# MODULES OF FINITE LENGTH OVER THEIR ENDOMORPHISM RINGS

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Given a ring R (associative, with 1) one can define the *endolength* of an R-module M to be its length when it is regarded in the natural way as an $\mathrm{End}_R(M)$-module, and thus one can consider the class of modules of finite endolength. The aim of this paper is to show that this is a useful concept. Briefly, the contents are as follows. In §§1-3 we cover some background machinery, in §§4-6 we discuss the modules of finite endolength for a general ring, and in §§7-9 we show how these modules control the behaviour of the finite length modules for noetherian and artin algebras. Although much of this paper has a survey nature, there are some new results proved here, the main ones being the characterization of the pure-injective modules which occur as the source of a left almost split map in §2, the character theory for modules of finite endolength in §5, and the characterization of the artin algebras with an indecomposable module of infinite length and finite endolength (a *generic* module) proved in §§8-9.

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## 1 THE FUNCTOR CATEGORY

If R is a ring, we denote by R-Mod the category of left R-modules, and by mod-R be the category of finitely presented (f.p.) right R-modules. We denote by D(R) the category of additive functors from mod-R to $\mathbb{Z}$-Mod (the category of abelian groups). This category has a very rich structure, and is an invaluable tool for the study of R-modules. First of all, it is an abelian category, with kernels, images and cokernels computed "pointwise". For example, a morphism of functors $f:\mathcal{F}\longrightarrow\mathcal{G}$ is by definition a natural transformation, so that for each f.p. module X there is a homomorphism $f_X:\mathcal{F}(X)\longrightarrow\mathcal{G}(X)$ of abelian groups. The kernel of f is then the functor which is defined on objects by $(\mathrm{Ker}\ f)(X) = \mathrm{Ker}(f_X)$, and similarly for the image and cokernel of f. It follows that a sequence $\mathcal{F}\longrightarrow\mathcal{G}\longrightarrow\mathcal{H}$ of functors is exact if and only if the sequence $\mathcal{F}(X)\longrightarrow\mathcal{G}(X)\longrightarrow\mathcal{H}(X)$ is exact for all f.p. modules X.

We shall also need to use the fact that D(R) is a Grothendieck category, which as far as we are concerned means that it has injective envelopes [G], and in particular, that we can do homological algebra in D(R). There are several very important classes of functors which we now list.

**1.1** The *representable functors* are the functors $(X,-) = \mathrm{Hom}_R(X,-)$ with X a f.p. right R-module. By Yoneda's lemma, $\mathrm{Hom}_{D(R)}((X,-),\mathcal{F})$ is isomorphic to $\mathcal{F}(X)$ for any functor $\mathcal{F}$, and it follows that $(X,-)$ is a projective object in D(R).

Given a functor $\mathcal{F}$, a family of finitely presented modules $(X_\lambda)_{\lambda\in\Lambda}$, and elements $f_\lambda \in \mathcal{F}(X_\lambda)$, one can consider the smallest subfunctor $\mathcal{G}$ of $\mathcal{F}$ such that $f_\lambda \in \mathcal{G}(X_\lambda)$ for all $\lambda\in\Lambda$. This is the subfunctor of $\mathcal{F}$ *generated by* the elements $f_\lambda$. Now the $f_\lambda$ determine maps $(X_\lambda,-)\longrightarrow\mathcal{F}$, and the functor $(X_\lambda,-)$ is generated by the identity endomorphism of $X_\lambda$, so that $\mathcal{G}$ is the image of the map $\coprod_{\lambda\in\Lambda} (X_\lambda,-)\longrightarrow\mathcal{F}$. Defining the notion of a *finitely generated* (f.g.) functor in the obvious way, the isomorphism

$$(X_1,-) \oplus \dots \oplus (X_n,-) \cong (X_1\oplus\dots\oplus X_n,-)$$

shows that a functor is finitely generated if and only if it is a quotient of a representable functor.

Finally let us observe that the representable functors are precisely the f.g. projective functors. Namely, any f.g. projective functor $\mathcal{F}$ is a summand of a representable functor $(X,-)$, and then the isomorphism $\mathrm{End}_{D(R)}((X,-)) \cong \mathrm{End}_R(X)^{op}$ shows that $\mathcal{F} \cong (Y,-)$ with Y a summand of X (so that it is f.p.).

**1.2** A functor $\mathcal{F}$ is said to be *coherent* if it is a quotient of a representable functor by a finitely generated subfunctor, or in other words, if it is finitely presented. If $\mathcal{F}$ has projective presentation

$$(Y,-) \xrightarrow{f} (X,-) \longrightarrow \mathcal{F} \longrightarrow 0, \qquad (\dagger)$$

then f is determined by a homomorphism $\alpha: X \longrightarrow Y$, and if Z is the cokernel of $\alpha$, then $\mathcal{F}$ actually has a projective resolution

$$0 \longrightarrow (Z,-) \longrightarrow (Y,-) \xrightarrow{f} (X,-) \longrightarrow \mathcal{F} \longrightarrow 0. \qquad (\ddagger)$$

The name "coherent" (rather than "finitely presented") is used because of the following well-known and important property, which implies that the category of coherent functors is closed under kernels, cokernels, images and extensions.

Lemma. A f.g. subfunctor of a coherent functor is coherent.

*Proof.* It suffices to prove this for representable functors. Now a f.g. subfunctor of $(X,-)$ is the image of a map f as in $(\dagger)$, and so the projective resolution $(\ddagger)$ shows that Im(f) is coherent.

**1.3** The simple functors in D(R) have been determined by Auslander [A3]. If X is a f.p. right R-module and J is a left ideal in $\mathrm{End}_R(X)$ one can define a subfunctor $(X,-)_J$ of $(X,-)$ via

$$(X,Y)_J = \{\theta \in \mathrm{Hom}_R(X,Y) \mid \phi \circ \theta \in J \text{ for all } \phi \in \mathrm{Hom}_R(Y,X)\}$$

If $\mathfrak{m}$ is a maximal left ideal in $\mathrm{End}_R(X)$ we set $\mathcal{S}_{X,\mathfrak{m}}=(X,-)/(X,-)_{\mathfrak{m}}$.

LEMMA. The functors $\mathcal{S}_{X,\mathfrak{m}}$ are simple, and every simple functor in D(R) is isomorphic to some $\mathcal{S}_{X,\mathfrak{m}}$.

*Proof*. If $\mathcal{F}$ is a subfunctor of $(X,-)$, then $J = \mathcal{F}(X)$ is a left ideal in $\mathrm{End}_R(X)$, and the inclusion $\mathcal{F} \subseteq (X,-)_J$ follows from the definition. If in addition $\mathcal{F}$ is a proper subfunctor of $(X,-)$ then J is a proper ideal, for if $1_X \in \mathcal{F}(X)$ then certainly $\mathcal{F} = (X,-)$. It follows that the maximal subfunctors of $(X,-)$ are precisely the functors $(X,-)_{\mathfrak{m}}$ with $\mathfrak{m}$ a maximal left ideal in $\mathrm{End}_R(X)$. Thus $\mathcal{S}_{X,\mathfrak{m}}$ is simple. Finally, one only has to observe that, by Yoneda's lemma, every simple functor is a quotient of a representable functor.

**1.4** If M is a left R-module, the tensor product functor $-\otimes M = -\otimes_R M$ is right exact. Conversely, if $\mathcal{F}$ is any right exact functor then $\mathcal{F} \cong -\otimes\mathcal{F}(R)$ where $\mathcal{F}(R)$ has its natural structure as a left R-module. It is easy to see that $\mathrm{Hom}_{D(R)}(-\otimes M, -\otimes N) \cong \mathrm{Hom}_R(M,N)$. The next property will be needed later.

LEMMA. $\mathrm{Ext}^1_{D(R)}(\mathcal{F}, -\otimes M) = 0$ for $\mathcal{F}$ coherent.

*Proof*. The functor $\mathcal{F}$ has a projective resolution (‡), so one can compute $\mathrm{Ext}^1$ as the cohomology of the complex

$$\mathrm{Hom}((X,-),-\otimes M) \longrightarrow \mathrm{Hom}((Y,-),-\otimes M) \longrightarrow \mathrm{Hom}((Z,-),-\otimes M).$$

This is, however, isomorphic to $X\otimes M \longrightarrow Y\otimes M \longrightarrow Z\otimes M$, so it is exact.

**1.5** An exact sequence $\xi : 0 \longrightarrow M \xrightarrow{\alpha} N \xrightarrow{\beta} L \longrightarrow 0$ of left R-modules is said to be *pure exact* (and $\alpha$ a *pure mono*, and $\mathrm{Im}(\alpha)$ a *pure submodule* of M) provided that the tensor product sequence

$$0 \longrightarrow X\otimes_R M \longrightarrow X\otimes_R N \longrightarrow X\otimes_R L \longrightarrow 0$$

is exact for every right R-module X. Of course it is always right exact. Since tensor products commute with direct limits, direct limits are exact, and every module is a direct limit of f.p. modules, it suffices for this to hold for f.p. modules X. In other words, $\xi$ is pure exact if and only if the sequence of functors $0 \longrightarrow -\otimes M \longrightarrow -\otimes N \longrightarrow -\otimes L \longrightarrow 0$ is exact in D(R).

**1.6** A left R-module M is said to be *pure-injective* if every pure exact sequence whose first term is M splits. This notion has many equivalents, for example that of algebraic compactness. For our purposes, however, we shall only need the following characterization [GJ2].

LEMMA. Up to isomorphism, the injectives in D(R) are the functors $-\otimes M$ with M pure-injective.

*Proof.* Let $\mathcal{F}$ be an injective functor. We show first that $\mathcal{F}$ is right exact. Namely, an exact sequence $X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ of f.p. right R-modules gives an exact sequence $0 \longrightarrow (Z,-) \longrightarrow (Y,-) \longrightarrow (X,-)$ of functors. Applying the exact functor $\mathrm{Hom}(-,\mathcal{F})$ and using Yoneda's lemma one sees that the sequence $\mathcal{F}(X) \longrightarrow \mathcal{F}(Y) \longrightarrow \mathcal{F}(Z) \longrightarrow 0$ is exact, as required. Thus $\mathcal{F} \cong -\otimes M$ where $M = \mathcal{F}(R)$. Now given any pure mono $M \longrightarrow N$, the morphism $-\otimes M \longrightarrow -\otimes N$ is mono, so split, so the map $M \longrightarrow N$ is split. Thus M is pure-injective.

Conversely, suppose that M is pure-injective. Since the category D(R) has injective envelopes, the functor $-\otimes M$ can be embedded in an injective functor $-\otimes N$. This gives a pure embedding $M \longrightarrow N$. Since M is pure-injective, M is a summand of N, so $-\otimes M$ is a summand of $-\otimes N$, and hence injective.

**1.7** Next we introduce another tool for studying modules, the subgroups of "finite definition" of a module. These were introduced by Gruson and Jensen, and by Zimmermann. In Azumaya's article [Az] they are called "finite matrix subgroups." An additive subgroup of a left R-module M is said to be of *finite definition* if it arises as the kernel $F_{X,x}(M)$ of a map

$$M \longrightarrow X \otimes_R M, \quad m \longmapsto x \otimes m$$

for some f.p. right R-module X, and some element $x \in X$. These subgroups are not necessarily R-submodules of M, but they are $\mathrm{End}_R(M)$-submodules.

There is an equivalent definition as follows. If $\mathcal{F}$ is a subfunctor of $(R,-)$ and M is a left R-module, then the space $F_{\mathcal{F}}(M) = \mathrm{Hom}((R,-)/\mathcal{F}, -\otimes M)$ can be regarded as the additive subgroup of $\mathrm{Hom}((R,-), -\otimes M)$ on the maps which annihilate $\mathcal{F}$. Moreover, by Yoneda's lemma, the last space can be identified with M. Thus $F_{\mathcal{F}}(M)$ is canonically an additive subgroup of M. The subgroups of M which arise in this way using f.g. subfunctors $\mathcal{F}$ of $(R,-)$ are the subgroups of M of finite definition. To see this one only has to note that any f.g. subfunctor of $(R,-)$ is the image of a map $(X,-) \longrightarrow (R,-)$ for some f.p. module X, and by Yoneda's lemma this map is determined by a map $R \longrightarrow X$, and hence by an element $x \in X$.

We list some basic properties of the subgroups of finite definition.

(1) The subgroups of finite definition form a lattice in M, for

$$F_{X,x}(M) \cap F_{Y,y}(M) = F_{X\oplus Y, x+y}(M), \text{ and}$$
$$F_{X,x}(M) + F_{Y,y}(M) = F_{Z,z}(M)$$

where Z is the cokernel of the map $R \longrightarrow X \oplus Y$ sending 1 to $x-y$, and z is the common image of x and y.

(2) If N is a pure submodule of M then $F_{X,x}(N) = N \cap F_{X,x}(M)$.

(3) If $(M_\lambda)_{\lambda\in\Lambda}$ is a family of modules then

$$F_{X,x}(\coprod_{\lambda\in\Lambda} M_\lambda) = \coprod_{\lambda\in\Lambda} F_{X,x}(M_\lambda), \text{ and}$$

$$F_{X,x}(\prod_{\lambda\in\Lambda} M_\lambda) = \prod_{\lambda\in\Lambda} F_{X,x}(M_\lambda),$$

the latter expression holding since tensor products $X\otimes_R-$ commute with products when X is a f.p. module.

## 2 INJECTIVE ENVELOPES OF SIMPLE FUNCTORS

Having determined the injective functors in the category D(R), it is worthwhile to characterize the injective envelopes of the simple functors. That is, to determine the indecomposable pure-injective modules M such that $-\otimes M$ has a simple subfunctor. This is not strictly necessary for our study of modules of finite endolength, but it is an important finiteness condition which will be relevant later.

**2.1** In this paragraph we compute the injective envelope of the simple functor $\mathcal{S}_{X,\mathfrak{m}}$.

LEMMA. Let X be a f.p. right R-module, $E = \mathrm{End}_R(X)$ and $\mathfrak{m}$ a maximal left ideal in E. If I is the injective envelope of the E-module $E/\mathfrak{m}$, and $M = \mathrm{Hom}_E(X, I)$, then the injective envelope of $\mathcal{S}_{X,\mathfrak{m}}$ is $-\otimes M$.

*Proof*. Let $\theta: E\longrightarrow I$ be an E-module map inducing an isomorphism from $E/\mathfrak{m}$ to $\mathrm{soc}_E(I)$, and let f be the morphism

$$(X,-) \longrightarrow \mathrm{Hom}_E(\mathrm{Hom}_R(-,X), I)$$

which when applied to a f.p. module Y sends a map $\phi \in \mathrm{Hom}_R(X,Y)$ to the map $\mathrm{Hom}_R(Y,X)\longrightarrow I$ which sends $\psi$ to $\theta(\psi\circ\phi)$. It is clear that the kernel of f is $(X,-)_{\mathfrak{m}}$, so that $\mathcal{S}_{X,\mathfrak{m}}$ embeds in $\mathrm{Hom}_E(\mathrm{Hom}_R(-,X), I)$. Now observe that the natural map

$$-\otimes M \longrightarrow \mathrm{Hom}_E(\mathrm{Hom}_R(-,X), I)$$

is an isomorphism since both functors are right exact and they agree on R. Thus there is an embedding of $\mathcal{S}_{X,m}$ in $-\otimes M$. Now M is pure-injective since $\mathrm{Hom}_R(-,M) \cong \mathrm{Hom}_E(X\otimes -, I)$ is exact on pure exact sequences, so that if $\xi: 0 \longrightarrow M \longrightarrow Y \longrightarrow Z \longrightarrow 0$ is pure, then the map $\mathrm{Hom}_R(Y,M) \longrightarrow \mathrm{Hom}_R(M,M)$ is onto, and hence $\xi$ splits. Finally

$$\begin{aligned} \mathrm{End}_R(M) &\cong \mathrm{Hom}_E(X\otimes_R M, I) \cong \\ &\cong \mathrm{Hom}_E(\mathrm{Hom}_E(\mathrm{Hom}_R(X,X),I),I) \cong \mathrm{End}_E(I) \end{aligned}$$

is a local ring. It follows that $-\otimes M$ is the injective envelope of the functor $\mathcal{S}_{X,m}$.

**2.2** Recall that a map $\alpha: M \longrightarrow N$ of R-modules is said to be *left almost split* if it is not a split mono, and any map $M \longrightarrow X$ which is not split mono factors through $\alpha$. Dually $\beta: N \longrightarrow M$ is *right almost split* if it is a not split epi, and any map $X \longrightarrow M$ which is not split epi factors through $\beta$. Taking X = M one sees that such maps can only exist if M has local endomorphism ring. Moreover, if $\alpha$ exists then it is not a pure mono if and only if M is pure-injective, while if $\beta$ exists then it is not a pure epi (the epi in a pure exact sequence) if and only if M is f.p.. Now Auslander has shown that if M is f.p. and has local endomorphism ring then there is a right almost split map terminating at M. this paragraph we treat the case of left almost split maps.

**2.3** Theorem. Let M be an indecomposable pure-injective R-module. The following statements are equivalent.

(1) M is the source of a left almost split map $\alpha: M \longrightarrow N$.

(2) $-\otimes M$ is the injective envelope of a simple functor.

(3) $M \cong \mathrm{Hom}_E(X,I)$ with X some f.p. right R-module, $E = \mathrm{End}_R(X)$ and I the injective envelope of a simple left E-module.

*Proof*. The equivalence (2) $\Leftrightarrow$ (3) follows from (1.3) and (2.1). Suppose that (1) holds and let $\mathcal{S}$ be the kernel of the morphism $-\otimes M \longrightarrow -\otimes N$. If $\mathcal{S} = 0$ then the inclusion of M in N is pure, and so a split mono since M is pure- injective, a contradiction. Now if $\mathscr{S}$ is not simple, say with proper non-zero subfunctor $\mathscr{F}$ then $(-\otimes M)/\mathscr{F}$ can be embedded in an injective functor $-\otimes L$, so there is a map $\theta: M \longrightarrow L$ inducing a map $-\otimes M \longrightarrow -\otimes L$ with kernel $\mathscr{F}$. It follows that $\theta$ is not a split mono and cannot factor through $\alpha$, a contradiction. Thus $\mathscr{S}$ is simple and (2) holds.

The proof of (2) $\Rightarrow$ (1) is essentially contained in [A3]. In order to sketch it we need to recall several facts. If X is a f.p. right R-module and $P \xrightarrow{f} Q \longrightarrow X \longrightarrow 0$ is a projective presentation of X with P and Q f.g., then the *transpose* Tr X of X relative to this projective presentation is defined to be the f.p. left R-module which is the cokernel of the map

$$\mathrm{Hom}_R(f,R) : \mathrm{Hom}_R(Q,R) \longrightarrow \mathrm{Hom}_R(P,R).$$

If Y is another f.p. right module, then

$$\underline{\mathrm{Hom}}_R(X,Y) \cong \underline{\mathrm{Hom}}_R(\mathrm{Tr}\ Y, \mathrm{Tr}\ X)$$

where $\underline{\mathrm{Hom}}$ denotes the group of homomorphisms modulo the subgroup of those which factor through a projective. In particular there is a ring isomorphism

$$\tau : \underline{\mathrm{End}}_R(X) \longrightarrow \underline{\mathrm{End}}_R(\mathrm{Tr}\ X)^{\mathrm{op}}.$$

Suppose (2), say $-\otimes M$ is the injective envelope of a simple $\mathscr{S}$. We construct a left almost split map with source M. As a first case, suppose that $\mathscr{S}(R) \neq 0$, so by Yoneda's lemma we know that $\mathscr{S}$ is a quotient of $(R,-)$, and hence that $\mathscr{S} \cong \mathscr{S}_{R,\mathfrak{m}}$ for some maximal left ideal $\mathfrak{m}$ in $\mathrm{End}(R_R) \cong R$. Now by (2.1) we know that $M \cong \mathrm{Hom}_R(R,I) \cong I$ is the injective envelope of a simple left R-module. In this case the projection $M \longrightarrow M/\mathrm{soc}_R(M)$ is a left almost split map. Thus we may assume that $\mathscr{S}(R) = 0$. Say $\mathscr{S} \cong \mathscr{S}_{X,\mathfrak{m}}$ where X is a f.p. right R-module and $\mathfrak{m}$ is a maximal left ideal in $E = \mathrm{End}_R(X)$, and so by (2.1) we may assume that $M = \mathrm{Hom}_E(X,I)$ where I is the injective envelope of $E/\mathfrak{m}$.

We now consider contravariant functors R-Mod $\longrightarrow$ $\mathbb{Z}$-Mod. By [A3] there are isomorphisms

$$\begin{aligned}\mathrm{Ext}^1_R(-,\mathrm{Hom}_E(X,I)) &\cong \mathrm{Hom}_E(\mathrm{Tor}^R_1(X,-),I)\\ &\cong \mathrm{Hom}_E(\underline{\mathrm{Hom}}_R(\mathrm{Tr}\ X,-),I), \qquad (\dagger)\end{aligned}$$

where for any left R-module Z, the group $\underline{\mathrm{Hom}}_R(\mathrm{Tr}\ X,Z)$ is considered as an E-module by means of the isomorphism $\tau$.

We construct a map

$$h : \mathrm{Hom}_R(-,\mathrm{Tr}\ X) \longrightarrow \mathrm{Hom}_E(\underline{\mathrm{Hom}}_R(\mathrm{Tr}\ X,-),I)$$

whose image is a simple functor. Since $\mathscr{S}(R) = 0$, any endomorphism of X which factors through a projective module belongs to $\mathfrak{m}$, and so $\mathfrak{m}$ descends to a maximal left ideal $\underline{\mathfrak{m}}$ in $\underline{\mathrm{End}}_R(X)$. Let $\sigma : \underline{\mathrm{End}}_R(X) \longrightarrow I$ be an E-module map inducing an isomorphism from $\underline{\mathrm{End}}_R(X)/\underline{\mathfrak{m}}$ onto the socle of I, and let $\theta$ be the composition

$$\mathrm{End}_R(\mathrm{Tr}\ X)^{op} \twoheadrightarrow \underline{\mathrm{End}}_R(\mathrm{Tr}\ X)^{op} \xrightarrow{\tau^{-1}} \underline{\mathrm{End}}_R(X) \xrightarrow{\sigma} I$$

We now define h by sending $\psi \in \mathrm{Hom}_R(Z,\mathrm{Tr}\ X)$ to the map $\underline{\mathrm{Hom}}_R(\mathrm{Tr}\ X,Z) \longrightarrow I$ which sends $\phi$ to $\theta(\psi\circ\phi)$. Clearly (Ker h)(Z) is equal to

$$\{\psi \in \mathrm{Hom}_R(Z,\mathrm{Tr}\ X) \mid \psi\circ\chi \in \mathrm{Ker}(\theta)\ \forall\ \chi \in \mathrm{Hom}_R(\mathrm{Tr}\ X,Z)\}$$

and since $\mathrm{Ker}(\theta)$ is a maximal right ideal in $\mathrm{End}_R(\mathrm{Tr}\ X)$, it follows that Ker h is a maximal subfunctor of $\mathrm{Hom}_R(-,\mathrm{Tr}\ X)$, just as in the proof of (1.3). Thus the image of h is indeed simple.

Let L be a module with $(\mathrm{Im}\ h)(L) \neq 0$, for example $L = \mathrm{Tr}\ X$ suffices. We can choose an extension

$$\xi : 0 \longrightarrow \mathrm{Hom}_E(X,I) \xrightarrow{\alpha} N \xrightarrow{\beta} L \longrightarrow 0$$

whose image in $\operatorname{Ext}^1_R(L,\operatorname{Hom}_E(X,I))$ generates the simple subfunctor of $\operatorname{Ext}^1_R(-,\operatorname{Hom}_E(X,I))$ corresponding to Im(h) under the isomorphism (†). From the long exact sequence for $\xi$ we obtain an exact sequence

$$\operatorname{Hom}_R(-,N)\xrightarrow{\operatorname{Hom}(-,\beta)}\operatorname{Hom}_R(-,L)\longrightarrow\operatorname{Ext}^1_R(-,\operatorname{Hom}_E(X,I))$$

and by definition the image of $1_L$ under the connecting map is $\xi$. Thus the cokernel of $\operatorname{Hom}(-,\beta)$ is isomorphic to Im(h) and hence is simple. The next lemma then shows that $\alpha:M\longrightarrow N$ is a left almost split map.

LEMMA [A3, Chapter II, Proposition 4.2]. Let $0\longrightarrow M\xrightarrow{\alpha}N\xrightarrow{\beta}L\longrightarrow 0$ be an exact sequence of left R-modules. If the cokernel of $\operatorname{Hom}(-,\beta)$ is a simple contravariant functor R-Mod $\longrightarrow$ $\mathbb{Z}$-Mod, then the map $\alpha$ is left almost split if and only if $\operatorname{End}_R(M)$ is local.

## 3 Σ-PURE-INJECTIVE MODULES

A module M is said to be *Σ-pure-injective* provided that every direct sum of copies of M is pure-injective. In this section we describe a very useful characterization of the Σ-pure-injective modules, one application of which is the fact that modules of finite endolength are always direct sums of indecomposable submodules, something which is not at all obvious.

**3.1** THEOREM. If M is an R-module, then the following statements are equivalent

(1) M has the dcc on subgroups of finite definition.

(2) M is Σ-pure-injective.

(3) Every product of copies of M is a direct sum of indecomposables with local endomorphism ring.

(4) Every product of copies of M is a direct sum of indecomposables of cardinality $\le \max(\aleph_0,\operatorname{card}(R))$.

We only prove (1)⇒(2) and (1)⇒(3), the implications which are most relevant for our study of modules of finite endolength. The reader can find the other implications, and indeed a host of other equivalent statements in [GJ1], [ZH], [JL, 8.1], [P, 3.2] and in Azumaya's article [Az].

*Proof of* (1)⇒(2). Since $F_{X,x}(\coprod_I M) = \coprod_I F_{X,x}(M)$ it is clear that $\coprod_I M$ also has the dcc on subgroups of finite definition, so by replacing M by $\coprod_I M$, it is enough to show that M is pure-injective, that is, that $-\otimes M$ is injective. In fact it suffices to prove that $\mathrm{Ext}^1_{D(R)}((R,-)/\mathcal{F}, -\otimes M) = 0$ for all subfunctors $\mathcal{F}$ of $(R,-)$, an analogue of Baer's criterion for injectivity. To see why, let $-\otimes M \longrightarrow -\otimes L$ be the injective envelope of $-\otimes M$, so there is an exact sequence

$$0 \longrightarrow -\otimes M \longrightarrow -\otimes L \longrightarrow -\otimes(L/M) \longrightarrow 0. \qquad (\dagger)$$

If $-\otimes M$ is not injective then $L/M \neq 0$, and choosing $x \in L/M$ one obtains a map $(R,-) \longrightarrow -\otimes(L/M)$. If this has kernel $\mathcal{F}$ then by assumption the pullback of ($\dagger$) via the mono $(R,-)/\mathcal{F} \longrightarrow -\otimes(L/M)$ is split. This means that $-\otimes M \oplus (R,-)/\mathcal{F}$ embeds in $-\otimes L$, which contradicts the minimality of $-\otimes L$.

Let $\xi : 0 \longrightarrow -\otimes M \longrightarrow \mathcal{E} \xrightarrow{\pi} (R,-)/\mathcal{F} \longrightarrow 0$ be an extension. We show that $\xi$ is split. We have the equality

$$\mathrm{Hom}((R,-)/\mathcal{F}, -\otimes M) = \bigcap_{\mathcal{G} \text{ f.g. } \leq \mathcal{F}} \mathrm{Hom}((R,-)/\mathcal{G}, -\otimes M)$$

and since the terms in the intersection are subgroups of finite definition of M, and M has the dcc on such subgroups, one can find a f.g. subfunctor $\mathcal{H}$ of $\mathcal{F}$ with

$$\mathrm{Hom}((R,-)/\mathcal{F}, -\otimes M) = \mathrm{Hom}((R,-)/\mathcal{H}, -\otimes M).$$

That is, $\mathcal{H}$ has the property that any map $(R,-) \longrightarrow -\otimes M$ which annihilates $\mathcal{H}$, also annihilates $\mathcal{F}$.

Now let $\mathcal{G}$ be a f.g. subfunctor of $\mathcal{F}$ containing $\mathcal{H}$. Since $(R,-)/\mathcal{G}$

is coherent, Lemma (1.4) shows that the pullback of $\xi$ with respect to the map $(R,-)/\mathcal{G} \twoheadrightarrow (R,-)/\mathcal{F}$ splits. Thus there is a map $f_{\mathcal{G}} : (R,-) \longrightarrow \mathcal{E}$ which annihilates $\mathcal{G}$, and such that the composition $\pi\circ f_{\mathcal{G}}$ is equal to the natural projection of $(R,-)$ onto $(R,-)/\mathcal{F}$. Now the difference $f_{\mathcal{G}} - f_{\mathcal{H}}$ actually maps $(R,-)$ into $-\otimes M$, and it annihilates $\mathcal{H}$, so that it also annihilates $\mathcal{F}$, and hence $f_{\mathcal{H}}$ annihilates $\mathcal{G}$. This shows that $f_{\mathcal{H}}$ annihilates any f.g. subfunctor of $\mathcal{F}$ containing $\mathcal{H}$. Thus $f_{\mathcal{H}}$ annihilates $\mathcal{F}$, so it induces a map $(R,-)/\mathcal{F} \longrightarrow \mathcal{E}$ splitting $\xi$, as required.

*Proof of* (1)⇒(3). Any product of copies of M also satisfies the hypothesis, so we only need to prove that M is a direct sum of indecomposables with local endomorphism rings. If U is a pure submodule of M then $F_{X,x}(U) = U \cap F_{X,x}(M)$, so U has the dcc on subgroups of finite definition, is pure-injective, and hence is a summand of M. Given an ascending chain of pure submodules of M, their union is again a pure submodule (since tensor products commute with direct limits), so if $0\neq x\in M$, by Zorn's lemma we can choose a pure submodule U, maximal with respect to the condition $x\notin U$. Now $M = U\oplus X$, and if X were to decompose this would contradict the maximality of U. Thus M has an indecomposable summand. Now let A be a maximal set of indecomposable submodules of M whose sum $U = \Sigma_{V\in A} V$ is direct and is pure in M. If $U\neq M$ then it is a summand, and its complement has an indecomposable summand which could be adjoined to A, a contradiction. Finally, because the category D(R) has injective envelopes, any indecomposable injective functor has local endomorphism ring, so any indecomposable pure-injective module has local endomorphism ring.

## 4 MODULES OF FINITE ENDOLENGTH

In this section we describe the basic properties of modules of finite endolength, most of which are deduced directly from the properties of $\Sigma$-pure-injective modules. We finish with some examples.

**4.1** Proposition. An R-module has finite endolength if and only if it has the acc and the dcc on subgroups of finite definition. In this case every $\mathrm{End}_R(M)$-submodule of M has finite definition.

This shows that the endolength of a module is the same as its pp-rank in the sense of [P]. The proposition follows directly from the lattice structure of the subgroups of finite definition and the next lemma.

Lemma. If M is a pure-injective R-module, then every cyclic $\mathrm{End}_R(M)$-submodule of M is an intersection of subgroups of finite definition.

*Proof*. An element $m \in M$ determines a morphism $(R,-) \longrightarrow -\otimes M$, and hence a monomorphism $\phi\colon (R,-)/\mathcal{F} \longrightarrow -\otimes M$ where $\mathcal{F}$ is the kernel. Recall that the space $\mathrm{Hom}((R,-)/\mathcal{F}, -\otimes M)$ can be regarded as a subgroup of M, and it is clearly an $\mathrm{End}_R(M)$-submodule. If $\theta \in \mathrm{Hom}((R,-)/\mathcal{F}, -\otimes M)$ then since $-\otimes M$ is injective there is a factorization $\theta = \alpha\circ\phi$ for some $\alpha\colon -\otimes M \longrightarrow -\otimes M$. Now $\alpha$ corresponds to an endomorphism of M, and this shows that $\mathrm{Hom}((R,-)/\mathcal{F}, -\otimes M)$ is the $\mathrm{End}_R(M)$-submodule of M generated by m. Finally, we have already used in the proof of (3.1) the fact that $\mathrm{Hom}((R,-)/\mathcal{F}, -\otimes M)$ is an intersection of subgroups of finite definition.

**4.2** Lemma. The endomorphism ring of a finite endolength module has nilpotent radical.

*Proof*. Nakayama's Lemma.

**4.3** Proposition. The class of finite endolength modules is closed under finite direct sums, and arbitrary products or direct sums of copies of one module. Moreover, if L is a pure submodule of a module M of finite endolength, then L is a direct summand and $\mathrm{endolen}(L) \le \mathrm{endolen}(M)$.

*Proof*. Most of this can be proved using elementary means, but having proved (4.1) this all follows from the equalities in (1.7). Recall that since a pure submodule has finite endolength it is pure-injective, and hence a summand.

**4.4** PROPOSITION. Indecomposable R-modules of finite endolength have local endomorphism rings and cardinality $\le \max(\aleph_0, \mathrm{card}(R))$.

*Proof*. This follows from (3.1).

**4.5** PROPOSITION. Every module of finite endolength is a direct sum of indecomposable modules of finite endolength. Conversely, such a direct sum has finite endolength if and only if there are only finitely many isomorphism classes of indecomposables involved.

The only part not contained in (3.1) is the "only if", which follows from the lemma below.

LEMMA. If $M_1,\dots,M_n$ are non-isomorphic indecomposable modules of finite endolength, then $\mathrm{endolen}(M_1\oplus\dots\oplus M_n) = \sum_{i=1}^{n} \mathrm{endolen}(M_i)$.

*Proof*. Since the $M_i$ have local endomorphism rings, the ring

$$\mathrm{End}_R(M_1\oplus\dots\oplus M_n)/\mathrm{rad}\ \mathrm{End}_R(M_1\oplus\dots\oplus M_n)$$

is isomorphic to the product $\prod_{i=1}^{n} \mathrm{End}_R(M_i)/\mathrm{rad}\ \mathrm{End}_R(M_i)$, and the assertion follows.

REMARKS. (1) Since the indecomposables have local endomorphism rings, the decomposition of a finite endolength module into indecomposable summands is essentially unique.

(2) A result of Garavaglia, see [P, Exercise 2, p200], shows that an indecomposable module M has finite endolength if and only if every product of copies of M is isomorphic to a direct sum of

copies of M. One direction is as follows. If M has finite endolength, then $\prod_I M$ is a direct sum of indecomposable modules, and it has endolength equal to that of M. Now since it has M as a summand, it cannot have any other indecomposable summands by the lemma above.

**4.6** PROPOSITION. If a module of finite endolength is either artinian or noetherian, then it has finite length.

This follows from a simple generalization of a theorem of Lenagan,

THEOREM. If a bimodule is artinian on one side and noetherian on the other, then it has finite length on each side.

*Proof.* Let ${}_R M_S$ be such a bimodule, artinian over R and noetherian over S. Since M has both chain conditions on sub-bimodules, to prove the theorem we may assume that M is simple as a bimodule. Now $\mathrm{soc}_R(M)$ is a non-zero sub-bimodule, so equal to M. Therefore ${}_R M$ is semisimple, and hence of finite length since it is artinian. Now the usual form of Lenagan's Theorem shows that $M_S$ has finite length, see [MR, 4.1.6] or [GW, 7.10].

**4.7** EXAMPLES. (1) If R is a ring without invariant basis number [C, §0.2], for example the endomorphism ring of an infinite dimensional vector space, then there are no non-zero modules M of finite endolength, for by definition $R^n \cong R^m$ for some $n \neq m$, and then $R^n \otimes_R M \cong R^m \otimes_R M$ as $\mathrm{End}_R(M)$-modules, but they have different lengths as such.

(2) The Jacobson density theorem shows that a simple R-module S has finite endolength if and only if $R/\mathrm{Ann}_R(S)$ is simple artinian.

(3) A ring R is said to be of *finite representation type* if it is

left artinian and has only finitely many isomorphism classes of finite length left R-modules. This is a left-right symmetric condition, and for such rings, every module is a direct sum of finitely generated modules [A2,RT]. It is proved in [P, 11.38], [ZHZ, Theorem 6] and [CB3, 1.2] that a ring R has finite representation type if and only if every R-module has finite endolength.

(4) There is a 1-1 correspondence between isomorphism classes of finite endolength modules M with $\mathrm{End}_R(M)$ a division ring, and isomorphism classes of ring-theoretic epimorphisms $\theta: R \longrightarrow S$ from R to a simple artinian ring S. The correspondence is given by sending M to the homomorphism

$$R \longrightarrow \mathrm{End}({}_{\mathrm{End}_R(M)}M),$$

and sending $\theta$ to the restriction of the simple S-module [R3].

(5) If R is a commutative ring, then the indecomposable modules of endolength 1 are precisely the quotient fields of factor rings R/P with P a prime ideal in R. More generally, if R is a prime Goldie ring then the restriction of the simple module for the simple artinian quotient ring of R is the unique faithful indecomposable module of finite endolength, see [CB3, 1.3]. It follows that if R is a noetherian ring or a PI ring, then the prime ideals in R can be identified with the indecomposable modules of finite endolength whose annihilator is prime. Thus, for example, a simple noetherian ring has a unique indecomposable module of finite endolength.

(6) If R is a Dedekind domain, then the indecomposable modules of finite endolength are the quotient field of R and the modules $R/m^n$ with m a maximal ideal. This is because, if I is a non-zero ideal in R, then R/I is an artinian principal ideal ring, so that the indecomposable R/I-modules have form $R/m^n$. More generally, if

R is an hereditary noetherian prime ring, then the only indecomposable module of finite endolength and infinite length is the restriction of the simple module of the simple artinian quotient ring of R. In this case, the proper factor rings of R have finite representation type.

## 5 CHARACTERS

Pure-injective modules have been extensively studied by model theorists using rather sophisticated ideas, see for example [P], but for a module M of finite endolength, it appears that many of these concepts can be encoded in simple numerical data, which we call the character of M. In general, by a *character* for mod-R we mean a function $\chi$ which assigns to each f.p. right R-module X a non-negative integer $\chi(X)$, and which satisfies the two conditions

(1) $\chi(X\oplus Y) = \chi(X) + \chi(Y)$ for all f.p. modules X, Y.

(2) $\chi(Z) \le \chi(Y) \le \chi(X) + \chi(Z)$ for any right exact sequence $X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ of f.p. modules.

This notion is adapted from Schofield's definition of a Sylvester module rank function [Sc], but we like the name "character" since they have many properties in common with group characters. We stress, however, that our characters are non-negative integer valued, and in no way involve traces. It is natural to call the number $\chi(R)$ the *degree* of $\chi$; if it is zero then condition (2) above shows that $\chi = 0$. If M is a left R-module of finite endolength, the assignment

$$\chi_M(X) = \mathrm{length}_{\mathrm{End}_R(M)}(X\otimes_R M)$$

defines the character $\chi_M$ of M. Its degree is the endolength of M.

**5.1** THEOREM. If $M_1,\ldots,M_n$ are non-isomorphic indecomposable R-modules of finite endolength then their characters are independent over $\mathbb{Z}$.

*Proof*. Suppose there is a relation $\sum_{i=1}^{n} a_i\chi_{M_i} = 0$ with $a_i \in \mathbb{Z}$. Let

$$M = \coprod_{a_i>0} a_i M_i, \qquad E = \prod_{a_i>0} \mathrm{End}_R(M_i)^{op}$$

so that M is naturally an R-E-bimodule, and let

$$N = \coprod_{a_i<0} (-a_i)M_i, \qquad F = \prod_{a_i<0} \mathrm{End}_R(M_i)^{op}$$

so that N is an R-F-bimodule. Thus the indecomposable summands of M and N are non-isomorphic, and the relation implies that

$$\mathrm{length}(X\otimes_R M_E) = \mathrm{length}(X\otimes_R N_F)$$

for all f.p. modules X. In particular, assuming that the relation is non-trivial, both M and N are non-zero. Recall that if X is a f.p. right R-module and $x\in X$ then the subgroup $F_{X,x}(M)$ is defined by an exact sequence

$$0 \longrightarrow F_{X,x}(M) \longrightarrow R\otimes_R M \longrightarrow X\otimes_R M \longrightarrow (X/xR)\otimes_R M \longrightarrow 0,$$

so we deduce that

$$\mathrm{length}(F_{X,x}(M)_E) = \mathrm{length}(F_{X,x}(N)_F). \tag{$\dagger$}$$

We construct a pair U,u such that $F_{U,u}(M)$ is a simple $\mathrm{End}_R(M)$-submodule of M and $F_{U,u}(N)$ is a simple $\mathrm{End}_R(N)$-submodule of N. By (4.1) one can choose a subgroup of finite definition $F_{Y,y}(M)$ which is simple as an $\mathrm{End}_R(M)$-module, and then $F_{Y,y}(N)$ is non-zero by ($\dagger$). Inside $F_{Y,y}(N)$ we can find a subgroup $F_{Z,z}(N)$ which is simple as an $\mathrm{End}_R(N)$-module. Setting $U = Y\oplus Z$ and $u = y+z$ one has $F_{U,u} = F_{Y,y} \cap F_{Z,z}$ so that $F_{U,u}(N) = F_{Y,y}(N)$ is a simple $\mathrm{End}_R(N)$-submodule of N. Also $F_{U,u}(M)$ is a submodule of $F_{Y,y}(M)$ and is non-zero by ($\dagger$), so is equal to $F_{Y,y}(M)$, and hence is a simple $\mathrm{End}_R(M)$-submodule.

Choose $0 \neq m \in F_{U,u}(M)$ and $0 \neq n \in F_{U,u}(N)$. We show that

$$m \in F_{X,x}(M) \quad \Leftrightarrow \quad n \in F_{X,x}(N)$$

for any pair $X, x$. Now

$$\begin{aligned} m \in F_{X,x}(M) \quad &\Leftrightarrow \quad F_{U,u}(M) \subseteq F_{X,x}(M) \\ &\Leftrightarrow \quad F_{X,x}(M) = F_{U,u}(M) + F_{X,x}(M), \end{aligned}$$

and $F_{U,u} + F_{X,x} = F_{Z,z}$ where $Z$ is the cokernel of the map $R \longrightarrow U \oplus X$ and $z$ is the common image of $u$ and $x$. Thus

$$m \in F_{X,x}(M) \quad \Leftrightarrow \quad \text{length}(F_{X,x}(M)_E) = \text{length}(F_{Z,z}(M)_E).$$

The same argument for $N$ and the property (†) then prove our assertion.

The element $m$ determines a map $(R,-) \longrightarrow -\otimes M$ whose kernel $\mathcal{K}$ is given by

$$\begin{aligned} \mathcal{K}(X) &= \{\theta \in \text{Hom}_R(R,X) \mid \theta(1) \otimes m = 0\} \\ &= \{\theta \in \text{Hom}_R(R,X) \mid m \in F_{X,\theta(1)}(M)\}. \end{aligned}$$

Similarly the element $n$ gives a map $(R,-) \longrightarrow -\otimes N$ whose kernel is also $\mathcal{K}$ by the statement above. Thus the injective envelope $-\otimes L$ of $(R,-)/\mathcal{K}$ embeds in both $-\otimes M$ and $-\otimes N$, so that $L$ is a summand of both $M$ and $N$. Since $L \neq 0$ this is impossible by the Krull-Schmidt theorem.

**5.2** THEOREM. Every character $\chi$ can be written as a sum

$$\chi = \chi_{M_1} + \dots + \chi_{M_n}$$

with the $M_i$ indecomposable modules of finite endolength.

The proof is given in several steps.

*Step 1.* We define a function on the coherent functors $\mathcal{F} \in D(R)$, which we again denote by $\chi$, as follows. If $\mathcal{F}$ has resolution

$$0 \longrightarrow (Z,-) \longrightarrow (Y,-) \longrightarrow (X,-) \longrightarrow \mathcal{F} \longrightarrow 0$$

for a right exact sequence $X \longrightarrow Y \longrightarrow Z \longrightarrow 0$, then we set

$$\chi(\mathcal{F}) = \chi(X) - \chi(Y) + \chi(Z).$$

This is a non-negative integer since $\chi$ is a character, and it is well-defined by the long form of Schanuel's Lemma. Note that $\chi((X,-)) = \chi(X)$, so this function can really be thought of as extending $\chi$. Standard arguments with projective resolutions show that if $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ is an exact sequence of coherent functors then $\chi(\mathcal{G}) = \chi(\mathcal{F}) + \chi(\mathcal{H})$.

*Step 2.* We extend the function of the previous paragraph to f.g. functors $\mathcal{F} \in D(R)$ by setting

$$\chi(\mathcal{F}) = \min \{ \chi(\mathcal{H}) \mid \mathcal{H} \text{ coherent}, \mathcal{H} \twoheadrightarrow \mathcal{F} \}.$$

This agrees with the first definition in case $\mathcal{F}$ is coherent because of the additivity of $\chi$ in that case.

If $\mathcal{G}$ is a f.g. subfunctor of $\mathcal{F}$ then $\chi(\mathcal{G}) \le \chi(\mathcal{F})$. Namely, there is a map $\theta : \mathcal{H} \twoheadrightarrow \mathcal{F}$ with $\mathcal{H}$ coherent and $\chi(\mathcal{H}) = \chi(\mathcal{F})$. Since $\mathcal{G}$ is f.g. one can find a f.g. subfunctor $\mathcal{K}$ of $\theta^{-1}(\mathcal{G})$ mapping onto $\mathcal{G}$. Now $\mathcal{K}$ is a f.g. subfunctor of $\mathcal{H}$, so is coherent. Thus $\chi(\mathcal{G}) \le \chi(\mathcal{K}) \le \chi(\mathcal{H}) = \chi(\mathcal{F})$.

*Step 3.* We extend the function $\chi$ to all functors $\mathcal{F} \in D(R)$, with $\chi$ now taking values in $\mathbb{N} \cup \{\infty\}$. Namely, we set

$$\chi(\mathcal{F}) = \max \{ \chi(\mathcal{G}) \mid \mathcal{G} \text{ f.g.}, \mathcal{G} \subseteq \mathcal{F} \}.$$

This agrees with the definition for f.g. functors by the observation above. We show that if $0 \longrightarrow \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H} \longrightarrow 0$ is exact, then

$$\chi(\mathcal{G}) = \chi(\mathcal{F}) + \chi(\mathcal{H}) \qquad (\dagger)$$

with the usual conventions if any term is $\infty$. Our proof involves a sequence of special cases. We have already observed "Case 0", that (†) holds when $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ are coherent.

*Case 1.* Suppose that all f.g. subfunctors of $\mathcal{G}$ and $\mathcal{H}$ are coherent. If $\mathcal{L}$ is a f.g. subfunctor of $\mathcal{G}$ then the sequence

$$0 \longrightarrow \alpha^{-1}(\mathcal{L}) \longrightarrow \mathcal{L} \longrightarrow \beta(\mathcal{L}) \longrightarrow 0$$

is exact. Since $\mathcal{L}$ and $\beta(\mathcal{L})$ are finitely generated, they are coherent by the assumption, and hence so is $\alpha^{-1}(\mathcal{L})$, and so by Case 0 we have

$$\chi(\mathcal{L}) = \chi(\alpha^{-1}(\mathcal{L})) + \chi(\beta(\mathcal{L})). \tag{‡}$$

Now $\alpha^{-1}(\mathcal{L})$, $\mathcal{L}$ and $\beta(\mathcal{L})$ are subfunctors of $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ respectively, so

$$\chi(\alpha^{-1}(\mathcal{L})) \le \chi(\mathcal{F}), \quad \chi(\mathcal{L}) \le \chi(\mathcal{G}), \quad \chi(\beta(\mathcal{L})) \le \chi(\mathcal{H}).$$

Moreover, by taking $\mathcal{L}$ large enough we can ensure that $\alpha^{-1}(\mathcal{L})$, $\mathcal{L}$ and $\beta(\mathcal{L})$ contain any given f.g. subfunctors of $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$. Now the equality (†) follows on taking the supremum of (‡) over all $\mathcal{L}$.

*Case 2.* Suppose that $\mathcal{G}$ is coherent. Since $\mathcal{H}$ is f.g. so one can find a surjection $\mathcal{L} \twoheadrightarrow \mathcal{H}$ with $\mathcal{L}$ coherent and $\chi(\mathcal{L}) = \chi(\mathcal{H})$. Form the pullback

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{K} & = & \mathcal{K} & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{L} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{H} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Note that $\mathcal{E}$ embeds in $\mathcal{G}\oplus\mathcal{L}$ which is coherent, so that f.g.

subfunctors of $\mathcal{E}$ are coherent. Thus $\chi(\mathcal{E}) = \chi(\mathcal{F}) + \chi(\mathcal{L})$ and $\chi(\mathcal{E}) = \chi(\mathcal{G}) + \chi(\mathcal{K})$, and so $\chi(\mathcal{G}) = \chi(\mathcal{F}) + \chi(\mathcal{H}) - \chi(\mathcal{K})$. Now if $\chi(\mathcal{K}) \neq 0$ then $\mathcal{K}$ has a f.g. (and hence coherent) subfunctor $\mathcal{I}$ with $\chi(\mathcal{I}) \neq 0$. But then $\mathcal{L}/\mathcal{I} \twoheadrightarrow \mathcal{H}$, so that

$$\chi(\mathcal{H}) \leq \chi(\mathcal{L}/\mathcal{I}) = \chi(\mathcal{L}) - \chi(\mathcal{I}) < \chi(\mathcal{L}),$$

a contradiction.

*Case 3.* Suppose that all f.g. subfunctors of $\mathcal{G}$ are coherent. The proof is the same as Case 1, using Case 2 to prove (‡).

*Case 4.* Suppose that $\mathcal{G}$ is finitely generated. We can find a surjection $\mathcal{L} \twoheadrightarrow \mathcal{G}$ with $\mathcal{L}$ coherent. This gives a commutative exact diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \mathcal{K} \arrow[r, equal] \arrow[d] & \mathcal{K} \arrow[d] & & \\
0 \arrow[r] & \mathcal{E} \arrow[r] \arrow[d] & \mathcal{L} \arrow[r] \arrow[d] & \mathcal{H} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d] & \mathcal{G} \arrow[r] \arrow[d] & \mathcal{H} \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

so that $\chi(\mathcal{L}) = \chi(\mathcal{E}) + \chi(\mathcal{H})$ and $\chi(\mathcal{L}) = \chi(\mathcal{K}) + \chi(\mathcal{G})$ since $\mathcal{L}$ is coherent, and $\chi(\mathcal{E}) = \chi(\mathcal{K}) + \chi(\mathcal{F})$ since $\mathcal{E}$ is a subfunctor of $\mathcal{L}$, so that f.g. subfunctors of $\mathcal{E}$ are coherent. Thus $\chi(\mathcal{G}) = \chi(\mathcal{F}) + \chi(\mathcal{H})$, as required.

*General case.* The proof is the same as in Case 1, now using Case 4 to prove (‡).

*Step 4.* If M is a module with the property that

$$\mathrm{Hom}(\mathcal{H}, -\otimes M) = 0 \text{ for all functors } \mathcal{H} \text{ with } \chi(\mathcal{H}) = 0 \quad (*)$$

then M has endolength $\leq \chi(R)$.

*Proof*. If not, then by (4.1) we can find a strictly increasing chain

$$F_{\mathcal{F}_0}(M) < F_{\mathcal{F}_1}(M) < \dots < F_{\mathcal{F}_d}(M) < F_{\mathcal{F}_{d+1}}(M)$$

of subgroups of finite definition, where $d = \chi(R)$ and the $\mathcal{F}_i$ are f.g. subfunctors of $(R,-)$. Moreover, we may assume that

$$\mathcal{F}_{d+1} \le \mathcal{F}_d \le \dots \le \mathcal{F}_1 \le \mathcal{F}_0,$$

if necessary by replacing $\mathcal{F}_i$ by $\sum_{j\ge i} \mathcal{F}_j$. Consider the coherent functors

$$(R,-)/\mathcal{F}_{d+1} \twoheadrightarrow (R,-)/\mathcal{F}_d \twoheadrightarrow \dots \twoheadrightarrow (R,-)/\mathcal{F}_1 \twoheadrightarrow (R,-)/\mathcal{F}_0.$$

Each of these functors is a quotient of $(R,-)$ so has $\chi$ bounded by $\chi(R)$, and they have $\chi$ decreasing, so at some stage there are two functors with the same $\chi$, say $(R,-)/\mathcal{F}_{i+1} \twoheadrightarrow (R,-)/\mathcal{F}_i$. Since the kernel $\mathcal{K}$ of this map has $\chi(\mathcal{K}) = 0$, we have an exact sequence

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathrm{Hom}((R,-)/\mathcal{F}_i, -\otimes M) & \longrightarrow & \mathrm{Hom}((R,-)/\mathcal{F}_{i+1}, -\otimes M) & \longrightarrow & \mathrm{Hom}(\mathcal{K}, -\otimes M) \\
& \| & & \| & & \| \\
& F_{\mathcal{F}_i}(M) & & F_{\mathcal{F}_{i+1}}(M) & & 0
\end{array}$$

which contradicts the assumption that the chain is strictly increasing.

*Step 5.* It follows from the considerations above and Lemma (4.5) that there are at most $\chi(R)$ non-isomorphic indecomposable modules with the property (*). Let them be $M_1, \dots, M_r$, and denote by $a_i$ the smallest value of $\chi(\mathcal{F})$ with $\mathcal{F}$ a non-zero subfunctor of $-\otimes M_i$. Since $-\otimes M_i$ has no subfunctors with $\chi = 0$ it follows that $a_i \ne 0$. We claim that for all $\mathcal{G}$ with $\chi(\mathcal{G}) < \infty$ one has

$$\chi(\mathcal{G}) = \sum_{i=1}^{r} a_i \ \mathrm{length}_{\,\mathrm{End}(M_i)}(\mathrm{Hom}(\mathcal{G}, -\otimes M_i)). \tag{\#}$$

On specializing to the case when $\mathcal{G} = (X,-)$ one deduces that

$$\chi(X) = \sum_{i=1}^{r} a_i \, \chi_{M_i}(X),$$

which proves the theorem. We first prove (#) in two special cases.

*Case 1.* Suppose that every non-zero subfunctor $\mathcal{H}$ of $\mathcal{G}$ has $\chi(\mathcal{H}) = \chi(\mathcal{G})$. Clearly we may assume that $\chi(\mathcal{G}) \neq 0$. The injective envelope $-\otimes M$ of $\mathcal{G}$ has the property (*) since if $\mathcal{H}$ is a functor with $\chi(\mathcal{H}) = 0$ and $\theta:\mathcal{H} \longrightarrow -\otimes M$ is a map, then $\mathcal{G} \cap \mathrm{Im}(\theta)$ is a subfunctor of $\mathcal{G}$ with $\chi = 0$, so is zero, and hence $\theta = 0$ since $\mathcal{G}$ is essential in $-\otimes M$. Thus M has finite endolength and is a direct sum of copies of the $M_i$. In particular there is a non-zero map from $\mathcal{G}$ to some $-\otimes M_i$. Now any non-zero map $\mathcal{G} \longrightarrow -\otimes M_j$ must be mono, so that $-\otimes M_j$ is the injective envelope of $\mathcal{G}$. Thus $\mathrm{Hom}(\mathcal{G}, -\otimes M_j) = 0$ for $j \neq i$, and the injective property for $-\otimes M_i$ shows that $\mathrm{Hom}(\mathcal{G}, -\otimes M_i)$ is simple as an $\mathrm{End}_R(M_i)$-module, proving (#), since clearly $a_i = \chi(\mathcal{G})$.

*Case 2.* Suppose that every subfunctor $\mathcal{H}$ of $\mathcal{G}$ has $\chi(\mathcal{H}) \in \{0, \chi(\mathcal{G})\}$. For any functor $\mathcal{F}$, the sum $T\mathcal{F}$ of the subfunctors $\mathcal{L} \subseteq \mathcal{F}$ with $\chi(\mathcal{L}) = 0$ is the unique largest subfunctor of $\mathcal{F}$ with $\chi = 0$. Thus $\mathcal{G}/T\mathcal{G}$ satisfies the condition of Case 1, and then (#) follows since $\chi(\mathcal{G}) = \chi(\mathcal{G}/T\mathcal{G})$ and $\mathrm{Hom}(\mathcal{G}, -\otimes M_i) \cong \mathrm{Hom}(\mathcal{G}/T\mathcal{G}, -\otimes M_i)$.

*General case.* We can filter $\mathcal{G}$ by subfunctors

$$0 = \mathcal{G}_0 < \mathcal{G}_1 < \dots < \mathcal{G}_h = \mathcal{G}$$

so that whenever $\mathcal{G}_i \leq \mathcal{H} \leq \mathcal{G}_{i+1}$ then $\chi(\mathcal{H}) \in \{\chi(\mathcal{G}_i), \chi(\mathcal{G}_{i+1})\}$. Now $\mathcal{G}_{i+1}/\mathcal{G}_i$ satisfies the condition of case 2, and (#) follows since both sides of (#) are additive on short exact sequences.

**5.3** Let us say that a non-zero character $\chi$ is *irreducible* if it cannot be written as a sum $\chi = \chi_1 + \chi_2$ with $\chi_1$ and $\chi_2$ non-zero characters. The previous two theorems may now be reformulated as follows.

COROLLARY. (1) The assignment $M \longmapsto \chi_M$ induces a bijection between the isomorphism classes of indecomposable modules of finite endolength and the irreducible characters.

(2) The irreducible characters are independent over $\mathbb{Z}$.

(3) Every character is a sum of irreducible characters.

**5.4** REMARK. The Sylvester module rank functions, upon which our characters are based, were used by Schofield [Sc] to classify suitable equivalence classes of ring homomorphisms from R to a simple artinian ring. Quite why these homomorphisms are related to finite endolength modules is not clear to us, except in the case of a ring epi, in which case Example (4.7)(4) applies.

## 6 DUALITY

In this section we use characters to define a 1-1 correspondence between the indecomposable left and right R-modules of finite endolength. This seems to be a special case of a duality studied by Herzog [H]. When he introduced Sylvester module rank functions in [Sc], Schofield observed that these were equivalent to Sylvester map rank functions, and that the latter were left-right symmetric for the ring R. In the context of characters this takes the following form. If $\chi$ is a character for mod-R we define a character $D\chi$ on f.p. left R-modules, so formally a character for mod-$R^{op}$, as follows. If X is a f.p. left R-module, let

$$P \xrightarrow{\alpha} Q \longrightarrow X \longrightarrow 0 \qquad (\dagger)$$

be a projective presentation of X. We define

$$(D\chi)(X) = \chi(Q^*) - \chi(P^*) + \chi(\mathrm{Coker}(\alpha^*))$$

where $(-)^* = \mathrm{Hom}_R(-,R)$ is the duality between f.g. projective left and right R-modules, so that $\mathrm{Coker}(\alpha^*)$ is the transpose of X with respect to the resolution (†). One can check that $D\chi$ is well-defined, and when this is done, that $DD\chi = \chi$, so that D defines a duality between the characters on mod-R and on mod-$R^{op}$. Note also that $\chi$ and $D\chi$ have the same degree.

**6.1** If $\chi = \chi_M$ then there is a much simpler expression for $D\chi$. Combined with the fact that every character is a sum of characters of this form, this gives an indirect proof that $D\chi$ is well-defined.

LEMMA. $D\chi_M(X) = \mathrm{length}_{\mathrm{End}_R(M)}(\mathrm{Hom}_R(X,M))$ for f.p. left R-modules X.

*Proof*. If (†) is a projective presentation, then the diagram

$$\begin{tikzcd}
 & & Q^* \otimes_R M \arrow[r] \arrow[d, equal] & P^* \otimes_R M \arrow[r] \arrow[d, equal] & \mathrm{Coker}(\alpha^*) \otimes_R M \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Hom}_R(X,M) \arrow[r] & \mathrm{Hom}_R(Q,M) \arrow[r] & \mathrm{Hom}_R(P,M) & &
\end{tikzcd}$$

commutes and has exact rows. Now count lengths.

**6.2** The duality D clearly induces a 1-1 correspondence between irreducible characters for mod-R and mod-$R^{op}$, and hence between the isomorphism classes of indecomposable left and right R-modules of finite endolength, say $M \longleftrightarrow DM$ with $\chi_{DM} = D\chi_M$. Note in particular that M and DM have the same endolength. The proposition below shows how to construct DM.

**6.3** PROPOSITION. If M is an indecomposable R-module of finite endolength, $E = \mathrm{End}_R(M)$, and I is the injective envelope of the unique simple left E-module, then $\mathrm{Hom}_E(M,I)$ is a direct sum of copies of DM.

*Proof*. Let $S = \mathrm{soc}_E(I)$ and $F = \mathrm{End}_E(I)$ so that $N = \mathrm{Hom}_E(M,I)$ is an F-R-bimodule. Since I is the injective envelope of S, the F-module $\mathrm{Hom}_E(S,I)$ is simple, and so $\mathrm{length}_F(\mathrm{Hom}_E(Z,I)) = \mathrm{length}_E(Z)$ for any E-module Z of finite length. In particular N has finite length over F. Now there is an isomorphism $\mathrm{Hom}_E(\mathrm{Hom}_R(-,M),I) \cong \mathrm{Hom}_E(M,I)\otimes_R-$ on f.p. left R-modules since both functors are right exact and they agree on R, and so

$$\begin{aligned}\mathrm{length}_E(\mathrm{Hom}_R(X,M)) &= \mathrm{length}_F(\mathrm{Hom}_E(\mathrm{Hom}_R(X,M),I)) \\ &= \mathrm{length}_F(N\otimes_R X)\end{aligned}$$

and hence $\mathrm{length}_F(N\otimes_R X) = D\chi_M(X) = \chi_{DM}(X)$. Now N is a direct sum of copies of DM by the lemma below.

**6.4** LEMMA. If N is an R-E-bimodule of finite length over E and $\chi_1,\dots,\chi_n$ are the characters of the isomorphism classes of indecomposable summands of N as an R-module, then there are positive integers $a_1,\dots,a_n$ with

$$\mathrm{length}_E(X\otimes_R N) = a_1\chi_1(X) + \dots + a_n\chi_n(X)$$

for all f.p. modules X.

*Proof*. Let $\chi_i$ be the character of an indecomposable module $M_i$, so by assumption we can write $N = \coprod_{i=1}^{n} \coprod_{I_i} M_i$ for non-empty index sets $I_i$. Since

$$\mathrm{End}_R(N)/\mathrm{rad}\ \mathrm{End}_R(N) \cong \prod_{i=1}^{n}\mathrm{End}_R(\coprod_{I_i}M_i)/\mathrm{rad}\ \mathrm{End}_R(\coprod_{I_i}M_i)$$

the simple $\mathrm{End}_R(N)$-modules which occur in a composition series for $X\otimes_R N$ have the form $S_i = \coprod_{I_i} T_i$ for some i, where $T_i$ is the unique simple $\mathrm{End}_R(M_i)$-module. Since $X\otimes_R N \cong \coprod_{i=1}^{n} \coprod_{I_i} X\otimes M_i$, the number of times that $S_i$ occurs in the composition series is

$$\mathrm{length}_{\mathrm{End}_R(M_i)}(X\otimes M_i) = \chi_i(X).$$

Letting $a_i = \mathrm{length}_E(S_i) > 0$, the equality follows.

**6.5** REMARK. If M is a f.p. module of finite endolength, then $\mathrm{Hom}_E(M,I)$ is indecomposable by (2.1), so is isomorphic to DM. It then follows from (2.3) that DM is the source of a left almost split map.

## 7 GENERIC MODULES

Recall that a ring R is called a *noetherian* (respectively *artin*) *algebra* if its centre Z(R) is a noetherian (respectively artinian) ring, and R is a f.g. Z(R)-module. The reason for considering noetherian algebras is that they have a good supply of modules of finite endolength, the finite length modules. Indeed, a module has finite length if and only if it has finite length as a Z(R)-module, for example by Lenagan's Theorem (4.6). The next proposition shows that amongst the finite endolength modules, there is an essentially unique finiteness condition.

**7.1** PROPOSITION. For an indecomposable finite endolength module M over a noetherian algebra, the following are equivalent

(1) M has finite length.

(2) M is finitely presented.

(3) M occurs as the source for a left almost split map.

Moreover these conditions are equivalent to the same conditions for DM.

*Proof.* (1)$\Leftrightarrow$(2) follows from Lenagan's Theorem (4.6). If (3) holds, then $M \cong \mathrm{Hom}_E(X,I)$ by (2.3). Now X is f.g. as a Z(R)-module so E is a noetherian Z(R)-algebra, and therefore I is artinian as a Z(R)-module. Thus M is artinian, and hence of finite length by (4.6). If M is f.p. then the same argument shows that DM has finite length, and then M is the source of a left almost split map by (6.5).

**7.2** In view of the above proposition, it is natural to pick out the indecomposable modules of finite endolength which have infinite length. We call these *generic modules*. We then say that a noetherian algebra R is *generically trivial* if it has no generic modules, is *generically tame* if for all d there are only finitely many isomorphism classes of generic modules of endolength d, and is *generically wild* if there is a generic module whose endomorphism ring is not a PI ring. Note that since the endomorphism ring $\mathrm{End}_R(M)$ of a generic module has nilpotent radical, it is a PI ring if and only if the division ring $\mathrm{End}_R(M)/\mathrm{rad}\,\mathrm{End}_R(M)$ is finite dimensional over its centre. The definitions above are in terms of generic left R-modules. By (6.2) the notions of generic triviality and generic tameness are left-right symmetric. We do not know if the same is true for generic wildness.

**7.3** We first consider the question of generic triviality. Recall that an artin algebra R is said to have *strongly unbounded representation type* if, for infinitely many d, there are infinitely many non-isomorphic indecomposable R-modules of length d. The important part is the existence of one such d, for then the existence of infinitely many d has been proved by Smalø [Sm]. This condition is, however, impossible if R is a finite ring. A more natural condition is to use finite length modules of bounded endolength. The following result is proved in §9.

THEOREM. If R is an artin algebra, then R has a generic module if and only if there are infinitely many non-isomorphic indecomposable finite length R-modules of some fixed endolength. If in addition the simple R-modules have infinite underlying sets, these statements are equivalent to R having strongly unbounded representation type.

The example of a discrete valuation ring shows that the equivalence fails for noetherian algebras without some extra assumptions, while an example of Ringel, see [CB3, 1.6], shows that it fails in general for artinian rings.

The Second Brauer-Thrall Conjecture, which is now proved, asserts that if a finite dimensional algebra over an algebraically closed field is of infinite representation type, then it has strongly unbounded representation type. Thus one has

COROLLARY. A finite dimensional algebra over an algebraically closed field has finite representation type if and only if it is generically trivial.

The natural extension of the Second Brauer-Thrall Conjecture is to ask whether this corollary remains true for noetherian algebras. Of course this reduces immediately to artin algebras, since in a generically trivial noetherian algebra every prime ideal must be maximal.

**7.4** If k is an algebraically closed field and R and S are f.g. k-algebras, let us say that a functor $F: S\text{-Mod} \longrightarrow R\text{-Mod}$ is a *representation embedding* if

(1) F sends indecomposable modules to indecomposable modules,

(2) F sends non-isomorphic modules to non-isomorphic modules,

(3) $F \cong M\otimes_S -$ where M is an R-S-bimodule which is f.g. projective as an S-module (and on which k acts centrally). Equivalently, F is an exact k-linear functor which preserves products and direct sums.

Clearly a representation embedding sends f.d. modules to f.d. modules, and sends modules of finite endolength to modules of finite endolength. A f.g. k-algebra R is said to be of *wild representation type* if there is representation embedding $k\langle x,y\rangle\text{-Mod} \longrightarrow R\text{-Mod}$, where $k\langle x,y\rangle$ is the free associative algebra on two generators. (This is the variant of Drozd's original definition of wild representation type used in [CB3].) In this case there is a representation embedding $S\text{-Mod} \longrightarrow R\text{-Mod}$ for any f.g. k-algebra S, namely if $S = k\langle x_1,\dots,x_n\rangle/I$ then the composition

$$S\text{-Mod} \longrightarrow k\langle x_1,\dots,x_n\rangle\text{-Mod} \xrightarrow{G} k\langle x,y\rangle\text{-Mod} \xrightarrow{F} R\text{-Mod},$$

is a representation embedding. Here G is the fully faithful representation embedding used by Brenner [B]. Thus, in some sense, R is at least as bad as S, for any S.

Now suppose that R is a f.d. k-algebra, with k still an algebraically closed field. By a *one-parameter family of R-modules of dimension* d, we mean the set of modules $\{M\otimes_{k[T]}k[T]/(T-\lambda)\,|\,\lambda\in k\}$, where M is an R-k[T]-bimodule, free of rank d over k[T]. We say that R is of *tame representation type* provided that for all d > 0 there are a finite number of such one-parameter families, such that every indecomposable R-module of dimension d is isomorphic to a module in one of these families. The following theorem is fundamental.

THEOREM OF DROZD. A f.d. algebra R is either tame or wild, and not both.

This is proved in [D], see also [CB1]. With the precise definitions used here, it is discussed in [CB3]. In [CB3] we have used the method of Drozd's Theorem to study generic modules. The basic results are

THEOREM. If R is a tame f.d. algebra, then R is generically tame. In this case, if M is a generic R-module then $\mathrm{End}_R(M)/\mathrm{rad}\,\mathrm{End}_R(M)$ is a rational function field in one variable over k, the ring $\mathrm{End}_R(M)$ is split over its radical, and any two splittings are conjugate.

There are additional results which show that in this case the generic modules act in some way as "function fields" or "generic points" for the one-parameter families of f.d. modules. It is this fact which explains the terminology of "generic module", and indeed our original interest in modules of finite endolength. We shall not explain these results here, but refer the reader to [CB3]. We point out, however, the following characterization of tame and wild representation type.

COROLLARY. A f.d. algebra R is either generically tame or generically wild, and not both.

*Proof.* In view of the theorem above, one only needs to prove that if R is wild, then it is generically wild and not generically tame. Now if R is wild then there is a representation embedding $F: k\langle x,y\rangle\text{-Mod} \longrightarrow R\text{-Mod}$. To see that R is not generically tame, observe that the images of the modules $k(x)[y]/(y-\lambda)$ with $\lambda \in k$ are non-isomorphic generic R-modules of bounded endolength. For the generically wildness of R, observe that if D is the universal skew field of fractions of $k\langle x,y\rangle$ then F(D) is a generic R-module, D embeds in $\mathrm{End}_R(F(D))$, but D is not a PI ring.

We conjecture that this corollary remains true for noetherian algebras.

**7.5** We finish this section with another question. Let us say that an artin algebra is *generically directed* if generic modules can never be involved in cycles $M_0 \longrightarrow M_1 \longrightarrow \ldots \longrightarrow M_n \longrightarrow M_0$ of non-zero non-isomorphisms between finite endolength indecomposables. In particular the endomorphism ring of any generic module is a division ring.

If R is a finite dimensional algebra over an algebraically closed field, R is generically directed, and R has a faithful generic module, is it true that R is either tame concealed or tubular in the sense of [R4]? Note that such an algebra must be tame, since if R is wild then one can easily construct generic modules whose endomorphism ring is not a division ring.

## 8 HEREDITARY ALGEBRAS

Let k be a field. In this section, by "algebra" we mean a k-algebra (not necessarily f.d.), and by "bimodule" we mean a bimodule on which k acts centrally. We prove a result which will be needed in the next section, but along the way we determine the behaviour of generic modules for f.d. hereditary algebras.

**8.1** The following lemma makes an assertion of Ringel more precise, allowing the argument used in [R1, §5.4] to be extended from the category of finite dimensional modules, to the category of all modules. Let R be an algebra and let X and Y be left R-modules which are finite dimensional and finitely presented. Suppose that $\mathrm{Hom}_R(Y,X) = 0$, $\mathrm{Hom}_R(X,Y) = 0$, and that $E = \mathrm{End}_R(X)^{op}$ and $F = \mathrm{End}_R(Y)^{op}$ are semisimple algebras. Since X is f.p. and Y is f.d., the E-F-bimodule $\mathrm{Ext}^1_R(X,Y)$ is f.d.. Let M be the F-E-bimodule $\mathrm{Hom}_F(\mathrm{Ext}^1_R(X,Y),F)$ and let S be the generalized upper triangular matrix algebra $\begin{pmatrix} F & M \\ 0 & E \end{pmatrix}$.

Lemma. There is an R-S-bimodule T, f.g. projective over S, inducing a fully faithful functor $T\otimes_S - : S\text{-Mod} \longrightarrow R\text{-Mod}$.

*Proof*. (1) Let P, P′ be E-modules and Q, Q′ be F-modules. Since X and Y are f.g. we have $\mathrm{Hom}_R(X\otimes_E P, X\otimes_E P') \cong \mathrm{Hom}_E(P,P')$, $\mathrm{Hom}_R(Y\otimes_F Q, Y\otimes_F Q') \cong \mathrm{Hom}_F(Q, Q')$, $\mathrm{Hom}_R(X\otimes_E P, Y\otimes_F Q') = 0$ and $\mathrm{Hom}_R(Y\otimes_F Q, X\otimes_E P') = 0$. Thus, if we are given short exact sequences of R-modules making up the rows of the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y\otimes_F Q & \longrightarrow & L & \longrightarrow & X\otimes_E P & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\psi} & & \\ 0 & \longrightarrow & Y\otimes_F Q' & \longrightarrow & L' & \longrightarrow & X\otimes_E P' & \longrightarrow & 0 \end{array}$$

then for any map $\theta \in \mathrm{Hom}_R(L,L')$ there are maps $\phi$, $\psi$ making the diagram commute. Moreover $\theta$ is uniquely determined by $\phi$, $\psi$ since if $\theta'$ also makes the diagram commute then $\theta-\theta'$ induces a map $X\otimes_E P \longrightarrow Y\otimes_F Q'$.

(2) The fact that X is finitely presented implies that

$$\mathrm{Ext}^1_R(X\otimes_E P, Y\otimes_F Q) \cong \mathrm{Hom}_E(P, \mathrm{Ext}^1_R(X,Y)\otimes_F Q)$$

and in particular that

$$\mathrm{Ext}^1_R(X, Y\otimes_F M) \cong \mathrm{Ext}^1_R(X,Y)\otimes_F M \cong \mathrm{Hom}_F(M,M). \tag{$\dagger$}$$

Let

$$\zeta : 0 \longrightarrow Y\otimes_F M \xrightarrow{p} N \xrightarrow{q} X \longrightarrow 0$$

be an extension corresponding to the identity endomorphism of M. Now the left and right hand terms of $\zeta$ are naturally R-E-bimodules, and since ($\dagger$) is an isomorphism of E-E-bimodules it follows that $e\zeta = \zeta e$ for all $e\in E$. Thus by (1), for each $e\in E$ there is a unique endomorphism $\theta$ of N making the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y\otimes_F M & \longrightarrow & N & \longrightarrow & X & \longrightarrow & 0 \\ & & \downarrow e & & \downarrow \theta & & \downarrow e & & \\ 0 & \longrightarrow & Y\otimes_F M & \longrightarrow & N & \longrightarrow & X & \longrightarrow & 0 \end{array}$$

commute. This gives N the structure of an R-E-bimodule in such a way that p and q are bimodule maps.

(3) Set $T = Y\oplus N$. We turn it into R-S-bimodule by defining

$$(y,n)\begin{bmatrix} f & m \\ 0 & e \end{bmatrix} = (yf, p(y\otimes m)+ne) \quad \text{for } (y,n)\in T \text{ and } \begin{bmatrix} f & m \\ 0 & e \end{bmatrix}\in S.$$

This is projective as an S-module since p is mono, and of course it is finitely generated over S since it is f.d..

(4) A left S-module U is determined by a triple $(P,Q,g)$ where P is an E-module, Q is an F-module, and g is an F-module map $M\otimes_E P \longrightarrow Q$. Namely set $P = e_{22}U$ and $Q = e_{11}U$. Now U has a projective presentation of the form

$$0 \longrightarrow (0, M\otimes_E P, 0) \longrightarrow (P, M\otimes_E P\oplus Q, \begin{bmatrix} 1 \\ 0 \end{bmatrix}) \longrightarrow U \longrightarrow 0,$$

and tensoring with T gives an exact sequence

$$0 \longrightarrow Y\otimes_F M\otimes_E P \longrightarrow N\otimes_E P \oplus Y\otimes_F Q \longrightarrow T\otimes_S U \longrightarrow 0.$$

Thus $T\otimes_S U$ fits in the pushout diagram

$$\begin{tikzcd}
0 \arrow[r] & Y\otimes_F M\otimes_E P \arrow[r] \arrow[d] & N\otimes_E P \arrow[r] \arrow[d] & X\otimes_E P \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Y\otimes_F Q \arrow[r] & T\otimes_S U \arrow[r] & X\otimes_E P \arrow[r] & 0
\end{tikzcd}$$

Clearly the lower exact sequence corresponds to the element g under the isomorphism $\mathrm{Hom}_F(M\otimes_E P, Q) \cong \mathrm{Ext}^1_R(X\otimes_E P, Y\otimes_F Q)$.

Now let $U' = (P', Q', g')$ be a second S-module. The S-module maps $\alpha: U \longrightarrow U'$ correspond to pairs $(\beta, \gamma)$ where $\beta \in \mathrm{Hom}_E(P, P')$, $\gamma \in \mathrm{Hom}_F(Q, Q')$ and such that $g'\circ(1\otimes\beta) = \gamma\circ g$. Such a map $\alpha$ gives a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & Y\otimes_F Q \arrow[r] \arrow[d, "1\otimes\gamma"'] & T\otimes_S U \arrow[r] \arrow[d, "1\otimes\alpha"'] & X\otimes_E P \arrow[r] \arrow[d, "1\otimes\beta"'] & 0 \\
0 \arrow[r] & Y\otimes_F Q' \arrow[r] & T\otimes_S U' \arrow[r] & X\otimes_E P' \arrow[r] & 0
\end{tikzcd}$$

and if $1\otimes\alpha = 0$, then $1\otimes\beta = 1\otimes\gamma = 0$, and hence $\beta = \gamma = 0$, so that $T\otimes_S -$ is faithful. Conversely, if $\theta \in \mathrm{Hom}_R(T\otimes_S U, T\otimes_S U')$ then by (1) this map induces a commutative diagram with maps $\beta$, $\gamma$. Now $(\beta, \gamma)$ is a homomorphism $\alpha: U \longrightarrow U'$ and $1\otimes\alpha = \theta$, so that $T\otimes_S -$ is full.

**8.2** Let us say that an algebra R is *strictly wild* if there are f.d. left R-modules X and Y, which are finitely presented, whose endomorphism rings are division algebras, with $\mathrm{Hom}_R(X, Y) = 0$, $\mathrm{Hom}_R(Y, X) = 0$, and the product

$$p = \dim_{\mathrm{End}_R(Y)} \mathrm{Ext}^1_R(X, Y) \;.\; \dim \mathrm{Ext}^1_R(X, Y)_{\mathrm{End}_R(X)}$$

equal to at least 5. The next result is due to Ringel [R1].

LEMMA. A finitely generated algebra R is strictly wild if and only if there is a finite extension field K of k and an

R-K<x,y>-bimodule T which is f.g. projective over K<x,y> and such that the tensor product functor $T\otimes_{K<x,y>}-$ : K<x,y>-Mod $\longrightarrow$ R-Mod is fully faithful.

*Proof*. Suppose first that there exists such a bimodule. As in (7.4), for any K-algebra S there is a fully faithful functor $M\otimes_S-$: S-Mod$\longrightarrow$R-Mod with M an R-S-bimodule, f.g. projective over S. Taking S to be strictly wild, and letting X and Y be the images of a pair of S-modules which make S strictly wild, one obtains a pair of f.d. R-modules whose Hom and $\mathrm{Ext}^1$ spaces satisfy the requirements above. Because R is f.g. and X, Y are finite dimensional, they are finitely presented. Thus R is strictly wild.

To prove that a strictly wild algebra R has a fully faithful tensor product functor K<x,y>-Mod $\longrightarrow$ R-Mod, it suffices to deal with algebras of the form $R = \begin{pmatrix} F & M \\ 0 & E \end{pmatrix}$ with E, F division algebras and $(\dim_F M)(\dim M_E) \geq 5$. Namely, in general, if R is strictly wild due to the existence of modules X, Y, then the algebra S constructed in (8.1) has this special form, so by assumption there is a suitable functor K<x,y>-Mod $\longrightarrow$ S-Mod. The composition of this functor with the functor S-Mod $\longrightarrow$ R-Mod given by Lemma (8.1) is a suitable tensor product functor K<x,y>-Mod $\longrightarrow$ S-Mod.

The proof that an algebra of the form $R = \begin{pmatrix} F & M \\ 0 & E \end{pmatrix}$ has a suitable functor K<x,y>-Mod $\longrightarrow$ R-Mod, follows part of the argument used in the proof of [R1, Theorem 2], working by induction on $d = \max\{\dim_K E, \dim_K F\}$ where K is the centre of the bimodule M (which is also the centre of R). If d=1, then E=F=K and $M=K^r$ with $r\geq 3$, and it is easy to find a suitable R-K<x,y>-bimodule, free of rank two over K<x,y>. Suppose, therefore that d>1. As in [R1, §5.3] one can find finite dimensional R-modules $A_1$, $A_2$ with each $\mathrm{End}_R(A_i)$ a division ring, $\dim_K \mathrm{End}_R(A_1) < d$, $\mathrm{Hom}_R(A_i, A_j) = 0$ for $i\neq j$ and $\mathrm{Ext}^1_R(A_i, A_j) \neq 0$ for all i,j. Now let $\mathfrak{U}$ be the full subcategory of R-Mod on the modules which have a finite filtration in which the quotients are isomorphic to $A_1$ or $A_2$. By [R1, §1.2 Theorem] this is an exact abelian subcategory of R-Mod, and since R is hereditary, the category $\mathfrak{U}$ is hereditary. It follows from this, and the fact that $\mathrm{Ext}^1_R(A_i,A_j) \neq 0$ for all i, j, that one can find objects X, Y of $\mathfrak{U}$ which are uniserial as objects of $\mathfrak{U}$, and with composition series in $\mathfrak{U}$ of the form

$$X \quad \begin{array}{l} A2 \\ A1 \end{array} \qquad\qquad Y \quad \begin{array}{l} A2 \\ A2 \\ A1 \\ A1 \end{array}$$

Now $\mathrm{End}_R(X)$ and $\mathrm{End}_R(Y)$ embed in $\mathrm{End}_R(A_1)$, $\mathrm{Hom}_R(X,Y) = \mathrm{Hom}_R(Y,X) = 0$, $\dim_{\mathrm{End}(Y)}\mathrm{Ext}^1_R(X,Y) \geq 3$, and $\dim \mathrm{Ext}^1_R(X,Y)_{\mathrm{End}(X)} \geq 2$, see [R1, §5.4]. Let S be the algebra constructed from X and Y in (8.1). By induction there is a functor $K\langle x,y\rangle$-Mod $\longrightarrow$ S-Mod, and its composition with the functor S-Mod $\longrightarrow$ R-Mod gives the desired tensor product functor from $K\langle x,y\rangle$-Mod to R-Mod.

**8.3** If R is a connected f.d. hereditary algebra, there is a bilinear form defined on the Grothendieck group $K_0(R)$ of f.d. left modules modulo short exact sequences given by

$$\langle M,N\rangle = \dim_k \mathrm{Hom}_R(X,Y) - \dim_k \mathrm{Ext}^1_R(X,Y),$$

and this induces a quadratic form $q_R$ on $K_0(R)\otimes_{\mathbb{Z}}\mathbb{Q}$. The algebra R is of finite representation type if $q_R$ is positive definite, it is said to be *tame hereditary* if $q_R$ is positive semidefinite but not positive definite, and *wild hereditary* if $q_R$ is indefinite. Note that these notions are purely combinatorial, but in case the base field k is algebraically closed they coincide with the notions discussed in §7, except for the fact that a tame hereditary algebra is necessarily of infinite representation type.

**8.4** THEOREM. A f.d. wild hereditary algebra is strictly wild.

*Proof.* If R has two simple modules this is clear. Thus suppose that R has $n > 2$ simple modules. We follows the argument of [R5, §1 Theorem] and use the terminology of that paper. Let $\Delta(R)$ be the species of R. If $R^{op}$ is strictly wild, then clearly so is R. If S is obtained from R by reflection at a sink in $\Delta(R)$ and R is strictly wild, then so is S, for by Lemma (8.2) there is a very wide choice of modules X and Y giving the strict wildness, so we may choose X and Y to be regular, and then they correspond to S-modules. Now the argument in [R5] shows that, up to duality and reflections, one of the three cases below occurs. We verify in each one that R is strictly wild.

*Case 1.* R is a one-point extension of a connected hereditary algebra S of infinite representation type, so $R = \begin{pmatrix} S & M \\ 0 & D \end{pmatrix}$ where D is a division algebra and M is a non-zero S-D-bimodule which is projective as a S-module. Now if Y is an S-module regarded as an R-module, X is the simple R-module corresponding to D, and P is the projective cover of X, then X has projective resolution $0 \to M \to P \to X \to 0$, so $\mathrm{Ext}^1_R(X,Y) \cong \mathrm{Hom}_S(M,Y)$. This can be made arbitrarily large with Y an indecomposable preprojective S-module by [R5, §1 Lemma 1]. It follows that R is strictly wild.

*Case 2.* R has species

$$\begin{array}{ccccccccc} & 2 & \leftarrow & 3 & \leftarrow & \cdots & \leftarrow & r & \\ \swarrow & & & & & & & & \nwarrow \\ 1 & & & & & & & & r+s \\ \nwarrow & & & & & & & & \swarrow \\ & r+1 & \leftarrow & r+2 & \leftarrow & \cdots & \leftarrow & r+s-1 & \end{array}$$

with $r\geq 2$, $s\geq 1$, and not all arrows trivially valued. Let $Y = P(1)$ and set $X = P(r+s)/\mathrm{soc}\, P(r+s)$. Note that the socle of $P(r+s)$ is the direct sum of $a'+b'$ copies of $P(1)$ where

$$a' = d'_{12}\, d'_{23} \cdots d'_{r-1,r}\, d'_{r,r+s}, \quad \text{and}$$

$$b' = d'_{1,r+1}\, d'_{r+1,r+2} \cdots d'_{r+s-1,r+s}.$$

Using the projective resolution $0 \to (a'+b')P(1) \to P(r+s) \to X \to 0$ one sees that $\dim_{\mathrm{End}(Y)} \mathrm{Ext}^1_R(X,Y) = a'+b'$. Moreover $\mathrm{End}_R(X) \cong$

$\mathrm{End}_R(P(r+s))$ so that $\dim_k \mathrm{End}_R(Y)/\dim_k \mathrm{End}_R(X) = a/a' = b/b'$, where $a$ and $b$ are defined in the same way as $a'$ and $b'$ but using the $d_{ij}$ instead of the $d'_{ij}$. Thus the product $p$ of (8.2) is equal to $(a'+b')(a+b)$. Now $a, a', b, b' \geq 1$, and by assumption not all are equal to 1, so we have $p \geq 6$.

*Case 3.* R has species

$$1 \xleftarrow{(a,b)} 2 \xleftarrow{(c,d)} 3.$$

with $abcd \geq 6$. Let $Y = P(1)$ and $X = P(3)/\mathrm{soc}\, P(3)$. Using the projective resolution $0 \longrightarrow bdP(1) \longrightarrow P(3) \longrightarrow X \longrightarrow 0$ one finds that the product $p$ of (8.2) is equal to $abcd$, so R is strictly wild.

**Corollary.** If R is a f.d. hereditary algebra, then R is generically trivial if and only if it has finite representation type. Moreover R is either generically tame or generically wild and not both.

*Proof.* We may assume that R is connected. If R is wild hereditary then it is generically wild, but not generically tame as in Corollary (7.4). If R is tame hereditary then Ringel [R3, §6] has proved (with an unnecessary extra hypothesis) that R has a unique generic module, and its endomorphism ring is a PI ring by [BGL, 6.12].

**Remark.** In the tame hereditary case the generic module has a rather interesting endomorphism ring E. For example, if R is the generalized triangular matrix ring $\begin{pmatrix} \mathbb{H} & \mathbb{H} \\ 0 & \mathbb{R} \end{pmatrix}$ with $\mathbb{H}$ the quaternion division ring, then E is the field $\mathbb{R}(X,Y \mid X^2+Y^2+1=0)$. More generally, if R is tame hereditary and R/rad R is separable over Z(R) then the endomorphism ring of the generic module is a division ring whose centre is a function field in one variable over Z(R) of genus zero. See [CB2] for more discussion.

**8.5** We also need to consider a class of non-noetherian rings which generalizes the free associative algebras. In his study of free ideal rings, Cohn has considered filtered rings

$$R_0 \subseteq R_1 \subseteq R_2 \subseteq \dots \subseteq R,$$

defined a 'weak algorithm' for such rings, and given a construction [C, §2.5] of all filtered rings which have a weak algorithm. Supposing that R is generated by $R_1$ this takes the following form. Let D be a division ring, let

$$0 \longrightarrow D \xrightarrow{e} X \xrightarrow{f} Y \longrightarrow 0 \qquad (\ddagger)$$

be an exact sequence of D-D-bimodules, and set $\pi = e(1)$. Then R is the ring $X^{\otimes D}/(\pi-1)$, where $X^{\otimes D} = D \oplus X \oplus X\otimes_D X \oplus \dots$ is the tensor ring. The terms in the filtration are the images of $X\otimes_D \dots \otimes_D X$, so that $R_0 = D$ and $R_1 = X$. In case D is a f.d. division algebra and X and Y are f.d. bimodules we call $X^{\otimes D}/(\pi-1)$ a *skew tensor algebra*.

**Lemma**. If R is a skew tensor algebra given by an exact sequence $(\ddagger)$ with $\dim_D Y \geq 2$, then R is strictly wild.

*Proof*. We begin by showing that there are at least two non-isomorphic R-modules S with $\dim_D S = 1$ (so that S is simple). An R-module is determined by a left D-vector space V and a D-module map $\psi : X\otimes_D V \longrightarrow V$ satisfying $\psi(\pi\otimes v) = v$. The modules we want, correspond to the case V = D, say with $S_\psi$ being the module determined by a map $\psi \in \mathrm{Hom}_D(X,D)$ with $\psi(\pi) = 1$. Now the algebraic group $D^\times = D\setminus\{0\}$ acts on the variety $W = \{\psi\in\mathrm{Hom}_D(X,D) \mid \psi(\pi)=1\}$ of such $\psi$ via $(d.\psi)(x) = \psi(xd)d^{-1}$, and the orbits correspond to the isomorphism classes of $S_\psi$. Since $\dim(W) = \dim_k Y \geq 2\dim_k D = 2\dim(D^\times)$, a dimension argument, or if k is finite a counting argument, shows that there must be at least two orbits, as required.

Next we construct an exact sequence of R-R-bimodules of the form

$$\eta\ :\ 0 \longrightarrow R\otimes_D Y\otimes_D R \xrightarrow{h} R\otimes_D R \xrightarrow{m} R \longrightarrow 0.$$

For m one takes the multiplication map. Let $\alpha$ be the map

$$X^{\otimes D} \otimes_D X \otimes_D X^{\otimes D} \longrightarrow R \otimes_D R,$$
$$u\otimes x\otimes v \longmapsto \overline{u\otimes x} \otimes \bar{v} - \bar{u} \otimes \overline{x\otimes v}$$

where the bar denotes reduction by the ideal $(\pi-1)$. It is easily seen that $\alpha$ induces a map h as above. It remains to show that $\eta$ is exact. Now Cohn has shown that if $\pi, x_1, \ldots, x_n$ is a left D-basis for X, then the (images in R of the) monomials in the $x_i$ form a left D-basis for R. Thus $R\otimes_D R$ is a free left R-module with basis the elements $1\otimes q$ (q a monomial), and so the elements $q\otimes q'$ (q, q′ monomials) form a left D-basis for $R\otimes_D R$. It follows that Ker(m) has as left D-basis the elements of the form $qx_i\otimes q' - q\otimes x_i q'$ with q and q′ monomials and $1\le i\le n$. These elements are the images under h of the elements $q\otimes f(x_i)\otimes q'$, and a similar argument shows that these form a left D-basis for $R\otimes_D Y\otimes_D R$, which proves that $\eta$ is exact.

Tensoring $\eta$ with any R-module gives a projective presentation of that module, so R is hereditary, which was already clear. Moreover it follows that if $S_1$ and $S_2$ are non-isomorphic modules with $\dim_D S_i = 1$ then the $S_i$ are finitely presented and $\mathrm{Ext}^1_R(S_i, S_j) \neq 0$ for any choice of i, j. Now the argument of [R1, §5.4] together with (8.2) shows that R is strictly wild.

**8.6** Lemma. Let R be a f.d. hereditary algebra or a skew tensor algebra. If R has infinite representation type, then

(1) If k is infinite then R has infinitely many non-isomorphic indecomposable modules of some fixed dimension.

(2) R has infinitely many non-isomorphic f.d. indecomposable modules of some fixed endolength.

(3) There is an R-k(T)-bimodule, indecomposable over R, and

finite dimensional over $k(T)$.

*Proof.* If $R$ is strictly wild, then this follows from Lemma (8.2). If $R$ is a skew tensor algebra corresponding to the exact sequence (‡) of (8.5) then we may assume that $\dim_D Y = 1$ so that $R$ is actually a skew polynomial ring $D[T;\varepsilon,\delta]$ with $\varepsilon$ an automorphism of $D/k$ and $\delta$ an $(\varepsilon,1)$-derivation of $D/k$. Now $R$ is an hereditary noetherian domain and a PI ring since it is f.g. as a module over the (non-central) subring $k[T]$. Therefore the centre $Z$ of $R$ is a Dedekind domain and $R$ is f.g. as a $Z$-module [MR, 13.9.16]. For (1) one takes the simple modules $R/(T-\lambda)$ with $\lambda\in k$. For (2) one takes the simple modules, since by the Nullstellensatz [MR, 13.10.3] there are infinitely many and they are f.d.. Let $Q$ be the simple artinian quotient ring of $R$. By Posner's Theorem $Q$ is f.d. over its centre which is the quotient field $K$ of $Z$. Now $Z$ is f.g. over $k$ by the Artin-Tate lemma [MR, 13.9.10], so $K$ is a finitely generated extension field of $k$ of transcendence degree 1. Therefore $K$ and $Q$ are finite dimensional over $k(T)$ for some $T \in K$. Now for the bimodule in (3) one can take the simple $Q$-module.

If $R$ is a f.d. hereditary algebra then we may assume that it is tame hereditary. Now (1) is contained in [DR, Theorem E], and (3) in [R2, Theorem 5.7] and [BGL, Corollary 6.12]. One knows, see for example [CB2], that there is a ring-theoretic epimorphism $R \longrightarrow S$ where $S$ is a classical hereditary order, finitely generated as a $k$-algebra. Now $S$ has infinitely many simple modules, and each one restricts to a f.d. indecomposable $R$-module of endolength bounded by the PI degree of $S$, proving (2).

## 9 LIFT CATEGORIES

In this section we prove Theorem (7.3). We use the method of matrix reductions, which enables an inductive proof, reducing to the hereditary case which we have solved in the previous section.

The proof of Drozd's Tame and Wild Theorem also uses matrix reductions, in the form of bocses, but these cannot be used with general artin algebras. Instead, we have introduced in [CB4] the notion of a "lift category" and used it to study artinian rings of finite representation type.

A *lift pair* $(R,\xi)$ consists of a ring R and an exact sequence

$$\xi : 0 \longrightarrow M \longrightarrow E \xrightarrow{\pi} R \longrightarrow 0$$

of R-R-bimodules, and the corresponding *lift category* $\xi(R)$ has as objects the pairs $(P,e)$ where P is a projective left R-module and e is a section for map $\pi\otimes 1: E\otimes_R P \longrightarrow P$, and as morphisms from $(P,e)$ to $(P',e')$ the R-module maps $\theta: P \longrightarrow P'$ which intertwine e and e'.

$$\begin{array}{ccc} P & \xrightarrow{e} & E\otimes_R P \\ \theta\downarrow & & \downarrow 1\otimes\theta \\ P' & \xrightarrow{e'} & E\otimes_R P' \end{array}$$

Let C be a commutative artinian local ring with maximal ideal $\mathfrak{m}$ and residue field k. We now consider C-algebras, and C is supposed to act centrally on all bimodules. We say that a lift pair $(R,\xi)$ is C-*algebraic* provided that R is an artin C-algebra and E is f.g. as a C-module.

Let $(R,\xi)$ be a C-algebraic lift pair and let $J = \operatorname{rad} R$. Let us say that an object $X = (P,e)$ in $\xi(R)$ is *sincere* if P/JP is a sincere R-module, so involves all simple R-modules. We define the *length* of X (over C) to be $\operatorname{length}_C(P/JP)$. If $R_X = \operatorname{End}_{\xi(R)}(X)^{op}$, then P is naturally an R-$R_X$-bimodule, and we define the *endolength*, endolen(X), of X to be the length of P/JP as an $R_X$-module. One might have defined the last two notions without reducing modulo JP, but the definitions given provide the useful numbers, and since R is artinian the finiteness of the length or endolength is independent of the definition. We say that X is *generic* if it is indecomposable, of finite endolength, but of infinite length.

A lift pair $(R,\xi)$ is said to be of *finite representation type* if there are only finitely many isomorphism classes of indecomposable objects in $\xi(R)$ and they all have finite length. In addition we consider the following conditions

(C1) $\xi(R)$ has infinitely many non-isomorphic indecomposable objects of some fixed length.

(C2) $\xi(R)$ has infinitely many non-isomorphic finite length objects of some fixed endolength.

(C3) $\xi(R)$ has an indecomposable object $X = (P,e)$ with a C-algebra map $C[T]_{mC[T]} \longrightarrow R_X$, such that $P/JP$ has finite length over $C[T]_{mC[T]}$.

(C4) $\xi(R)$ has a generic object.

Note that $C[T]_{mC[T]}$ is an artinian local ring with residue field $k(T)$ and that (C3) $\Rightarrow$ (C4). Also (C1) $\Rightarrow$ (C2), but (C1) is never possible if k is finite. We adopt the *convention* that when we talk of (C1)-(C4) below, we exclude (C1) in case k is finite.

**9.1** Lemma. If $(R,\xi)$ is a C-algebraic lift pair, R is semisimple, and the first term M of $\xi$ is a simple bimodule or zero, then (C1)-(C4) are equivalent to $(R,\xi)$ being of infinite representation type.

*Proof*. Since R is semisimple, $mR = 0$, so that R is a k-algebra. By [CB4, 2.1] the category $\xi(R)$ is equivalent to A-Mod where $A = (E^*)^{\otimes R}/(\pi-1)$, and this equivalence preserves endolength. Moreover by [CB4, 2.2] (or at least its proof) this algebra is either a f.d. hereditary algebra, or the product of a semisimple artinian ring and a matrix ring over a skew tensor algebra. The result thus follows from Lemma (8.6). Note that since $C[T]_{mC[T]}$ is a local ring with residue field $k(T)$, any $k(T)$-module is naturally a $C[T]_{\mathfrak{m}C[T]}$-module.

**9.2** Let $(R,\xi)$ be a C-algebraic lift pair with exact sequence

$$\xi : 0 \longrightarrow M \longrightarrow E \xrightarrow{\pi} R \longrightarrow 0.$$

Let $N \subseteq M$ be a maximal sub-bimodule, and let $J = \operatorname{rad} R$. One can form lift pairs $(R,\xi_N)$ and $(R/J,\xi_{NJ})$ with

$$\xi_N : 0 \longrightarrow \bar{M} \longrightarrow \bar{E} \longrightarrow R \longrightarrow 0$$

where $\bar{M} = M/N$ and $\bar{E} = E/N$, and

$$\xi_{NJ} : 0 \longrightarrow \bar{M}/[\bar{M}\cap(\bar{E}J+J\bar{E})] \longrightarrow \bar{E}/(\bar{E}J+J\bar{E}) \xrightarrow{\pi_{NJ}} R/J \longrightarrow 0$$

and there are functors

$$\xi(R) \xrightarrow{\sigma_N} \xi_N(R) \xrightarrow{\rho_J} \xi_{NJ}(R/J)$$

defined as follows. If $X = (P,e)$ belongs to $\xi(R)$, then $\sigma_N(X) = (P,\bar{e})$ where $\bar{e}$ is the composition $P\xrightarrow{e}E\otimes_R P\longrightarrow\bar{E}\otimes_R P$, and if $\theta:(P,e)\longrightarrow(P',e')$ is a morphism then $\sigma_N(\theta)$ is the same R-module map, considered now as a morphism from $(P,\bar{e})$ to $(P',\bar{e}')$. If $Z = (P,f) \in \xi_N(R)$ then $\rho_J(Z) = (R/J\otimes_R P,\bar{f})$ where $\bar{f}$ is the composition

$$R/J\otimes_R P \xrightarrow{1\otimes f} R/J\otimes_R\bar{E}\otimes_R P \xrightarrow{p} \bar{E}/(\bar{E}J+J\bar{E})\otimes_R P$$

with p the natural projection, and if $\theta:(P,f)\longrightarrow(P',f')$ is a morphism then $\rho_J(\theta) = 1\otimes\theta$. It is shown in [CB4, 3.1 and 4.1] that $\sigma_N$ and $\rho_J$ are both dense and reflect isomorphisms, and that $\rho_J$ is full, so that it is a representation equivalence.

One can apply (9.1) to the lift pair $(R/J,\xi_{NJ})$, and in the next two paragraphs we investigate the consequences for $(R,\xi)$.

**9.3** In this paragraph we treat the case when $(R/J,\xi_{NJ})$ is of infinite representation type. We begin with some lemmas.

**Lemma a.** Let R be an artin C-algebra and S a C-algebra such that S/mS is a separably generated extension field of k. If N is an $R\otimes_C S$-module which is projective as an R-module, then it is projective as an $R\otimes_C S$-module.

*Proof.* Let J = rad R and let L be the image of $J\otimes_C S$ in $R\otimes_C S$. Since R is artinian, m annihilates R/J, so that

$$(R\otimes_C S)/L \cong (R/J)\otimes_C S \cong (R/J)\otimes_k (S/mS),$$

and by the assumption on S/mS, this is semisimple. Since also L is nilpotent, it is the radical of $R\otimes_C S$. Now $R\otimes_C S$ is a semiprimary ring, so the module N has a projective cover, a map $\alpha: P \twoheadrightarrow N$ with P projective and $\mathrm{Ker}(\alpha)$ superfluous in P, and moreover a submodule of any $R\otimes_C S$-module M is superfluous if and only if it is contained in LM. Thus $\mathrm{Ker}(\alpha) \subseteq LP = JP$, and since R is semiprimary it follows that $\mathrm{Ker}(\alpha)$ is superfluous as an R-submodule of P. However $\alpha$ splits as an R-module map, so $\mathrm{Ker}(\alpha) = 0$, and hence N is a projective $R\otimes_C S$-module.

**Lemma b.** Let R be an artin C-algebra and let S be a C-algebra which is projective over C, and with S/mS a separably generated extension field of k. If N is an $R\otimes_C S$-module, then its projective cover as an $R\otimes_C S$-module is a projective cover as an R-module. In particular this assertion holds for $S = C[T]_{mC[T]}$.

*Proof.* As in Lemma a, the ring $R\otimes_C S$ is semiprimary with $\mathrm{rad}(R\otimes_C S)$ the image of $J\otimes_C S$, and if $\alpha: P \twoheadrightarrow N$ is a projective cover then $\mathrm{Ker}(\alpha)$ is superfluous as an R-submodule of P. Since S is projective as a C-module, P is a projective R-module, and hence $\alpha$ is an R-module projective cover. For $S = C[T]_{mC[T]}$ note that $S/mS \cong k(T)$ is separably generated over k, and that S is flat over C, and hence projective since C is artinian.

**Lemma c.** Let A be an artin C-algebra. If k is a perfect field then A has a C-subalgebra S with S + rad A = A, S ∩ rad A = rad S and rad $S = \mathfrak{m}S$.

*Proof.* Since $S = A$ satisfies the first two inequalities, one can choose $S$ to be a $C$-subalgebra of $A$, with $\text{length}_C(S)$ minimal, amongst those subalgebras satisfying the first two equalities. Since $\mathfrak{m}S$ is nilpotent it is contained in rad $S$, and for a contradiction we may suppose that $S/\mathfrak{m}S$ is not semisimple. Now $S/\mathfrak{m}S$ is a finite dimensional $k$-algebra, so split over its radical by the Wedderburn-Malcev Theorem. Taking a splitting $T/\mathfrak{m}S$, one obtains a strictly smaller subalgebra $T$ with $T + \text{rad}\, S = S$, $T \cap \text{rad}\, S = \mathfrak{m}S$ and $\text{rad}\, T \subseteq \mathfrak{m}S$, so that $T$ also satisfies the first two inequalities. A contradiction.

Proposition. Under the hypotheses of (9.2), if $(R/J,\xi_{NJ})$ is of infinite representation type, then (C1)-(C4) hold for $(R,\xi)$.

*Proof.* By (9.1) the conditions (C1)-(C4) hold for $(R/J,\xi_{NJ})$. We deduce them for $(R,\xi)$. Note first that (C1) holds because any object in $\xi_{NJ}(R/J)$ lifts to one in $\xi(R)$ of the same length, and (C4) follows from (C3), so we only need to prove (C2) and (C3). The problem is to lift the objects in $\xi_{NJ}(R/J)$ to objects in $\xi(R)$ *of the same endolength*. The approach for both of these is similar, but differs because in the case of (C3) we also need a homomorphism from $C[T]_{\mathfrak{m}C[T]}$ to the endomorphism ring of the lifted object. This is compensated for by the fact that $C[T]_{\mathfrak{m}C[T]}$ is projective over $C$.

(C2) In view of the implication (C1)⇒(C2) we only need to prove this when $k$ is finite, and hence a perfect field. Let $X = (Q,g)$ be an indecomposable object in $\xi_{NJ}(R/J)$ of finite length and with endomorphism ring $F$. Let $\alpha: P \twoheadrightarrow Q$ be the projective cover of $Q$ as an $R$-module and let

$$A = \{\theta \in \text{End}_R(P) \mid \theta(\text{Ker}(\alpha)) \subseteq \text{Ker}(\alpha) \text{ and } \bar{\theta} \in F\}$$

where $\bar{\theta}$ is the endomorphism of $Q$ induced by $\theta$ under the assumption that $\theta(\mathrm{Ker}(\alpha)) \subseteq \mathrm{Ker}(\alpha)$. The assignment $\theta \longmapsto \bar{\theta}$ is a homomorphism $A \longrightarrow F$ and the induced map $A/\mathrm{rad}\, A \longrightarrow F/\mathrm{rad}\, F$ is an isomorphism since $\alpha$ is a projective cover. Let S be the subalgebra of A chosen by Lemma c. Now P can be regarded as an $R \otimes_C S$-module and it is projective over $R \otimes_C S$ by Lemma a, since S has the property that $S/mS \cong F/\mathrm{rad}\, F$ is a finite extension field of k, hence separable. Now in the diagram

$$\begin{tikzcd} P \arrow[r, "\alpha"] \arrow[d, dotted, "f"'] & Q \arrow[d, "g"] \\ E \otimes_R P \arrow[r, "p\otimes\alpha"] & \bar{E}/(\bar{E}J + J\bar{E}) \otimes_R Q \end{tikzcd}$$

with p the projection, the modules all have natural structures as $R \otimes_C S$-modules, and the maps are all $R \otimes_C S$-module maps. Since $p \otimes \alpha$ is epi, there is a map f making the diagram commute. Also, the map $\pi \otimes 1 : E \otimes_R P \longrightarrow P$ has a section $e_0$ as an $R \otimes_C S$-module map, and if $e : P \longrightarrow E \otimes_R P$ is defined by

$$e = f + e_0 \circ (1 - (\pi \otimes 1) \circ f)$$

then $Y = (P, e)$ is an object in $\xi(R)$, and it has image X in $\xi_{NJ}(R/J)$. Moreover the fact that e is an $R \otimes_C S$-module map means that S is contained in $\mathrm{End}_{\xi(R)}(Y)$. Now the isomorphism $S/\mathrm{rad}\, S \longrightarrow A/\mathrm{rad}\, A \longrightarrow F/\mathrm{rad}\, F$ shows that X and Y have the same endolength. Finally (C2) follows by using the infinite family of objects X of the same endolength.

(C3) Since $\xi_{NJ}(R/J)$ satisfies (C3) it has an indecomposable object $X = (Q, g)$, a map $C[T]_{mC[T]} \longrightarrow \mathrm{End}(X)$, and with Q of finite length over $C[T]_{mC[T]}$. By Lemma b the projective cover P of Q as an $R \otimes_C C[T]_{mC[T]}$-module is a projective cover as an R-module. Now, as in the verification of (C2) one can lift the $R \otimes_C C[T]_{mC[T]}$-module map $P \longrightarrow Q \longrightarrow \bar{E}/(\bar{E}J + J\bar{E}) \otimes Q$ to a map $f : P \longrightarrow E \otimes P$, and this can then be adjusted to give an object $Y = (P, e)$ in $\xi(R)$. Since e is an $R \otimes_C C[T]_{mC[T]}$-module map, there is a natural map $C[T]_{mC[T]} \longrightarrow \mathrm{End}_{\xi(R)}(Y)$. Now $P/JP \cong Q$ has finite length over

$C[T]_{mC[T]}$, and Y has image X in $\xi_{NJ}(R/J)$ so it is indecomposable. Thus (C3) holds.

**9.4** Continuing with the hypotheses of (9.2) suppose now that the lift pair $(R/J,\xi_{NJ})$ has finite representation type. Let $X = (P,\bar{e})$ be an object in $\xi_N(R)$ of finite length, and which is the direct sum of exactly n non-isomorphic indecomposable objects. Let

$$R_X = \mathrm{End}_{\xi_N(R)}(X)^{op}$$

and let $\xi_X$ be defined via the pullback of $R_X$-$R_X$-bimodules

$$\begin{tikzcd}
\xi_X : 0 \arrow[r] & M_X \arrow[r] \arrow[d, equal] & E_X \arrow[r] \arrow[d, "\alpha"] & R_X \arrow[r] \arrow[d, "\beta"] & 0 \\
0 \arrow[r] & \mathrm{Hom}_R(P,N\otimes_R P) \arrow[r] & \mathrm{Hom}_R(P,E\otimes_R P) \arrow[r] & \mathrm{Hom}_R(P,E/N\otimes_R P) \arrow[r] & 0
\end{tikzcd}$$

where $\beta$ is the map sending 1 to $\bar{e}$. Thus $(R_X,\xi_X)$ is a lift pair. Let $\tau_X:\xi_X(R_X)\longrightarrow\xi(R)$ be the functor defined as follows. If $Y = (Q,g)$ belongs to $\xi_X(R_X)$ then $\tau_X(Y) = (P\otimes_{R_X}Q,h)$, where h is the composition

$$P\otimes_{R_X}Q \xrightarrow{1\otimes g} P\otimes_{R_X}E_X\otimes_{R_X}Q \xrightarrow{1\otimes\alpha\otimes 1} P\otimes_{R_X}\mathrm{Hom}_R(P,E\otimes_R P)\otimes_{R_X}Q$$
$$\xrightarrow{ev\otimes 1} E\otimes_R P\otimes_{R_X}Q,$$

and ev is the evaluation map, and if $\theta:(Q,g)\longrightarrow(Q',g')$ is a morphism then $\tau_X(\theta) = 1\otimes\theta$. It is shown in [CB4, 4.2] that $\tau_X$ is fully faithful, and that it induces an equivalence from $\xi_X(R_X)$ to the full subcategory of $\xi(R)$ on those objects whose image under $\sigma_N$ is a summand of a direct sum of copies of X. Now the representation equivalence $\rho_J$ shows that $\xi_N(R)$ has only finitely many non-isomorphic indecomposable objects, and if X is the direct sum of all of them, then $\tau_X$ is an equivalence (since there is a direct sum preserving representation equivalence from $\xi_N(R)$ to the category of modules for an algebra of finite representation type, so that every object is a direct sum of indecomposables).

**Lemma.** If $Y = (Q,g)$ is an object in $\xi_X(R_X)$ then

$$\mathrm{endolen}(Y) \le \mathrm{endolen}(\tau_X(Y)) \le \mathrm{length}_C(X).\mathrm{endolen}(Y).$$

If in addition $Y$ is sincere, then either $\mathrm{endolen}(Y)$ is less than $\mathrm{endolen}(\tau_X(Y))$, or $\mathrm{length}_C(M_X) < \mathrm{length}_C(M)$.

*Proof*. Since $\tau_X$ is fully faithful,

$$\mathrm{endolen}(\tau_X(Y)) = \mathrm{length}(P/JP \otimes_{R_X} Q_F) \quad \text{where}$$
$$F = \mathrm{End}_{\xi_X(R_X)}(Y)^{op}.$$

If $S$ is a simple right $R_X$-module, then $S$ is a summand of $R_X/J_X$, where $J_X = \mathrm{rad}\ R_X$, so that

$$\mathrm{length}(S \otimes_{R_X} Q_F) \le \mathrm{length}(R_X/J_X \otimes_{R_X} Q_F) = \mathrm{endolen}(Y).$$

Taking a composition series of $P/JP$ as a right $R_X$-module one obtains

$$\begin{aligned}\mathrm{endolen}(\tau_X(Y)) &\le \mathrm{length}(P/JP_{R_X}).\mathrm{endolen}(Y) \\ &\le \mathrm{length}_C(X).\mathrm{endolen}(Y),\end{aligned}$$

which is the second inequality. Now $P/JP$ is a sincere right $R_X$-module, since $P$ is faithful, so sincere as a right $R_X$-module, but for each $r$, $J^rP/J^{r+1}P$ is a quotient of $(J^r/J^{r+1}) \otimes_{R/J} P/JP$, which is a summand of a direct sum of copies of $P/JP$. On the other hand, since the indecomposable summands of $X$ are non-isomorphic, $R_X$ is basic, that is, $R_X/J_X$ is isomorphic as a right $R_X$-module to the direct sum of one copy of each simple. It follows that

$$\mathrm{length}(R_X/J_X \otimes_{R_X} Q_F) \le \mathrm{length}(P/JP \otimes_{R_X} Q_F). \qquad (\dagger)$$

which is the first inequality.

Now suppose that Y is sincere, so that $Q/J_X Q$ is a sincere $R_X$-module, and hence $S \otimes_{R_X} Q \neq 0$ for any simple right $R_X$-module S. If the inequality (†) is not strict, then P/JP must have length exactly n as an $R_X$-module, that is, endolength(X) = n. Now the functor $\rho_J : \xi_N(R) \longrightarrow \xi_{NJ}(R/J)$ is a representation equivalence, and by [CB4, 2.1] there is an equivalence $\xi_{NJ}(R/J) \longrightarrow$A-Mod, where

$$A = ([\bar{E}/(\bar{E}J+J\bar{E})]^{*})^{\otimes R/J}/(\pi_{NJ}-1)$$

is a f.d. hereditary k-algebra of finite representation type. Both of these functors preserve endolength, so the image X′ of X in A-Mod has endolength n. Recall that X is a direct sum of exactly n non-isomorphic indecomposable summands, so there is a decomposition $X' = U_1 \oplus \ldots \oplus U_n$ into non-isomorphic indecomposable summands. Now the endolength of the direct sum is the sum of the endolengths by Lemma (4.5), so all $U_i$ must have endolength 1, and therefore be simple. This means that P/JP is a direct sum of distinct simples, and hence P is a summand of R. Since $M_X = \mathrm{Hom}_R(P, N \otimes_R P)$ this implies that

$$\mathrm{length}_C(M_X) \leq \mathrm{length}_C(N) < \mathrm{length}_C(M),$$

as required.

**9.5** THEOREM. (C1)-(C4) are equivalent for C-algebraic lift pairs.

*Proof.* Let $(R, \xi)$ be such a lift pair. We assume that (C2) or (C4) holds, and wish to prove that the rest hold. Thus for some d there is either a generic object G of endolength d, or an infinite family $(N_\lambda)_{\lambda \in \Lambda}$ of finite length indecomposable objects of endolength d. We use induction on d and $\mathrm{length}_C(M)$.

If M = 0, then $\xi(R) \cong$ R-Proj has finite representation type, which is impossible under our assumption. Thus one can pick a maximal sub-bimodule $N \subseteq M$ and make the constructions of (9.2).

If $\xi_{NJ}(R/J)$ has infinite type the conclusions are given by Proposition (9.3), so suppose that $\xi_{NJ}(R/J)$ has finite type.

For X as in (9.4) the functor $\tau_X: \xi_X(R_X) \longrightarrow \xi(R)$ is fully faithful, and in case X is the direct sum of all indecomposables this is an equivalence. By choosing X carefully we can ensure that either G or infinitely many of the $N_\lambda$ are the images under $\tau_X$ of *sincere* objects in $\xi_X(R_X)$. Now by Lemma (9.4) the lift pair $(R_X, \xi_X)$ either satisfies the hypotheses for some $d' < d$, or for d but with $\mathrm{length}_C(M_X) < \mathrm{length}_C(M)$. By the induction, the lift pair $(R_X, \xi_X)$ satisfies (C1)-(C4), and then the second inequality in Lemma (9.4) ensures that $(R, \xi)$ satisfies (C1)-(C4).

**9.6** Let A be an artin C-algebra with radical L, and let $(R, \xi)$ be the lift pair with $R = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$, and

$$\xi : 0 \longrightarrow \begin{pmatrix} 0 & L \\ 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} A & L \\ 0 & A \end{pmatrix} \longrightarrow \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \longrightarrow 0.$$

Clearly $(R, \xi)$ is C-algebraic. By [CB4, 1.7] there is an equivalence between $\xi(R)$ and the category $P^1(A)$ of triples $(P', P'', \alpha)$ with $P'$ and $P''$ projective A-modules and $\alpha: P' \longrightarrow P''$ a map with $\mathrm{Im}(\alpha) \subseteq LP''$. If $P^2(A)$ denotes the subcategory of $P^1(A)$ on the triples with $\mathrm{Ker}(\alpha) \subseteq LP'$, then every object in $P^1(A)$ is the direct sum of an object in $P^2(A)$ and a triple of the form $(P', 0, 0)$, and the functor $P^2(A) \longrightarrow A\text{-Mod}$ sending $(P', P'', \alpha)$ to $\mathrm{Coker}(\alpha)$ is a representation equivalence.

**Lemma.** Let $X = (P, e) \in \xi(R)$ correspond to an object in $P^2(A)$ with image $N \in A\text{-Mod}$ under the cokernel functor. If S is a C-subalgebra of $\mathrm{End}_{\xi(R)}(X)^{op}$, and $J = \mathrm{rad}\, R$, then

$$\mathrm{length}(N_S) \le \mathrm{length}(A_A).\mathrm{length}(P/JP_S) \qquad \text{and}$$

$$\mathrm{length}(P/JP_S) \le (\mathrm{length}(A_A) + 1).\mathrm{length}(N_S).$$

Moreover $\mathrm{endolen}(X) \le (\mathrm{length}(A_A) + 1).\mathrm{endolen}(N)$.

*Proof.* Let X correspond to the triple $(P', P'', \alpha)$ in $P^2(A)$, so there is an exact sequence

$$P' \xrightarrow{\alpha} P'' \longrightarrow N \longrightarrow 0 \qquad (\ddagger)$$

of A-S-bimodules. Now $R = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$ and $P = \begin{pmatrix} P' \\ P'' \end{pmatrix}$, so that

$$\text{length}(P/JP_S) = \text{length}(P'/LP'_S) + \text{length}(P''/LP''_S).$$

If T is a simple right A-module, then $\text{length}(T \otimes_A P''_S) \leq \text{length}(P''/LP''_S)$, so a composition series of $A_A$ gives

$$\text{length}(P''_S) \leq \text{length}(A_A).\text{length}(P''/LP''_S) \qquad (*)$$

and the first inequality follows. Now X corresponds to an object in $P^2(A)$, so the exact sequence ($\ddagger$) is a minimal projective presentation, and thus $P''/LP'' \cong N/LN$ and $P'/LP' \cong \text{Im}(\alpha)/L\,\text{Im}(\alpha)$, a subquotient of $P''$. Therefore

$$\text{length}(P''/LP''_S) \leq \text{length}(N_S)$$

and then ($*$) implies that

$$\text{length}(P'/LP'_S) \leq \text{length}(A_A).\text{length}(N_S),$$

giving the second inequality. Finally, taking $S = \text{End}_{\mathcal{E}(R)}(X)^{op}$, the fact that the cokernel functor is full means that $\text{endolen}(N) = \text{length}(N_S)$.

THEOREM. Let A be an artin C-algebra. Consider the following statements.

(1) For some $d \in \mathbb{N}$ there are infinitely many non-isomorphic indecomposable A-modules of length d over C.

(2) For some $d \in \mathbb{N}$ there are infinitely many non-isomorphic indecomposable A-modules which are of endolength d, and have finite length over C.

(3) There is an $A$-$C[T]_{\mathfrak{m}C[T]}$-bimodule, indecomposable over A, and of finite length over $C[T]_{\mathfrak{m}C[T]}$.

(4) A has a generic module.

Then (2)-(4) are equivalent. If in additiion the field k is infinite, then they are also equivalent to (1).

*Proof.* If A satisfies one of (1)-(4), then it satisfies (2) or (4), and by the lemma, the lift pair $(R,\xi)$ satisfies (C2) or (C4). By Theorem (9.5) the lift pair satisfies (C1)-(C4). Now there are only finitely many indecomposable objects in $P^1(A)$ which do not belong to $P^2(A)$, and they all correspond to objects in $\xi(R)$ of finite length over C. Thus the lemma enables one to deduce (1)-(4).

REMARK. Theorem (7.3) follows on reducing to the case when the artin algebra R is connected, and then setting C = Z(R). If the simple R-modules have infinite underlying sets, then k is infinite. Note that a family of R-modules has bounded length if and only if it has bounded length over C.

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