theta = (\theta)_{i,j} \in \mathrm{End}_A(M_{2n})$. To examplify the general method, we first consider relations arising from the commutativity relations in positions $(i,j)$ with $|i-j| \leq 1$.
We obtain (258) - (262) from the commutativity relation $e\theta = \theta e$, and

$f\theta = \theta f$ gives (263) - (267):

$$\begin{aligned}
\theta_{2h+1,2h-1} &= \theta_{2h+1,2h} + \theta_{2h+2,2h}, & 1 \le h \le n-1, && (258)\\
\theta_{2h+1,2h+1} &= \theta_{2h+1,2h+1} + \theta_{2h+2,2h+1}, & 0 \le h \le n-1, && (259)\\
\theta_{2h+1,2h+1} &= \theta_{2h+1,2h+2} + \theta_{2h+2,2h+2}, & 0 \le h \le n-1, && (260)\\
\theta_{2h,2h-1} &= 0, & 1 \le h \le n, && (261)\\
\theta_{2h,2h+1} &= 0, & 1 \le h \le n-1, && (262)\\
\theta_{2h,2h-2} &= \theta_{2h,2h-1} + \theta_{2h+1,2h-1}, & 2 \le h \le n, && (263)\\
\theta_{2h,2h} &= \theta_{2h,2h} + \theta_{2h+1,2h}, & 1 \le h \le n, && (264)\\
\theta_{2h,2h} &= \theta_{2h,2h+1} + \theta_{2h+1,2h+1}, & 1 \le h \le n-1, && (265)\\
\theta_{2h+1,2h+2} &= 0, & 0 \le h \le n-1, && (266)\\
\theta_{2h+1,2h} &= 0, & 1 \le h \le n-1, && (267)
\end{aligned}$$

We first analyse the entries being equal to zero:
Note that (259) and (261) give the same zero entries. Furthermore, (264) gives $\theta_{2h+1,2h} = 0$ for $1 \le h \le n$. This is identical with (267). Equation (261) results in the diagonal below the main one being zero with respect to the even rows. The entries in the diagonal above the main one are zero by (262) (even rows) and (266) (odd rows). Equation (267) results in the entries being zero in odd rows in the diagonal below the main one.
Moreover, these entries simplify some of the other equations. After inserting (261), equation (263) reads

$$\theta_{2h,2h-2} = \theta_{2h+1,2h-1}, \quad 2 \le h \le n, \tag{268}$$

which results in pairwise equal entries between even rows $i$ and odd rows $i+1$ on the second diagonal below the main one. On the other hand, (258) simplifies to

$$\theta_{2h+1,2h-1} = \theta_{2h+2,2h}, \quad 1 \le h \le n-1 \tag{269}$$

by (267). It follows that we have pairwise same entries on the second diagonal below the main one between odd rows $i$ and even rows $i+1$. Thus, we obtain by (268) and (269) that all entries of the second diagonal below the main one are equal to each other.
Similarly, by (266), we can consider

$$\theta_{2h+1,2h+1} = \theta_{2h+2,2h+2}, \quad 0 \le h \le n-1 \tag{270}$$

instead of (260). Hence, we obtain pairwise equal entries on the main diagonal between odd rows $i$ and even rows $i+1$. Equation (261) simplifies (265) to

$$\theta_{2h,2h} = \theta_{2h+1,2h+1}, \quad 1 \le h \le n-1. \tag{271}$$

It follows that there are pairwise equal entries on the main diagonal between even rows $i$ and odd rows $i+1$. Hence, (270) and (271) result in the main diagonal consisting of equal entries.

Let us now consider entries $(i,j)$ with $|i-j| \geq 2$. The commutativity relation $f\theta = \theta f$ results in (272) - (279), $e\theta = \theta e$ in (280) - (287):

$$\begin{aligned}
\theta_{i,i+l} &= 0, & i \text{ odd}, l \geq 2 \text{ odd}, && (272)\\
\theta_{i,i+l-1} &= 0, & i \text{ odd}, l \geq 2 \text{ even}, && (273)\\
\theta_{i,i-l} &= 0, & i \text{ odd}, l \geq 2 \text{ odd}, i-l \neq 1 && (274)\\
\theta_{i,i-l-1} &= 0, & i \text{ odd}, l \geq 2 \text{ even}, i-l \neq 1 && (275)\\
\theta_{i,i+l} &= \theta_{i,i+l} + \theta_{i+1,i+l}, & i \text{ even}, l \geq 2 \text{ even}, && (276)\\
\theta_{i,i+l-1} &= \theta_{i,i+l} + \theta_{i+1,i+l}, & i \text{ even}, l \geq 2 \text{ odd}, && (277)\\
\theta_{i,i-l} &= \theta_{i,i-l} + \theta_{i+1,i-l}, & i \text{ even}, l \geq 2 \text{ even}, i-l \neq 1 && (278)\\
\theta_{i,i-l-1} &= \theta_{i,i-l} + \theta_{i+1,i-l}, & i \text{ even}, l \geq 2 \text{ odd}, i-l \neq 1 && (279)\\
\theta_{i,i+l} &= 0, & i \text{ even}, l \geq 2 \text{ odd}, && (280)\\
\theta_{i,i+l-1} &= 0, & i \text{ even}, l \geq 2 \text{ even}, && (281)\\
\theta_{i,i-l} &= 0, & i \text{ even}, l \geq 2 \text{ odd}, && (282)\\
\theta_{i,i-l-1} &= 0, & i \text{ even}, l \geq 2 \text{ even}, && (283)\\
\theta_{i,i+l} &= \theta_{i,i+l} + \theta_{i+1,i+l}, & i \text{ odd}, l \geq 2 \text{ even}, && (284)\\
\theta_{i,i+l-1} &= \theta_{i,i+l} + \theta_{i+1,i+l}, & i \text{ odd}, l \geq 2 \text{ odd}, && (285)\\
\theta_{i,i-l} &= \theta_{i,i-l} + \theta_{i+1,i-l}, & i \text{ odd}, l \geq 2 \text{ even}, && (286)\\
\theta_{i,i-l-1} &= \theta_{i,i-l} + \theta_{i+1,i-l}, & i \text{ odd}, l \geq 2 \text{ odd}, . && (287)
\end{aligned}$$

Note at first that (274) and (275) give the same zero entries, as well as (282) and (283) do. Moreover, (280) is covered by (281), and (272) is covered by (273).
It follows by (274) and (282) that the entries of the minor diagonals which start in even rows with index greater than 2, are zero. That is, (274) yields that all enries $\theta_{k,h}$ with $k$ odd, $h$ even, in the mentioned diagonals in the lower triangular part of $\theta$ are zero. On the other hand, (282) yields the same for the respective entries $\theta_{k,h}$ with $k$ even and $h$ odd. We obtain a similar result for the upper triangular part of $\theta$: (273) and (281) imply that the entries in the minor diagonals which start in even columns, are zero. We have by (273) that the entries $\theta_{k,h}$ with $k$ odd, $h$ even in the respective diagonals, are zero. Similarly, (281) shows that the entries $\theta_{k,h}$ with $k$ even, $h$ odd in the respective diagonals are zero.
Some of the above equations also simplify to some additional zero relations: (276), (278), (284), (286) can be rewritten in the same order

as follows:

$$\theta_{i+1,i+l} = 0, \qquad i \text{ even}, l \geq 2 \text{ even}, \tag{288}$$
$$\theta_{i+1,i-l} = 0, \qquad i \text{ even}, l \geq 2 \text{ even}, i-l \neq 1 \tag{289}$$
$$\theta_{i+1,i+l} = 0 \qquad i \text{ odd}, l \geq 2 \text{ even}, \tag{290}$$
$$\theta_{i+1,i-l} = 0 \qquad i \text{ odd}, l \geq 2 \text{ even}. \tag{291}$$

Relation (289) gives the same zero entries as (274), and (290) the same as (281). Similarly, (291) coincides with (283), and (288) gives a subset of (273).
Equation (277) simplifies to the following by (284):

$$\theta_{i,i+l-1} = \theta_{i+1,i+l}, \quad i \text{ even}, l \geq 2 \text{ odd}. \tag{292}$$

Thus, it implies that in the upper triangular part there exist pairwise equal entries between odd rows $i$ and even rows $i+1$ in the diagonals, counting from the third diagonal above the main diagonal on.
Equation (285) is simplified by (273) to

$$\theta_{i,i+l-1} = \theta_{i+1,i+l}, \quad i \text{ odd}, l \geq 2 \text{ odd}. \tag{293}$$

It follows from (293) that from the second diagonal on, in the upper triangular part, there are pairwise equal entries in the diagonals between odd rows $i$ and even rows $i+1$. Hence, by (292) and (293), the entries of any diagonal starting in an odd column in the upper triangular part of $\theta$ are all equal to each other, counting from the third diagonal on.
We obtain similar results for the lower triangular part: equation (286) simplifies (279) to

$$\theta_{i,i-l-1} = \theta_{i+1,i-l}, \quad i \text{ even}, l \geq 2 \text{ odd}, i-l \neq 1. \tag{294}$$

This relation gives pairwise equal entries in the diagonals in the lower triangular part between even rows $i$ and odd rows $i+1$, counting from the third diagonal on. Also, (274) inserted into (287) results in

$$\theta_{i,i-l-1} = \theta_{i+1,i-l}, \quad i \text{ odd}, l \geq 2 \text{ odd}, \tag{295}$$

giving pairwise equal entries in the diagonals in the lower triangular part, from the fourth diagonal on, between odd rows $i$ and even rows $i+1$. Thus, together with (294), we obtain that on any diagonal in the lower triangular part any entry is equal to another.
Finally, we consider the special entry $(2n, j)$. We have so far that

$$\theta_{2n,j} = \begin{cases} 0 & \text{if } j \text{ odd}, \\ \theta_{2n,j} & \text{if } j \text{ even}. \end{cases} \tag{296}$$

where $\theta_{2n,j}$ is possibly non-zero for $j$ even. Recall also, that any diagonal ending in an entry $\theta_{2n,j}$ for $j$ even consists of entries which are equal to each other. We now analyse the commutativity relation $f\theta = \theta f$ for the index $(2n, j)$ separately. We obtain that

$$(f\theta)_{2n,j} = \sum_{k=1}^{2n} f_{2n,k}\theta_{k,j} = f_{2n,2n}\theta_{2n,j} = \theta_{2n,j}, \tag{297}$$

$$(\theta f)_{2n,j} = \sum_{k=1}^{2n} f_{2n,k}\theta_{k,j} = \begin{cases} \theta_{2n,j} f_{j,j} = \theta_{2n,j} & \text{if } j \text{ even },\\ \theta_{2n,j-1} f_{j-1,j} = \theta_{2n,j-1} & \text{if } j \text{ odd}, j \neq 1. \end{cases} \tag{298}$$

Summarising, we obtain that

$$\theta_{2n,j} = \begin{cases} \theta_{2n,j} & \text{if } j \text{ even},\\ \theta_{2n,j-1} & \text{if } j \text{ odd}, j \neq 1. \end{cases} \tag{299}$$

It follows that any minor diagonal in the lower triangular part of $\theta$ is zero. In particular, we have that $\theta$ is an upper triangular matrix with every second diagonal, starting to count from the main one on, is zero. Any $\theta \in \mathrm{End}_A(M_{2n})$ is thus of the form of an upper triangular matrix as in Lemma 6.3. It follows by the same lemma that the only idempotents in $\mathrm{End}_A(M_{2n})$ are given by 0 and 1. Hence, it is local and indecomposability of $M$ follows.

2) For case a), we consider $e$ and $f$ in terms of the following matrices:

$$f = \begin{pmatrix} \boxed{0} & & & \\ & \boxed{\begin{matrix} 1 & 1 \\ 0 & 0 \end{matrix}} & & \\ & & \ddots & \\ & & & \boxed{\begin{matrix} 1 & 1 \\ 0 & 0 \end{matrix}} \end{pmatrix}, \quad e = \begin{pmatrix} \boxed{\begin{matrix} 1 & 1 \\ 0 & 0 \end{matrix}} & & & \\ & \ddots & & \\ & & \boxed{\begin{matrix} 1 & 1 \\ 0 & 0 \end{matrix}} & \\ & & & \boxed{1} \end{pmatrix}.$$

Exploiting again commutativity of $f$ and $e$ with $\theta \in \mathrm{End}_A(M_{2n+1})$, in particular $(e\theta)_{2n,j} = (\theta e)_{2n,j}$ gives that $\theta$ is of upper triangular form as described in Lemma 6.3. Indecomposability of $M_{2n+1}$ follows.
The cases b) - d) follow analogously.

Finally, we show that the modules are pairwise non-isomorphic. To this end, we show that they coincide with modules from [Bre74], case (ii) (dimension $2n$) or case (iii) (dimension $2n + 1$). We already know that the modules in the statement are indecomposable, hence, it is enough to show that the dimensions of the subspaces coincide and that the intersections of them do. Therefore, we assume an ordering as in (255).

1a) $M_{2n}$ corresponds to case (ii), (12)(34):

$$\begin{aligned}
\operatorname{im}(f) &= \langle b_2, b_4, \dots, b_{2n}\rangle, && |\operatorname{im}(f)| = n,\\
\ker(e) &= \langle b_1, b_2 - b_3, b_4 - b_5, \dots, b_{2n-2} - b_{2n-1}\rangle, && |\ker(f)| = n,\\
\operatorname{im}(e) &= \langle b_1, b_3, \dots, b_{2n-1}\rangle, && |\operatorname{im}(e)| = n,\\
\ker(e) &= \langle b_1 - b_2, b_3 - b_4, \dots, b_{2n-1} - b_{2n}\rangle, && |\ker(e)| = n,\\
\operatorname{im}(e) \cap \ker(f) &= \langle b_1\rangle \neq 0
\end{aligned}$$

1b) $M_{2n}$ corresponds to case (ii), (34). We use that $\ker(\bar{f}) = \operatorname{im}(f)$ and $\operatorname{im}(\bar{f}) = \ker(f)$ with $f$ given as in a). We also have $e$ given as in a). It follows that

$$\operatorname{im}(e) \cap \ker(f) = \langle b_1\rangle \neq 0.$$

1c) $M_{2n}$ corresponds to case (ii), (12): We have $f$ acting as in a), and $\bar{e}$ acting as $1 - e$ with $e$ given as in a). Thus, we obtain that

$$\ker(e) \cap \ker(f) = \langle b_1\rangle \neq 0.$$

1d) $M_{2n}$ corresponds to case (ii), $\iota$: We have $\bar{f} = 1 - f$ and $\bar{e} = 1 - e$ with $e$ and $f$ as in a). Applying $\ker(\bar{f}) = \operatorname{im}(f)$ and $\operatorname{im}(\bar{f}) = \ker(f)$ as before, and same for $e$, we obtain that

$$\operatorname{im}(f) \cap \ker(e) = \langle b_1\rangle \neq 0.$$

2a) $M_{2n+1}$ corresponds to case (iii), (13):

$$\begin{aligned}
\operatorname{im}(f) &= \langle b_2, b_4, \dots, b_{2n}\rangle, && |\operatorname{im}(f)| = n,\\
\ker(f) &= \langle b_1, b_2 - b_3, b_4 - b_5, \dots, b_{2n} - b_{2n+1}\rangle, && |\ker(f)| = n+1,\\
\operatorname{im}(e) &= \langle b_1, b_3, \dots, b_{2n+1}\rangle, && |\operatorname{im}(e)| = n+1,\\
\ker(e) &= \langle b_1 - b_2, b_3 - b_4, \dots, b_{2n-1} - b_{2n}\rangle, && |\ker(e)| = n,\\
\operatorname{im}(e) \cap \ker(f) &= \langle b_1\rangle \neq 0.
\end{aligned}$$

2b) $M_{2n+1}$ corresponds to case (iii), (23): We have $e$ given as in a), and $\bar{f} = 1 - f$ with $f$ as in a). It follows that

$$\operatorname{im}(f) \cap \operatorname{im}(e) \neq 0.$$

2c) $M_{2n+1}$ corresponds to case (iii), (14): apply that $\bar{e} = 1 - e$ with $e$ as in a). Moreover, we have that $f$ is given as in a). Hence, we obtain that

$$\ker(f) \cap \ker(e) \neq 0$$

2d) $M_{2n+1}$ corresponds to case (iii), (24): As before, we use $\bar{f} = 1 - f$ and $\bar{e} = 1 - e$ giving the correspondence $\ker(\bar{f}) = \operatorname{im}(f)$, $\operatorname{im}(\bar{f}) = \ker(f)$, and similar for $e, \bar{e}$. This yields that

$$\ker(e) \cap \operatorname{im}(f) \neq 0.$$

$\square$

**Proposition 6.5.** *The modules in Theorem 6.4, 1a) - d) correspond to the modules $F(U_s(g,p))$, $s = 1,2,3,4$, for $p \in \mathbb{N}\backslash\{0\}$ even, considered as $\mathrm{k}\langle e, f \mid e^2 = e, f^2 = f\rangle$–modules. Similarly, the modules in 2a)-d) correspond to the modules $F(U_s(g,p))$, $s = 1,2,3,4$, for $p \in \mathbb{N}\backslash\{0\}$ odd, considered as $\mathrm{k}\langle e, f \mid e^2 = e, f^2 = f\rangle$–modules.*

*Proof.* Let $w_{\mathbb{Z}}$ be a symmetric band with periodic part $\hat{w}_p = \varepsilon^* u \eta^* u^{-1}$. Denote by $g_u$ the corresponding $\mathfrak{L}$–chain with two double ends. Then we have that $x_1 \sim x_1 \mathrel{\hat{=}} \varepsilon^*$ and $x_m \sim x_m \mathrel{\hat{=}} \eta^*$. We consider the composite $\mathfrak{L}$–chain $g_u^{[p]}$ reduced to its joints and its start and end link:

$$x_1 - \cdots - \overleftarrow{x_m \sim x_m} - \cdots - \overleftarrow{x_1 \backsim x_1} - \cdots - \overleftarrow{x_k \backsim x_k} - \cdots - x_{\bar{k}}$$

where

$$k = \begin{cases} 1 & \text{if } p \text{ odd,} \\ m & \text{if } p \text{ even,} \end{cases}$$

$$\bar{k} = \begin{cases} 1 & \text{if } p \text{ even,} \\ m & \text{if } p \text{ odd.} \end{cases}$$

In particular, we have that $\bar{k} = \{1, m\}\backslash\{k\}$. We add the action of $\psi_s \in \Psi(g_u)$ on the joints for $p$ odd:

$$\begin{array}{lccccccccccccccc}
 & x_1 & - \cdots - & \overleftarrow{x_m} & \sim & \overleftarrow{x_m} & - \cdots - & \overleftarrow{x_1} & \backsim & \overleftarrow{x_1} & - \cdots - & \overleftarrow{x_1} & \backsim & \overleftarrow{x_1} & - \cdots - & x_m \\
s = 1\text{:} & \text{-1} & & 1 & & \text{-1} & & 1 & & \text{-1} & & 1 & & \text{-1} & & 1 \\
s = 2\text{:} & 1 & & 1 & & \text{-1} & & \text{-1} & & 1 & & \text{-1} & & 1 & & 1 \\
s = 3\text{:} & \text{-1} & & \text{-1} & & 1 & & 1 & & \text{-1} & & 1 & & \text{-1} & & \text{-1} \\
s = 4\text{:} & 1 & & \text{-1} & & 1 & & \text{-1} & & 1 & & \text{-1} & & 1 & & \text{-1}
\end{array}$$

Applying Lemma 5.14 and Lemma 5.9 results in the following for $F(U_s(g_u^{[p]}))$ in terms of strings:

$$s = 1: \quad \varepsilon{=}0 \circlearrowright \xleftarrow{\eta=-1} \xleftarrow{\varepsilon=-1} \qquad \cdots \xleftarrow{\varepsilon=-1} \circlearrowleft \eta{=}1$$

$$s = 2: \quad \bar{\varepsilon}{=}0 \circlearrowright \xleftarrow{\eta=-1} \xleftarrow{\bar{\varepsilon}=-1} \qquad \cdots \xleftarrow{\bar{\varepsilon}=-1} \circlearrowleft \eta{=}1$$

$$s = 3: \quad \varepsilon{=}0 \circlearrowright \xleftarrow{\bar{\eta}=-1} \xleftarrow{\varepsilon=-1} \qquad \cdots \xleftarrow{\varepsilon=-1} \circlearrowleft \bar{\eta}{=}1$$

$$s = 4: \quad \bar{\varepsilon}{=}0 \circlearrowright \xleftarrow{\bar{\eta}=-1} \xleftarrow{\bar{\varepsilon}=-1} \qquad \cdots \xleftarrow{\bar{\varepsilon}=-1} \circlearrowleft \bar{\eta}{=}1$$

| $s$ | $p$ even | $p$ odd |
|---|---|---|
| 1 | 1a) | 2a) |
| 2 | 1b) | 2b) |
| 3 | 1c) | 2c) |
| 4 | 1d) | 2d) |

Table 3: Cases 1 and 2 with respect to $s$

With $f = \varepsilon$ and $e = \eta$ and a small adjustment on the signs of the basis elements, we obtain the wanted correspondence to the modules described in Theorem 6.4, 2a)-d). Similarly, we obtain the correspondence for $p$ even and 1a)-d). In detail, the correspondence is given as in Table 3. □

**Remark 6.6.** *Let $w_{\mathbb{Z}}$ be a symmetric band with periodic part $\hat{w}_p = \varepsilon^* u \eta^* u^{-1}$. Applying Proposition 6.5 to $g_u$ yields by Theorem 5.21 (iv.ii) that the modules from Theorem 6.4 correspond to modules of the form $M_{0,1,x,y}(t^{[p]})$, $x \in \{\varepsilon, \bar{\varepsilon}\}$, $y \in \{\eta, \bar{\eta}\}$, $t \in (\Phi^{\mathrm{d}}_{\mathrm{ud}})^{-1}(u)$.*

### 6.3 Interpretation of case (i) in terms of bands

Let $w_{\mathbb{Z}}$ be a symmetric band with periodic part $\hat{w}_p = \varepsilon^* u \eta^* u^{-1}$. Denote by $g_{w_{\mathbb{Z}}}$ the corresponding $\mathfrak{L}$–cycle and by $U(g_{w_{\mathbb{Z}}}, \varphi)$ its representation. Let $v'_{\mathbb{Z}} \in \left(\tilde{\Psi}^{\Sigma}_{\mathrm{ud}}\right)^{-1}(w_{\mathbb{Z}})$ be weakly consistent such that $F(U(g_{w_{\mathbb{Z}}}, \varphi)) = M(v'_{\mathbb{Z}}, V)$. We denote its periodic parts by $\hat{v'}^{(i)}_p = \varepsilon t \eta t^{-1}$ for all $i \in \mathbb{Z}$. We know by Lemma 5.9, 5.11 and 5.15 that $M(v'_{\mathbb{Z}}, V)$ is of the following form:

$$\begin{tikzcd} V_0 & V_1 \arrow[l, "t"'] & V_2 \arrow[l, "\eta=-1"'] \arrow[r, "t"] & V_3 \arrow[lll, bend left, "\varepsilon=-F_\varphi^{-1}"] \end{tikzcd} \tag{300}$$

where any inverse special letter $v_j$ in $t$ and $t^{-1}$ is of the form $\bar{\varepsilon}^{-1}$ sending $b_{j-1}$ to $-b_j$, and any direct special letter $v_j$ is of the form $\varepsilon$ sending $b_j$ to $-b_{j-1}$. Changing any $\bar{\varepsilon}^{-1}$ to $\varepsilon^{-1}$ will change $v_1$ to acting as $(-1)^{\bar{\omega}+1} F_\varphi^{-1}$ with $\bar{\omega}$ describing the number of inverse special letters in $\hat{v}^{(i)}_p$. Denote by $v_{\mathbb{Z}} \in \left(\Phi^{\mathrm{d}}_{\mathrm{ud}}\right)^{-1}(w_{\mathbb{Z}})$ the word obtained from $v'_{\mathbb{Z}}$ after this action. For now, set $B = (-1)^{\bar{\omega}+1} F_\varphi^{-1}$. We can rewrite (300) as follows (cf. Section 6.1):

$$\begin{tikzcd} V_0 \oplus V_3 \arrow[loop below, "\varepsilon=\begin{pmatrix} 0 & 0 \\ B & 1 \end{pmatrix}"] & V_1 \oplus V_2 \arrow[l, "t"'] \arrow[loop below, "\eta=\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}"] \end{tikzcd} \tag{301}$$

We have that the above $V_i$'s are disjoint copies of $V$. Assume that $V$ is of dimension $n$. For $V \oplus V$ we consider its standard basis $e_1, \dots, e_{2n}$. We obtain for (301):

$$\begin{aligned}
\operatorname{im}(\eta) &= \langle e_1, \dots, e_n \rangle, \\
\ker(\eta) &= \langle e_1 + e_{n+1}, \dots, e_n + e_{2n} \rangle, \\
\operatorname{im}(\varepsilon) &= \langle e_1, \dots, e_n \rangle, \\
\ker(\varepsilon) &= \langle B^{-1}(e_1) - e_{n+1}, \dots, B^{-1}(e_n) - e_{2n} \rangle = \langle e_1 - B(e_{n+1}), \dots, e_n - B(e_{2n}) \rangle.
\end{aligned}$$

With those details, we can rewrite (301) in terms of the four subspace problem:

$$\begin{tikzcd}
 & & V \oplus V & & \\
\operatorname{im}(\eta) \arrow[urr, hook, "\begin{pmatrix}1\\0\end{pmatrix}"] & \operatorname{im}(\varepsilon) \arrow[ur, hook, "\begin{pmatrix}0\\1\end{pmatrix}"'] & & \ker(\eta) \arrow[ul, hook', "\begin{pmatrix}1\\1\end{pmatrix}"] & \ker(\varepsilon) \arrow[ull, hook', "\begin{pmatrix}1\\-B\end{pmatrix}"']
\end{tikzcd} \tag{302}$$

**Theorem 6.7.** *Let $w_{\mathbb{Z}}$ be a symmetric band and let $v_{\mathbb{Z}} \in \left(\Phi^{\mathrm{d}}_{\mathrm{ud}}\right)^{-1}(w_{\mathbb{Z}})$ be weakly consistent with $\hat{v}_p^{(i)} = \varepsilon t \eta t^{-1}$ for all $i \in \mathbb{Z}$ and such that $F(U(g_{w_{\mathbb{Z}}}, \varphi)) \cong M(v_{\mathbb{Z}}, V)$. Then $V \oplus V$ with $e = \varepsilon$ and $f = \eta$ described as in (302) is an indecomposable $\mathrm{k}\langle e, f \mid e^2 = e, f^2 = f \rangle$–module.*

*Proof.* We show that (302) fulfills the conditions of case (i) in [Bre74, §5]. Lemma 5.40 yields that if $\delta_0(g_{w_{\mathbb{Z}}})$ is odd, $\bar{\omega}$ is even and, vice versa, if $\delta_0(g_{w_{\mathbb{Z}}})$ is even, $\bar{\omega}$ is odd. Thus, we obtain that

$$\begin{aligned}
B = (-1)^{\bar{\omega}+1} F_{\varphi}^{-1} &= \begin{cases} -F_{\varphi}^{-1} & \text{if } \bar{\omega} \text{ is even,} \\ \phantom{-}F_{\varphi}^{-1} & \text{if } \bar{\omega} \text{ is odd,} \end{cases} \\
&= \begin{cases} -F_{\varphi}^{-1} & \text{if } \delta_0(g_{w_{\mathbb{Z}}}) \text{ is odd,} \\ \phantom{-}F_{\varphi}^{-1} & \text{if } \delta_0(g_{w_{\mathbb{Z}}}) \text{ is even.} \end{cases}
\end{aligned}$$

Recall the meaning of $\delta_0(g_{w_{\mathbb{Z}}})$ in the context of $\overline{\mathfrak{X}}$–represenations (cf. Chapter 4): if $\delta_0(g_{w_{\mathbb{Z}}})$ is odd, then $\varphi \neq t, t-1$, and if $\delta_0(g_{w_{\mathbb{Z}}})$ is even, then $\varphi \neq t, t+1$. This implies for $F_{\varphi}^{-1}$ that 0 and 1 are not its eigenvalues for $\delta_0(g_{w_{\mathbb{Z}}})$ odd. Similary, 0 and $-1$ are not its eigenvalues for $\delta_0(g_{w_{\mathbb{Z}}})$ even. It follows that $-B$ does not have eigenvalue 0 or 1 in both cases.
It remains to see that (302) is part of the list given by Brenner. To this end, we observe that (302) can be obtained from the module with $\pi = \iota$ in Brenner's list by applying the permutations $(14)(12)(34)$ from right to left. Note that $(34)(14)(14) = (34)$.
Recall that we can consider $V$ as $\mathrm{k}[x]$–module for $-B \in \operatorname{End}(V)$ (Remark

6.2). By definition of $-B$ in terms of $\varphi = \varphi_0^n$, it follows that $V \cong \mathrm{k}[x]/(\varphi_0^n)$ as $\mathrm{k}[x]$–modules. The Fundamental Structure Theorem for finitely generated modules over a principal ideal domain yields that $V$ is indecomposable as $\mathrm{k}[x]$–module ([Jac85, §3.8]). $\square$

**Corollary 6.8.** *The modules of the form $M(v_{\mathbb{Z}}, V)$ for $v_{\mathbb{Z}}$ weakly consistent with $\mathrm{dir}(v_{1+kp}) = \mathrm{dir}(v_{m+2+kp}) = 1$ and $\Psi_{\mathrm{ud}}^{\Sigma}(v_{\mathbb{Z}}) = w_{\mathbb{Z}}$ for $w_{\mathbb{Z}}$ a symmetric band, $V \oplus V$ an indecomposable $\mathrm{k}\langle e, f \mid e^2 = e, f^2 = f\rangle$–module, give a complete list of pairwise non-isomorphic indecomposable modules as described in case (i).*

Theorem 6.7 and Theorem 6.5 enable us to reformulate Theorem 5.49 and finally give a classification of the finite dimensional modules of a clannish algebras in terms of Crawley-Boevey's conjecture from [CB88]. To this end, we recall the notation for the modules of concern as described in Section 3.4:
We denote by $\mathcal{V}_i$ a complete set of all finite dimensional, pairwise non-isomorphic indecomposable $\mathrm{mod}\,\mathcal{C}_i$–modules, where

$$\mathcal{C}_i = \begin{cases} \mathrm{k} & \text{if } i = 1, \\ \mathrm{k}[f \mid f^2 = f] & \text{if } i = 2, \\ \mathrm{k}[T, T^{-1}] & \text{if } i = 3, \\ \mathrm{k}\langle e, f \mid e^2 = e, f^2 = f\rangle & \text{if } i = 4. \end{cases}$$

For easier notation, we denote by $\mathcal{W}_1$ the set of asymmetric strings, by $\mathcal{W}_2$ the set of symmetric strings, by $\mathcal{W}_3$ the set of asymmetric bands and by $\mathcal{W}_4$ the set of symmetric bands.
Let $w \in \mathcal{W}_1$, $V$ be a $\mathcal{C}_1$-module. Then we denote by $\mathcal{M}_1(w, V)$ the following module:

$$V_0 \xleftarrow{w_1^{\kappa_1}} V_1 \xleftarrow{w_2^{\kappa_2}} V_2 \xleftarrow{w_3^{\kappa_3}} \cdots \xleftarrow{w_n^{\kappa_n}} V_n\ ,$$

where

$$\kappa_i = \begin{cases} 1 & \text{if } (w[<i])^{-1} > w[>i], \\ -1 & \text{else,} \end{cases}$$

for all $i \in \mathrm{I}$ with $w_i$ a special letter, and where the $V_i$'s are disjoint copies of $V$. The direction $\kappa_i$ on any ordinary letter $w_i$ is given as in $w$.
Let $w = u\varepsilon^* u^{-1} \in \mathcal{W}_2$ and let $V$ be a $\mathcal{C}_2$–module. Then we denote by $\mathcal{M}_2(w, V)$ the following module:

$$\begin{tikzcd} V_0 & V_1 \arrow[l, "w_1^{\kappa_1}"'] & V_2 \arrow[l, "w_2^{\kappa_2}"'] & \cdots \arrow[l, "w_3^{\kappa_3}"'] & V_m \arrow[l, "w_m^{\kappa_m}"'] \arrow[loop right, "\varepsilon=f"] \end{tikzcd}\ ,$$

where

$$\kappa_i = \begin{cases} 1 & \text{if } (w[< i])^{-1} > w[> i], \\ -1 & \text{else,} \end{cases}$$

for all $1 \le i \le m$, with $w_i$ a special letter, and where the $V_i$'s are disjoint copies of $V$. The direction $\kappa_i$ on any ordinary letter $w_i$ is given as in $w$.
Let $w_{\mathbb{Z}} \in \mathcal{W}_3$ be of period $p$, an let $V$ be a $\mathcal{C}_3$–module. We denote by $\mathcal{M}_3(w_{\mathbb{Z}}, V)$ the module

$$\begin{tikzcd} V_0 \arrow[rrrr, bend right, "w_p^{\kappa_p}"'] & V_1 \arrow[l, "w_1^{\kappa_1}"'] & V_2 \arrow[l, "w_2^{\kappa_2}"'] & \cdots \arrow[l, "w_3^{\kappa_3}"'] & V_{p-1} \arrow[l, "w_{p-1}^{\kappa_{p-1}}"'] \end{tikzcd} ,$$

where

$$\kappa_i = \begin{cases} 1 & \text{if } (w_{\mathbb{Z}}[< i])^{-1} > w_{\mathbb{Z}}[> i], \\ -1 & \text{else,} \end{cases}$$

for $i \in \mathbb{Z}$ with $w_i$ a special letter, and where the $V_i$'s are disjoint copies of $V$. The direction $\kappa_i$ on any ordinary letter $w_i$ is given as in $w_{\mathbb{Z}}$.
Let $w_{\mathbb{Z}} \in \mathcal{W}_4$ be of period $p$ with periodic part $\hat{w}_p = \varepsilon^* u \eta^* u^{-1}$, and let $V$ be a $\mathcal{C}_4$–module. We denote by $\mathcal{M}_4(w_{\mathbb{Z}}, V)$ the module

$$\begin{tikzcd} V_0 \arrow[loop left, "\varepsilon=e"] & V_1 \arrow[l, "w_2^{\kappa_2}"'] & V_2 \arrow[l, "w_3^{\kappa_3}"'] & \cdots \arrow[l, "w_4^{\kappa_4}"'] & V_m \arrow[l, "w_{m+1}^{\kappa_{m+1}}"'] \arrow[loop right, "\eta=f"] \end{tikzcd} ,$$

where

$$\kappa_i = \begin{cases} 1 & \text{if } (w_{\mathbb{Z}}[< i])^{-1} > w_{\mathbb{Z}}[> i], \\ -1 & \text{else,} \end{cases}$$

for all $2 \le i \le m+1$ with $w_i$ a special letter, and where the $V_i$'s are disjoint copies of $V$. The direction $\kappa_i$ on any ordinary letter $w_i$ is given as in $w_{\mathbb{Z}}$.
Our final classification result reads as follows:

**Theorem 6.9** (Main Theorem - skewed-gentle algebras)**.** *Let $\Lambda$ be a skewed-gentle algebra. The modules of the form $\mathcal{M}_i(w, V)$, $i = 1, 2, 3, 4$, with $w$ running through $\mathcal{W}_i$ and $V$ running through $\mathcal{V}_i$, give a complete list of finite dimensional, pairwise non-isomorphic indecomposable modules of $\Lambda$.*

*Proof.* The result follows from Theorem 5.49, Theorem 6.7, Theorem 6.5. □

**Theorem 6.10** (Main Theorem - clannish algebras)**.** *Let $\Lambda$ be a clannish algebra. The modules of the form $\mathcal{M}_i(w, V)$, $i = 1, 2, 3, 4$, with $w$ running through $\mathcal{W}_i$ and $V$ running through $\mathcal{V}_i$, give a complete list of finite dimensional, pairwise non-isomorphic indecomposable modules of $\Lambda$.*

*Proof.* The proof follows from Theorem 6.9 by Section 4.3. □

**Remark 6.11.** *It follows from the proofs, that the letters in $\mathcal{M}_2(w, V)$ and $\mathcal{M}_4(w_{\mathbb{Z}}, V)$ are described by $u_1^{\kappa_1}, \dots, u_m^{\kappa_m}$, with $\kappa_i$ as described above.*
*Furthermore, we see that the modules $\mathcal{M}_i(w, V)$ and $\mathcal{M}_i(w_{\mathbb{Z}}, V)$ correspond to modules which are described by weakly consistent words in the alphabet $\Gamma_{\mathrm{d}}(\Lambda)$. One exception is given by the modules $\mathcal{M}_4(w_{\mathbb{Z}}, V)$ of which some correspond to words in $\Sigma_{\mathrm{d}}(\Lambda)$.*

We see that Theorem 6.10 gives a classification of the finite dimensional modules for a clannish algebra over an arbitrary field. Thus, we can confirm Crawley-Boevey's conjecture made in [CB88].
Furthermore, we are able to describe the indecomposable modules explicitely in terms of strings and bands by creating directed words from the asymmetric and symmetric strings and bands, either by using our conventions on the symmetry axes or by involving the categories $\mathcal{C}_i$.

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