Two applications of the functorial filtration method

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String Algebras

Let k be a field, Q a quiver, not necessarily finite.
Let ρ be a set of zero relations, that is, paths of length ≥ 2.

A **string algebra** is an algebra $\Lambda = kQ/(\rho)$ satisfying:

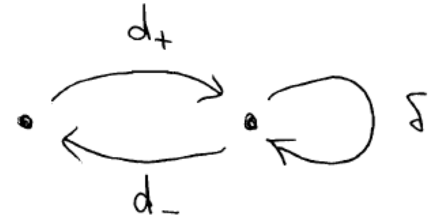
(a) Any vertex of Q is the head of at most two arrows and the tail of at most two arrows, and

(b) Given any arrow y in Q, there is at most one path xy of length 2 with $xy \notin \rho$ and at most one path yz of length 2 with $yz \notin \rho$.

Examples

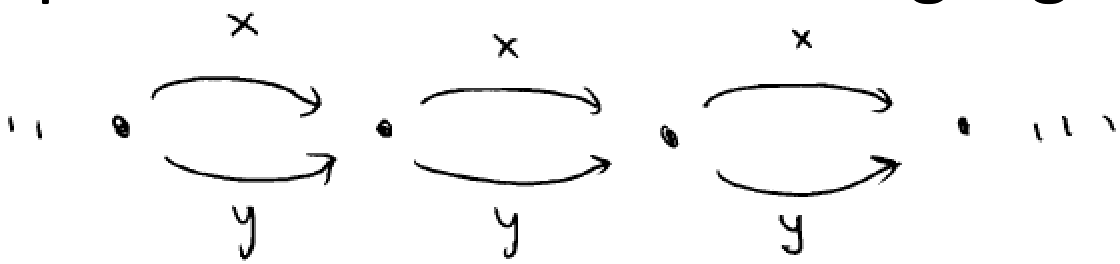
- $k[x,y]/(xy)$ =

- Finite-dimensional version: $k[x,y]/(xy,x^n,y^n)$
- Gelfand-Ponomarev:



$\delta d_+=0$, $d_-\delta=0$
$d^n=0$, $(d_-d_+)^n=0$

- Graded $k[x,y]/(xy)$-modules correspond to representations of the string algebra:



$xy = yx = 0$

Classical Theorem

Theorem. The indecomposable finite-dimensional modules for a finite-dimensional string algebra are classified as “string” and “band” modules

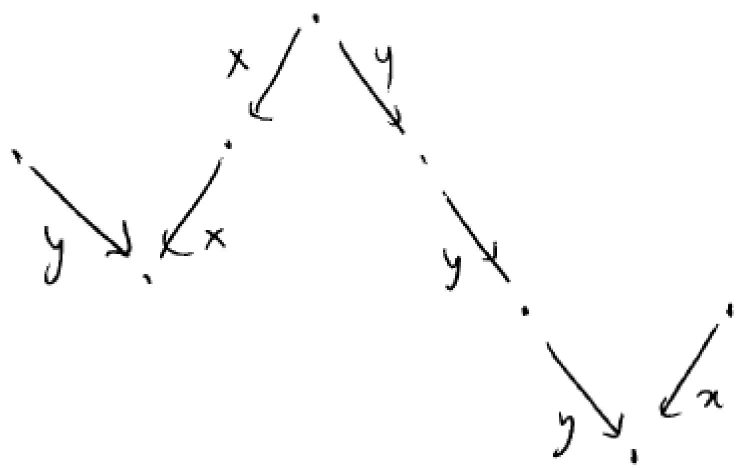
(Special cases due to Gelfand and Ponomarev, Ringel; general case observed by Donovan and Freislich, Wald and Waschbüsch, Butler and Ringel)

String modules

A string module *M(C)* is given by a walk *C* in the quiver

- The walk can reverse direction, but not along the same arrow
- The walk mustn't pass through any zero relations
- The vertices give basis elements of the module *M(C)*
- The arrows show the action of the algebra

Example for the algebra $k[x,y]/(xy)$



Band modules

These are the modules $M(C) \otimes_{k[T,T^{-1}]} V$

where

C is a doubly infinite periodic word
$M(C)$ is the corresponding string module
T acts on $M(C)$ as the shift automorphism
V is a f.d. indecomposable $k[T,T^{-1}]$-module

INDECOMPOSABLE REPRESENTATIONS OF THE LORENTZ GROUP

I.M. Gel'fand and V.A. Ponomarev

Let L be the Lie algebra of the Lorentz group or, what is the same, of the group $SL(2,C)$. We denote by L_k the Lie algebra of its maximal compact subgroup, that is, of $SU(2)$. Let M_i be the finite-dimensional irreducible L_k-modules (the finite-dimensional representations of L_k). Consider an L-module M. The authors call M a *Harish-Chandra module* if, regarded as L_k-module, it can be written as a sum

$$M = \bigoplus_i M_i$$

of finite-dimensional irreducible L_k-modules M_i. Here, for each M_{i_0}, only finitely many L_k-submodules equivalent to M_{i_0} are supposed to occur in the decomposition of M.

A Harish-Chandra module is called indecomposable if it cannot be decomposed into the direct sum of L-submodules. In this paper the indecomposable Harish-Chandra modules over L are completely described. We find that there are two types of indecomposable Harish-Chandra modules. The modules of the first type are the non-singular Harish-Chandra modules and are defined by the following invariants: an integer $2l_0 (l_0 \geqslant 0)$, a complex number l_1, and an integer n. The first two of these invariants are already known as invariants of the irreducible representations of the Lorentz group (see [2]). The case of non-singular modules has been investigated earlier by Zhelobenko [3] from a somewhat different approach.

The case of singular Harish-Chandra modules is of the greatest interest. The solution of this problem reduces to a non-trivial problem of linear algebra, which is investigated in detail in Chapter 2. The invariants of singular indecomposable modules are, as before, numbers l_0, l_1, $l_0 \geqslant 0$, $2l_0$ integral and $2l_0 - |l_1|$ integral.

Uspehi Mat. Nauk 1968, English translation: Russian Math. Surveys

Chapter II

In this chapter we discuss the problem of linear algebra to which we were led in Chapter I in trying to classify the singular modules. At the Mathematical Congress of 1967 Szekeres communicated to us a similar solution. However, we do not know his method, and therefore we are not aware whether the more general problem we need may be solved in the same way. We use here MacLane's notion of relations, which is a generalization of the notion of a linear map. It is remarkable that to work with relations is easier than with linear maps.

And so the problem is the following. We consider two linear spaces P_1 and P_2 and three linear maps:

$$d_+ : \quad P_1 \longrightarrow P_2, \qquad d_- : \quad P_2 \longrightarrow P_1, \qquad \delta : \quad P_2 \longrightarrow P_2, \qquad \delta : \quad P_1 \longrightarrow 0$$

such that $\delta d_+ = 0$, $d_- \delta = 0$, δ and $d_- d_+$ are nilpotent. We have to bring this system to a canonical form.

If we introduce the space $P = P_1 \oplus P_2$ and consider in it the maps a and b given by the matrices

$$a = \begin{pmatrix} 0 & d_+ \\ d_- & 0 \end{pmatrix}, \qquad b = \begin{pmatrix} 0 & 0 \\ 0 & \delta \end{pmatrix},$$

then $ab = ba = 0$, and a and b are nilpotent.

Extract from page 26 of Gelfand and Ponomarev

Math. Ann. 214, 19—34 (1975)

The Indecomposable Representations of the Dihedral 2-Groups

Claus Michael Ringel

Let K be a field. We will give a complete list of the normal forms of pairs a, b of endomorphisms of a K-vector space such that $a^2 = b^2 = 0$. Thus, we determine the modules over the ring $R = K\langle X, Y\rangle/(X^2, Y^2)$ which are finite dimensional as K-vector spaces; here (X^2, Y^2) stands for the ideal generated by X^2 and Y^2 in the free associative algebra $K\langle X, Y\rangle$ in the variables X and Y.

If G is the dihedral group of order $4q$ (where q is a power of 2) generated by the involutions g_1 and g_2, and if the characteristic of K is 2, then the group algebra KG is a factor ring of R, and the KG-modules ${}_{KG}M$ which have no non-zero projective submodule correspond to the K-vector spaces (take the underlying space of ${}_{KG}M$) together with two endomorphisms a and b (namely multiplication by $g_1 - 1$ and $g_2 - 1$, respectively) such that, in addition to $a^2 = b^2 = 0$, also $(ab)^q = (ba)^q = 0$ is satisfied.

We use the methods of Gelfand and Ponomarev developped in their joint paper on the representations of the Lorentz group, where they classify pairs of endomorphisms a, b such that $ab = ba = 0$. The presentation given here follows closely the functorial interpretation of the Gelfand-Ponomarev result by Gabriel, which he exposed in a seminar at Bonn, and the author would like to thank him for many helpful conversations.

§1 The apparatus of relations and the construction of stabilized sequences.

1. Definition: Let $\mathbb{C}^n$ be a n-dimensional complex vector space. Then we mean by a relation any subspace of $\mathbb{C}^n \oplus \mathbb{C}^n$.

To a linear map a: $\mathbb{C}^n \longrightarrow \mathbb{C}^n$ there corresponds the subspace of pairs (x, ax) in $\mathbb{C}^n \oplus \mathbb{C}^n$, where x ranges over $\mathbb{C}^n$. Thus, a relation is a generalization of a linear map. In particular, the zero map θ: $\mathbb{C}^n \to 0$ corresponds to the relation θ, consisting of the pairs $(x, 0)$, where x ranges over $\mathbb{C}^n$.

The identity map $I: \mathbb{C}^n \to \mathbb{C}^n$ corresponds to the relation I, consisting of the pairs (x, x), as x ranges over $\mathbb{C}^n$.

We denote the relation corresponding to a linear map a by the same symbol a.

If A is a relation consisting of some set of pairs (x, y), then the set of all pairs (y, x) defines a new subspace of $\mathbb{C}^n \oplus \mathbb{C}^n$, called the inverse relation $A^{\#}$. It is easy to verify that if a is an invertible linear map, then $a^{\#} = a^{-1}$.

We define the dimension of a relation A as the dimension of the subspace A of $\mathbb{C}^n \oplus \mathbb{C}^n$. It is denoted by dim $A = m(A)$.

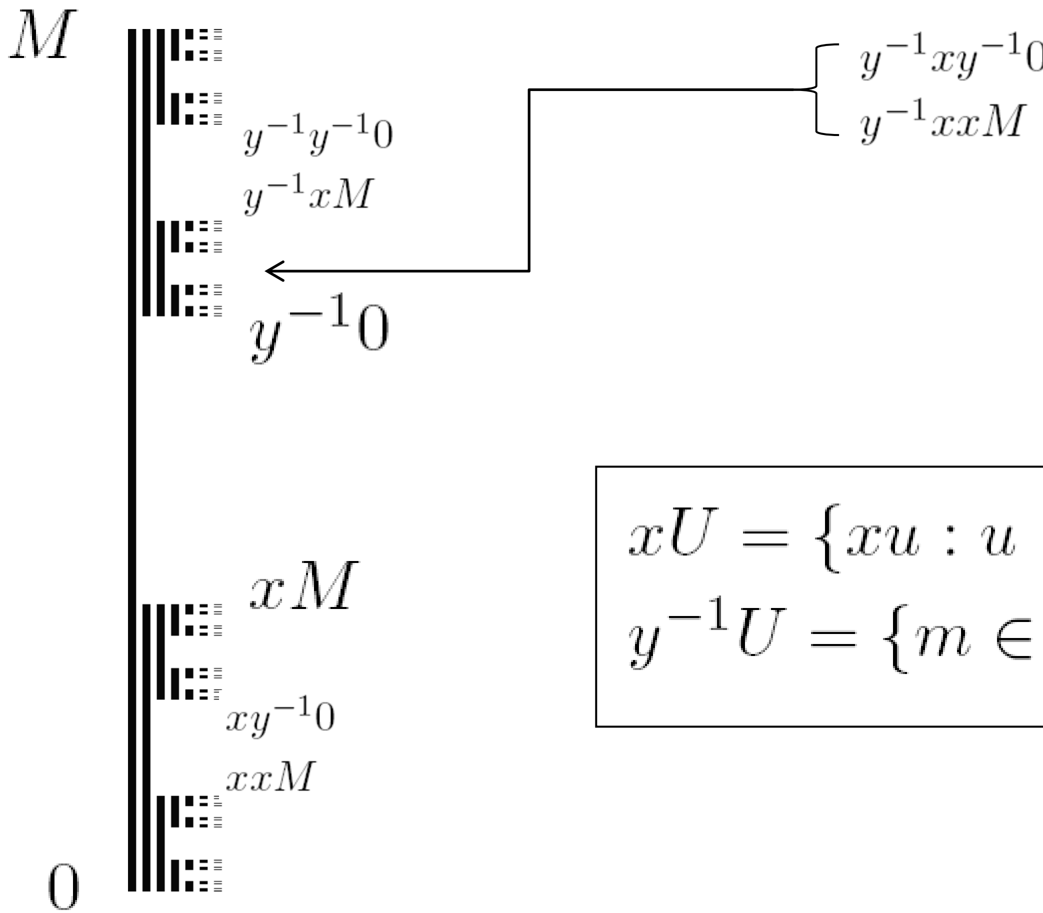
Let A and B be two relations. Their product AB is defined in the following manner:[1] $(x, z) \in AB$ if and only if there exists a y such that $(x, y) \in B$ and $(y, z) \in A$.

It is easy to verify that if the relations are defined by linear maps, then this definition coincides with the generally accepted definition of multiplication of maps.

Extract from page 27 of Gelfand and Ponomarev

"Cantor set"-type filtration

Vector subspaces of a $k[x,y]/(xy)$-module M



$$xU = \{xu : u \in U\}$$
$$y^{-1}U = \{m \in M : ym \in U\}$$

First new development: Finitely generated/controlled modules

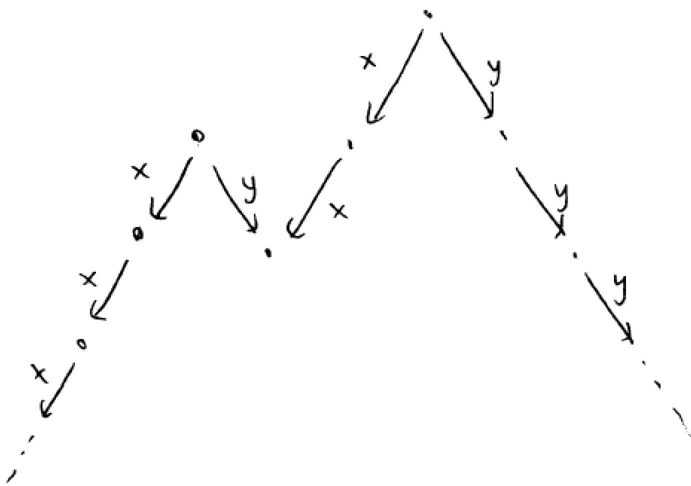
Definition. A module M is **finitely controlled** if for all a in the algebra, aM is contained in a finitely generated submodule.

Theorem. Any finitely controlled module for a string algebra is a direct sum of string and band modules.

These summands are indecomposable and uniquely determined up to isomorphism.

Example: $k[x,y]/(xy)$

The finitely generated modules which are not finite-dimensional come from infinite strings, such as



(For this commutative algebra the classification is not new, e.g. work of L. S. Levy.)

New ideas needed

- Results about relations on infinite-dimensional vector spaces.
- Right definition and properties of functors corresponding to infinite words (limit points in the Cantor set).
- Given M, the functorial filtration argument gives a submodule N which is a direct sum of string and band modules, such that M/N is ‘primitive torsion’. Need to adjust N to make it equal to M.

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Topological data analysis

From Wikipedia, the free encyclopedia

Topological data analysis (TDA) is a new area of study aimed at having applications in areas such as data mining and computer vision. The main problems are:

1. how one infers high-dimensional structure from low-dimensional representations; and
2. how one assembles discrete points into global structure.

The human brain can easily extract global structure from representations in a strictly lower dimension, i.e. we infer a 3D environment from a 2D image from each eye. The inference of global structure also occurs when converting discrete data into continuous images, e.g. dot-matrix printers and televisions communicate images via arrays of discrete points.

The main method used by topological data analysis is:

1. Replace a set of data points with a family of simplicial complexes, indexed by a proximity parameter.
2. Analyse these topological complexes via algebraic topology — specifically, via the theory of persistent homology.[1]
3. Encode the persistent homology of a data set in the form of a parameterized version of a Betti number which is called a **persistence diagram** or **barcode**.[1]

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Notices of the AMS, January 2011

WHAT IS...

Persistent Homology?

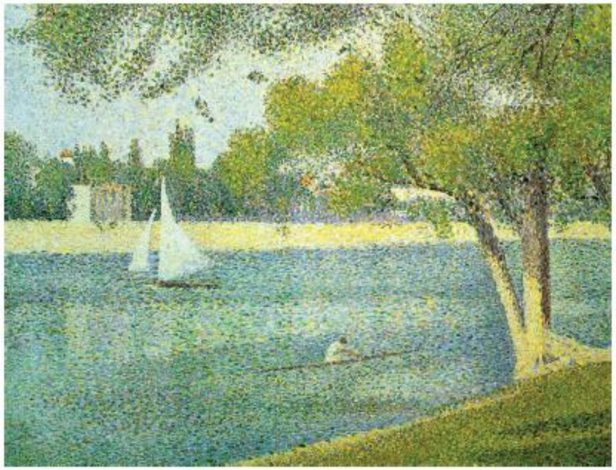
Shmuel Weinberger

In memory of my friend, Partha Niyogi (1967–2010)

Consider the art of Seurat or a piece of old newsprint. The eye, or the brain, performs the marvelous task of taking the sense data of individual points and assembling them into a coherent image of a continuum—it infers the continuous from the discrete.

Difficult issues of a similar sort occur in many problems of data analysis. One might have samples that are chosen nonuniformly (e.g., not filling a grid), and, moreover, one is constantly plagued by problems of noise—the data can be corrupted in various ways.

Pure mathematicians have problems of this sort as well. One is often interested in inferring properties of an enveloping space from a discrete object within it or, in reverse, seeking commonalities of all the discrete subobjects of a given continuous one. To give one example, this theme is a central one in geometric group theory, in which a typical



The Seine at La Grande Jatte by Georges Seurat.

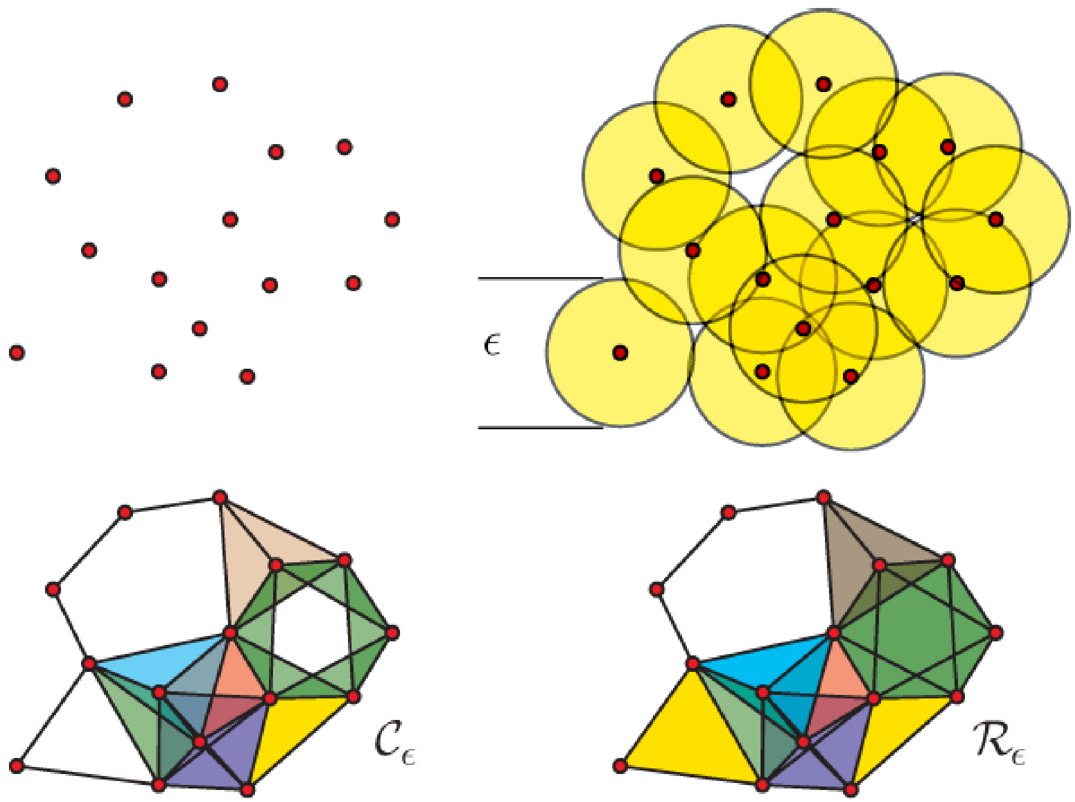


FIGURE 2. A fixed set of points [upper left] can be completed to a Čech complex $\mathcal{C}_\epsilon$ [lower left] or to a Rips complex $\mathcal{R}_\epsilon$ [lower right] based on a proximity parameter ϵ [upper right]. This Čech complex has the homotopy type of the $\epsilon/2$ cover ($S^1 \vee S^1 \vee S^1$), while the Rips complex has a wholly different homotopy type ($S^1 \vee S^2$).

R. Ghrist, Barcodes: the persistent topology of data, Bulletin of the AMS 2008

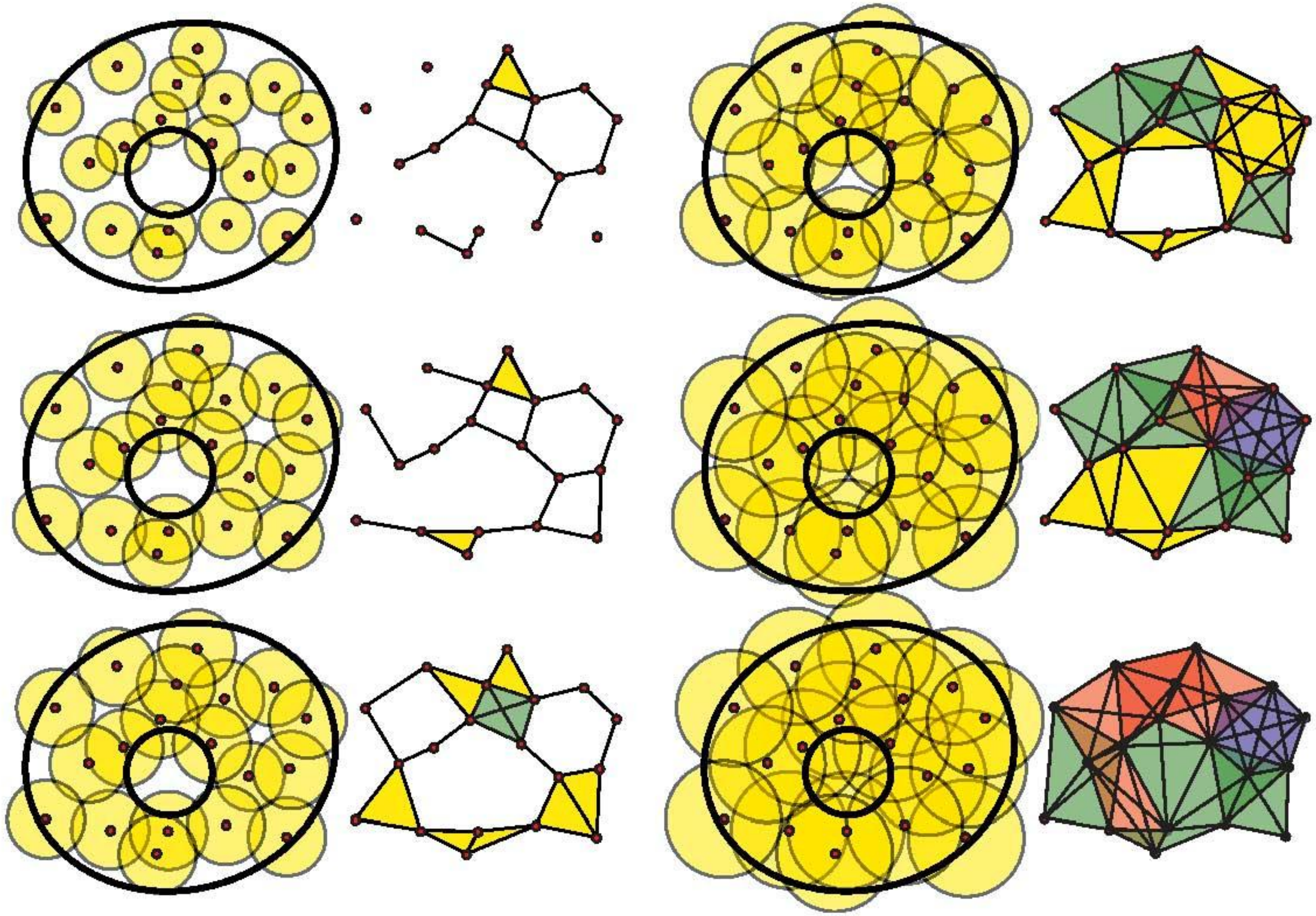


FIGURE 3. A sequence of Rips complexes for a point cloud data set representing an annulus. Upon increasing ϵ, holes appear and disappear. Which holes are real and which are noise?

R. Ghrist, Barcodes: the persistent topology of data, Bulletin of the AMS 2008

Persistence modules

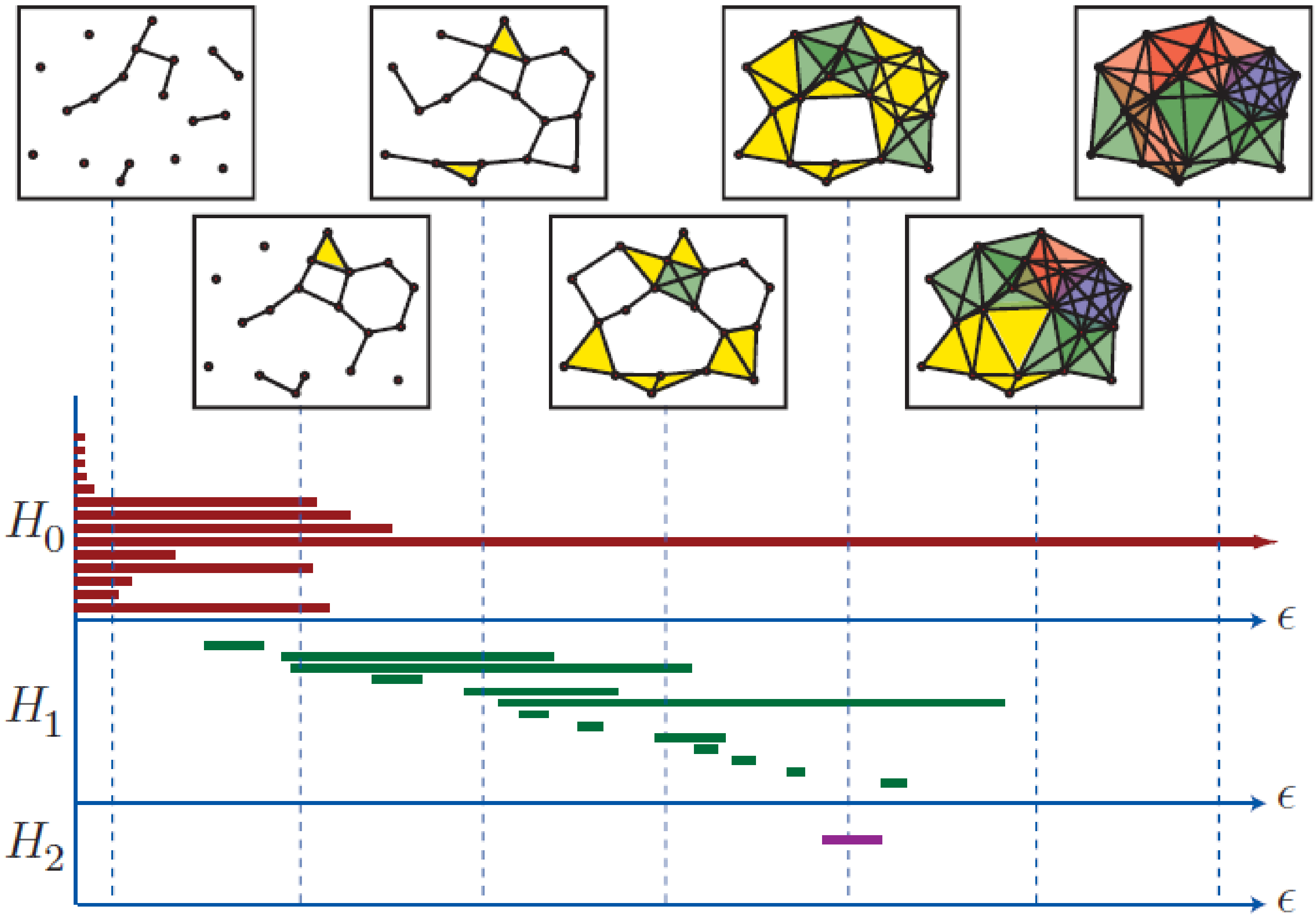
Let R be a totally ordered set. A **persistence module** V is a functor from R (considered as a category) to vector spaces. Thus it is given by vector spaces V_t for $t \in R$ and linear maps $\rho_{ts} : V_s \to V_t$ for $s \leq t$.

Associated to any interval I in R (=convex subset) there is an **interval module** $V = k_I$ with

$$V_t = \begin{cases} k & (t \in I) \\ 0 & (t \notin I). \end{cases}$$

In simple cases a persistence module can be written as a direct sum of interval modules. The set of intervals is the **barcode**.

Persistent homology gives a persistence module indexed by $\mathbb{R}^+$ with $V_t = H_n(X_t, k)$, where $X_t = \{\text{union of balls of radius } t\}$, and with ρ_{ts} induced by the inclusion $X_s \subseteq X_t$.



R. Ghrist, Barcodes: the persistent topology of data, Bulletin of the AMS 2008

Decomposition result
(This version joint with V. de Silva and F. Chazal)

Assume that R is separable in the order topology (can weaken).

Theorem. If V has the following descending chain conditions, then it is a direct sum of interval modules

dcc on images: for all $t \geq s_1 > s_2 > \ldots$ the following chain stabilizes

$$V_t \supseteq \mathrm{Im}(\rho_{ts_1}) \supseteq \mathrm{Im}(\rho_{ts_2}) \supseteq \ldots$$

dcc on sufficient bounded kernels: for all $t \in R$ and $v \in V_t$ there is $s \leq t$ such that (i) $v \in \mathrm{Im}(\rho_{ts})$ and (ii) for all $t < \ldots < r_2 < r_1$ the following chain stabilizes

$$V_s \supseteq \mathrm{Ker}(\rho_{r_1 s}) \supseteq \mathrm{Ker}(\rho_{r_2 s}) \supseteq \ldots$$

Special case 1. If V is **pointwise finite-dimensional**, that is, all V_t are finite dimensional, then V is a direct sum of interval modules.

Observable category

A persistence module V is **ephemeral** if $\rho_{ts} = 0$ for $s < t$ (Same as semisimple!)

Suppose R is a dense order. Then the ephemeral modules form a Serre subcategory and we define

$$\textbf{Observable category} = \frac{\text{Persistence modules}}{\text{Ephemeral modules}}$$

Special case 2. If V is a **q-tame** persistence module, that is, ρ_{ts} has finite rank for $s < t$, then $\operatorname{rad} V$ is a direct sum of interval modules.

Thus V decomposes in the observable category into interval modules.

The filtrations

An interval is given by two **cuts**

$$c = (c^-, c^+), \quad R = c^- \cup c^+, \quad s \in c^-, t \in c^+ \Rightarrow s < t.$$

Given $t \in R$, the cuts c with $t \in c^+$ give subspaces of V_t:

$$\begin{aligned} \mathrm{Im}^+_{ct} &= \bigcup_{s \in c^-} \mathrm{Im}(\rho_{ts}), \\ \mathrm{Im}^-_{ct} &= \bigcap_{\substack{s \in c^+ \\ s \leq t}} \mathrm{Im}(\rho_{ts}). \end{aligned}$$

Using kernels, the cuts c with $t \in c^-$ give subspaces of V_t.

End