Infinite-dimensional modules in the representation theory of finite-dimensional algebras

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Normally, when one studies the representation theory of a finite-dimensional associative algebra, one restricts to studying its finite-dimensional modules. After all, this includes the simple modules, ideals, the indecomposable injective modules, and so on. However, infinite-dimensional modules do occasionally arise, either as objects of interest in themselves, or because of their relationship with infinite families of finite-dimensional modules. These notes correspond closely to a series of three introductory talks on infinite-dimensional modules that I gave in Trondheim in Summer 1996. My aim was:

(1) To give a selection of examples showing how infinite-dimensional modules can arise naturally when studying finite-dimensional algebras.

(2) To outline the general theory of infinite-dimensional modules, in particular the notion of purity and the connection with the model theory of modules. The high points are the various finiteness conditions on algebraically compact modules, and the Ziegler spectrum.

(3) To show how this general theory applies to the special cases of tame and/or hereditary algebras. In particular for tame algebras, to show how certain infinite-dimensional 'generic modules' control the behaviour of finite-dimensional modules, and for hereditary algebras to show how infinite-dimensional 'stones' are related to properties of the general representation of a given dimension.

These three objectives corresponded to the three lectures, and they correspond to the three sections in this article.

Throughout, we restrict to studying finite-dimensional associative algebras (with 1) over an algebraically closed field K, and write D for duality with the field. Except where stated, all modules are left modules. We write A-mod for the category of finite-dimensional A-modules, and A-Mod for the category of all A-modules. (Many of the results remain true for artin algebras, provided that one replaces the phrase 'finite-dimensional module' by 'finitely generated module'. In

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fact the general theory expounded in Section 2 applies to arbitrary rings, but 'finite-dimensional' sometimes needs to be replaced by 'finitely presented', and sometimes by 'module with a left almost split map', so there are complications.)

This article contains a couple of original results, the properties of tree modules in Section 1.4, the characterization of coherent functors in Section 2.1 (which enables the definition used in these notes), and the characterization of definable subcategories in Section 2.3. (The last two were first presented in lectures I gave at Bielefeld University in 1993.) In addition, I sketch a proof of the classification of the indecomposable algebraically compact modules for a tame hereditary algebra, which does not seem to be fully written down in the literature, and a proof of Schofield's classification of the stones for a hereditary algebra. Although first mentioned by Schofield in 1991, at the time of writing there is no other written version of this result. The proof given here is based on lectures given by Schofield in Krippen, Germany, in 1995. I would like to thank Henning Krause, who visited Leeds while I was preparing my lectures.

1. Examples of infinite-dimensional modules

In this section we explain a few standard facts, and give various examples showing how infinite-dimensional modules can actually arise for finite-dimensional algebras.

1.1. Kronecker modules. The *Kronecker algebra* is the path algebra of the Kronecker quiver $\bullet \rightrightarrows \bullet$. Its modules, *Kronecker modules*, are therefore given by vector spaces and linear maps

$$U \underset{\phi}{\overset{\theta}{\rightrightarrows}} V.$$

Of course, in most situations where Kronecker modules arise, the vector spaces will be finite dimensional. Sometimes, however, it is essential to consider infinite-dimensional Kronecker modules.

Example. Aronszajn and others initiated the study of infinite-dimensional Kronecker modules because of their applications to functional analysis. Let V be a Hilbert space, for example $L^2(0,1)$, and let T be an unbounded operator on V, for example the differential operator $(Tf)(x) = x^2\frac{df}{dx}$. Observe that T is not everywhere defined, but only on a dense subspace U of V. This defines an infinite-dimensional Kronecker module

$$X \quad = \quad U \underset{\text{inc}}{\overset{T}{\rightrightarrows}} V$$

with base field $K = \mathbb{C}$. Typically one wants to find the eigenvalues of T, so the $\lambda \in K$ such that $Tf = \lambda f$ for some $f \in U$. This can be reformulated as $\mathrm{Hom}(R_\lambda, X) \neq 0$, where R_λ is the module

$$R_\lambda \quad = \quad K \underset{1}{\overset{\lambda}{\rightrightarrows}} K,$$

and in this way one can introduce the algebraic and homological properties of Kronecker modules.

EXAMPLE. In addition to the R_λ ($\lambda \in K$) one should consider the module

$$R_\infty \quad = \quad K \underset{0}{\overset{1}{\rightrightarrows}} K.$$

Now a Kronecker module X is said to be

□ *torsion-free* if $\mathrm{Hom}(R_\lambda, X) = 0$ for all $\lambda \in K \cup \{\infty\}$, and

□ *divisible* if $\mathrm{Ext}^1(R_\lambda, X) = 0$ for all $\lambda \in K \cup \{\infty\}$.

It turns out that there are no non-zero finite-dimensional modules which are both torsion-free and divisible, but there are infinite-dimensional ones. Using the standard projective resolution of a Kronecker module, one finds that

$$U \underset{\phi}{\overset{\theta}{\rightrightarrows}} V$$

is torsion-free divisible if and only if ϕ is invertible and $\theta - \lambda\phi$ invertible for all $\lambda \in K$. The first of these conditions enables us to identify U and V, with ϕ the identity map. Now the second condition, that $\theta - \lambda$ is invertible for all λ, implies that $f(\theta)$ is invertible for all nonzero polynomials $f(T) \in K[T]$. Thus U becomes a vector space over the rational function field $K(T)$ on setting

$$\frac{f(T)}{g(T)}\, u = g(\theta)^{-1} f(\theta)(u).$$

We conclude that a module is torsion-free divisible if and only if it is isomorphic to

$$U \underset{1}{\overset{T}{\rightrightarrows}} U$$

for some $K(T)$-vector space U.

1.2. Indecomposable decompositions. Any finite-dimensional module is isomorphic to a direct sum of indecomposables, essentially uniquely. Does this generalize to infinite-dimensional modules?

First we need infinite direct sums. If X_i ($i \in I$) is a family of modules, then their direct sum is

$$\bigoplus_{i\in I} X_i = \{(x_i) \in \prod_{i\in I} X_i \mid \text{all but finitely many } x_i \text{ are zero}\}.$$

Whereas the product satisfies $\mathrm{Hom}(X, \prod_i Y_i) = \prod_i \mathrm{Hom}(X, Y_i)$, the direct sum satisfies $\mathrm{Hom}(\oplus_i X_i, Y) = \prod_i \mathrm{Hom}(X_i, Y)$. One writes X^I and $X^{(I)}$ for the product and direct sum of copies of X indexed by I.

THEOREM (Krull-Remak-Schmidt-Azumaya). *If a module M decomposes as a direct sum of indecomposables,*

$$M = \bigoplus_{i\in I} M_i\,,$$

each with local endomorphism ring, then any indecomposable direct summand of M is isomorphic to some M_i, and in any decomposition of M as a direct sum of indecomposables, the terms are in 1-1 correspondence with the M_i.

However, even for Kronecker modules there is pathological behaviour. Such behaviour was originally found by Corner for abelian groups, and it was adapted to $K[T]$-modules, the Kronecker algebra, and other algebras by Brenner and Ringel [**3**].

EXAMPLE. For the Kronecker algebra:
□ There is a nonzero module with no indecomposable direct summand.
□ There are indecomposables L, M, M' with $L \oplus M \cong L \oplus M'$ but $M \not\cong M'$.
□ If $q \geq 2$ there is a module M with $M^i \cong M^j \Leftrightarrow i \equiv j \pmod q$.

For algebras of finite representation type, however, there is no pathology whatsoever. The following result is due to Auslander [**2**] and independently to Ringel and Tachikawa [**29**].

THEOREM. *If A has finite representation type, then any indecomposable module is finite dimensional (so has local endomorphism ring), and any module is a direct sum of indecomposables.*

1.3. Endofinite modules. A vector subspace X of an A-module M is *fully invariant* if $\theta(X) \subseteq X$ for any A-module endomorphism of M, or in other words, if X is a submodule of M when it is considered in the natural way as an $\mathrm{End}(M)$-module. One says that M is *endofinite* if it has the ascending and descending chain condition on its fully invariant subspaces, that is, if $\mathrm{length}_{\mathrm{End}(M)} M < \infty$. Clearly finite-dimensional modules are endofinite, but there are also infinite-dimensional endofinite modules. Nevertheless, the general theory shows that every endofinite module is a (possibly infinite) direct sum of indecomposable endofinites, and these have local endomorphism ring. Thus the Krull-Remak-Schmidt-Azumaya Theorem applies.

EXAMPLE. If L/K is a field extension (transcendental, since K is algebraically closed), then $A^L = A \otimes_K L$ is a finite-dimensional L-algebra. Now if M is a finite-dimensional A^L-module, then it is automatically an A-module by restriction, and as such it is infinite-dimensional, but endofinite. It is of interest to determine the indecomposable direct summands. In particular if A is the Kronecker algebra and $L = K(T)$, then the regular module

$$K(T) \underset{1}{\overset{T}{\rightrightarrows}} K(T)$$

for A^L is indecomposable as an A-module since its endomorphism ring is easily seen to be $\mathrm{End}_{K[T]}(K(T)) \cong K(T)$, and this is a local ring. Clearly the direct sums of copies of this module are the torsion-free divisible modules.

EXAMPLE. For an algebra A and $n \geq 0$ there is an affine scheme $\mathrm{mod}(A, n)$ of A-module structures on K^n, see for example [**10**]. If R is the coordinate ring of $\mathrm{mod}(A, n)$, then there is a universal A-R-bimodule ${}_A M_R$, free of rank n over R, which specializes to each of these A-module structures when tensored with the simple R-modules. If the total quotient ring S of R is artinian (for example, if $\mathrm{mod}(A, n)$ is reduced) then $M \otimes_R S$ is an endofinite A-module, and presumably its structure describes the general behaviour of n-dimensional A-modules.

REMARK. For any algebra A the endofinite A-modules can be described using certain integer-valued functions. By definition, a *character* χ is a function from the set of finite-dimensional A-modules to $\mathbb{N}$ with

1. $\chi(X \oplus Y) = \chi(X) + \chi(Y)$, and
2. $\chi(Z) \leq \chi(Y) \leq \chi(X) + \chi(Z)$ for any exact sequence $X \to Y \to Z \to 0$.

A character is said to be *irreducible* if it cannot be written in a non-trivial way as a sum of other characters. In [**8**], I proved that every character can be written in a unique way as a sum of irreducibles, and that the indecomposable endofinite modules correspond 1-1 to irreducible characters via

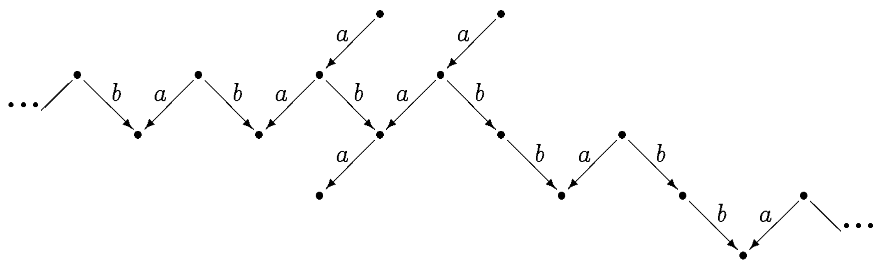
$$M \mapsto \chi_M \quad \text{with} \quad \chi_M(X) = \text{length}_{\text{End}(M)} \text{Hom}(X, M).$$

It must be mentioned that this character theory is very similar to a theory of Schofield [**30**, Theorem 7.12], in which equivalence classes of homomorphisms from A to a simple artinian ring are in 1-1 correspondence with 'Sylvester rank functions'. In particular, the fact that every character is uniquely a sum of irreducibles is already implicit in Schofield's work. Nevertheless, the character theory seems more appropriate here, being more module-theoretic.

1.4. Tree modules. Suppose that $A = KQ/I$ where Q is a quiver and I is an ideal in the path algebra KQ. By a *tree over* A we mean a map of quivers $F : \Gamma \to Q$ with the following properties

1. Γ is a connected quiver, which is a tree, so has no cycles, even unoriented. However, Γ may be infinite.
2. F is *unramified*, meaning that for each vertex $i \in \Gamma$ and arrow $a \in Q$ with head (respectively tail) at $F(i)$, there is at most one arrow $b \in F^{-1}(a)$ with head (respectively tail) at i.
3. No path in Γ is sent to a path occuring (with non-zero coefficient) in any element of I.

A tree over A can be described by drawing the quiver Γ and labelling each vertex and arrow with its image in Q. For example, the picture



describes a tree over the algebra $A = K\langle a, b\rangle/(a, b)^4$. (In this case Q has only one vertex, so there is no need to label the vertices of Γ.)

If F is a tree over A then there is a corresponding *tree module* K_F with a basis element x_i for each vertex i in Γ, and the action of A given as follows. If i is a vertex in Γ and p is a path in Q, then $px_i = 0$ if there is no path in Γ lifting p and starting at i. On the other hand, if there is such a path, then it is uniquely determined, and $px_i = x_j$ if it terminates at vertex j.

THEOREM. *If $F : \Gamma \to Q$ is a tree over A, then K_F is an indecomposable A-module. If $F' : \Gamma' \to Q$ is another tree over A then $K_F \cong K_{F'}$ if and only if there is a quiver isomorphism $\theta : \Gamma \to \Gamma'$ with $F' \circ \theta = F$.*

This has been known for some time in case Γ is finite (so that K_F is a finite-dimensional module), see Gabriel [**11**, 3.5]. Indecomposability has also been proved by Krause [**19**] in case Γ is infinite, but at most two arrows are incident at any vertex in Γ. To solve the general case we use that the group algebra of a free group has no non-trivial idempotents. First we need a lemma.

If $F : \Gamma \to Q$, $F' : \Gamma' \to Q$ are trees over A, then by a *partial map* $\theta : F \rightsquigarrow F'$ we mean a quiver isomorphism $\theta : D_\theta \to R_\theta$ satisfying $F' \circ \theta = F|_{D_\theta}$, where D_θ is a non-empty connected (hence full) subquiver of Γ which is closed under predecessors (i.e. if $r \to s$ is an arrow in Γ and $s \in D_\theta$, then $r \in D_\theta$), and R_θ is a non-empty connected (hence full) subquiver of Γ' which is closed under successors (i.e. if $r \to s$ is an arrow in Γ' and $r \in R_\theta$, then $s \in R_\theta$).

Given vertices $r \in \Gamma$ and $s \in \Gamma'$, we write $r \rightsquigarrow s$ to mean that there is a partial map $\theta : F \to F'$ with $r \in D_\theta$, $s \in R_\theta$ and $s = \theta(r)$. This relation has the following properties.

(A) If $r \in \Gamma$ and $s \in \Gamma'$ are vertices, then there is at most one partial map $\theta : F \to F'$ inducing $r \rightsquigarrow s$.

(B) If $F'' : \Gamma'' \to Q$ is another tree over A, and $r \in \Gamma$, $s \in \Gamma'$ and $t \in \Gamma''$ are vertices, and $r \rightsquigarrow s \rightsquigarrow t$, then $r \rightsquigarrow t$. Indeed if $\theta : F \to F'$ and $\phi : F' \to F''$ are the corresponding partial maps, then there is a partial map $\psi : F \to F''$ with $D_\psi = D_\theta \cap \theta^{-1}(D_\phi)$ and $\psi = \phi \circ \theta|_{D_\psi}$.

(C) If $r \in \Gamma$ and $s \in \Gamma'$ are vertices with $r \rightsquigarrow s$ and $s \rightsquigarrow r$, then the corresponding partial maps are inverse isomorphisms between Γ and Γ'. Namely, the construction of (B) gives a partial map from F to F sending r to r, but by (A) it is the identity map.

A partial map $\theta : F \rightsquigarrow F'$ induces a linear map $\alpha_\theta : K_F \to K_{F'}$ sending x_r to $x_{\theta(r)}$ if $r \in D_\theta$, and to zero if $r \notin D_\theta$. It is easy to see that α_θ is an A-module map. Given a vertex $r \in \Gamma$ and an element $x \in K_F$ we write $c_r(x)$ for the coefficient of x_r in x.

LEMMA. *Any A-module map $\beta : K_F \to K_{F'}$ can be written uniquely as a (possibly infinite) linear combination*

$$\beta = \sum_{\theta : F \rightsquigarrow F'} \lambda_\theta \alpha_\theta$$

with $\lambda_\theta \in K$, such that for each vertex $r \in \Gamma$, there are only finitely many non-zero λ_θ with $r \in D_\theta$. In particular, if $c_s(\beta(x_r)) \neq 0$ then $r \rightsquigarrow s$.

PROOF. The finiteness condition ensures that $\sum_\theta \lambda_\theta \alpha_\theta$ is well-defined, and the uniqueness follows from property (A). Let $\beta : K_F \to K_{F'}$ be an arbitrary homomorphism. Because F and F' are unramified, given $a : r \to t$ in Γ and $u \in \Gamma'$ with $c_u(\beta(x_t)) \neq 0$, there must be an arrow $b : s \to u$ in Γ' with $F(a) = F'(b)$ and $c_s(\beta(x_r)) = c_u(\beta(x_t))$. Dually, given $b : s \to u$ in Γ' and $r \in \Gamma$ with $c_s(\beta(x_r)) \neq 0$, there must be an arrow $a : r \to t$ in Γ with $F(a) = F'(b)$ and $c_u(\beta(x_t)) = c_s(\beta(x_r))$.

It follows that if $r \in \Gamma$ and $s \in \Gamma'$ are vertices with $c_s(\beta(x_r)) \neq 0$, then there is a partial map θ inducing $r \rightsquigarrow s$ and with $c_{\theta(t)}(\beta(x_t)) = c_s(\beta(x_r))$ for all $t \in D_\theta$. The lemma follows. $\square$

Proof of the Theorem. Let G be the group of automorphisms of Γ over Q, so consisting of those automorphisms g of Γ with $F \circ g = F$. The fact that F is unramified implies that G acts freely on the tree Γ, so it is free. We consider each element of G as a partial map $F \to F$. By property (C) and the lemma, $\mathrm{End}(K_F) = S \oplus J$ where

$$S = \{\beta \in \mathrm{End}(K_\Gamma) \mid \beta \text{ is of the form } \sum_{\theta \in G} \lambda_\theta \alpha_\theta\}, \quad \text{and}$$

$$J = \{\beta \in \mathrm{End}(K_\Gamma) \mid \ c_s(\beta(x_r)) = 0 \text{ for all } r, s \in \Gamma \text{ with } s \rightsquigarrow r.\}$$

Evidently S is a subalgebra of $\mathrm{End}(K_F)$ and J is an ideal. Let $\beta \in \mathrm{End}(K_\Gamma)$ be a non-trivial idempotent. Now $\overline{\beta} \in \mathrm{End}(K_\Gamma)/J \cong S \cong KG$, and since G is a free group, its group algebra has no non-trivial idempotents. Thus $\overline{\beta} = 0$ or 1, and replacing β by $1 - \beta$ if necessary, we obtain a non-zero idempotent $\beta \in J$. Choose a vertex $r \in \Gamma$ with $\beta(x_r) \neq 0$, say

$$\beta(x_r) = \lambda_1 x_{r_1} + \ldots + \lambda_n x_{r_n}$$

with $0 \neq \lambda_j \in K$ and vertices $r_j \in \Gamma$ with $r \rightsquigarrow r_j$. Since the relation $\rightsquigarrow$ is transitive, by reordering we may assume that r_1 is minimal, so that if $r_j \rightsquigarrow r_1$, then also $r_1 \rightsquigarrow r_j$. Since $\beta^2 = \beta$, for some j we have

$$c_{r_1}(\beta(x_{r_j})) \neq 0$$

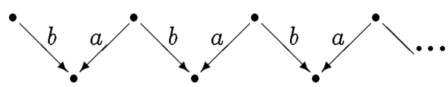
so $r_j \rightsquigarrow r_1$. Thus also $r_1 \rightsquigarrow r_j$, but this contradicts the fact that $\beta \in J$. Thus K_F has no non-trivial idempotent endomorphisms, so is indecomposable.

Now suppose that γ is an isomorphism $K_F \to K_{F'}$. Let $r \in \Gamma$ be a vertex, and write

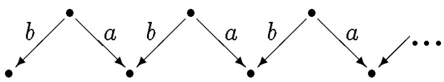
$$\gamma(x_r) = \lambda_1 x_{r_1} + \ldots + \lambda_n x_{r_n}$$

with $0 \neq \lambda_j \in K$ and vertices $r_j \in \Gamma'$ with $r \rightsquigarrow r_j$. Now $x_r = \sum_{j=1}^n \lambda_j \gamma^{-1}(x_{r_j})$ and each term in the sum is a linear combination of x_s with $r_j \rightsquigarrow s$. Thus for some j we have $r \rightsquigarrow r_j \rightsquigarrow r$, so Γ and Γ' are isomorphic over Q. $\square$

Remark. Associated to a tree $F : \Gamma \to Q$ over A there is also a completed tree module $\hat{K}_F$. If K_F is considered as the set of functions from the vertex set of Γ to K with finite support, then $\hat{K}_F$ is the set of functions without restriction on the support. Another construction of it is as follows. Reversing all arrows in Γ and Q, we get a quiver map $F^{op} : \Gamma^{op} \to Q^{op}$ which is a tree over the opposite algebra A^{op}. Then $\hat{K}_F \cong D(K_{F^{op}})$. Note, however, that $\hat{K}_F$ need not be indecomposable. For example if A is the path algebra of the Kronecker quiver, with the arrows labelled a and b, then $\hat{K}_{F_1}$ is decomposable, where F_1 is the tree



whereas $\hat{K}_{F_2}$ is indecomposable, where F_2 is the tree



1.5. Limits. A system of modules and maps $X_1 \to X_2 \to X_3 \to \dots$ has *direct* (or *inductive*) *limit*

$$\varinjlim X_i = (\bigcup X_i)/\sim$$

where $\sim$ is the equivalence relation which identifies $x \in X_i$ with $x' \in X_j$ if they have the same image in some X_k $(k \geq i, j)$. If the maps are 1-1 then the direct limit is the union of the modules. Dually, a system $\to Y_3 \to Y_2 \to Y_1$ has *inverse* (or *projective*) *limit*

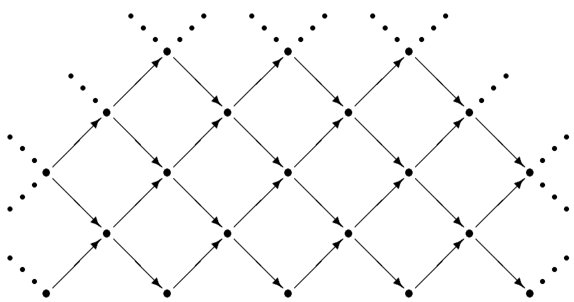
$$\varprojlim Y_i = \{(y_i) \in \prod Y_i \mid y_{i+1} \mapsto y_i \text{ in } Y_i \text{ for all } i\}.$$

More generally one needs the notions of direct and inverse limits for systems of modules and maps indexed by a filtered poset or category, but we shall not define these here. To work with limits, one needs the following formulas

$$\begin{aligned} \operatorname{Hom}(\varinjlim X_i, M) &\cong \varprojlim \operatorname{Hom}(X_i, M) \\ \operatorname{Hom}(M, \varprojlim Y_i) &\cong \varprojlim \operatorname{Hom}(M, Y_i) \\ \operatorname{Hom}(N, \varinjlim X_i) &\cong \varinjlim \operatorname{Hom}(N, X_i) \quad \text{for } N \text{ finite dimensional.} \end{aligned}$$

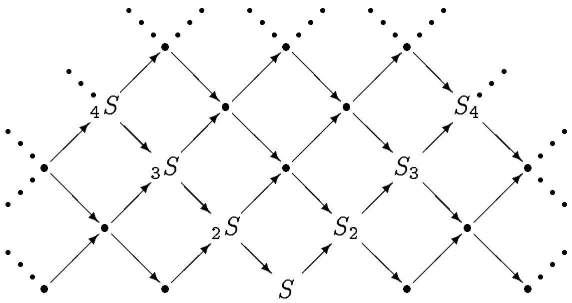
In fact the last of these characterizes the set of finite-dimensional modules. It is also crucial to know that a direct limit of exact sequences is exact, but an inverse limit need not be.

EXAMPLE. Suppose that Δ is a connected component of the Auslander-Reiten quiver which is *quasi-serial*, meaning that it does not contain any projective or injective module, and for any indecomposable module X in Δ the middle term E of the Auslander-Reiten sequence $0 \to D\operatorname{Tr} X \to E \to X \to 0$ is either indecomposable, or can be decomposed into two indecomposable summands E_1, E_2 with $\dim E_1 < \dim X < \dim E_2$ and $\dim E_1 < \dim D\operatorname{Tr} X < \dim E_2$. Any quasi-serial component either has shape $\mathbb{Z}A_\infty$



or it is a *tube* $\mathbb{Z}A_\infty/(\tau^d)$ of *width* $d \geq 1$, obtained by identifying each vertex and arrow with the one d places to the left.

An indecomposable module X in a quasi-serial component is *quasi-simple* if the middle term E in its Auslander-Reiten sequence is indecomposable. These are exactly the modules on the edge of the component. If S is a quasi-simple, there are rays starting and ending at S, and we define the *Prüfer* module S_∞ to be the direct limit of a chain of irreducible maps $S \to S_2 \to S_3 \to \dots$, and the *adic* module $\hat{S}$ to be the inverse limit of a chain of irreducible maps $\dots \to {}_3S \to {}_2S \to S$.



It may seem that these modules depend on the choice of irreducible maps, but this is not the case, for example using [**25**, Corollary 4.2] and its dual.

As an example, for the path algebra of the Kronecker quiver the modules R_λ ($\lambda \in K \cup \{\infty\}$) are quasi-simple, all in tubes of width 1. Now the Prüfer module $(R_0)_\infty$ is the tree module for the tree F_1 of the last section, while the adic module $\hat{R}_0$ is the completed tree module for the tree F_2.

THEOREM (Krause). *If S is quasi-simple then S_∞ is indecomposable.*

PROOF. Krause's proof [**20**] uses a functorial argument due to Auslander. Here we reformulate it naively. Identify S and each S_i as a submodule of S_∞, and write i_n for the inclusion $S \to S_n$. Let us say that a map $\theta : S \to M$ *extends indefinitely* if for all n, it can be extended to a map $S_n \to M$ (that is, it factors through i_n). We need two properties pertaining to this notion:

(a) If T is a non-zero submodule of S_n and $p : S_n \to S_n/T$ is the natural projection, then $pi_n : S \to S_n/T$ extends indefinitely. By induction it suffices to prove that there is a map $g : S_{n+1} \to S_n/T$ with $gi_{n+1} = pi_n$ (for then g is surjective, so S_n/T can also be identified as a quotient S_{n+1}/T', etc). Now the Auslander-Reiten sequence starting at S_n has the form

$$0 \to S_n \xrightarrow{\binom{j}{q}} S_{n+1} \oplus (S_n/S) \to \operatorname{Tr} DS_n \to 0$$

where j is the inclusion and q the natural projection. Since p is not a split monomorphism it factors as $p = gj + hq$ where $g : S_{n+1} \to S_n/T$ and $h : S_n/S \to S_n/T$. Then $qi_n = 0$, so $pi_n = gji_n = gi_{n+1}$, as required.

(b) The inclusion $i_n : S \to S_n$ doesn't extend indefinitely. Namely, suppose that $k > n$ and there is a map $\phi : S_k \to S_n$ with $i_n = \phi i_k$. Then $i_n = \psi i_n$ where ψ is the composite of ϕ with the inclusion $S_n \to S_k$. Now ϕ is not a split epimorphism, so ψ is not invertible, so it must be nilpotent. This is impossible since $i_n = \psi i_n = \psi^2 i_n = \dots$.

Now suppose that S_∞ is decomposable, say as $U \oplus V$, and write π_U and π_V for the corresponding projections onto U and V. Choose n sufficiently large to ensure that S_n meets both U and V non-trivially, and then choose k so as to ensure that $\pi_U(S_n), \pi_V(S_n) \subseteq S_k$. Now $i_k : S \to S_k$ factors as

$$S \xrightarrow{s \mapsto (\overline{s}, \overline{s})} S_n/(S_n \cap U) \oplus S_n/(S_n \cap V) \xrightarrow{(\overline{s}, \overline{t}) \mapsto \pi_V(s) + \pi_U(t)} S_k.$$

By (a) the left hand map extends indefinitely, but by (b) the composite does not. This is impossible. □

2. Purity and model theory of modules

In this section we describe the ingredients in a general theory of infinite-dimensional modules, culminating in the notion of the Ziegler spectrum. Actually there are two different ways to study such modules.

On the one hand, one can study modules using first-order logic. Indeed an A-module is nothing more than a 'model' of a theory in a suitable 'language of A-modules'. With this approach, the study of modules is a part of model theory.

On the other hand, one can study modules using algebra. In particular the 'functor category' $(A\text{-mod}, \mathrm{Ab})$ of all additive functors from A-mod to the category of abelian groups, turns out to be an abelian category, so one can apply the concepts of homological algebra.

For example the Σ-algebraically compact modules (mentioned in Section 2.4 below), can be studied in model theory with the notion of a 'totally transcendental theory', or in algebra with the notion of a Σ-injective object of the functor category. With my own algebraic background, while quite ready to use results proved with model-theoretic techniques, I am always happy to see a proof using functor categories.

2.1. Coherent functors. An additive functor $F : A\text{-Mod} \to \mathrm{Ab}$ is said to be *coherent* if it commutes with direct limits and products. The coherent functors form an abelian category $C(A)$, whose morphisms are the natural transformations of functors.

Examples. The following are coherent functors:
1. The *n-fold forgetful functor* $\mathrm{Forget}^n : M \mapsto M \oplus \ldots \oplus M$ (n copies).
2. The representable functor $\mathrm{Hom}(X, -)$ for X finite dimensional.
3. The tensor product functor $N \otimes_A -$ for N a finite-dimensional right module.
4. More generally, $\mathrm{Ext}^i(X, -)$ and $\mathrm{Tor}_i(N, -)$ for X and N finite dimensional.

The definition we have given of a coherent functor differs from the usual one: normally one only considers functors on the category of finite-dimensional modules, and then the coherent ones are those which are cokernels of a morphism between two representable functors. The next result, however, shows that the two concepts are equivalent.

Lemma 1. *If $\theta : X \to Y$ is a map between finite-dimensional modules, then the functor $M \mapsto \mathrm{Coker}(\mathrm{Hom}(Y, M) \to \mathrm{Hom}(X, M))$ is coherent. Moreover, any coherent functor is isomorphic to one of these.*

PROOF. It is straightforward that this construction defines a coherent functor, as one sees by using the fact that a direct limit or a product of exact sequences is exact. Now suppose that F is a coherent functor. First, observe that it suffices to find a finite-dimensional module X and an element $\xi \in F(X)$ such that the map $\mathrm{Hom}(X,M) \to F(M)$, $h \mapsto F(h)(\xi)$ is surjective for all modules M. Namely, in this case the functor $G(M) = \mathrm{Ker}(\mathrm{Hom}(X,M) \to F(M))$ is also coherent, so by the same argument there is Y and an element $\theta \in G(Y)$ such that the map $\mathrm{Hom}(Y,M) \to G(M)$ is surjective. Now θ is actually a map $X \to Y$, and $F(M) \cong \mathrm{Coker}(\mathrm{Hom}(Y,M) \to \mathrm{Hom}(X,M))$. Second, observe that one only needs to check that $\mathrm{Hom}(X,M) \to F(M)$ is surjective for M finite dimensional, for an arbitrary module is the direct limit of its finite-dimensional submodules M_μ, and if each of the maps $\mathrm{Hom}(X,M_\mu) \to F(M_\mu)$ is surjective, then so is the direct limit $\mathrm{Hom}(X,M) \to F(M)$.

We now consider the product $\prod_{\lambda,\phi} F(M_\lambda)$, in which M_λ runs through all isomorphism classes of finite-dimensional A-modules, and ϕ runs through all elements of each $F(M_\lambda)$. This product contains a canonical element c, whose λ,ϕ component is ϕ itself. Now we have

$$c \in \prod_{\lambda,\phi} F(M_\lambda) \cong F(\prod_{\lambda,\phi} M_\lambda) \cong \varinjlim F(X_\nu)$$

where the direct limit is over the finite-dimensional submodules X_ν of $\prod_{\lambda,\phi} M_\lambda$. Thus $c = F(\psi)(\xi)$ for some module X_ν and some $\xi \in F(X_\nu)$, where ψ is the inclusion $\psi : X_\nu \hookrightarrow \prod_{\lambda,\phi} M_\lambda$. Clearly ξ is a suitable element. $\square$

LEMMA 2. *If $a = (a_{ij})$ is a $p \times q$ matrix of elements of A, and $n \le q$, then*

$$F_a(M) = \{(z_1,\dots,z_n) \in M^n \mid \exists z_{n+1},\dots,z_q \in M \text{ with } \sum_{j=1}^{q} a_{ij} z_j = 0 \text{ for all } i\}$$

defines a coherent subfunctor F_a of Forget^n*, and any coherent subfunctor of* Forget^n *arises in this way.*

PROOF. It is easy to see that this defines a coherent subfunctor. Conversely, by the previous result any coherent subfunctor of Forget^n is the image of a morphism $\mathrm{Hom}(X,-) \to \mathrm{Forget}^n$, where X is some finite-dimensional module. Identifying Forget^n with $\mathrm{Hom}(A^n,-)$, Yoneda's lemma shows that this morphism is induced by a homomorphism $A^n \to X$. Now extending this to a surjection $A^q \to X$ for some $q \ge n$, there is a projective resolution $A^p \to A^q \to X \to 0$. Now the map $A^p \to A^q$ is given by a $q \times p$ matrix of elements of A, and its transpose induces the original subfunctor. $\square$

The previous result provides a connection between coherent functors and the 'language of A-modules', which is used to study modules using the methods of model theory. Given elements $a_{ij} \in A$, the string

$$(\exists z_{n+1})(\exists z_{n+2})\dots(\exists z_q)(a_{11}z_1 + \dots + a_{1q}z_q = 0 \wedge \dots \wedge a_{p1}z_1 + \dots + a_{pq}z_q = 0)$$

is an example of a *formula* in the language of A-modules. Its features are:

□ It has 'free' variables $z_1,\dots,z_n$.

□ It has 'quantified' variables $z_{n+1}, \dots, z_q$.
□ Elements of the module (other than 0) only enter as variables.
□ Elements of A only enter as constants.

This is a *positive primitive* formula; in a general formula there can be other logical symbols, for example $\neg$, $\vee$ and $\forall$, but still elements of the module only enter as variables and elements of A only enter as constants. If a formula has n free variables, then it has a solution set in M^n. Clearly in this way the positive primitive formulas correspond to the coherent subfunctors of Forget^n.

As is usual in model theory, one would like to prove 'elimination of quantifiers', meaning that if M is a module and φ is a formula with n free variables, then the solution set of φ in M^n can also be defined by a formula without quantified variables. Unfortunately this is not true; the best that can be done is given by the following theorem. There are proofs in [**17**], [**24**] and [**32**].

THEOREM (Baur, Monk). *Given a module M and a formula φ, there is a boolean combination ψ of positive primitive formulas such that φ and ψ have the same solution sets.*

A formula without free variables is called a *sentence*, so that in any module, a sentence is either true or false. For example, if $A = K[\epsilon]/(\epsilon^2)$ is the ring of dual numbers, then a module is semisimple if the sentence $(\forall x)(\epsilon x = 0)$ is true, and it is free if the sentence $(\forall x)(\exists y)((x = \epsilon y) \vee \neg(\epsilon x = 0))$ is true.

One says that two modules are *elementarily equivalent* if they satisfy exactly the same sentences. In fact in the theorem above, the formula ψ only depends on φ and the sizes of the sets $F(M)$ with F coherent. Moreover, because of our standing assumption that A is an algebra over an algebraically closed (hence infinite) field, it turns out to only depend on whether or not $F(M) = 0$. Thus there is the following consequence.

COROLLARY. *Two modules M and M' are elementarily equivalent if and only if $F(M) = 0 \Leftrightarrow F(M') = 0$ for all coherent functors F.*

2.2. Purity. When dealing with infinite dimensional modules, it is possible to have an exact sequence $0 \to L \to M \to N \to 0$ which is not split, but which is very close to being split. One says that it is a *pure-exact sequence*, and that the image of L in M is a *pure submodule*, if the following equivalent conditions hold:

(a) $0 \to F(L) \to F(M) \to F(N) \to 0$ is exact for all coherent functors F.
(b) $0 \to Y \otimes L \to Y \otimes M \to Y \otimes N \to 0$ is exact for any finite-dimensional right module Y (and hence also for Y infinite dimensional).
(c) $0 \to \text{Hom}(N, X) \to \text{Hom}(M, X) \to \text{Hom}(L, X) \to 0$ is exact for any finite-dimensional module X. That is, any map from L to a finite-dimensional module factors through the map $L \to M$.
(d) $0 \to \text{Hom}(X, L) \to \text{Hom}(X, M) \to \text{Hom}(X, N) \to 0$ is exact for any finite-dimensional module X. In other words, any map from a finite-dimensional module to N factors through the map $M \to N$.

Here condition (b) is the one most commonly seen in references as the definition of a pure submodule. Conditions (c) and (d) show that if any module in a pure-exact sequence is finite-dimensional, then the sequence must actually be split exact.

Very briefly, to see the equivalence, we have (a)⇒(b) on taking F to be the tensor product functor; (b)⇒(c) on dualizing the tensor product sequence for $Y = DX$ and using the isomorphism $D(DX \otimes M) \cong \mathrm{Hom}(M, X)$; (c)⇒(d) by Auslander-Reiten theory, for there is an exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}(N, D\,\mathrm{Tr}\,X) \to \mathrm{Hom}(M, D\,\mathrm{Tr}\,X) \to \mathrm{Hom}(L, D\,\mathrm{Tr}\,X) \to \\ \to D\,\mathrm{Hom}(N, X) \to D\,\mathrm{Hom}(M, X) \to D\,\mathrm{Hom}(L, X) \to 0; \end{aligned}$$

(d)⇒(a) using a resolution $0 \to \mathrm{Hom}(Z, -) \to \mathrm{Hom}(Y, -) \to \mathrm{Hom}(X, -) \to F \to 0$ of F (where Z is the cokernel of the map $X \to Y$), applying each term to the exact sequence, and then using the snake lemma. Many other equivalent conditions for a pure-exact sequence can be found, for example, in [**17**, Theorem 6.4] and [**26**, §1F].

Examples.

1. If $L_1 \subseteq L_2 \subseteq \dots$ are direct summands of M, then $L = \bigcup L_i$ need not be a direct summand of M, but it is a pure submodule. Namely, if F is a coherent functor, then each sequence $0 \to F(L_i) \to F(M) \to F(M/L_i) \to 0$ is exact, hence so is their direct limit $0 \to F(L) \to F(M) \to F(M/L) \to 0$.
2. A direct sum of modules $\oplus_i X_i$ is a pure submodule of the product $\prod_i X_i$, for any finite direct sum is a summand of the product.
3. Every module can be embedded as a pure submodule in a product of finite-dimensional modules. If $\theta_\lambda : M \to N_\lambda$ $(\lambda \in \Lambda)$ is a complete list of all maps from M to a finite-dimensional module (up to isomorphism), then clearly any map from M to a finite-dimensional module factors through the map $M \to \prod_\lambda N_\lambda$, so this embeds M as a pure submodule of $\prod_\lambda N_\lambda$.

2.3. Definable subcategories. Let $\mathcal{C}$ be a full subcategory of A-Mod. We say that $\mathcal{C}$ is *definable* if the following equivalent conditions hold:

(i) $\mathcal{C}$ is closed under products, direct limits and pure submodules
(ii) $\mathcal{C}$ is closed under elementary equivalence and direct summands.
(iii) $\mathcal{C}$ is defined by the vanishing of some set of coherent functors.

It follows that $\mathcal{C}$ is also closed under direct sums.

Proof of equivalence. (i)⇒(ii) A theorem of Frayne [**4**, Corollary 4.3.13] implies that if M and M' are elementarily equivalent, then M' is elementarily embedded in (hence a pure submodule of) some ultrapower of M. Now if $\mathcal{F}$ is an ultrafilter on a set I then the ultrapower $\prod_I M/\mathcal{F}$ is the direct limit over all $J \in \mathcal{F}$ of the powers M^J. Thus if $M \in \mathcal{C}$ then so is M'. (ii)⇒(iii) is the corollary in Section 2.1. (iii)⇒(i) follows from properties of coherent functors. □

Remark. There are three other concepts equivalent to definable subcategories.

I. Complete theories of modules. These are maximal consistent sets of sentences. The corresponding definable subcategory consists of all direct summands of models of the theory. (In general this subcategory need not be closed under products, but in our case it is, since it follows from the fact that A contains an infinite field.)

II. Serre subcategories in $C(A)$. A Serre subcategory $\mathcal{S}$ is a full subcategory with the property that $F' \in \mathcal{S} \Leftrightarrow F, F'' \in \mathcal{S}$ for any exact sequence $0 \to F \to F' \to F'' \to 0$ of coherent functors. The corresponding definable subcategory consists of the modules on which all $F \in \mathcal{S}$ vanish. This correspondence is due to I. Herzog [**16**].

III. Closed subsets of the Ziegler spectrum. We discuss this in Section 2.5.

EXAMPLES. The following are definable subcategories:

1. The *perpendicular category* $X^{\perp} = \{M \mid \mathrm{Hom}(X, M) = \mathrm{Ext}^1(X, M) = 0\}$ for X a finite-dimensional module. This is because $\mathrm{Hom}(X, -)$ and $\mathrm{Ext}^1(X, -)$ are coherent.
2. $\{M \mid \mathrm{Hom}(M, X) = 0\}$ is definable for X finite dimensional, since there is an isomorphism $\mathrm{Hom}(M, X) \cong D(DX \otimes M)$, and the functor $DX \otimes -$ is coherent.
3. $\{M \mid \mathrm{proj.\,dim}\, M \leq n\}$ is definable for $n \geq 0$, for $\mathrm{proj.\,dim}\, M \leq n$ if and only if $\mathrm{Tor}^{n+1}(X, M) = 0$ for all finite-dimensional modules X, and the functors $\mathrm{Tor}^{n+1}(X, -)$ are coherent.

REMARK. Any collection $\mathcal{A}$ of modules can be closed up to form a definable subcategory, which we denote by $\overline{\mathcal{A}}$. In the case when $\mathcal{A}$ consists of only one module M, we write $\overline{M}$. In fact every definable category occurs as some $\overline{M}$, for if $\mathcal{C}$ is definable, and F_λ runs through the isomorphism classes of coherent functors not vanishing on $\mathcal{C}$, choose $M_\lambda \in \mathcal{C}$ with $F_\lambda(M_\lambda) \neq 0$. Then $\mathcal{C} = \overline{\prod_\lambda M_\lambda}$.

Given any collection $\mathcal{A}$ of finite-dimensional indecomposable modules, it should be considered a standard problem to describe $\overline{\mathcal{A}}$. For example this is of interest if $\mathcal{A}$ is an Auslander-Reiten component, or part of a component. (It may be that Ringel's sewing procedure for Auslander-Reiten components of special biserial algebras is given by sewing two components when their closures intersect, see [**28**].)

PROPOSITION. *If $\mathcal{A}$ is a collection of finite-dimensional indecomposables, then*

1. *$\overline{\mathcal{A}}$ contains no other finite-dimensional indecomposables, and*
2. *if $\mathcal{A}$ is infinite then $\overline{\mathcal{A}}$ must contain infinite-dimensional indecomposables.*

PROOF. The first assertion follows from Auslander-Reiten theory. If X is a finite-dimensional indecomposable module, then the simple functor S defined by

$$S(M) = \mathrm{Hom}(X, M)/\{\theta : X \to M \text{ not a split monomorphism}\}$$

is coherent. Now S vanishes on $\mathcal{A}$, so it vanishes on $\overline{\mathcal{A}}$. The second assertion uses the compactness of the Ziegler spectrum, discussed later. $\square$

2.4. Algebraically compact modules. If M is a module and F is a coherent subfunctor of Forget^1, then $F(M)$ is a subgroup of M (even an $\mathrm{End}(M)$-submodule of M). The subgroups which arise in this way (for some F) are called *subgroups of finite definition* of M. They form a lattice, which is denoted $\mathrm{Latt}(M)$.

A module M is said to be *algebraically compact* provided that any covering of M by complements of cosets of subgroups of finite definition has a finite subcovering. We relate this definition to more standard ones with the following lemma.

LEMMA. *M is algebraically compact if and only if it has the following property: the natural map*

$$M \to \varprojlim M/M_\lambda, \quad x \mapsto (M_\lambda + x)$$

is surjective for any family M_λ ($\lambda \in \Lambda$) of subgroups of finite definition of M, which is filtered in the sense that for any $\lambda, \mu \in \Lambda$ there is $\nu \in \Lambda$ with $M_\nu \subseteq M_\lambda \cap M_\mu$.

PROOF. Suppose that M has the given property for filtered families, and has a covering by complements of cosets C_i of subgroups M_i of finite definition. If this covering has no finite subcovering, then for any finite subset $F \subseteq I$ the intersection $C_F = \bigcap_{i \in F} C_i$ is non-empty, and so it is a coset of $M_F = \bigcap_{i \in F} M_i$, which is a subgroup of finite definition. As F varies, the M_F form a filtered family, and the C_F define an element of the inverse limit, so there is $x \in M$ such that each $C_F = M_F + x$. But this is impossible since M is covered by the complements of the C_i.

Now suppose that M is algebraically compact and M_λ is a filtered family. An element of the inverse limit is given by cosets C_λ with $C_\mu \subseteq C_\lambda$ whenever $M_\mu \subseteq M_\lambda$. If this element is not in the image of M, then

$$M = \bigcup_\lambda (M \setminus C_\lambda) = (M \setminus C_{\lambda_1}) \cup \cdots \cup (M \setminus C_{\lambda_r})$$

using the finite subcovering property. Now there is $M_\mu \subseteq \bigcap M_{\lambda_i}$, and so $C_\mu \subseteq C_{\lambda_i}$ for each i, which is impossible. $\square$

Using [**17**, Corollary 7.4], the lemma shows that our definition agrees with the usual definition of an algebraically compact module, and then there are many equivalent properties. The most important one being that M is algebraically compact if and only if it is *pure-injective*, which means that any pure-exact sequence $0 \to M \to E \to N \to 0$ splits. Using the example at the end of Section 2.2 it is equivalent that M is a direct summand of a product of finite-dimensional modules.

REMARK. There is a whole hierarchy of finiteness conditions on $\mathrm{Latt}(M)$ which are reflected in properties of the module M and the definable subcategory $\overline{M}$. Starting from the strongest, we have:

(1) M is endofinite if and only if $\mathrm{Latt}(M)$ has the ascending and descending chain conditions. To see this one uses that every $\mathrm{End}(M)$-submodule is a sum of intersections of subgroups of finite definition, see [**8**, Proposition 4.1].

(2) M is Σ-*algebraically compact* (meaning that $M^{(I)}$ is algebraically compact for all sets I) if and only if $\mathrm{Latt}(M)$ has the descending chain condition. There are many equivalent conditions for this, for example that every power M^I is a direct sum of indecomposables with local endomorphism ring. See [**17**, Theorem 8.1].

(3) M has *elementary Krull dimension* if the Krull dimension of $\mathrm{Latt}(M)$ exists as an ordinal number. It is equivalent that $\mathrm{Latt}(M)$ contains no subset isomorphic to $(\mathbb{Q}, \leq)$. This notion was introduced by Garavaglia [**12**] and later studied by

Ziegler [**32**]. Various properties are known, for example if M is a non-zero module with elementary Krull dimension, then $\overline{M}$ contains an indecomposable endofinite module. Recently Krause [**21**] has given a proof of this using functor categories.

(4) In [**32**], Ziegler introduced the notion of the *width* of a lattice, and showed that if M is a non-zero algebraically compact module and $\mathrm{Latt}(M)$ has bounded width, then M must have an indecomposable direct summand.

EXAMPLES. The following are algebraically compact modules.

1. Any finite-dimensional or endofinite module is algebraically compact.
2. The dual of an arbitrary module is algebraically compact.
3. If $\ldots \to X_2 \to X_1 \to X_0$ is a system of finite-dimensional modules then $\varprojlim X_i \cong D(\varinjlim DX_i)$ so it is algebraically compact. In particular, $\hat{S}$ is algebraically compact for any quasi-simple module S.
4. If M has an endomorphism ϕ with finite-dimensional kernel which is locally nilpotent, so $M = \bigcup \mathrm{Ker}(\phi^n)$, then M is Σ-algebraically compact. Indeed M is an A-$K[T]$-bimodule with T acting as ϕ, and as a $K[T]$-module it is torsion with finite-dimensional socle. Thus it is an artinian $K[T]$-module, so as an A-module it must have the descending chain condition on subgroups of finite definition.
5. (Krause [**21**]) If S is a quasi-simple module in a tube, then S_∞ is algebraically compact, for if the tube has width d then for all n there is an exact sequence $0 \to S_d \to S_{nd} \to S_{(n-1)d} \to 0$. Moreover, as n varies, these exact sequences are compatible, and taking the direct limit gives rise to an exact sequence $0 \to S_d \to S_\infty \to S_\infty \to 0$. Now the endomorphism of S_∞ is locally nilpotent with finite-dimensional kernel, so the previous example applies.

2.5. The Ziegler spectrum. The *Ziegler spectrum* $\mathrm{Zg}\, A$ is defined to be the set of isomorphism classes of indecomposable algebraically compact modules. Any indecomposable algebraically compact module has cardinality at most $2^{\mathrm{card}\, A}$, so there are no set-theoretic problems. The Ziegler spectrum is useful because every module is elementarily equivalent to a direct sum of indecomposable algebraically compact modules [**32**, Corollary 6.9], and it follows that any definable subcategory is uniquely determined by the indecomposable algebraically compact modules that it contains.

THEOREM (Ziegler). *The sets* $(F) = \{M \in \mathrm{Zg}\, A \mid F(M) \neq 0\}$ *with* F *a coherent functor, form a base of open sets for a topology on* $\mathrm{Zg}\, A$. *With this topology,* $\mathrm{Zg}\, A$ *is compact. More generally, all of the sets* (F) *are compact.*

It follows immediately that a subset of the Ziegler spectrum is closed if and only if it is the set of indecomposable algebraically compact modules in some definable subcategory. Ziegler's proof of the theorem [**32**] uses model-theoretic language. A more algebraic proof using functor categories has been given by Herzog [**16**]. See also Krause [**22**].

REMARK. By Auslander-Reiten theory the open points of $\mathrm{Zg}\, A$ are the finite-dimensional indecomposable modules. Now compactness implies that the closure of an infinite set of finite-dimensional indecomposables must contain an infinite-dimensional indecomposable.

REMARK. The closed points of $\operatorname{Zg} A$ are not fully understood. However, the result below implies that if M is indecomposable endofinite then $\{M\}$ is closed in $\operatorname{Zg} A$. On the other hand, if $\{M\}$ is a closed subset of $\operatorname{Zg} A$ and M has elementary Krull dimension, then the result mentioned in Section 2.4 implies that M is endofinite.

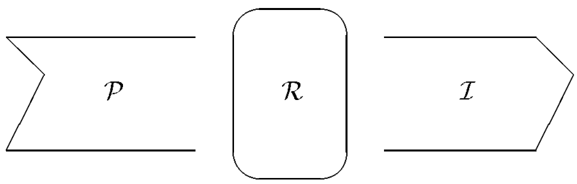
PROPOSITION (Garavaglia). *An indecomposable module M is endofinite if and only $\overline{M}$ consists of the direct sums of copies of M.*

PROOF. If M is endofinite, of endolength n, then any chain of subgroups of finite definition of M has length at most n. This carries over to any module N in $\overline{M}$, so N has endolength at most n, so is a direct sum of indecomposable endofinite modules. Now the module $M \oplus N$ also has endolength at most n, so all indecomposable direct summands of N must be isomorphic to M, see [**8**, §4.5].

If $\overline{M}$ consists of the direct sums of copies of M then M is Σ-algebraically compact, since one of the characterizations of such modules is that every product of copies of M is a direct sum of modules of bounded cardinality, see [**17**, Theorem 8.1]. Thus $\operatorname{Latt}(M)$ has the descending chain condition, so M must have elementary Krull dimension. Now by the property mentioned in Section 2.4, $\overline{M}$ must contain an indecomposable endofinite module, but the only possible such module is M. $\square$

3. Tame and/or hereditary algebras

3.1. Tame hereditary algebras. Suppose that A is a finite-dimensional (connected) tame hereditary algebra. Assuming that A is basic, it must be the path algebra of an extended Dynkin quiver, for example the Kronecker quiver. Recall that the Auslander-Reiten quiver of A decomposes into three parts



where $\mathcal{P}$ is the set of indecomposable preprojective modules, $\mathcal{I}$ is the set of indecomposable preinjective modules, and $\mathcal{R}$ is the set of indecomposable regular modules. Moreover $\mathcal{R}$ consists of a family of tubes indexed by $K \cup \{\infty\}$. (For more details, see [**27**].)

A module M is said to be *torsion-free* if $\operatorname{Hom}(\mathcal{R}, M) = 0$, or equivalently if $\operatorname{Hom}(S, M) = 0$ for all quasi-simple modules S. It is said to be *divisible* if $\operatorname{Ext}^1(\mathcal{R}, M) = 0$, or equivalently if $\operatorname{Ext}^1(S, M) = 0$ for all quasi-simple modules S. These notions agree with the definitions given in Section 1.1 for Kronecker modules.

THEOREM. *The indecomposable algebraically compact A-modules are:*

1. *The finite-dimensional indecomposable modules.*
2. *The modules $\hat{S}$ and S_∞ for each quasi-simple module S.*
3. *A unique indecomposable torsion-free divisible module. It is endofinite.*

Infinite-dimensional modules for tame hereditary algebras were first studied by Ringel [**26**], generalizing earlier work of Aronszajn and Fixman [**1**] on Kronecker modules. Using this, a classification of the indecomposable algebraically compact modules was given by Okoh [**23**], but it didn't fully pin down the adic modules. Finally, the full classification was obtained by Prest [**24**]. (It was also implicit in work of Geigle [**13**].)

Here we sketch a proof of the theorem, freely using the properties of finite-dimensional modules for a tame hereditary algebra. For example, the layout of the picture above indicates that there are no non-zero homomorphisms from $\mathcal{R}$ to $\mathcal{P}$ or from $\mathcal{I}$ to $\mathcal{P}$ or $\mathcal{R}$. We also need the fact that for any tube T, any map from a module in $\mathcal{P}$ to one in $\mathcal{I}$ factors through a module whose indecomposable direct summands lie in T. In addition we need to use the fact that the category of finite dimensional regular modules is an abelian category, closed under images, kernels and cokernels, in which every object is uniserial, and with the simple objects being the quasi-simple modules. In particular there are no non-zero maps between modules in different tubes.

We need the Auslander-Reiten translations, which for a hereditary algebra can be defined by $\tau M = \mathrm{Tor}_1(DA, M)$ and $\tau^- M = \mathrm{Ext}^1(DA, M)$. They are adjoint functors, so that $\mathrm{Hom}(M, \tau N) \cong \mathrm{Hom}(\tau^- M, N)$. In addition there are the Auslander-Reiten formulas

$$D\,\mathrm{Ext}^1(X, M) \cong \mathrm{Hom}(M, \tau X) \quad \text{and} \quad \mathrm{Ext}^1(M, X) \cong D\,\mathrm{Hom}(\tau^- X, M)$$

for X finite dimensional. Finally, it is not hard to see that even for infinite-dimensional modules there are isomorphisms $M \cong \tau^- \tau M$ if $\mathrm{Hom}(M, A) = 0$, and $M \cong \tau\tau^- M$ if $\mathrm{Hom}(DA, M) = 0$.

Lemma 1. *If M is an A-module, then*
□ $\mathrm{Hom}(M, \mathcal{P}) = 0$ *if and only if M has no direct summand in $\mathcal{P}$.*
□ $\mathrm{Hom}(\mathcal{I}, M) = 0$ *if and only if M has no direct summand in $\mathcal{I}$.*

Proof. We prove the first of these two statements. Clearly, if $\mathrm{Hom}(M, \mathcal{P}) = 0$ then M has no summand in $\mathcal{P}$. For the converse, we prove by induction that if n is the smallest integer with $\mathrm{Hom}(M, \tau^{-n} P) \neq 0$ for some projective P, then M has a direct summand in $\mathcal{P}$. If $n = 0$ then there is a non-zero map from M to a projective. Since A is hereditary, the image of this map is projective, and then the map onto this projective must split, so M has a projective direct summand. If $n > 0$ then

$$\begin{aligned} \mathrm{Hom}(M, \tau^{-n} P) &\cong \mathrm{Hom}(\tau^- \tau M, \tau^{-n} P) && \text{since } \mathrm{Hom}(M, A) = 0 \\ &\cong \mathrm{Hom}(\tau M, \tau\tau^{-n} P) && \text{since } \tau^-, \tau \text{ are adjoint} \\ &\cong \mathrm{Hom}(\tau M, \tau^{-(n-1)} P). \end{aligned}$$

Thus by induction τM has a direct summand in $\mathcal{P}$, and hence so does M. □

LEMMA 2. $\overline{\mathcal{R} \cup \mathcal{I}} = \{M \mid \mathrm{Hom}(M, \mathcal{P}) = 0\}$.

PROOF. The right hand side is definable and contains $\mathcal{R} \cup \mathcal{I}$, so it also contains $\overline{\mathcal{R} \cup \mathcal{I}}$. For the reverse inclusion, suppose that $\mathrm{Hom}(M, \mathcal{P}) = 0$. Now M embeds purely in a product N of finite-dimensional modules, and since each of these is a finite product of indecomposables, N is a product of finite-dimensional indecomposables. Collecting terms we have $N \cong N_1 \oplus N_2$ where N_1 is a product of modules in $\mathcal{P}$ and N_2 is a product of modules in $\mathcal{R} \cup \mathcal{I}$. Now $\mathrm{Hom}(M, N_1) = 0$, so M must embed purely in N_2. Thus $M \in \overline{\mathcal{R} \cup \mathcal{I}}$. $\square$

This lemma is used in the proof of the next, which is the key result in Prest's proof.

LEMMA 3. $\overline{\mathcal{R}} = \{N \mid \mathrm{Hom}(N, \mathcal{P}) = \mathrm{Hom}(\mathcal{I}, N) = 0\}$.

PROOF. The right hand side is definable and contains $\mathcal{R}$, so it also contains $\overline{\mathcal{R}}$. For the reverse inclusion, suppose that N belongs to the right hand side, but is not in $\overline{\mathcal{R}}$. Thus there is a coherent functor F with $F(R) = 0$ for all $R \in \mathcal{R}$ but with $F(N) \neq 0$. Now

$$F(-) \cong \mathrm{Coker}(\mathrm{Hom}(Y, -) \xrightarrow{\mathrm{Hom}(\theta,-)} \mathrm{Hom}(X, -))$$

for some map $\theta : X \to Y$ of finite-dimensional modules. Write $X = X_1 \oplus X_2$ with the indecomposable direct summands of X_1 in $\mathcal{P} \cup \mathcal{R}$ and those of X_2 in $\mathcal{I}$, and let θ_i be the restriction of θ to X_i. Defining

$$F'(-) = \mathrm{Coker}(\mathrm{Hom}(Y, -) \xrightarrow{\mathrm{Hom}(\theta_1,-)} \mathrm{Hom}(X_1, -)),$$

one sees immediately that if M is a module with $\mathrm{Hom}(X_2, M) = 0$, then $F(M) = 0$ if and only if $F'(M) = 0$. It follows that $F'(R) = 0$ for all $R \in \mathcal{R}$ but $F'(N) \neq 0$.

Now if I is preinjective then $F'(I) = 0$. Namely, since any map from a preprojective to a preinjective factors through a regular module, any map $\phi : X_1 \to I$ factors through a regular module, say as $X_1 \to R \to I$. Now $F'(R) = 0$, which means that $X_1 \to R$ factors through θ_1. But this implies that ϕ factors through θ_1, so $F'(I) = 0$.

Thus $\{M \mid F'(M) = 0\}$ contains $\mathcal{R} \cup \mathcal{I}$, and since it is definable, it contains $\overline{\mathcal{R} \cup \mathcal{I}}$, and hence by the previous lemma it contains N. But we have already seen that $F'(N) \neq 0$. $\square$

LEMMA 4. *If S is a quasi-simple module then S_∞ is divisible and $\hat{S}$ is torsion-free. If T is another quasi-simple module then* $\mathrm{Hom}(T, S_\infty)$ *is 1-dimensional if $T \cong S$, and zero otherwise, while* $\mathrm{Ext}^1(T, \hat{S})$ *is 1-dimensional if $T \cong \tau^- S$, and zero otherwise.*

PROOF. We use the fact that the category of finite-dimensional regular modules is an abelian category, in which every indecomposable object is uniserial, and in which the quasi-simple modules are the simple objects. Now if T is quasi-simple then

$$\mathrm{Hom}(S_\infty, T) \cong \varprojlim \mathrm{Hom}(S_n, T) \quad \text{and} \quad \mathrm{Hom}(T, \hat{S}) \cong \varprojlim \mathrm{Hom}(T, {}_nS)$$

and the uniserial structure implies that the maps in these inverse limit systems are zero, so the inverse limits are themselves zero. Also

$$\operatorname{Hom}(T, S_\infty) \cong \varinjlim \operatorname{Hom}(T, S_n)$$

from which the assertion for $\operatorname{Hom}(T, S_\infty)$ follows.

Finally, the isomorphism $\operatorname{Hom}(X, DY) \cong \operatorname{Hom}(Y, DX)$ induces an isomorphism on Ext^1, and so

$$\operatorname{Ext}^1(T, \hat{S}) \cong \operatorname{Ext}^1((DS)_\infty, DT),$$

where DT and DS are quasi-simple modules for the opposite algebra of A, which is tame hereditary again, and we have used the isomorphism $\hat{S} \cong D((DS)_\infty)$. Now

$$\operatorname{Ext}^1((DS)_\infty, DT) \cong D\operatorname{Hom}(\tau^- DT, (DS)_\infty)$$

by the Auslander-Reiten formula, and this last space is 1-dimensional or zero, according to whether or not $\tau^- DT \cong DS$, or equivalently whether $T \cong \tau^- S$. □

In these notes we have only discussed algebraically compact modules for finite-dimensional algebras, but there are no problems defining such modules for any ring, and there are short proofs of the next result in [**32**] and [**17**].

LEMMA 5. *The indecomposable algebraically compact modules for a Dedekind domain Λ are*

1. *The finite length indecomposables $\Lambda/\mathfrak{m}^n$, with $\mathfrak{m}$ a maximal ideal in Λ.*
2. *The $\mathfrak{m}$-adic modules $\hat{\Lambda}_\mathfrak{m} = \varprojlim \Lambda/\mathfrak{m}^n$ and Prüfer modules $\Lambda_{\mathfrak{m}^\infty} = \varinjlim \Lambda/\mathfrak{m}^n$.*
3. *The field of fractions of Λ.*

PROOF OF THE THEOREM. We use the perpendicular category

$$X^\perp = \{M \mid \operatorname{Hom}(X, M) = \operatorname{Ext}^1(X, M) = 0\}$$

associated to a finite-dimensional module X, which is a definable subcategory. If M and N are two modules in $X^\perp$ then the image of any map $\theta : M \to N$ is also in $X^\perp$, and then using the long-exact sequence for $\operatorname{Hom}(X, -)$ it follows that $\operatorname{Ker}\theta$ and $\operatorname{Coker}\theta$ also belong to $X^\perp$.

We need the following observation: if the indecomposable direct summands of X include all quasi-simples in some tube T, then $X^\perp$ contains no indecomposable preprojective or preinjective modules. This follows from the fact that any map from a preprojective module to a preinjective module factors through a direct sum of modules in T.

We show first that A has a unique indecomposable torsion-free divisible module, and it is endofinite. We have seen this in Section 1.1 for the Kronecker algebra. If A is not Morita equivalent to the Kronecker algebra then there is a quasi-simple module S in a tube of width at least two, and the category $S^\perp$ is equivalent to the module category for a new tame hereditary algebra B, see for example [**14**] or [**6**]. Now the Grothendieck group of B has smaller rank than A, so by induction B has a unique indecomposable torsion-free divisible module, and hence so does A. Moreover this module is endofinite by Garavaglia's characterization.

Next we check that the modules in the list are indecomposable and algebraically compact. If S is quasi-simple then $\hat{S}$ is indecomposable, for by Lemma 4 if it decomposes then one summand is divisible, and thus by the Auslander-Reiten formula has no non-zero map to a regular module. This is impossible since $\hat{S}$ embeds in $\prod_n S$. By a similar argument (or the theorem in Section 1.5) the modules S_∞ are indecomposable. We have already observed at the end of Section 2.4 that $\hat{S}$ and S_∞ are algebraically compact.

It remains to check that the list contains all indecomposable algebraically compact modules. We prove this by induction on the rank of the Grothendieck group of A. Let M be an infinite-dimensional indecomposable algebraically compact module.

Suppose first that $\mathrm{Hom}(S, M) \neq 0$ for some quasi-simple module S. Choose two tubes T_1 and T_2 not containing S, and let X_i be the direct sum of all quasi-simples in T_i. Since there are no non-zero maps between modules in distinct tubes, every module in $\mathcal{R}$ belongs either to $X_1^\perp$ or to $X_2^\perp$. If F_1 and F_2 are the corresponding closed subsets of the Ziegler spectrum, then by Lemmas 1 and 3 we have $M \in \overline{\mathcal{R}} = F_1 \cup F_2$ in the Ziegler spectrum. Thus, say, $M \in X_1^\perp$. Now a non-zero map $S \to M$ can be extended to a map $S_\infty \to M$ by the argument of [**25**, 4.1]. Moreover, this map must be injective, for the kernel of the map $S_n \to M$ is in $X_1^\perp$, so is regular, and then if it is non-zero it must contain S. Thus there is an exact sequence

$$\xi : 0 \to S_\infty \to M \to N \to 0.$$

Now N belongs to $X_1^\perp$, so $\mathrm{Hom}(\mathcal{I}, N) = 0$. If $S_\infty \to Y$ is a non-zero map to a finite-dimensional indecomposable module then Y must be preinjective, and the pushout of ξ splits since $\mathrm{Ext}^1(N, Y) \cong D\,\mathrm{Hom}(\tau^- Y, N) = 0$. Thus ξ is pure-exact, and hence split. This implies that $M \cong S_\infty$.

Thus we may suppose that $\mathrm{Hom}(S, M) = 0$ for all quasi-simple S, or in other words, M is torsion-free.

If the algebra is the Kronecker algebra, let U_1 and U_2 be two quasi-simple modules, for example the modules R_0 and R_∞ of Section 1.1. Then each $U_i^\perp$ is equivalent to $K[T]$-Mod. By the argument used above, M either belongs to $U_1^\perp$ or to $U_2^\perp$, and it corresponds to an algebraically compact $K[T]$-module, for example using the characterization of algebraically compact modules as those for which the summation map $M^{(I)} \to M$ extends to a map $M^I \to M$, see [**17**, Theorem 7.1]. Now the the indecomposable algebraically compact $K[T]$-modules are known by Lemma 5, and each one corresponds to an A-module listed in the theorem.

Finally if the algebra is not Kronecker, one can find distinct quasi-simples S_1 and S_2 in a tube of width at least two. Now every indecomposable regular module is in one of the categories $E_i = \{N \mid \mathrm{Ext}^1(S_i, N) = 0\}$, so by the Ziegler spectrum argument, M belongs to one of these two categories, say E_1. Since M is also torsion-free it is in $S_1^\perp$. Now this category is equivalent to B-Mod for some tame hereditary algebra B whose Grothendieck group has smaller rank. Moreover M corresponds to an algebraically compact B-module. Now by induction the indecomposable algebraically compact B-modules are known, and it is easy to see that the corresponding A-modules are listed in the theorem. $\square$

3.2. General tame algebras. An algebra is *tame* if for all d, its indecomposable modules of dimension d can be parametrized by a finite number of curves. Adapting Drozd's Tame and Wild theorem, I proved [**5, 7**]:

THEOREM. *If A has tame representation type then*

1. *For each d, all but finitely many indecomposable modules of dimension d belong to tubes.*
2. *For each d, there are only finitely many infinite-dimensional indecomposable endofinite modules of endolength d.*

The infinite-dimensional indecomposable endofinite modules which occur in the second part of the theorem are called *generic modules*. The question of how tubes correspond to generic modules remained open, but it is now partially answered by the following correspondence. (The first part is due to Krause [**21**]).

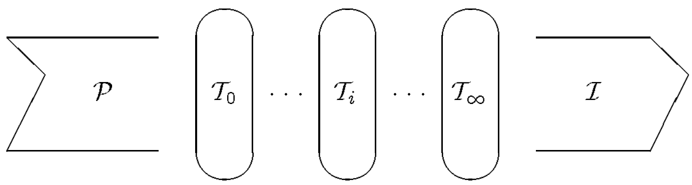
THEOREM. *The closure of any tube contains at least one generic module. Conversely, for a tame algebra, every generic module is in the closure of a tube (indeed, infinitely many tubes).*

PROOF. If S is quasi-simple in a tube, then S_∞ has the descending chain condition on subgroups of finite definition, so it has elementary Krull dimension, see Section 2.4. Thus $\overline{S_\infty}$ contains an indecomposable endofinite module G. But S_∞ is an infinite-dimensional indecomposable module, so G must be infinite dimensional.

For the converse, I proved in [**7**] that if G is a generic module for a tame algebra, then there is an A-$K[T]$-bimodule M, finitely generated free over $K[T]$, such that $G \cong M \otimes K(T)$, and such that for all but finitely many $\lambda \in K$, the modules $M \otimes K[T]/(T-\lambda)^n$ $(n \geq 1)$ form a tube. Now the closure of this tube contains G. $\square$

REMARK. One hope with infinite-dimensional modules was to find an elementary proof of the second Brauer-Thrall conjecture, that an algebra of infinite representation type has strongly unbounded representation type. This is the closest yet: any algebra with a tube must have strongly unbounded representation type.

EXAMPLE. If A is a tubular algebra (in the sense of [**27**]) then its Auslander-Reiten quiver has the structure



where $\mathcal{P}$ is the set of indecomposable preprojective modules, $\mathcal{I}$ is the set of indecomposable preinjectives, and for each rational number $0 \leq i \leq \infty$ there is a family of tubes $\mathcal{T}_i$ indexed by $K \cup \{\infty\}$ (except that the families $\mathcal{T}_0$ and $\mathcal{T}_\infty$ also have some components containing projectives and injectives).

By the argument we used for tame hereditary algebras, the indecomposable algebraically compact modules in each $\overline{\mathcal{T}_i}$ $(0 < i < \infty)$ are the finite-dimensional modules, the Prüfer and adic modules associated to quasi-simples, and one generic module. We shall not attempt here to analyse the indecomposable algebraically compact modules in $\overline{\mathcal{T}_0}$ and $\overline{\mathcal{T}_\infty}$, but we note that each contains one generic module. In this way one obtains all generic modules. Presumably there are indecomposable algebraically compact modules in $\overline{\bigcup \mathcal{T}_i}$ which are not in any $\overline{\mathcal{T}_i}$ (certainly this is the case if the base field is countable), but no examples seem to be known.

3.3. General hereditary algebras. If A is an arbitrary finite-dimensional hereditary algebra, then it is hopeless to try and classify all indecomposable algebraically compact A-modules or even all endofinite modules. However, a theorem of Schofield classifies the *stones*, that is, the indecomposable endofinite modules M which satisfy $\mathrm{Ext}^1(M, M) = 0$. (The endomorphism ring of any such module is a division ring, by the argument of the Happel-Ringel Lemma [**15**, Lemma 4.1])

Assuming that A is basic, we can write it as a path algebra $A = KQ$, where Q is a quiver without oriented cycles. Let the vertices in Q be labelled $1, \dots, n$, so any finite-dimensional module X has dimension vector $\underline{\dim} X \in \mathbb{N}^n$. Its i-th component is $\dim \mathrm{Hom}(P_i, X)$, where P_i is the indecomposable projective module for vertex i. One says that $\alpha \in \mathbb{N}^n$ is a *Schur root* if it is the dimension vector of a finite-dimensional module X with $\mathrm{End}(X) = K$. A vector $\alpha \in \mathbb{N}^n$ is *indivisible* if its coordinates have no common divisor.

THEOREM (Schofield). *The stones are in 1-1 correspondence with indivisible Schur roots, with a stone M corresponding to the vector β whose components are $\beta_i = \mathrm{length}_{\mathrm{End}(M)} \mathrm{Hom}(P_i, M)$.*

This theorem was first mentioned by Schofield in 1991, and he outlined a proof of it in a lecture in Krippen, Germany, in 1995. The proof here is based upon that lecture, except that we use the language of characters, as in Section 1.3, rather than Schofield's 'Sylvester rank functions' [**30**].

If $\beta \in \mathbb{N}^n$, then the function χ_β defined on finite-dimensional modules by $\chi_\beta(X) = \max\{\langle \underline{\dim} X/Y, \beta\rangle \mid Y \subseteq X \text{ is a submodule}\}$ is a character, where $\langle -, -\rangle$ is the Ringel form for Q. One defines $\hom(X, \beta)$ (respectively $\mathrm{ext}(X, \beta)$) to be the minimal value of $\dim \mathrm{Hom}(X, Z)$ with Z ranging over all modules of dimension vector β (respectively $\dim \mathrm{Ext}^1(X, Z)$). This is also the value taken for Z in a dense open subset of the variety of representations of dimension β. A finite-dimensional module X is said to be *β-semistable* if $\langle \underline{\dim} X, \beta\rangle = 0$ and $\langle \underline{\dim} Y, \beta\rangle \geq 0$ for all submodules $Y \subseteq X$. The following two facts are proved in [**9**]. (Actually, only one direction of (b) is mentioned there, but the converse follows from (a).)

(a) $\chi_\beta(X) = \lim_{r\to\infty} \frac{1}{r} \hom(X, r\beta)$.

(b) X is β-semistable if and only if $\hom(X, r\beta) = \mathrm{ext}(X, r\beta) = 0$ for some $r > 0$.
In the proof outlined by Schofield, property (a) was not used, and property (b) was derived from a classification of the semi-invariants of representations of quivers.

LEMMA 1. *The characters of stones are exactly the χ_β which are irreducible.*

PROOF. If χ is a character, we write $\underline{\dim}\,\chi$ for the vector with $(\underline{\dim}\,\chi)_i = \chi(P_i)$, so that $\chi(P) = \langle \underline{\dim}\, P, \underline{\dim}\,\chi\rangle$ for any projective module P. Then

$$\chi(X) \geq \max\{\langle \underline{\dim}\, X/Y, \underline{\dim}\,\chi\rangle \mid Y \subseteq X\}$$

for any module X (for if $0 \to P' \to P \to X/Y \to 0$ is a projective resolution, then $\chi(X) \geq \chi(X/Y) \geq \chi(P) - \chi(P') = \langle \underline{\dim}\, X/Y, \underline{\dim}\,\chi\rangle$). Clearly the χ_β are exactly the 'extremal characters' for which this inequality is always an equality. Writing an extremal character as a sum of irreducibles, it is clear that every summand must also be extremal. Thus, in order to prove the lemma it suffices to show that if M is an indecomposable endofinite module then χ_M is extremal if and only if M is a stone. Note first that if X is any finite-dimensional module, then

$$\langle \underline{\dim}\, X, \underline{\dim}\,\chi_M\rangle = \mathrm{length}_{\mathrm{End}(M)} \mathrm{Hom}(X, M) - \mathrm{length}_{\mathrm{End}(M)} \mathrm{Ext}^1(X, M),$$

for both sides are additive on short exact sequences, and the assertion holds by definition for X projective.

Suppose that M is a stone. Let Y be the kernel of the map $X \to M^r$ given by a set of $\mathrm{End}(M)$-module generators of $\mathrm{Hom}(X, M)$. Clearly we have $\mathrm{Hom}(X/Y, M) \cong \mathrm{Hom}(X, M)$, and since X/Y embeds in M^r, the equality $\mathrm{Ext}^1(M^r, M) = 0$ implies that $\mathrm{Ext}^1(X/Y, M) = 0$. Thus

$$\langle \underline{\dim}\, X/Y, \underline{\dim}\,\chi_M\rangle = \mathrm{length}\, \mathrm{Hom}(X/Y, M) - \mathrm{length}\, \mathrm{Ext}^1(X/Y, M) = \chi_M(X),$$

so that χ_M is extremal.

Conversely, suppose that $\mathrm{Ext}^1(M, M) \neq 0$. Thus there is a non-split exact sequence $0 \to M \to E \to M \to 0$, and since M is algebraically compact, this sequence cannot be pure-exact. Thus there is a finite-dimensional submodule X of M such that the inclusion $X \to M$ doesn't factor through E, and hence $\mathrm{Ext}^1(X, M)$ is non-zero. Now for any submodule $Y \subseteq X$, we have $\mathrm{length}\, \mathrm{Ext}^1(X/Y, M) \geq 0$, strict for $Y = 0$, and $\mathrm{length}\, \mathrm{Hom}(X/Y, M) \leq \mathrm{length}\, \mathrm{Hom}(X, M)$, strict for $Y \neq 0$ (since the inclusion of X in M doesn't factor through X/Y). Thus

$$\langle \underline{\dim}\, X/Y, \underline{\dim}\,\chi_M\rangle = \mathrm{length}\, \mathrm{Hom}(X/Y, M) - \mathrm{length}\, \mathrm{Ext}^1(X/Y, M) < \chi_M(X).$$

Since this holds for all Y, the character χ_M cannot be extremal. $\square$

LEMMA 2. *If β is a Schur root then ${}^\perp\beta = \{\xi \in \mathbb{Q}^n \mid \langle \xi, \beta\rangle = 0\}$ is the $\mathbb{Q}$-span of the dimension vectors of β-semistable modules.*

PROOF. We may assume that β is sincere (i.e. every coordinate is strictly positive). Namely, if $\beta_i = 0$, then by fact (b) the projective module P_i is β-semistable. Now if $\xi \in {}^\perp\beta$ then so is $\xi - \xi_i\, \underline{\dim}\, P_i$, and this vector has support contained in the quiver obtained by deleting vertex i. The claim then follows by induction.

Since β is a Schur root, it is the dimension vector of a module X with trivial endomorphism algebra. Moreover, by [**31**, Theorem 6.1] we may assume that $\langle \underline{\dim}\, Y, \beta\rangle - \langle \beta, \underline{\dim}\, Y\rangle > 0$ for all non-zero proper submodules Y of X.

We first check the lemma in case $X \cong P_i$ is projective. In this case the inverse translates $\tau^- S_j$ of simple modules S_j $(j \neq i)$ are β-semistable, as are the projectives

P_j with $\operatorname{Hom}(P_i, P_j) = 0$. Thus, if Φ is the Coxeter transformation, then the $\mathbb{Q}$-span contains all vectors $\Phi^{-1}(\epsilon_j)$ $(j \neq i)$, where ϵ_j is the coordinate vector. The assertion follows.

Now assume that X is not projective. Thus $\gamma = \beta + \Phi(\beta) = \beta + \underline{\dim}\, \tau X \in \mathbb{N}^n$. Now if Y is a non-zero proper submodule of X then the equality $\langle \alpha, \Phi(\beta)\rangle = -\langle \beta, \alpha\rangle$ gives

$$\langle \underline{\dim}\, X/Y, \gamma\rangle = \langle \beta, \underline{\dim}\, Y\rangle - \langle \underline{\dim}\, Y, \beta\rangle < 0.$$

Suppose that $\xi \in {}^{\perp}\beta$. By rescaling, we suppose that the components of ξ are integral. For $m \in \mathbb{N}$ sufficiently large, the vector $\delta = \Phi(\xi) + m\gamma$ is sincere and $\langle \underline{\dim}\, X/Y, \delta\rangle < 0$ for any non-zero proper submodule Y of X. Since also $\langle \beta, \delta\rangle = 0$, this means that X is δ-semistable, and hence by fact (b) there is a module Z of dimension $\ell\delta$ $(\ell \geq 1)$ with $\operatorname{Hom}(X, Z) = \operatorname{Ext}^1(X, Z) = 0$. Since X is sincere, Z cannot have an injective summand. Then $\operatorname{Hom}(\tau^- Z, X) = \operatorname{Ext}^1(\tau^- Z, X) = 0$, so $\tau^- Z$ is β-semistable, and it has dimension $\ell(\xi + m\Phi^{-1}(\gamma))$.

We apply this first with $\xi = 0$ to see that $\Phi^{-1}(\gamma)$ belongs to the $\mathbb{Q}$-span, and then to deduce that any $\xi \in {}^{\perp}\beta$ belongs to the $\mathbb{Q}$-span, as required. $\square$

Proof of the theorem. By the first lemma it suffices to prove that χ_β is irreducible if and only if β is an indivisible Schur root.

Suppose that χ_β irreducible. If β is not a Schur root then by the canonical decomposition [**18**, §4] one can write $\beta = \gamma + \delta$ with $\operatorname{ext}(\gamma, \delta) = \operatorname{ext}(\delta, \gamma) = 0$. Thus for any $r > 0$ we have $\operatorname{ext}(r\gamma, r\delta) = \operatorname{ext}(r\delta, r\gamma) = 0$, and this means that the general representation of Q of dimension $r\beta$ is a direct sum of representations of dimensions $r\gamma$ and $r\delta$. Thus $\hom(X, r\beta) = \hom(X, r\gamma) + \hom(X, r\delta)$, so by fact (a) we have $\chi_\beta = \chi_\gamma + \chi_\delta$. Thus β must be a Schur root, and it is indivisible, for if $\beta = m\gamma$ then $\chi_\beta(X) = \lim_{r\to\infty} \frac{1}{r} \hom(X, rm\gamma) = m\chi_\gamma(X)$.

Now suppose that β is an indivisible Schur root, but that χ_β is reducible. Write it as sum of irreducibles. By the extremal property used in the proof of Lemma 1 the summands are extremal, so $\chi_\beta = \chi_\gamma + \chi_\delta + \dots$ with $\gamma, \delta, \dots$ indivisible Schur roots.

Now any β-semistable module is γ-semistable, for if X is β-semistable, then $\chi_\beta(X) = 0$. Thus $\chi_\gamma(X) = \chi_\delta(X) = \dots = 0$, and in particular

$$\langle \underline{\dim}\, X, \gamma\rangle, \langle \underline{\dim}\, X, \delta\rangle, \ldots \leq 0.$$

However, the sum of all these terms is $\langle \underline{\dim}\, X, \beta\rangle = 0$, so each term must be zero, and therefore X is γ-semistable. Now ${}^{\perp}\beta \subseteq {}^{\perp}\gamma$ by the second lemma, but the Ringel form $\langle -, -\rangle$ is nondegenerate, so γ and β must be multiples of each other. Thus $\beta = \gamma$, so χ_β is irreducible. $\square$

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