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THE OBSERVABLE STRUCTURE OF PERSISTENCE MODULES

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(*communicated by J.F. Jardine*)

Abstract

In persistent topology, q-tame modules appear as a natural and large class of persistence modules indexed over the real line for which a persistence diagram is definable. However, unlike persistence modules indexed over a totally ordered finite set or the natural numbers, such diagrams do not provide a complete invariant of q-tame modules. The purpose of this paper is to show that the category of persistence modules can be adjusted to overcome this issue. We introduce the *observable category* of persistence modules: a localization of the usual category, in which the classical properties of q-tame modules still hold but where the persistence diagram is a complete isomorphism invariant and all q-tame modules admit an interval decomposition.

1. Introduction

1.1. Discrete persistence modules

Topological persistence [**9**, **17**] may be introduced with the observation that a nested sequence of topological spaces

$$X_0 \xrightarrow{\subseteq} X_1 \xrightarrow{\subseteq} \cdots \xrightarrow{\subseteq} X_n$$

gives rise to a sequence of vector spaces and linear maps

$$\mathrm{H}(X_0) \longrightarrow \mathrm{H}(X_1) \longrightarrow \cdots \longrightarrow \mathrm{H}(X_n)$$

upon computing homology with coefficients in a field k. In general, a diagram of vector spaces and linear maps

$$V_0 \longrightarrow V_1 \longrightarrow \cdots \longrightarrow V_n$$

is called a *persistence module* indexed by $\{0, 1, \ldots, n\}$. Any such diagram can be expressed as a direct sum of certain indecomposable diagrams called *interval modules* [**17**], parametrized by intervals $[p, q] \subseteq \{0, 1, \ldots, n\}$. The interval module $V = k_I$

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associated to an interval I is defined by

$$V_i = \begin{cases} k & \text{if } i \in I, \\ 0 & \text{otherwise,} \end{cases}$$

with the maps $k \to k$ set equal to 1 (all other maps being necessarily zero).

The number of direct summands $\mathrm{m}_{p,q}$ of each type $k_{[p,q]}$ is independent of the specific decomposition, by a suitable version of the Krull–Schmidt theorem or by appeal to an explicit invariant formula such as

$$\mathrm{m}_{p,q} = \dim \left[\frac{\mathrm{im}(V_p \to V_q) \cap \ker(V_q \to V_{q+1})}{\mathrm{im}(V_{p-1} \to V_q) \cap \ker(V_q \to V_{q+1})} \right].$$

As a result, the collection of numbers $(\mathrm{m}_{p,q} \mid 0 \leqslant p \leqslant q \leqslant n)$ is a complete invariant of the persistence module, and an invariant of the initial topological data. It is typically expressed as a *barcode* or *persistence diagram* [**9**, **17**].

1.2. Persistence modules for the real line

The purpose of this short paper is to address some issues that arise when attempting to follow the same thought process for persistence modules indexed by the real line. Here are the main points of divergence:

- Not every persistence module is decomposable into interval modules.
- Nonetheless, there are easily described classes of persistence module for which a persistence diagram is definable. We favour the class of q-tame modules [**4**], which are characterized by having finite-rank structure maps (Section 1.4). Despite the existence of the persistence diagram, it turns out that not every q-tame persistence module is decomposable into interval modules.
- The persistence diagram is not a complete invariant. Two non-isomorphic q-tame persistence modules may have the same persistence diagram. This is true even if we use a more refined invariant, the decorated persistence diagram [**4**].

To be fair, there are ways of working around these problems [**4**, **13**]. What we offer here is the suggestion that the awkwardness dissipates completely if we make a small adjustment to the category of persistence modules that we work in.

The adjustment is motivated by the following principle: whereas persistence modules carry information at many different scales simultaneously, what matters most is how the information persists across scales (through the structure maps). Features that exist over a short range are regarded as relatively unimportant. In topological data analysis, such short-term information may arise from noisy sampling, for instance. In the extreme case, we have the *ephemeral* features: non-zero features that are supported at exactly one index value. Standard practice is to regard these as statistically meaningless.

Our proposal is to build this principle—of ignoring ephemeral information—directly into the category of modules. The mechanism for doing so is Serre localization. The resulting *observable category* of persistence modules turns out to be beautifully behaved. In this category, persistence modules $k_{[p,q]}$, $k_{[p,q)}$, $k_{(p,q]}$, $k_{(p,q)}$ associated to different intervals with the same endpoints are isomorphic. Every q-tame module has an interval decomposition. The persistence diagram is a complete isomorphism invariant for q-tame modules. Finally, there is a very clean description of the morphisms in this category (something that is not always available for such constructions).

1.3. Basic definitions

Let $(R, \leqslant)$ be a totally ordered set. The category **Pers** of persistence modules over R (or 'indexed by R') is defined as follows. Here are the objects:

- A *persistence module* V is a functor from R, considered in the natural way as a category, to the category of vector spaces. Thus it consists of vector spaces V_t for $t \in R$ and linear maps $\rho_{ts} \colon V_s \to V_t$ for $s \leqslant t$ called *structure maps*, which satisfy $\rho_{ts} = \rho_{tu}\rho_{us}$ for all $s \leqslant u \leqslant t$ and $\rho_{tt} = 1_{V_t}$ for all t.

Here are the morphisms. We give two equivalent formulations:

- A *morphism* $\phi \colon V \to W$ is a natural transformation between functors. Thus, it is a collection of linear maps $\phi_t \colon V_t \to W_t$ such that $\phi_t \rho_{ts} = \sigma_{ts}\phi_s$ for all $s \leqslant t$. (The maps σ_{ts} are the structure maps for W.)
- A *morphism* $\phi \colon V \to W$ is a collection of linear maps $\phi_{ts} \colon V_s \to W_t$ defined for $s \leqslant t$, such that $\phi_{ts} = \sigma_{tv}\phi_{vu}\rho_{us}$ whenever $s \leqslant u \leqslant v \leqslant t$.

The translation between the two formulations is given by $\phi_{st} = \phi_t \rho_{ts} = \sigma_{ts}\phi_s$ in one direction, and $\phi_t = \phi_{tt}$ in the other. In what follows, we favour the second formulation.

Remark 1.1. A natural generalization is to replace the category of vector spaces with some other category [**1**]. The interval decomposition results are specific to the theory of vector spaces, but the localization results are valid somewhat more generally. Another natural generalization is to allow the indexing set R to be some other poset. For instance, $\mathbb{R}^n$ with its standard partial order is used in the theory of multidimensional persistence [**3**]. For our purposes R will always be totally ordered.

By an *interval* in R we mean a non-empty subset I of R with the property that $s \leqslant u \leqslant t$ with $s, t \in I$ implies $u \in I$. The corresponding *interval module* $V = k_I$ is defined by setting $V_t = k$ for $t \in I$, $V_t = 0$ for $t \notin I$, and $\rho_{ts} = 1$ for $s, t \in I$ with $s \leqslant t$ (all other maps necessarily being zero).

Example 1.2. Let $p, q \in R$ with $p < q$. We define closed, half-open and open intervals

$$[p,q] = \{t \in R \mid p \leqslant t \leqslant q\}, \qquad [p,q) = \{t \in R \mid p \leqslant t < q\},$$
$$(p,q] = \{t \in R \mid p < t \leqslant q\}, \qquad (p,q) = \{t \in R \mid p < t < q\},$$

with endpoints p, q. Not all intervals in R need be of this type (for example, when $R = \mathbb{Q}$ there exist singleton intervals, unbounded intervals, and intervals with one or two irrational endpoints).

Lemma 1.3. *Interval modules are indecomposable: they cannot be expressed as a nontrivial direct sum of submodules.*

Proof. The endomorphism ring of an interval module is isomorphic to k. Indeed, for any endomorphism $\phi = (\phi_{ts})$ the non-trivial terms (those with $s, t \in I$) are scalars and, indeed, must be equal to the same scalar. The projection maps in a direct-sum decomposition would be idempotent endomorphisms, but k has no nontrivial idempotents. □

With this in mind, the natural question is whether every persistence module over a total order R decomposes as a direct sum of interval modules. The answer is yes when R is finite or the natural numbers [**16**]; and also yes in the special case of modules which are finite-dimensional at each index, assuming that R has a countable subset which is dense in R in the order topology [**7**]. But in general there are persistence modules which do not decompose into intervals, such as $\hat{V}$ in Example 4.7, due to Webb [**16**].

1.4. Tame persistence modules

Of particular importance are the *q-tame* persistence modules [**4**], defined by the condition that $\operatorname{rank}(\rho_{ts})$ be finite whenever $s < t$. Here are some standard examples, indexed by the real line:

- Let X be a locally compact polyhedron and let $f\colon X \to \mathbb{R}$ be a proper continuous map which is bounded below. Then $(\mathrm{H}_*(f^{-1}(-\infty, t]))_{t\in\mathbb{R}}$ is q-tame. This includes the case where f is the distance from a compact subset $A \subset \mathbb{R}^n$ in any norm. The result is a slight variant of [**4**, Section 3.9].
- Let X be a totally bounded metric space. Then the Vietoris–Rips and intrinsic Čech filtered complexes on X have q-tame persistent homology [**5**].

Many of these q-tame examples fail to be *pointwise finite-dimensional*: there are index values where $\dim(V_t)$ is infinite. For an extreme case, Droz [**8**] has constructed a compact metric space whose Vietoris–Rips homology is uncountably infinite-dimensional at all values of t in an interval of positive length.

Our main results are summarized in the following theorem, which collates Corollaries 2.15 and 3.8, Theorem 3.9, Example 2.21 and Propositions 2.23, 4.2 and 4.3. We state it here only for $R = \mathbb{R}$, but some parts hold more generally.

Theorem 1.4. *There is a quotient category* **Obs** *of the category of persistence modules over* $\mathbb{R}$*, with the following properties:*

(i) The property of a persistence module being q-tame, the undecorated diagram of a persistence module, and the interleaving distance between two persistence modules depend only on the image of the module or modules in **Obs***.*

(ii) Any q-tame persistence module, on passing to its image in **Obs***, decomposes as a direct sum of interval modules. The list of summands is essentially unique, and is determined by the persistence diagram.*

This theorem 'explains' the goodish behaviour of q-tame persistence modules and their persistence diagrams in the usual framework: it is the pullback of their good behaviour in the observable category.

1.5. Prerequisites

Much of this paper is self-contained. In particular the definition of the observable category **Obs** is straightforward and requires no special technology. However, there are certain ingredients that we need to import from elsewhere.

Serre localization

Familiarity with abelian categories [**10, 12**] is recommended but not strictly necessary to understand most of this paper: we construct **Obs** and establish its status as a quotient category of **Pers** quite directly. That said, it's worth keeping in mind that our construction is an instance of a general procedure known as *Serre localization* [**11**, **15**]. This is a way of forming the quotient of an abelian category **A** by a full subcategory **C** whose objects are to be regarded as 'small'. Subobjects and quotient objects of a small object are required to be small, as are extensions of a small object by a small object. For instance, if $\mathbf{A} = \{\text{abelian groups}\}$ then the full subcategory $\mathbf{C} = \{\text{finite abelian groups}\}$ satisfies this condition. Localization renders invertible every morphism whose kernel and cokernel are small, so in particular the small objects become isomorphic to the zero object. In the present work, **Obs** is the Serre localization of **Pers** with respect to the subcategory **Eph** of *ephemeral* modules (Section 2.1).

Module decomposition
In order to show that q-tame persistence modules are interval-decomposable in **Obs**, we provide in Section 3.1 an interval decomposition theorem in **Pers** valid for persistence modules that satisfy certain conditions. The theorem is an adaptation of the main result in [**7**]. Our presentation is not self-contained; the technical proof in that section is intended to be read in conjunction with the original paper. We use the notation from [**7**] without further explanation and give only the necessary changes.

Grothendieck categories
At the end of Section 3.2, we need to know that interval decompositions in **Obs** are essentially unique. For this we use the fact that it is a *Grothendieck category*: an abelian category which has a generator and which satisfies Grothendieck's (AB5) condition. These conditions enable the study of homological algebra for objects in the category. The category of modules over a ring—and in particular the category of vector spaces over a field k—is perhaps the simplest example of a Grothendieck category. Since functor categories inherit this property from the codomain category [**10**, Theorem 14.2], it follows that **Pers** is a Grothendieck category. In turn, its localization **Obs** is a Grothendieck category. The Krull–Remak–Schmidt–Azumaya theorem then gives the uniqueness that we seek.

The rest of the paper is organized as follows. In Section 2 we define and study the 'observable' category **Obs**. In Section 3 we study interval decompositions. In Section 4 we apply our results to the motivating case of persistence modules over the real line.

2. The observable category

For this section we make the standing assumption that $(R, \leqslant)$ is a total order that is *dense*: for every $s < t$ there exists an intermediate element $s < u < t$.

2.1. Ephemeral modules

Following [**4**], we say that a persistence module is *ephemeral* if $\rho_{ts} = 0$ whenever $s < t$. Let **Eph** denote the full subcategory of **Pers** whose objects are the ephemeral modules.

Definition 2.1. A morphism ϕ between persistence modules is called a *weak isomorphism* if $\operatorname{Ker}\phi$ and $\operatorname{Coker}\phi$ are both ephemeral.

In Section 2.2 we will construct a category **Obs** and show that it equivalent to the Serre quotient category [**11, 15**] obtained from **Pers** by inverting all weak isomorphisms. The following lemma reassures us that this is a sensible thing to do.

Lemma 2.2. *The full subcategory of ephemeral modules satisfies the condition of Serre: given a short exact sequence of persistence modules*

$$0 \longrightarrow V' \xrightarrow{\iota} V \xrightarrow{\pi} V'' \longrightarrow 0,$$

either statement

1. *V is ephemeral*
2. *V' and V'' are both ephemeral*

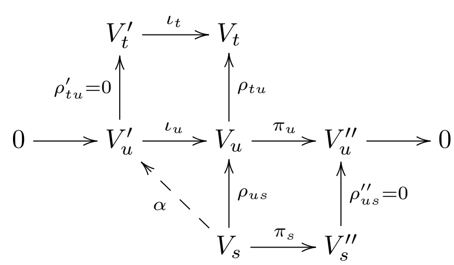
implies the other.

The Serre condition ensures that the class of weak isomorphisms is closed under composition, thanks to the exact sequence

$$0 \longrightarrow \operatorname{Ker}\phi \longrightarrow \operatorname{Ker}\psi\phi \longrightarrow \operatorname{Ker}\psi \longrightarrow \operatorname{Coker}\phi \longrightarrow \operatorname{Coker}\psi\phi \longrightarrow \operatorname{Coker}\psi \longrightarrow 0$$

for a composable pair of maps $V \xrightarrow{\phi} V' \xrightarrow{\psi} V''$.

Proof. If V is ephemeral, then clearly so are V' and V''. Conversely, suppose V' and V'' are ephemeral and $s < t$. Since the total order is dense, there exists u with $s < u < t$. Now consider the following diagram:



Since $\pi_u\rho_{us} = \rho''_{us}\pi_s = 0$ and the middle row is exact, there is a map α with $\rho_{us} = \iota_u\alpha$. Then $\rho_{ts} = \rho_{tu}\rho_{us} = \rho_{tu}\iota_u\alpha = \iota_t\rho'_{tu}\alpha = 0$. Thus V is ephemeral. $\square$

Example 2.3. If the total order is not dense then the ephemeral subcategory is not Serre. For $s < t$ with no intermediate element, the sets $\{s\}$, $\{t\}$ and $\{s, t\}$ are intervals. The short exact sequence of interval modules

$$0 \longrightarrow k_{\{t\}} \longrightarrow k_{\{s,t\}} \longrightarrow k_{\{s\}} \longrightarrow 0$$

has ephemeral outer terms and a non-ephemeral middle term.

2.2. Observable morphisms

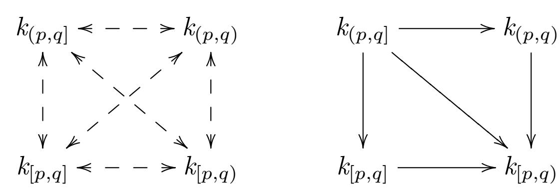
The quotient **Pers** $\xrightarrow{\pi}$ **Pers/Eph** that we wish to construct is characterized by the following universal property [**11**]: first, the functor π carries weak isomorphisms to isomorphisms; second, any other functor **Pers** $\to$ **C** that carries weak isomorphisms to isomorphisms factorizes uniquely through π.

Our plan is to define a category **Obs** and a functor $\mathbf{Pers} \xrightarrow{\pi} \mathbf{Obs}$ explicitly, and then verify the universal property. In this way, $\mathbf{Obs} = \mathbf{Pers}/\mathbf{Eph}$ (where '=' means 'is a category equivalent to').

Definition 2.4. An *observable morphism* (or *obs-morphism*) of persistence modules $\phi^\circ \colon V \dashrightarrow W$ is a collection of maps $\phi_{ts} \colon V_s \to W_t$ defined for $s < t$ (strictly less than), such that $\phi_{ts} = \sigma_{tv}\phi_{vu}\rho_{us}$ whenever $s \leqslant u < v \leqslant t$. Composition of obs-morphisms is defined as follows, using the fact that the index set R is a dense order. If $\phi^\circ \colon V \dashrightarrow W$ and $\psi^\circ \colon W \dashrightarrow X$ are obs-morphisms, then we define $(\psi^\circ\phi^\circ)_{ts} = \psi_{tu}\phi_{us}$ for any u with $s < u < t$. This is well-defined since if $s < u < v < t$ then $\psi_{tv}\phi_{vs} = \psi_{tv}\sigma_{vu}\phi_{us} = \psi_{tu}\phi_{us}$. Every persistence module V has an obs-identity $1_V^\circ = (\rho_{ts} \mid s < t)$ extracted from its structure maps.

Definition 2.5. The category of persistence modules and obs-morphisms is called the *observable category* of persistence modules, **Obs**. It comes with a functor $\mathbf{Pers} \xrightarrow{\pi} \mathbf{Obs}$ which keeps the objects the same and maps each morphism $\phi = (\phi_{ts} \mid s \leqslant t)$ to an obs-morphism $\pi(\phi) = \phi^\circ = (\phi_{ts} \mid s < t)$ by forgetting the terms ϕ_{tt}.

Example 2.6. Between every ordered pair among the four interval modules $k_{(p,q)}$, $k_{[p,q)}$, $k_{(p,q]}$ and $k_{[p,q]}$ there is a nonzero obs-morphism defined by setting $\phi_{ts} = 1$ wherever domain and range both equal k. It follows that the four interval modules are isomorphic in **Obs**. This contrasts with the situation in **Pers** where nonzero maps exist only between certain pairs. The situation is summarized as follows:



In general there is a nonzero obs-morphism $k_I \dashrightarrow k_J$ if and only if $\inf(J) \leqslant \inf(I) < \sup(J) \leqslant \sup(I)$ (these limits being interpreted in the completion of R).

Example 2.7. For a non-singleton interval I, the obs-endomorphism ring of the interval module k_I is isomorphic to k. (The proof of Lemma 1.3 applies directly. The 'non-singleton' condition guarantees that there is at least one non-trivial ϕ_{ts}.)

Example 2.8. If V is ephemeral then $1_V^\circ = 0$ and therefore every obs-morphism to or from V is zero. Thus V is zero (that is, both initial and terminal) in **Obs**.

In the remainder of this subsection we show **Obs** is equivalent to the localized category **Pers/Eph**, by establishing that $\mathbf{Pers} \xrightarrow{\pi} \mathbf{Obs}$ satisfies the universal property described above. Here is the first part of the universal property:

Theorem 2.9. *If $\phi \colon V \to W$ is a weak isomorphism then ϕ° is invertible in* **Obs**.

Proof. We construct an inverse $\psi^\circ = (\psi_{st} \mid s < t)$ as follows. Given $s < t$, select an intermediate index u.

Since $\operatorname{Coker}\phi$ is ephemeral, the composition of $\sigma_{us} \colon W_s \to W_u$ with the natural map $W_u \to \operatorname{Coker}\phi_{uu}$ is zero. Thus σ_{us} factors as a map $\omega_{us} \colon W_s \to \operatorname{Im}\phi_{uu}$ followed by the inclusion of $\operatorname{Im}\phi_{uu}$ into W_u.

Dually, since $\operatorname{Ker}\phi$ is ephemeral, the composition of the inclusion $\operatorname{Ker}\phi_{uu} \to V_u$ and $\rho_{tu}\colon V_u \to V_t$ is zero. Thus there is an induced morphism $\tau_{tu}\colon \operatorname{Im}\phi_{uu} \to V_t$ whose composition with the natural map $V_u \to \operatorname{Im}\phi_{uu}$ is ρ_{tu}.

We define $\psi_{ts} = \tau_{tu}\omega_{us}$. It is straightforward to verify that this construction does not depend on the choice of intermediate element u, and that it defines an obs-morphism $\psi^\circ\colon W \dashrightarrow V$ that is inverse to $\phi^\circ\colon V \dashrightarrow W$. □

Definition 2.10. Let V be a persistence module. Define a persistence module $\overline{V}$ by setting

$$\overline{V}_t = \operatorname{colim}(V_s \mid s < t)$$

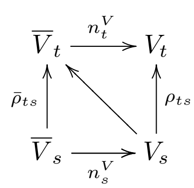
at each index t. The structure maps $\bar{\rho}_{ts}$ are defined using the universal property of colimits. The universal property also generates the following maps:

- A morphism $n^V : \overline{V} \to V$, induced by the maps $(\rho_{ts} \mid s < t)$.
- A morphism $\bar{\phi}\colon \overline{V} \to \overline{W}$ for every obs-morphism $\phi^\circ\colon V \dashrightarrow W$.

This last operation respects composition and identities, so 'bar' is a functor **Obs** $\to$ **Pers**. One can show that this is a left adjoint for π.

Proposition 2.11. *Each $n^V : \overline{V} \to V$ is a weak isomorphism.*

Proof. For every $s < t$ we have a commutative diagram:



From this we see that $\bar{\rho}_{ts}$ carries $\operatorname{Ker}(n_s^V)$ to zero, while ρ_{ts} carries V_s to $\operatorname{Im}(n_t^V)$ and hence to zero in $\operatorname{Coker}(n_t^V)$. Thus $\operatorname{Ker}(n^V)$ and $\operatorname{Coker}(n^V)$ are ephemeral. □

Remark 2.12. Similarly, the functor π has a right adjoint defined on objects by $\underline{V}_t = \lim(V_u \mid u > t)$, and there is a weak isomorphism $u^V : V \to \underline{V}$.

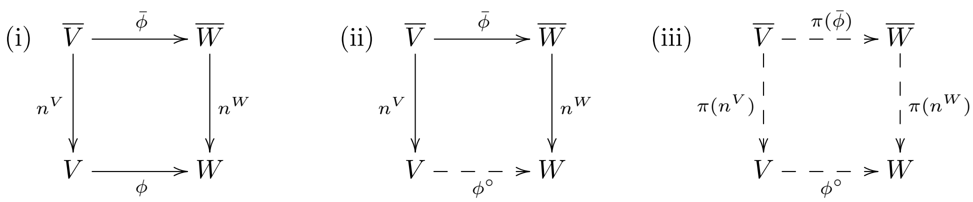
Example 2.13. If $V = k_{(p,q)}$, $k_{[p,q)}$, $k_{(p,q]}$ or $k_{[p,q]}$ then $\overline{V} = k_{(p,q]}$ and $\underline{V} = k_{[p,q)}$. All five morphisms in Example 2.6 are instances of n^V or u^V. They become invertible in **Obs**.

Now we prove the second part of the universal property.

Theorem 2.14. *If $F\colon$ **Pers** $\to$ **C** is a functor that carries weak isomorphisms to isomorphisms, then there is a unique functor $G\colon$ **Obs** $\to$ **C** such that $F = G\pi$.*

Proof. Since **Obs** has the same objects as **Pers**, it follows that G is uniquely defined and satisfies $F = G\pi$ on objects. It remains to consider morphisms.

Let $\phi^\circ \colon V \dashrightarrow W$ be an obs-morphism. We have a mixed-category diagram (ii)



which commutes after applying π to the top three morphisms (iii). By assumption, $F(n^V)$ is invertible and we are forced to define

$$G(\phi^\circ) = F(n_W)F(\bar{\phi})F(n_V)^{-1}.$$

Since $\overline{\psi\phi} = \bar{\psi}\bar{\phi}$ it follows that G, defined in this way, is indeed a functor.

Now suppose $\phi^\circ = \pi(\phi)$ for some morphism $\phi \colon V \to W$. Then we have a commutative diagram (i) in **Pers** to which we apply F to get $F(\phi)F(n^V) = F(n^W)F(\bar{\phi})$. Since $F(n^V)$ is invertible we deduce

$$F(\phi) = F(n_W)F(\bar{\phi})F(n_V)^{-1} = G(\phi^\circ).$$

Hence $F = G\pi$ on morphisms. □

Theorems 2.9 and 2.14 together constitute the following result:

Corollary 2.15. Obs = Pers/Eph. □

Remark 2.16. The results of Sections 2.1 and 2.2, including Corollary 2.15 in particular, remain valid when the category of vector spaces is replaced by any abelian category with colimits. A verbatim reading of the two sections (omitting Examples 2.3, 2.6, 2.7, 2.13 and Remark 2.12) effects the generalization immediately.

2.3. Observable invariants

Because there are more isomorphisms in the observable category, there are fewer isomorphism invariants. In this subsection we consider which quantities and constructions 'make sense' in the observable category. A function on persistence modules is a *strict invariant* if it is invariant under isomorphisms in **Pers**; it is an *observable invariant* if it is invariant under obs-isomorphisms.

Example 2.17. Let $t \in R$. Then $\mathrm{rk}_t(V) = \dim(V_t)$ is a strict invariant but not an observable invariant of the persistence module V.

Example 2.18. Let $s < t$. Then $\mathrm{rk}_{st}(V) = \mathrm{rank}(\rho_{ts} \colon V_s \to V_t)$ is a strict invariant but not an observable invariant of the persistence module V.

Example 2.19. Let $s < t$. Then each of the four 'limiting ranks'

$$\mathrm{rk}_{[st]}(V) = \mathrm{rank}(\overline{V}_s \to \underline{V}_t), \qquad \mathrm{rk}_{[st)}(V) = \mathrm{rank}(\overline{V}_s \to \overline{V}_t),$$
$$\mathrm{rk}_{(st]}(V) = \mathrm{rank}(\underline{V}_s \to \underline{V}_t), \qquad \mathrm{rk}_{(st)}(V) = \mathrm{rank}(\underline{V}_s \to \overline{V}_t)$$

is an observable invariant. We have $\mathrm{rk}_{[st]} \leqslant \{\mathrm{rk}_{st}, \mathrm{rk}_{[st)}, \mathrm{rk}_{(st]}\} \leqslant \mathrm{rk}_{(st)}$.

Proof. The limiting ranks are observable because 'bar' and 'underbar' are functors **Obs** $\to$ **Pers**. The factorization

$$\overline{V}_s \longrightarrow V_s \longrightarrow \underline{V}_s \longrightarrow \overline{V}_t \longrightarrow V_t \longrightarrow \underline{V}_t$$

implies the given inequalities. □

Remark 2.20. For a q-tame persistence module we have the following formulæ:

$$\begin{aligned} \mathrm{rk}_{[st]}(V) &= \max\left(\mathrm{rk}_{ab}(V) \mid a < s < t < b\right), \\ \mathrm{rk}_{(st)}(V) &= \min\left(\mathrm{rk}_{ab}(V) \mid s < a < b < t\right). \end{aligned}$$

Example 2.21. The property of being q-tame is observable.

Proof. Since $\mathrm{rk}_{ab} \leqslant \mathrm{rk}_{[st]} \leqslant \mathrm{rk}_{st}$ whenever $a < s < t < b$, it follows that V is q-tame if and only if $\mathrm{rk}_{[st]}(V) < \infty$ whenever $s < t$. This criterion is observable. □

The *order topology* on R has basis given by the following *basic open sets*:

$$\begin{aligned} (s,t) &= \{x \in R : s < x < t\}, & (s,\infty) &= \{x \in R : s < x\}, \\ (-\infty,t) &= \{x \in R : x < t\}, & (-\infty,\infty) &= R. \end{aligned}$$

An *open interval* in R is an interval which is open in the order topology. Note that any basic open set is an open interval, provided it is non-empty, but there may be others, such as $\mathbb{Q} \cap (0,\sqrt{2})$ for $R = \mathbb{Q}$. The *interior* of any subset X of R is the union of all basic open sets contained in X.

The reader may easily verify the following lemma.

Lemma 2.22. *In a dense total order, an interval has empty interior if and only if it is a singleton. If two intervals I, J have the same non-empty interior, then that interior includes all basic open sets whose endpoints lie in $I \cup J$.* □

Proposition 2.23. *In a dense total order, interval modules k_I, k_J are obs-isomorphic if and only if the intervals I, J have the same interior.*

Proof. If the interiors of the intervals differ, then there is a basic open set (s,t) contained in one of I, J but not the other. Then $\mathrm{rk}_{(st)}(k_I) \neq \mathrm{rk}_{(st)}(k_J)$ so the interval modules are not obs-isomorphic.

Conversely, suppose I, J have the same interior N. If N is empty then k_I, k_J are ephemeral and therefore obs-isomorphic. Otherwise, define an obs-morphism $\phi^\circ \colon k_I \dashrightarrow k_J$ by setting $\phi_{ts} = 1$ whenever $s \in I$ and $t \in J$ (and zero otherwise, by necessity). To verify that this is an obs-morphism, we must show that

$$\phi_{ts} = \rho^J_{tv} \phi_{vu} \rho^I_{us}$$

whenever $s \leqslant u < v \leqslant t$. This risks failure only when $s \in I$, $t \in J$ (otherwise both sides are automatically zero), and in that case Lemma 2.22 implies

$$\begin{aligned} u \in \{s\} \cup (s,t) &\subseteq \{s\} \cup N \subseteq I, \\ v \in \{t\} \cup (s,t) &\subseteq \{t\} \cup N \subseteq J, \end{aligned}$$

so $\rho^J_{tv}\phi_{vu}\rho^I_{us} = 1 = \phi_{ts}$ as required. Define $\psi^\circ \colon k_J \dashrightarrow k_I$ symmetrically. To verify that $\psi^\circ\phi^\circ$ is the obs-identity on k_I, we must show that

$$\rho^I_{ts} = \psi_{tu}\phi_{us}$$

whenever $s < u < t$. This risks failure only when $s, t \in I$ (otherwise both sides are automatically zero), and in that case $u \in (s,t) \subseteq N \subseteq J$ so $\psi_{tu}\phi_{us} = 1 = \rho^I_{ts}$ as required. Symmetrically, $\phi^\circ\psi^\circ$ is the obs-identity on k_J. Thus k_I, k_J are obs-isomorphic. □

3. Interval decomposition

In this section $(R, \leqslant)$ is a total order. Recall that R is said to be a dense order if for every $s < t$ there is an intermediate element $s < u < t$. We say that an interval I in R is *left separable* if it has a countable subset $S \subseteq I$ such that for all $t \in I$ there is $s \in S$ with $s \leqslant t$. (It is equivalent that I equipped with the left order topology is a separable topological space.) Clearly $\mathbb{R}$ is dense and any interval I in $\mathbb{R}$ is left separable, so all the results in this section apply for the real line.

3.1. Decomposition of persistence modules with chain conditions

In this subsection we prove a mild generalization of the main result of [**7**]. In the next subsection we apply it to q-tame persistence modules.

Definition 3.1. Let V be a persistence module over a total order R.

(i) One says that V has the *descending chain condition on images* provided that for all $t, s_1, s_2, \dots \in R$ with $t \geqslant s_1 > s_2 > \cdots$, the chain

$$V_t \supseteq \mathrm{Im}(\rho_{ts_1}) \supseteq \mathrm{Im}(\rho_{ts_2}) \supseteq \cdots$$

stabilizes [**7**].

(ii) Given $s, t \in R$ with $s \leqslant t$, we say that V_s has the *descending chain condition on t-bounded kernels* provided that for all $r_1, r_2, \dots \in R$ with $t < \cdots < r_2 < r_1$, the chain

$$V_s \supseteq \mathrm{Ker}(\rho_{r_1 s}) \supseteq \mathrm{Ker}(\rho_{r_2 s}) \supseteq \cdots$$

stabilizes. Applying ρ_{ts}, it is equivalent that the chain

$$\mathrm{Im}(\rho_{ts}) \supseteq \mathrm{Im}(\rho_{ts}) \cap \mathrm{Ker}(\rho_{r_1 t}) \supseteq \mathrm{Im}(\rho_{ts}) \cap \mathrm{Ker}(\rho_{r_2 t}) \supseteq \cdots$$

stabilizes.

(iii) We say that V has the *descending chain condition on sufficient bounded kernels* provided that for all $t \in R$ and $0 \neq v \in V_t$, there exists $s \leqslant t$ such that $v \in \mathrm{Im}(\rho_{ts})$ and V_s has the descending chain condition on t-bounded kernels.

Note that condition (iii) holds if V has the descending chain condition on kernels, as considered in [**7**], since one can then take $s = t$. The following theorem thus generalizes [**7**, Theorem 1.2].

Theorem 3.2. *Suppose that R is a total order with the property that any interval in R is left separable. Then any persistence module with the descending chain condition on images and on sufficient bounded kernels is a direct sum of interval modules.*

For the proof we freely use the notation and results of [**7**]. The hypothesis in that paper that R have a countable subset which is dense in the order topology was only used in [**7**, Lemma 3.2], but it is stronger than is required (for example consider $\mathbb{R}^2$ with the lexicographic ordering), so we have replaced it here with the left separability hypothesis on intervals.

Suppose that V has the descending chain condition on images. Of the results in [**7**], Lemmas 2.1(a) and 2.2 hold, all results in Sections 3–6 hold, and Lemma 7.1(a) holds. What fails is Lemma 2.1(b). Then in Lemma 7.1(b) the set is disjoint, but needn't strongly cover V_t. The following is a partial replacement for Lemma 2.1(b).

Lemma 3.3. *Let $s \leqslant t$ and suppose that V_s has the descending chain condition on t-bounded kernels. Suppose that c is a cut with $t \in c^-$ and $c^+ \neq \emptyset$. Then $\operatorname{Im}(\rho_{ts}) \cap \operatorname{Ker}^+_{ct} = \operatorname{Im}(\rho_{ts}) \cap \operatorname{Ker}(\rho_{rt})$ for some $r \in c^+$.*

Proof. Suppose that $\operatorname{Im}(\rho_{ts}) \cap \operatorname{Ker}^+_{ct} \neq \operatorname{Im}(\rho_{ts}) \cap \operatorname{Ker}(\rho_{rt})$ for all $r \in c^+$. Since c^+ is non-empty, we can choose $r_1 \in c^+$. Since $\operatorname{Im}(\rho_{ts}) \cap \operatorname{Ker}^+_{ct} \neq \operatorname{Im}(\rho_{ts}) \cap \operatorname{Ker}(\rho_{r_1 t})$ there must be some r_2 with $\operatorname{Im}(\rho_{ts}) \cap \operatorname{Ker}(\rho_{r_2 t})$ strictly contained in $\operatorname{Im}(\rho_{ts}) \cap \operatorname{Ker}(\rho_{r_1 t})$. Similarly, since $\operatorname{Im}(\rho_{ts}) \cap \operatorname{Ker}^+_{ct} \neq \operatorname{Im}(\rho_{ts}) \cap \operatorname{Ker}(\rho_{r_2 t})$, there must be some r_3 with $\operatorname{Im}(\rho_{ts}) \cap \operatorname{Ker}(\rho_{r_3 t})$ strictly contained in $\operatorname{Im}(\rho_{ts}) \cap \operatorname{Ker}(\rho_{r_2 t})$, and so on. But then the chain

$$\operatorname{Im}(\rho_{ts}) \cap \operatorname{Ker}(\rho_{r_1 t}) \supset \operatorname{Im}(\rho_{ts}) \cap \operatorname{Ker}(\rho_{r_2 t}) \supset \operatorname{Im}(\rho_{ts}) \cap \operatorname{Ker}(\rho_{r_3 t}) \supset \cdots$$

does not stabilize. □

Proof of Theorem 3.2. Suppose that V has the descending chain condition on images and on sufficient bounded kernels. As in [**7**, §5], one obtains submodules W_I of V for each interval I.

For $t \in R$, as in the proof of [**7**, Theorem 1.2] there are sections (F^-_{It}, F^+_{It}) for I an interval which contains t, where

$$F^{\pm}_{It} = \operatorname{Im}^-_{\ell t} + \operatorname{Ker}^{\pm}_{ut} \cap \operatorname{Im}^+_{\ell t},$$

satisfying

$$F^+_{It} = W_{It} \oplus F^-_{It}. \tag{$*$}$$

These sections no longer need to cover V_t, but they are still disjoint, so by the argument in [**7**, Lemma 6.1] the sum of the W_{It} is a direct sum.

Thus we obtain a submodule $\bigoplus_I W_I$ of V. By [**7**, Lemma 5.3] this submodule is a direct sum of interval modules. We need to show it is equal to V. Assume for a contradiction that there is $t \in R$ and an element $v \in V_t$ not in $\bigoplus_I W_{It}$. By assumption there is $s \leqslant t$ such that $v \in \operatorname{Im}(\rho_{ts})$ and V_s has the descending chain condition on t-bounded kernels.

Let $X = (\bigoplus_I W_{It}) \cap \operatorname{Im}(\rho_{ts})$. Since $v \in \operatorname{Im}(\rho_{ts})$ but $v \notin X$, we have $\operatorname{Im}(\rho_{ts}) \not\subseteq X$. Thus by [**7**, Lemma 7.1(a)] there is a cut ℓ with $t \in \ell^+$ and

$$X + \operatorname{Im}^-_{\ell t} \cap \operatorname{Im}(\rho_{ts}) \neq X + \operatorname{Im}^+_{\ell t} \cap \operatorname{Im}(\rho_{ts}).$$

This inequality can only happen if $\operatorname{Im}(\rho_{ts}) \not\subseteq \operatorname{Im}^-_{\ell t}$, so $s \notin \ell^-$, and hence $s \in \ell^+$. Thus $\operatorname{Im}^+_{\ell t} \subseteq \operatorname{Im}(\rho_{ts})$. Thus the inequality simplifies to

$$X + \operatorname{Im}^-_{\ell t} \neq X + \operatorname{Im}^+_{\ell t}.$$

Let $Y = X + \operatorname{Im}^-_{\ell t}$. Clearly $\operatorname{Im}^+_{\ell t} \not\subseteq Y$. Define

$$u^- = \{r \in R : r < t \text{ or } r \geqslant t \text{ and } \operatorname{Ker}(\rho_{rt}) \cap \operatorname{Im}^+_{\ell t} \subseteq Y\}, \text{ and}$$
$$u^+ = \{r \in R : r \geqslant t \text{ and } \operatorname{Ker}(\rho_{rt}) \cap \operatorname{Im}^+_{\ell t} \not\subseteq Y\}.$$

Then u is a cut and $t \in u^-$.

Now $\mathrm{Ker}^-_{ut} \cap \mathrm{Im}^+_{\ell t} \subseteq Y$ since

$$\mathrm{Ker}^-_{ut} \cap \mathrm{Im}^+_{\ell t} = \bigcup_{\substack{r \in u^- \\ t \leqslant r}} \mathrm{Ker}(\rho_{rt}) \cap \mathrm{Im}^+_{\ell t}$$

and by the definition of u^-, each term in the union is contained in Y. We show that $\mathrm{Ker}^+_{ut} \cap \mathrm{Im}^+_{\ell t} \nsubseteq Y$. This is clear if u^+ is empty, for then $\mathrm{Ker}^+_{ut} = V_t$. Thus suppose that u^+ is non-empty. Since V_s has the descending chain condition on t-bounded kernels, by Lemma 3.3 there is some $r \in u^+$ such that $\mathrm{Ker}^+_{ut} \cap \mathrm{Im}(\rho_{ts}) = \mathrm{Ker}(\rho_{rt}) \cap \mathrm{Im}(\rho_{ts})$. By taking the intersection with $\mathrm{Im}^+_{\ell t} \subseteq \mathrm{Im}(\rho_{ts})$, we obtain

$$\mathrm{Ker}^+_{ut} \cap \mathrm{Im}^+_{\ell t} = \mathrm{Ker}(\rho_{rt}) \cap \mathrm{Im}^+_{\ell t}$$

and by the definition of u^+ we have $\mathrm{Ker}(\rho_{rt}) \cap \mathrm{Im}^+_{\ell t} \nsubseteq Y$.

Now since $t \in u^-$ and $t \in \ell^+$, the cuts u and ℓ define an interval I which contains t. As already observed $(*)$, we have

$$W_{It} \oplus (\mathrm{Im}^-_{\ell t} + \mathrm{Ker}^-_{ut} \cap \mathrm{Im}^+_{\ell t}) = \mathrm{Im}^-_{\ell t} + \mathrm{Ker}^+_{ut} \cap \mathrm{Im}^+_{\ell t}\,.$$

It follows that $W_{It} \subseteq \mathrm{Im}^+_{\ell t} \subseteq \mathrm{Im}(\rho_{ts})$, so $W_{It} \subseteq X$. Then

$$\begin{aligned} Y &= Y + \mathrm{Ker}^-_{ut} \cap \mathrm{Im}^+_{\ell t} \\ &= X + \mathrm{Im}^-_{\ell t} + \mathrm{Ker}^-_{ut} \cap \mathrm{Im}^+_{\ell t} \\ &= X + W_{It} + \mathrm{Im}^-_{\ell t} + \mathrm{Ker}^-_{ut} \cap \mathrm{Im}^+_{\ell t} \\ &= X + \mathrm{Im}^-_{\ell t} + \mathrm{Ker}^+_{ut} \cap \mathrm{Im}^+_{\ell t} \\ &= Y + \mathrm{Ker}^+_{ut} \cap \mathrm{Im}^+_{\ell t}, \end{aligned}$$

a contradiction. Thus $V = \bigoplus_I W_I$. □

3.2. Decomposition of q-tame modules

In this section we prove an interval decomposition theorem for q-tame persistence modules in the observable category for a total order which is dense and has the property that all intervals are left separable.

Definition 3.4. The *radical* of a persistence module V is the submodule $\mathrm{rad}\, V$ of V defined by

$$(\mathrm{rad}\, V)_t = \sum_{s<t} \mathrm{Im}(\rho_{ts}).$$

By construction, it is the smallest submodule of V such that $(V/\mathrm{rad}\, V)$ is ephemeral. We say that V is *radical* if $V = \mathrm{rad}\, V$.

Observe that if V is a q-tame persistence module, then V has the descending chain condition on images and V_s has the descending chain condition on t-bounded kernels for all $s < t$. If in addition V is radical, it follows that V has the descending chain condition on sufficient bounded kernels. Thus Theorem 3.2 gives:

Corollary 3.5. *If every interval in R is left separable, then any radical q-tame persistence module is a direct sum of interval modules.*

Now suppose that R is a dense order. In this case $\mathrm{rad}\,\mathrm{rad}\, V = \mathrm{rad}\, V$ for any V, so $\mathrm{rad}\, V$ is a radical persistence module. Clearly any submodule of a q-tame persistence module is again q-tame. Thus we obtain:

Corollary 3.6. *Suppose R is dense and every interval in R is left separable. If V is a q-tame persistence module, then* $\operatorname{rad} V$ *is a direct sum of interval modules.*

Example 3.7. If R is the set of real numbers, the product of the interval modules associated to the intervals $[-1/n, 1/n]$ with $n \geqslant 1$ is q-tame, and its radical is the direct sum of the interval modules for the intervals $(-1/n, 1/n]$. Neither of these modules satisfies the hypothesis for the decomposition theorem of [**7**] (specifically, they fail the descending chain condition on kernels).

Suppose again that R is a dense order. Since the observable category **Obs** is identified with the quotient category **Pers**/**Eph**, and the functor $\pi\colon \mathbf{Pers} \to \mathbf{Obs}$ has a right adjoint, it follows that **Eph** is a localizing subcategory in the sense of [**12**, p. 372]. Therefore, **Obs** is a Grothendieck category by [**12**, Proposition 9, p. 378] and π commutes with direct sums. Thus direct sums exist in **Obs**, and are given in the same way as in **Pers**: by taking the direct sum of the vector spaces for each point of R.

For any persistence module V, the inclusion $\operatorname{rad} V \to V$ is a weak isomorphism. (In fact, $\operatorname{rad} V$ is the image of the weak isomorphism $n^V : \overline{V} \to V$ from Section 2.2). Thus we reach our main goal:

Corollary 3.8. *Suppose R is dense and every interval in R is left separable. If V is a q-tame persistence module, then V is isomorphic in* **Obs** *to a direct sum of interval modules.*

This decomposition is in fact essentially unique. There is a version of the Krull–Remak–Schmidt–Azumaya Theorem for Grothendieck categories, see [**2**, §6.7] or [**14**, §4.8]. It says that if an object is written as a direct sum of objects in two different ways, and if each summand has local endomorphism ring, then the terms in the two sums can be paired off in such a way that corresponding summands are isomorphic. In particular, since by Example 2.7 interval modules (for non-singleton intervals) have obs-endomorphism ring equal to k, which is a local ring, the Krull–Remak–Schmidt–Azumaya Theorem and Proposition 2.23 give the following result.

Theorem 3.9. *Over a dense total order, if a persistence module is isomorphic in* **Obs** *to a direct sum of interval modules in two different ways, then the non-singleton intervals in each sum can be paired off in such a way that corresponding intervals have the same interior.* □

4. Real-parameter persistence modules

We return to the motivating case of persistence modules indexed by $\mathbb{R}$.

4.1. Interleavings and diagrams

Persistence modules over the real line are codified and studied using their persistence diagrams. The principal results are the stability theorem [**4**, **6**] and Lesnick's isometry theorem [**4**, **13**]. We review these results now.

Two persistence modules V, W are compared by finding *interleavings* between them. An *ϵ-interleaving* is specified by collections of maps $\phi_{ts}\colon V_s \to W_t$ and

$\psi_{ts}\colon W_s \to V_t$, defined for $t \geqslant s + \epsilon$, such that the equations

$$\phi_{ts} = \sigma_{tv}\phi_{vu}\rho_{us}, \qquad \psi_{ts} = \rho_{tv}\psi_{vu}\sigma_{us},$$
$$\rho_{ts} = \psi_{tu}\phi_{us}, \qquad \sigma_{ts} = \phi_{tu}\psi_{us}$$

are satisfied whenever they are defined. It is immediate that

- an isomorphism is the same thing as a 0-interleaving;
- an obs-isomorphism restricts to ϵ-interleavings for all $\epsilon > 0$.

The interleaving distance between two persistence modules is defined thus:

$$\mathrm{d_i}(V, W) = \inf\left(\epsilon \mid \text{there exists an } \epsilon\text{-interleaving between } V,\, W\right).$$

It is an extended pseudometric, taking values in $[0, \infty]$. The triangle inequality results from the fact that interleavings can be composed (adding the respective ϵ-values). If V, W are obs-isomorphic then $\mathrm{d_i}(V, W) = 0$, so the 'pseudo' is necessary.

Associated to a persistence module V is its *persistence measure* [**4**]. This is a function defined on rectangles $[a, b] \times [c, d]$ by the formula

$$\mu_V([a,b] \times [c,d]) \\ = \text{multiplicity of } \big(0 \longrightarrow k \longrightarrow k \longrightarrow 0\big) \text{ in } \big(V_a \longrightarrow V_b \longrightarrow V_c \longrightarrow V_d\big).$$

We require $-\infty \leqslant a < b \leqslant c < d \leqslant +\infty$, so the rectangle lies in the extended closed half-plane

$$\overline{\mathbb{H}} = \{(p, q) \mid -\infty \leqslant p \leqslant q \leqslant +\infty\}.$$

We set $V_{-\infty} = V_{+\infty} = 0$ to interpret the extreme cases. The measure is additive with respect to splitting a rectangle into a finite number of smaller rectangles, and therefore (being nonnegative) it is monotone with respect to inclusions of rectangles.

The *undecorated diagram* $\mathrm{dgm}(V)$ of a persistence module V is a multiset in the extended open half-plane

$$\mathbb{H} = \{(p, q) \mid -\infty \leqslant p < q \leqslant +\infty\}.$$

The diagram is defined, following [**4**], by its multiplicity function[1]

$$\mathrm{m}_V(p, q) = \min\left(\mu_V([a,b] \times [c,d]) \mid a < p < b < c < q < d\right).$$

We temporarily allow $-\infty < -\infty$ and $+\infty < +\infty$ when selecting a and d. Because of monotonicity, the minimum can be interpreted as a limit over a decreasing sequence of rectangles that contain (p, q) in their interior. The set of values (p, q) of finite multiplicity is an open subset $\mathbb{F}_V \subseteq \mathbb{H}$, called the *finite interior* of V. Within the finite interior, the undecorated diagram is locally finite. These facts are known:

- If V is q-tame then $\mathbb{F}_V = \mathbb{H}$.
- If V is q-tame and decomposable into intervals, then the undecorated diagram records exactly the endpoints of the intervals in the decomposition.[2]

There is also a 'decorated diagram' which discriminates between open, closed and half-open intervals.

[1]In other words, $\mathrm{m}_V(p, q)$ specifies the multiplicity of the element (p, q) in the multiset $\mathrm{dgm}(V)$.
[2]Thus in this case $\mathrm{m}_V(p, q)$ is exactly analogous to $\mathrm{m}_{p,q}$ of Section 1.1.

Two diagrams $\mathrm{dgm}(V)$, $\mathrm{dgm}(W)$ may be compared using the *bottleneck distance*. Let $\sim$ denote a partial matching between the points of $\mathrm{dgm}(V)$ and $\mathrm{dgm}(W)$ in the respective finite interiors. The cost of the partial matching is

$$\mathrm{cost}(\sim) = \sup \begin{cases} \mathrm{d}^\infty(v,w) & \text{matched pairs } v \sim w, \\ \mathrm{d}^\infty(v, \overline{\mathbb{H}} - \mathbb{F}_W) & \text{unmatched } v, \\ \mathrm{d}^\infty(w, \overline{\mathbb{H}} - \mathbb{F}_V) & \text{unmatched } w, \\ \mathrm{d}^\infty_{\mathrm{haus}}(\overline{\mathbb{H}} - \mathbb{F}_V, \overline{\mathbb{H}} - \mathbb{F}_W), & \end{cases}$$

where d^∞ is the extended metric $\mathrm{d}^\infty((p_1,q_1),(p_2,q_2)) = \max(|p_1 - p_2|, |q_1 - q_2|)$ and $\mathrm{d}^\infty_{\mathrm{haus}}$ is the corresponding Hausdorff distance between subsets. The bottleneck distance between diagrams is defined

$$\mathrm{d_b}(\mathrm{dgm}(V), \mathrm{dgm}(W)) = \inf(\mathrm{cost}(\sim) \mid \sim \text{ is a partial matching}).$$

One can show that the infimum is attained using a compactness argument.[3]

Theorem 4.1 (Stability and isometry [**4**, **6**, **13**])**.** *For arbitrary persistence modules V, W over the real line, we have*

$$\mathrm{d_b}(\mathrm{dgm}(V), \mathrm{dgm}(W)) \leqslant \mathrm{d_i}(V, W).$$

If V, W are q-tame then equality holds. □

4.2. Results in the observable category

We now transport our discussion to the observable category.

Proposition 4.2. *The interleaving distance is observable.*

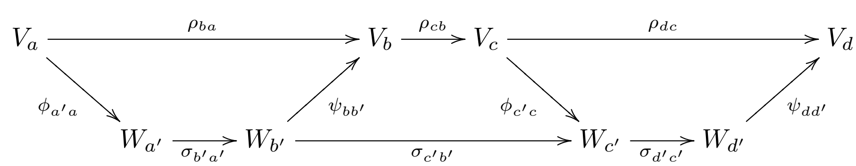
Proof. We know that $\mathrm{d_i}(V, V') = 0$ whenever V, V' are obs-isomorphic. If also W, W' are obs-isomorphic, then $\mathrm{d_i}(V', W') = \mathrm{d_i}(V, W)$ by the triangle inequality. □

Proposition 4.3. *The undecorated persistence diagram is observable.*

Proof. Let $\phi^\circ \colon V \dashrightarrow W$ be an obs-isomorphism with inverse $\psi^\circ \colon W \dashrightarrow V$. We will show that $\mathrm{m}_V(p,q) = \mathrm{m}_W(p,q)$ for all points (p,q). Let a, b, c, d be values attaining the minimum in the definition of $\mathrm{m}_V(p,q)$, and select a', b', c', d' such that

$$a < a' < p < b' < b < c < c' < q < d' < d.$$

Thus (p,q) lies in the interior of $[a', b'] \times [c', d']$ which lies in the interior of $[a,b] \times [c,d]$. From the commutative diagram



it follows by applying monotonicity (to the eight-term chain of vector spaces) that

[3] A sequence of partial matchings with cost converging to δ can be refined to a subsequence which stabilizes for any particular point v or w, thanks to the local finiteness of the diagrams, and therefore to a subsequence which stabilizes for each of the countably many points of $\mathrm{dgm}(V)$ and $\mathrm{dgm}(W)$. The limit is a well-defined partial matching with cost at most δ. Compare [**4**, Theorem 4.10].

$\mu_W([a',b'] \times [c',d']) \leqslant \mu_V([a,b] \times [c,d])$, therefore $\mathrm{m}_W(p,q) \leqslant \mathrm{m}_V(p,q)$. The reverse inequality follows symmetrically. □

Corollary 4.4. *The stability and isometry theorem for persistence modules over the real line is meaningful and true in the observable category.* □

There is a particularly clean structure theory for q-tame modules in **Obs**.

Theorem 4.5. *Let V, W be q-tame persistence modules over the real line. The following statements are equivalent:*

(a) V and W are obs-isomorphic.

(b) The interleaving distance between V and W is zero.

(c) The undecorated persistence diagrams of V and W are equal.

Proof. We have seen (a) ⇒ (b).

(b) ⇒ (c): The stability theorem implies that the bottleneck distance between the diagrams is zero. Since q-tame persistence modules have locally finite diagrams, it follows that the diagrams are equal.

(c) ⇒ (a): Being q-tame, the modules V, W are obs-isomorphic to direct sums of interval modules. We may assume that the intervals are open and nonempty; then the intervals are determined by the persistence diagrams, so the two direct sums are isomorphic. □

We finish by showing what happens when we drop q-tameness.

Example 4.6. We construct a pair of persistence modules V, W whose interleaving distance is zero but which are not obs-isomorphic. Let K be a compact subset of the half-plane with no isolated points, and let X, Y be countable dense subsets of K. If $X \neq Y$ then

$$V = \bigoplus_{(p,q)\in X} k_{(p,q)} \quad \text{and} \quad W = \bigoplus_{(p,q)\in Y} k_{(p,q)}$$

are not obs-isomorphic, by Theorem 3.9. Now let $\epsilon > 0$. Select a bijection $f\colon X \to Y$ that moves points by at most ϵ. Each matched pair of summands $k_{(p,q)}$, $k_{f(p,q)}$ is ϵ-interleaved, so V, W are ϵ-interleaved. Thus the interleaving distance between V and W is zero. □

Example 4.7. We construct a persistence module V indexed by $\mathbb{R}$ which is not obs-isomorphic to a direct sum of interval modules. Let $\hat{V}$ be a persistence module indexed by $\mathbb{Z}$ that is not isomorphic to a direct sum of interval modules. For instance, we can set

$$\hat{V}_n = \{\text{sequences } (x_1, x_2, \dots) \text{ in } k\}, \qquad \text{for } n \geqslant 0,$$

$$\hat{V}_n = \{\text{sequences } (x_1, x_2, \dots) \text{ in } k \text{ with } x_1 = x_2 = \cdots = x_{|n|} = 0\}, \quad \text{for } n \leqslant 0,$$

and set each $\hat{\rho}_{nm}$ to be the canonical inclusion map, following Webb [**16**].

Define V by setting $V_t = \hat{V}_{\lfloor t \rfloor}$ and $\rho_{ts} = \hat{\rho}_{\lfloor t \rfloor \lfloor s \rfloor}$. Certainly V cannot decompose into interval modules because that would induce an interval decomposition of $\hat{V}$. We show that the same is true for any module W obs-isomorphic to V. To show this, let $\hat{W}$ be the module indexed by $\mathbb{Z}$ defined by

$$\hat{W}_n = \operatorname{Im}(W_{n+(1/5)} \to W_{n+(3/5)}),$$

with structure maps induced by those of W. Then any direct-sum decomposition of W induces a direct sum decomposition of $\hat{W}$, and interval module summands of W become interval module summands of $\hat{W}$. Meanwhile, thanks to the obs-isomorphism between V, W we have a commutative diagram:

$$\begin{array}{ccccccccc}
V_n & & \longrightarrow & & V_{n+(2/5)} & & \longrightarrow & & V_{n+(4/5)} \\
& \searrow & & \nearrow & & \overset{*_n}{\searrow} & & \nearrow & \\
& & W_{n+(1/5)} & & \longrightarrow & & W_{n+(3/5)} & &
\end{array}$$

The top row is just $\hat{V}_n = \hat{V}_n = \hat{V}_n$, and it follows that the map labelled $*_n$ induces an isomorphism between $\hat{V}_n$ and $\hat{W}_n$. From the diagram

$$\begin{array}{ccccc}
V_{m+(2/5)} & & \longrightarrow & & V_{n+(2/5)} & & \\
& \overset{*_m}{\searrow} & & & & \overset{*_n}{\searrow} & \\
& & W_{m+(3/5)} & & \longrightarrow & & W_{n+(3/5)}
\end{array}$$

we see that the structure maps agree under these isomorphisms. We conclude that $\hat{V}, \hat{W}$ are isomorphic. An interval decomposition of W would induce an interval decomposition of $\hat{V}$ which, by assumption, does not exist. □

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