Geometry of the moment map for representations of quivers

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Abstract. We study the moment map associated to the cotangent bundle of the space of representations of a quiver, determining when it is flat, and giving a stratification of its Marsden-Weinstein reductions. In order to do this we determine the possible dimension vectors of simple representations of deformed preprojective algebras. In an appendix we use deformed preprojective algebras to give a simple proof of much of Kac's Theorem on representations of quivers in characteristic zero.

Keywords: Quiver, Representation, Moment map, Preprojective algebra

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1. Introduction

Let K be an algebraically closed field and let Q be a quiver with vertex set I. Representations of Q of dimension vector $\alpha \in \mathbb{N}^I$ are given by elements of the space

$$\operatorname{Rep}(Q,\alpha) = \bigoplus_{a\in Q} \operatorname{Mat}(\alpha_{h(a)} \times \alpha_{t(a)}, K),$$

where $h(a)$ and $t(a)$ are the head and tail vertices of an arrow $a \in Q$; isomorphism classes correspond to orbits of the group

$$\mathrm{G}(\alpha) = \left(\prod_{i\in I} \mathrm{GL}(\alpha_i, K)\right) / K^*$$

acting by conjugation. Using the trace pairing there is an identification of the cotangent bundle

$$\mathrm{T}^* \operatorname{Rep}(Q,\alpha) \cong \operatorname{Rep}(\overline{Q},\alpha),$$

where $\overline{Q}$ is the *double* of Q, obtained by adjoining a reverse arrow $a^* : j \to i$ for each arrow $a : i \to j$ in Q.

We consider the *moment map* $\mu_\alpha : \operatorname{Rep}(\overline{Q},\alpha) \to \operatorname{End}(\alpha)_0$ defined by

$$\mu_\alpha(x)_i = \sum_{\substack{a\in Q\\ h(a)=i}} x_a x_{a^*} - \sum_{\substack{a\in Q\\ t(a)=i}} x_{a^*} x_a,$$

where

$$\mathrm{End}(\alpha)_0 = \{(\theta_i) \mid \sum_{i\in I} \mathrm{tr}(\theta_i) = 0\} \subseteq \mathrm{End}(\alpha) = \bigoplus_{i\in I} \mathrm{Mat}(\alpha_i, K).$$

If one uses the trace pairing to identify $\mathrm{End}(\alpha)_0$ with the dual of the Lie algebra of $\mathrm{G}(\alpha)$, then this is a moment map in the usual sense. (Identifying $\mathrm{Rep}(\overline{Q}, \alpha)$ with its tangent space at any point, the natural symplectic form on the cotangent bundle corresponds to the form

$$\omega(x, y) = \sum_{a\in Q} \big(\mathrm{tr}(x_a y_{a^*}) - \mathrm{tr}(x_{a^*} y_a)\big)$$

on $\mathrm{Rep}(\overline{Q}, \alpha)$. Now if $\theta \in \mathrm{End}(\alpha)$, and $f : \mathrm{Rep}(\overline{Q}, \alpha) \to K$ is defined by $f(x) = \sum_i \mathrm{tr}(\theta_i \mu_\alpha(x)_i)$, then $\mathrm{d}f_x(y) = \omega([\theta, x], y)$ for $x, y \in \mathrm{Rep}(\overline{Q}, \alpha)$, where $[\theta, x]$ is defined by $[\theta, x]_a = \theta_{h(a)} x_a - x_a \theta_{t(a)}$ for any $a \in \overline{Q}$.)

Now the elements of $\mathrm{End}(\alpha)_0$ which are invariant under $\mathrm{G}(\alpha)$ acting by conjugation are those whose components are scalar matrices. We identify them with the $\lambda \in K^I$ which have $\lambda \cdot \alpha = \sum_{i\in I} \lambda_i \alpha_i$ equal to zero. In this paper we study the fibres $\mu_\alpha^{-1}(\lambda)$ and the quotients $\mu_\alpha^{-1}(\lambda) /\!/ \mathrm{G}(\alpha)$. These are Marsden-Weinstein reductions [15], except that we work with schemes rather than manifolds.

This moment map has been considered before. Kronheimer [11] constructed the Kleinian singularities and their deformations in this way from the extended Dynkin quivers (see also [2, 5]). Later, Lusztig [14, Section 12] used the nilpotent cone of $\mu_\alpha^{-1}(0)$ in his geometric construction of the negative part of the quantum group of type Q, for any quiver Q without loops. Finally Nakajima [16, 17, 18] used the moment map to define some *quiver varieties* and used these in a geometric construction of integrable representations of Kac-Moody Lie algebras. In the first of his papers he used hyper-Kähler quotients to define a family $\mathfrak{M}_\zeta$, and this family includes $\mu_\alpha^{-1}(\lambda) /\!/ \mathrm{G}(\alpha)$ with $K = \mathbb{C}$ by [16, Theorem 3.1]. In his later papers he used geometric invariant theory quotients, and $\mu_\alpha^{-1}(0) /\!/ \mathrm{G}(\alpha)$ appears as the variety $\mathfrak{M}_0(\mathbf{v}, 0)$ in [18, §3].

Kac [7, 8] has shown that the dimension vectors of indecomposable representations of Q are exactly the positive roots for Q, and that the number of parameters of indecomposable representations of dimension α is given by the function

$$p(\alpha) = 1 + \sum_{a\in Q} \alpha_{t(a)} \alpha_{h(a)} - \alpha \cdot \alpha,$$

where $\alpha\cdot\alpha = \sum_{i\in I} \alpha_i^2$. After some preliminaries in Sections 2 and 3, we use Kac's Theorem in Section 4 to compute the dimension of $\mu_\alpha^{-1}(\lambda)$ and then use his 'canonical decomposition' to prove the following result.

THEOREM 1.1. *If $\alpha \in \mathbb{N}^I$ then the following are equivalent*

(1) μ_α is a flat morphism.

(2) $\mu_\alpha^{-1}(0)$ has dimension $\alpha \cdot \alpha - 1 + 2p(\alpha)$.

(3) $p(\alpha) \geq \sum_{t=1}^r p(\beta^{(t)})$ for any decomposition $\alpha = \beta^{(1)} + \cdots + \beta^{(r)}$ with the $\beta^{(t)}$ positive roots.

(4) $p(\alpha) \geq \sum_{t=1}^r p(\beta^{(t)})$ for any decomposition $\alpha = \beta^{(1)} + \cdots + \beta^{(r)}$ into nonzero $\beta^{(t)} \in \mathbb{N}^I$.

The *deformed preprojective algebra* introduced by M. P. Holland and the author [5] (see also [3]) is the algebra defined for $\lambda \in K^I$ by

$$\Pi^\lambda = K\overline{Q}/(\sum_{a \in Q}[a, a^*] - \sum_{i \in I} \lambda_i e_i),$$

where $K\overline{Q}$ is the path algebra of $\overline{Q}$, the trivial path at vertex i is denoted e_i, and $[a, a^*]$ is the commutator $aa^* - a^*a$.

Clearly if $\lambda \in K^I$ and $\lambda \cdot \alpha = 0$, then $\mu_\alpha^{-1}(\lambda)$ is identified with the space of representations of Π^λ of dimension vector α. Now the closed orbits of $\mathrm{G}(\alpha)$ on $\mathrm{Rep}(\overline{Q}, \alpha)$ correspond to isomorphism classes of semisimple representations of $\overline{Q}$ of dimension α. (For example take $\theta = 0$ in [9, Proposition 3.2].) Thus the closed orbits of $\mathrm{G}(\alpha)$ on $\mu_\alpha^{-1}(\lambda)$ correspond to isomorphism classes of semisimple representations of Π^λ of dimension α. Of these, the orbits on which $\mathrm{G}(\alpha)$ acts freely are those corresponding to a simple representation of Π^λ. Our main result is as follows.

THEOREM 1.2. *For $\lambda \in K^I$ and $\alpha \in \mathbb{N}^I$ the following are equivalent*

(1) There is a simple representation of Π^λ of dimension vector α.

(2) α is a positive root, $\lambda \cdot \alpha = 0$, and $p(\alpha) > \sum_{t=1}^r p(\beta^{(t)})$ for any decomposition $\alpha = \beta^{(1)} + \cdots + \beta^{(r)}$ with $r \geq 2$ and $\beta^{(t)}$ a positive root with $\lambda \cdot \beta^{(t)} = 0$ for all t.

In this case $\mu_\alpha^{-1}(\lambda)$ is a reduced and irreducible complete intersection of dimension $\alpha \cdot \alpha - 1 + 2p(\alpha)$, and the general element of $\mu_\alpha^{-1}(\lambda)$ is a simple representation of Π^λ.

The special case $\lambda = 0$ answers some questions of Nakajima. In [17, Problem 4.6], in the situation where Q has no loops, Nakajima asks whether if Q is connected and non-Dynkin then Π^0 has a simple representation which is not one-dimensional. This is true, for in Theorem

1.2 one can take α to be any minimal imaginary root. In [18, Question after Lemma 4.9], he asks which elements of the fundamental region are dimension vectors of simple representations of Π^0. The answer is given by Theorems 1.2 and 8.1.

Henceforth we write Σ_λ for the set of α satisfying the conditions in part (2) of Theorem 1.2. In Section 5 we study the set Σ_λ, and provide another characterization of it. In Section 6 we use Kac's Theorem again to prove that $\mu_\alpha^{-1}(\lambda)$ is irreducible of dimension $\alpha \cdot \alpha - 1 + 2p(\alpha)$ for $\alpha \in \Sigma_\lambda$. We then use Schofield's theory of general representations of quivers to show that the general element of $\mu_\alpha^{-1}(\lambda)$ is a simple representation. This proves (2) $\Longrightarrow$ (1). The implication (1) $\Longrightarrow$ (2) is more complicated and is proved in Sections 7 to 10.

If $\alpha \in \Sigma_\lambda$, how many simple representations of dimension α are there? The $\mathrm{G}(\alpha)$-orbit of a simple representation has dimension $\alpha\cdot\alpha-1$. Thus if α is a real root (so $p(\alpha) = 0$), there is a unique simple representation up to isomorphism, while if α is an imaginary root (so $p(\alpha) > 0$), there are infinitely many non-isomorphic simple representations.

Now suppose that K has characteristic zero. In Section 11 we study the affine quotient schemes $\mu_\alpha^{-1}(\lambda) /\!/ \mathrm{G}(\alpha)$. Recall that the points of this quotient are in 1-1 correspondence with the closed orbits, so with isomorphism classes of semisimple representations of Π^λ of dimension α. Given a semisimple representation X, we can decompose it into its simple components

$$X = X_1^{\oplus k_1} \oplus \cdots \oplus X_r^{\oplus k_r}$$

where the X_t are non-isomorphic simples. If $\beta^{(t)}$ is the dimension vector of X_t, we say that X has *representation type*

$$\tau = (k_1, \beta^{(1)}; \ldots; k_r, \beta^{(r)}).$$

For τ to occur as the representation type of a semisimple representation of dimension α, clearly one must have $\alpha = k_1\beta^{(1)} + \cdots + k_r\beta^{(r)}$ and $\beta^{(t)} \in \Sigma_\lambda$ for all t. In addition, although the $\beta^{(t)}$ need not be distinct, any real root can occur as at most one of the $\beta^{(t)}$.

THEOREM 1.3. *If τ is a representation type, then the set of semisimple representations of type τ is an irreducible locally closed subset of $\mu_\alpha^{-1}(\lambda) /\!/ \mathrm{G}(\alpha)$ of dimension $\sum_{t=1}^r 2p(\beta^{(t)})$.*

This has the following consequence.

COROLLARY 1.4. *If $\lambda \in K^I$ and $\alpha \in \Sigma_\lambda$ then $\mu_\alpha^{-1}(\lambda) /\!/ \mathrm{G}(\alpha)$ is a reduced and irreducible scheme of dimension $2p(\alpha)$.*

Finally, in an appendix we show how deformed preprojective algebras can be used to give a simple proof of much of Kac's Theorem in case the base field has characteristic zero. In particular, we give an explicit construction of the indecomposable representations whose dimension vector is a real root.

Preliminary versions of these results (with $\lambda = 0$) were first announced at a conference on Geometry and Quivers in Hamburg in November 1996. I should like to thank the organisers O. Riemenschneider and P. Slodowy for inviting me to attend the meeting. I would also like to thank M. P. Holland for some useful discussions.

Remarks added in April 2000 (after writing the paper [4]). We would like to explain some additional applications of the results in this paper to the study of Nakajima's quiver varieties.

Let Q_0 be a quiver with vertex set I. In case Q_0 has no oriented cycles this is to correspond to an orientation Ω of a graph (I, E) as in [18, Section 3.1]. For $\mathbf{v}, \mathbf{w} \in \mathbb{N}^I$, let $\mathbf{M}(\mathbf{v}, \mathbf{w})$ be the space

$$\mathrm{Rep}(\overline{Q_0}, \mathbf{v}) \oplus \bigoplus_{k\in I} \mathrm{Mat}(\mathbf{v}_k \times \mathbf{w}_k, K) \oplus \bigoplus_{k\in I} \mathrm{Mat}(\mathbf{w}_k \times \mathbf{v}_k, K).$$

There is a natural action of the group $G_{\mathbf{v}} = \prod_{k\in I} \mathrm{GL}(\mathbf{v}_k, K)$ and a moment map

$$\mu : \mathbf{M}(\mathbf{v}, \mathbf{w}) \to \bigoplus_{k\in I} \mathrm{Mat}(\mathbf{v}_k, K)$$

whose k-th component sends (B, i, j) to

$$\sum_{\substack{a\in Q_0\\ h(a)=k}} B_a B_{a^*} - \sum_{\substack{a\in Q_0\\ t(a)=k}} B_{a^*} B_a + \sum_{k\in I} i_k j_k.$$

One of the spaces that Nakajima considers is

$$\mathfrak{M}_0(\mathbf{v}, \mathbf{w}) = \mu^{-1}(0) \,/\!/\, G_{\mathbf{v}}.$$

Let Q be the quiver obtained from Q_0 by adjoining a new vertex ∞ and $\mathbf{w}_k$ arrows from ∞ to k for each $k \in I$; let α be the dimension vector for Q whose restriction to I is equal to $\mathbf{v}$ and with $\alpha_\infty = 1$. By dividing the matrices in $\mathrm{Mat}(\mathbf{v}_k \times \mathbf{w}_k, K)$ into their columns, and the matrices in $\mathrm{Mat}(\mathbf{w}_k \times \mathbf{v}_k, K)$ into their rows, one can identify

$$\mathbf{M}(\mathbf{v}, \mathbf{w}) \cong \mathrm{Rep}(\overline{Q}, \alpha), \quad G_{\mathbf{v}} \cong \mathrm{G}(\alpha).$$

Moreover μ corresponds to the usual moment map μ_α, so we have

$$\mathfrak{M}_0(\mathbf{v}, \mathbf{w}) \cong \mu_\alpha^{-1}(0) \,/\!/\, \mathrm{G}(\alpha).$$

Thus the set $\mathfrak{M}_0^{\mathrm{reg}}(\mathbf{v},\mathbf{w})$ of [18, §3.v] is non-empty if and only if $\alpha \in \Sigma_0$.

The other space that Nakajima considers is the *quiver variety*

$$\mathfrak{M}(\mathbf{v},\mathbf{w}) = \mu^{-1}(0) /\!/ (G_{\mathbf{v}}, \chi_0) \cong \mu_\alpha^{-1}(0) /\!/ (\mathrm{G}(\alpha), \chi)$$

in the notation of [9], where $\chi_0 : G_{\mathbf{v}} \to K^*$ is the character defined by $\chi_0(g) = \prod_{k\in I} \det(g_k^{-1})$, and χ is the corresponding character of $\mathrm{G}(\alpha)$. This is a smooth variety. We say that a representation of Π^0 of dimension β is *v-cogenerated* (where v is a vertex with $\beta_v = 1$) if it has no non-zero subrepresentation which is zero at v. This is dual to the notion of 'v-generated' of [4, Section 2]. By [18, Lemma 3.8] the points of $\mathfrak{M}(\mathbf{v},\mathbf{w})$ are in 1-1 correspondence with isomorphism classes of ∞-cogenerated representations of Π^0 of dimension α.

Now assume that K is the field $\mathbb{C}$ of complex numbers. It is claimed in [18, Theorem 6.2] that $\mathfrak{M}(\mathbf{v},\mathbf{w})$ is connected, but this is retracted in [19, Section 7.5]. Nakajima has, however, mentioned to the author that connectivity can be recovered, and the following argument is perhaps similar to what he had in mind.

Define λ by $\lambda_k = -1$ for $k \in I$ and $\lambda_\infty = \sum_{k\in I} \mathbf{v}_k$. Thus $\lambda \cdot \alpha = 0$, but $\lambda \cdot \beta \neq 0$ for all $0 < \beta < \alpha$. If α is a root then trivially $\alpha \in \Sigma_\lambda$, so Theorem 1.2 implies that $\mu_\alpha^{-1}(\lambda) /\!/ \mathrm{G}(\alpha)$ is non-empty and irreducible. On the other hand, if α is not a root, then Theorem 1.2 implies that there is no representation of Π^λ of dimension α, so that $\mu_\alpha^{-1}(\lambda) /\!/ \mathrm{G}(\alpha)$ is empty. Now there is a bijection

$$\mu_\alpha^{-1}(\lambda) /\!/ \mathrm{G}(\alpha) \to \mu_\alpha^{-1}(0) /\!/ (\mathrm{G}(\alpha), \chi) \cong \mathfrak{M}(\mathbf{v},\mathbf{w}).$$

which is continuous for the analytic topology. (See [16, §§3,4] and [4, §3].) It follows that $\mathfrak{M}(\mathbf{v},\mathbf{w})$ is either non-empty connected or empty, according to whether α is a root for Q or not.

2. Notation and reflection functors

Let Q be a quiver with vertex set I and let K be an algebraically closed field. In this section we introduce some standard notation, recall the reflection functors, and determine the effect of reflection functors on the fibres $\mu_\alpha^{-1}(\lambda)$.

We call elements of $\mathbb{Z}^I$ (or sometimes $\mathbb{R}^I$) *vectors*, and write ϵ_i for the coordinate vector at a vertex i. We partially order $\mathbb{Z}^I$ via $\alpha \geq \beta$ if $\alpha_i \geq \beta_i$ for all i, and we write $\alpha > \beta$ to mean that $\alpha \geq \beta$ and $\alpha \neq \beta$. We say that α is *sincere* if $\alpha_i > 0$ for all i.

The *Ringel form* on $\mathbb{Z}^I$ is defined by

$$\langle \alpha, \beta \rangle = \sum_{i\in I} \alpha_i \beta_i - \sum_{a\in Q} \alpha_{t(a)} \beta_{h(a)}.$$

Let $(\alpha, \beta) = \langle\alpha, \beta\rangle + \langle\beta, \alpha\rangle$ be its symmetrization. The corresponding quadratic form $q(\alpha) = \langle\alpha, \alpha\rangle = \frac{1}{2}(\alpha, \alpha)$ is the *Tits form*, and we have $p(\alpha) = 1 - q(\alpha)$. The *fundamental region* is the set of $0 \neq \alpha \in \mathbb{N}^I$ with connected support and with $(\alpha, \epsilon_i) \leq 0$ for every vertex i.

If i is a loopfree vertex (so $q(\epsilon_i) = 1$), there is a reflection $s_i : \mathbb{Z}^I \to \mathbb{Z}^I$ defined by $s_i(\alpha) = \alpha - (\alpha, \epsilon_i)\epsilon_i$. The *real roots* (respectively *imaginary roots*) are the elements of $\mathbb{Z}^I$ which can be obtained from the coordinate vector at a loopfree vertex (respectively $\pm$ an element of the fundamental region) by applying some sequence of reflections at loopfree vertices.

There is a reflection $r_i : K^I \to K^I$ which is dual to s_i. It is defined by $r_i(\lambda)_j = \lambda_j - (\epsilon_i, \epsilon_j)\lambda_i$. It satisfies $r_i(\lambda) \cdot \alpha = \lambda \cdot s_i(\alpha)$ for all α.

We say that the reflection at a loopfree vertex i is *admissible* for the pair (λ, α) if $\lambda_i \neq 0$. Let $\sim$ be the smallest equivalence relation on $K^I \times \mathbb{Z}^I$ with $(\lambda, \alpha) \sim (r_i(\lambda), s_i(\alpha))$ whenever the reflection at i is admissible for (λ, α).

If the reflection at i is admissible for (λ, α) then by [5, §5] there is a reflection functor from representations of Π^λ to representations of $\Pi^{r_i(\lambda)}$ which acts as as s_i on dimension vectors. (In fact these reflection functors were discovered earlier, by Rump [21].)

We briefly describe the construction. Assume for simplicity that no arrow in Q has tail at i, and let $H = \{a \in Q \mid h(a) = i\}$. Suppose that V is a representation of Π^λ, given by vector spaces V_j for each vertex j and linear maps $V_a : V_{t(a)} \to V_{h(a)}$ for each arrow $a \in \overline{Q}$. Define

$$V_\oplus = \bigoplus_{a \in H} V_{t(a)},$$

and let $\mu_a : V_{t(a)} \to V_\oplus$ and $\pi_a : V_\oplus \to V_{t(a)}$ be the canonical inclusions and projections. Define $\mu : V_i \to V_\oplus$ and $\pi : V_\oplus \to V_i$ by

$$\mu = \sum_{a \in H} \mu_a V_{a^*}, \quad \pi = \frac{1}{\lambda_i} \sum_{a \in H} V_a \pi_a.$$

The relations for Π^λ ensure that $\pi\mu = 1_{V_i}$, so that $\mu\pi$ is an idempotent endomorphism of $V_\oplus$. By definition the reflection functor sends V to the representation V' of $\Pi^{r_i(\lambda)}$ given by vector spaces $V'_j = V_j$ for $j \neq i$ and $V'_i = \mathrm{Im}(1 - \mu\pi)$, and by linear maps $V'_a = V_a$ and $V'_{a^*} = V'_a$ for $a \in Q$ with $h(a) \neq i$, and

$$V'_a = -\lambda_i(1 - \mu\pi)\mu_a : V'_{t(a)} \to V'_i, \quad V'_{a^*} = \pi_a|_{V'_i} : V'_i \to V'_{t(a)}$$

for $a \in H$.

We use the reflection functors to relate the schemes $\mu_\alpha^{-1}(\lambda)$ and $\mu_{s_i(\alpha)}^{-1}(r_i(\lambda))$ (equipped with their scheme structure as fibres of the moment map). For our geometric arguments all schemes are quasiprojective over K, and all points are closed points.

LEMMA 2.1. *If $0 \neq \nu \in K$ and m, n are non-negative integers, then the projection from*

$$\begin{aligned}\mathcal{S} = \{(X, X^*, Y, Y^*) \mid XX^* = \nu 1, YY^* = -\nu 1, X^*X - Y^*Y = \nu 1\} \\ \subseteq \mathrm{Mat}(n \times (n+m), K) \times \mathrm{Mat}((n+m) \times n, K) \times \\ \mathrm{Mat}(m \times (n+m), K) \times \mathrm{Mat}((n+m) \times m, K)\end{aligned}$$

to

$$\begin{aligned}\mathcal{X} = \{(X, X^*) \mid XX^* = \nu 1\} \\ \subseteq \mathrm{Mat}(n \times (n+m), K) \times \mathrm{Mat}((n+m) \times n, K)\end{aligned}$$

is a principal $\mathrm{GL}(m, K)$*-bundle. Moreover the natural scheme structures on $\mathcal{S}$ and $\mathcal{X}$ given by the indicated relations are reduced.*

Proof. By rescaling X and Y one can replace the equations by $XX^* = 1$, $YY^* = 1$, and $X^*X + Y^*Y = 1$, so the matrices define inverse isomorphisms between K^{n+m} and $K^n \oplus K^m$. The result is now standard.

LEMMA 2.2. *Suppose given a pair (λ, α) with $\lambda \cdot \alpha = 0$. If i is a loopfree vertex with $\lambda_i \neq 0$ then there is a scheme T and morphisms*

$$\mu_\alpha^{-1}(\lambda) \xleftarrow{f} T \xrightarrow{g} \mu_{s_i(\alpha)}^{-1}(r_i(\lambda))$$

where the map f is a principal $\mathrm{GL}(s_i(\alpha)_i, K)$*-bundle and g is a principal* $\mathrm{GL}(\alpha_i, K)$*-bundle. In particular $\mu_\alpha^{-1}(\lambda)$ and $\mu_{s_i(\alpha)}^{-1}(r_i(\lambda))$ have the same number of irreducible components, and*

$$\dim \mu_{s_i(\alpha)}^{-1}(r_i(\lambda)) - s_i(\alpha) \cdot s_i(\alpha) = \dim \mu_\alpha^{-1}(\lambda) - \alpha \cdot \alpha.$$

Proof. We suppose for simplicity that no arrow in Q has tail at i. We can do this because the deformed preprojective algebra Π^λ does not depend on the orientation of Q, see [5, Lemma 2.2]. (If a were an arrow with tail at i we could reverse it by sending x_a to x_{a^*} and x_{a^*} to $-x_a$ for $x \in \mathrm{Rep}(\overline{Q}, \alpha)$.) Let $H = \{a \in Q \mid h(a) = i\}$.

Let Q' be the quiver obtained from Q by deleting all arrows in H, and let $R' = \mathrm{Rep}(\overline{Q'}, \alpha)$. Letting $n = \alpha_i$ and

$$m = s_i(\alpha)_i = -\alpha_i + \sum_{a \in H} \alpha_{t(a)},$$

one can combine the matrices for the arrows incident at i into block matrices, and identify

$$\mathrm{Rep}(\overline{Q},\alpha) \cong R' \times \mathrm{Mat}(n \times (n+m), K) \times \mathrm{Mat}((n+m) \times n, K),$$

so that if $x \in \mathrm{Rep}(\overline{Q},\alpha)$ corresponds to a triple (x', X, X^*) then

$$\mu_\alpha(x)_i = \sum_{a \in H} x_a x_{a^*} = XX^*.$$

Also one can identify

$$\mathrm{Rep}(\overline{Q}, s_i(\alpha)) \cong R' \times \mathrm{Mat}(m \times (n+m), K) \times \mathrm{Mat}((n+m) \times m, K)$$

and if y corresponds to (x', Y, Y^*) then

$$\mu_{s_i(\alpha)}(y)_i = \sum_{a \in H} y_a y_{a^*} = YY^*.$$

We now apply Lemma 2.1 with $\nu = \lambda_i$ to obtain a principal $\mathrm{GL}(m,K)$-bundle

$$f' : R' \times \mathcal{S} \to R' \times \mathcal{X} \cong \{x \in \mathrm{Rep}(\overline{Q},\alpha) \mid \mu_\alpha(x)_i = \lambda_i 1\},$$

where $\mathcal{S}$ and $\mathcal{X}$ are as in Lemma 2.1. Exchanging the role of the X's and Y's, we also obtain a principal $\mathrm{GL}(n,K)$-bundle

$$g' : R' \times \mathcal{S} \to \{y \in \mathrm{Rep}(\overline{Q}, s_i(\alpha)) \mid \mu_{s_i(\alpha)}(y)_i = r_i(\lambda)_i 1\}.$$

To show that f' and g' restrict to give a scheme T and principal bundles f and g, we need to show that for each vertex $j \neq i$ and each $z \in R' \times \mathcal{S}$ we have

$$\mu_\alpha(f'(z))_j - \lambda_j 1 = \mu_{s_i(\alpha)}(g'(z))_j - r_i(\lambda)_j 1$$

in $\mathrm{Mat}(\alpha_j, K)$.

Now if $x = f'(z)$ and $y = g'(z)$ then the relation $X^*X - Y^*Y = \lambda_i 1$ for $\mathcal{S}$ implies that $x_{a^*}x_a - y_{a^*}y_a = \lambda_i 1$ for any $a \in H$. Also $x_a = y_a$ for any arrow a not incident at i, so that $x_{a^*}x_a - y_{a^*}y_a = 0$ if $a \in Q$ and $h(a) \neq i$. Thus, if j is a vertex different from i, we have

$$\sum_{\substack{a \in Q \\ t(a)=j}} x_{a^*}x_a = \sum_{\substack{a \in Q \\ t(a)=j}} y_{a^*}y_a + N\lambda_i 1,$$

where N is the number of arrows from j to i. Clearly we also have

$$\sum_{\substack{a \in Q \\ h(a)=j}} x_a x_{a^*} = \sum_{\substack{a \in Q \\ h(a)=j}} y_a y_{a^*}$$

since $j \neq i$. It follows that

$$\mu_\alpha(x)_j - \mu_{s_i(\alpha)}(y)_j = -N\lambda_i 1 = (\lambda_j - r_i(\lambda)_j)1,$$

as required.

3. Lifting representations from Q to Π^λ

Let Q be a quiver with vertex set I and let $\lambda \in K^I$. In this section we determine which representations of Q lift to representations of Π^λ. That is, for $\alpha \in \mathbb{N}^I$ we determine the image of the projection $\pi : \mu_\alpha^{-1}(\lambda) \to \operatorname{Rep}(Q, \alpha)$. (For Dynkin quivers this problem has been studied by Rump [21]. His methods are, however, quite different.) In addition, if U is a constructible subset of $\operatorname{Im}(\pi)$ which is $\mathrm{G}(\alpha)$-stable (that is, a union of $\mathrm{G}(\alpha)$-orbits), we relate the dimension of $\pi^{-1}(U)$ to the number of parameters of $\mathrm{G}(\alpha)$ on U. Recall that if X is a scheme, G is an algebraic group acting on X, and U is a constructible subset of X which is G-stable, then the *number of parameters* (or *modularity*) of G on U, is defined by

$$\dim_G U = \max_d \left(\dim(U \cap X_d) + d - \dim G\right)$$

where X_d is the locally closed subset of X consisting of those points whose stabilizer has dimension d, so which have orbit of dimension $\dim G - d$.

LEMMA 3.1. *If $x = (x_a)_{a\in Q} \in \operatorname{Rep}(Q, \alpha)$, then there is an exact sequence*

$$0 \to \operatorname{Ext}^1(x, x)^* \to \operatorname{Rep}(Q^{op}, \alpha) \xrightarrow{c} \operatorname{End}(\alpha) \xrightarrow{t} \operatorname{End}(x)^* \to 0$$

where c sends $(y_{a^*}) \in \operatorname{Rep}(Q^{op}, \alpha)$ to $\sum_{a\in Q}[x_a, y_{a^*}]$ and t sends (θ_i) to the linear map $\operatorname{End}(x) \to K$ sending (ϕ_i) to $\sum_i \operatorname{tr}(\theta_i\phi_i)$.

Proof. This is just a fuller statement of [5, Lemma 4.2].

LEMMA 3.2. *If $\lambda \in K^I$ and x is a representation of Q which lifts to Π^λ, then $\sum_i \lambda_i \operatorname{tr}(\theta_i) = 0$ for any $\theta \in \operatorname{End}(x)$.*

Proof. Applying Lemma 3.1, since x lifts, one deduces that λ is in the image of c, so in the kernel of t.

THEOREM 3.3. *If $\lambda \in K^I$ then a representation of Q lifts to a representation of Π^λ if and only if the dimension vector β of any direct summand satisfies $\lambda \cdot \beta = 0$. Moreover, if $x \in \operatorname{Rep}(Q, \alpha)$ does lift, then $\pi^{-1}(x) \cong \operatorname{Ext}^1(x, x)^*$*

Proof. If the representation lifts, and there is a direct summand of dimension β then letting θ be the projection onto this summand, we have $\lambda \cdot \beta = 0$ by Lemma 3.2.

For the converse, it suffices to prove the liftability of any indecomposable x whose dimension vector α satisfies $\lambda \cdot \alpha = 0$. Now any endomorphism θ of x is the sum of a nilpotent matrix and a scalar matrix, so $\sum_i \lambda_i \operatorname{tr}(\theta_i) = 0$. Thus, considering λ as an element of $\operatorname{End}(\alpha)$, it is in the kernel of the map t of Lemma 3.1. Thus λ is in the image of c, and this gives a lift to Π^λ.

LEMMA 3.4. *If U is a $\mathrm{G}(\alpha)$-stable constructible subset of $\operatorname{Rep}(Q,\alpha)$ contained in the image of π, then*

$$\dim \pi^{-1}(U) = \dim_{\mathrm{G}(\alpha)} U + \alpha \cdot \alpha - q(\alpha).$$

If in addition U is a $\mathrm{G}(\alpha)$-orbit, then $\pi^{-1}(U)$ is irreducible of dimension $\alpha \cdot \alpha - q(\alpha)$.

Proof. By partitioning U we may suppose that all representations $x \in U$ have endomorphism ring of dimension e. Now if $x \in U$ then by Theorem 3.3 the fibre $\pi^{-1}(x)$ is isomorphic to $\operatorname{Ext}^1(x,x)^*$, so has dimension $e - q(\alpha)$ by Lemma 3.1. Thus $\dim \pi^{-1}(U) = \dim U + e - q(\alpha)$ On the other hand, each orbit of $\mathrm{G}(\alpha)$ on U has dimension $\dim \mathrm{G}(\alpha) + 1 - e$, so $\dim_{\mathrm{G}(\alpha)} U = \dim U - 1 + e - \dim \mathrm{G}(\alpha)$. The dimension formula follows.

Now suppose in addition that $U = \mathrm{G}(\alpha)x$. Since $\dim_{\mathrm{G}(\alpha)} U = 0$ the inverse image $\pi^{-1}(U)$ has dimension $\alpha \cdot \alpha - q(\alpha)$. It remains to prove that it is irreducible. Observe that $\mathrm{G}(\alpha)$ acts on $\mu_\alpha^{-1}(\lambda)$ and π is equivariant. Now if $\pi^{-1}(U)$ is not irreducible one can find nonempty disjoint $\mathrm{G}(\alpha)$-stable open subsets Z_1, Z_2. But $\pi(Z_i) = U$, so $\pi^{-1}(x) \cap Z_i$ $(i = 1, 2)$ are non-empty disjoint open subsets of $\pi^{-1}(x)$, which is impossible since $\pi^{-1}(x)$ is irreducible.

4. Application of Kac's Theorem

Let Q be a quiver with vertex set I. Kac's Theorem [7, 8] asserts that the dimension vectors of indecomposable representations of Q are exactly the positive roots for Q. Moreover, if α is a positive real root then there is a unique indecomposable representation of dimension α, while if α is a positive imaginary root then $\dim_{\mathrm{G}(\alpha)} I(\alpha) = p(\alpha)$ where $I(\alpha) \subseteq \operatorname{Rep}(Q,\alpha)$ is the set of indecomposable representations.

We need some properties of $\dim_G$ which are easy to prove using Chevalley's Theorems.

LEMMA 4.1. *Let X be a scheme on which an algebraic group G acts. Suppose that $Z \subseteq Y \subseteq X$ are constructible subsets, with Y being G-stable and Z being H-stable, where H is a closed subgroup of G. If $Y = GZ$ and the intersection of Z with any G-orbit in Y is a finite union of H-orbits, then $\dim_H Z = \dim_G Y$.*

LEMMA 4.2. *Suppose that algebraic groups G_i act on schemes X_i. If $Y_i \subseteq X_i$ are G_i-stable constructible subsets, then setting $G = \prod_i G_i$ and $Y = \prod_i Y_i$, we have $\dim_G Y = \sum_i \dim_{G_i} Y_i$.*

For arbitrary α, suppose that $\alpha = \beta^{(1)} + \cdots + \beta^{(r)}$ is a decomposition of α as a sum of positive roots for Q, and let $I(\beta^{(1)}, \ldots, \beta^{(r)})$ be the subset of $\mathrm{Rep}(Q, \alpha)$ consisting of the representations whose indecomposable summands have dimension $\beta^{(t)}$. Clearly this is a $\mathrm{G}(\alpha)$-stable constructible set.

LEMMA 4.3. *If $\alpha = \beta^{(1)} + \cdots + \beta^{(r)}$ with the $\beta^{(t)}$ positive roots, then*

$$\dim_{\mathrm{G}(\alpha)} I(\beta^{(1)}, \ldots, \beta^{(r)}) = \sum_{t=1}^{r} p(\beta^{(t)}).$$

Proof. Let $R' = \mathrm{Rep}(Q, \beta^{(1)}) \times \cdots \times \mathrm{Rep}(Q, \beta^{(r)})$, and consider it as a subset of $\mathrm{Rep}(Q, \alpha)$ using block-diagonal matrices. Let I' be the constructible subset of R' consisting of the elements in which each representation of dimension $\beta^{(t)}$ is indecomposable. By the Krull-Schmidt Theorem, Lemma 4.1 applies to the subsets

$$I' \subseteq I(\beta^{(1)}, \ldots, \beta^{(r)}) \subseteq \mathrm{Rep}(Q, \alpha)$$

with H the subgroup of $\mathrm{G}(\alpha)$ corresponding to the product $\prod_t \mathrm{G}(\beta^{(t)})$. Thus

$$\dim_{\mathrm{G}(\alpha)} I(\beta^{(1)}, \ldots, \beta^{(r)}) = \dim_H I' = \sum_t \dim_{\mathrm{G}(\beta^{(t)})} I(\beta^{(t)})$$

by Lemma 4.2, and this is $\sum_t p(\beta^{(t)})$ by Kac's Theorem.

THEOREM 4.4. *Given a pair (λ, α) with $\lambda \cdot \alpha = 0$, we have*

$$\dim \mu_\alpha^{-1}(\lambda) = \alpha \cdot \alpha - q(\alpha) + m,$$

where m is the maximum value of $\sum_{t=1}^{r} p(\beta^{(t)})$ where $r \geq 1$ and $\alpha = \beta^{(1)} + \cdots + \beta^{(r)}$ is a decomposition with each $\beta^{(t)}$ a positive root and $\lambda \cdot \beta^{(t)} = 0$.

Proof. Let $\pi : \mu_\alpha^{-1}(\lambda) \to \mathrm{Rep}(Q,\alpha)$ be the projection. We decompose $\mathrm{Rep}(Q,\alpha)$ as a union of sets of the form $I(\beta^{(1)},\dots,\beta^{(r)})$, and consider the inverse images $\pi^{-1}(I(\beta^{(1)},\dots,\beta^{(r)}))$. If some $\beta^{(t)}$ has $\lambda\cdot\beta^{(t)} \neq 0$ then this inverse image is empty. Otherwise, by Lemmas 3.4 and 4.3 this inverse image has dimension $\sum_{t=1}^r p(\beta^{(t)}) + \alpha\cdot\alpha - q(\alpha)$. The result follows.

We now turn to the proof of Theorem 1.1. We use Kac's 'canonical decomposition'. (See [8, Section 1.18].)

LEMMA 4.5. *If $\alpha \in \mathbb{N}^I$ has canonical decomposition $\alpha = \beta^{(1)} + \cdots + \beta^{(r)}$ with $r \geq 2$, then $p(\alpha) < \sum_t p(\beta^{(t)})$.*

Proof. This holds since $(\beta^{(s)},\beta^{(t)}) \geq 0$ for $s \neq t$ by [8, Proposition 1.20].

Proof. (of Theorem 1.1) Let $d = \alpha\cdot\alpha - 1 + 2p(\alpha)$, the relative dimension of μ_α.

(1) $\Longrightarrow$ (2) Since μ_α is flat, its image U is an open subset of $\mathrm{End}(\alpha)_0$. Now apply [6, Corollaire 6.1.4] to the map $\mathrm{Rep}(\overline{Q},\alpha) \to U$. Clearly $0 \in U$, so $\mu_\alpha^{-1}(0)$ has dimension d.

(2) $\Longrightarrow$ (3) Follows from Theorem 4.4.

(3) $\Longrightarrow$ (4) If $p(\alpha) < \sum_t p(\beta^{(t)})$ for some decomposition $\alpha = \beta^{(1)} + \cdots + \beta^{(r)}$, then Lemma 4.5 shows that the inequality remains true when we replace each $\beta^{(t)}$ by all the terms in its canonical decomposition. But now the terms are positive roots.

(4) $\Longrightarrow$ (1) By Lemma 4.5 the canonical decomposition of α can only have one term. It follows that α is a Schur root. (See [8, Section 1.18].) This means that there is a representation of Q of dimension α whose endomorphism algebra is the base field K. If $x \in \mathrm{Rep}(Q,\alpha)$ is such a representation, then the map c of Lemma 3.1 has 1-dimensional cokernel. Since $\mathrm{Im}(c)$ is clearly contained in $\mathrm{End}(\alpha)_0$, it follows that $\mathrm{Im}(c) = \mathrm{End}(\alpha)_0$. It follows that any element of $\mathrm{End}(\alpha)_0$ is the image under the moment map $\mu_\alpha : \mathrm{Rep}(\overline{Q},\alpha) \to \mathrm{End}(\alpha)_0$ of an element of $\mathrm{Rep}(\overline{Q},\alpha)$ whose restriction to Q is equal to x. In particular the moment map is surjective. We consider its fibres $\mu_\alpha^{-1}(\phi)$ with $\phi \in \mathrm{End}(\alpha)_0$. Let $\tilde{\pi} : \mu_\alpha^{-1}(\phi) \to \mathrm{Rep}(Q,\alpha)$ be the projection. Now if U is a constructible $\mathrm{G}(\alpha)$-stable subset of $\mathrm{Rep}(Q,\alpha)$ then

$$\dim \tilde{\pi}^{-1}(U) \leq \dim_{\mathrm{G}(\alpha)} U + \alpha\cdot\alpha - q(\alpha).$$

by the same argument as Lemma 3.4. It follows by Lemma 4.3 and the hypothesis that $\mu_\alpha^{-1}(\phi)$ has dimension at most d. Clearly, in fact, it is equidimensional of dimension d. Now [6, Proposition 6.1.5] implies that μ_α is flat.

5. Properties of the set Σ_λ

Throughout this section Q is a quiver with vertex set I. We prove some combinatorial results about the set Σ_λ which are needed later. In the course of this, we obtain another characterization of Σ_λ, Theorem 5.6.

We write R_λ^+ for the set of positive roots α with $\lambda \cdot \alpha = 0$. Thus Σ_λ is the set of $\alpha \in R_\lambda^+$ with the property that $p(\alpha) > \sum p(\beta^{(t)})$ for any decomposition $\alpha = \beta^{(1)} + \dots + \beta^{(r)}$ with $r \geq 2$ and all $\beta^{(t)} \in R_\lambda^+$. We write $\mathbb{N}R_\lambda^+$ for the set of sums of elements of R_λ^+ (including 0).

LEMMA 5.1. *Given any pair (λ, α) with $\alpha \in \mathbb{N}R_\lambda^+$, if i is a vertex with $\lambda_i = 0$ and $(\alpha, \epsilon_i) > 0$, then $\alpha - \epsilon_i \in \mathbb{N}R_\lambda^+$.*

Proof. Since $(\alpha, \epsilon_i) > 0$ there cannot be a loop at i, and therefore there is a reflection at i, although it is not admissible. Now α is a sum of positive roots $\sum_{t=1}^r \gamma^{(t)}$. If any $\gamma^{(t)}$ is equal to ϵ_i then we're done. Otherwise all $s_i(\gamma^{(t)})$ are positive roots, so in R_λ^+. Thus $s_i(\alpha) = \alpha - (\alpha, \epsilon_i)\epsilon_i \in \mathbb{N}R_\lambda^+$. Now adding on a suitable number of copies of $\epsilon_i \in R_\lambda^+$, it follows that $\alpha - \epsilon_i \in \mathbb{N}R_\lambda^+$.

LEMMA 5.2. *If $(\lambda, \alpha) \sim (\lambda', \alpha')$ then*

(1) $\alpha \in R_\lambda^+$ if and only if $\alpha' \in R_{\lambda'}^+$.

(2) $\alpha \in \mathbb{N}R_\lambda^+$ if and only if $\alpha' \in \mathbb{N}R_{\lambda'}^+$.

(3) $\alpha \in \Sigma_\lambda$ if and only if $\alpha' \in \Sigma_{\lambda'}$.

Proof. It suffices to prove (1), for then the other parts follow. Consider the admissible reflection at a loopfree vertex i with $\lambda_i \neq 0$. Now if α is a positive root, then so is $s_i(\alpha)$, except when $\alpha = \epsilon_i$. However, this case cannot occur since $\lambda \cdot \epsilon_i \neq 0$, so that $\epsilon_i \notin R_\lambda^+$.

LEMMA 5.3. *Given any pair (λ, α) with $\alpha \in \mathbb{N}R_\lambda^+$, there is an equivalent pair (λ', α') with the property that $(\alpha', \epsilon_i) \leq 0$ whenever $\lambda'_i \neq 0$.*

Proof. Amongst all equivalent pairs, choose (λ', α') with α' minimal. This is possible since Lemma 5.2(2) ensures that $\alpha' \geq 0$. Now if $\lambda'_i \neq 0$ and there is a loop at i then $(\alpha', \epsilon_i) \leq 0$ is automatic, while if $\lambda'_i \neq 0$ and i is loopfree then $(\alpha', \epsilon_i) \leq 0$, for otherwise the pair $(r_i(\lambda'), s_i(\alpha'))$ is smaller.

LEMMA 5.4. *Suppose that $0 \neq \alpha \in \mathbb{N}R_\lambda^+$ and $(\alpha, \epsilon_i) \leq 0$ for all vertices i with $\lambda_i \neq 0$. If $(\beta, \alpha - \beta) \leq -2$ whenever $\beta, \alpha - \beta$ are nonzero and in $\mathbb{N}R_\lambda^+$, then α is either a coordinate vector or in the fundamental region.*

Proof. Suppose that α is not a coordinate vector. We have $(\alpha, \epsilon_i) \leq 0$ for all i, for if $(\alpha, \epsilon_i) > 0$ then we must have $\lambda_i = 0$. Now the inequality $(\alpha, \epsilon_i) > 0$ implies that i is loopfree, so $(\epsilon_i, \epsilon_i) = 2$. Thus

$$(\alpha - \epsilon_i, \epsilon_i) = (\alpha, \epsilon_i) - 2 > -2.$$

This contradicts the hypotheses, since $\alpha - \epsilon_i \in \mathbb{N}R_\lambda^+$ by Lemma 5.1.

Next, the support quiver of α is connected. By assumption $\alpha \in \mathbb{N}R_\lambda^+$, so we can write $\alpha = \sum_{t=1}^r \gamma^{(t)}$ with the $\gamma^{(t)} \in R_\lambda^+$. Now supposing that the support of α is a disjoint union $C \cup D$ with no arrows connecting C to D, then each $\gamma^{(t)}$ has support contained in either C or D. Letting β be the sum of the $\gamma^{(t)}$ with support contained in C gives $(\beta, \alpha - \beta) = 0$, contrary to the assumption.

Thus α is in the fundamental region.

LEMMA 5.5. *If $0 \neq \alpha \in \mathbb{N}R_\lambda^+$ and $(\beta, \alpha - \beta) \leq -2$ whenever $\beta, \alpha - \beta$ are nonzero and in $\mathbb{N}R_\lambda^+$, then $\alpha \in R_\lambda^+$.*

Proof. By Lemma 5.2 we may replace the pair (λ, α) by any equivalent pair. Thus by Lemma 5.3 we may suppose that $(\alpha, \epsilon_i) \leq 0$ whenever $\lambda_i \neq 0$. Now by the previous lemma α is either a coordinate vector or in the fundamental region. Thus it is in R_λ^+.

We now have another description of the set Σ_λ.

THEOREM 5.6. *If $\alpha \in \mathbb{N}^I$ then $\alpha \in \Sigma_\lambda$ if and only if $0 \neq \alpha \in \mathbb{N}R_\lambda^+$ and $(\beta, \alpha - \beta) \leq -2$ whenever $\beta, \alpha - \beta$ are nonzero and in $\mathbb{N}R_\lambda^+$.*

Proof. Suppose first that $\alpha \in \Sigma_\lambda$. Clearly we have $0 \neq \alpha \in \mathbb{N}R_\lambda^+$. We prove that $(\beta, \alpha - \beta) \leq -2$ whenever $\beta, \alpha - \beta$ are nonzero and in $\mathbb{N}R_\lambda^+$. For a contradiction, suppose that $(\beta, \alpha - \beta) \geq -1$ with $\beta, \alpha - \beta$ nonzero and in $\mathbb{N}R_\lambda^+$. It follows that $p(\alpha) \leq p(\beta) + p(\alpha - \beta)$. This gives a decomposition of the form

$$\alpha = \sum_{t=1}^r \beta^{(t)}, \quad 0 \neq \beta^{(t)} \in \mathbb{N}R_\lambda^+, \quad p(\alpha) \leq \sum_{t=1}^r p(\beta^{(t)})$$

with $r = 2$. Choose a decomposition of this type with r maximal. Now each term $\beta^{(t)}$ in this sum is nonzero, and belongs to $\mathbb{N}R_\lambda^+$. By maximality, if $\gamma, \beta^{(t)} - \gamma$ are nonzero and in $\mathbb{N}R_\lambda^+$, then $p(\beta^{(t)}) > p(\gamma) + p(\beta^{(t)} - \gamma)$, so $(\gamma, \beta^{(t)} - \gamma) < -1$, and hence $\beta^{(t)} \in R_\lambda^+$ by Lemma 5.5. Now this decomposition contradicts the fact that $\alpha \in \Sigma_\lambda$.

For the converse, suppose that $0 \neq \alpha \in \mathbb{N}R_\lambda^+$ and $(\beta, \alpha - \beta) \leq -2$ whenever $\beta, \alpha - \beta$ are nonzero and in $\mathbb{N}R_\lambda^+$. By Lemma 5.5 we have $\alpha \in R_\lambda^+$. Assuming that $\alpha \notin \Sigma_\lambda$, there is a decomposition $\alpha = \sum_{t=1}^r \beta^{(t)}$

with $\beta^{(t)} \in R_\lambda^+$ and with $p(\alpha) \leq \sum_{t=1}^r p(\beta^{(t)})$. It follows that $q(\alpha) - \sum_{t=1}^r q(\beta^{(t)}) \geq 1 - r$, so

$$\sum_{t=1}^{r} (\beta^{(t)}, \alpha - \beta^{(t)}) = \sum_{t \neq k} (\beta^{(t)}, \beta^{(k)}) = 2\Big(q(\alpha) - \sum_{t=1}^{r} q(\beta^{(t)})\Big) \geq 2 - 2r.$$

This implies that $(\beta^{(t)}, \alpha - \beta^{(t)}) > -2$ for some t, contrary to the assumption.

Note in particular that $\mathbb{N}R_0^+ = \mathbb{N}^I$, giving the following simple description of Σ_0.

COROLLARY 5.7. *If $\alpha \in \mathbb{N}^I$ then $\alpha \in \Sigma_0$ if and only if $\alpha > 0$ and $(\beta, \alpha - \beta) \leq -2$ whenever $\beta \in \mathbb{N}^I$ and $0 < \beta < \alpha$.*

Combining Lemmas 5.2, 5.3, 5.4 and Theorem 5.6, we have proved:

THEOREM 5.8. *If $\alpha \in \Sigma_\lambda$ then there is an equivalent pair (λ', α') with α' either the coordinate vector at a loopfree vertex or in the fundamental region. The first case occurs if α is a real root; the second case if α is an imaginary root.*

6. Existence of simple representations

Let Q be a quiver with vertex set I. In this section we prove the implication $(2) \Longrightarrow (1)$ of Theorem 1.2.

LEMMA 6.1. *If X is an equidimensional scheme, Y is an irreducible scheme and $f : X \to Y$ is a dominant morphism with all fibres irreducible of constant dimension d, then X is irreducible.*

Proof. If X is not irreducible, one can find disjoint irreducible open subsets Z, Z'. Now the restriction of f to Z is a map $Z \to \overline{f(Z)}$ whose fibres have dimension at most d, so $d + \dim \overline{f(Z)} \geq \dim Z = \dim X = d + \dim Y$, so $\overline{f(Z)} = Y$, and for the general point $y \in Y$ the fibre $Z \cap f^{-1}(y)$ has dimension d. Similarly, for the general point $y \in Y$ the fibre $Z' \cap f^{-1}(y)$ has dimension d. But $f^{-1}(y)$ is irreducible of dimension d, so these two sets must intersect. A contradiction since Z, Z' are disjoint.

Recall that a representation is said to be a *brick* if its endomorphism algebra is the base field K. We denote by $B(\alpha) \subseteq \mathrm{Rep}(Q, \alpha)$ the set of bricks for Q of dimension α.

If α is a dimension vector in the fundamental region and $q(\alpha) < 0$ then by Kac's Lemma 1 (see [8, Section 1.10]), the set $B(\alpha)$ is a dense open subset of $\mathrm{Rep}(Q, \alpha)$, we have $\dim_{\mathrm{G}(\alpha)} B(\alpha) = p(\alpha)$, and $\dim_{\mathrm{G}(\alpha)}(I(\alpha) \setminus B(\alpha)) < p(\alpha)$.

On the other hand, if α is in the fundamental region but $q(\alpha) = 0$, then there need not be any bricks. In this case the support quiver of α is extended Dynkin, α is a multiple of the minimal imaginary root δ, and we have the following result.

LEMMA 6.2. *If Q is an extended Dynkin quiver with minimal imaginary root δ and $\alpha = m\delta$ with $m \geq 1$, then every indecomposable representation of Q of dimension α has endomorphism algebra of dimension m, and $I(\alpha)$ is an irreducible locally closed subset of* $\mathrm{Rep}(Q, \alpha)$ *with* $\dim_{\mathrm{G}(\alpha)} I(\alpha) = 1$.

Proof. Of course the fact that $\dim_{\mathrm{G}(\alpha)} I(\alpha) = 1$ is one of the things that needs to be verified during the proof of Kac's Theorem.

The indecomposable representations of Q of dimension α are known by the representation theory of extended Dynkin quivers. They all belong to the tubular family T of [20, §3.6 (5), (6)]. Recall from [20, §3.1] that T is a serial abelian category. Its simple objects are called *simple regular modules.*

We claim that an indecomposable in T, say in a tube of rank r, has dimension α if and only if it has a composition series in T of length mr. Namely, suppose Q has no oriented cycles. (The case of an oriented cycle follows by [20, §3.6 (6)].) Inspecting the proof of [20, Theorem 3.4], we see that the tube contains a module $W_0(\rho)$ of length r and with a composition series involving each simple regular module in the tube. By the proof of [20, §3.6 (5)], the module W_0 corresponds to the maximal root for the corresponding Dynkin quiver, so $W_0(\rho)$ has dimension δ. The claim follows.

It follows from this description that all indecomposables of dimension α have endomorphism algebra of dimension m. Now $I(\alpha)$ is locally closed by [10, §2.5 Proposition]. It is a union of infinitely many $\mathrm{G}(\alpha)$-orbits. We show that each orbit is contained in an irreducible open subset of $I(\alpha)$ whose complement is a finite union of orbits. This implies the irreducibility of $I(\alpha)$.

If U is a finite set of simple regular modules, the *perpendicular category* is the full subcategory

$$U^{\perp} = \{M \mid \mathrm{Hom}(S, M) = \mathrm{Ext}^1(S, M) = 0 \text{ for all } S \in U\}.$$

of the category of KQ-modules. Using the fact that the tubes are standard, and the Auslander-Reiten formula [20, §2.4 (5)] we see that an indecomposable of dimension α is in $U^{\perp}$ if and only if its regular socle (in T) is not in U.

We consider the orbit corresponding to an be an indecomposable module X of dimension α. Choose a finite collection U of simple regular modules with the properties that (a) U does not contain the regular socle of X; (b) at most one simple regular module in each tube is not in U; (c) if Q has no oriented cycles then there is a unique tube which has all its simple regular modules in U, if Q is an oriented cycle then no tube has all its simple regular modules in U. As in [3, Lemma 11.1], there is a homomorphism

$$\theta : KQ \to \mathrm{Mat}(N, K[x])$$

(where $N = \sum_i \delta_i$) such that restriction induces an equivalence from the category of $\mathrm{Mat}(N, K[x])$-modules to $U^\perp$. Thus there is a KQ-$K[x]$-bimodule L, free of rank N over $K[x]$, such that the tensor product functor $L \otimes_{K[x]} -$ is an equivalence from $K[x]$-modules to $U^\perp$. It follows that as $\lambda \in K$ varies, the modules $L \otimes_{K[x]} K[x]/(x-\lambda)^m$ run through all indecomposables in $U^\perp$ of dimension α. Choosing generators of L, this induces a morphism $\phi : K \to \mathrm{Rep}(Q, \alpha)$ from the affine line, whose image meets all $\mathrm{G}(\alpha)$-orbits in $I(\alpha)$ in $U^\perp$. Now consider the map

$$\mathrm{G}(\alpha) \times K \to \mathrm{Rep}(Q, \alpha), \quad (g, \lambda) \mapsto g\phi(\lambda).$$

The image is contained in $I(\alpha)$, it contains the orbit for X, it is $\mathrm{G}(\alpha)$-stable, and it omits only finitely many orbits, so it is open in $I(\alpha)$. Since $\mathrm{G}(\alpha) \times K$ is irreducible, so is the image.

LEMMA 6.3. *If (λ, α) is a pair with $\alpha \in \Sigma_\lambda$ then $\mu_\alpha^{-1}(\lambda)$ is irreducible of dimension $d = \alpha \cdot \alpha - 1 + 2p(\alpha)$. In particular it is a complete intersection.*

Proof. By Theorem 5.8 and Lemma 2.2 we may reduce to the case where α is either a coordinate vector, or in the fundamental region. If α is a coordinate vector at a loopfree vertex, the result is trivial, so we suppose that α is in the fundamental region.

By Theorem 4.4 the space $\mu_\alpha^{-1}(\lambda)$ has dimension d. Moreover, since d is the relative dimension of μ_α, it is equidimensional of dimension d. It remains to prove that it is irreducible.

Let π be the projection $\mu_\alpha^{-1}(\lambda) \to \mathrm{Rep}(Q, \alpha)$. Thus the image of π is given by Theorem 3.3, and any nonempty fibre $\pi^{-1}(x)$ is isomorphic to $\mathrm{Ext}^1(x, x)^*$, so is irreducible.

As in Theorem 4.4 we write $\mu_\alpha^{-1}(\lambda)$ as a union of sets of the form $\pi^{-1}(I(\beta^{(1)}, \ldots, \beta^{(r)}))$. All except $\pi^{-1}(I(\alpha))$ have dimension strictly smaller than d.

Suppose first that $q(\alpha) < 0$. As mentioned before Lemma 6.2, the set $B(\alpha)$ of bricks is a dense open subset of $\mathrm{Rep}(Q, \alpha)$. Now the set

$\pi^{-1}(I(\alpha) \setminus B(\alpha))$ has dimension less than d. Thus it suffices to prove that $\pi^{-1}(B(\alpha))$ is irreducible. This space is open in $\mu_\alpha^{-1}(\lambda)$, so it is equidimensional of dimension d. Moreover every fibre of the map $\pi^{-1}(B(\alpha)) \to B(\alpha)$ is irreducible. Therefore $\pi^{-1}(B(\alpha))$ is irreducible by Lemma 6.1.

If $q(\alpha) = 0$ and α is indivisible the same argument holds. The set $B(\alpha)$ of bricks is a dense open subset, and there are only finitely many other orbits of indecomposables.

Finally suppose that $q(\alpha) = 0$ and α is divisible. Thus the support of α is extended Dynkin with minimal positive imaginary root δ, and $\alpha = m\delta$ for some $m \geq 2$. Now $\lambda \cdot \delta \neq 0$, for the decomposition $\alpha = \delta + \cdots + \delta$ contradicts the fact that $\alpha \in \Sigma_\lambda$. However $\lambda \cdot \alpha = 0$, so the only possibility is that the field K has characteristic $p > 0$ and m is a multiple of p. Now in fact $m = p$, for otherwise the decomposition $\alpha = p\delta + \cdots + p\delta$ contradicts the fact that $\alpha \in \Sigma_\lambda$.

Now the image of π is contained in the set of representations of Q with no summand of dimension $k\delta$ with $k < p$. Thus it consists of $I(\alpha)$ and only finitely many other orbits. Now $\pi^{-1}(I(\alpha))$ is obtained from $\mu_\alpha^{-1}(\lambda)$ by removing the inverse images of finitely many orbits. These inverse images have dimension strictly less than d. It follows that $\pi^{-1}(I(\alpha))$ is equidimensional of dimension d, and by the same argument $\pi^{-1}(I(\alpha))$ is irreducible, hence so is $\mu_\alpha^{-1}(\lambda)$.

LEMMA 6.4. *Given a pair (λ, α) with $\lambda \cdot \alpha = 0$, if $\beta \leq \alpha$ then the set of elements of $\mu_\alpha^{-1}(\lambda)$ such that the corresponding representation of Π^λ has a subrepresentation of dimension vector β is closed.*

Proof. If $\mathrm{Gr}(k, n)$ denotes the Grassmannian of k-dimensional subspaces of an n-dimensional space, then the set of pairs consisting of an element of $\mu_\alpha^{-1}(\lambda)$ and a subrepresentation of dimension β is a closed subset of

$$\mu_\alpha^{-1}(\lambda) \times \prod_{i \in I} \mathrm{Gr}(\beta_i, \alpha_i).$$

Since Grassmannians are projective, its image under the projection onto $\mu_\alpha^{-1}(\lambda)$ is closed. (See [22, Lemma 3.1].)

LEMMA 6.5. *Given a pair (λ, α) with $\lambda \cdot \alpha = 0$, if $x \in \mu_\alpha^{-1}(\lambda)$ corresponds to a representation of Π^λ which is a brick (that is, has endomorphism algebra equal to the base field K) then $\mu_\alpha^{-1}(\lambda)$ is smooth at x.*

Proof. It suffices to prove that μ_α is smooth at x. Now this holds by [3, Lemma 10.3]. Alternatively, note that x corresponds to a brick if and only if x has trivial stabilizer in $\mathrm{G}(\alpha)$, and the claim is standard differential geometry.

LEMMA 6.6. *Let Q be an extended Dynkin quiver with minimal imaginary root δ and $\alpha = m\delta$ with $m \geq 1$. Let $\beta \in \mathbb{N}^I$. If the general element of $I(\alpha)$ has subrepresentations of dimension β and $\alpha - \beta$, then β is a multiple of δ.*

Proof. If Q has no oriented cycles then clearly β must have defect zero, so the subrepresentations of dimensions β and $\alpha - \beta$ must be regular. Now the general element of $I(\alpha)$ is in a homogeneous tube, so all regular subrepresentations have dimension a multiple of δ. If Q is an oriented cycle then the same argument works, for the general element of $I(\alpha)$ involves m copies of a simple representation of Q of dimension δ.

THEOREM 6.7. *If (λ, α) is a pair with $\alpha \in \Sigma_\lambda$ then $\mu_\alpha^{-1}(\lambda)$ is a reduced and irreducible complete intersection of dimension $\alpha \cdot \alpha - 1 + 2p(\alpha)$, and the general element of $\mu_\alpha^{-1}(\lambda)$ is a simple representation of Π^λ.*

Proof. By Lemma 6.3, $\mu_\alpha^{-1}(\lambda)$ is irreducible of the right dimension.By Lemma 6.4, the simple representations are an open subset of $\mu_\alpha^{-1}(\lambda)$, so to show that the general element is simple it suffices to prove the existence of one simple representation of dimension α. Now because the reflection functors of [5] are equivalences, we may assume as in Lemma 6.3 that α is a coordinate vector or in the fundamental region. Clearly there is a simple representation if α is a coordinate vector, so assume that α is in the fundamental region.

Assume for a contradiction that there is no simple representation. The irreducibility of $\mu_\alpha^{-1}(\lambda)$ implies that there is some β such that the general representation of Π^λ of dimension α has a subrepresentation of dimension β. Then by Lemma 6.4 this holds for every representation of Π^λ of dimension α.

First suppose that $q(\alpha) < 0$ or α is indivisible. Then α is a Schur root. Thus the general representation of Q of dimension α is indecomposable, so extends to a representation of Π^λ, and hence has a subrepresentation of dimension β. Similarly, the general representation of Q^{op} of dimension α has a subrepresentation of dimension β. Considering duals, this implies that the general representation of Q of dimension α has a subrepresentation of dimension $\alpha - \beta$. Now by [22, Theorem 3.4] the general representation of dimension α decomposes as a direct sum of representations of dimension β and $\alpha - \beta$, contrary to the fact that α is a Schur root.

Now suppose that $q(\alpha) = 0$ and α is divisible. As in Lemma 6.3 the support of α is extended Dynkin with minimal positive imaginary root δ and $\alpha = p\delta$ where K has characteristic $p > 0$ and $\lambda \cdot \delta \neq 0$. Now any element of $I(\alpha)$ extends to a representation of Π^λ, and hence has a subrepresentation of dimension β. Similarly, by considering duals and the opposite quiver, any element of $I(\alpha)$ has a subrepresentation of dimension $\alpha - \beta$. Now by Lemma 6.6 we have $\beta = k\delta$ with $0 < k < p$. But then $\lambda \cdot \beta \neq 0$, contradicting the fact that there are representations of Π^λ of dimension β.

Finally, since the general element x of $\mu_\alpha^{-1}(\lambda)$ is a simple representation, it is a brick, and hence by Lemma 6.5 it is a smooth point. Thus $\mu_\alpha^{-1}(\lambda)$ is generically reduced. Since it is also a complete intersection, hence Cohen-Macaulay, it is reduced.

7. The set F_λ

In this section Q is a quiver with vertex set I. If $\lambda \in K^I$, recall that R_λ^+ is the set of positive roots α with $\lambda \cdot \alpha = 0$. We define F_λ to be the set of $\alpha \in R_\lambda^+$ with the property that $(\alpha', \epsilon_i) \leq 0$ for any $(\lambda', \alpha') \sim (\lambda, \alpha)$ and any vertex i with $\lambda'_i = 0$. It is a sort of fundamental region with respect to λ. (Of course F_0 is precisely the fundamental region.) We prove that if there is a simple representation of Π^λ of dimension α, then either (λ, α) is equivalent to a pair (λ', α') with α' the coordinate vector of a loopfree vertex, or $\alpha \in F_\lambda$.

By definition, if $(\lambda, \alpha) \sim (\lambda', \alpha')$ then $\alpha \in F_\lambda$ if and only if $\alpha' \in F_{\lambda'}$. Now Lemma 5.3 immediately implies the following result.

LEMMA 7.1. *If $\alpha \in F_\lambda$ then there is an equivalent pair $(\lambda', \alpha') \sim (\lambda, \alpha)$ with α' in the fundamental region. In particular α is an imaginary root.*

LEMMA 7.2. *If Π^λ has a simple representation of dimension α and i is a vertex, then either $\alpha = \epsilon_i$, or $\lambda_i \neq 0$, or $(\alpha, \epsilon_i) \leq 0$.*

Proof. For simplicity we may suppose that no arrow has tail at i. Suppose that $\lambda_i = 0$. Let V be a simple representation of dimension vector α, with vector space V_j at each vertex j. Letting $V_\oplus = \oplus V_{t(a)}$, where the sum is over all arrows in Q with head at i, the linear maps in V combine to give maps

$$V_i \underset{\phi}{\overset{\theta}{\rightleftarrows}} V_\oplus$$

with $\phi\theta = 0$.

Now if $\mathrm{Ker}(\theta) \neq 0$ then V has a nonzero subrepresentation W where $W_i = \mathrm{Ker}(\theta)$ and $W_j = 0$ for all $j \neq i$.

On the other hand if $\mathrm{Im}(\phi) \neq V_i$ then V has a proper subrepresentation W with $W_i = \mathrm{Im}(\phi)$ and $W_j = V_j$ for all $j \neq i$.

Thus, assuming that V is simple and $\alpha \neq \epsilon_i$, we deduce that θ is injective and ϕ is surjective. Since $\phi\theta = 0$ the map ϕ induces a surjection $V_\oplus / \operatorname{Im}(\theta) \to V_i$, and hence $\dim V_\oplus \geq 2 \dim V_i$. Thus $(\alpha, \epsilon_i) = 2 \dim V_i - \dim V_\oplus \leq 0$.

LEMMA 7.3. *If there is a simple representation of* Π^λ *of dimension* α *then there is a pair* $(\lambda', \alpha') \sim (\lambda, \alpha)$ *with* α' *a coordinate vector or in the fundamental region. In particular* α *is a root.*

Proof. If $(\lambda', \alpha') \sim (\lambda, \alpha)$ then, because of the reflection functors, there is a simple representation of $\Pi^{\lambda'}$ of dimension α'. In particular $\alpha' > 0$, so we can choose a pair (λ', α') with α' minimal. Now either α' is a coordinate vector, or in the fundamental region. Namely, supposing that α' is not a coordinate vector, since there is a simple representation of dimension α', it has connected support. Thus it suffices to prove that $(\alpha', \epsilon_i) \leq 0$ for any vertex i. This is true if there is a loop at i, so we may suppose that i is loopfree. If $\lambda_i = 0$ then $(\alpha', \epsilon_i) \leq 0$ by Lemma 7.2. If $\lambda_i \neq 0$ then the reflection at i is admissible, and $(\alpha', \epsilon_i) \leq 0$ by the minimality of α'.

LEMMA 7.4. *If there is a simple representation of* Π^λ *of dimension* α *then either* (λ, α) *is equivalent to a pair* (λ', α') *with* α' *the coordinate vector of a loopfree vertex, or* $\alpha \in F_\lambda$.

Proof. Supposing that there is no equivalent pair (λ', α') with α' the coordinate vector of a loopfree vertex, we show that $\alpha \in F_\lambda$. Of course α is a root by Lemma 7.3. If $(\lambda', \alpha') \sim (\lambda, \alpha)$ then there is a simple representation of $\Pi^{\lambda'}$ of dimension α'. Now if i is a vertex with $\lambda'_i = 0$ then either there is a loop at i, in which case $(\alpha', \epsilon_i) \leq 0$ automatically, or if there is no loop at i, then $(\alpha', \epsilon_i) \leq 0$ by Lemma 7.2.

8. Classification of $F_\lambda \setminus \Sigma_\lambda$

Let Q be a quiver with vertex set I. It follows from Theorem 6.7 and Lemma 7.4 that the set of imaginary roots in Σ_λ is a subset of F_λ. In this section we show that this is quite close to being an equality. Not only is this a good way of determining the elements of Σ_λ (especially when $\lambda = 0$, so there are no admissible reflections), it is also essential for the proof of our characterization of the dimension vectors of simple representations of Π^λ.

THEOREM 8.1. *If* (λ, α) *is a pair with* $\alpha \in F_\lambda \setminus \Sigma_\lambda$*, then after first passing to an equivalent pair, and then passing to the support quiver of*

α and the corresponding restrictions of λ and α, one of the following cases holds:

(I) Q is extended Dynkin with minimal positive imaginary root δ, and either $\lambda \cdot \delta = 0$ and $\alpha = m\delta$ with $m \geq 2$ or, if the field K has characteristic $p > 0$, $\lambda \cdot \delta \neq 0$ and $\alpha = m'p\delta$ with $m' \geq 2$.

(II) I is a disjoint union $\mathcal{J} \cup \mathcal{K}$, with $\sum_{i \in \mathcal{K}} \lambda_i \alpha_i = 0$, there is a unique arrow with one end in $\mathcal{J}$ and the other in $\mathcal{K}$, say connecting vertices $j \in \mathcal{J}$ and $k \in \mathcal{K}$, and $\alpha_j = \alpha_k = 1$.

(III) I is a disjoint union $\mathcal{J} \cup \mathcal{K}$, there is a unique arrow with one end in $\mathcal{J}$ and the other in $\mathcal{K}$, say connecting vertices $j \in \mathcal{J}$ and $k \in \mathcal{K}$, $\alpha_j = 1$, the restriction of Q to $\mathcal{K}$ is extended Dynkin with extending vertex k and minimal positive imaginary root δ, $\lambda \cdot \delta = 0$, and the restriction of α to $\mathcal{K}$ is a multiple $m\delta$ with $m \geq 2$.

(Recall that if Q is an extended Dynkin quiver and δ is its minimal imaginary root, then an *extending vertex* is a vertex i with $\delta_i = 1$.)

The proof of this theorem takes the rest of this section. Throughout, we assume that $\alpha \in F_\lambda$. In particular α is a root, so if it is sincere then Q is connected. We say that $\beta \in \mathbb{N}^I$ is a (-1)-*vector* for the pair (λ, α) if $\beta, \alpha - \beta \in \mathbb{N}R_\lambda^+$ and $(\beta, \alpha - \beta) = -1$. We say that β is a *divisor* for (λ, α) if it is a (-1)-vector, $(\beta, \epsilon_i) \leq 0$ for every vertex i, and $(\alpha - \beta, \epsilon_i) \leq 0$ whenever $(\beta, \epsilon_i) = 0$. If β is a divisor for (λ, α), then

$$-1 = (\beta, \alpha - \beta) = \sum_i (\alpha - \beta)_i (\beta, \epsilon_i),$$

and all terms in this sum are ≤ 0. Thus there is a vertex j, which we call the *critical vertex* for β, with $(\beta, \epsilon_j) = -1$ and $(\alpha - \beta)_j = 1$, and for every other vertex i one has $(\beta, \epsilon_i) = 0$ or $(\alpha - \beta)_i = 0$.

LEMMA 8.2. *If $q(\alpha) < 0$ then $(\beta, \alpha - \beta) < 0$ for any β with β and $\alpha - \beta$ both nonzero and in $\mathbb{N}R_\lambda^+$.*

Proof. Suppose that $(\beta, \alpha - \beta) \geq 0$ with $\beta, \alpha - \beta$ nonzero and in $\mathbb{N}R_\lambda^+$. By Lemma 7.1 there is an equivalent pair (λ', α') with α' in the fundamental region. Applying the same sequence of reflections to β gives a vector β' with $(\beta', \alpha' - \beta') \geq 0$ and β' and $\alpha' - \beta'$ both nonzero. Now

$$q(\alpha') = q(\alpha' - \beta') + q(\beta') + (\beta', \alpha' - \beta') \geq q(\alpha' - \beta') + q(\beta')$$

so by [10, Lemma 2, p123] the support quiver of α' is extended Dynkin and α' is a multiple of the minimal imaginary root. But this implies that $q(\alpha) = q(\alpha') = 0$, a contradiction.

LEMMA 8.3. *If $\alpha \in F_\lambda \setminus \Sigma_\lambda$ and $q(\alpha) < 0$ then there is a (-1)-vector β for (λ, α).*

Proof. Combine the Lemma 8.2 with Theorem 5.6.

LEMMA 8.4. *If $\alpha \in F_\lambda \setminus \Sigma_\lambda$, $q(\alpha) < 0$ and β is a (-1)-vector for (λ, α), then there is an equivalent pair $(\lambda', \alpha') \sim (\lambda, \alpha)$ which has a divisor β' satisfying $\beta' \leq \beta$.*

Proof. Amongst all (-1)-vectors β' for all pairs (λ', α') equivalent to (λ, α), choose β' to be minimal with $\beta' \leq \beta$. Then choose an equivalent pair (λ', α') with α' minimal amongst those having β' as a (-1)-vector.

We claim that $(\beta', \epsilon_i) \leq 0$ for every vertex i. This is automatic if there is a loop at i, so we may suppose that i is loopfree, and for a contradiction suppose that $(\beta', \epsilon_i) > 0$. We divide into two cases according to whether or not $\lambda'_i = 0$.

Suppose that $\lambda'_i \neq 0$. This ensures that $\epsilon_i \notin R^+_{\lambda'}$, so any positive root in $R^+_{\lambda'}$ remains a positive root on applying the reflection s_i. Thus $s_i(\beta')$ and $s_i(\alpha' - \beta')$ are in $\mathbb{N}R^+_{r_i(\lambda')}$, and hence $s_i(\beta')$ is a (-1)-vector for $(r_i(\lambda'), s_i(\alpha'))$. Now since $(\beta', \epsilon_i) > 0$ it follows that $s_i(\beta')$ is strictly smaller than β', a contradiction.

Suppose on the other hand that $\lambda'_i = 0$. The vector $\beta' - \epsilon_i$ is in $\mathbb{N}R^+_{\lambda'}$ by Lemma 5.1. It is also nonzero, for if $\beta' = \epsilon_i$ then

$$-1 = (\epsilon_i, \alpha' - \epsilon_i) = (\alpha', \epsilon_i) - 2,$$

which is impossible since $(\alpha', \epsilon_i) \leq 0$ because $\alpha \in F_\lambda$. Now $\beta' - \epsilon_i$ is a (-1)-vector for (λ', α'), since

$$\begin{aligned}(\beta' - \epsilon_i, \alpha' - \beta' + \epsilon_i) &= (\beta', \alpha' - \beta') - (\epsilon_i, \epsilon_i) - (\alpha', \epsilon_i) + 2(\beta', \epsilon_i) \\ &\geq -1 - 2 - 0 + 2 = -1,\end{aligned}$$

so $(\beta' - \epsilon_i, \alpha' - \beta' + \epsilon_i) = -1$ by Lemma 8.2. This contradicts the minimality of β'. Thus the claim is proved.

Finally, suppose that i is a vertex with $(\beta', \epsilon_i) = 0$. If $\lambda'_i = 0$, then $(\alpha' - \beta', \epsilon_i) = (\alpha', \epsilon_i) \leq 0$ since $\alpha \in F_\lambda$. On the other hand, if $\lambda'_i \neq 0$ then the reflection at i is admissible for (λ', α'), but s_i has no effect on β', so if $(\alpha' - \beta', \epsilon_i) > 0$ then β' is a (-1)-vector for $(r_i(\lambda'), (s_i\alpha'))$, contradicting the minimality of α'. Thus β' is a divisor for (λ', α').

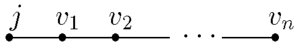
LEMMA 8.5. *Let β be a divisor for (λ, α), and let j be the critical vertex for β. Suppose that ξ is a vector whose components are non-negative real numbers, with support contained in the support of $\alpha - \beta$, with $\xi_j = 0$, $(\xi, \epsilon_j) = -1$ and with (ξ, ϵ_i) non-negative and integer-valued for every vertex $i \neq j$. Then there is at most one vertex i at which (ξ, ϵ_i) is strictly positive, and at this vertex $(\xi, \epsilon_i) = 1$.*

Proof. If i is a vertex with $\xi_i \neq 0$ then $i \neq j$ and by assumption $(\alpha - \beta)_i \neq 0$, so $(\beta, \epsilon_i) = 0$, and hence $(\alpha - \beta, \epsilon_i) \leq 0$. It follows that $(\alpha - \beta, \xi) \leq 0$. Now since $(\alpha - \beta)_j = 1$ we have

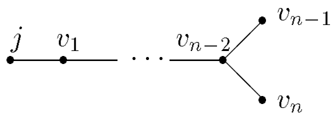
$$(\alpha - \beta, \xi) = \sum_i (\alpha - \beta)_i (\xi, \epsilon_i) = -1 + \sum_{i \neq j} (\alpha - \beta)_i (\xi, \epsilon_i)$$

Now the terms in this last sum are non-negative integers, so at most one term is nonzero. Now if $(\xi, \epsilon_i) > 0$ then certainly $\xi_i \neq 0$, so by hypothesis $(\alpha - \beta)_i \neq 0$, and hence the corresponding term in the sum is nonzero.

LEMMA 8.6. *Let β be a divisor for (λ, α), and let j be the critical vertex for β. Suppose there are vertices v_i $(1 \leq i \leq n)$ and the only arrows connected to the v_i are of the following form (the orientation of the arrows is irrelevant): either*



with $n \geq 1$, or



with $n \geq 3$. Then $(\alpha - \beta)_{v_i} = 0$ for some i.

Proof. Supposing otherwise, we obtain a contradiction using Lemma 8.5. In the first case take ξ to be the vector with $\xi_{v_i} = i$ for all i, and ξ zero at all other vertices. In the second case take ξ to be the vector with $\xi_{v_i} = i$ for $i \leq n-2$, $\xi_{v_{n-1}} = \xi_{v_n} = (n-1)/2$, and ξ zero at all other vertices.

LEMMA 8.7. *Suppose that β is a divisor for (λ, α), and let j be the critical vertex for β. Let Q' be an extended Dynkin subquiver of Q contained in the support of $\alpha - \beta$, and let δ be its minimal positive imaginary root. If for any vertex $i \in Q'$ we define*

$$s_i = \sum_{\substack{a \in Q \setminus Q' \\ h(a) = i}} \beta_{t(a)} + \sum_{\substack{a \in Q \setminus Q' \\ t(a) = i}} \beta_{h(a)},$$

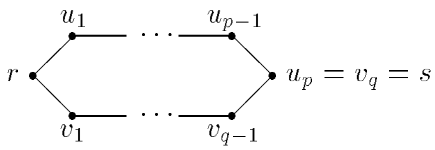
then either $j \notin Q'$ and $s_i = 0$ for all vertices $i \in Q'$, or $j \in Q'$ and $\delta_j = \sum_{i \in Q'} \delta_i s_i$.

Proof. For any vertex i in Q' we have $(\beta, \epsilon_i) = (\beta|_{Q'}, \epsilon_i)_{Q'} - s_i$. Thus $(\beta, \delta) = -\sum_i \delta_i s_i$. Since $(\alpha - \beta)_i \neq 0$ for all i in Q', we have $(\beta, \epsilon_i) = 0$ for any $i \neq j$. The result follows.

LEMMA 8.8. *If Q' is an extended Dynkin quiver and r and s are distinct extending vertices, then there is no vector γ with integer components, with $(\gamma, \epsilon_r) = 1$, and with $(\gamma, \epsilon_i) = 0$ for all $i \neq r, s$.*

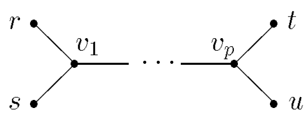
Proof. Adding a suitable multiple of the minimal positive imaginary root δ we may assume that $\gamma_r = 0$.

Suppose that Q' is of type $\tilde{A}_n$. Thus Q' has shape

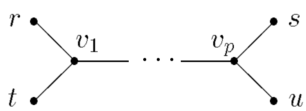


with $p, q \geq 1$. Now if $\gamma_{u_1} = x$ and $\gamma_{v_1} = y$ then the hypotheses imply that $x + y = -1$ and $\gamma_{u_i} = ix$ and $\gamma_{v_i} = iy$ for all i. Thus $px = qy$, so x and y have the same sign. But then the equality $x + y = -1$ is impossible for x, y integers.

Next suppose that Q' is of type $\tilde{D}_n$, in which case there are two possibilities for the location of r and s. The first possibility is



Now the hypotheses imply that $\gamma_{v_1} = -1$, but also $\gamma_{v_p} = 2\gamma_t = 2\gamma_u$, and then $\gamma_{v_p} = \gamma_{v_{p-1}} = \cdots = \gamma_{v_1}$, so γ_{v_1} is even, a contradiction. The second possibility is



in which case $\gamma_{v_1} = -1$, but also $\gamma_{v_1} = 2\gamma_t$, a contradiction.

Finally, suppose that Q' is of type $\tilde{E}_n$. For type $\tilde{E}_6$, the components of γ on the arm containing r are successively $0, -1, -2$ (so -2 at the central vertex), but considering the arm not containing r or s, if the component of γ at the tip is x, then the components on the arm are $x, 2x, 3x$. Thus we need $3x = -2$, which is impossible. For type $\tilde{E}_7$, the components of γ on the arm containing r are successively $0, -1, -2, -3$, but considering the shortest arm, if the component of γ at the tip is x, then the component at the centre is $2x$. Thus we need $2x = -3$, which is impossible. Note that $\tilde{E}_8$ doesn't occur since it has only one extending vertex.

LEMMA 8.9. *Suppose that β is a divisor for (λ, α), that j is the critical vertex for β, and that β and $\alpha - \beta$ are both sincere. If Q' is an extended*

Dynkin subquiver of Q, then j is contained in Q', and it is not an extending vertex for Q'.

Proof. If j is not in Q' then by Lemma 8.7 we have $s_i = 0$ for all i. Since β is sincere this implies that any arrow with one vertex in Q' is contained in Q'. Since Q is connected we must have $Q = Q'$, but then j is in Q', a contradiction. Thus j is in Q'.

Now suppose that j is an extending vertex for Q', that is, $\delta_j = 1$, where δ is the minimal positive imaginary root for Q'. Thus by Lemma 8.7 there is a unique arrow a in $Q \setminus Q'$ with one end in Q', say at vertex ℓ. The other end cannot be in Q', say it is at vertex k. Then also $\beta_k = 1$ and $\delta_\ell = 1$. Now $\ell = j$, for otherwise by considering the restriction of β to Q' we obtain a contradiction by Lemma 8.8.

Now $0 = (\beta, \epsilon_k) \le 2\beta_k - \beta_j - t = 2 - \beta_j - t$, where t is the sum of all terms β_i with i a vertex not in Q' connected by an arrow to k. Thus β_j is 1 or 2.

If $\beta_j = 2$ then $t = 0$, so there are no arrows, apart from a incident at k. Thus there is a linear quiver of length 1 attached to j, contrary to Lemma 8.6.

On the other hand, if $\beta_j = 1$ then $t = 1$, so k must be connected to a unique vertex u_1 not in Q', and $\beta_{u_1} = 1$. Now the condition $(\beta, \epsilon_{u_1}) = 0$ implies that u_1 must be connected to a unique vertex $u_2 \neq k$ and $\beta_{u_2} = 1$. Repeating in this way gives an infinite collection of distinct vertices $k, u_1, u_2, \ldots$. This is impossible.

Thus j cannot be an extending vertex for Q'.

LEMMA 8.10. *If β is a divisor for (λ, α) and β and $\alpha - \beta$ are both sincere then Q is a star with three arms.*

Proof. Since every vertex of the extended Dynkin quiver of type $\tilde{A}_n$ is an extending vertex, by Lemma 8.9 the quiver Q must be a tree.

Suppose that Q' is a subquiver of Q which is extended Dynkin of type $\tilde{D}_n$, and let δ be the minimal positive imaginary root for Q'. By Lemma 8.9, j must be contained in Q' and it is not an extending vertex. Thus j is on the trunk of Q', and $\delta_j = 2$. By Lemma 8.6, there must be arrows in Q connecting to vertices on both sides of j, so by Lemma 8.7 there are two such arrows, they attach to extending vertices $k, \ell \in Q'$, and we have $s_k = s_\ell = 1$. Let m be the vertex in Q' connected to k, and let p be the other extending vertex in Q' connected to m (or in case Q' is of type $\tilde{D}_4$, let p be one of the other extending vertices with $p \neq \ell$). Since $(\beta, \epsilon_p) = 0$, we have $\beta_m = 2\beta_p$, so β_m is even. On the other hand, since $(\beta, \epsilon_k) = 0$ we have $\beta_m + s_k = 2\beta_k$, so β_m is odd, a contradiction.

Thus Q contains no subquiver of type $\tilde{D}_n$, and so it is a star with three arms.

LEMMA 8.11. *If β is a divisor for (λ, α) then β and $\alpha - \beta$ cannot both be sincere.*

Proof. Supposing that β and $\alpha - \beta$ are both sincere, we derive a contradiction. By Lemma 8.10, the quiver Q is a star with three arms. Moreover, j must be at the tip of one of the arms by Lemma 8.6. Note that Q is not Dynkin or extended Dynkin since $(\beta, \beta) = \sum_i \beta_i(\beta, \epsilon_i) = -\beta_j < 0$. On deleting the vertex j, however, the quiver must be Dynkin by Lemma 8.9.

We say that Q has type (p, q, r) if the arm containing j involves p arrows and the other two arms involve q and r arrows respectively. Let k and ℓ be the vertices at the tips of the second and third arms.

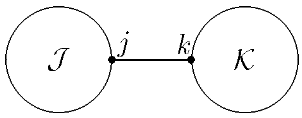
If Q has type $(1, q, r)$, let ξ be the vector which is 0 at j and 1 at every other vertex. This gives a contradiction by Lemma 8.5.

If Q has type $(2, 1, r)$ for some r, let ξ be the vector which is 0 at j, 1 at k and the vertex adjacent to j, and 2 at all other vertices. This gives a contradiction by Lemma 8.5.

If Q has type $(2, q, r)$ with $q, r \geq 2$, then j is an extending vertex for a subquiver of type $\tilde{E}_6$. This is impossible by Lemma 8.9.

Finally suppose that Q has type (p, q, r) with $p \geq 3$. Now Q must contain an extended Dynkin subquiver Q'. By Lemma 8.9, Q' must contain j, but the condition $p \geq 3$ forces j to be an extending vertex for Q'. This is impossible.

LEMMA 8.12. *Suppose β is a divisor for (λ, α). Assume α is sincere but β is not. Then $\beta_j = 0$, and decomposing I as a disjoint union $\mathcal{J} \cup \mathcal{K}$ where $\mathcal{K}$ is the support of β and $\mathcal{J}$ is the set of vertices where β vanishes, there is a unique arrow connecting $\mathcal{J}$ to $\mathcal{K}$. It connects j to some vertex $k \in \mathcal{K}$ with $\beta_k = 1$.*



In addition there is a vertex $\ell \in \mathcal{K}$ (possibly equal to k) with the property that $\beta_\ell = 1$, $(\alpha - \beta, \epsilon_\ell) = -1$, and $(\alpha - \beta, \epsilon_i) = 0$ for all $i \in \mathcal{K}$ with $i \neq \ell$.

Proof. Since Q is connected, at least one arrow a connects $\mathcal{J}$ to $\mathcal{K}$. If its vertex in $\mathcal{J}$ is i, then clearly $(\beta, \epsilon_i) < 0$, so $i = j$. Now $(\beta, \epsilon_j) = -1$, so there can be no other arrows between j and $\mathcal{K}$, and if k is the end of a in $\mathcal{K}$, then $\beta_k = 1$.

Observe that $(\alpha - \beta, \epsilon_i) \leq 0$ for all $i \neq j$, since this is part of the definition of a divisor if $(\beta, \epsilon_i) = 0$, while if $(\beta, \epsilon_i) \neq 0$ then we must have $(\alpha - \beta)_i = 0$, and the assertion is clear. Now since $\beta_j = 0$ we have

$$-1 = (\beta, \alpha - \beta) = \sum_{i \neq j} \beta_i (\alpha - \beta, \epsilon_i)$$

and in this sum all terms are ≤ 0. Thus exactly one term is -1, say corresponding to the vertex $i = \ell$, and all other terms are zero. The result follows.

LEMMA 8.13. *If Q' is a Dynkin quiver it is not possible to find vertices r and s (possibly equal) and vectors β and γ with integer components, satisfying*

$$\beta_r = 1, \quad (\beta, \epsilon_s) = 1, \quad (\beta, \epsilon_i) = 0 \textit{ for } i \neq s$$
$$\gamma_s = 1, \quad (\gamma, \epsilon_r) = 1, \quad (\gamma, \epsilon_i) = 0 \textit{ for } i \neq r.$$

Proof. First observe that $r \neq s$, for otherwise $(\beta, \beta) = 1$, which is impossible since $(\beta, \beta) = 2q(\beta)$ is even.

Suppose there are vertices and arrows

$$\overset{v_1}{\bullet} \!-\!\!-\!\! \overset{v_2}{\bullet} \!-\!\!-\!\!-\!\! \cdots \!-\!\!-\!\!-\!\! \overset{v_n = r}{\bullet}$$

with $n \geq 2$, no other arrows attached to the v_i $(i < n)$, and all $v_i \neq s$. Then the conditions $(\beta, \epsilon_{v_i}) = 0$ imply by induction that $\beta_{v_i} = i\beta_{v_1}$. This is impossible since $\beta_r = 1$. Similarly the configuration with r and s interchanged cannot occur. Thus one of three cases occurs. We eliminate each one in turn.

(1) Q' is of type A_n and r and s are the opposite tips. Starting at the vertex r, the components of β must be $1, 2, 3, \ldots, n$. But then $(\beta, \epsilon_s) = 2n - (n-1) = n + 1 \neq 1$, a contradiction.

(2) Q' is a star with three arms, and r and s occur on the same arms, with one of them at the tip. Without loss of generality, assume that s is at the tip. Letting $x = \beta_s$, working inwards from the vertex s the components of β must be $x, 2x - 1, 3x - 2, \ldots$. Now if there are p arrows between r and s we have $1 = \beta_r = (p+1)x - p$, so $x = 1$. Thus the component of β at the centre of the star is also 1, but considering either of the other arms, this is impossible.

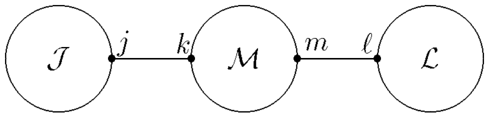
(3) Q' is a star with three arms, and r and s occur as tips of different arms. Let the arm containing r contain p arrows, and let the arm containing s contain q arrows. Now, starting from r, the components of β on the arm containing r are $1, 2, \ldots, p+1$. If $x = \beta_s$ then the components of β on the arm containing s are $x, 2x-1, 3x-2, \ldots, (q+1)x - q$. Thus $p + 1 = (q+1)x - q$. Solving for x this implies that $x \geq 2$ (since it is an integer), and then $p + 1 = q(x-1) + x > q + 1$. Thus $p > q$. A similar argument with r and s interchanged gives $q > p$. Contradiction.

LEMMA 8.14. *If β is non-sincere divisor for (λ, α), α is sincere and $\alpha - \beta$ is not sincere, then (λ, α) is of type* (II).

Proof. If β and $\alpha - \beta$ have disjoint support then it is easy to deduce from Lemma 8.12 that (λ, α) is of type (II). Thus, supposing that the supports of β and $\alpha - \beta$ intersect, we need to derive a contradiction.

We decompose $\mathcal{K}$ as the disjoint union of $\mathcal{L}$, the set of vertices in $\mathcal{K}$ at which $\alpha - \beta$ vanishes, and $\mathcal{M}$, the intersection of $\mathcal{K}$ with the support of $\alpha - \beta$.

Since Q is connected there is at least one arrow b connecting $\mathcal{L}$ to $\mathcal{M}$. If its vertex in $\mathcal{L}$ is i, then clearly $(\alpha - \beta, \epsilon_i) < 0$, so i is the vertex ℓ appearing in Lemma 8.12. Now $(\alpha - \beta, \epsilon_i) = -1$ so there can be no other arrows between ℓ and $\mathcal{M}$, and if m is the end of b in $\mathcal{M}$ then $(\alpha - \beta)_m = 1$. Thus Q decomposes as follows (except that possibly $k = m$).



Let Q' be the restriction of Q to $\mathcal{M}$. Now any subquiver of Q' must be connected by an arrow to a vertex at which β is nonzero, so Q' cannot contain an extended Dynkin subquiver by Lemma 8.7. Thus Q' is Dynkin, and considering the restrictions of β and $\alpha - \beta$ to Q', one gets a contradiction by Lemma 8.13.

LEMMA 8.15. *If β is non-sincere divisor for (λ, α) and $\alpha - \beta$ is sincere, then (λ, α) is of type* (III).

Proof. Let Q' be the restriction of Q to $\mathcal{K}$. Since $\alpha - \beta$ is sincere, we have $(\beta, \epsilon_i) = 0$ for all $i \neq j$, and hence $(\beta, \epsilon_i)_{Q'} = 0$ for all $i \in Q'$. Now Q is connected, and hence so is Q', and then since β has support Q', it follows from [10, Lemma 1, p123] that Q' is extended Dynkin and β is a multiple of the minimal positive imaginary root δ for Q'. Now $\beta_k = \beta_\ell = 1$, so $\beta = \delta$ and k and ℓ are extending vertices for Q'. By Lemma 8.8 we have $k = \ell$. Let γ be the restriction of $\alpha - \beta$ to Q'. Then $(\gamma, \epsilon_i)_{Q'} = 0$ for all $i \in Q'$, so γ is a multiple of δ. The result follows.

Proof. (of Theorem 8.1) Suppose that $\alpha \in F_\lambda \setminus \Sigma_\lambda$. Since α is an imaginary root, $q(\alpha) \leq 0$. Suppose first that $q(\alpha) = 0$. By passing to an equivalent pair, we may assume by Lemma 7.1 that α is in the fundamental region. Since $q(\alpha) = 0$, this implies by [10, Lemma 1, p123] that the support of α is extended Dynkin and α is a multiple of the minimal imaginary root δ, say $\alpha = m\delta$. If $\lambda \cdot \delta = 0$ then clearly $m \geq 2$, for otherwise $\alpha \in \Sigma_\lambda$. On the other hand, if $\lambda \cdot \delta \neq 0$ then since $\lambda \cdot \alpha = 0$ the field K must have characteristic $p > 0$ and m is a multiple of p, say $m = m'p$. Now $m' \geq 2$ for otherwise $\alpha \in \Sigma_\lambda$. Thus we are in the situation of case (I).

Thus suppose that $q(\alpha) < 0$. We replace (λ, α) by an equivalent pair to ensure that α has support as small as possible. Then we pass to the support quiver Q' of α and the restrictions (λ', α') of λ and α. Clearly $\alpha' \in F_{\lambda'} \setminus \Sigma_{\lambda'}$. Observe that if we replace (λ', α') by any equivalent pair (λ'', α''), then α'' is sincere (that is, has support Q'), and (λ'', α'') can equally well be obtained from (λ, α) by applying the reflections first, and then passing to the support quiver.

Now by Lemma 8.3 there is a (-1)-vector β, for (λ', α'), and hence a divisor β' for some equivalent pair (λ'', α'') by Lemma 8.4. Now β' and $\alpha'' - \beta'$ cannot both be sincere by Lemma 8.11. Thus either β' is a non-sincere divisor, or we obtain a non-sincere divisor for some pair equivalent to (λ'', α'') on applying Lemma 8.4 to the (-1)-vector $\alpha'' - \beta'$ for (λ'', α''). Thus case (II) or (III) holds by Lemmas 8.14 and 8.15.

9. Nonexistence of certain simple representations

In this section we prove the following result. This is used in the next section to complete the proof of Theorem 1.2.

THEOREM 9.1. *Let Q' be an extended Dynkin quiver, let k be an extending vertex for Q', and let Q be the quiver obtained from Q' by adjoining one vertex j and one arrow $b : j \to k$. Let I be the vertex set of Q, let $\delta \in K^I$ be the minimal positive imaginary root for Q', and let $\alpha = \epsilon_j + m\delta$, where $m \geq 2$. If $\lambda \in K^I$ satisfies $\lambda_j = 0$ and $\lambda \cdot \delta = 0$, then there is no simple representation of Π^λ of dimension vector α.*

Throughout this section we assume that $Q', Q, I, j, k, b, \delta, \alpha, m$ and λ are as in the theorem.

LEMMA 9.2. *If $\alpha = \beta^{(1)} + \cdots + \beta^{(r)}$ with $\beta^{(t)} \in \mathbb{N}^I \setminus \{0\}$ for each t, then $\sum_{t=1}^r p(\beta^{(t)}) \leq p(\alpha)$, with equality exactly when all but one of the $\beta^{(t)}$ are equal to δ.*

Proof. Reordering, we may suppose that $\beta_j^{(1)} = 1$ and $\beta_j^{(t)} = 0$ for $t \neq 1$. Letting $\gamma = \beta^{(1)} - \epsilon_j$, we have

$$\sum_t p(\beta^{(t)}) = \gamma_k - q(\gamma) + \sum_{t \neq 1} p(\beta^{(t)})$$

Using the fact that the restriction of q to Q' is positive semidefinite with radical $\mathbb{Z}\delta$, one can easily see that $p(\beta^{(t)}) \leq \beta_k^{(t)}$ for $t \neq 1$, with equality only possible if $\beta_k^{(t)} = 0$ or $\beta^{(t)} = \delta$. Thus

$$\sum_t p(\beta^{(t)}) \leq \alpha_k - q(\gamma) = m - q(\gamma) \leq m = p(\alpha).$$

Now to have equality we must have $q(\gamma) = 0$ and each $\beta^{(t)}$ $(t \neq 1)$ either equal to δ, or vanishing at k. But the condition $q(\gamma) = 0$ implies that γ is a multiple of δ, and hence $\sum_{t\neq 1} \beta^{(t)}$ is also a multiple of δ. This is impossible unless each of the terms is equal to δ.

Let $\sigma : \mathrm{Rep}(Q, \alpha) \to \mathrm{Rep}(Q', m\delta)$ be the projection. If U is a $\mathrm{G}(m\delta)$-stable subset of $\mathrm{Rep}(Q', m\delta)$, then clearly $\sigma^{-1}(U)$ is a $\mathrm{G}(\alpha)$-stable subset of $\mathrm{Rep}(Q, \alpha)$.

LEMMA 9.3. *If U is a non-empty open subset of* $\mathrm{Rep}(Q', m\delta)$ *which is* $\mathrm{G}(m\delta)$*-stable, then* $\dim_{\mathrm{G}(\alpha)}(\mathrm{Rep}(Q, \alpha) \setminus \sigma^{-1}(U)) < p(\alpha)$.

Proof. For a dimension vector γ, we write $B(\gamma) \subseteq \mathrm{Rep}(Q, \gamma)$ for the set of bricks, and $I(\gamma)$ for the set of indecomposable representations. We claim that for $s \geq 0$ the vector $\gamma = \epsilon_j + s\delta$ is a Schur root, and

$$\dim_{\mathrm{G}(\gamma)}(I(\gamma) \setminus B(\gamma)) < p(\gamma).$$

If $s = 0$ this is trivial. If $s \geq 2$ then γ is in the fundamental region, and the assertion follows from Kac [8, §1.10, Lemma 1]. Finally, if $s = 1$ then γ is obtained from the dimension vector δ by a reflection functor, see for example [8, §1.7], and the assertion follows from the fact (which we also need later) that δ is a Schur root, and

$$\dim_{\mathrm{G}(\delta)}(I(\delta) \setminus B(\delta)) < p(\delta).$$

Indeed, $I(\delta) \setminus B(\delta)$ contains only finitely many orbits.

Now we decompose $\mathrm{Rep}(Q, \alpha)$ into sets $I(\beta^{(1)}, \ldots, \beta^{(r)})$ as in Section 4. We need to prove that

$$\dim_{\mathrm{G}(\alpha)}(I(\beta^{(1)}, \ldots, \beta^{(r)}) \setminus \sigma^{-1}(U)) < p(\alpha).$$

By Lemmas 9.2 and 4.3 we only need to consider the sets $I(\epsilon_j + s\delta, \delta, \ldots, \delta)$ for $0 \leq s \leq m$ (where there are $m - s$ copies of δ). Now by the claim above and the argument of Lemma 4.3 it suffices to prove that

$$\dim_{\mathrm{G}(\alpha)}(B_s \setminus \sigma^{-1}(U)) < p(\alpha),$$

where $B(\beta^{(1)}, \ldots, \beta^{(r)})$ denotes the subset of $I(\epsilon_j + s\delta, \delta, \ldots, \delta)$ in which the indecomposable summands are bricks.

Let $R'_s = \mathrm{Rep}(Q, \epsilon_j + s\delta) \times \mathrm{Rep}(Q, \delta) \times \cdots \times \mathrm{Rep}(Q, \delta)$, considered as a subset of $\mathrm{Rep}(Q, \alpha)$ using block-diagonal matrices. Let B'_s be the open subset of R'_s consisting of the elements in which each representation is a brick. Let H the subgroup of $\mathrm{G}(\alpha)$ corresponding to the product $\mathrm{G}(\epsilon_j + s\delta) \times \mathrm{G}(\delta) \times \cdots \times \mathrm{G}(\delta)$. By Lemma 4.1 we need to prove that $\dim_H(B'_s \setminus \sigma^{-1}(U)) < p(\alpha)$. Since H acts freely on B'_s this reduces to a question of dimension, and since B'_s is irreducible of dimension

$$\begin{aligned}\dim \mathrm{Rep}(Q, \epsilon_j + s\delta) + (m-s) \dim \mathrm{Rep}(Q, \delta) \\ = \dim H + p(\epsilon_j + s\delta) + (m-s)p(\delta) = \dim H + p(\alpha),\end{aligned}$$

and $\sigma^{-1}(U)$ is an open subset, it suffices to prove that B'_s meets $\sigma^{-1}(U)$. In other words we need that $\sigma(B'_s)$ meets U.

Now the canonical decomposition for dimension vector $m\delta$ is of the form $\delta + \cdots + \delta$, so U contains a representation which is a direct sum of bricks of dimension δ, and then since U is $\mathrm{G}(m\delta)$-stable, it meets $P = \mathrm{Rep}(Q', s\delta) \times \mathrm{Rep}(Q', \delta) \times \cdots \times \mathrm{Rep}(Q', \delta)$. Also the map $B'_s \to P$ consists of an open inclusion followed by the projection, so the image $\sigma(B'_s)$ is open in P. Since P is irreducible, the two non-empty open subsets $\sigma(B'_s)$ and $U \cap P$ must intersect. Thus $\sigma(B'_s)$ meets U, as required.

Let $\pi : \mu_\alpha^{-1}(\lambda) \to \mathrm{Rep}(Q, \alpha)$ be the projection.

LEMMA 9.4. *Under the map $\sigma\pi$, any irreducible component of $\mu_\alpha^{-1}(\lambda)$ dominates* $\mathrm{Rep}(Q', m\delta)$.

Proof. Let V be an irreducible component of $\mu_\alpha^{-1}(\lambda)$. Clearly V is $\mathrm{G}(\alpha)$-stable, so $\sigma\pi(V)$ is $\mathrm{G}(m\delta)$-stable. Let U be the complement of the closure of $\sigma\pi(V)$, and for a contradiction suppose that U is non-empty. By Lemma 9.3 we have $\dim_{\mathrm{G}(\alpha)}(\mathrm{Rep}(Q, \alpha) \setminus \sigma^{-1}(U)) < p(\alpha)$. Now $V \subseteq \pi^{-1}(\mathrm{Rep}(Q, \alpha) \setminus \sigma^{-1}(U))$, so by Lemma 3.4 we have

$$\dim V < p(\alpha) + \alpha \cdot \alpha - q(\alpha) = \dim \mathrm{Rep}(\overline{Q}, \alpha) - \dim \mathrm{End}(\alpha)_0.$$

This is impossible since $\mu_\alpha^{-1}(\lambda)$ is a fibre of the moment map, so every irreducible component has dimension at least $\dim \mathrm{Rep}(\overline{Q}, \alpha) - \dim \mathrm{End}(\alpha)_0$.

Recall that a ring epimorphism $A \to B$ is said to be *pseudoflat* if $\mathrm{Tor}_1^A(B, B) = 0$. This is relevant because of [3, Theorem 0.7].

LEMMA 9.5. *If $N = \sum_i \delta_i$ then there is a pseudoflat epimorphism $\theta : KQ' \to \mathrm{Mat}(N, K[x])$ such that the general representation of Q' of dimension $m\delta$ is the restriction of a* $\mathrm{Mat}(N, K[x])$*-module,*

Proof. This is standard. See [3, Lemma 11.1].

LEMMA 9.6. *If $A \to B$ is a pseudoflat epimorphism of K-algebras, and M is a left A-module, then the map*

$$\begin{pmatrix} A & M \\ 0 & K \end{pmatrix} \longrightarrow \begin{pmatrix} B & B \otimes_A M \\ 0 & K \end{pmatrix}$$

is a pseudoflat epimorphism.

Proof. By [1, Proposition 5.2] it suffices to observe that the diagram

$$\begin{array}{ccc} \begin{pmatrix} A & 0 \\ 0 & K \end{pmatrix} & \longrightarrow & \begin{pmatrix} B & 0 \\ 0 & K \end{pmatrix} \\ \downarrow & & \downarrow \\ \begin{pmatrix} A & M \\ 0 & K \end{pmatrix} & \longrightarrow & \begin{pmatrix} B & B \otimes_A M \\ 0 & K \end{pmatrix} \end{array}$$

is a pushout in the category of rings.

LEMMA 9.7. *Suppose that f and g are endomorphisms of a vector space V of dimension $m \geq 2$. If the commutator $[f, g]$ has rank at most one, then V has a non-trivial proper subspace invariant under f and g.*

Proof. Replacing f by $f - \xi 1$ for some eigenvalue ξ of f, we may suppose that f is singular. Also we may suppose that $f \neq 0$, for otherwise one can take an invariant subspace for g. Let $v_1, \ldots, v_r$ be a basis of $\mathrm{Im}(f)$, and extend it to a basis $v_1, \ldots, v_m$ of V. Let $w_1, \ldots, w_s$ be a basis of $\mathrm{Ker}(f)$, and extend it to a basis $w_1, \ldots, w_m$ of V. With respect to these bases, we compute the matrices of f and g. With the rows and columns indexed by the v_i, let g take the block form

$$\begin{pmatrix} X & Y \\ Z & W \end{pmatrix},$$

and with the rows and columns indexed by the w_i, let g take the block form

$$\begin{pmatrix} P & Q \\ R & S \end{pmatrix}.$$

Now with the rows indexed by by the v_i and the columns indexed by the w_i, the map f takes block form

$$\begin{pmatrix} 0 & C \\ 0 & 0 \end{pmatrix}$$

with C invertible, and then $[f, g]$ takes the form

$$\begin{pmatrix} CR & CS - XC \\ 0 & -ZC \end{pmatrix}.$$

Now the rank one hypothesis implies that $CR = 0$ or $ZC = 0$, so that $R = 0$ or $Z = 0$. In the first case $\mathrm{Ker}(f)$ is an invariant subspace; in the second case $\mathrm{Im}(f)$ is invariant.

Proof. (of Theorem 9.1) Choose a pseudoflat epimorphism θ as in Lemma 9.5. By Lemma 9.6 it induces a pseudoflat epimorphism

$$\phi : KQ = \begin{pmatrix} KQ' & KQ'e_k \\ 0 & K \end{pmatrix} \longrightarrow \begin{pmatrix} \mathrm{Mat}(N, K[x]) & \mathrm{Mat}(N, K[x])\theta(e_k) \\ 0 & K \end{pmatrix}.$$

Denote the right hand algebra by R. Now the fact that $\delta_k = 1$ implies that $\mathrm{Mat}(N, K[x])\theta(e_k)$ is an indecomposable projective $\mathrm{Mat}(N, K[x])$-module. Thus R is Morita equivalent to

$$\begin{pmatrix} K[x] & K[x] \\ 0 & K \end{pmatrix} \cong KQ'',$$

where Q'' is the quiver with two vertices j, k and arrows $b : j \to k$ and $a : k \to k$.

Identify $\lambda \in K^I$ with the corresponding element of $K \otimes_{\mathbb{Z}} K_0(KQ)$, and then identify Π^λ with the algebra $\Pi^\lambda(KQ)$ as in [3, Theorem 0.2]. Now ϕ induces a map $\phi_\lambda : \Pi^\lambda \to \Pi^{\phi_*(\lambda)}(R)$, and by [3, Theorem 0.7] the diagram

$$\begin{array}{ccc} KQ & \xrightarrow{\phi} & R \\ \downarrow & & \downarrow \\ \Pi^\lambda & \xrightarrow{\phi_\lambda} & \Pi^{\phi_*(\lambda)}(R) \end{array}$$

is a pushout in the category of rings.

Now suppose that there is a simple representation of Π^λ of dimension vector α. Since the simple representations form an open subset of $\mu_\alpha^{-1}(\lambda)$, it follows by Lemma 9.4 that the set of simple representations dominates $\mathrm{Rep}(Q', m\delta)$. Thus there is a simple representation S whose restriction to Q' is the restriction by θ of a $\mathrm{Mat}(N, K[x])$-module. Thus the restriction of S to Q is the restriction by ϕ of an R-module. Now since the diagram above is a pushout, it follows that S is naturally a $\Pi^{\phi_*(\lambda)}(R)$-module, and clearly it must be simple. We show that this is impossible.

By [3, Corollary 5.5] the ring $\Pi^{\phi_*(\lambda)}(R)$ is Morita equivalent to $\Pi^\mu(Q'')$, where $\mu_j = \mu_k = 0$ by [3, Lemma 11.2]. Moreover S corresponds to a simple representation T of $\Pi^\mu(Q'')$ of dimension vector γ with $\gamma_j = 1$ and $\gamma_k = m$. Now the arrows a and a^* are endomorphisms of the vector space T_k with commutator equal to b^*b. Since $\dim T_j = 1$ it follows that this commutator has rank at most one, so by Lemma

9.7, T_k has a non-trivial proper subspace invariant under a and a^*. Now this subspace and its image under b^* are a non-trivial proper subrepresentation of T. This is a contradiction.

10. Dimension vectors of simple representations

Let Q be a quiver with vertex set I. In this section we complete the proof of Theorem 1.2. All that remains is to prove the implication (1) $\Longrightarrow$ (2), that is, if α is the dimension vector of a simple representation of Π^λ then $\alpha \in \Sigma_\lambda$. Thus suppose there is a simple representation of Π^λ of dimension α, and for a contradiction assume that $\alpha \notin \Sigma_\lambda$. Observe that there cannot be an equivalent pair (λ', α') with α' a coordinate vector, for then clearly $\alpha' \in \Sigma_{\lambda'}$, a contradiction by Lemma 5.2. Thus by Lemma 7.4 we have $\alpha \in F_\lambda$, and so Theorem 8.1 applies. Thus we may assume that we are in a situation as in (I), (II) or (III), and to obtain a contradiction it suffices to show that in each case there is no simple representation.

Case (I). By [3] there is a Conze embedding

$$\Pi^\lambda \to \mathrm{Mat}(N, K\langle x, y \mid xy - yx = \lambda \cdot \delta\rangle)$$

where $N = \sum_i \delta_i$. If $\lambda \cdot \delta = 0$ this embedding shows that Π^λ satisfies the identities of $N \times N$ matrices, so any simple representation has dimension at most N. Thus Π^λ cannot have a simple representation of dimension vector $m\delta$ with $m \geq 2$. If $\lambda \cdot \delta \neq 0$ and K has characteristic $p > 0$ then $K\langle x, y \mid xy - yx = \lambda \cdot \delta\rangle$ embeds in $\mathrm{Mat}(p, K[x^p, y])$. Thus Π^λ satisfies the identities of $pN \times pN$ matrices, so any simple representation has dimension at most pN. Thus Π^λ cannot have a simple representation of dimension vector $m'p\delta$ with $m' \geq 2$.

Case (II). Since up to isomorphism Π^λ does not depend on the orientation of Q, we may assume that the arrow connecting j and k is $b : j \to k$. Suppose that V is a representation of Π^λ of dimension α. Let V_i be the vector space corresponding to vertex i and let V_a be the linear map corresponding to an arrow a. Now at any vertex i we have

$$\sum_{h(a)=i} V_a V_{a^*} - \sum_{t(a)=i} V_{a^*} V_a = \lambda_i 1_{V_i}.$$

Taking traces and summing over all vertices $i \in \mathcal{K}$, almost all terms cancel, and one obtains $\mathrm{tr}(V_b V_{b^*}) = 0$. Now since $\alpha_j = \alpha_k = 1$ this implies that $V_b = 0$ or $V_{b^*} = 0$. In the first case $\oplus_{i\in\mathcal{J}} V_i$ is a subrepresentation of V; in the second case $\oplus_{i\in\mathcal{K}} V_i$ is a subrepresentation of V. Thus V is not simple, as required.

Case (III). Suppose that V is a representation of Π^λ of dimension α. As in case (II) we may assume that the arrow connecting j and k is $b : j \to k$, and furthermore $\mathrm{tr}(V_b V_{b^*}) = 0$. Thus $\mathrm{tr}(V_{b^*} V_b) = 0$, and since $\alpha_j = 1$ this implies that $V_{b^*} V_b = 0$.

Let Q'' be the quiver obtained from Q by deleting all vertices in $\mathcal{J}$ except j, and all arrows with head and tail in $\mathcal{J}$. Let α'' be the restriction of α to $\mathcal{K} \cup \{j\}$, and let λ'' be the vector with $\lambda''_j = 0$ and $\lambda''_i = \lambda_i$ for $i \in \mathcal{K}$. In view of the observation above, the restriction V'' of V to $\overline{Q''}$ is a representation of the deformed preprojective algebra $\Pi^{\lambda''}$ for the quiver Q'', of dimension vector α''. Now by Theorem 9.1, the representation V'' cannot be simple, so it has a non-trivial proper subrepresentation W. Now V_j is one-dimensional, so either $W_j = 0$ or $W_j = V_j$. In the first case W can be extended to a subrepresentation of V by defining $W_i = 0$ for all $i \in \mathcal{J} \setminus \{j\}$; in the second case W can be extended to a subrepresentation of V by defining $W_i = V_i$ for all $i \in \mathcal{J} \setminus \{j\}$. Thus V is not simple, as required.

11. Quotient schemes

In this section K is an algebraically closed field of characteristic zero, and Q is a quiver with vertex set I.

Proof. (of Theorem 1.3) By [12, Theorem 2] the quotient scheme $\mathrm{Rep}(\overline{Q}, \alpha) /\!/ \mathrm{G}(\alpha)$ is a disjoint union of locally closed strata according to the representation type of the semisimple representations. Now the quotient $\mu_\alpha^{-1}(\lambda) /\!/ \mathrm{G}(\alpha)$ can be identified with a closed subset of $\mathrm{Rep}(\overline{Q}, \alpha) /\!/ \mathrm{G}(\alpha)$, so the semisimple representations of a given type τ form a locally closed subset $S(\tau)$. Suppose that τ is the type

$$\tau = (k_1, \beta^{(1)}; \ldots ; k_r, \beta^{(r)}).$$

and consider the subset Z of

$$\mu^{-1}_{\beta^{(1)}}(\lambda) \times \cdots \times \mu^{-1}_{\beta^{(r)}}(\lambda)$$

consisting of those tuples $(x_1, \ldots, x_r)$ with the x_t corresponding to pairwise non-isomorphic simple representations. Clearly Z is an open subset and $S(\tau)$ is the image of the map

$$f : Z \to \mu_\alpha^{-1}(\lambda) /\!/ \mathrm{G}(\alpha)$$

sending $(x_1, \ldots, x_r)$ to the direct sum of the x_t with multiplicities. Thus $S(\tau)$ is irreducible. Now the group $H = \mathrm{G}(\beta^{(1)}) \times \cdots \times \mathrm{G}(\beta^{(r)})$ acts freely on Z, and any fibre of f is a finite union of H-orbits. Thus

$$\dim S(\tau) = \dim Z - \dim H = \sum_{t=1}^{r} 2p(\beta^{(t)}),$$

as required.

Proof. (of Corollary 1.4) Since $\mu_\alpha^{-1}(\lambda)$ is reduced and irreducible, so is the quotient $\mu_\alpha^{-1}(\lambda) /\!/ \mathrm{G}(\alpha)$. Now the stratum of simple representations has dimension $2p(\alpha)$, and all other strata have strictly smaller dimension.

Remark 11.1. If (λ, α) is a pair with $\lambda \cdot \alpha = 0$ but $\lambda \cdot \beta \neq 0$ for all $0 < \beta < \alpha$, then clearly $\alpha \in \Sigma_\lambda$ if and only if it is a positive root. If it is a positive root then every element of $\mu_\alpha^{-1}(\lambda)$ must be a simple representation of Π^λ by [5, Lemma 4.1]. Thus $\mu_\alpha^{-1}(\lambda)$ is smooth by Lemma 6.5, and the map

$$\mu_\alpha^{-1}(\lambda) \to \mu_\alpha^{-1}(\lambda) /\!/ \mathrm{G}(\alpha)$$

is a principal étale fibre space for the group $\mathrm{G}(\alpha)$ by Luna's slice theorem [13, §III.1, Corollaire 1]. It follows in this case that $\mu_\alpha^{-1}(\lambda) /\!/ \mathrm{G}(\alpha)$ is smooth. It would be interesting to know about the singularities of $\mu_\alpha^{-1}(\lambda)$ and $\mu_\alpha^{-1}(\lambda) /\!/ \mathrm{G}(\alpha)$ for general λ and α.

Appendix. Application to Kac's Theorem

In this appendix we show how the lifting results of section 3 can be used to give a simple proof of part of Kac's Theorem assuming that the base field K has characteristic zero. Recall [8, Section 1.10] that the proof of Kac's Theorem uses two key lemmas

KAC'S LEMMA 1. *If α is in the fundamental region and $q(\alpha) < 0$ then the set $B(\alpha)$ of bricks (representations with endomorphism algebra equal to K) is a dense open subset of* $\operatorname{Rep}(Q, \alpha)$ *(so* $\dim_{G(\alpha)} B(\alpha) = p(\alpha)$*) and* $\dim_{\mathrm{G}(\alpha)}(I(\alpha) \setminus B(\alpha)) < p(\alpha)$.

KAC'S LEMMA 2. *The number of indecomposable representations of dimension α (if it is finite) and* $\dim_{\mathrm{G}(\alpha)} I(\alpha)$ *are independent of the orientation of Q.*

Kac's proof of Lemma 1 is quite natural and straightforward. On the other hand, his proof of Lemma 2 is roundabout, and involves reducing to finite fields and then using counting arguments. It would be nice to avoid Lemma 2, or find a direct proof of it.

PROPOSITION A.1. *If $\alpha \in \mathbb{Z}^I$ then α is a positive root if and only if for the general element of $\{\lambda \in K^I \mid \lambda \cdot \alpha = 0\}$ there is an indecomposable representation of Π^λ of dimension α.*

Proof. Let $S(\alpha)$ be the statement that for the general element of $\{\lambda \in K^I \mid \lambda \cdot \alpha = 0\}$ there is an indecomposable representation of Π^λ of dimension α.

Since K has characteristic zero, if i is a loopfree vertex and α is not a multiple of ϵ_i, then the general element of the set $\{\lambda \in K^I \mid \lambda \cdot \alpha = 0\}$ has $\lambda_i \neq 0$. For such λ there is a a reflection functor relating representations of Π^λ of dimension α and representations of $\Pi^{r_i(\lambda)}$ of dimension $s_i(\alpha)$. It follows that $S(\alpha)$ holds if and only if $S(s_i(\alpha))$ holds.

Note also that if i is a loopfree vertex and α is not a multiple of ϵ_i, then α is a positive root if and only if $s_i(\alpha)$ is a positive root.

Now, by applying a sequence of reflections to reduce α, it suffices to prove the theorem in the following three cases.

(1) α is a multiple of the coordinate vector at a loopfree vertex, say $\alpha = k\epsilon_i$. In this case α is a positive root if and only if $k = 1$, and also clearly $S(\alpha)$ holds if and only if $k = 1$.

(2) α is in the fundamental region. In this case α is a positive root. Also, by Kac's Lemma 1 and the theory of extended Dynkin quivers, there is an indecomposable representation of Q of dimension α, and by Theorem 3.3 this lifts to a representation of Π^λ for any λ with $\lambda \cdot \alpha = 0$. Thus $S(\alpha)$ holds.

(3) α has disconnected support, or a strictly negative component. In this case α is not a positive root, and $S(\alpha)$ is false.

COROLLARY A.2. *If there is an indecomposable representation of Q of dimension α then α is a positive root.*

Proof. By Theorem 3.3 this representation lifts to an indecomposable representation of Π^λ for any λ with $\lambda \cdot \alpha = 0$. Thus $S(\alpha)$ holds.

Remark A.3. Suppose that $K = \mathbb{C}$ and Q has no oriented cycles. In this case Schofield [23] has used Euler characteristics to construct the positive part of the Kac-Moody Lie algebra associated to Q. In the course of his proof he shows that if α is a positive root then there is an indecomposable representation of Q of dimension α. This result and Corollary A.2 give a proof of Kac's characterization which completely avoids finite fields.

If α is a positive real root, then the unique indecomposable representation of Q of dimension α may be constructed as follows. Choose a sequence of reflections

$$\epsilon_i = \alpha^{(0)}, \alpha^{(1)}, \ldots, \alpha^{(m)} = \alpha$$

with i a loopfree vertex, $\alpha^{(t)} = s_{i_t}(\alpha^{(t-1)})$ for $t \geq 1$, and $\alpha^{(t)}$ not a coordinate vector for $t \geq 1$. Let $\lambda^{(0)} \in K^I$ be the vector with $\lambda_i^{(0)} = 0$ and $\lambda_j^{(0)} = 1$ for all $j \neq i$, and define $\lambda^{(t)} = r_{i_t}(\lambda^{(t-1)})$ for $t \geq 1$.

PROPOSITION A.4. *With the hypotheses above, the reflection at i_t is admissible for $(\lambda^{(t)}, \alpha^{(t)})$ for all t. Moreover, there is a unique indecomposable representation of Q of dimension α, and it may be obtained from the trivial representation of $\Pi^{\lambda^{(0)}}$ of dimension ϵ_i by applying successively the reflection functors at the vertices i_t, and then restricting the resulting representation of $\Pi^{\lambda^{(m)}}$ to Q.*

Proof. Since K has characteristic zero, $\lambda^{(0)} \cdot \beta \neq 0$ for any root β which is not equal to $\pm\epsilon_i$ (for some component β_j with $j \neq i$ must be nonzero, and all components have the same sign). It follows that $\lambda^{(t)} \cdot \beta \neq 0$ for any root β which is not equal to $\pm\alpha^{(t)}$. In particular $\lambda^{(t)} \cdot \epsilon_{i_t} \neq 0$ for $t \geq 1$. Thus the reflections are admissible.

Now the reflection functors give an equivalence between representations of $\Pi^{\lambda^{(0)}}$ of dimension ϵ_i, of which there is only one, and representations of $\Pi^{\lambda^{(m)}}$ of dimension α. Thus there is a unique representations of $\Pi^{\lambda^{(m)}}$ of dimension α, up to isomorphism.

Now the restriction of this representation to Q is indecomposable, for if it had an indecomposable direct summand of dimension β, then by Theorem 3.3 one has $\lambda^{(m)} \cdot \beta = 0$. But this is impossible since β is a root, not equal to $\pm\alpha$.

Finally, for uniqueness, observe that any indecomposable representation of Q of dimension α lifts to a representation of $\Pi^{\lambda^{(m)}}$ since $\lambda^{(m)} \cdot \alpha = 0$. Since there is only one representation of $\Pi^{\lambda^{(m)}}$, it follows that there is only one indecomposable representation of Q.

Finally we turn to Kac's Lemma 2. We have an elementary proof of it for *indivisible* dimension vectors, that is, vectors whose components have no common divisor.

PROPOSITION A.5. *If $\alpha \in \mathbb{N}^I$ is indivisible then the number of isomorphism classes of indecomposable representations of dimension α (if finite), and the number of parameters* $\dim_{\mathrm{G}(\alpha)} I(\alpha)$*, are independent of the orientation of Q.*

Proof. Since α is indivisible, the general element of $\{\lambda \in K^I \mid \lambda \cdot \alpha = 0\}$ has $\lambda \cdot \beta \neq 0$ for all $\beta \in \mathbb{N}^I$ with $0 < \beta < \alpha$. Choose λ with this property. Clearly a representation $x \in \mathrm{Rep}(Q, \alpha)$ lifts to a representation of Π^λ if and only if it is indecomposable. Thus by Lemma 3.4 we have

$$\dim \mu_\alpha^{-1}(\lambda) = \dim_{\mathrm{G}(\alpha)} I(\alpha) + \alpha \cdot \alpha - q(\alpha).$$

Moreover, if there are only m isomorphism classes of indecomposables, then $\mu_\alpha^{-1}(\lambda)$ is a disjoint union of m irreducible locally closed subsets of dimension $\alpha \cdot \alpha - q(\alpha)$, so it has m irreducible components (the closures of these subsets). Finally it suffices to note that up to isomorphism the scheme $\mu_\alpha^{-1}(\lambda)$ does not depend on the orientation of Q (see for example [5, Lemma 2.2]).

Remark A.6. Clearly the proposition holds for general α if one instead uses the set $E(\alpha)$ of representations of dimension α with the property that any direct summand has dimension proportional to α. In fact, using these methods it is possible to prove all of Kac's Theorem without using Kac's Lemma 2, except the existence (and number of parameters) of indecomposable representations of dimension α with α a divisible positive root with $q(\alpha) = 0$.

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