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MATRIX PROBLEMS AND DROZD'S THEOREM

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This paper is intended to introduce the reader to the idea of reductions for matrix problems. First, we introduce a new formulation for such matrix problems, bimodule problems, generalizing a notion of Drozd. This is then related to more established languages, including partitioned matrix problems, differential biquivers, and bocses. Finally, several applications of the reductions are given, including Drozd's Tame and Wild Theorem.

1. Introduction

The aim of this paper is to discuss several (more or less) equivalent formulations of matrix problems and their reductions. Before moving on to general frameworks, however, it is worthwhile to start at the beginning: the problem of putting a matrix over a field k into normal form by elementary row and column operations. Of course there are the two standard examples, which we give names, the reasons for which will become apparent later.

Edge reduction: putting a matrix into the form $\left[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right]$ (where the 1 is an identity matrix and 0 denotes an arbitrary rectangular matrix of zeros) using arbitrary elementary row and column operations; and

Loop reduction: over an algebraically closed field k, putting a square

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matrix into Jordan Normal Form using elementary row operations and their inverse column operations simultaneously.

More generally, one can consider a *partitioned matrix problem* in which the matrix is divided into blocks, and only prescribed elementary transformations may be used. The following two examples taken from [14] serve well to illustrate this.

PROBLEM 1. Put a matrix $[*|*]$ partitioned into two blocks (denoted by $*$'s) into normal form using any row operations, but column operations only within each block; and

PROBLEM 2. The same, but also allowing the addition of multiples of columns in the right-hand block to columns in the left-hand block.

In fact, these problems are very easily solved. For Problem 2, put the right-hand block into normal form using edge reduction to obtain $\left[\begin{smallmatrix} * \\ * \end{smallmatrix}\middle|\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ where we have divided the left-hand block into the corresponding smaller blocks. Adding multiples of columns on the right to the left, we can set the top of these to zero: $\left[\begin{smallmatrix} 0 \\ * \end{smallmatrix}\middle|\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ (this operation will be called *regularization* later). Finally, use edge reduction on the remaining $*$ to obtain

$$\left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Similarly for Problem 1, use edge reduction on the left-hand block to obtain $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\middle|\begin{smallmatrix} * \\ * \end{smallmatrix}\right]$ and now observe that the problem of putting the $*$'s into canonical form is precisely the transpose of Problem 2. (One can do any row operation on the top $*$ provided one simultaneously does the inverse operation on the first column.)

Although this approach works for Problems 1 and 2, in general the situation is much more complex. One needs a systematic method for solving such problems. The first such formulation, given by Kleiner and Roiter [15, 11], was via *differential graded categories*, and in particular used rather special such categories, namely *free triangular* ones. Roiter subsequently gave another formulation, with the notion of a *bocs* [14], but again the bocses must be rather special: *normal*, free and triangular. In the next section we introduce yet another formulation of these problems, using a *bimodule* over a category, equipped with a *derivation*. This seems to be the most elegant formulation (at least for theoretical purposes). We shall then give a form, *differential biquivers*, which is most suitable for calculations. This is precisely a restatement of normal free triangular (and linear) bocses. The last language we shall discuss is that of bocses, which is still needed for the proof of Drozd's Tame and Wild Theorem [7,3]. Finally, in §5 we give some applications.

2. Bimodule problems

Let k be a commutative ring, which will usually be a perfect (or even algebraically closed) field. Recall that a category is called a *k-category* if its morphism spaces are k-modules, and composition of morphisms is k-bilinear. We do not demand that the morphism spaces are finitely generated over k. A functor between k-categories is called a *k-functor* provided it is k-linear. In the sequel all categories will be k-categories, and all functors, k-functors. We denote the category of all k-modules by $\mathrm{Mod}\,k$, and the subcategory of finitely generated k-modules by $\mathrm{mod}\,k$.

DEFINITION. If K and L are k-categories, then

(1) a *left K-module* is a (covariant) k-functor $K^{\mathrm{op}} \to \mathrm{Mod}\,k$,
(2) a *right L-module* is a k-functor $L \to \mathrm{Mod}\,k$, and
(3) a *K-L-bimodule* is a k-bilinear functor $K^{\mathrm{op}} \times L \to \mathrm{Mod}\,k$.

If M is a K-L-bimodule, X, X' and Y, Y' are objects in K and L respectively, $m \in M(X, Y)$, $a \in K(X', X)$ and $b \in L(Y, Y')$, then the element $mM(a, b)$ of $M(X', Y')$ can conveniently be denoted by amb. Thus modules and bimodules over categories are in many ways like ordinary modules and bimodules.

DEFINITION. If K is a k-category and M is a K-K-bimodule, then a *derivation* i: $K \to M$ is given by k-linear maps i: $K(X, Y) \to M(X, Y)$ for each pair of objects X, Y in K, such that whenever $a \in K(X, Y)$ and $b \in K(Y, Z)$ then $i(ab) = ai(b) + i(a)b$.

DEFINITION. By a *bimodule problem* we mean a triple (K, M, i) where K is a k-category, M is a K-K-bimodule and i: $K \to M$ is a derivation.

DEFINITION. By a *representation* of (K, M, i), or a *matrix over* (K, M, i) we mean a pair, denoted by Xm, consisting of an object X in K and an element $m \in M(X, X)$.

We turn the set of matrices over (K, M, i) into a category $\mathrm{Mat}(K, M, i)$ by defining the morphisms from Xm to $X'm'$ to be the $f \in K(X, X')$ with $mf - fm' = i(f)$, and composing them using the composition in K. Of course the identity morphisms are in $\mathrm{Mat}(K, M, i)$ since $i(1_X) = 0$ for each object X. Note that f is an isomorphism in $\mathrm{Mat}(K, M, i)$ if and only if it is an isomorphism in K, for suppose that f has inverse f^{-1} in K, then $fi(f^{-1}) + i(f)f = i(ff^{-1}) = 0$, so $i(f^{-1}) = -f^{-1}i(f)f^{-1}$, and hence $m'f^{-1} - f^{-1}m = i(f^{-1})$, which just says that f^{-1} is a morphism from $X'm'$ to Xm.

We also want to consider some especially nice types of bimodule problems. Let us call a problem *Krull–Schmidt* if the following additional assumptions are satisfied: the ring k is a field, the spaces $K(X, Y)$ and $M(X, Y)$ are all finite-dimensional over k, and K is a *Krull–Schmidt category* [13]: it has finite direct sums (coproducts), and split idempotents, that is, if $e \in K(X, X)$ has $e^2 = e$, then there is an object Y and morphisms $f \in K(X, Y)$ and $g \in K(Y, X)$ with $fg = e$ and $gf = 1_Y$.

Of course in this case Mat(K, M, i) is also a Krull–Schmidt category. It clearly has direct sums, so suppose $e\colon Xm \to Xm$ is idempotent. Since idempotents split in K there is an object Y and $f \in K(X, Y)$ and $g \in K(Y, X)$ with $fg = e$ and $gf = 1_Y$. It is easy to check that f and g give morphisms between Xm and Yn where $n = gmf - gi(f)$.

We describe a Krull–Schmidt bimodule problem by drawing a picture as follows. For each isomorphism class of indecomposable objects in K, choose a representative X and draw a vertex labelled X. For each pair X, Y of vertices take a basis of $M(X, Y)$ and draw a solid arrow $X \to Y$ for each basis element. Also, if $X \neq Y$ take a basis of $K(X, Y)$ and draw a dotted arrow $X \dashrightarrow Y$ for each basis element. If $X = Y$ take a basis of $K(X, X)$ which includes 1_X, and draw a dotted loop at X for each basis element except 1_X. To complete the description of the bimodule problem we must specify multiplication tables for K and the left and right actions of K on M, and specify $i(a)$ for each dotted arrow a.

2.1. Examples

Problem 1 can now be formulated as

$$Y \xleftarrow{a} Z \xrightarrow{b} W$$

with $i = 0$, so K has 3 indecomposable objects with trivial endomorphism rings and no maps between them. Thus $K = (\mathrm{mod}\,k) \times (\mathrm{mod}\,k) \times (\mathrm{mod}\,k)$ or $\mathrm{mod}\,k \times k \times k$. A matrix over (K, M, i) is given by an object X of K and an element $m \in M(X, X)$. Writing

$$X = Y \oplus \ldots \oplus Y \oplus Z \oplus \ldots \oplus Z \oplus W \oplus \ldots \oplus W$$

we see that $M(X, X)$ looks like

$$\begin{bmatrix}
M(Y, Y) & \ldots & M(Y, Y) & M(Y, Z) & \ldots & M(Y, Z) & M(Y, W) & \ldots & M(Y, W) \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
M(Y, Y) & \ldots & M(Y, Y) & M(Y, Z) & \ldots & M(Y, Z) & M(Y, W) & \ldots & M(Y, W) \\
M(Z, Y) & \ldots & M(Z, Y) & M(Z, Z) & \ldots & M(Z, Z) & M(Z, W) & \ldots & M(Z, W) \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
M(Z, Y) & \ldots & M(Z, Y) & M(Z, Z) & \ldots & M(Z, Z) & M(Z, W) & \ldots & M(Z, W) \\
M(W, Y) & \ldots & M(W, Y) & M(W, Z) & \ldots & M(W, Z) & M(W, W) & \ldots & M(W, W) \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
M(W, Y) & \ldots & M(W, Y) & M(W, Z) & \ldots & M(W, Z) & M(W, W) & \ldots & M(Y, W)
\end{bmatrix}$$

and since $M(Z, Y)$ and $M(Z, W)$ are one-dimensional, and the other spaces are zero, the element m is given by a matrix of the form

$$\left[\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline A & 0 & B \\ \hline 0 & 0 & 0 \end{array}\right]$$

where the A and B are block matrices over k. Similarly an element $\theta \in K(X, X')$ is given by a block matrix

$$\left[\begin{array}{c|c|c} \theta_Y & 0 & 0 \\ \hline 0 & \theta_Z & 0 \\ \hline 0 & 0 & \theta_W \end{array}\right]$$

and the condition for θ to be a morphism from Xm to $X'm'$ is $m\theta = \theta m'$, which reduces to

$$A\theta_Y = \theta_Z A' \quad \text{and} \quad B\theta_W = \theta_Z B'.$$

For an isomorphism θ, this just says that $[A \mid B]$ is obtained from $[A' \mid B']$ by elementary column operations within each block, and simultaneous elementary row operations.

Problem 2 can be formulated as shown in Fig. 1, with $b\varphi = a$. This time

$$\begin{tikzcd} & Z \arrow[dl, "a"'] \arrow[dr, "b"] & \\ Y & & W \arrow[ll, dashed, "\varphi"] \end{tikzcd}$$

Fig. 1

K has been enlarged, so $\theta \in K(X, X)$ is given by a block matrix

$$\left[\begin{array}{c|c|c} \theta_Y & 0 & 0 \\ \hline 0 & \theta_Z & 0 \\ \hline \theta_\varphi & 0 & \theta_W \end{array}\right]$$

and now the condition that $m\theta = \theta m'$ is equivalent to

$$A\theta_Y + B\theta_\varphi = \theta_Z A' \quad \text{and} \quad B\theta_W = \theta_Z B'$$

which means that columns of B can be added to A.

2.2. Drozd's bimodule problems

One special type of bimodule problem has already been considered by Drozd. Let A and B be categories and let N be an A-B-bimodule. Set $K = A \times B$ and consider the K-K-bimodule ${}_K N_K$ obtained by restriction via the projections of K onto A and B. We denote $\mathrm{Mat}(K, {}_K N_K, 0)$ by $R(N)$. Pictorially, these bimodule problems look as in Fig. 2 where all solid arrows go

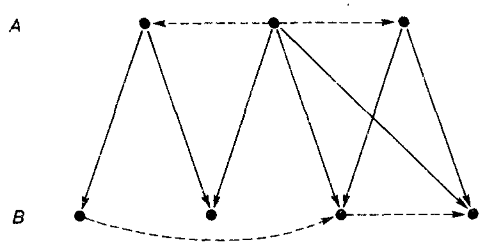

Fig. 2

from the top to the bottom, and all dotted arrows start and finish in the same half.

EXAMPLE 1. Our Problems 1 and 2 are of this type.

EXAMPLE 2. Let $(K, |-|)$ be a vector space category, so K is a Krull–Schmidt category and $|-|$ is a functor from K to $\mathrm{mod}\, k$. If N is the $\mathrm{mod}\, k$-K-bimodule with $N(-, -) = \hom_k(-, |-|)$, then $R(N)$ is just the category $\check{\mathscr{U}}(K, |-|)$ of representations of $(K, |-|)$; see [13, §2.5].

EXAMPLE 3. Let Λ be a finite-dimensional k-algebra, $A = B = \mathrm{proj}(\Lambda)$, the category of finite-dimensional projective left Λ-modules, and $N = \mathrm{rad}\,\mathrm{proj}(\Lambda)$ regarded as an A-B-bimodule in the obvious way (where $\mathrm{rad}\, C$ denotes the radical of the category C, see for example [13, §2.2]). Here an object of $R(N)$ is given by two projective Λ-modules P, Q and a Λ-module map $f\colon P \to \mathrm{rad}\, Q$, and a morphism from $f\colon P \to Q$ to $f'\colon P' \to Q'$ is given by Λ-module maps $P \to P'$ and $Q \to Q'$ giving a commutative square

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d] & Q \arrow[d] \\ P' \arrow[r, "g"'] & Q' \end{tikzcd}$$

The functor from $R(N)$ to $\mathrm{mod}\,\Lambda$ taking $f\colon P \to Q$ to $\mathrm{coker}(f)$ gives a representation equivalence between the full subcategory of $R(N)$ on the objects with no direct summand of the form $0\colon P \to 0$ and $\mathrm{mod}\,\Lambda$, and hence the representation theory of Λ is phrased as a bimodule problem.

2.3. The reduction lemma

The main reduction lemma is the following:

Reduction Lemma. *Let (K, M, i) be a bimodule problem, N a subbimodule of M and $\bar{i}$: $K \to M/N$ the derivation induced by i. Let $L = \mathrm{Mat}(K, M/N, \bar{i})$. If we consider N as an L-L-bimodule ${}_LN_L$ by using the forgetful functor $L \to K$, then there is a derivation i': $L \to {}_LN_L$ and an equivalence F: $\mathrm{Mat}(L, {}_LN_L, i') \to \mathrm{Mat}(K, M, i)$.*

Proof. For each element $q \in (M/N)(X, Y)$, choose a lifting $\alpha(q)$ in $M(X, Y)$, and define i' by sending $\theta \in L(Xq, X'q')$ to

$$i(\theta) + \theta\alpha(q') - \alpha(q)\theta \in ({}_LN_L)(Xq, X'q') = N(X, X').$$

Define F by sending the object $(Xq)n$ to $X(\alpha(q)+n)$ and θ to θ. It is trivial to check that F is an equivalence.

Example. Recall that Problem 1 has been phrased as a bimodule problem (K, M, i) which looks like

$$Y \xleftarrow{a} Z \xrightarrow{b} W.$$

Let N be the subbimodule of M generated by a, so that N is

$$Y \xleftarrow{a} Z \qquad W$$

and M/N is

$$Y \qquad Z \xrightarrow{\bar{b}} W.$$

We shall see later that $L = \mathrm{Mat}(K, M/N, 0)$ has indecomposable objects and morphisms described by Fig. 3 where $m \in (M/N)(Z \oplus W, Z \oplus W)$ is of the

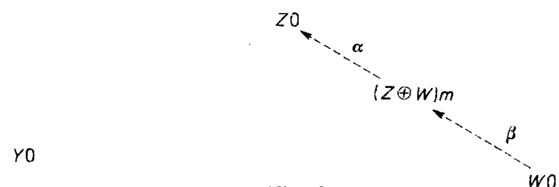

Fig. 3

form $\begin{bmatrix} 0 & \bar{b} \\ 0 & 0 \end{bmatrix}$. Now ${}_LN_L(Z0, Y0) = N(Z, Y) = ka$ and ${}_LN_L((Z \oplus W)m, Y0) = N(Z \oplus W, Y) = k\begin{bmatrix} a \\ 0 \end{bmatrix}$; but for example ${}_LN_L(Y0, Z0) = N(Y, Z) = 0$. Thus the bimodule problem $(L, {}_LN_L, i')$ looks as shown in Fig. 4. In fact i' is zero and

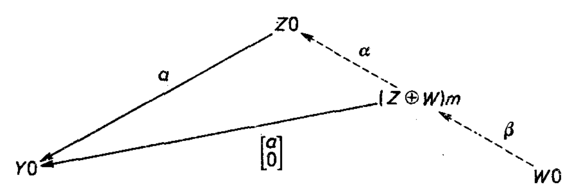

Fig. 4

$\alpha a = \begin{bmatrix} a \\ 0 \end{bmatrix}$. Now if Xn is an indecomposable matrix over this, and $W0$ is a summand of X, then X actually equals $W0$ (and n must of course be zero). If we replace L by its full subcategory on the objects without $W0$ as a summand,

and use the restrictions of ${}_L N_L$ and i', then the only indecomposable matrix we lose is precisely this one, while the remaining matrix problem is the transpose of Problem 2 (the arrows are reversed). Of course this is what happened to the partitioned matrices in the introduction.

2.4. Computation of $\mathrm{Mat}(K, M, i)$ **when** M **is semisimple**

If we want to use the reduction lemma to replace a matrix problem (K, M, i) by $(L, {}_L N_L, i')$, then we need to compute $L = \mathrm{Mat}(K, M/N, \bar{i})$, and the simplest case is when N is maximal, so that M/N is simple. In this section we show how to compute $\mathrm{Mat}(K, M, i)$ for M semisimple. Throughout this section we need the following assumptions: (K, M, i) is a Krull–Schmidt bimodule problem and k is a perfect field.

Since k is perfect, $K(X, X)$ splits over its radical for each object X in K, and so there is a subcategory S of K with the same objects, and such that $K = S \oplus \mathrm{rad}\, K$. Namely, choose splittings for a complete set of representatives of the indecomposable objects in K, and extend this to K by fixing a presentation of each object in K as a direct sum of these indecomposables. Clearly S is a semisimple category, that is, there are no nonzero morphisms between nonisomorphic indecomposables, and the endomorphism rings of indecomposables are division rings. Note also that in this case $\mathrm{rad}\, M = (\mathrm{rad}\, K) M + M (\mathrm{rad}\, K)$.

LEMMA. *There is a derivation* $j\colon K \to M$ *with* $j(S) = 0$ *and* $\mathrm{Mat}(K, M, i)$ *equivalent to* $\mathrm{Mat}(K, M, j)$.

Proof. Since k is perfect, for each indecomposable object X, $S(X, X)$ is separable over k, and hence $i|_{S(X,X)}\colon S(X, X) \to M(X, X)$ is an inner derivation. By fixing presentations of the objects in S as direct sums of indecomposables one deduces that $i|_S\colon S \to {}_S M_S$ is an inner derivation, that is, for each object X in K there is an element $m_X \in M(X, X)$ such that $i(s) = s m_Y - m_X s$ for $s \in S(X, Y)$. Define j by

$$j(a) = i(a) + m_X a - a m_Y \qquad (a \in K(X, Y)),$$

and an equivalence from $\mathrm{Mat}(K, M, i)$ to $\mathrm{Mat}(K, M, j)$ sending Xm to $X(m + m_X)$ and θ to θ.

REGULARIZATION. *Let* (K, M, i) *be a bimodule problem with* M *semisimple, and suppose that* $K = S \oplus \mathrm{rad}\, K$ *with* $i(S) = 0$. *Then there is an equivalence* F: $\mathrm{Mat}(\ker(i), M/\mathrm{im}(i), 0) \to \mathrm{Mat}(K, M, i)$.

Proof. Since M is semisimple, $(\mathrm{rad}\, K)M = M(\mathrm{rad}\, K) = 0$, and it follows that the restriction of i to $\mathrm{rad}\, K$ is a bimodule map. Since $\mathrm{im}(i) = i(S) + i(\mathrm{rad}\, K) = i(\mathrm{rad}\, K)$, this is a subbimodule of M, and hence the left-hand side is well-defined. Since M is semisimple, the projection

$M \to M/\mathrm{im}(i)$ is split, say by α. It is easy to check that the functor F defined by $F(Xq) = X\alpha(q)$ for $q \in (M/\mathrm{im}(i))(X, X)$ and $F(\theta) = \theta$ is an equivalence.

For example,

$$\mathrm{Mat}(\bullet \underset{\varphi}{\overset{a}{\rightrightarrows}} \bullet \;\; i(\varphi) = a) \cong \mathrm{Mat}(\bullet \;\; \bullet).$$

Thus we only need to compute $\mathrm{Mat}(K, M, 0)$ with M simple.

LEMMA. *Let* $(K, M, 0)$ *be a bimodule problem with* M *semisimple,* $K = S \oplus \mathrm{rad}\, K$ *and denote* $\mathrm{Mat}(S, {}_S M_S, 0)$ *by* L. *Then* $\mathrm{Mat}(K, M, 0)$ *is equivalent to* $L \oplus {}_L(\mathrm{rad}\, K)_L$, *the category with the same objects as* L, *with morphism spaces* $\hom(Xm, X'm') = L(Xm, X'm') \oplus (\mathrm{rad}\, K)(X, X')$, *and multiplication given by* $(\theta, \varphi)(\theta', \varphi') = (\theta\theta', \theta\varphi' + \theta'\varphi + \varphi\varphi')$.

Proof. Define $F\colon L \oplus {}_L(\mathrm{rad}\, K)_L \to \mathrm{Mat}(K, M, 0)$ by $F(Xm) = Xm$ and $F((\theta, \varphi)) = \theta + \varphi$. It is easy to see that F is an equivalence.

We have reduced further, and only need to compute $\mathrm{Mat}(K, M, 0)$ when K is semisimple and M is simple. In the general case of M semisimple this corresponds to the representations of a species (possibly with oriented cycles) and is rather complicated, but if k is algebraically closed and M is simple there are just two possibilities:

Edge reduction. If the bimodule problem has no dotted arrows (that is, K is semisimple) and the only solid arrow is an edge, i.e. not a loop, so it looks like

$$\overset{X}{\bullet} \xrightarrow{a} \overset{Y}{\bullet} \quad \overset{Z_1}{\bullet} \quad \overset{Z_2}{\bullet} \quad \cdots \quad \overset{Z_n}{\bullet},$$

then the category of matrices over it is

$$\overset{X0}{\bullet} \dashleftarrow \overset{W}{\bullet} \dashleftarrow \overset{Y0}{\bullet} \quad \overset{Z_1 0}{\bullet} \quad \overset{Z_2 0}{\bullet} \quad \cdots \quad \overset{Z_n 0}{\bullet}$$

where $W = (X \oplus Y)m$ and $m \in M(X \oplus Y, X \oplus Y)$ looks like $\left[\begin{smallmatrix} 0 & a \\ 0 & 0 \end{smallmatrix}\right]$.

Loop reduction. If the only arrow is a solid loop at X, then $\mathrm{Mat}(K, M, 0)$ is equivalent to $L \oplus \mathrm{mod}\, k[x]$ where L is obtained from K by deleting all objects with X as a direct summand.

3. Differential biquivers

In this section we give another formulation for Krull–Schmidt bimodules (K, M, i) with K having only finitely many indecomposable objects, and when the field k is algebraically closed. Our aim is two-fold. Firstly, we want to introduce the notion of a *differential*, which is the link with bocses in §4; and secondly, we want to show that there are practical algorithms for performing reductions (suitable for implementation on a computer, for example).

DEFINITION. A *biquiver* Q is a quiver with two types of arrows, *solid* and *dotted*. (Of course this is the same as the picture we drew before.)

By defining the degree of a path to be the number of dotted arrows involved, the path algebra kQ becomes a graded algebra.

DEFINITION. We call a linear map $d\colon kQ \to kQ$ a (*linear triangular*) *differential* d (and then we call (Q, d) a *differential biquiver*) provided that

(1) d raises degrees by 1 and $d^2 = 0$,
(2) $d(e) = 0$ if e is a trivial path,
(3) $d(ab) = d(a)b + (-1)^i a d(b)$ if a is a path of degree i,
(4) (linearity) the differential of each arrow is a linear combination of paths of length at most 2, and
(5) (triangularity) the arrows in Q can be ordered so that the differential of any arrow only involves strictly smaller arrows.

Note that in view of (3), the differential is completely specified by giving the images of arrows, and by (2) and (3), the differential of a path is always a linear combination of paths with the same start and end.

As an example, consider the biquiver of Fig. 5 with differential d defined by

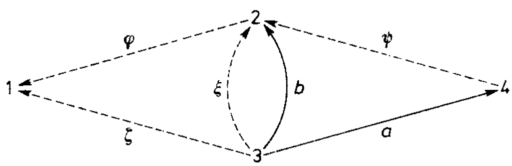

Fig. 5

$d(b) = -\xi + a\psi$, $d(\zeta) = \xi\varphi$, and d zero on the other arrows. We use this example to explain how to define the category $R(Q, d)$ of *representations* of (Q, d).

The objects in $R(Q, d)$ are given by representations V of the quiver of solid arrows, so are specified by a vector space for each vertex and a linear map for each solid arrow (see Fig. 6).

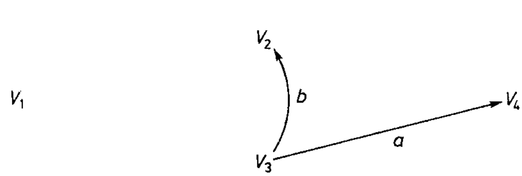

Fig. 6

The dotted arrows are used to increase the number of possible morphisms, namely $R(Q, d)$ contains the usual category of representations of the quiver of solid arrows as a subcategory with the same objects, but which is in general not full. Draw two copies of the quiver of solid arrows; draw arrows ω_i connecting the two vertices corresponding to i, and draw the dotted arrows stretched between the two copies (see Fig. 7).

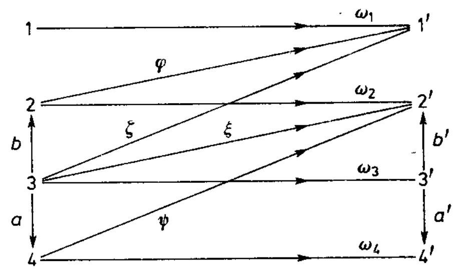

Fig. 7

A *morphism* F from V to V' is a representation of this quiver whose restriction to the left-hand half is V, to the right-hand half is V', and which satisfies relations of the form $a\omega_j - \omega_i a = d(a)$ for each solid arrow $a\colon i \to j$ (where the paths in $d(a)$ are regarded as going from the left-hand side of the diagram to the right). In our example the relations are

$$a\omega_4 - \omega_3 a' = d(a) = 0,$$

$$b\omega_2 - \omega_3 b' = d(b) = -\xi + a\psi.$$

Similarly the product of two morphisms is constructed from the differentials of the dotted arrows. Namely, in a product $F'' = FF'$ one has $\omega_i'' = \omega_i \omega_i'$ for each vertex i, and if $\varphi\colon i \dashrightarrow j$ is a dotted arrow, say with differential $d(\varphi) = \sum \lambda_m \varphi_m \psi_m$ ($\lambda_m \in k$ and φ_m, ψ_m dotted arrows), then $\varphi'' = \varphi\omega_i' - \omega_j \varphi' + \sum \lambda_m \varphi_m \psi_m'$.

3.1. Relationship with bimodule problems

In this section we exhibit the correspondence between differential biquivers and bimodule problems, generalizing the construction given by Drozd [7] of the bocs corresponding to one of Drozd's bimodule problems.

Let k be an algebraically closed field, (K, M, i) a Krull–Schmidt bimodule problem, $K = S \oplus J$ with $i(S) = 0$ as in §2.4, and suppose that K has only finitely many indecomposable objects $X_1, \ldots, X_n$. For each $1 \leqslant i, j \leqslant n$, let $M_{ij} = M(X_i, X_j)$ and $J_{ij} = J(X_i, X_j)$, and let A_{ij} and Φ_{ij} be bases for DM_{ij} and DJ_{ij} respectively (where $D(-) = \hom_k(-, k)$). Construct a biquiver Q on the vertices $1, \ldots, n$, whose set of solid arrows from i to j is A_{ij} and whose set of dotted arrows from i to j is Φ_{ij}. (Of course this is just the picture we drew before.) We shall define a linear map $d\colon kQ \to kQ$.

Clearly we may identify DM_{ij} (respectively DJ_{ij}) with the subspace of the path algebra kQ spanned by the solid (respectively dotted) arrows from i to j.

The multiplication map $m_{ijm}\colon J_{ij}\otimes_k J_{jm}\to J_{im}$ has as dual Dm_{ijm} a map from DJ_{im} to $D(J_{ij}\otimes J_{jm})$. Now this second space can be identified with $DJ_{ij}\otimes_k DJ_{jm}$, and hence with the subspace of kQ spanned by the paths consisting of a dotted arrow from i to j followed by a dotted arrow from j to m. We define d on dotted arrows by

$$d(\varphi)=\sum_{j=1}^{n} Dm_{ijm}(\varphi) \qquad \text{for } \varphi\in\Phi_{im}.$$

Similarly the duals of the derivation $i_{im}\colon J_{im}\to M_{im}$ and the multiplication maps $l_{ijm}\colon J_{ij}\otimes M_{jm}\to M_{im}$ and $r_{ijm}\colon M_{ij}\otimes J_{jm}\to M_{im}$ can be identified with maps into kQ. We then define d on solid arrows via

$$d(a)=Di_{im}(a)+\sum_{j=1}^{n}\left(Dl_{ijm}(a)-Dr_{ijm}(a)\right) \qquad \text{for } a\in A_{im}.$$

Finally, define d on arbitrary paths using the rules that d is zero on the trivial paths, and $d(ab)=d(a)b+(-1)^i a d(b)$ if a is a path of degree i. Then one has

LEMMA. *d is a differential, which is linear by construction, and triangular provided that the bases A_{ij} and Φ_{ij} have been chosen to respect the radical series of M and K.*

To show that d is a differential it suffices to prove that $d^2(\varphi)$ and $d^2(a)$ are zero. Writing out the appropriate equations and dualizing back, this is precisely equivalent to the axioms that K is associative, i is a derivation, the two actions of K on M commute, and that they are actions. In fact, since this is an equivalence, if (Q, d) is an arbitrary differential biquiver, then one can reverse this procedure and construct a bimodule problem.

PROPOSITION. *$\mathrm{Mat}(K, M, i)$ and $R(Q, d)$ are equivalent.*

Proof. We construct an equivalence F from $\mathrm{Mat}(K, M, i)$ to $R(Q, d)$. Any object X in K is a direct sum of the X_i, and if $X\cong\bigoplus_{i=1}^{n} X_i^{n_i}$ then we can choose morphisms $e_{ij}\in S(X_i, X)$ and $p_{ij}\in S(X, X_i)$ $(1\leqslant i\leqslant n,\ 1\leqslant j\leqslant n_i)$ such that

$$e_{ij}p_{rs}=\begin{cases}0, & (i,j)\neq(r,s)\\ 1_{X_i}, & (i,j)=(r,s),\end{cases} \qquad \sum_{i=1}^{n}\sum_{j=1}^{n_i} p_{ij}e_{ij}=1_X.$$

We define F by sending a matrix Xm over (K, M, i) to the representation V of (Q, d) whose vector space V_i at vertex i is k^{n_i}, and if $a\in A_{ij}$ then in the representation V it is given by a matrix with entries $a_{st}=a(e_{is}mp_{jt})$. If $\theta\colon Xm\to X'm'$ is a morphism in $\mathrm{Mat}(K, M, i)$, then $F(\theta)$ is the morphism in

$R(Q, d)$ given by linear maps ω_i: $V_i \to V_i'$ for each i, defined by $(\omega_i)_{st} 1_{X_i} = \pi_S(e_{is}\theta p_{it}')$ and φ: $V_i \to V_j'$ for $\varphi \in \Phi_{ij}$ defined by $\varphi_{st} = \varphi(\pi_J(e_{is}\theta p_{jt}'))$, where π_S and π_J are the projections of K onto S and J. It is tedious but not essentially difficult to verify that F is an equivalence.

3.2. Reductions of differential biquivers

The reduction algorithm for bimodule problems now gives the following operations on a differential biquiver (Q, d).

Regularization. If there is a solid arrow a in Q with $d(a)$ being a nonzero linear combination of dotted arrows, say $d(a) = \sum \lambda_i \varphi_i$ with the $\lambda_i \neq 0$ and φ_1 being maximal amongst the φ_i with respect to the triangularity, then construct (Q', d') with $R(Q, d) \cong R(Q', d')$ as follows. The quiver Q' is the same as Q but with a and φ_1 deleted, while d' is the same as the restriction of d, but whenever $d(x)$ has a term which involves φ_1, this occurrence is replaced by

$$-\frac{1}{\lambda_1} \sum_{i \neq 1} \lambda_i \varphi_i,$$

and whenever a term involves a, it is deleted.

Edge reduction. If there is a solid arrow a: $i \to j$ with differential zero, and which is an edge, i.e. $i \neq j$, one can again construct (Q', d'). This time the construction is slightly more complicated, so we shall only give an example. We hope that the interested reader can obtain the reduction either by using bimodule problems, or by examining the references [15, 11, 14]. For the example (Q, d) we gave before, after reducing the edge a, one obtains the situation of Fig. 8 with $d'(\psi_1) = \beta\psi_2$, $d'(\xi_2) = \alpha\xi_1$, $d'(\zeta_1) = \xi_1\varphi$,

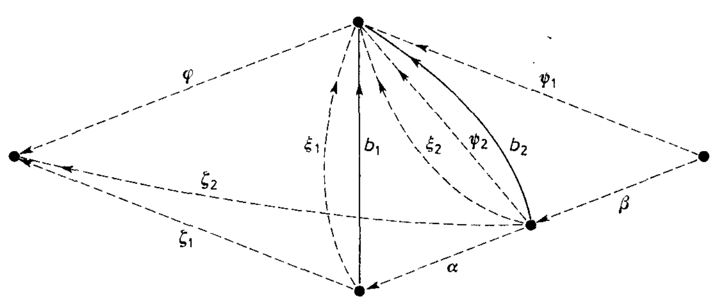

Fig. 8

$d'(\zeta_2) = \xi_2\varphi + \alpha\zeta_1$, $d'(b_1) = -\xi_1$ and $d'(b_2) = \psi_2 - \xi_2 - \alpha b_1$. (Now one can regularize b_1, and then b_2 to obtain a differential biquiver with no solid arrows, so the vertices correspond to the indecomposable representations of (Q, d).)

Loop reduction. If there is a solid arrow a: $i \to i$ with differential zero and which is a loop, then there is a reduction of the bimodule problem which would correspond to a differential biquiver but for the fact that it would have infinitely many vertices—corresponding to the different possible Jordan blocks for a in any representation. By restricting to a finite subset, however, one obtains (Q', d') and a fully faithful functor from $R(Q', d')$ to the full subcategory of $R(Q, d)$ on the representations in which a only involves this finite set of Jordan blocks. See §5.2 for an example of this reduction.

4. Bocses

We first recall the notion of *tensor products* over categories. If M is an A-B-bimodule and N a B-C-bimodule, then there is an A-C-bimodule $M \otimes_B N$ solving the usual universal problem. If $\theta\colon A \to B$ is a functor and M is an A-C-bimodule we denote $B \otimes_A M$ by ${}^B M$, and similarly on the right-hand side.

A pair (A, V) is called a *bocs* provided that A is a category and V is an A-coalgebra, that is, V is an A-A-bimodule equipped with A-A-bimodule maps $\varepsilon\colon V \to A$ (the counit) and $\mu\colon V \to V \otimes_A V$ (the comultiplication), satisfying the usual counitary and coassociativity laws $\mu \circ (1 \otimes \varepsilon) = \mu \circ (\varepsilon \otimes 1) = \mathrm{id}_V$ and $\mu \circ (1 \otimes \mu) = \mu \circ (\mu \otimes 1)$. The kernel of ε is denoted by $\bar{V}$.

The *principal bocs* (A, A) is defined by $\varepsilon = \mathrm{id}_A$ and μ the isomorphism $A \to A \otimes_A A$.

The category $R(A, V)$ of *representations* of the bocs (A, V) has as objects the finite-dimensional left A-modules, and as morphisms $F \to G$ the A-module maps $f\colon V \otimes_A F \to G$. If $g\colon V \otimes_A G \to H$ is another morphism, the product fg is defined by

$$V \otimes_A F \xrightarrow{\mu \otimes 1} V \otimes_A V \otimes_A F \xrightarrow{1 \otimes f} V \otimes_A G \xrightarrow{g} H.$$

Clearly $R(A, A)$ is just $\operatorname{mod} A$, the category of finite-dimensional left A-modules.

A *morphism* of bocses $(\theta_0, \theta_1)\colon (A, V) \to (B, W)$ is defined by a functor $\theta_0\colon A \to B$ and an A-A-bimodule map $\theta\colon V \to {}_A W_A$, where ${}_A W_A$ is the A-A-bimodule obtained from W by restricting on each side using θ_0, and which preserves the coalgebra structure in the obvious way. Such a morphism induces a functor

$$(\theta_0, \theta_1)^*\colon R(B, W) \to R(A, V)$$

whose effect on objects is restriction via θ_0.

4.1. Operations on bocses

If (A, V) is a bocs and $\theta\colon A \to B$ is a functor, one can define a B-coalgebra structure on ${}^B V^B = B \otimes_A V \otimes_A B$, and obtain a bocs $(B, {}^B V^B)$, and a morphism $\theta_I = (\theta, \theta_1)\colon (A, V) \to (B, {}^B V^B)$. The *raison d'être* for bocses is

LEMMA [2]. *θ_I^* is fully faithful.*

In fact the proof of this is rather easy; see [3]. Let us stress why this is so important. Given a bocs (A, V), the idea is to simplify it by constructing a category B and functor θ: $A \to B$ such that res_θ: $\mathrm{mod}\,B \to \mathrm{mod}\,A$ is dense. In this case θ_I^* is an equivalence, and we can replace (A, V) by $(B, {}^BV^B)$.

In order to construct such functors one uses pushouts. If A' is a subcategory of A and θ': $A' \to B'$ is a functor, there is a pushout diagram

$$(*) \qquad \begin{array}{ccc} A' & \xrightarrow{\theta'} & B' \\ \downarrow & & \downarrow \\ A & \xrightarrow{\theta} & A \end{array}$$

Now if $\mathrm{res}_{\theta'}$: $\mathrm{mod}\,B' \to \mathrm{mod}\,A'$ is dense, then the pushout property ensures that so is res_θ. For example if A' is of finite representation type one can let θ': $A' \to B'$ be a semisimple approximation [1]: for categories this means that if $M_1, \ldots, M_n$ are a complete set of representatives of the indecomposable finite-dimensional right A'-modules then B' is a product of n copies of $\mathrm{mod}\,k$, and $\theta' = (M_1, \ldots, M_n)$.

4.2. Freely generated categories and bimodules

In general, pushouts of the form $(*)$ are rather difficult to compute, and there is also the even more difficult problem of computing ${}^BV^B$. However, when A is freely generated over A' and $\bar{V}$ is freely generated as an A-A-bimodule, the calculations are possible. Let us first make these notions precise.

Definition. An A-A-bimodule M is *freely generated by* $m_1, \ldots, m_n$ with $m_i \in M(X_i, Y_i)$ provided that the natural map

$$\bigoplus_{i=1}^{n} A(-, X_i) \otimes_k A(Y_i, -) \to M$$

is an isomorphism. A category A is *freely generated by* $a_1, \ldots, a_n$ *over* A', a subcategory of A with the same objects, provided that the a_i freely generate an A'-A'-subbimodule T of A, and the natural functor from $T^{\otimes}$ to A is an isomorphism, where $T^{\otimes}$ is the tensor category of T over A' defined by

$$T^{\otimes} = \bigoplus_{n=0}^{\infty} \underbrace{T \otimes_{A'} T \otimes_{A'} \cdots \otimes_{A'} T}_{n}.$$

In this case B is freely generated over B' by $\theta(a_1), \ldots, \theta(a_n)$ and one can make computations with ${}^BV^B$. For example

Lemma [3, Lemma 4.1]. *If A is freely generated over A', the counit ε is onto, the kernel $\bar{V}$ of ε is projective as an A-A-bimodule and $J' = \ker(B' \otimes_{A'} B' \to B')$ is a projective B'-B'-bimodule, then the kernel of the induced bocs ${}^BV^B$ is isomorphic to ${}^BJ'^B \oplus {}^B\bar{V}^B$.*

4.3. The bocs of a differential biquiver
as follows. Let A' be a Krull–Schmidt category whose indecomposable objects correspond to the vertices of Q, and with $A'(X, X) = k1_X$ and $A'(X, Y) = 0$ for X and Y distinct vertices. Let A be the category freely generated over A' by elements $a \in A(X, Y)$ corresponding to the solid arrows a: $X \to Y$ in Q, let $\bar{V}$ be the A-A-bimodule freely generated by elements $\varphi \in \bar{V}(X, Y)$ for dotted arrows φ: $X \dashrightarrow Y$, and let $V = A \oplus \bar{V}$ as an A'-A-bimodule. We turn (A, V) into a bocs by defining the left A-module structure on V via

$$a \cdot (b, v) = (ab, d(a)b + av)$$

for a solid arrow a: $X \to Y$, $b \in A(Y, Z)$ and $v \in \bar{V}(Y, Z)$ (where $d(a)$ can be regarded as an element of $\bar{V}$), the counit ε via $\varepsilon(a, v) = a$, and comultiplication μ via

$$\mu(a, 0) = (1_X, 0) \otimes (a, 0)$$

for $a \in A(X, Y)$, and

$$\mu(0, \varphi) = (0, \varphi) \otimes (1_Y, 0) + (1_X, 0) \otimes (0, \varphi) + d(\varphi)$$

where φ: $X \dashrightarrow Y$ and $d(\varphi)$ can be regarded as an element of $\bar{V} \otimes_A \bar{V}$.

The bocses that one obtains this way are precisely the *normal free triangular linear bocses* [14]. Of course the operations of regularization, edge reduction and loop reduction carry over quite naturally to such bocses. For example, if a is a solid edge with differential zero, let $A'' = \langle A', a \rangle$ and use the semisimple approximation $A'' \to B''$.

Let us finally observe how to phrase the representation theory of an algebra using bocses. In §2.2, we have shown that an algebra gives rise to a special type of bimodule problem. In §3.1 we have seen that bimodule problems lead to differential biquivers, and now we have shown how to phrase differential biquivers as bocses.

5. Applications

5.1. Problems of finite representation type

Given a matrix problem, say a differential biquiver (Q, d), which is known to be of finite representation type, that is, $R(Q, d)$ contains only finitely many nonisomorphic indecomposables, the triangularity ensures that there is always a solid arrow whose differential involves only dotted arrows. If the differential is nonzero, the arrow can be regularized; if the differential is zero, then the arrow is an edge—otherwise there are infinitely many nonisomorphic indecomposables—and it can be reduced. Since each vertex corresponds to an indecomposable representation, and each edge reduction adds a new vertex, one can do no more than

$$\text{Number of indecomposables} - \text{Initial number of vertices}$$

edge reductions. Since also each regularization reduces the number of arrows in the biquiver, this procedure must terminate, and it can only do so if eventually there are no solid arrows in the biquiver. However, at this stage, the vertices correspond to the indecomposable representations, the dotted arrows correspond to bases of the spaces of noninvertible morphisms between them, and the differentials give their multiplication table, so $R(Q, d)$ is completely described (up to equivalence). Of course it is also easy to explicitly compute the representations of the original differential biquiver by reversing this process.

However, appealing that this might be, a sample computation for the quiver $\mathbf{E}_6$

$$\begin{tikzcd} & & 6 \arrow[d, "e"] & & \\ 1 \arrow[r, "a"] & 2 \arrow[r, "b"] & 3 \arrow[r, "c"] & 4 \arrow[r, "d"] & 5 \end{tikzcd}$$

involves 30 edge reductions (of course), 225 regularizations, and involves biquivers with up to 621 arrows. For $\mathbf{E}_7$ it takes 56 edge reductions and 842 regularizations on biquivers with up to 2146 arrows. Although performing the operations in a different sequence will change these numbers, we do not expect that they can be reduced substantially.

Incidentally, given an arbitrary differential biquiver (Q, d) of finite representation type, when this procedure is reversed in order to determine the indecomposable representations of (Q, d) it turns out that they are described by trees (and hence by matrices only involving the numbers 0 and 1). For example (and to indicate what we mean), the maximal sincere indecomposable representation of $\mathbf{E}_6$ is given by the tree

$$\begin{tikzcd} 1 \arrow[d, "a"'] & 6 \arrow[r, "e"] \arrow[d, "e"'] & 3 & 2 \arrow[d, "b"] \\ 2 \arrow[r, "b"] & 3 \arrow[d, "c"'] & 6 \arrow[l, "e"'] \arrow[r, "e"] & 3 \arrow[d, "c"] \\ & 4 \arrow[r, "d"] & 5 & 4 \end{tikzcd}$$

where a vertex labelled with i corresponds to a basis element of the vector space at vertex i of the representation of $\mathbf{E}_6$, and an arrow labelled with x corresponds to an identity in the matrix for x; the other elements being zero.

5.2. Computing representations of a fixed size

In this section we want to give an example to indicate to what extent the reductions can be used to classify the representations of a given size for a matrix problem of arbitrary representation type. We shall use differential biquivers and consider the three-dimensional representations of the free associative algebra $k\langle x, y\rangle$, that is, the differential biquiver (Q, d) of Fig. 9

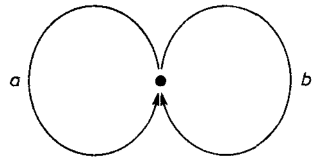

Fig. 9

with $d = 0$. We must begin by reducing a loop, and there are the following possibilities for the Jordan normal form of b:

(1) $J_3(\lambda)$,
(2) $J_2(\lambda) \oplus J_1(\lambda)$,
(3) $J_1(\lambda)^3$,
(4) $J_2(\lambda) \oplus J_1(\mu)$ ($\lambda \neq \mu$),
(5) $J_1(\lambda)^2 \oplus J_1(\mu)$ ($\lambda \neq \mu$),
(6) $J_1(\lambda) \oplus J_1(\mu) \oplus J_1(\nu)$ (λ, μ, ν distinct).

One must divide into cases, and we shall consider case (2). After reducing b, one obtains the differential biquiver (Q', d') of Fig. 10 with differential given (for the

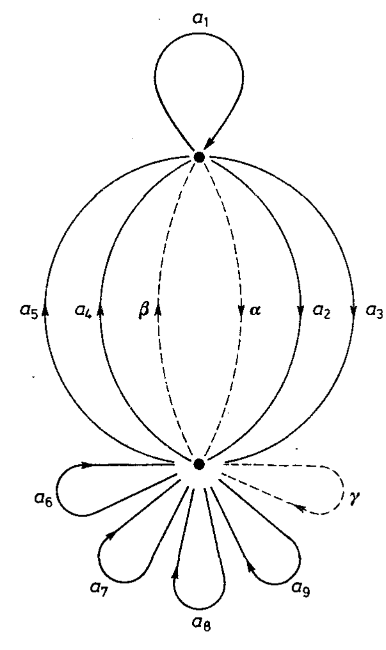

Fig. 10

solid arrows) by

$$\begin{aligned}
d'(a_1) &= \alpha a_4 - a_3\beta, & d'(a_6) &= -a_4\alpha - a_7\gamma,\\
d'(a_2) &= \alpha a_6 - a_1\alpha - a_3\gamma, & d'(a_7) &= 0,\\
d'(a_3) &= \alpha a_7, & d'(a_8) &= \beta a_2 + \gamma a_6 - a_5\alpha - a_9\gamma,\\
d'(a_4) &= -a_7\beta, & d'(a_9) &= \beta a_3 + \gamma a_7.\\
d'(a_5) &= \beta a_1 + \gamma a_4 - a_9\beta,
\end{aligned}$$

The vertices of this biquiver correspond to the different types of Jordan blocks: the upper one being $J_1(\lambda)$ and the lower $J_2(\lambda)$; and a representation of this which is n-dimensional at the top, and m-dimensional at the bottom corresponds to a representation of (Q, d) in which a involves n copies of $J_1(\lambda)$ and m of $J_2(\lambda)$. Thus we are concerned with the representations of this which are one-dimensional at each vertex.

Next we reduce the loop a_7. In a representation of the appropriate dimension this is a 1×1 matrix, so when we perform the reduction we only need to consider the Jordan block $J_1(\mu)$. In this case the effect of the reduction is to delete the arrow a_7 and replace each occurrence of a_7 in a differential by μ. Thus we now have

$$\begin{aligned}
d'(a_3) &= \mu\alpha, & d'(a_6) &= -a_4\alpha - \mu\gamma,\\
d'(a_4) &= -\mu\beta, & d'(a_9) &= \beta a_3 + \mu\gamma.
\end{aligned}$$

Next one picks a solid arrow whose differential only involves dotted arrows, for example a_4. We need to divide into cases according to whether or not its differential is zero; in this case, whether $\mu = 0$ or $\mu \neq 0$. In the first case one would reduce a_4, but we shall just consider the latter, so we regularize a_4, which deletes a_4 and β. Also we regularize a_3, deleting a_3 and α. We obtain the differential biquiver with arrows γ, a_1, a_2, a_5, a_6, a_8 and a_9, and with differentials

$$\begin{aligned}
d'(a_1) &= 0, & d'(a_6) &= -\mu\gamma,\\
d'(a_2) &= 0, & d'(a_8) &= \gamma a_6 - a_9\gamma,\\
d'(a_5) &= 0, & d'(a_9) &= \mu\gamma.
\end{aligned}$$

Now regularizing a_6 leaves only the solid arrows a_1, a_2, a_5, a_8 and a_9, all with differential zero. Reducing a_1, a_8 and a_9 in the same way as a_7, introduces new parameters ν, ξ and ϱ, and leaves just a_2 and a_5. Now reduce the edge a_5, to obtain the differential biquiver of Fig. 11 with differential

$$\begin{aligned}
d(\delta) = d(\varepsilon) = d(b_4) &= 0, & d(b_2) &= b_4\delta,\\
d(b_1) &= b_3\delta - \varepsilon b_2, & d(b_3) &= -\varepsilon b_4.
\end{aligned}$$

Now a representation which has dimension n_i at vertex i corresponds to a representation of the starting biquiver of dimension $n_1 + n_2$ at the top and $n_2 + n_3$ at the bottom, so the possibilities are $\underline{n} = (1, 0, 1)$ or $\underline{n} = (0, 1, 0)$.

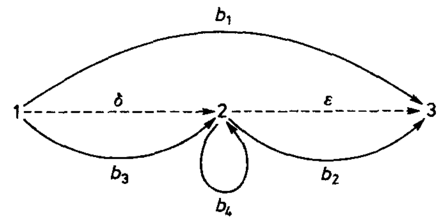

Fig. 11

Again one divides into cases, in each of which one deletes the vertices on which $\underline{n}$ is zero. Continuing only with the second case, one obtains the differential biquiver consisting only of the loop b_4, and after reducing this with parameter σ, it has no solid arrows. Thus we are finished: there is a unique representation of this with the required dimension, namely the simple representation. Backtracking one obtains an indecomposable three-dimensional representation of $k\langle x, y\rangle$ given by the matrices

$$x = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 1 & \lambda \end{bmatrix} \quad \text{and} \quad y = \begin{bmatrix} \nu & \sigma & 0 \\ 0 & 0 & \mu \\ 1 & \xi & \varrho \end{bmatrix}$$

for any λ, μ, ν, ξ, ϱ, σ with $\mu \neq 0$.

The general algorithm. For the inductive step we are given the following data:

(1) a locally closed subset X of affine m-space, i.e. one defined by polynomial equalities and inequalities,

(2) a differential biquiver (Q, d) over the field $k(x_1, \ldots, x_m)$ of rational functions, which is defined at every element of X,

(3) the dimension vector of the representations we must classify, that is, a vector specifying the dimensions of the spaces V_i.

Step 1. Delete any vertex where the dimension vector is zero. This ensures that the procedure must terminate, since provided that one is considering representations with a fixed dimension vector, which is nonzero at each vertex, the operations all reduce the *norm* of the representation: the sum over the solid arrows of the product of the components of the dimension vector at each end (or for bimodule problems, the dimension of $M(X, X)$).

Step 2. Let a be a solid arrow whose differential only involves dotted arrows, say

$$d(a) = \lambda_1 \varphi_1 + \lambda_2 \varphi_2 + \ldots + \lambda_n \varphi_n$$

and the φ_i are arranged in decreasing order relative to the triangularity of the differential. Divide X into the subsets

$$X_0 = \{x \in X \mid \lambda_1(x) = \ldots = \lambda_n(x) = 0\},$$

$$X_i = \{x \in X \mid \lambda_1(x) = \ldots = \lambda_{i-1}(x) = 0,\ \lambda_i(x) \neq 0\} \quad (1 \leqslant i \leqslant n).$$

For the $x \in X_i$ $(i \neq 0)$, regularize a and use the induction. Thus we may replace X by X_0 and suppose that $d(a) = 0$.

Step 3. Use edge or loop reduction on a. Divide into cases according to the possible dimension vectors and/or Jordan normal form structure as in the example. In the case of loop reduction, replace X by $X \times \mathbf{A}^m$ if m parameters have been introduced.

Remarks. (1) The output of this procedure is a complete list of representations of the given dimension vector, each one specified by matrices whose entries are 0, 1, or a parameter, with the parameters running through a locally closed subset of an affine m-space. Of course it will not determine whether such subsets are nonempty, let alone give parametrizations for them.

(2) There are some problems of uniqueness. For example in case (6) of the very first reduction, the obvious symmetry of exchanging λ and μ will be lost later on. This may (not very prettily) be avoided by choosing a total ordering on the field (arbitrarily) and demanding $\lambda < \mu < \nu$.

5.3. Drozd's Tame and Wild Theorem

Let k be an algebraically closed field. We recall some definitions.

DEFINITION. A finite-dimensional algebra Λ is *wild* if there is a finitely generated Λ-$k\langle x, y\rangle$-bimodule M which is free as a right $k\langle x, y\rangle$-module and such that the functor $F(-) = M \otimes_{k\langle x,y\rangle} -$ from the category of finite-dimensional $k\langle x, y\rangle$-modules to $\mathrm{mod}(\Lambda)$ preserves indecomposability and isomorphism classes.

DEFINITION. A finite-dimensional algebra Λ is *tame* if, for each $d > 0$, there are a finite number of finitely generated Λ-$k[x]$-bimodules M_i which are free as right $k[x]$-modules such that every indecomposable Λ-module of dimension d is isomorphic to $M_i \otimes_{k[x]} N$ for some i and some simple $k[x]$-module N.

If Λ is tame then for each $d > 0$, in the affine variety $\mathrm{mod}_d(\Lambda)$ of Λ-module structures on k^d, there is a one-dimensional subvariety which meets every isomorphism class of indecomposable modules. If Λ is wild, then for each n, there is some d such that in $\mathrm{mod}_d(\Lambda)$ there is a locally closed subset of dimension n of nonisomorphic indecomposable modules. Thus an algebra cannot be both tame and wild. In [7] Drozd proved the remarkable fact that

THEOREM (Drozd). *Every finite-dimensional algebra is either tame or wild.*

The proof uses a new type of bocs, which Drozd calls *almost free*. In [3] we have repeated the argument, using a variation on this called a *layered bocs*. We outline how the proof goes in [3].

DEFINITION. Let (A, V) be a bocs. We say that (A, V) is *layered* if there is a collection $(A'; \omega; a_1, \ldots, a_n; v_1, \ldots, v_m)$ where:

(1) A' is a subcategory of A on the same objects, and it is a *minimal category*: a skeletal category equivalent to $\mathrm{proj}(R)$ where R is a finite product of k-algebras, each of which is either k or of the form $k[x, f(x)^{-1}]$ for some nonzero polynomial f.

(2) ω is a *grouplike*: it is an A'-A'-bimodule map $A' \to {}_{A'}V_{A'}$ inducing a bocs morphism (i, ω): $(A', A') \to (A, V)$ where i is the inclusion of A' in A.

(3) ω is a *reflector* which means that $(i, \omega)^*$ reflects isomorphisms, that is, if f is a morphism in $R(A, V)$, and $(i, \omega)^*$ (f) is an isomorphism, then so is f.

(4) The a_i (respectively v_j) are *indecomposable elements* of A (respectively $\bar{V}$): they are elements of $A(X_i, Y_i)$ (respectively $\bar{V}(Z_j, W_j)$) for indecomposable objects X_i, Y_i (respectively Z_j, W_j) of A.

(5) The v_j freely generate $\bar{V}$ as an A-A-bimodule.

(6) The a_i freely generate A over A'.

(7) Define the *differential* d: ${}_{A'}A_{A'} \to {}_{A'}\bar{V}_{A'}$ by $d(a) = a\omega(1_Y) - \omega(1_X)a$ for $a \in A(X, Y)$. We demand that for each i, $d(a_i)$ is contained in the A_i-A_i-subbimodule of $\bar{V}$ generated by all the v_j's, where $A_i = \langle A', a_1, \ldots, a_{i-1} \rangle$.

To set up an induction using the norm (as in the last section), suppose that the bocs (A, V) is not wild (in an appropriate sense). Consider a_1. If it has differential zero and it is an edge, then the edge reduction operation still works provided that $A'(X_1, X_1)$ and $A'(Y_1, Y_1)$ are both equal to k. However, there is the following

PROPOSITION (Drozd). *(A, V) is wild in the following two cases:*

(1) *$d(a_1) = 0$, and either $A'(X_1, X_1) \neq k$ or $A'(Y_1, Y_1) \neq k$; and*

(2) *$A'(X_1, X_1) \neq k$ and $A'(Y_1, Y_1) \neq k$ and $d(a_1) = r \cdot v_1$, for some noninvertible r in $R = A'(X_1, X_1) \otimes_k A'(Y, Y)^{\mathrm{op}}$.*

If a_1 has differential zero and it is a loop, then by the proposition, $A'(X_1, X_1) = k$, so one can replace A' by $\langle A', a_1 \rangle$. Now suppose that a_1 has nonzero differential. If its differential was a k-linear combination of the v_i, then it would be possible to regularize it. However, this need not be the case, since it is in general an R-linear combination of them. For example, if $A'(X_1, X_1) = k[x]$ and $v_1 \in \bar{V}(X_1, Y_1)$, then it is possible that $d(a_1) = xv_1$.

If V is a vector space and x an automorphism, then by Fitting's Lemma there is a decomposition $V = V_0 \oplus V_1$ with x acting as an automorphism on V_0, and nilpotent on V_1. More generally, there is

Partial loop reduction. Let X be an indecomposable object in A, $A'(X, X) = k[x]$, and r a positive integer. Let A'' be the category obtained from A' by localizing x, so $A''(X, X) = k[x, x^{-1}]$, let $B' = A'' \times (\text{mod}\, k)^r$, and let θ': $A' \to B'$ be the functor whose first component is the inclusion $A' \to A''$, and whose ith component to mod$k$ is the i-dimensional indecomposable representation of A' in which x is a nilpotent Jordan block. If θ: $A \to B$ is the pushout of θ', then the induced bocs $(B, {}^B V^B)$ is layered.

After applying this reduction one can identify a_1 and v_1 in the new bocs and still $d(a_1) = xv_1$. But now x is invertible in $B'(X_1, X_1)$, so one can replace v_1 by xv_1 and then regularize. In general, one must be more sophisticated and use part (2) of the proposition, but essentially the same procedure is possible. In this way one obtains

THEOREM. *If (A, V) is a layered bocs which is not wild, and $d > 0$, then there are categories $B_1, \ldots, B_n$ and functors θ_i: $A \to B_i$ such that*

(1) *the bocses (B_i, W_i) induced by the θ_i are minimal, that is, they are layered with a collection of the form $(B; \omega; \ ; w_1, \ldots, w_m)$; and*

(2) *every representation of (A, V) with dimension at most d is isomorphia to $(\theta_i)^*_?(N)$ for some i and some representation N of (B_i, W_i).*

From this result, by observing that minimal bocses have a rather simple representation theory, not only can one deduce the Tame and Wild Theorem, but also there is control on the maps, and one obtains

THEOREM [3]. *If Λ is tame, then*

(1) *for all $d > 0$, all but a finite number of isomorphism classes of indecomposable Λ-modules of dimension d are isomorphic to their Auslander–Reiten translates;*

(2) *for all $d > 0$, all but a finite number of isomorphism classes of indecomposable Λ-modules of dimension d lie in homogeneous tubes* [13, §3.1];

(3) *there are at most countably many other components in the Auslander–Reiten quiver; and*

(4) *every Auslander–Reiten component contains only finitely many isomorphism classes of indecomposables of each dimension.*

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