Lectures on
Representations of Quivers

by

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A 'quiver' is a directed graph, and a representation is defined by a vector space for each vertex and a linear map for each arrow. The theory of representations of quivers touches linear algebra, invariant theory, finite dimensional algebras, free ideal rings, Kac-Moody Lie algebras, and many other fields.

These are the notes for a course of eight lectures given in Oxford in spring 1992. My aim was the classification of the representations for the Euclidean diagrams $\tilde{A}_n$, $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$. It seemed ambitious for eight lectures, but turned out to be easier than I expected.

The Dynkin case is analysed using an argument of J.Tits, P.Gabriel and C.M.Ringel, which involves actions of algebraic groups, a study of root systems, and some clever homological algebra. The Euclidean case is treated using the same tools, and in addition the Auslander-Reiten translations τ,τ^-, and the notion of a 'regular uniserial module'. I have avoided the use of reflection functors, Auslander-Reiten sequences, and case-by-case analyses.

The prerequisites for this course are quite modest, consisting of the basic notions about rings and modules; a little homological algebra, up to Ext^1 and long exact sequences; the Zariski topology on $\mathbb{A}^n$; and maybe some ideas from category theory.

In the last section I have listed some topics which are the object of current research. I hope these lectures are a useful preparation for reading the papers listed there.

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§1. Path algebras

Once and for all, we fix an algebraically closed field k.

DEFINITIONS.

(1) A quiver $Q = (Q_0, Q_1, s, t: Q_1 \longrightarrow Q_0)$ is given by

a set Q_0 of vertices, which for us will be $\{1,2,\ldots,n\}$, and

a set Q_1 of arrows, which for us will be finite.

An arrow ρ starts at the vertex $s(\rho)$ and terminates at $t(\rho)$. We sometimes indicate this as $s(\rho) \xrightarrow{\rho} t(\rho)$.

(2) A non-trivial path in Q is a sequence $\rho_1 \ldots \rho_m$ $(m \geq 1)$ of arrows which satisfies $t(\rho_{i+1}) = s(\rho_i)$ for $1 \leq i < m$. Pictorially

$$\bullet \xleftarrow{\rho_1} \bullet \xleftarrow{\rho_2} \ldots \xleftarrow{\rho_m} \bullet$$

This path starts at $s(\rho_m)$ and terminates at $t(\rho_1)$. For each vertex i we denote by e_i the trivial path which starts and terminates at i. We use the notation $s(x)$ and $t(x)$ to denote the starting and terminating vertex of a path x. Note that the arrows in a path are ordered in the same way as one orders a composition of functions.

(3) The path algebra kQ is the k-algebra with basis the paths in Q, and with the product of two paths x,y given by

$$xy = \begin{cases} \text{obvious composition} & (\text{if } t(y)=s(x)) \\ 0 & (\text{else}) \end{cases}$$

This is an associative multiplication.

For example if Q is the quiver $1 \xrightarrow{\rho} 2 \xrightarrow{\sigma} 3$ then kQ has basis the paths $e_1, e_2, e_3, \rho, \sigma$ and $\sigma\rho$. The product $\sigma\rho$ of the paths σ and ρ is the path $\sigma\rho$. On the other hand the product $\rho\sigma$ is zero. Some other products are $\rho\rho=0$, $e_1\rho=0$, $e_2\rho=\rho$, $\rho e_1=\rho$, $e_3(\sigma\rho)=\sigma\rho$, $e_1e_1=e_1$, $e_1e_2=0$, etc.

EXAMPLES.

(1) If Q consists of one vertex and one loop, then $kQ \cong k[T]$. If Q has one vertex and r loops, then kQ is the free associative algebra on r letters.

(2) If there is at most one path between any two points, then kQ can be identified with the subalgebra

$$\{C \in M_n(k) \mid C_{ij}=0 \text{ if no path from } j \text{ to } i\}$$

of $M_n(k)$. If Q is $1 \longrightarrow 2 \longrightarrow \ldots \longrightarrow n$ this is the lower triangular matrices.

IDEMPOTENTS. Set $A=kQ$.

(1) The e_i are orthogonal idempotents, ie $e_i e_j = 0$ $(i \neq j)$, $e_i^2 = e_i$.

(2) A has an identity given by $1 = \sum_{i=1}^{n} e_i$.

(3) The spaces Ae_i, e_jA, and e_jAe_i have as bases the paths starting at i and/or terminating at j.

(4) $A = \oplus_{i=1}^{n} Ae_i$, so each Ae_i is a projective left A-module.

(5) If X is a left A-module, then $\mathrm{Hom}_A(Ae_i,X) \cong e_iX$.

(6) If $0 \neq f \in Ae_i$ and $0 \neq g \in e_iA$ then $fg \neq 0$.

PROOF. Look at the longest paths x,y involved in f,g. In the product fg the coefficient of xy cannot be zero.

(7) The e_i are primitive idempotents, ie Ae_i is a indecomposable module.

PROOF. If $\mathrm{End}_A(Ae_i) \cong e_iAe_i$ contains idempotent f, then $f^2=f=fe_i$, so $f(e_i-f)=0$. Now use (6).

(8) If $e_i \in Ae_jA$ then $i=j$.

PROOF. Ae_jA has as basis the paths passing through the vertex j.

(9) The e_i are inequivalent, ie $Ae_i \not\cong Ae_j$ for $i \neq j$.

PROOF. Thanks to (5), inverse isomorphisms give elements $f \in e_iAe_j$, $g \in e_jAe_i$ with $fg=e_i$ and $gf=e_j$. This contradicts (8).

PROPERTIES OF PATH ALGEBRAS.

These are exercises, but some are rather testing.

(1) A is finite dimensional $\Leftrightarrow$ Q has no oriented cycles.

(2) A is prime (ie $IJ\neq 0$ for two-sided ideals $I,J\neq 0$) $\Leftrightarrow$ $\forall i,j$ $\exists$ path i to j.

(3) A is left (right) noetherian $\Leftrightarrow$ if there is an oriented cycle through i, then only one arrow starts (terminates) at i.

(4) rad A has basis {paths i to j | there is no path from j to i}.

(5) The centre of A is $k\times k\times\ldots\times k[T]\times k[T]\times\ldots$, with one factor for each connected component C of Q, and that factor is k[T] $\Leftrightarrow$ C is an oriented cycle.

REPRESENTATIONS.

We define a category Rep(Q) of representations of Q as follows.

A <u>representation</u> X of Q is given by a vector space X_i for each $i\in Q_0$ and a linear map $X_\rho:X_{s(\rho)}\longrightarrow X_{t(\rho)}$ for each $\rho\in Q_1$.

A <u>morphism</u> $\theta:X\longrightarrow X'$ is given by linear maps $\theta_i:X_i\longrightarrow X'_i$ for each $i\in Q_0$ satisfying $X'_\rho\theta_{s(\rho)} = \theta_{t(\rho)}X_\rho$ for each $\rho\in Q_1$.

The <u>composition</u> of θ with $\phi:X'\longrightarrow X''$ is given by $(\phi\circ\theta)_i = \phi_i\circ\theta_i$.

EXAMPLE. Let S(i) be the representation with

$$S(i)_j = \begin{cases} k & (j=i) \\ 0 & (\text{else}) \end{cases} \qquad S(i)_\rho=0 \text{ (all } \rho\in Q_1).$$

EXERCISE. It is very easy to compute with representations. For example let Q be the quiver $\bullet\longleftarrow\bullet\longrightarrow\bullet$, and let X and Y be the representations

$$k\xleftarrow{1}k\xrightarrow{1}k \qquad\qquad k\xleftarrow{1}k\longrightarrow 0.$$

Show that Hom(X,Y) is one-dimensional, and that Hom(Y,X)=0.

LEMMA. The category Rep(Q) is equivalent to kQ-Mod.

PROOF. We only give the construction. If $\mathcal{X}$ is a kQ-module, define a representation X with

$$X_i = e_i\mathcal{X}$$
$$X_\rho(x) = \rho x = e_{t(\rho)}\rho x \in X_{t(\rho)} \quad \text{for } x\in X_{s(\rho)}.$$

If X is a representation, define a module $\mathcal{X}$ via

$\mathcal{X}=\oplus_{i=1}^{n} X_i$. Let $X_i \xrightarrow{\varepsilon_i} \mathcal{X} \xrightarrow{\pi_i} X_i$ be the canonical maps.

$$\rho_1\ldots\rho_m x = \varepsilon_{t(\rho_1)} X_{\rho_1} \ldots X_{\rho_m} \pi_{s(\rho_m)}(x)$$
$$e_i x = \varepsilon_i \pi_i(x),$$

It is straightforward, but tedious, to check that these are inverses and that morphisms behave, etc. We can now use the same letter for a module and the corresponding representation, ignoring the distinction.

EXAMPLE. Under this correspondence, the representations S(i) are simple modules. Moreover, if Q has no oriented cycles, it is easy to see that the S(i) are the only simple modules.

DEFINITIONS.

(1) The <u>dimension</u> <u>vector</u> of a finite dimensional kQ-module X is the vector $\underline{\dim}\ X \in \mathbb{N}^n$, with

$$(\underline{\dim}\ X)_i = \dim X_i = \dim e_iX = \dim \mathrm{Hom}(Ae_i,X).$$

Thus $\dim X = \sum_{i=1}^{n} (\underline{\dim}\ X)_i$.

(2) The <u>Euler</u> <u>form</u> is $\langle\alpha,\beta\rangle = \sum_{i=1}^{n} \alpha_i\beta_i - \sum_{\rho\in Q_1} \alpha_{s(\rho)}\beta_{t(\rho)}$ for $\alpha,\beta \in \mathbb{Z}^n$. This is a bilinear form on $\mathbb{Z}^n$.

(3) The <u>Tits</u> <u>form</u> is $q(\alpha) = \langle\alpha,\alpha\rangle$. This is a quadratic form on $\mathbb{Z}^n$.

(4) The <u>Symmetric</u> <u>bilinear</u> <u>form</u> is $(\alpha,\beta) = \langle\alpha,\beta\rangle + \langle\beta,\alpha\rangle$.

THE STANDARD RESOLUTION.

Let $A=kQ$. If X is a left A-module, there is an exact sequence

$$0 \longrightarrow \bigoplus_{\rho\in Q_1} Ae_{t(\rho)}\otimes_k e_{s(\rho)}X \xrightarrow{f} \bigoplus_{i=1}^{n} Ae_i\otimes_k e_i X \xrightarrow{g} X \longrightarrow 0$$

where $g(a\otimes x) = ax$ for $a\in Ae_i$, $x\in e_iX$, and

$f(a\otimes x) = a\rho\otimes x - a\otimes\rho x$ for $a\in Ae_{t(\rho)}$ and $x\in e_{s(\rho)}X$

in $s(\rho)$ $t(\rho)$ component.

PROOF. Clearly $g\circ f=0$ and g is onto. If ξ is an element of the middle term of the sequence, we can write it uniquely in the form

$$\xi = \sum_{i=1}^{n} \sum_{\text{paths } a \text{ with } s(a)=i} a\otimes x_a \qquad (x_a\in e_{s(a)}X \text{ almost all zero})$$

and define degree(ξ) = length of the longest path a with $x_a\neq 0$.

If a is a non-trivial path with $s(a)=i$, then we can express it as a product $a=a'\rho$ with ρ an arrow with $s(\rho)=i$, and a' another path. Viewing $a'\otimes x_a$ as an element in the ρ'th component of left hand term, we have

$$f(a'\otimes x_a) = a\otimes x_a - a'\otimes\rho x_a.$$

We claim that ξ + Im(f) always contains an element of degree 0. Namely, if ξ has degree $d>0$, then

$$\xi - f\Big(\sum_{i=1}^{n} \sum_{\text{paths } a \text{ with } s(a)=i \text{ and length } d} a'\otimes x_a\Big)$$

has degree < d, so the claim follows by induction.

Im(f)=Ker(g): If $\xi\in\mathrm{Ker}(g)$, let $\xi'\in\xi+\mathrm{Im}(f)$ have degree zero. Thus

$$0 = g(\xi) = g(\xi') = g(\textstyle\sum_i e_i\otimes x'_{e_i}) = \sum x'_{e_i}$$

Now this belongs to $\bigoplus_{i=1}^{n} X_i$, so each term in the final sum must be zero. Thus $\xi'=0$, and the assertion follows.

Ker(f)=0: we can write an element $\xi\in\mathrm{Ker}(f)$ in the form

$$\xi = \textstyle\sum_{\rho\in Q_1} \sum_{\text{paths } a \text{ with } s(a)=t(\rho)} a\otimes x_{\rho,a} \quad (x_{\rho,a}\in e_{s(\rho)}X \text{ almost all } 0).$$

Let a be a path of maximal length such that $x_{\rho,a}\neq 0$ (some ρ). Now

$$f(\xi) = \textstyle\sum_\rho\sum_a a\rho\otimes x_{\rho,a} - \sum_\rho\sum_a a\otimes\rho x_{\rho,a}$$

so the coefficient of $a\rho$ in $f(\xi)$ is $x_{\rho,a}$. A contradiction.

CONSEQUENCES.

(1) If X is a left A-module, then proj.dim $X \leq 1$, ie $\mathrm{Ext}^i(X,Y)=0$ $\forall Y, i\geq 2$.

PROOF. f and g are A-module maps and $Ae_i\otimes V$ is isomorphic to the direct sum of dim V copies of Ae_i, so is a projective left A-module. Thus the standard resolution is a projective resolution for X.

(2) A is <u>hereditary</u>, ie if $X\subseteq P$ with P projective, then X is projective.

PROOF. $\mathrm{Ext}^1(X,Y) \cong \mathrm{Ext}^2(P/X,Y) = 0$ $\forall Y$.

(3) If X,Y are f.d., then $\dim \mathrm{Hom}(X,Y) - \dim \mathrm{Ext}^1(X,Y) = \langle \underline{\dim}\, X, \underline{\dim}\, Y\rangle$.

PROOF. Apply $\mathrm{Hom}(-,Y)$ to the standard resolution:

$$0\longrightarrow \mathrm{Hom}(X,Y)\longrightarrow \mathrm{Hom}(\oplus_i Ae_i\otimes_k e_i X,Y)\longrightarrow \mathrm{Hom}(\oplus_\rho Ae_{t(\rho)}\otimes_k e_{s(\rho)} X,Y)\longrightarrow \mathrm{Ext}^1(X,Y)\longrightarrow 0.$$

Now $\dim \mathrm{Hom}(Ae_i\otimes e_j X,Y) = (\dim e_j X)(\dim \mathrm{Hom}(Ae_i,Y)) = (\underline{\dim}\, X)_j(\underline{\dim}\, Y)_i$.

(4) If X is f.d., then $\dim \mathrm{End}(X) - \dim \mathrm{Ext}^1(X,X) = q(\underline{\dim}\, X)$.

PROOF. Put $X=Y$ in (3).

REMARK.

Let i be a vertex in Q and suppose that either no arrows start at i, or no arrows terminate at i. Let Q' be the quiver obtained by reversing the direction of every arrow connected to i. We say that Q' is obtained from Q be <u>reflecting</u> at the vertex i. The two categories $\mathrm{Rep}(Q)$ and $\mathrm{Rep}(Q')$ are closely related, by means of so-called <u>reflection</u> functors. See I.N.Bernstein, I.M.Gelfand and V.A.Ponomarev, Coxeter functors and Gabriel's Theorem, Uspekhi Mat. Nauk. 28 (1973), 19-33, English Translation Russ. Math. Surveys, 28 (1973), 17-32.

§2. Bricks

In this section we consider finite dimensional left A-modules with A an hereditary k-algebra. In particular the results hold when A is a path algebra. We recall the Happel-Ringel Lemma and another lemma due to Ringel.

INDECOMPOSABLE MODULES.
Recall Fitting's Lemma, that X is indecomposable $\Leftrightarrow$ End(X) is a local ring, ie $\mathrm{End}(X) = k1_X + \mathrm{rad}\,\mathrm{End}(X)$, since the field k is algebraically closed.

Any module can be written as a direct sum of indecomposable modules, and by the Krull-Schmidt Theorem the isomorphism types of the summands and their multiplicities are uniquely determined.

We say that X is a brick if End(X)=k. Thus a brick is indecomposable.

LEMMA 1. Suppose X,Y are indecomposable. If $\mathrm{Ext}^1(Y,X)=0$ then any non-zero map $\theta:X\longrightarrow Y$ is mono or epi.

PROOF. We have exact sequences

$$\xi:0\longrightarrow \mathrm{Im}(\theta)\longrightarrow Y\longrightarrow \mathrm{Cok}(\theta)\longrightarrow 0 \quad\text{and}\quad \eta:0\longrightarrow \mathrm{Ker}(\theta)\longrightarrow X\longrightarrow \mathrm{Im}(\theta)\longrightarrow 0.$$

From $\mathrm{Ext}^1(\mathrm{Cok}(\theta),\eta)$ we get

$$\ldots \longrightarrow \mathrm{Ext}^1(\mathrm{Cok}(\theta),X) \xrightarrow{f} \mathrm{Ext}^1(\mathrm{Cok}(\theta),\mathrm{Im}(\theta)) \longrightarrow 0.$$

so $\xi = f(\zeta)$ for some ζ. Thus there is commutative diagram

$$\begin{array}{ccccccccc}
\zeta:0 & \longrightarrow & X & \xrightarrow{\alpha} & Z & \longrightarrow & \mathrm{Cok}(\theta) & \longrightarrow & 0 \\
 & & \beta\downarrow & & \gamma\downarrow & & \| & & \\
\xi:0 & \longrightarrow & \mathrm{Im}(\theta) & \xrightarrow{\delta} & Y & \longrightarrow & \mathrm{Cok}(\theta) & \longrightarrow & 0
\end{array}$$

Now the sequence

$$0 \longrightarrow X \xrightarrow{\begin{pmatrix}\alpha\\ \beta\end{pmatrix}} Z\oplus \mathrm{Im}(\theta) \xrightarrow{(\gamma\ \ -\delta)} Y \longrightarrow 0$$

is exact, so splits since $\mathrm{Ext}^1(Y,X)=0$.

If $\mathrm{Im}(\theta)\neq 0$ then X or Y is summand of $\mathrm{Im}(\theta)$ by Krull-Schmidt. But if θ is not mono or epi, then $\dim \mathrm{Im}(\theta) < \dim X, \dim Y$, a contradiction.

SPECIAL CASE. If X is indecomposable with no self-extensions (ie $\mathrm{Ext}^1(X,X)=0$), then X is a brick.

LEMMA 2. If X is indecomposable, not a brick, then X has a submodule and a quotient which are bricks with self-extensions.

PROOF. It suffices to prove that if X is indecomposable and not a brick then there is a proper submodule $U\subset X$ which is indecomposable and with self-extensions, for if U is not a brick one can iterate, and a dual argument deals with the case of a quotient.

Pick $\theta\in\mathrm{End}(X)$ with $I=\mathrm{Im}(\theta)$ of minimal dimension $\neq 0$. We have $I\leq\mathrm{Ker}(\theta)$, for X is indecomposable and not a brick so θ is nilpotent. Now $\theta^2=0$ by minimality. Let $\mathrm{Ker}(\theta) = \oplus_{i=1}^r K_i$ with K_i indecomposable, and pick j such that the composition $\alpha : I\hookrightarrow\mathrm{Ker}(\theta)\twoheadrightarrow K_j$ is non-zero. Now α is mono, for the map $X\twoheadrightarrow I\xrightarrow{\alpha}K_j\hookrightarrow X$ has image $\mathrm{Im}(\alpha)\neq 0$ so α mono by minimality.

We have $\mathrm{Ext}^1(I,K_j)\neq 0$, for otherwise the pushout

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \oplus_{i=1}^r K_i & \longrightarrow & X & \longrightarrow & I & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & K_j & \longrightarrow & Y & \longrightarrow & I & \longrightarrow & 0 \end{array}$$

splits, and it follows that K_j is summand of X, a contradiction. Now K_j has self-extensions since α induces an epi $\mathrm{Ext}^1(K_j,K_j)\twoheadrightarrow\mathrm{Ext}^1(I,K_j)$. Finally take $U=K_j$.

§3. The variety of representations

In this section Q is a quiver and A=kQ. We define the variety of representations of Q of dimension vector $\alpha\in\mathbb{N}^n$, and describe some elementary properties. We use elementary dimension arguments from algebraic geometry. The properties we need are listed below.

ALGEBRAIC GEOMETRY.
$\mathbb{A}^r$ is affine r-space with the Zariski topology. We consider locally closed subsets U in $\mathbb{A}^r$, ie subsets U which are open in their closure $\bar{U}$.

A non-empty locally closed subset U is irreducible if any non-empty subset of U which is open in U, is dense in U. The space $\mathbb{A}^r$ is irreducible.

The dimension of a non-empty locally closed subset U is

$$\sup\{n \mid \exists\ Z_0\subset Z_1\subset\ldots\subset Z_n \text{ irreducible subsets closed in } U\}.$$

We have dim U = dim $\bar{U}$; if $W=U\cup V$ then dim W = max{dim U,dim V}; the space $\mathbb{A}^r$ has dimension r.

If an algebraic group G acts on $\mathbb{A}^r$, then the orbits $\mathcal{O}$ are locally closed; $\bar{\mathcal{O}}\setminus\mathcal{O}$ is a union of orbits of dimension strictly smaller than dim $\mathcal{O}$; and if $x\in\mathcal{O}$ then $\dim\mathcal{O}=\dim G-\dim \mathrm{Stab}_G(x)$.

DEFINITIONS. Let Q be a quiver and $\alpha\in\mathbb{N}^n$. We define

$$\mathrm{Rep}(\alpha) = \prod_{\rho\in Q_1} \mathrm{Hom}_k(k^{s(\rho)},k^{t(\rho)}).$$

This is isomorphic to $\mathbb{A}^r$ where $r = \sum_{\rho\in Q_1} \alpha_{t(\rho)}\alpha_{s(\rho)}$.

An element $x\in\mathrm{Rep}(\alpha)$ gives a representation R(x) of Q with $R(x)_i = k^{\alpha_i}$ for $1\le i\le n$, and $R(x)_\rho = x_\rho$ for $\rho\in Q_1$.

We define $GL(\alpha) = \prod_{i=1}^n GL(\alpha_i,k)$. This is open in $\mathbb{A}^s$ where $s = \sum_{i=1}^n \alpha_i^{\,2}$.

THE ACTION.

$GL(\alpha)$ acts on $Rep(\alpha)$ by conjugation. Explicitly

$$(gx)_\rho = g_{t(\rho)} x_\rho g_{s(\rho)}^{-1}$$

for $g \in GL(\alpha)$ and $x \in Rep(\alpha)$.

If $x, y \in Rep(\alpha)$, then the set of A-module isomorphisms $R(x) \longrightarrow R(y)$ can be identified with $\{g \in GL(\alpha) \mid gx = y\}$. It follows that

(1) $Stab_{GL(\alpha)}(x) \cong Aut_A(R(x))$.

(2) There is a 1-1 correspondence between isoclasses of representations X with dimension vector α and orbits, given by $\mathcal{O}_X = \{x \in Rep(\alpha) \mid R(x) \cong X\}$. To see this we only need to realize that every representation of dimension vector α is isomorphic to some $R(x)$, which follows on choosing a basis.

REMARKS.

(1) <u>Invariant</u> <u>Theory</u> is about polynomial and rational maps $\phi : Rep(\alpha) \longrightarrow k$ which are constant on $GL(\alpha)$-orbits. For example, if $a = \rho_1 \ldots \rho_m$ is an oriented cycle, we have a polynomial invariant

$$f_a(x) = \mathrm{Trace}(x_{\rho_1} x_{\rho_2} \ldots x_{\rho_m}),$$

and more generally if $\chi_\theta(T)$ is the characteristic polynomial of θ, we have

$$f_{ai}(x) = \text{Coefficient of } T^i \text{ in } \chi_{x_{\rho_1} x_{\rho_2} \ldots x_{\rho_m}}(T).$$

(2) If char $k=0$, then any polynomial invariant can be expressed as a polynomial in the f_a. This has been proved by Sibirski and Procesi in case Q has only one vertex, and in general can be found in L.Le Bruyn & C.Procesi, Semisimple representations of quivers, Trans. Amer. Math. Soc. 317 (1990), 585-598.

(3) If char $k \geq 0$ and Q has only one vertex, any polynomial invariant can be expressed as a polynomial in the f_{ai}. This is recent work of S.Donkin. Presumably the restriction on Q is unnecessary.

LEMMA 1. dim Rep(α) - dim $\mathcal{O}_X$ = dim $\mathrm{End}_A(X)$ - $q(\alpha)$ = dim $\mathrm{Ext}^1(X,X)$.

PROOF. Say $X\cong R(x)$. We have

$$\dim \mathcal{O}_X = \dim GL(\alpha) - \dim \mathrm{Stab}(x) = \dim GL(\alpha) - \dim \mathrm{Aut}_A(X)$$

Now $GL(\alpha)$ is non-empty and open in $\mathbb{A}^s$, so dense, so dim $GL(\alpha) = s$.
Similarly $\mathrm{Aut}_A(X)$ is non-empty and open in $\mathrm{End}_A(X)$, so dense, so dim Aut(x) = dim End(X). The assertion follows.

CONSEQUENCES.

(1) If $\alpha\neq 0$ and $q(\alpha)\leq 0$, then there are infinitely many orbits in Rep(α).
PROOF. $\mathrm{End}_A(X)\neq 0$ so dim $\mathcal{O}_X$ < dim Rep(α).

(2) $\mathcal{O}_X$ is open $\Leftrightarrow$ X has no self-extensions.
PROOF. By the lemma, $\mathrm{Ext}^1(X,X)=0$ $\Leftrightarrow$ dim $\mathcal{O}_X$=dim Rep(α) $\Leftrightarrow$ dim $\overline{\mathcal{O}}_X$=dim Rep(α).
If dim $\overline{\mathcal{O}}_X$=dim Rep(α) then $\overline{\mathcal{O}}_X$=Rep(α), since a proper closed subset of an irreducible subset has strictly smaller dimension. Now $\mathcal{O}_X$ is open in Rep(α) since it is locally closed. Conversely, if $\mathcal{O}_X$ is open in Rep(α) then $\overline{\mathcal{O}}_X$=Rep(α) since Rep(α) is irreducible. Thus their dimensions are certainly equal.

(3) There is at most one module without self-extensions of dimension α (up to isomorphism).
PROOF. If $\mathcal{O}_X\neq\mathcal{O}_Y$ are open, then $\mathcal{O}_X\subseteq \mathrm{Rep}(\alpha)\setminus\mathcal{O}_Y$, and so $\overline{\mathcal{O}}_X\subseteq\mathrm{Rep}(\alpha)\setminus\mathcal{O}_Y$, which contradicts the irreducibility of Rep(α).

LEMMA 2. If $\xi:0\longrightarrow U\longrightarrow X\longrightarrow V\longrightarrow 0$ is a non-split exact sequence, then $\mathcal{O}_{U\oplus V} \subseteq \overline{\mathcal{O}}_X\setminus\mathcal{O}_X$.

PROOF. For each vertex i, identify U_i as a subspace of X_i. Choose bases of the U_i and extend to bases of X_i. Then $X\cong R(x)$ with

$$x_\rho = \begin{pmatrix} u_\rho & w_\rho \\ 0 & v_\rho \end{pmatrix}$$

with $U\cong R(u)$ and $V\cong R(v)$. For $0\neq\lambda\in k$ define $g_\lambda\in GL(\alpha)$ via $(g_\lambda)_\rho = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}$. Then

$$(g_\lambda x)_\rho = \begin{pmatrix} u_\rho & \lambda w_\rho \\ 0 & v_\rho \end{pmatrix}$$

so the closure of $\mathcal{O}_X$ contains the point with matrices

$$\begin{pmatrix} u_\rho & 0 \\ 0 & v_\rho \end{pmatrix}$$

which corresponds to $U \oplus V$.

Finally $\mathrm{Hom}(\xi, U)$ gives an exact sequence

$$0 \longrightarrow \mathrm{Hom}(V,U) \longrightarrow \mathrm{Hom}(X,U) \longrightarrow \mathrm{Hom}(U,U) \xrightarrow{f} \mathrm{Ext}^1(V,U),$$

so

$$\dim \mathrm{Hom}(V,U) - \dim \mathrm{Hom}(X,U) + \dim \mathrm{Hom}(U,U) - \dim \mathrm{Im}(f) = 0,$$

but $f(1_U)=\xi\neq 0$, so $\dim \mathrm{Hom}(X,U) \neq \dim \mathrm{Hom}(U\oplus V,U)$, and hence $X \not\cong U\oplus V$.

CONSEQUENCES.

(1) If $\mathcal{O}_X$ is an orbit in $\mathrm{Rep}(\alpha)$ of maximal dimension, and $X=U\oplus V$, then $\mathrm{Ext}^1_A(V,U)=0$.

PROOF. If there is non-split extension $0 \longrightarrow U \longrightarrow E \longrightarrow V \longrightarrow 0$ then $\mathcal{O}_X \subseteq \overline{\mathcal{O}}_E \setminus \mathcal{O}_E$, so $\dim \mathcal{O}_X < \dim \mathcal{O}_E$.

(2) If $\mathcal{O}_X$ is closed then X is semisimple.

REMARKS.

(1) Suppose Q has no oriented cycles. Let $z\in\mathrm{Rep}(\alpha)$ be the element with all matrices $z_\rho=0$. We can easily show that z is in the closure of every orbit, and it follows that there are no non-constant polynomial invariants. Moreover, an orbit $\mathcal{O}_X$ is closed $\Leftrightarrow$ X is semisimple, for the only semisimple module of dimension α is $R(z)$, and $\{z\}$ is clearly a closed orbit.

(2) If Q is allowed to have oriented cycles, $x,x'\in\mathrm{Rep}(\alpha)$ and $R(x)$ and $R(x')$ are non-isomorphic semisimple modules, then there is a polynomial invariant ϕ (of the form f_{ai}) with $\phi(x)\neq\phi(y)$. In case Q has only one vertex and char k=0 this is proved in §12.6 of M.Artin, On Azumaya algebras and finite dimensional representations of rings, J.Algebra 11 (1969), 532-563, but it seems to be true in general. It follows that $\mathcal{O}_X$ is closed $\Leftrightarrow$ X is semisimple.

§4. Dynkin and Euclidean diagrams

In this section we give the classification of graphs into Dynkin, Euclidean, and 'wild' graphs, and in the first two cases we study the corresponding root system.

DEFINITIONS.

Let Γ be finite graph with vertices $\{1,..,n\}$. We allows loops and multiple edges, so that Γ is given by any set of natural numbers

$$n_{ij} = n_{ji} = \text{the number of edges between } i \text{ and } j.$$

Let $q(\alpha) = \sum_{i=1}^{n} \alpha_i^2 - \sum_{i\le j} n_{ij}\alpha_i\alpha_j$

Let $(-,-)$ be the symmetric bilinear form on $\mathbb{Z}^n$ with

$$(\varepsilon_i,\varepsilon_j) = \begin{cases} -n_{ij} & (i\neq j) \\ 2 - 2n_{ii} & (i=j) \end{cases}$$

where ε_i is the i^{th} coordinate vector.

Note that knowledge of any one of Γ, q or $(-,-)$ determines the others, since $q(\alpha) = \frac{1}{2}(\alpha,\alpha)$ and $(\alpha,\beta) = q(\alpha+\beta)-q(\alpha)-q(\beta)$.

If Q is a quiver and Γ is its underlying graph, then $(-,-)$ and q are the same as before. The bilinear form $\langle -,-\rangle$, however, depends on the orientation of Q.

We say q is <u>positive</u> <u>definite</u> if $q(\alpha)>0$ for all $0\neq\alpha\in\mathbb{Z}^n$.

We say q is <u>positive</u> <u>semi-definite</u> if $q(\alpha)\ge 0$ for all $\alpha\in\mathbb{Z}^n$.

The <u>radical</u> of q is $\mathrm{rad}(q) = \{\alpha\in\mathbb{Z}^n \mid (\alpha,-) = 0\}$.

We have a partial ordering on $\mathbb{Z}^n$ given by $\alpha\le\beta$ if $\beta-\alpha \in \mathbb{N}^n$.

We say that $\alpha\in\mathbb{Z}^n$ is <u>sincere</u> if each component is non-zero.

LEMMA. If Γ is connected and $\beta\ge 0$ is a non-zero radical vector, then β is sincere and q is positive semi-definite. For $\alpha\in\mathbb{Z}^n$ we have

$$q(\alpha)=0 \Leftrightarrow \alpha\in\mathbb{Q}\beta \Leftrightarrow \alpha\in\mathrm{rad}(q).$$

PROOF. By assumption $0 = (\varepsilon_i,\beta) = (2-2n_{ii})\beta_i - \sum_{j\neq i} n_{ij}\,\beta_j$.

If $\beta_i=0$ then $\sum_{j\neq i} n_{ij}\,\beta_j = 0$, and since each term in ≥ 0 we have $\beta_j=0$ whenever there is an edge i—j. Since Γ is connected it follows that $\beta=0$, a contradiction. Thus β is sincere. Now

$$\sum_{i<j} n_{ij}\,\frac{\beta_i\beta_j}{2}\left(\frac{\alpha_i}{\beta_i}-\frac{\alpha_j}{\beta_j}\right)^2$$

$$= \sum_{i<j} n_{ij}\,\frac{\beta_j}{2\beta_i}\,\alpha_i^2 \;-\; \sum_{i<j} n_{ij}(-\alpha_i\alpha_j) \;+\; \sum_{i<j} n_{ij}\,\frac{\beta_i}{2\beta_j}\,\alpha_j^2$$

$$= \sum_{i\neq j} n_{ij}\,\frac{\beta_j}{2\beta_i}\,\alpha_i^2 \;+\; \sum_{i<j} n_{ij}\alpha_i\alpha_j$$

$$= \sum_i (2-2n_{ii})\beta_i\,\frac{1}{2\beta_i}\,\alpha_i^2 \;+\; \sum_{i<j} n_{ij}\alpha_i\alpha_j = q(\alpha).$$

It follows that q is positive semi-definite. If $q(\alpha)=0$ then $\alpha_i/\beta_i=\alpha_j/\beta_j$ whenever there is an edge i——j, and since Γ is connected it follows that $\alpha\in\mathbb{Q}\beta$. If $\alpha\in\mathbb{Q}\beta$ then $\alpha\in\mathrm{rad}(q)$ since $\beta\in\mathrm{rad}(q)$ by assumption. Finally if $\alpha\in\mathrm{rad}(q)$ then certainly $q(\alpha)=0$.

CLASSIFICATION. Suppose Γ is connected.

(1) If Γ is Dynkin then q is positive definite. By definition the Dynkin diagrams are:

A_n •—•—• ... —• D_n •—•—• ... —• (two end vertices joined to the second vertex)

E_6 •—•—•—•—• (with • above the middle vertex) E_7 •—•—•—•—•—• (with • above the third vertex) E_8 •—•—•—•—•—•—• (with • above the third vertex)

(2) If Γ is Euclidean, then q is positive semi-definite and $\mathrm{rad}(q)=\mathbb{Z}\delta$. By definition the Euclidean diagrams are as below. We have marked each vertex i with the value of δ_i. Note that δ is sincere and $\delta\geq 0$.

$\tilde{A}_m$ (cycle: 1—1— … —1—1—1— … —1—1) $(m\geq 0)$ $\tilde{D}_m$ 1,1 — 2—2 ··· —2 — 1,1 $(m\geq 4)$ $(n=m+1$ vertices$)$

$\tilde{E}_6$ 1—2—3—2—1 (with 2—1 above the 3) $\tilde{E}_7$ 1—2—3—4—3—2—1 (with 2 above the 4) $\tilde{E}_8$ 2—4—6—5—4—3—2—1 (with 3 above the 6)

Note that $\tilde{A}_0$ has one vertex and one loop, and $\tilde{A}_1$ has two vertices joined by two edges.

(3) Otherwise, there is a vector $\alpha\geq 0$ with $q(\alpha)<0$ and $(\alpha,\varepsilon_i)\leq 0$ for all i.

PROOF.

(2) By inspection the given vector δ is radical, eg if there are no loops or multiple edges, we need to check that

$$2\delta_i = \sum_{\text{neighbours } j \text{ of } i} \delta_j.$$

Now q is positive semi-definite by the lemma. Finally, since some $\delta_i=1$,

$$\mathrm{rad}(q) = \mathbb{Q}\delta \cap \mathbb{Z}^n = \mathbb{Z}\delta.$$

(1) Embed the Dynkin diagram in the corresponding Euclidean diagram $\tilde{\Gamma}$, and note that the quadratic form for $\tilde{\Gamma}$ is strictly positive on non-zero, non-sincere vectors.

(3) It is not hard to show that Γ has a Euclidean subgraph Γ', say with radical vector δ. If all vertices of Γ are in Γ' take $\alpha=\delta$. If i is a vertex not in Γ', connected to Γ' by an edge, take $\alpha=2\delta+\varepsilon_i$

EXTENDING VERTICES.

If Γ is Euclidean, a vertex e is called an <u>extending</u> vertex if $\delta_e=1$. Note

(1) There always is an extending vertex.

(2) The graph obtained by deleting e is the corresponding Dynkin diagram.

NOW SUPPOSE that Γ is Dynkin or Euclidean, so q is positive semi-definite.

ROOTS.

We define $\Delta = \{\alpha\in\mathbb{Z}^n \mid \alpha\neq 0,\ q(\alpha)\leq 1\}$, the set of <u>roots</u>.

A root α is <u>real</u> if $q(\alpha)=1$ and <u>imaginary</u> if $q(\alpha)=0$.

REMARK.

One can define roots for any graph Γ, and more generally for valued graphs (in which situation the Dynkin diagrams B_n, C_n, F_4, G_2 also arise). In case the graph has no loops, this can be found in Kac's book on infinite dimensional Lie algebras. In case there are loops, the definition can be found in V.G.Kac, Some remarks on representations of quivers and infinite root systems, in Springer Lec. Notes 832.

PROPERTIES.

(1) Each ε_i is a root.

(2) If $\alpha\in\Delta\cup\{0\}$, so are $-\alpha$ and $\alpha+\beta$ with $\beta\in\mathrm{rad}(q)$.

PROOF. $q(\beta\pm\alpha) = q(\beta)+q(\alpha)\pm(\beta,\alpha) = q(\alpha)$.

(3) $\{\text{imaginary roots}\} = \begin{cases} \varnothing & \text{(Dynkin)} \\ \{r\delta \mid 0\neq r\in\mathbb{Z}\} & \text{(Euclidean)} \end{cases}$

PROOF. Use the lemma.

(4) Every root α is positive or negative.

PROOF. Let $\alpha = \alpha^+-\alpha^-$ where $\alpha^+,\alpha^-\geq 0$ are non-zero and have disjoint support. Clearly we have $(\alpha^+,\alpha^-)\leq 0$, so that

$$1 \geq q(\alpha) = q(\alpha^+) + q(\alpha^-) - (\alpha^+,\alpha^-) \geq q(\alpha^+) + q(\alpha^-).$$

Thus one of α^+,α^- is an imaginary root, and hence is sincere. This means that the other is zero, a contradiction.

(5) If Γ is Euclidean then $(\Delta\cup\{0\})/\mathbb{Z}\delta$ is finite.

PROOF. Let e be an extending vertex. If α is a root with $\alpha_e=0$, then $\delta-\alpha$ and $\delta+\alpha$ are roots which are positive at the vertex e, and hence are positive roots. Thus

$$\{\alpha\in\Delta\cup\{0\}|\alpha_e=0\} \subseteq \{\alpha\in\mathbb{Z}^n|-\delta\leq\alpha\leq\delta\}$$

which is finite. Now if $\beta\in\Delta\cup\{0\}$ then $\beta-\beta_e\delta$ belongs to the finite set $\{\alpha\in\Delta\cup\{0\}|\alpha_e=0\}$.

(6) If Γ is Dynkin then Δ is finite.

PROOF. Embed Γ in the corresponding Euclidean graph $\tilde{\Gamma}$ with extending vertex e. We can now view a root α for Γ as a root for $\tilde{\Gamma}$ with $\alpha_e=0$, so the result follows from (5).

§5. Finite representation type

In this section we combine almost everything that we have done so far in order to prove Gabriel's Theorem. The proof given here is due to J.Tits, P.Gabriel, and the key step to C.M.Ringel, Four papers on problems in linear algebra, in I.M.Gelfand, 'Representation Theory', London Math. Soc. Lec. Note Series 69 (1982).

THEOREM 1. Suppose Q is a quiver with underlying graph Γ Dynkin. The assignment $X \longmapsto \underline{\dim}\, X$ induces a bijection between the isoclasses of indecomposable modules and the positive roots of q.

PROOF.
If X is indecomposable, then X is a brick, for otherwise by §2 Lemma 2 there is $Y \subseteq X$ a brick with self-extensions, and then

$$0 < q(\underline{\dim}\, Y) = \dim \mathrm{End}(Y) - \dim \mathrm{Ext}^1(Y,Y) \leq 0.$$

If X is indecomposable then it has no self-extensions and $\underline{\dim}\, X$ is a positive root, for $0 < q(\underline{\dim}\, X) = 1 - \dim \mathrm{Ext}^1(X,X)$.

If X,X' are two indecomposables with the same dimension vector, then $X \cong X'$ by §3 Lemma 1.

If α is a positive root, then there is an indecomposable X with $\underline{\dim}\, X = \alpha$. To see this, pick an orbit $\mathcal{O}_X$ of maximal dimension in $\mathrm{Rep}(\alpha)$. If X decomposes, $X = U \oplus V$ then $\mathrm{Ext}^1(U,V) = \mathrm{Ext}^1(V,U) = 0$ by §3 Lemma 2. Thus

$$\begin{aligned} 1 = q(\alpha) &= q(\underline{\dim}\, U) + q(\underline{\dim}\, V) + \langle \underline{\dim}\, U, \underline{\dim}\, V \rangle + \langle \underline{\dim}\, V, \underline{\dim}\, U \rangle \\ &= q(\underline{\dim}\, U) + q(\underline{\dim}\, V) + \dim \mathrm{Hom}(U,V) + \dim \mathrm{Hom}(V,U) \geq 2, \end{aligned}$$

a contradiction.

THEOREM 2. If Q is a connected quiver with graph Γ, then there are only finitely many indecomposable representations $\Leftrightarrow$ Γ is Dynkin.

PROOF. If Γ is Dynkin then the indecomposables correspond to the positive roots, and there are only a finite number of roots.

Conversely, suppose there are only a finite number of indecomposables. Any module is a direct sum of indecomposables, so it follows that there are only finitely many isoclasses of modules of dimension α for all $\alpha\in\mathbb{N}^n$. Thus there are only finitely many orbits in $\mathrm{Rep}(\alpha)$. By §3 Lemma 1 we have $q(\alpha)>0$ for $0\neq\alpha\in\mathbb{N}^n$. Now the classification of graphs shows that Γ is Dynkin.

§6. More homological algebra

FROM NOW ON we suppose that Q is a quiver without oriented cycles, so the path algebra $A=kQ$ is finite dimensional. We still consider f.d. A-modules. We study the properties of projective, injective, non-projective, and non-injective modules. We give a little bit of Auslander-Reiten theory.

DUALITIES.

(1) If X is a left or right A-module, then $DX = \mathrm{Hom}_k(X,k)$, $\mathrm{Hom}(X,A)$ and $\mathrm{Ext}^1(X,A)$ are all A-modules on the other side.

(2) D is duality between left and right A-modules.
PROOF. $\mathrm{Hom}(X,Y)\cong\mathrm{Hom}(DY,DX)$ and $DDX\cong X$.

(3) D gives a duality between injective left modules and projective right modules.
PROOF. $\mathrm{Ext}^1(DX,DY) \cong \mathrm{Ext}^1(Y,X)$. This is zero for all Y if and only if DX is projective, if and only if X is injective.

(4) $\mathrm{Hom}(-,A)$ gives a duality between projective left modules and projective right modules.
PROOF. If P is a summand of A^n then $\mathrm{Hom}(P,A)$ is a summand of $\mathrm{Hom}(A^n,A)\cong A^n$, so is projective. Now the map $P\longrightarrow\mathrm{Hom}(\mathrm{Hom}(P,A),A)$ is an iso for all P, since it is for $P=A$.

(5) The <u>Nakayama</u> <u>functor</u> $\nu(-) = D\mathrm{Hom}(-,A)$ gives an equivalence from projective left modules to injective left modules. The inverse functor is $\nu^{-}(-) = \mathrm{Hom}(D(-),A) \cong \mathrm{Hom}(DA,-)$.

(6) $\mathrm{Hom}(X,\nu P) \cong D\mathrm{Hom}(P,X)$ for X,P left A-modules, P projective.
PROOF. The composition

$$\mathrm{Hom}(P,A)\otimes_A X \cong \mathrm{Hom}(P,A)\otimes_A \mathrm{Hom}(A,X) \longrightarrow \mathrm{Hom}(P,X)$$

is an isomorphism, since it is for $P=A$. Thus

$$D\mathrm{Hom}(P,X) \cong \mathrm{Hom}_k(\mathrm{Hom}(P,A)\otimes_A X,k) \cong \mathrm{Hom}(X,\mathrm{Hom}_k(\mathrm{Hom}(P,A),k)) = \mathrm{Hom}(X,\nu P).$$

DEFINITION. The Auslander-Reiten translate of a left A-module X is $\tau X = D\mathrm{Ext}^1(X,A)$. We also define $\tau^- X = \mathrm{Ext}^1(DX,A) \cong \mathrm{Ext}^1(DA,X)$.

If $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ is an exact sequence then since A is hereditary there are long exact sequences

$$0 \longrightarrow \tau L \longrightarrow \tau M \longrightarrow \tau N \longrightarrow \nu L \longrightarrow \nu M \longrightarrow \nu N \longrightarrow 0$$

$$0 \longrightarrow \nu^- L \longrightarrow \nu^- M \longrightarrow \nu^- N \longrightarrow \tau^- L \longrightarrow \tau^- M \longrightarrow \tau^- N \longrightarrow 0.$$

LEMMA 1. $\mathrm{Hom}(Y,\tau X) \cong D\mathrm{Ext}^1(X,Y) \cong \mathrm{Hom}(\tau^- Y,X)$.
(Thus τ^- is left adjoint to τ)

PROOF. Let $0 \longrightarrow P \longrightarrow Q \longrightarrow X \longrightarrow 0$ be a projective resolution. The sequence

$$\tau Q \longrightarrow \tau X \longrightarrow \nu P \longrightarrow \nu Q$$

is exact, and $\tau Q=0$, so we have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(Y,\tau X) \arrow[r] & \mathrm{Hom}(Y,\nu P) \arrow[r] \arrow[d, equal] & \mathrm{Hom}(Y,\nu Q) \arrow[d, equal] \\
0 \arrow[r] & D\mathrm{Ext}^1(X,Y) \arrow[r] & D\mathrm{Hom}(P,Y) \arrow[r] & D\mathrm{Hom}(Q,Y)
\end{tikzcd}$$

and hence $\mathrm{Hom}(Y,\tau X) \cong D\mathrm{Ext}^1(X,Y)$. The other isomorphism is dual.

LEMMA 2. Let X be indecomposable.
(1) If X is non-projective then $\mathrm{Hom}(X,P)=0$ for P projective, and $\tau^-\tau X \cong X$.
(2) If X is non-injective then $\mathrm{Hom}(I,X)=0$ for I injective, and $\tau\tau^- X \cong X$.

PROOF OF (1). If $\theta: X \longrightarrow P$ is non-zero, then $\mathrm{Im}(\theta)$ is projective since A is hereditary. Now $X \twoheadrightarrow \mathrm{Im}(\theta)$ is epi, so $\mathrm{Im}(\theta)$ is summand of X. But X is indecomposable so $X \cong \mathrm{Im}(\theta)$, a contradiction.

Let $0 \longrightarrow P \longrightarrow Q \longrightarrow X \longrightarrow 0$ be a projective resolution. Now

$$0 \longrightarrow \tau X \longrightarrow \nu P \longrightarrow \nu Q \longrightarrow \nu X$$

is exact, and $\nu X=0$ since $\mathrm{Hom}(X,A)=0$. Thus we have a commutative diagram

$$\begin{tikzcd}
\nu^- \nu P \arrow[r] \arrow[d, equal] & \nu^- \nu Q \arrow[r] \arrow[d, equal] & \tau^- \tau X \arrow[r] & \tau^- \nu P \\
P \arrow[r] & Q \arrow[r] & X \arrow[r] & 0
\end{tikzcd}$$

with exact rows. Since νP is injective, $\tau^- \nu P=0$, and hence $\tau^- \tau X \cong X$.

LEMMA 3. τ and τ^- give inverse bijections

$$\text{non-projective indecomposables} \underset{\tau^-}{\overset{\tau}{\rightleftarrows}} \text{non-injective indecomposables}$$

PROOF. Let X be a non-projective indecomposable, and write τX as a direct sum of indecomposables, say $\tau X = \oplus_{i=1}^{r} Y_i$. Each Y_i is non-injective, since otherwise $\mathrm{Hom}(Y_i,\tau X)=0$ by Lemma 1. By part (2) of Lemma 2 it follows that each $\tau^-(Y_i)\neq 0$. By part (1) of Lemma 2 we have $X \cong \tau^-\tau X \cong \oplus_{i=1}^{r}\tau^-(Y_i)$, and since X is indecomposable we must have r=1. Thus τX is a non-injective indecomposable. Dually for τ^-.

REMARKS.
(1) For any f.d. algebra there are more complicated constructions τ,τ^- giving the bijection above, which involve D and a <u>transpose</u> operator Tr. In general, however, τ and τ^- are not functors, Lemma 1 needs to be modified, and Lemma 2 is nonsense.

(2) If X is indecomposable and non-projective, then $\mathrm{Ext}^1(X,\tau X) \cong D\mathrm{End}(X)$, and this space contains a special element, the map $f\in \mathrm{Hom}_k(\mathrm{End}(X),k)$ with $f(1_X)=1$ and $f(\mathrm{rad}\,\mathrm{End}(X))=1$. The corresponding short exact sequence $0\longrightarrow\tau X\longrightarrow E\longrightarrow X\longrightarrow 0$ is an <u>Auslander-Reiten</u> <u>sequence</u>, which has very special properties.

Auslander-Reiten sequences exist for any f.d. algebra, and (under the name 'almost split sequences' and together with the transpose) have been defined and studied by M.Auslander & I.Reiten, Representation theory of artin algebras III,IV,V,VI, Comm. in Algebra, 3(1975) 239-294, 5(1977) 443-518, 5(1977) 519-554, 6(1978) 257-300.

(3) The translate τ can also be defined as a product of reflection functors, see the remark in §1 and the paper by Bernstein, Gelfand and Ponomarev. The equivalence of the two definition was proved by S.Benner and M.C.R.Butler, The equivalence of certain functors occuring in the representation theory of artin algebras and species, J. London Math. Soc., 14 (1976), 183-187.

INDECOMPOSABLE PROJECTIVES AND INJECTIVES.

(1) The modules $P(i) = Ae_i$ are a complete set of non-isomorphic indecomposable projective left A-modules.

PROOF. The e_i are inequivalent primitive idempotents and $A = \oplus_{i=1}^{n} Ae_i$. Now use Krull-Schmidt.

(2) The modules $I(i) = \nu(P(i)) = D(e_i A)$ are a complete set of non-isomorphic indecomposable injective left A-modules.

PROOF. Use Hom(-,A) and D.

(3) $<\underline{\dim}\, P(i),\alpha> = \alpha_i = <\alpha,\underline{\dim}\, I(i)>$ for any α.

PROOF. If X has dimension α, then

$$<\underline{\dim}\, P(i),\alpha> = \dim \mathrm{Hom}(P(i),X) - \dim \mathrm{Ext}^1(P(i),X) = \dim e_i X = \alpha_i.$$
$$<\alpha,\underline{\dim}\, I(i)> = \dim \mathrm{Hom}(X,I(i)) = \dim \mathrm{Hom}(P(i),X) = \alpha_i.$$

(4) The vectors $\underline{\dim}\, P(i)$ are a basis of $\mathbb{Z}^n$. The $\underline{\dim}\, I(i)$ are a basis of $\mathbb{Z}^n$.

PROOF. The module S(i) with dimension vector ε_i has a projective resolution $0\longrightarrow P_1\longrightarrow P_0\longrightarrow S(i)\longrightarrow 0$ and an injective resolution $0\longrightarrow S(i)\longrightarrow I_0\longrightarrow I_1\longrightarrow 0$.

COXETER TRANSFORMATION.

(1) There is an automorphism $c:\mathbb{Z}^n\longrightarrow\mathbb{Z}^n$ with $\underline{\dim}\, \nu P = - c(\underline{\dim}\, P)$ for P projective.

PROOF. Define c via $c(\underline{\dim}\, P(i)) = - \underline{\dim}\, I(i)$.

(2) If X is indecomposable and non-projective then $\underline{\dim}\, \tau X = c(\underline{\dim}\, X)$.

PROOF. Let $0\longrightarrow P\longrightarrow Q\longrightarrow X\longrightarrow 0$ be a projective resolution. We have an exact sequence $0\longrightarrow \tau X\longrightarrow \nu P\longrightarrow \nu Q\longrightarrow 0$ and so

$$\underline{\dim}\, \tau X = \underline{\dim}\, \nu P - \underline{\dim}\, \nu Q = -c(\underline{\dim}\, P - \underline{\dim}\, Q) = -c(\underline{\dim}\, X).$$

(3) $<\alpha,\beta> = -<\beta,c\alpha> = <c\alpha,c\beta>$.

PROOF. $<\underline{\dim}\, P(i),\beta> = <\beta,\underline{\dim}\, I(i)> = -<\beta,c(\underline{\dim}\, P(i))>$.

(4) $c\alpha=\alpha \Leftrightarrow \alpha\in \mathrm{rad}(q)$.

PROOF. $<\beta,\alpha-c\alpha>=<\beta,\alpha>-<\beta,c\alpha>=(\beta,\alpha)$.

REMARK. When τ is written as a product of reflections, one sees that the Coxeter transformation is a <u>Coxeter</u> <u>element</u> in the sense of Coxeter groups.

§7. Euclidean case. Preprojectives and preinjectives

FROM NOW ON we set $A=kQ$ where Q is a quiver without oriented cycles and with underlying graph Γ Euclidean. We denote by δ the minimal positive imaginary root for Γ. In this section we describe the three classes of preprojective, regular and preinjective modules.

DEFINITIONS. If X is indecomposable, then

(1) X is <u>preprojective</u> $\Leftrightarrow \tau^{i}X=0$ for $i>>0$ $\Leftrightarrow X=\tau^{-m}P(j)$ some $m\geq 0$, j.

(2) X is <u>preinjective</u> $\Leftrightarrow \tau^{-i}X=0$ for $i>>0$ $\Leftrightarrow X=\tau^{m}I(j)$ some $m\geq 0$, j.

(3) X is <u>regular</u> $\Leftrightarrow \tau^{i}X\neq 0$ for all $i\in\mathbb{Z}$.

We say a decomposable module X is preprojective, preinjective or regular if each indecomposable summand is.

The <u>defect</u> of a module X is $\langle\delta,\underline{\dim}\, X\rangle = -\langle\underline{\dim}\, X,\delta\rangle$.

LEMMA 1. There is $N>0$ such that $c^{N}\underline{\dim}\, X = \underline{\dim}\, X$ for regular X.

PROOF. Recall that $c\alpha=\alpha$ if and only if α is radical, and that $q(c\alpha)=q(\alpha)$. Thus c induces a permutation of the finite set $\Delta\cup\{0\}/\mathbb{Z}\delta$. Thus there is some $N>0$ with c^{N} the identity on $\Delta\cup\{0\}/\mathbb{Z}\delta$. Since $\varepsilon_{i}\in\Delta$ it follows that c^{N} is the identity on $\mathbb{Z}^{n}/\mathbb{Z}\delta$.

Let $c^{N}\underline{\dim}\, X - \underline{\dim}\, X = r\delta$. An induction shows that $c^{iN}\underline{\dim}\, X = \underline{\dim}\, X + ir\delta$ for all $i\in\mathbb{Z}$. If $r<0$ this is not positive for $i>>0$, so X must be preprojective. If $r>0$ this is not positive for $i<<0$, so X is preinjective. Thus $r=0$.

LEMMA 2. If X is indecomposable, then X is preprojective, regular or preinjective according as the defect of X is -ve, zero or +ve.

PROOF. If X is preprojective then defect < 0, since

$$\langle \underline{\dim}\ \tau^{-m}P(j),\delta\rangle=\langle c^{-m}(\underline{\dim}\ P(j)),\delta\rangle=\langle \underline{\dim}\ P(j),c^{m}\delta\rangle=\langle \underline{\dim}\ P(j),\delta\rangle=\delta_i>0.$$

Similarly preinjectives have defect > 0. If X is regular with dimension vector α, then $c^N\alpha=\alpha$. Let $\beta=\alpha+..+c^{N-1}\alpha$. Clearly $c\beta=\beta$, so that $\beta=r\delta$. Now

$$0 = \langle\beta,\delta\rangle = \sum_{i=0}^{N-1} \langle c^i\alpha,\delta\rangle = N\langle\alpha,\delta\rangle,$$

so $\langle\alpha,\delta\rangle=0$, ie X has defect zero.

LEMMA 3. Let X,Y be indecomposable.
(1) If Y is preprojective and X is not, then Hom(X,Y)=0 and $\mathrm{Ext}^1(Y,X)=0$.
(2) If Y is preinjective and X is not, then Hom(Y,X)=0 and $\mathrm{Ext}^1(X,Y)=0$.

PROOF. (1) As X is not preprojective, $X\cong\tau^{-i}\tau^{i}X$ for $i\geq 0$. Thus

$$\mathrm{Hom}(X,Y) \cong \mathrm{Hom}(\tau^{-i}\tau^{i}X,Y) \cong \mathrm{Hom}(\tau^{i}X,\tau^{i}Y) = 0 \text{ for } i>>0.$$

Also $\mathrm{Ext}^1(Y,X) \cong D\mathrm{Hom}(\tau^{-}X,Y) = 0$. (2) is dual.

REMARK. We draw a picture

preprojectives	regulars	preinjectives

by drawing a dot for each indecomposable module. We draw the projectives at the extreme left, then the modules $\tau^{-}P(j)$, then the $\tau^{-2}P(j)$, etc. We draw the injectives at the extreme right, then the modules $\tau I(j)$, then $\tau^2 I(j)$, etc. Finally we draw all the regular indecomposables in the middle.

The lemma above, and §6 Lemma 2, say that non-zero maps tend to go from the left to the right in the picture.

LEMMA 4. If α is a positive real root, and either $<\alpha,\delta>\neq 0$ or $\alpha\leq\delta$, then there is a unique indecomposable of dimension α. It is a brick.

PROOF. If Y is a brick with self-extensions then $q(\underline{\dim}\ Y)\leq 0$ so Y is regular and of dimension $\geq\delta$.

If X is indecomposable of dimension α, then it is a brick, for otherwise it has submodule and quotient which are regular of dimension $\geq\delta$. This is impossible for either X has dimension $\alpha\leq\delta$, or X is preprojective (so there is no such submodule), or it is preinjective (so there is no such quotient). By assumption $q(\alpha)=1$, so X has no self-extensions, and the uniqueness follows by the open orbit argument.

For the existence of an indecomposable of dimension vector α, pick an orbit $\mathcal{O}_X$ in $\mathrm{Rep}(\alpha)$ of maximal dimension. If X decomposes, $X=U\oplus V$, then

$$1 = q(\alpha) = q(\underline{\dim}\ U) + q(\underline{\dim}\ V) + \dim \mathrm{Hom}(U,V) + \dim \mathrm{Hom}(V,U).$$

Thus, $q(\underline{\dim}\ U)=0$, say, so $\underline{\dim}\ U\in\mathbb{Z}\delta$. Now $\underline{\dim}\ V\notin\mathbb{Z}\delta$ for otherwise $\underline{\dim}\ X\in\mathbb{Z}\delta$ and then $q(\alpha)=0$. Thus $q(\underline{\dim}\ V)=1$ and therefore the Hom spaces must be zero. Thus $<\underline{\dim}\ V,\underline{\dim}\ U>=0$, so $<\underline{\dim}\ V,\delta>=0$. Since also $<\underline{\dim}\ U,\delta>=0$ we have $<\alpha,\delta>=0$. Now $\underline{\dim}\ U\in\mathbb{Z}\delta$, so $\delta\leq\alpha$, which contradicts the assumption on α.

§8. Euclidean case. Regular modules

In this section we study the category of regular modules. We show that its behaviour is completely determined by certain 'regular simple' modules.

PROPERTIES OF REGULAR MODULES.

(1) If $\theta:X\longrightarrow Y$ with X,Y regular, then Im(θ) is regular.

PROOF. Im(θ)⊆Y, so it has no preinjective summand. Also $X\twoheadrightarrow \mathrm{Im}(\theta)$, so it has no preprojective summand.

(2) In the situation above Ker(θ) and Coker(θ) are also regular.

PROOF. $0\longrightarrow \mathrm{Ker}(\theta)\longrightarrow X\longrightarrow \mathrm{Im}(\theta)\longrightarrow 0$ is exact, so Ker(θ) has defect zero. Now Ker(θ)⊆X, so Ker(θ)=preprojectives⊕regulars. If there were any preprojective summand, then the defect would have to be negative. Similarly for Coker(θ).

(3) If $0\longrightarrow X\longrightarrow Y\longrightarrow Z\longrightarrow 0$ is exact and X,Z are regular, then so is Y.

PROOF. The long exact sequence shows Hom(Z,Preproj) = 0 = Hom(Preinj,Z).

(4) The regular modules form an extension-closed abelian subcategory of the category of all modules.

(5) τ and τ^- are inverse equivalences on this category.

DEFINITION.

A module X is <u>regular simple</u> if it is regular, and has no proper non-zero regular submodule. Equivalently if defect(X)=0, and defect(Y)<0 ∀ 0<Y<X.

PROPERTIES. Let X be regular simple, $\underline{\dim}\, X = \alpha$.

(1) X is a brick, so α is a root.

(2) $\tau^i X$ is regular simple for all $i \in \mathbb{Z}$.

(3) $\tau X \cong X \Leftrightarrow \alpha$ is an imaginary root.

PROOF. If $\tau X \cong X$ then $c\alpha = \alpha$ so α is radical. Conversely, if $q(\alpha)=0$, then $\mathrm{Hom}(X,\tau X) \cong D\mathrm{Ext}^1(X,X) \neq 0$, so $X \cong \tau X$ since X and τX regular simple.

(4) $\tau^N X \cong X$.

PROOF. We may assume α is a real root. Now $\langle\alpha, c^N\alpha\rangle = \langle\alpha,\alpha\rangle = 1$, so $\mathrm{Hom}(X,\tau^N X) \neq 0$, so $X \cong \tau^N X$.

DEFINITION.

X is <u>regular</u> <u>uniserial</u> if there are regular submodules

$$0 = X_0 \subset X_1 \subset \ldots \subset X_r = X$$

and these are the ONLY regular submodules of X. We say X has <u>regular</u> <u>composition</u> <u>factors</u> $X_1, X_2/X_1, \ldots, X_r/X_{r-1}$ (which are clearly regular simples), <u>regular</u> <u>length</u> r, <u>regular</u> <u>socle</u> X_1 and <u>regular</u> <u>top</u> X/X_{r-1}.

LEMMA 1. If X is regular uniserial, S is regular simple, and $\xi: 0 \longrightarrow S \longrightarrow E \xrightarrow{f} X \longrightarrow 0$ is non-split, then E is regular uniserial.

PROOF. It suffices to prove that if $U \subseteq E$ is regular and U is not contained in S, then $S \subseteq U$. Thus $f(U) \neq 0$, so $T \subseteq f(U)$ where T is the regular socle of X, and so $f^{-1}(T) = S + U \cap f^{-1}(T)$.

Since $\tau^- S$ is regular simple the inclusion $T \hookrightarrow X$ gives an isomorphism $\mathrm{Hom}(\tau^- S, T) \longrightarrow \mathrm{Hom}(\tau^- S, X)$. Thus it gives an isomorphism

$$\mathrm{Ext}^1(X,S) \cong D\mathrm{Hom}(\tau^- S, X) \cong D\mathrm{Hom}(\tau^- S, T) \cong \mathrm{Ext}^1(T,S),$$

so the pullback sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S & \longrightarrow & f^{-1}(T) & \longrightarrow & T & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & S & \longrightarrow & E & \longrightarrow & X & \longrightarrow & 0 \end{array}$$

is non-split. Now we have $f^{-1}(T) = S + U \cap f^{-1}(T)$, and this cannot be a direct sum, so $S \cap U \cap f^{-1}(T) \neq 0$. It follows that $S \subseteq U$.

LEMMA 2. For each regular simple T and $r\geq 1$ there is a unique regular uniserial module with regular top T and regular length r. Its regular composition factors are (from the top) $T, \tau T, \ldots, \tau^{r-1}T$.

PROOF. Induction on r. Suppose X is regular uniserial of regular length r with regular top T and regular socle $\tau^{r-1}T$. Let S be regular simple. Now

$$\mathrm{Ext}^1(X,S) \cong \mathrm{Hom}(\tau^- S,X) \cong \mathrm{Hom}(\tau^- S,\tau^{r-1}T) \cong \begin{cases} k & (S\cong\tau^r T) \\ 0 & (\text{else}) \end{cases}$$

so there is a non-split sequence $\xi: 0\longrightarrow Y\longrightarrow E\longrightarrow X\longrightarrow 0$ if and only if $S\cong\tau^r T$, and in this case, since the space of extensions is 1-dimensional, any non-zero $\xi\in\mathrm{Ext}^1(X,S)$ gives rise to the same module E. It is regular uniserial by the previous lemma.

THEOREM. Every indecomposable regular module X is regular uniserial.

PROOF. Induction on dim X. Let $S\leq X$ be a regular simple submodule of X. By induction $X/S = \oplus_{i=1}^{r} Y_i$ is a direct sum of regular uniserials. Now

$$\mathrm{Ext}^1(X/S,S) \cong \bigoplus_{i=1}^{r} \mathrm{Ext}^1(Y_i,S), \quad 0\longrightarrow S\longrightarrow X\longrightarrow X/S\longrightarrow 0 \;\longleftrightarrow\; (\xi_i)$$

Since X is indecomposable, all $\xi_i\neq 0$. Now

$$\mathrm{Ext}^1(Y_i,S) \cong \begin{cases} k & (\text{if } Y_i \text{ has regular socle } \tau^- S) \\ 0 & (\text{else}) \end{cases}$$

so all Y_i have regular socle $\tau^- S$.

If $r=1$ then X is regular uniserial, so suppose $r\geq 2$, for contradiction. We may assume that $\dim Y_1\leq \dim Y_2$, and then (by Lemma 2, or more simply, by the dual of Lemma 2), there is a map $f:Y_1\hookrightarrow Y_2$. This map induces an isomorphism $\mathrm{Ext}^1(Y_2,S)\longrightarrow\mathrm{Ext}^1(Y_1,S)$ so we can use f to adjust the decomposition of X/S to make one component ξ_i zero, a contradiction. Explicitly we write $X/S=Y_1'\oplus Y_2\oplus\ldots\oplus Y_r$ with $Y_1'=\{y_1+\lambda f(y_1) \mid y_1\in Y_1\}$ for some $\lambda\in k$. We leave the details as an exercise.

DEFINITION.

Given a τ-orbit of regular simples, the corresponding <u>tube</u> consists of the indecomposable regular modules whose regular composition factors belong to this orbit.

PROPERTIES.

(1) Every regular indecomposable belongs to a unique tube.

(2) Every indecomposable in a tube has the same period p under τ.

PROOF. If X is regular uniserial with regular top T and regular length r, then $\tau^i X$ is regular uniserial with regular top $\tau^i T$ and regular length r. If $\tau^i T \cong T$ we must have $\tau^i X \cong X$.

(3) If the regular simples in a tube of period p are $S_i = \tau^i S$, then the modules in the tube can be displayed as below. The symbol obtained by stacking various S_i's is the corresponding regular uniserial. We indicate the inclusion of the maximal proper regular submodule Y of X by $Y \hookrightarrow X$, and the map of X onto the quotient Z of X by its regular socle as $X \twoheadrightarrow Z$. The translation τ acts as a shift to the left, and the two vertical dotted lines must be identified.

§9. Euclidean case. Regular simples and roots

In this section we show that the tubes are indexed by the projective line, and that the dimension vectors of indecomposable representations are precisely the positive roots for Γ.

CONSTRUCTION.

Let e be an extending vertex, P=P(e), p=$\underline{\dim}$ P. Clearly $\langle p,p\rangle=1=\langle p,\delta\rangle$.
By §7 Lemma 4 there is a unique indecomposable L of dimension $\delta+p$.

P and L are preprojective, are bricks, and have no self-extensions.
Hom(L,P)=0 for if $\theta:L\longrightarrow P$ then Im θ is a summand of L, a contradiction.
$\mathrm{Ext}^1(L,P)=0$ since $\langle\underline{\dim}\, L,\underline{\dim}\, P\rangle=\langle p+\delta,p\rangle=\langle p,p\rangle-\langle p,\delta\rangle=0$.
dim Hom(P,L)=2 since $\langle p,p+\delta\rangle = 2$.

LEMMA 1. If $0\neq\theta\in\mathrm{Hom}(P,L)$ then θ is mono, Coker θ is a regular indecomposable of dimension δ, and reg.top$(\mathrm{Coker}\ \theta)_e\neq 0$.

PROOF. Suppose θ is not mono. Now Ker θ and Im θ are preprojective (since they embed in P and L), and so they have defect ≤ -1. Now the sequence $0\longrightarrow \mathrm{Ker}\ \theta\longrightarrow P\longrightarrow \mathrm{Im}\ \theta\longrightarrow 0$ is exact, so

$$-1 = \mathrm{defect}(P) = \mathrm{defect}(\mathrm{Ker}\ \theta) + \mathrm{defect}(\mathrm{Im}\ \theta) \leq -2,$$

a contradiction.

Let X=Coker θ, and consider $\xi: 0\longrightarrow P\xrightarrow{\theta} L\longrightarrow X\longrightarrow 0$. Apply Hom(-,P) to get $\mathrm{Ext}^1(X,P)=k$. Apply Hom(-,L) to get Hom(X,L)=0. Apply Hom(X,-) to get X a brick.

If X has regular top T, then

$$\dim T_e = \dim \mathrm{Hom}(P,T) = \langle p,\underline{\dim}\, T\rangle = \langle p+\delta,\underline{\dim}\, T\rangle = \dim \mathrm{Hom}(L,T) \neq 0.$$

LEMMA 2. If X is regular, $X_e \neq 0$ then Hom(Coker θ,X)$\neq$0 for some $0\neq\theta\in$Hom(P,L).

PROOF. Ext^1(L,X)=0, so

$$\dim \mathrm{Hom}(L,X) = \langle p+\delta, \underline{\dim}\, X\rangle = \langle p, \underline{\dim}\, X\rangle = \dim \mathrm{Hom}(P,X) \neq 0.$$

Let α,β be a basis of Hom(P,L). These give maps a,b:Hom(L,X)$\longrightarrow$Hom(P,X).

If a is an iso, let λ be an eigenvalue of $a^{-1}b$ and set $\theta=\beta-\lambda\alpha$.

If a is non-iso, set $\theta=\alpha$.

Either way, there is $0\neq\phi\in$Hom(L,X) with $\phi\circ\theta=0$. Thus $\bar{\phi}$:Coker $\theta\longrightarrow$X.

LEMMA 3. If X is regular simple of period p, then

$$\underline{\dim}\, X + \underline{\dim}\, \tau X + \ldots + \underline{\dim}\, \tau^{p-1}X = \delta.$$

PROOF. Let $\underline{\dim}$ X=α.

If $\alpha_e\neq 0$ there is a map Coker $\theta\longrightarrow$X which must be onto.

If α_e=0 then $\delta-\alpha$ is a root, and $(\delta-\alpha)_e=1$, so $\delta-\alpha$ is a positive root.

Either way $\alpha\leq\delta$.

If $\alpha=\delta$ then X$\cong\tau$X, so we are done. Thus we may suppose α is a real root. Now $\delta-\alpha$ is a real root, and $\langle\delta,\delta-\alpha\rangle=0$, so by §7 Lemma 4 there is a regular brick Y of dimension $\delta-\alpha$. Now

$\langle\alpha,\delta-\alpha\rangle = -1$, so $0 \neq Ext^1(X,Y) \cong D\mathrm{Hom}(Y,\tau X)$, so reg.top(Y) $\cong \tau X$

$\langle\delta-\alpha,\alpha\rangle = -1$, so $0 \neq Ext^1(Y,X) \cong D\mathrm{Hom}(\tau^- X,Y)$, so reg.socle(Y) $\cong \tau^- X$.

It follows that Y must at least involve $\tau X,\tau^2 X,\ldots,\tau^{p-1}X$, so

$$\underline{\dim}\, X + \underline{\dim}\, \tau X + \ldots + \underline{\dim}\, \tau^{p-1}X \leq \delta.$$

Also the sum is invariant under c, so is a multiple of δ.

CONSEQUENCES.

(1) All but finitely many regular simples have dimension δ, so all but finitely many tubes have period one. This follows from §7 Lemma 4.

(2) Each tube contains a unique module in the set

Ω = {isoclasses of indecomposable X with $\underline{\dim}$ X=δ and reg.top$(X)_e\neq 0$}.

(3) If X is indecomposable regular, then

$\underline{\dim}\, X\in\mathbb{Z}\delta$ $\Leftrightarrow$ the period of X divides regular length of X, and

$\underline{\dim}\, X\leq\delta$ $\Leftrightarrow$ regular length X $\leq$ period of X $\Leftrightarrow$ X is a brick.

THEOREM 1. The assignment $\theta \longmapsto \mathrm{Coker}\ \theta$ gives a bijection $\mathbb{P}\mathrm{Hom}(P,L) \longrightarrow \Omega$, so the set of tubes is indexed by the projective line.

PROOF. If U is indecomposable regular of dimension δ and $\mathrm{reg.top}(U)_e \neq 0$, then there is a map $\mathrm{Coker}\ \theta \longrightarrow U$ for some θ. This map must be epi, since any proper regular submodule of U is zero at e. Thus the map is an isomorphism.

If $0 \neq \theta, \theta' \in \mathrm{Hom}(P,L)$ and $\mathrm{Coker}\ \theta \cong \mathrm{Coker}\ \theta'$, then

$$\mathrm{Hom}(L,P) \longrightarrow \mathrm{Hom}(L,L) \longrightarrow \mathrm{Hom}(L,\mathrm{Coker}\ \theta) \longrightarrow \mathrm{Ext}^1(L,P)=0$$

so the composition $L \longrightarrow \mathrm{Coker}\ \theta' \cong \mathrm{Coker}\ \theta$ lifts to map $g:L \longrightarrow L$. Thus one obtains a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & P \arrow[r, "\theta'"] \arrow[d, "f"] & L \arrow[r] \arrow[d, "g"] & \mathrm{Coker}\ \theta' \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & P \arrow[r, "\theta"] & L \arrow[r] & \mathrm{Coker}\ \theta \arrow[r] & 0
\end{tikzcd}$$

Now f,g are non-zero multiples of identity, so $\theta=\lambda\theta'$ with $0\neq\lambda\in k$.

THEOREM 2.

(1) If X is indecomposable then $\underline{\dim}$ X is a root.

(2) If α is positive imaginary root there are ∞ly many indecs with $\underline{\dim}\ X=\alpha$.

(3) If α is positive real root there is a unique indec with $\underline{\dim}\ X=\alpha$.

PROOF.

(1) If X is a brick, this is clear. If X is not a brick, it is regular. Let X have period p and regular length rp+q with $1<q\leq p$. The submodule Y with regular length q is a brick, and so $\underline{\dim}\ X = \underline{\dim}\ Y + r\delta$ is a root.

(2) $\alpha=r\delta$. If T is a tube of period p, then the indecomposables in T of regular length rp have dimension $r\delta$. There are infinitely many tubes.

(3) We know there is a unique indecomposable of dimension α if $\langle\alpha,\delta\rangle\neq 0$ or $\alpha\leq\delta$, so suppose $\langle\alpha,\delta\rangle=0$ and write $\alpha=r\delta+\beta$ with $0\leq\beta\leq\delta$ a real root. There is a unique regular indecomposable Y of dimension β, say of period p, and regular length q. Let X be the regular uniserial containing Y and with regular length rp+q. Clearly $\underline{\dim}\ X = r\delta + \underline{\dim}\ Y = \alpha$. It is easy to see that this is the only indecomposable of dimension α.

REMARKS.

(1) For the Kronecker quiver $\tilde{A}_1$

$$\bullet \rightrightarrows \bullet,$$

the regular simples all have period one. They are

$$k \underset{\mu}{\overset{\lambda}{\rightrightarrows}} k \qquad \lambda:\mu \in \mathbb{P}^1.$$

(2) For the 4-subspace quiver, $\tilde{D}_4$ with the following orientation

$$\begin{tikzcd} & & \bullet & & \\ \bullet \arrow[urr] & \bullet \arrow[ur] & & \bullet \arrow[ul] & \bullet \arrow[ull] \end{tikzcd}$$

the real regular simples have period 2, and have dimension vectors

$$\begin{smallmatrix}1\\1100\end{smallmatrix}\ \begin{smallmatrix}1\\0011\end{smallmatrix} \qquad \begin{smallmatrix}1\\1001\end{smallmatrix}\ \begin{smallmatrix}1\\0110\end{smallmatrix} \qquad \begin{smallmatrix}1\\1010\end{smallmatrix}\ \begin{smallmatrix}1\\0101\end{smallmatrix}$$

The regular simples of dimension vector δ are

$$\begin{tikzcd} & & k^2 & & \\ k \arrow[urr] & k \arrow[ur] & & k \arrow[ul] & k \arrow[ull] \end{tikzcd} \quad \text{with maps } \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\\lambda\end{pmatrix} \quad \text{where } \lambda\in k,\ \lambda\neq 0,1.$$

(3) One can find lists of regular simples in the tables in the back of V.Dlab & C.M.Ringel, Indecomposable representations of graphs and algebras, Mem. Amer. Math. Soc., 173 (1976). For the different graphs Γ the tubes with period $\neq 1$ have period as follows

$\tilde{\mathbb{A}}_m$ p,q if p>0 arrows go clockwise and q>0 go anticlockwise.

$\tilde{\mathbb{D}}_m$ m-2,2,2

$\tilde{\mathbb{E}}_6$ 3,3,2

$\tilde{\mathbb{E}}_7$ 4,3,2

$\tilde{\mathbb{E}}_8$ 5,3,2

One always has $\sum_{\text{tubes}}(\text{period}-1) = n-2$, which can be proved with a little more analysis.

§10. Further topics

In this section I want to list some of the topics which have attracted interest in the past, and which are areas of present research. The lists of papers are only meant to be pointers: you should consult the references in the listed papers for more information.

(1) Kac’s Theorem: for any quiver the dimension vectors of the indecomposables are the positive roots of the graph.

V.Kac, Infinite root systems, representations of graphs and invariant theory I,II, Invent. Math 56 (1980), 57-92, J. Algebra 77 (1982), 141-162.

V.Kac, Root systems, representations of quivers and invariant theory, in Springer Lec. Notes 996 (1983), 74-108.

H.Kraft & Ch.Riedtmann, Geometry of representations of quivers, in Representations of algebras (ed. P.Webb) London Math. Soc. Lec. Note Series 116 (1986), 109-145.

(2) Invariant theory and geometry for the action of the group $GL(\alpha)$ on the variety $Rep(\alpha)$.

C.Procesi, The invariant theory of $n\times n$ matrices, Adv. Math. 19(1976), 306-381.

C.M.Ringel, The rational invariants of the tame quivers, Invent. Math., 58(1980), 217-239.

L.Le Bruyn & C.Procesi, Semisimple representations of quivers, Trans. Amer. Math. Soc. 317 (1990), 585-598.

Ch.Riedtmann & A.Schofield, On open orbits and their complements, J.Algebra 130 (1990), 388-411.

A.Schofield, Semi-invariants of quivers, J. London Math. Soc. 43 (1991), 385-395.

A.Schofield, Generic representations of quivers, preprint.

(3) Construction of the Lie algebra and quantum group of type Γ from the representations of a quiver with graph Γ.

C.M.Ringel, Hall polynomials for the representation-finite hereditary algebras, Adv. Math. 84 (1990), 137-178.

C.M.Ringel, Hall algebras and quantum groups, Invent. Math. 101 (1990), 583-592.

G.Lusztig, Quivers, perverse sheaves, and quantized enveloping algebras, J. Amer. Math. Soc. 4 (1991), 365-421.

A.Schofield, Quivers and Kac-Moody Lie algebras, preprint.

(4) Auslander-Reiten theory for wild quivers. In particular, the behaviour of the functions dim Hom$(X,\tau^{-1}Y)$ for fixed X,Y.

C.M.Ringel, Finite dimensional hereditary algebras of wild representation type, Math.Z., 161 (1978), 235-255.
V.Dlab and C.M.Ringel, Eigenvalues of Coxeter transformations and the Gelfand-Kirillov dimension of the preprojective algebras, Proc. Amer. Math. Soc. 83 (1981), 228-232.
D.Baer, Wild hereditary artin algebras and linear methods, Manuscripta Math. 55 (1986), 68-82.
O.Kerner, Tilting wild algebras, J.London Math.Soc, 39(1989), 29-47.
J.A.de la Peña & M.Takane, Spectral properties of Coxeter transformations and applications, Arch. Math. 55 (1990), 120-134.
O.Kerner & F.Lukas, Regular modules over wild hereditary algebras, preprint.

(5) Tame algebras of global dimension 2, but with properties analogous to those of path algebras: the tame concealed and tubular algebras.

C.M.Ringel, Tame algebras and integral quadratic forms, Springer Lec. Notes 1099 (1984).
C.M.Ringel, Representation theory of finite-dimensional algebras, in Representations of algebras (ed. P.Webb) London Math. Soc. Lec. Note Series 116 (1986), 7-79.
I.Assem & A.Skowronski, Algebras with cycle finite derived categories, Math. Ann., 280 (1988), 441-463.

(6) Interpretation of the representation theory of quivers as non-commutative algebraic geometry.

H.Lenzing, Curve singularities arising from the representation theory of tame hereditary artin algebras, in Springer Lec.Notes 1177 (1986), 199-231.
W.Geigle & H.Lenzing, A class of weighted projective curves arising in representation theory of finite dimensional algebras, in Springer Lec. Notes 1273 (1987), 265-297.

(7) Tame hereditary algebras when the field is not algebraically closed, and the more ring-theoretic aspects of hereditary algebras.

V.Dlab & C.M.Ringel, Indecomposable representations of graphs and algebras, Mem. Amer. Math. Soc., 173 (1976).
C.M.Ringel, Representations of K-species and bimodules, J. Algebra, 41 (1976), 269-302.
V.Dlab & C.M.Ringel, Real subspaces of a vector space over the quaternions, Can. J. Math. 30 (1978), 1228-1242.
A.Schofield, Universal localization for hereditary rings and quivers, in Springer Lec. Notes 1197 (1986).
D.Baer, W.Geigle & H.Lenzing, The preprojective algebra of a tame hereditary Artin algebra, Comm. Algebra 15 (1987), 425-457.
W.Crawley-Boevey, Regular modules for tame hereditary algebras, Proc. London Math. Soc., 62 (1991), 490-508.

(8) Infinite dimensional representations, and contrast with the theory of abelian groups.

C.M.Ringel, Infinite dimensional representations of finite dimensional hereditary algebras, Symposia Math., 23 (1979), 321-412.

F.Okoh, Indecomposable pure-injective modules over hereditary artin algebras of tame type, Commun. in Algebra 8 (1980), 1939-1941.

F.Okoh, Separable modules over finite-dimensional algebras, J. Algebra, 116 (1988), 400-414.

A.Dean & F.Zorzitto, Infinite dimensional representations of $\tilde{D}_4$, Glasgow Math. J. 32 (1990), 25-33.

F.Lukas, A class of infinite-rank modules over tame hereditary algebras, preprint.

D.Happel & L.Unger, A family of infinite-dimensional non-selfextending bricks for wild hereditary algebras, preprint