LECTURES ON REPRESENTATION THEORY AND INVARIANT THEORY

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Errata (Darij Grinberg)

§1

- On page 7 in the PDF (a. k. a. page 4 according to the numbering in the text), in the proof of Lemma 4, you write that

$$\dim_{\mathbb{C}} \operatorname{End}_R(M) \cong \sum\nolimits_S (\dim_{\mathbb{C}} \operatorname{Hom}(S, M))^2 .$$

  The $\cong$ sign should be an $=$ sign here.

§2

- On page 10 in the PDF (a. k. a. page 7 according to the numbering in the text), in the Example, I think you should explain that $(a_1^{m_1}, a_2^{m_2}, ..., a_k^{m_k})$ means the partition $\left(\underbrace{a_1, a_1, ..., a_1}_{m_1 \text{ times}}, \underbrace{a_2, a_2, ..., a_2}_{m_2 \text{ times}}, ..., \underbrace{a_k, a_k, ..., a_k}_{m_k \text{ times}}, 0, 0, ...\right)$ (where $a_1 \geq a_2 \geq ... \geq a_k$ are nonnegative integers).

- On page 12 in the PDF (a. k. a. page 9 according to the numbering in the text), in the proof of Lemma 2, you write: "and can find c_2 such that $c_2 c_1 \Sigma'_\mu$ have the same numbers in each of the first two rows". Here, "$c_2 c_1 \Sigma'_\mu$" should be "Σ_λ and $c_2 c_1 \Sigma'_\mu$".

- On page 12 in the PDF (a. k. a. page 9 according to the numbering in the text), in the proof of Lemma 2, you write: "such that $\Sigma_\lambda = c' \Sigma'_\mu$ have the same numbers in each row". The equality sign is inappropriate here; I think it should be an "and".

- On page 13 in the PDF (a. k. a. page 10 according to the numbering in the text), in the proof of Lemma 4 (the part (2) $\Longrightarrow$ (1)), you write $rh(\Sigma_\lambda) c = \varepsilon_\sigma h(\Sigma_\lambda)$. Presumably ε_σ means ε_c here.

- On page 14 in the PDF (a. k. a. page 11 according to the numbering in the text), in the proof of Lemma 8, there is a "be" too much in "there are be two integers in the same row".

- In the same sentence, Σ_μ should be Σ'_μ.

§4

- On page 22 in the PDF (a. k. a. page 19 according to the numbering in the text), in the proof of Lemma 3, a word "is" is missing (in: "In particular it zero").
- On page 24 in the PDF (a. k. a. page 21 according to the numbering in the text), in the proof of Lemma 4, after the words "where the sum extends over all sequences $(\alpha_1, \alpha_2, ...)$ of non-negative integers with only finitely many non-zero terms", it should be mentioned that n means $\alpha_1 + \alpha_2 + ...$ in the sum (since you are no longer summing over n).
- On page 26 in the PDF (a. k. a. page 23 according to the numbering in the text), in the proof of Lemma 7, you write: "Now $g\sigma g^{-1} \in R$ if and only g is in a coset $g_i R$ with $g_i \sigma g_i^{-1} \in R$". Here, $g\sigma g^{-1}$ should be replaced by $g^{-1}\sigma g$ and $g_i \sigma g_i^{-1}$ by $g_i^{-1}\sigma g_i$. The same mistake is repeated several times below. In particular, $g'^{-1}g$ (in $g'^{-1}g \in c_{S_n}(\sigma)$) should be gg'^{-1} instead.
- It is rather ambiguous how a term like a/bc has to be understood (the two possibilities being $(a/b)\,c$ and $a/(bc)$). For instance, when you write $\theta(\alpha) = 1^{\alpha_1}2^{\alpha_2}...\alpha_1!\alpha_2!.../\lambda_1!\lambda_2!...\,|\alpha \cap R|$, you mean $\theta(\alpha) = (1^{\alpha_1}2^{\alpha_2}...\alpha_1!\alpha_2!.../(\lambda_1!\lambda_2!...))\,|\alpha \cap R|$. But when you write $\lambda_1!/1^{\alpha_{11}}2^{\alpha_{12}}...\alpha_{11}!\alpha_{12}!...$, you mean $\lambda_1!/(1^{\alpha_{11}}2^{\alpha_{12}}...\alpha_{11}!\alpha_{12}!...)$.
- On page 27 in the PDF (a. k. a. page 24 according to the numbering in the text), Lemma 8 claims that $\mathbb{C}S_n h_\mu$ is a submodule of $\mathbb{C}S_n r_\lambda$ if and only if $\mu = \lambda$. But what you actually mean (and what your proof shows) is that $\mathbb{C}S_n h_\mu$ is *isomorphic to* a submodule of $\mathbb{C}S_n r_\lambda$ if and only if $\mu = \lambda$. That's quite a difference. (In general, $\mathbb{C}S_n h_\lambda$ is not a submodule of $\mathbb{C}S_n r_\lambda$.)

§5

- In the Theorem, you require the partition λ to have "exactly m parts". I think $\leq m$ parts is enough.

§6

- On page 31 in the PDF (a. k. a. page 28 according to the numbering in the text), in part (2) of the "Properties", the letter f should always be replaced by n.
- On page 33 in the PDF (a. k. a. page 30 according to the numbering in the text), in part (2) of the "Properties", replace "are a" by "are".
- On page 34 in the PDF (a. k. a. page 31 according to the numbering in the text), in part (2) of the "Examples", you should replace .. by ... (this typo appears twice in $\sum_{i_1 \leq .. \leq i_n} c_{i_1,..,i_n}$).
- In the Remark on page 35 in the PDF (a. k. a. page 32 according to the numbering in the text), you write: "Now for $v \in V$ we have $(P\phi) \circ \Delta = n!\phi$." I don't think you need the "for $v \in V$" part here.

- In the Remark on page 35 in the PDF (a. k. a. page 32 according to the numbering in the text), the X_n should be an X_m in the formula

$$= \sum_j n!\phi_j (X_1, ..., X_n) f_j = n!\phi (X_1 e_1 + ... + X_m e_m).$$

§7

- On page 37 in the PDF (a. k. a. page 34 according to the numbering in the text), you prove Lemma 3 using Hilbert's Nullstellensatz. This is reasonable for pedagogical purposes, but there is an alternative and much simpler argument: The ring of regular functions on $\mathbb{A}^n$ is $R = \mathbb{C}[X_1, X_2, ..., X_n]$. If X and Y are the zero sets of ideals I and J in R, then the assumption is that at each point $(a_1, a_2, ..., a_n) \in \mathbb{C}^n$, either all polynomials from I or all polynomials from J evaluate to 0. If I and J are both non-zero, then we can pick two polynomials $0 \neq i \in I$ and $0 \neq j \in J$. Now at each point $(a_1, a_2, ..., a_n) \in \mathbb{C}^n$, we have either $i(a_1, a_2, ..., a_n) = 0$ or $j(a_1, a_2, ..., a_n) = 0$; consequently, $(ij)(a_1, a_2, ..., a_n) = 0$, and thus the polynomial ij must be zero (because it evaluates to 0 at each point), which contradicts the fact that R is an integral domain.

§8

- On page 40 in the PDF (a. k. a. page 37 according to the numbering in the text), the proof of Lemma 2 begins with "$S_i = \operatorname{Aut}\{1, ..., i\}$ and $S_j = \operatorname{Aut}\{1', ..., n'\}$". The n' should be a j' here.

§9

- On page 43 in the PDF (a. k. a. page 40 according to the numbering in the text), Lemma 2 (4) requires $n \neq n'$.
- On page 45 in the PDF (a. k. a. page 42 according to the numbering in the text), in the proof of Lemma 4, you write:

$$u_0 + \alpha u_1 + \alpha^2 u_2 + ... + \alpha^N u_N = \alpha^i u_0 + \alpha^i u_1 + \alpha^i u_2 + ... + \alpha^i u_N.$$

  The α^i on the right hand side should all be α^n.

§10

- On page 48 in the PDF (a. k. a. page 45 according to the numbering in the text), you write

$$T^n V = \bigoplus_\lambda \mathbb{C}S_n h_\lambda \oplus \operatorname{Hom}_{\mathbb{C}S_n} (\mathbb{C}S_n h_\lambda, T^n V)$$

  (this is the very first formula on the page). The $\oplus$ sign here should be a $\otimes$ sign.

- On page 48 in the PDF (a. k. a. page 45 according to the numbering in the text), $(i_1, ..., i_m)$ should be $(i_1, ..., i_n)$ in the formula

$$\left\{(i_1, ..., i_m) \mid 1 \leq i_1 \leq m, \ ..., \ 1 \leq i_n \leq m \text{ and } i_{g^{-1}(j)} = i_j \text{ for each } j\right\}.$$

§11

- On page 52 in the PDF (a. k. a. page 49 according to the numbering in the text), there is a full stop too much in the formula

$$\mathbb{C}[U]^G = \{f \in \mathbb{C}[U] \mid gf = f\}.$$

  Besides, it would be helpful to add "for all $g \in G$" after the "$gf = f$" part.

- On page 52 in the PDF (a. k. a. page 49 according to the numbering in the text), the full stop in the formula

$$\mathbb{C}[X_1, ..., X_n] \to \mathbb{C}[U], \qquad X_i \mapsto \xi_i.$$

  is (sorry...) pointless.

- On page 54 in the PDF (a. k. a. page 51 according to the numbering in the text), you write

$$\theta : \mathbb{C}[Y_1, .., Y_n, Z] \to \mathbb{C}[X_1, .., X_n]^{A_n}.$$

  Here, .. should be ... two times.

- On page 54 in the PDF (a. k. a. page 51 according to the numbering in the text), you write "$f \in A_n$". This should be $\sigma \in A_n$.

- On page 54 in the PDF (a. k. a. page 51 according to the numbering in the text), in the proof, you are using Hilbert's Nullstellensatz to see that f_a is divisible by $X_1 - X_2$. There is an easier way to do this: Consider f_a as a polynomial in the variable X_1 over the ring $\mathbb{C}[X_2, X_3, ..., X_n]$. Then, this polynomial has X_2 as a root (because if we substitute X_2 for X_1 in the polynomial f_a, we obtain $f_a(X_2) = f_a(X_2, X_2, X_3, ..., X_n) = 0$, since f_a is alternating), and thus is divisible by $X_1 - X_2$ (since whenever P is a polynomial in a variable X_1 over some commutative ring R and $r \in R$ is a root of P, the polynomial P is divisible by $X_1 - r$), qed.

- On page 56 in the PDF (a. k. a. page 53 according to the numbering in the text), you state that "The $\mathbb{C}$-algebra map $\mathbb{C}[X] \to \mathbb{C}[U]^{\mathrm{SL}_2(\mathbb{C})}$, $X \mapsto$ disc is an isomorphism". Here, $\mathrm{SL}_2(\mathbb{C})$ is probably supposed to mean $\mathrm{SL}_m(\mathbb{C})$.

- On page 57 in the PDF (a. k. a. page 54 according to the numbering in the text), in the first line, "any matrix is congruent" should be "any symmetric matrix is congruent".

- On page 58 in the PDF (a. k. a. page 55 according to the numbering in the text), in the formula

$$a_0^s Q'(a_0, ...a_n) = a_n^t Q(a_0, ..., a_n),$$

  there is a comma missing before the a_n on the left hand side.

§12

- On page 61 in the PDF (a. k. a. page 58 according to the numbering in the text), in the "2nd way", you write: "If $f \in \operatorname{Hom}_{\mathbb{C}G}(S^n U, W)$, then [...]" but you actually mean "If $f \in \operatorname{Hom}_{\mathbb{C}G,n}(U, W)$, then [...]".

- On page 64 in the PDF (a. k. a. page 61 according to the numbering in the text), you write:
$$= \operatorname{Tr}(\theta_{i_k}...\theta_{i_1}) \operatorname{Tr}(\theta_{j_1}...\theta_{j_1}) ...$$
The first of the two θ_{j_1}'s here should be θ_{j_ℓ} instead.

- On page 66 in the PDF (a. k. a. page 63 according to the numbering in the text), in the proof, you write:
$$\mu_\sigma(\phi_1 \otimes ... \otimes \phi_r \otimes v_1 \otimes ... \otimes v_r) = \phi_1\left(v_{\sigma^{-1}(1)}\right) ... \phi_n\left(v_{\sigma^{-1}(n)}\right).$$
Both n's on the right hand side should be r's.

- On page 66 in the PDF (a. k. a. page 63 according to the numbering in the text), I don't like the idea of using the letter δ for some element of $\det^{-1}$, because δ denotes the surjective map $\operatorname{Hom}_{\mathbb{C}G}(T^n U, W) \to \operatorname{Hom}_{\mathbb{C}G,n}(U, W)$ defined in the Remark after Lemma 2, which is induced by the map $\delta : U \to T^n U$, $u \mapsto u \otimes u \otimes ... \otimes u$.

- On page 67 in the PDF (a. k. a. page 64 according to the numbering in the text), you write $\left[v_{\sigma^{-1}(1)}...v_{\sigma^{-1}(m)}\right]$, $\left[v_{\sigma^{-1}(m+1)}...v_{\sigma^{-1}(2m)}\right]$, $[v_{i_1}...v_{i_m}]$ and $[\phi_{j_1}...\phi_{j_m}]$. To keep notations consistent, these should be $\left[v_{\sigma^{-1}(1)}, ..., v_{\sigma^{-1}(m)}\right]$, $\left[v_{\sigma^{-1}(m+1)}, ..., v_{\sigma^{-1}(2m)}\right]$, $[v_{i_1}, ..., v_{i_m}]$ and $[\phi_{j_1}, ..., \phi_{j_m}]$.

§13

- On page 69 in the PDF (a. k. a. page 66 according to the numbering in the text), in part (3), you notice that $\operatorname{Hom}_{\mathbb{C}G}(T^2(T^2 V^*), \mathbb{C})$ is spanned by the polynomial invariants sending $(\phi_1 \otimes \phi_2) \otimes (\phi_3 \otimes \phi_4)$ to
$$[\phi_1, \phi_2][\phi_3, \phi_4], \qquad [\phi_1, \phi_3][\phi_2, \phi_4], \qquad [\phi_1, \phi_4][\phi_2, \phi_3],$$
and the corresponding covariants send (f, v) to
$$0, \qquad -\frac{1}{2}\operatorname{disc}(f), \qquad \frac{1}{2}\operatorname{disc}(f).$$
I think the third of these should be $-\frac{1}{2}\operatorname{disc}(f)$ rather than $\frac{1}{2}\operatorname{disc}(f)$, but I may be mistaken.

- On page 69 in the PDF (a. k. a. page 66 according to the numbering in the text), in Example (1), you write $\partial(f, g) \diagup \partial(X_1, X_2)$. I think what you mean is usually denoted by $\det(\partial(f, g) \diagup \partial(X_1, X_2))$, while $\partial(f, g) \diagup \partial(X_1, X_2)$ is a matrix.

- On page 72 in the PDF (a. k. a. page 69 according to the numbering in the text), in the proof of $a_r \neq 0$, you set $h = X_2^r$. I am pretty positive that you want $h = X_2^n$ here.

- On page 72 in the PDF (a. k. a. page 69 according to the numbering in the text), in the proof of $a_r \neq 0$, you write

  $$\tau_r (g, h) = \frac{1}{r!} \frac{\partial^r g}{\partial X_1^r} \frac{\partial^r h}{\partial X_2^r} = \frac{1}{r!} q (q-1) ... (q-r+1) X_1^{q-r} n (n-1) ... (n-q+1) X_2^{n-r}.$$

  The factor $(n-q+1)$ should be $(n-r+1)$.

- On page 72 in the PDF (a. k. a. page 69 according to the numbering in the text), in Lemma 3, $\psi (g)$ should be $\psi (f)$.

- On page 74 in the PDF (a. k. a. page 71 according to the numbering in the text), the first formula should end with a comma and not with a full stop. Same for the fourth formula on this page.

- On page 75 in the PDF (a. k. a. page 72 according to the numbering in the text), you write:

  $$\mathrm{ev}^x H^y t^z D^w \in R^G_{x+2y+3z+4w, 3x+2y+3w}.$$

  The $3x + 2y + 3w$ should be $3x + 2y + 3z$.

- The table at the very end of your text is wrong, due to mistakes in Sylvester's computations. While the table gives correct values for binary forms of degree ≤ 6, the values for $n > 6$ cannot be trusted. For example, for $n = 7$, the entry 124 is too small (124 generators are not enough to generate all covariants). On the other hand, if von Gall's paper "Das vollständige Formensystem der binären Form 7ter Ordnung" (Math. Ann. 31 (1888), pp. 318–336) is right, the number 69 for $n = 8$ it is too large (i. e., there are 69 covariants that generate the whole covariant ring, but 69 is not the minimal number with this property). See also `http://arxiv.org/abs/math/0612113` and Olver's "Classical Invariant Theory" p. 40.