Horn's problem and semi-stability for quiver representations

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1 Introduction

Given a Hermitian $n \times n$ matrix H, we write $\mathrm{Eig}(H) = \{\nu_1, \nu_2, \ldots, \nu_n\}$ for the eigenvalues of H, which we list in decreasing order, and repeat each value according to its multiplicity. We are interested in the following problem: Suppose, we have given three Hermitian matrices $H(1), H(2), H(3)$ with $H(1) + H(2) + H(3) = \mu\, \mathbb{1}_{\mathbb{C}^n}$. What can be said about the possible eigenvalues $\mathrm{Eig}(H(s))$?

Fulton [3] has explained a recent complete solution of Horn's conjecture, which answers this question. The first key step in the solution was taken by Klyachko [5], who used the correspondence between symplectic quotients and geometric invariant theory quotients to convert questions about eigenvalues into Schubert calculus. A final step was taken by Knutson and Tao [6], who proved a certain saturation property for Littlewood-Richardson coefficients. Another proof of the saturation property was recently given by Derksen and Weyman [2] using properties of semi-invariants for representations of a certain quiver.

In this expository note we show that most of the argument, from Klyachko onwards, can be formulated naturally in terms of quiver representations. We use the theory developed by Schofield [7],[8], King [4] and others, as well as the observations in [2] and [3].

2 Semi-stability for representations of quivers

2.1 (Notation) Let k be an algebraically closed field. Denote by $Q = (Q_0, Q_1, t, h)$ be a finite *quiver*, i.e. Q_0 is the set vertices, Q_1 the set of arrows, and the maps $t, h\colon Q_1 \longrightarrow Q_0$ determine the orientation of the arrows. Thus we have $ta \xrightarrow{a} ha$ for each $a \in Q_1$.

We write $K_0(Q) := \mathbb{Z}^{Q_0}$ with the canonical basis $(\varepsilon_x)_{x\in Q_0}$. Moreover $K_0(Q)^* := \mathrm{Hom}_{\mathbb{Z}}(K_0(Q), \mathbb{Z})$.

A representation V of Q is given by a collection of vector spaces $(V_x)_{x\in Q_0}$ and a collection of linear maps $(V_a)_{a\in Q_1}$ with $V_a \in \mathrm{Hom}_k(V_{ta}, V_{ha})$. The *dimension vector* $\mathbf{dim}\, V \in K_0(Q)$ is given by $(\mathbf{dim}\, V)(x) := \dim_k V_x$.

For representations V, W we define

$$\mathrm{Hom}_Q(V, W) := \{(\varphi_x)_{x\in Q_0} \in \oplus_{x\in Q_0} \mathrm{Hom}(V_x, W_x) \mid \varphi_{ha} V_a = W_a \varphi_{ta} \text{ for all } a \in Q_1\}$$

Thus we obtain the Abelian category of representations of Q. Will assume always Q without oriented cycles.

2.2 (Canonical exact sequence and Ringel form) For two representations V and W of Q we have the following canonical exact sequence:

$$0 \longrightarrow \mathrm{Hom}_Q(V,W) \xrightarrow{\iota_W^V} \oplus_{x\in Q_0} \mathrm{Hom}(V_x, W_x) \xrightarrow{\delta_W^V}$$
$$\oplus_{a\in Q_1} \mathrm{Hom}(V_{ta}, W_{ha}) \xrightarrow{\pi_W^V} \mathrm{Ext}_Q(V,W) \longrightarrow 0 \quad (1)$$

with

$$\delta_W^V((\varphi_x)_{x\in Q_0} := (\varphi_{ha}V_a - W_a\varphi_{ta})_{a\in Q_1} \text{ and } \pi_W^V((\psi_a)_{a\in Q_1}) := [V \hookrightarrow E(\psi) \twoheadrightarrow W]$$

where $E_a(\psi) = \begin{pmatrix} V_a & \psi_a \\ 0 & W_a \end{pmatrix}$.

We define the Ringel form for $\alpha, \beta \in K_0(Q)$ by

$$\langle \alpha, \beta \rangle = \sum_{x\in Q_0} \alpha(x)\beta(x) - \sum_{a\in Q_1} \alpha(ha)\beta(ta) \quad (2)$$

and find by (1) that $\langle \mathbf{dim}\, V, \mathbf{dim}\, W \rangle = \dim \mathrm{Hom}_Q(V,W) - \dim \mathrm{Ext}_Q(V,W)$. For $\alpha \in K_0(Q)$ we get $\langle \alpha, - \rangle \in K_0(Q)^*$.

2.3 (Semi-invariants) For a dimension vector $\beta \in K_0(Q)$ we consider the affine space

$$\mathrm{Rep}(Q,\beta) := \oplus_{a\in Q_1} \mathrm{Hom}(\mathrm{k}^{\beta(ta)}, \mathrm{k}^{\beta(ha)})$$

with the actions of the algebraic groups $\mathrm{GL_k}(\beta) := \prod_{x\in Q_0} \mathrm{GL_k}(\beta(x))$ and $\mathrm{SL_k}(\beta) := \prod_{x\in Q_0} \mathrm{SL_k}(\beta(x))$ by conjugation. Thus $\mathrm{GL_k}(\beta)$ orbits correspond to isoclasses of representations of Q with dimension vector β.

We call the ring of $\mathrm{SL_k}(\beta)$-invariants

$$\mathrm{SI}(Q,\beta) := \mathrm{k}[\mathrm{Rep}(Q,\beta)]^{\mathrm{SL_k}(\beta)}$$

also the ring of $\mathrm{GL_k}(\beta)$-semi-invariants, since it has a weight space decomposition

$$\mathrm{SI}(Q,\beta) = \oplus_{\sigma\in K_0(Q)^*} \mathrm{SI}(Q,\beta)$$

with

$$\mathrm{SI}(Q,\beta)_\sigma = \{f \in \mathrm{k}[\mathrm{Rep}(Q,\beta) \mid g\cdot f = \prod_{x\in Q_0} \det(g_x)^{\sigma(\varepsilon_x)} f \text{ for all } g \in \mathrm{GL_k}(\beta)\}$$

We have identified $K_0(Q)^*$ with the character group of $\mathrm{GL_k}(\beta)$. Note, that $\mathrm{SI}(Q,\beta)_\sigma \neq 0$ implies $\sigma(\beta) = 0$, since the diagonal elements $t \cdot \mathbb{1}_{\mathrm{GL_k}(\beta)}$ act trivially on $\mathrm{Rep}(Q,\beta)$.

Proposition (Schofield) *Suppose $\alpha,\beta \in K_0(Q)$ are dimension vectors with $\langle\alpha,\beta\rangle = 0$. Fix $V \in \mathrm{Rep}(Q,\alpha)$, then the polynomial*

$$d^V : \mathrm{Rep}(Q,\beta) \longrightarrow \mathrm{k}, W \mapsto \det(\delta^V_W)$$

is an element of $\mathrm{SI}(Q,\beta)_{\langle\alpha,-\rangle}$, and $d^V(W) = 0$ if and only if $\mathrm{Hom}_Q(V,W) \neq 0$.

Remark: In the above proposition we have $\mathrm{Ext}_Q(V,W) = 0$ if and only if $\mathrm{Hom}_Q(V,W) = 0$ since $\langle\alpha,\beta\rangle = 0$.

2.4 (Semi-stability) For $\sigma \in K_0(Q)^*$ and a dimension vector $\beta \in K_0(Q)$, an element $W \in \mathrm{Rep}(Q,\beta)$ is called *σ-semi-stable*, if there exists $f \in \mathrm{SI}(Q,\beta)_{n\sigma}$ with $f(W) \neq 0$ for some $n \geq 1$. Note, that the set of σ-semi-stable points $\mathrm{Rep}(Q,\beta)^{\mathrm{s.s}}_\sigma$ is open in $\mathrm{Rep}(Q,\beta)$.

For dimension vectors $\beta,\beta' \in K_0(Q)$ we write $\beta' \hookrightarrow \beta$ if a general representation of dimension β has a subrepresentation of dimension β'. Similarly, we write

$$\mathrm{ext}(\alpha,\beta) = \min\{\dim \mathrm{Ext}_Q(V,W) \mid V \in \mathrm{Rep}(Q,\alpha),\ W \in \mathrm{Rep}(Q,\beta)\}$$

Note, that $\mathrm{ext}(\alpha,\beta) = 0$ implies that general representations of dimension α resp. β admit only trivial extensions.

The implications (a)$\Longleftrightarrow$(b)$\Longleftrightarrow$(c) of the following theorem appear (independently) in [2] and [10]. Anyway, we include the proof since it is so easy.

Theorem *Let $\alpha,\beta \in K_0(Q)$ be dimension vectors with $\langle\alpha,\beta\rangle = 0$. Then the following are equivalent:*

(a) There exists a $\langle\alpha,-\rangle$ semi-stable representation in $\mathrm{Rep}(Q,\beta)$.

(b) $\langle\alpha,\beta'\rangle \leq 0$ for all $\beta' \hookrightarrow \beta$.

(c) $\mathrm{ext}(\alpha,\beta) = 0$.

(d) For some $V \in \mathrm{Rep}(Q,\alpha)$ we have $0 \neq d^V \in \mathrm{SI}(Q,\beta)_{\langle\alpha,-\rangle}$.

In case $\mathrm{k} = \mathbb{C}$, these conditions are equivalent to

(e) There exists a representation $W \in \mathrm{Rep}(Q,\beta)$ with

$$\sum_{\substack{a\in Q_1\\ ta=x}} W_a^* W_a - \sum_{\substack{a\in Q_1\\ ha=x}} W_a W_a^* = \langle\alpha,\varepsilon_x\rangle\, \mathbf{1}_{\mathbb{C}^{\beta(x)}} \tag{3}$$

for all $x \in Q_0$, where $W_a^* \in \mathrm{Hom}(\mathbb{C}^{\beta(ha)},\mathbb{C}^{\beta(ta)})$ denotes the adjoint of W_a with respect to the standard Hermitian product on $\mathbb{C}^n$.

Proof: (a)$\Longrightarrow$(b) By hypothesis, a general representation is $\langle\alpha,-\rangle$ semi-stable, thus (b) holds by Proposition 3.1 from [4].

(b)$\Longrightarrow$(c). This is [8, 5.4], using $\langle\alpha,\beta\rangle = 0$.

(c)$\Longrightarrow$(d). This is 2.3, since by hypothesis we find $V \in \mathrm{Rep}(Q,\alpha)$ and $W \in \mathrm{Rep}(Q,\beta)$ with $\mathrm{Ext}_Q(V,W) = 0 = \mathrm{Hom}_Q(V,W)$.

(d)$\Longrightarrow$(a) is trivial.

The equivalence of (a) and (e) is essentially [4, 6.5]. □.

2.5 (General subrepresentations) Let $\mathcal{S} := \mathbb{Z}^{\mathbb{N}}$ and

$$\mathcal{P}(r,b) := \{\lambda = (\lambda_1, \lambda_2, \ldots) \in \mathcal{S} \mid b-r \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0,\ \lambda_i = 0 \text{ for } i > r\}$$

We write $\mathrm{Gr}\left(\begin{smallmatrix} b \\ r \end{smallmatrix}\right)$ for the Grassmannian of r-dimensional subspaces of k^b. For a given flag $F_.$ with

$0 = F_0 \subset F_1 \subset \cdots \subset F_b = \mathrm{k}^b$ and a partition $\lambda \in \mathcal{P}(r,b)$ we have the Schubert variety

$$\Omega_\lambda(F_.) := \{L \in \mathrm{Gr}\left(\begin{smallmatrix} b \\ a \end{smallmatrix}\right) \mid \dim(L \cap F_{b+i-\lambda_i}) \geq i \text{ for } 1 \leq i \leq r\}$$

and we write σ_λ for the corresponding class in the intersection ring $A^* \mathrm{Gr}\left(\begin{smallmatrix} b \\ r \end{smallmatrix}\right)$. Recall, that the classes σ_λ with $\lambda \in \mathcal{P}(r,d)$ form a $\mathbb{Z}$-basis of $A^* \mathrm{Gr}\left(\begin{smallmatrix} b \\ r \end{smallmatrix}\right)$, and we agree $\sigma_\lambda = 0 \in A^* \mathrm{Gr}\left(\begin{smallmatrix} b \\ r \end{smallmatrix}\right)$ for $\lambda \notin \mathcal{P}(r,b)$.

For dimension-vectors β, ρ with $\rho \leq \beta$ we define $\mathrm{Gr}\binom{\beta}{\rho} := \prod_{i \in Q_0} \mathrm{Gr}\binom{\beta(i)}{\rho(i)}$ and note that the intersction ring $A^* \mathrm{Gr}\binom{\beta}{\rho}$ is canonically isomorphic to $\otimes_{i \in Q_0} A^* \mathrm{Gr}\binom{\beta(i)}{\rho(i)}$. Thus we can write $\sigma_\lambda^{(x)} := 1 \otimes \cdots \otimes \sigma_\lambda \otimes \cdots \otimes 1 \in A^* \mathrm{Gr}\binom{\beta}{\rho}$ for $\lambda \in \mathcal{P}(\rho(x), \beta(x))$.

For tecnical resons we set

$$\mathcal{Q}(\rho,\beta) := \times_{a \in Q_1} \mathcal{P}(\rho(ta), \beta(ta)) \subset \mathcal{S}^{Q_1}$$

and for $(\lambda(a))_{a \in Q_1} \in \mathcal{Q}(\rho,\beta)$ we define $(\tilde{\lambda}(a))_{a \in Q_1} \in \mathcal{S}^{Q_1}$ by

$$\tilde{\lambda}_i(a) := \beta(ha) - \rho(ha) - \lambda_{\rho(ta)+1-i}(a) \text{ if } 1 \leq i \leq \rho(ta)$$

and $\tilde{\lambda}_i(a) = 0$ else. Finally we let

$$\mathcal{Q}'(\rho,\beta) := \{(\lambda(a))_{a \in Q_1} \in \mathcal{Q}(\rho,\beta) \mid \tilde{\lambda}(a) \in \mathcal{P}(\rho(ha), \beta(ha)) \text{ for all } a \in Q_1\}$$

Since the Littlewood-Richardson coefficients are nonnegative, we have the following version of the Theorem in [1].

Proposition *For dimension vectors $\rho, \beta \in K_0(Q)$ we have $\rho \hookrightarrow \beta$ if and only if for some $\lambda = (\lambda(a))_{a \in Q_1} \in \mathcal{Q}'(\rho,\beta)$ we have*

$$0 \neq \prod_{\substack{a \in Q_1 \\ ta = x}} \sigma_{\lambda(a)} \prod_{\substack{a \in Q_1 \\ ha = x}} \sigma_{\tilde{\lambda}(a)} \in A^* \mathrm{Gr}\binom{\beta(x)}{\rho(x)} \ \text{ for all } x \in Q_0 \qquad (4)$$

Proof: By the Theorem in [1] we have

$$\begin{aligned} \rho \hookrightarrow \beta \Longleftrightarrow \quad 0 \neq & \prod_{a \in Q_1} \left(\sum_{\lambda \in \mathcal{P}(\rho(ta), \beta(ta))} \sigma_\lambda^{(ta)} \sigma_{\tilde{\lambda}}^{(ha)} \right) \\ = & \sum_{\lambda \in \mathcal{Q}(\rho,\beta)} \left(\prod_{a \in Q_1} \sigma_{\lambda(a)}^{(ta)} \sigma_{\tilde{\lambda}(a)}^{(ha)} \right) \\ = & \sum_{\lambda \in \mathcal{Q}'(\rho,\beta)} \left(\prod_{x \in Q_0} \Big(\prod_{\substack{a \in Q_1 \\ ta = x}} \sigma_{\lambda(a)}^{(x)} \prod_{\substack{a \in Q_1 \\ ha = x}} \sigma_{\tilde{\lambda}(a)}^{(x)} \Big) \right) \end{aligned}$$

Now, by the Littlewood-Richardson rule, for each $\lambda \in \mathcal{P}'(\rho,\beta,Q)$ we have

$$\prod_{x\in Q_0} \Big(\prod_{\substack{a\in Q_1\\ ta=x}} \sigma^{(x)}_{\lambda(a)} \prod_{\substack{a\in Q_1\\ ha=x}} \sigma^{(x)}_{\tilde{\lambda}(a)} \Big) = \sum_{\mu\in\mathcal{P}(\rho,\beta)} c^{\mu}_{\lambda} \prod_{x\in Q_0} \sigma^{(x)}_{\mu(x)} \tag{5}$$

for some $c^{\mu}_{\lambda} \geq 0$, if we set $\mathcal{P}(\rho,\beta) := \times_{x\in Q_0}\mathcal{P}(\rho(x),\beta(x))$. Thus, $\rho \hookrightarrow \beta$ iff (5) gives a nonzero expression for some $\lambda \in \mathcal{Q}'(\rho,\beta)$. This is clearly the case iff (4) holds. □

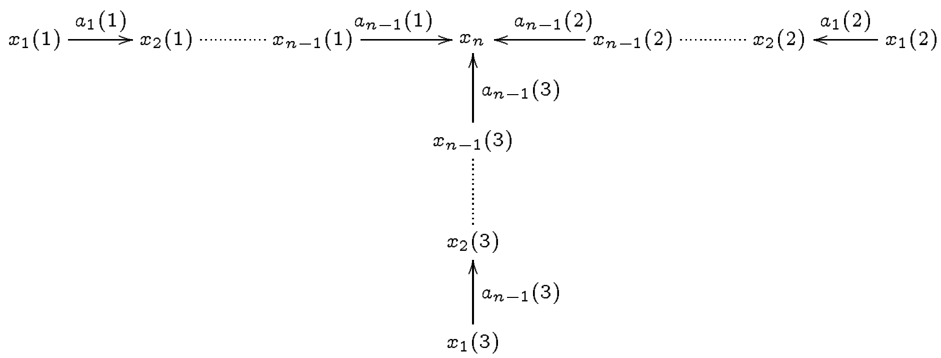
Remark: Suppose, at $x \in Q_1$ end exactly one arrow a and starts exactly one arrow b. If we have $\lambda \in \mathcal{Q}'(\rho,\beta)$, then condition (4) is for x equivalent to

$$\lambda(b) \subseteq \hat{\lambda}(a) \text{ where}$$

$$\hat{\lambda}(a) := \begin{cases} ((\beta(x)-\rho(x))^{\rho(x)-\rho(ta)}, \lambda_1(a),\ldots,\lambda_{\rho(ta)}(a)) & \text{if } \rho(x)\geq\rho(ta) \\ (\lambda_{1+(\rho(ta)-\rho(x))}(a),\ldots,\lambda_{\rho(ta)}(a)) & \text{else} \end{cases}$$

3 The star quiver

3.1 (Notation) We consider for $n \in \mathbb{N}$ the following quiver $Q^{(n)}$:



and the dimension vector $\beta_n \in K_0(Q^{(n)})$ defined by $\beta_n(x_i(s)) = i$ for $1 \leq i \leq n-1$, $s = 1,2,3$, and $\beta(x_n) = n$.

Moreover, we choose partitions $\nu(s) = (\nu_1(s),\ldots,\nu_n(s))$ with $\nu_n(s) = 0$ for $s = 1,2,3$, and suppose $\mu := \frac{1}{n}\sum_s\sum_{i=1}^n \nu_i(s) \in \mathbb{N}_0$. We associate to ν a dimension vector α_ν by setting

$$\alpha_\nu(x_i(s)) := \nu_1(s) - \nu_{i+1}(s) \qquad 1\leq i\leq n-1,\ s = 1,2,3$$
$$\alpha_\nu(x_n) := \sum_s \nu_1(s) - \mu$$

note, that in fact $\alpha_\nu \geq 0$. The corresponding weight $\sigma_\nu := \langle\alpha_\nu, -\rangle$ is given by $\sigma_\nu(\varepsilon_{x_i(s)}) = \nu_i(s) - \nu_{i+1}(s)$ for $1 \leq i \leq n-1$ and $\sigma_\nu(x_n) = -\mu$.

3.2 (Semi-invariants and $\mathrm{GL}_{\mathbb{C}}(n)$-modules) We keep the above notation and state the following result from [2] in a for us convenient way:

Proposition *The* $\mathrm{GL}_{\mathbb{C}}(n)$*-module* $\otimes_{s=1}^{s} S_{\nu(s)}(\mathbb{C}^n)$ *has a summand of type* $S_{\mu^n}(\mathbb{C}^n)$ *if and only if* $\mathrm{SI}(Q^{(n)}, \beta_n)_{\sigma_\nu} \neq 0$

3.3 (General submodules) Let I be a subset of $\{1, 2, \ldots, n\}$, with r elements $i_1, i_2, \ldots, i_r$ in increasing order. We assign to I a partition $\lambda(I)$ with $\lambda_j(I) := n-r+j-i_j$. Note, that the corresponding Schubert cell in $\mathrm{Gr}\binom{n}{r}$ with respect to a given flag $F_.$ is

$$\Omega^0_{\lambda(I)}(F_.) = \{L \in \mathrm{Gr}\binom{n}{r} \mid \dim(L \cap F_l) = j \text{ for } i_j \leq l < i_{j+1} \text{ and } 0 \leq j \leq r\}$$

if we take $i_0 := 0,\ i_{r+1} = n+1$.

We denote by P^n_r the set of triples $I = (I(1), I(2), I(3))$ of r-element subsets of $\{1, 2, \ldots, n\}$. For $I \in P^n_r$ we write the elements of $I(s)$ as $(i_1(s), \ldots, i_r(s))$ in increasing order. We define for $I \in P^n_r$ a dimension vector $\beta_I \in K_0(Q^{(n)})$ by $\beta_I(x_n) = r$ and

$$\beta_I(x_l(s)) := j \text{ for } i_j(s) \leq l < i_{j+1}(s) \text{ and } 0 \leq j \leq n,\ \ s = 1, 2, 3$$

Next we define

$$S^n_r := \{I \in P^n_r \mid \sum_s \sum_{j=1}^{r} \lambda_j(I(s)) = r(n-r) \text{ and } 0 \neq \prod_s \sigma_{\lambda(I(s))} \in A^* \mathrm{Gr}\binom{n}{r}\}$$

Note, that by the first condition in S^n_r the second means $\prod_s \sigma_{\lambda(I(s)} = d\sigma_{\lambda(1,2,\ldots,r)}$ for some $d \in \mathbb{N}$.

Proposition *With the above notation we have:*

(a) For $P \in P^n_r$ *we have* $\beta_I \hookrightarrow \beta_n$ *if and only if* $0 \neq \prod_s \sigma_{\lambda(I(s))} \in A^* \mathrm{Gr}\binom{r}{n}$.

(b) Let $\beta' \hookrightarrow \beta_n$ *with* $\beta'(x_n) = r$*, then there exists* $I \in S^n_r$ *with* $\beta_I(x_i(s)) \geq \beta'(x_i(s))$ *for all* $1 \leq i \leq n-1$ *and* $1 \leq s \leq 3$.

Proof: (a) We apply the proposition in 2.5 to our special situation. First, for $I \in P^n_r$ take

$$\lambda(a_l(s)) := (i_j(s) - j, i_{j-1} - j - 1, \ldots i_1(s) - 1)$$

for $i_j(s) \leq l < i_{j+1}(s)$ and $0 \leq j \leq r$, $1 \leq s \leq 3$. This shows, that $\prod_s \lambda(I(s)) \neq 0$ implies $\beta_I \hookrightarrow \beta_n$ Conversely, we see from the remark in 2.5, that any possible choice of the $\lambda(a_i(s))$ with $\sigma_{\tilde{\lambda}(a_i(s))}\sigma_{\lambda(a_{i+1}(s))} \neq 0$ implies $\lambda(I(s)) \subseteq \tilde{\lambda}(a_{n-1}(s))$. Thus the condition is also necessary.

(b) Note first, that by passage to the closure of Schubert-cells we find for each $I \in P^n_r$ with $\prod_s \sigma_{\lambda(I(s))} \neq 0$ some $I' \in S^n_r$ with $\beta_I(x_i(s)) \leq \beta_{I'}(x_i(s))$ for all i, s. The rest is clear. □

Remark: It is a straightforward calculation, that for $I \in P^n_r$ we have

$$\langle \alpha_\nu, \beta_I \rangle = (\sum_s \sum_{i \in I(s)} \nu_i(s)) - r\mu$$

3.4 (Linear algebra) Suppose, we have a Hermitian matrix $H \in \mathbb{C}^{n\times n}$ with eigenvalues $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_n = 0$. Then

$$H = U \begin{pmatrix} \nu_1 & & 0 \\ & \ddots & \\ 0 & & \nu_n \end{pmatrix} U^*$$

for some unitary matrix U. If we set then

$$H_{n-1} := U \begin{pmatrix} \sqrt{\nu_1} & & 0 \\ & \ddots & \\ & & \sqrt{\nu_{n-1}} \\ 0 & \cdots & 0 \end{pmatrix} \text{ and } H_i := \begin{pmatrix} \sqrt{\nu_1 - \nu_{i+1}} & & 0 \\ & \ddots & \\ & & \sqrt{\nu_i - \nu_{i+1}} \\ 0 & \cdots & 0 \end{pmatrix}$$

for $1 \leq i \leq n-2$ then $H_{n-1}H_{n-1}^* = H$ and

$$H_i^* H_i - H_{i-1} H_{i-i}^* = (\nu_i - \nu_{i+1})\, \mathbb{1}_{\mathbb{C}^i} \text{ for } 2 \leq i \leq n-1 \tag{6}$$

$$H_1^* H_1 = \nu_1 - \nu_2 \tag{7}$$

Conversely, if we have matrices $H_j \in \mathbb{C}^{(j+1)\times j}$ for $1 \leq j \leq (n-1)$ that fulfill (6) and (7), then $H_{n-1}H_{n-1}^*$ is a Hermitian matrix with eigenvalues $\nu_1, \nu_2, \ldots, \nu_n = 0$. This last property follows from the fact that if A and B are $n \times n$ matrices, then the traces of the powers of AB and BA are eqal, so that AB and BA have the same characteristic polynomial.

3.5 (Application to Horn's problem) By the observations in this section, we can interpret the equivalent statement in the following corollary as different characterizations for the existence of a σ_ν-semi stable representation of dimension β_n for the quiver $Q^{(n)}$.

Corollary 1 *Let $\nu_1(s) \geq \nu_2(s) \geq \cdots \geq \nu_n(s)$ be integers for $s = 1, 2, 3$, and suppose $\mu := \frac{1}{n}\sum_s \sum_i \nu_i(s) \in \mathbb{Z}$. Then the following are equivalent:*

(a) There exist Hermitian matrices $H(s) \in \mathbb{C}^{n\times n}$ with eigenvalues $\nu_1(s) \geq \cdots \geq \nu_n(s)$ for $s = 1, 2, 3$ with

$$\sum_s H(s) = \mu \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

(b) For all $I \in S_r^n$ and $1 \leq r < n$ we have

$$\frac{1}{r}\left(\sum_s \sum_{i\in I(s)} \mu_i(s)\right) \leq \mu$$

(c) The $\mathrm{GL}_\mathrm{k}(n)$-module $\otimes_{s=1}^3 S_{\nu(s)}(\mathbb{C}^n)$ has a summand isomorphic to $S_{\mu^n}(\mathbb{C}^n)$.

Proof: Clearly, each of the conditions is equivalent to the corresponding condition for ν', with $\nu'_i(s) := \nu_i(s) - \nu_n(s)$ and $\mu' := \mu - \sum_s \nu_n(s)$. Thus we may assume $\nu_n(s) = 0$ for $s = 1, 2, 3$. Now, we apply Theorem in 2.4 to the situation of $Q^{(n)}$ and $\alpha_\nu, \beta_n \in K_0(Q^{(n)})$ as defined in 3.1.

Now, by 3.4, condition (a) above is equivalent to condition (e) of the theorem. By the proposition and remark in 3.3, condition (b) is equivalent to condition (b) of the theorem. By the proposition in 3.2 condition (c) is equivalent to condition (d) in the theorem. □

Remark: The moduli space of $\langle \alpha_\nu, - \rangle$-semi-stable representations, [4]

$$\mathrm{Proj}(\oplus_{n \in \mathbb{N}_0} \mathrm{SI}(Q^{(n)}, \beta_n)_{\langle n\alpha_\nu, - \rangle})$$

should be interesting. It follows from recent work of A. Schofield [9], that this space is a rational variety. It is not hard, to identify it with the GIT-quotient for triples of flags with respect to the linearization associated to $(\nu(1), \nu(2), \nu(3))$, see also [5].

If we set $U^n_r := \{I \in P^n_r \mid \sum_s \sum_{i=1}^r \lambda_i(I(s)) = r(n-r)\}$ we obtain the following recursive description of S^n_r, which is basically Horn's original definition.

Corollary 2 *For $I \in U^n_r$ we have $I \in S^n_r$ if and only if*

$$\frac{1}{q}\left(\sum_s \sum_{j \in J(s)} \lambda_j(I(s))\right) \leq (n-r)$$

for all $J \in S^r_q$ and $1 \leq q < r$.

Proof: It follows by an codimension argument, that for $I \in U^n_r$ we have

$$\prod_s \sigma_{\lambda(I(s))} \neq 0 \Longleftrightarrow \prod_s \sigma_{\lambda(I(s))} = d\sigma_{(n-r)^r}$$

for some integer $d \geq 1$. Since the multiplication of classes $\sigma_\lambda \in A^* \,\mathrm{Gr}\binom{n}{r}$ is determined by Littlewood-Richardson coefficients, we may use the equivalence of (b) and (c) in Corollary 1 above. □

Corollary 3 *The conditions (a) and (b) in Corollary 1 remain equivalent, if we allow the $\nu_i(s) \in \mathbb{R}$, and $\mu \in \mathbb{R}$.*

This is the same argument as in the proof of Proposition 7 in [3].

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