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Rigid integral representations of quivers

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Dedicated to the memory of Maurice Auslander

We study representations of a quiver by means of finitely generated free modules for a principal ideal domain. In particular we determine all representations X with $\mathrm{Ext}^1(X,X)=0$.

1. Introduction

Let Q be a finite quiver and let R be a principal ideal domain. By an *RQ-lattice* we mean a representation of Q by means of finitely generated free R-modules; equivalently it is an RQ-module which is finitely generated and free over R. If Q has vertex set $\{1,\dots,n\}$, the rank vector $\underline{\mathrm{rank}}\, X \in \mathbb{N}^n$ of a lattice X gives the rank of the R-module attached to each vertex. A lattice is *rigid* if $\mathrm{Ext}^1(X,X)=0$, and is *exceptional* if in addition $\mathrm{End}(X)=R$. A lattice is *absolutely indecomposable* if $X^K = X\otimes_R K$ is an indecomposable KQ-module for each homomorphism $R\to K$ to an algebraically closed field.

In this paper we use an action of the braid group to describe the exceptional lattices. We then use exceptional lattices to compute all rigid lattices. At the end we prove some special cases of the following conjecture: for each positive real root $\alpha\in\mathbb{N}^n$ there is a unique absolutely indecomposable lattice of rank vector α.

2. Exceptional lattices

By an *R-field* we mean a homomorphism of R into a field K. We normally only consider algebraically closed R-fields, meaning that K is algebraically closed. We write M^K for $M\otimes_R K$. If M is an RQ-module, then we consider M^K as a KQ-module. Since R is a principal ideal domain, a f.g. R-module M is free if and

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only if M^K has constant dimension for all algebraically closed R-fields K, and a homomorphism $\theta : M \to N$ between f.g. free R-modules is an isomorphism (resp. an epimorphism, resp. a split monomorphism) if and only if $\theta^K : M^K \to N^K$ is an isomorphism (resp. an epimorphism, resp. a monomorphism) for all K.

LEMMA 1. *If X is an RQ-lattice, then it has projective dimension at most 1 and $\operatorname{Ext}^1_{RQ}(X,Y)^K \cong \operatorname{Ext}^1_{KQ}(X^K,Y^K)$ for all RQ-modules Y and algebraically closed R-fields K. In particular X is rigid if and only if X^K is a rigid KQ-module for all algebraically closed R-fields K.*

PROOF. There is a standard exact sequence

$$0 \to RQ \otimes_S B \otimes_S RQ \to RQ \otimes_S RQ \to RQ \to 0.$$

where S is the R-subalgebra of RQ with basis the trivial paths $e_1, \dots, e_n$, and B is the free R-submodule of RQ with basis the arrows. Applying $- \otimes_{RQ} X$ gives

$$0 = \operatorname{Tor}_1^{RQ}(RQ, X) \to RQ \otimes_S B \otimes_S X \to RQ \otimes_S X \to X \to 0,$$

a projective resolution. The statement about extensions holds since if P is a f.g. projective RQ-module then $\operatorname{Hom}_{RQ}(P,Y)^K \cong \operatorname{Hom}_{KQ}(P^K, Y^K)$. □

For $\alpha, \beta \in \mathbb{N}^n$ the Ringel form is defined by

$$\langle \alpha, \beta \rangle = \sum_{i=1}^{n} \alpha_i \beta_i - \sum_{a:i \to j} \alpha_i \beta_j,$$

and $\operatorname{ext}(\alpha,\beta) \in \mathbb{N}$ is defined inductively by $\operatorname{ext}(\alpha, 0) = \operatorname{ext}(0, \beta) = 0$ and

$$\operatorname{ext}(\alpha,\beta) = \max\left\{ -\langle \alpha', \beta' \rangle \;\middle|\; \begin{array}{ll} 0 \le \alpha' \le \alpha, & \operatorname{ext}(\alpha', \alpha - \alpha') = 0 \\ 0 \le \beta' \le \beta, & \operatorname{ext}(\beta - \beta', \beta') = 0 \end{array} \right\}.$$

By the results of **[S3]** and **[C2]** this is the general value of $\dim \operatorname{Ext}^1_{KQ}(M,N)$ with M and N running through the varieties of KQ-modules of dimension vectors α and β, where K is any algebraically closed field. In particular if M and N are rigid KQ-modules then $\dim \operatorname{Ext}^1_{KQ}(M,N) = \operatorname{ext}(\alpha,\beta)$ since they correspond to open orbits in these varieties.

LEMMA 2. *If X and Y are rigid lattices with rank vectors α and β, then $\operatorname{Ext}^1_{RQ}(X,Y)$ and $\operatorname{Hom}_{RQ}(X,Y)$ are free R-modules. They have ranks $\operatorname{ext}(\alpha,\beta)$ and $\hom(\alpha,\beta) = \langle \alpha,\beta \rangle + \operatorname{ext}(\alpha,\beta)$ respectively.*

PROOF. For all algebraically closed R-fields K the modules X^K and Y^K are rigid KQ-modules of dimension vectors α and β, so $\dim \operatorname{Ext}^1_{KQ}(X^K, Y^K) = \operatorname{ext}(\alpha,\beta)$ by the remarks above. By Lemma 1 it follows that $\operatorname{Ext}^1_{RQ}(X,Y)$ is free over R of rank $\operatorname{ext}(\alpha,\beta)$. If $0 \to P \to P' \to X \to 0$ is a projective resolution of X then the long exact sequence

$$0 \to \operatorname{Hom}_{RQ}(X,Y) \to \operatorname{Hom}_{RQ}(P',Y) \to \operatorname{Hom}_{RQ}(P,Y) \to \operatorname{Ext}^1_{RQ}(X,Y) \to 0$$

consists of free R-modules, so remains exact on tensoring with K. It follows that the natural map $\mathrm{Hom}_{RQ}(X,Y)^K \to \mathrm{Hom}_{KQ}(X^K,Y^K)$ is an isomorphism. Now $\dim \mathrm{Hom}_{KQ}(X^K,Y^K) = \hom(\alpha,\beta)$ since the Ringel form gives the difference in the dimensions of Hom and Ext^1, and the assertion about $\mathrm{Hom}_{RQ}(X,Y)$ follows. □

LEMMA 3. *There is at most one exceptional lattice of each rank vector.*

PROOF. Say X and Y are exceptional lattices of rank α. Now $\hom(\alpha,\alpha) = 1$ since $\mathrm{End}_{RQ}(X) \cong R$, and then $\mathrm{Hom}_{RQ}(X,Y) \cong R$ since R is a principal ideal domain. Let $\theta : X \to Y$ be a generator. If K is an algebraically closed R-field, then the map $\theta^K : X^K \to Y^K$ is nonzero. Since there is at most one rigid KQ-module of any given dimension vector we have $X^K \cong Y^K$. Now $\mathrm{Hom}_{KQ}(X^K,Y^K)$ is one-dimensional, so θ^K is an isomorphism. It follows that θ is an isomorphism. □

By an *exceptional sequence* $(X_1,\dots,X_r)$ of length r we mean a sequence of exceptional lattices with $\mathrm{Hom}(X_i,X_j) = \mathrm{Ext}^1(X_i,X_j) = 0$ for $i > j$. If (X,Y) is an exceptional sequence and X and Y have ranks α,β, then the mutations L_XY and R_YX are defined as follows. We write D for $\mathrm{Hom}_R(-,R)$, and use Lemma 2 to ensure that the universal constructions exist.

(1) If $\mathrm{Hom}(X,Y) = 0$ then L_XY and R_YX are defined by the universal exact sequences

$$0 \to Y \to L_XY \to X \otimes_R \mathrm{Ext}^1(X,Y) \to 0$$
$$0 \to Y \otimes_R D\,\mathrm{Ext}^1(X,Y) \to R_YX \to X \to 0.$$

(2) Suppose that $\mathrm{Hom}(X,Y) \neq 0$. If $\beta \geq \langle\alpha,\beta\rangle\alpha$ then L_XY is the cokernel of the universal map

$$X \otimes_R \mathrm{Hom}(X,Y) \to Y,$$

and otherwise it is the kernel. If $\alpha \geq \langle\alpha,\beta\rangle\beta$ then R_YX is the kernel of the universal map

$$X \to Y \otimes_R D\,\mathrm{Hom}(X,Y),$$

and otherwise it is the cokernel.

LEMMA 4. (L_XY, X) *and* (Y, R_YX) *are exceptional sequences, and*

$$\underline{\mathrm{rank}}\, L_XY = \pm(\beta - \langle\alpha,\beta\rangle\alpha), \qquad \underline{\mathrm{rank}}\, R_YX = \pm(\alpha - \langle\alpha,\beta\rangle\beta).$$

PROOF. Passing to an algebraically closed R-field K, these are the standard constructions of the mutations in **[C1]**, cf. **[Ru]**. The assertion follows. For example if $\mathrm{Hom}(X,Y) \neq 0$ and $\alpha \geq \langle\alpha,\beta\rangle\beta$ we have an exact sequence

$$0 \to R_YX \to X \to Y \otimes_R D\,\mathrm{Hom}(X,Y) \to C \to 0,$$

and hence

$$X^K \to [Y \otimes_R D\,\mathrm{Hom}(X,Y)]^K \to C^K \to 0.$$

Now the standard mutation is

$$0 \to R_{Y^K} X^K \to X^K \to Y^K \otimes D_K \operatorname{Hom}(X^K, Y^K) \to 0$$

where D_K is the duality with K. Thus $C^K = 0$, so C is zero. It follows that

$$(R_Y X)^K \cong R_{Y^K} X^K,$$

so $R_Y X$ is an exceptional lattice of rank $\alpha - \langle \alpha, \beta \rangle \beta$. □

Recall that the braid group on r strings is generated by $\sigma_1, \dots, \sigma_{r-1}$ subject to the relations $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $i \neq j \pm 1$ and $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$. By Lemma 3 and [**C1**] we obtain

LEMMA 5. *The braid group on r strings acts naturally on the set of exceptional sequences of length r via the assignments*

$$\sigma_i(X_1, \dots, X_r) = (X_1, \dots, X_{i-1}, X_{i+1}, R_{X_{i+1}} X_i, X_{i+2}, \dots, X_r)$$
$$\sigma_i^{-1}(X_1, \dots, X_r) = (X_1, \dots, X_{i-1}, L_{X_i} X_{i+1}, X_i, X_{i+2}, \dots, X_r).$$

Recall that $\alpha \in \mathbb{N}^n$ is a *real Schur root* if over an algebraically closed field there is an exceptional representation of dimension vector α. By Kac's canonical decomposition the real Schur roots can be characterized as the $\alpha \in \mathbb{N}^n$ with $\langle \alpha, \alpha \rangle = 1$ and with no non-trivial decomposition $\alpha = \beta + \gamma$ with $\beta, \gamma \in \mathbb{N}^n$ and $\operatorname{ext}(\beta, \gamma) = \operatorname{ext}(\gamma, \beta) = 0$, see [**K1**, §4]. This characterization shows that the real Schur roots do not depend on the field.

THEOREM 1. *The assignment $X \mapsto \underline{\operatorname{rank}}\, X$ induces a 1-1 correspondence between exceptional lattices and real Schur roots.*

PROOF. Thanks to lemma 3 we only need to show that the rank vectors of the exceptional lattices are the real Schur roots. Let K be an algebraically closed R-field. If X is an exceptional lattice of rank α then X^K is an exceptional KQ-module, so α is a real Schur root. On the other hand, if α is a real Schur root then there is an exceptional KQ-module of dimension α. This implies that the full subquiver of Q on the support of α has no oriented cycles (for otherwise one can construct representations of dimension α on which the trace of the oriented cycle is arbitrary, but the variety of representations of dimension α has a dense orbit, so the trace must be constant). Thus, passing to the support of α we may suppose that Q has no oriented cycles. Suppose the vertices are ordered so that the sequence of projectives $(KQe_1, \dots, KQe_n)$ is an exceptional sequence. By the result of [**C1**] there is an element g of the braid group on n strings such that $g(KQe_1, \dots, KQe_n)$ includes the exceptional KQ-module of dimension α. Then $g(RQe_1, \dots, RQe_n)$ includes an exceptional lattice of rank α. □

3. Rigid modules

An RQ-module X is *rigid* if $\mathrm{Ext}^1(X,X)=0$. We can now give a complete description of the rigid RQ-modules which are f.g. over R.

THEOREM 2. (1) *Any rigid RQ-module, f.g. over R, is a lattice.*

(2) *Any rigid lattice is a direct sum of exceptional lattices, and the terms in this direct sum are unique up to isomorphism and reordering.*

(3) *There is at most one rigid lattice of each rank vector, and the rank vectors which arise in this way do not depend on R.*

PROOF. (1) Let X be the module, and consider it as a representation of Q, with a f.g. R-module X_i for each vertex i, and a homomorphism $X_a : X_i \to X_j$ for each arrow $a : i \to j$. Since R is a principal ideal domain we can choose a decomposition of each X_i as a direct sum of copies of R and R/π^r with π prime and $r \geq 1$. Let $0 \to X_i \to E_i \to X_i \to 0$ be the direct sum of a split exact sequence $0 \to R \to R^2 \to R \to 0$ for each occurrence of R as a summand of X_i, and a non-split exact sequence

$$0 \to R/\pi^r \xrightarrow{\binom{1}{\pi}} R/\pi^{r-1} \oplus R/\pi^{r+1} \xrightarrow{(-\pi\ 1)} R/\pi^r \to 0$$

for each occurrence of R/π^r as a summand of X_i. If $a : i \to j$ is an arrow then the map $X_a : X_i \to X_j$ can be completed to a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X_i & \longrightarrow & E_i & \longrightarrow & X_i & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle X_a} & & \downarrow{\scriptstyle E_a} & & \downarrow{\scriptstyle X_a} & & \\ 0 & \longrightarrow & X_j & \longrightarrow & E_j & \longrightarrow & X_j & \longrightarrow & 0. \end{array}$$

To prove this we may suppose that X_i and X_j are indecomposable.

If $X_i = R$ then the top sequence splits, X_a factors through the epi $E_j \to X_j$, and it is easy to construct E_a. On the other hand, if $X_i = R/\pi^r$ and the map X_a is nonzero then X_j must be of the form R/π^s. Now X_a is multiplication by $x \in R$ with $\pi^r x \in \pi^s R$, and one can take E_a with matrix $\left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right)$.

We have constructed an exact sequence of representations $0 \to X \to E \to X \to 0$, and by assumption this splits. Thus each short exact sequence $0 \to X_i \to E_i \to X_i \to 0$ splits, so each X_i must be free. This proves the result.

(2) First uniqueness. If X can be written as a direct sum of exceptionals lattices $Y_1 \oplus \cdots \oplus Y_r$ and K is an algebraically closed R-field then X^K is the direct sum of indecomposables $Y_1^K \oplus \cdots \oplus Y_r^K$. By the Krull-Schmidt theorem the Y_i^K are uniquely determined, so by Theorem 1 the Y_i are uniquely determined.

For the existence of a decomposition, fix an algebraically closed R-field F and let $M_1, \dots, M_r$ be the non-isomorphic indecomposable summands of X^F, say

$$X^F \cong M_1^{m_1} \oplus \cdots \oplus M_r^{m_r}.$$

The M_i are exceptional and by [**HR**, Corollary 4.2] they can be ordered so that $(M_1, \dots, M_r)$ is an exceptional sequence. Let $(X_1, \dots, X_r)$ be the sequence of exceptional lattices with the same ranks. It is an exceptional sequence by Lemma 2. For any algebraically closed R-field K we have

$$X^K \cong (X_1^K)^{m_1} \oplus \cdots \oplus (X_r^K)^{m_r}$$

since both sides are rigid of the same dimension vector. Also

$$\operatorname{rank} \operatorname{Hom}(X_r, X) = \dim \operatorname{Hom}(X_r^K, X^K) = m_r > 0.$$

Let $\theta : X_r \otimes_R \operatorname{Hom}(X_r, X) \to X$ be the universal map and let C be its cokernel. Now the map θ^K can be identified with the universal map

$$X_r^K \otimes_K \operatorname{Hom}(X_r^K, X^K) \to X^K$$

which is the inclusion of $(X_r^K)^{m_r}$ as a direct summand of X^K. Since this holds for all K, the map θ is mono and C is free over R. Also

$$C^K \cong \operatorname{Coker}(\theta^K) \cong (X_1^K)^{m_1} \oplus \cdots \oplus (X_{r-1}^K)^{m_{r-1}}$$

so $\operatorname{Ext}^1(C^K, C^K) = 0$ and $\operatorname{Ext}^1(C^K, X_r^K) = 0$, and hence C is rigid and $\operatorname{Ext}^1(C, X_r) = 0$. Thus $X \cong X_r \otimes_R \operatorname{Hom}(X_r, X) \oplus C$, and by induction X is a direct sum of exceptional lattices.

(3) If X and Y are rigid lattices, both of rank α, and if K is an algebraically closed R-field, then X^K and Y^K are rigid KQ-modules of dimension α, so isomorphic. As in the uniqueness part of (2), the assertion $X \cong Y$ follows on considering the decompositions of X and Y into exceptional lattices. The rank vectors which arise can be characterized as the sums $\alpha_1 + \cdots + \alpha_r$ of real Schur roots with $\operatorname{ext}(\alpha_i, \alpha_j) = 0$ for all i, j. □

Remark 1. We can reformulate some of the previous theorem. Fix an R-field $R \to K$ and let M be an KQ-module. By an *R-form* of M we mean an RQ-lattice X and an isomorphism $X^K \to M$. We say that an R-form is *rigid* if it is so as an RQ-lattice. We say that two R-forms X, Y are *conjugate* if there is an isomorphism $\theta : X \to Y$. In this case θ^K is an isomorphism $X^K \to Y^K$, and identifying both sides with M we obtain $\phi \in \operatorname{Aut}_{KQ}(M)$ making the diagram

$$\begin{array}{ccc} X^K & \longrightarrow & M \\ {\scriptstyle \theta^K}\downarrow & & \downarrow{\scriptstyle \phi} \\ Y^K & \longrightarrow & M \end{array}$$

commute.

Observe that in case the homomorphism $R \to K$ is mono the R-forms of M can be identified with RQ-submodules of M, and two R-forms X, Y are conjugate if and only if there is $\phi \in \operatorname{Aut}_{KQ}(M)$ with $\phi(X) = Y$.

Now Theorem 2 implies that any rigid KQ-module has a rigid R-form, unique up to conjugacy.

Remark 2. The theory of perpendicular categories extends to integral representations. If Q has no oriented cycles, X is an exceptional lattice of rank α, and $X^\perp$ is the category of RQ-modules M with $\mathrm{Hom}(X, M) = \mathrm{Ext}^1(X, M) = 0$, then there is an equivalence RQ'-Mod $\to X^\perp$ for some quiver Q' with $n - 1$ vertices and no oriented cycles. To prove this, recall that if K is an algebraically closed R-field then the category of KQ-modules perpendicular to X^K is equivalent to KQ'-Mod for some quiver Q' with $n-1$ vertices [**S1**]. Now the dimension vectors $p_1, \dots, p_{n-1}$ of the indecomposable relative projectives in $(X^K)^\perp$ do not depend on K since they can be characterized as the real Schur roots p with $\hom(\alpha, p) = \mathrm{ext}(\alpha, p) = 0$ and $\mathrm{ext}(p, \beta) = 0$ for all real Schur roots β with $\hom(\alpha, \beta) = \mathrm{ext}(\alpha, \beta) = 0$. Also the dimension vectors $s_1, \dots, s_{n-1}$ of the corresponding simple objects in $(X^K)^\perp$ are independent of K, since they can be deduced from the Cartan matrix $(\langle p_i, p_j\rangle)_{ij}$. Let P_i and S_i be the exceptional lattices of rank vectors p_i and s_i. Now $P = P_1 \oplus \cdots \oplus P_{n-1}$ is rigid, belongs to $X^\perp$, and is a generator for $X^\perp$ (for if $M \in X^\perp$ then $M^K \in (X^K)^\perp$, so P^K generates M^K). Thus P is a f.g. projective generator for $X^\perp$, so $X^\perp$ is equivalent to $\mathrm{End}(P)^{\mathrm{op}}$-Mod. Now $\mathrm{Hom}(P_i, S_i) \cong R$, so we can choose an R-module generator $\theta_i : P_i \to S_i$. Since θ_i^K is an epimorphism for all K, with relative projective kernel, it follows that θ_i is an epimorphism with kernel isomorphic to a direct sum of copies of the P_j. Moreover the number of copies of P_j in a direct sum decomposition of $\mathrm{Ker}(\theta_i)$ is the number of arrows $i \to j$ in Q'. Thus there is a natural map $RQ' \to \mathrm{End}(P)^{\mathrm{op}}$, which is an isomorphism, since it is so on inducing to any R-field.

Remark 3. If Q is Dynkin then the lattices X which are direct sums of exceptionals can be characterized as those with $\dim \tau_K^n X^K$ and $\dim \tau_K^{-n} X^K$ independent of K for all $n \in \mathbb{N}$, where τ_K and τ_K^- are the Auslander-Reiten translates for KQ. For a proof one uses the functors $\tau(-) = D\,\mathrm{Ext}^1(-, RQ)$ and $\tau^-(-) = \mathrm{Ext}^1(D(-), RQ)$.

4. A conjecture

One can conjecture that for each positive real root $\alpha \in \mathbb{N}^n$ there is a unique absolutely indecomposable lattice of rank vector α.

(1) In case R is an algebraically closed field the conjecture is part of Kac's Theorem, see [**K2**], to which we also refer for the definition of real roots in case the quiver has loops. For R an arbitrary field the conjecture holds by a Galois-theoretic argument of Schofield [**S2**].

(2) In case α is a real Schur root the conjecture follows from Theorem 1. If X is a lattice of rank α then by definition X is absolutely indecomposable if and only if X^K is indecomposable for each algebraically closed R-field K. Since α is a real root, by Kac's theorem the exceptional KQ-module of dimension α is the unique indecomposable KQ-module of dimension α. Thus X is absolutely

indecomposable if and only if each X^K is exceptional, so if and only if X is exceptional.

(3) In case Q has only two vertices the conjecture holds: if either vertex has a loop there is nothing to do, since there is at most one positive real root, while if there are no oriented cycles then all positive real roots are real Schur roots. It only remains to deal with the quiver Γ_{ab} with $a > 0$ arrows from 1 to 2 and $b > 0$ arrows from 2 to 1. In this case the claim follows from Ringel's work [**Ri**]. All we need is the following version of Ringel's reflection functor σ_S.

Let S be an exceptional lattice. Let $\mathfrak{M}_S^S$ be the category of lattices X such that for all algebraically closed R-fields K, $\operatorname{Ext}^1(S^K, X^K) = \operatorname{Ext}^1(X^K, S^K) = 0$ and X^K has no summand which can be embedded in, or is a quotient of, a direct sum of copies of S^K. Write $\mathfrak{M}_S^S/S$ for the category obtained by fatoring out those maps which factor through a direct sum of copies of S. Let $\mathfrak{M}_{-S}^{-S}$ be the category of lattices X with $\operatorname{Hom}(S^K, X^K) = \operatorname{Hom}(X^K, S^K) = 0$ for all K. If $X \in \mathfrak{M}_S^S$, it is easy to see that $\operatorname{Hom}(S, X)$ and $\operatorname{Hom}(X, S)$ are both free, $\operatorname{Hom}(S, X)^K \cong \operatorname{Hom}(S^K, X^K)$, and $\operatorname{Hom}(X, S)^K \cong \operatorname{Hom}(X^K, S^K)$. Now the universal maps $\theta : S \otimes_R \operatorname{Hom}(S, X) \to X$ and $\phi : X \to S \otimes_R D\operatorname{Hom}(X, S)$ are a mono- and an epimorphism, since they are on inducing to any R-field [**Ri**, Lemma 2], and the composition $\phi\theta$ is zero, since it is zero over any R-field, [**Ri**, p470]. The assertion is that the functor $\sigma_S(X) = \operatorname{Ker}(\phi)/\operatorname{Im}(\theta)$ defines an equivalence $\mathfrak{M}_S^S/S \to \mathfrak{M}_{-S}^{-S}$, cf. [**Ri**, Proposition 2]. The proof is straightforward.

(4) In case Q is Dynkin or Euclidean the conjecture holds: in order to deal with positive real roots which are not real Schur roots one can use perpendicular categories to reduce to $\tilde{A}_n$ with cyclic orientation, and thence to the quiver Γ_{11}.

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