Exercises to “Introduction to Analysis on Graphs”

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1. Let $\Gamma = (V, E)$ be a simple, connected, locally finite graph. The diameter of $\Gamma = (V, E)$ is defined by
$$\operatorname{diam} \Gamma = \sup_{x,y \in V} d(x, y).$$
   (a) Prove that Γ is finite if and only if $\operatorname{diam} \Gamma < \infty$.

   (b) Prove that Γ is a complete graph if and only if $\operatorname{diam} \Gamma = 1$.

   (c) Prove that, for any vertex x and for any positive integer $n \le \frac{1}{2} \operatorname{diam} \Gamma$, there is a vertex $y \in V$ such that $d(x, y) = n$.

Solution. (a) If Γ is finite then
$$d(x, y) = \max_{x,y \in V} d(x, y) < \infty.$$
Assume that Γ is infinite. Fix a point $x_0 \in V$ and consider the balls
$$B_n = \{x \in V : d(x, x_0) \le n\}.$$
Since Γ is locally finite, the sets B_n are finite. Hence, $V \setminus B_n$ is non-empty for any n. This means that, for any n, there is $x \in V$ such that $d(x, x_0) > n$, which implies $\operatorname{diam} \Gamma = \infty$

(b) Since in K_n we have $x \sim y$ for all distinct vertices x, y, it follows that $d(x, y) = 1$ and, hence, $\operatorname{diam} K_n = 1$. Conversely, let $\operatorname{diam} \Gamma = 1$. By part (a), Γ is finite, let $|V| = n$. Also, $\operatorname{diam} \Gamma = 1$ implies that $d(x, y)$ can take only two values: 0 (if $x = y$) and 1 (if $x \ne y$). Hence, $d(x, y) = 1$ for any two distinct points, which means $x \sim y$ and $\Gamma = K_n$.

(c) Let us first show that there is $z \in V$ such that $d(x, z) \ge n$. Assume from the contrary that $d(x, z) < n$ for all $z \in V$. Then, for any two points $z_1, z_2 \in V$,
$$d(z_1, z_2) \le d(z_1, x) + d(x, z_2) < 2n.$$
If the graph is finite then this implies that
$$\operatorname{diam} \Gamma = \max_{z_1,z_2 \in V} d(z_1, d_2) < 2n$$
which contradicts the assumption that $n \le \frac{1}{2} \operatorname{diam} \Gamma$. If graph Γ is infinite then
$$\sup_{z_1,z_2 \in V} d(z_1, z_2) = \operatorname{diam} \Gamma = \infty$$
which contradicts $d(z_1, z_2) < 2n$.

Now fix a point z with the property that $m := d(x, z) \ge n$ and chose a shortest path $\{x_k\}_{k=1}^m$ connecting x and z. Since
$$x = x_0 \sim x_1 \sim ... \sim x_{m-1} \sim x_m = z,$$

it follows that x and x_n are connected by a path of length n, whence $d(x, x_n) \leq n$. We claim that $d(x, x_n) = n$ (this will finish the proof). Indeed, if $d(x, x_n) < n$ then by the triangle inequality we obtain

$$d(x, y) \leq d(x, x_n) + d(x_n, z) < n + (m - n) = m,$$

which contradicts $d(x, y) = m$.

2. Let (V, E) be a simple finite graph.
   (a) Assume that there exists a vertex path on (V, E) that contains every edge exactly once (an Euler walk). Prove that the set

   $$M = \{\deg(x) : x \in V\}$$

   either contains no odd number or contains exactly two odd numbers.

   (b) Show that the graph on Fig. 1 has no Euler walk.

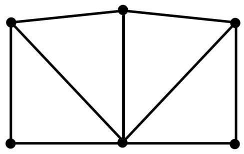


Figure 1: The graph of Königsberg bridges

Solution. (a) Let $\{x_k\}_{k=0}^n$ be an Euler walk, so that the sequence of edges $\{x_{k-1}x_k\}_{k=1}^n$ contains every edge of the graph exactly once. Let x be one of the vertices x_k. If $x \neq x_0$ and $x \neq x_n$ then $\deg(x)$ must be even. Indeed, let $k_1, ..., k_l$ be all the numbers such that

$$x = x_{k_1} = x_{k_2} = ... = x_{k_l}. \tag{1}$$

Then all the edges adjacent to x are

$$x_{k_i-1}x_{k_i} \text{ and } x_{k_i}x_{k_i+1}, \quad i = 1, ..., l \tag{2}$$

that is, $\deg(x) = 2l$.

Similarly, if $x = x_0 = x_n$ then $\deg(x)$ is even. Consider the case $x = x_0 \neq x_n$. Let $k_1, ..., k_l$ be all positive numbers such that (1) holds. Then (2) gives all the edges adjacent to x except for x_0x_1. Hence, $\deg(x) = 2l + 1$. Similarly, $\deg(x)$ is odd in the case $x = x_n \neq x_0$. Combining all the cases, we see that $\deg(x)$ is either always even or odd at two vertices, which finishes the proof.

(b) This graph contains 4 vertices with odd degree. Hence, it has no Euler walk.

3. A simple connected graph is called bipartite if it admits a coloring of its vertices into two colors, say, black and white, so that the vertices of the same color are not connected by an edge (for example, the set of fields of a chessboard is a bipartite graph). Prove that a simple connected graph is bipartite if and only if it contains no cycle C_n with odd n.

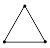
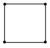
   Hint: Chose the color of a vertex x depending on the distance $d(x, x_0)$ where x_0 is a fixed vertex.

   Solution. Chose an arbitrary vertex x_0 and color any vertex x as follows: it is white if $d(x, x_0)$ is even, and black otherwise. Let us prove that if x, y two vertices of the same color then they are not connected. Let for certainty x and y be white so that both distances $a = d(x, x_0)$ and $b = d(y, x_0)$

are even. Then there exists path $\{x_k\}_{k=0}^{a}$ connecting x_0 and x, and a path $\{y_k\}_{k=0}^{b}$ connecting x_0 and y. If $x \sim y$ then we obtain a closed path

$$x_0 \sim x_1 \sim ... \sim x_a = x \sim y = y_b \sim y_{b-1} \sim ... \sim y_0 = x_0,$$

that is, a cycle of length $a+b+1$. However, $a+b+1$ is an odd number, and the graph contains no cycles of this length. Hence, $x \not\sim y$, which was to be proved. The case when x and y are both black is similar.

4. A graph (V_1, E_1) is said to be a *subgraph* of (V, E) if $V_1 \subset V$ and $E_1 \subset E$. Two graphs (V_1, E_1) and (V_2, E_2) are said to be *isomorphic* if there is a bijection $\varphi : V_1 \hookrightarrow V_2$ that preserves edges, that is, $(\varphi(x), \varphi(y)) \in E_2$ if and only if $(x, y) \in E_1$. Given two graphs $\Gamma = (V, E)$ and $\Gamma_1 = (V_1, E_1)$ denote by $N(\Gamma, \Gamma_1)$ the number of distinct subgraphs of Γ that are isomorphic to Γ_1. For example, let K_n be a complete graph with n vertices and C_n be a cycle with n vertices. Then $N(\Gamma, K_1)$ is the number of vertices of graph Γ, $N(\Gamma, K_2)$ is the number of edges of Γ, $N(\Gamma, K_3)$ is the number of complete triangles  to be found in Γ, $N(\Gamma, C_4)$ is the number of 4-cycles  to be found in Γ.

   Evaluate $N(K_n, C_k)$ for all $n \geq k \geq 3$.

   Solution. Let $x_1 x_2 ... x_k$ be a k-cycle in K_n. The point x_1 can be chosen in n ways, the point x_2 - in $n-1$ ways, etc. Hence, there are $n(n-1)...(n-k+1)$ choices altogether. However, in this argument we have counted each cycle $2k$ times. Indeed, let $x_1...x_k$ and $y_1...y_k$ be two identical cycles. Then y_1 may be equal to any of the points $x_1, ..., x_k$. Having identified y_1 with some x_i, we have left with two possibilities for y_2: this must be either x_{i-1} or x_{i+1}. Having identified y_1 and y_2 with x_i and x_j, all other points $y_3, ..., y_k$ have unique identifications with x_i's. Hence, the above number of *labeled* cycles has to be divided by $2k$ so that

$$N(K_n, C_k) = \frac{n(n-1)...(n-k+1)}{2k}.$$

5. Let $K_{n,m}$ be a complete bipartite graph. Evaluate $N(K_{n,m}, C_k)$ for all $k \geq 3$.

   Solution. Let V_1 and V_2 be the partition of $K_{n,m}$ so that $|V_1| = n$, $|V_2| = m$ and two vertices x, y are connected if and only if they belong to different parts V_i. Any cycle $x_1 x_2 ... x_k$ is a sequence of vertices such that $x_i \sim x_{i+1}$ and $x_k \sim x_1$. Equivalently, x_i and x_{i+1} must belong to different parts, and so do x_1 and x_k. This implies that the number k of vertices in the cycle must be even. That is, if k is odd then $N(K_{n,m}, C_k) = 0$. Assume further that k is even. By renaming the vertices of a cycle, we can always assume that $x_1 \in V_1$, then $x_2 \in V_2$, $x_3 \in V_1$,..., $x_k \in V_2$. It follows that each part V_i contains $k/2$ elements of the cycles. Hence, we must have the inequality

$$\min(n, m) \geq k/2. \tag{3}$$

   In other words, if $k/2 > \min(n, m)$ then $N(K_{n,m}, C_k) = 0$. We proceed under the assumption (3). The value of x_1 can be chosen in n ways, the value of x_2 - in m ways, then the value of x_3 - in $n-1$ ways (because x_1 is already chosen in V_1), etc. Therefore, altogether the number of choices is

$$nm(n-1)(m-1)(n-2)(m-2)...(n-k/2+1)(m-k/2+1).$$

   However, some of the cycles counted this way may be the same. For examples, the cycles $x_1 x_2 .... x_n$ and $x_3 x_4 .... x_k x_1 x_2$ are obviously the same but each of them was counted separately in the above argument. Suppose we have two cycles $x_1...x_k$ and $y_1...y_k$ constructed as above. When do they coincide? Since $y_1 \in V_1$, the point y_1 may coincide with any of the points $x_1, x_3, .., x_{k-1}$, which gives $k/2$ possibilities. Let $y_1 = x_i$. Then the point y_2 can coincides with x_{i+1} or x_{i-1} which gives two possibilities. Once we have identified the points y_1 and y_2 with x_i and x_j, the rest points $y_3, ..., y_k$ have unique identifications with x_l's. Hence, each cycle was counted above $2 \cdot k/2 = k$ times. Therefore,

$$N\left(K_{n,m}, C_k\right) = \frac{nm\left(n-1\right)\left(m-1\right)\left(n-2\right)\left(m-2\right)...\left(n-k/2+1\right)\left(m-k/2+1\right)}{k}.$$

6. A complete m-partite graph $K_{n_1,...,n_m}$ is defined as follows. It has $n_1 + ... + n_m$ vertices that are split into m groups $V_1, ..., V_m$ such that $|V_k| = n_k$, and two vertices x, y are connected if and only if they belong to different groups V_k. Prove that if $n_1 = ... = n_m = n$ then the graph $K_{n,...,n}$ is a Cayley graph.

   Solution. Consider the group $\mathbb{Z}_n \times \mathbb{Z}_m$ and denote its elements by (α, i) where $\alpha \in \mathbb{Z}_n$ and $i \in \mathbb{Z}_m$. Then $(\alpha, i) \sim (\beta, j)$ if and only if $i \neq j$, which can be stated as follows: $(\alpha, i) \sim (\beta, j)$ if and only if

   $$(\alpha, i) - (\beta, j) \in S$$

   where $S = \mathbb{Z}_n \times (\mathbb{Z}_m \setminus \{0\})$. Hence, $K_{\underbrace{n, ..., n}_{m}}$ is the Cayley graph of $\mathbb{Z}_n \times \mathbb{Z}_m$ with the edge generating set S.

7. Prove that the numbers $a_j = N\left(K_{n_1,...,n_m}, K_j\right)$ satisfy the following identity:

   $$\prod_{i=1}^{m}\left(z - n_i\right) = z^n - a_1 z^{n-1} + a_2 z^{n-2} + ... + \left(-1\right)^m a_m$$

   for all complex z.

   Solution. Any complete graph K_j that is a subgraph of $K_{n_1,...,n_m}$ has j vertices in different partitions V_k. If we fix j partitions $V_{i_1}, ..., V_{i_j}$ where $i_1 < i_2 < ... < i_j$ then they give rise to $n_{i_1} n_{i_2} ... n_{i_j}$ complete graphs. Hence, the total number of K_j subgraphs is

   $$a_j = \sum_{i_1 < i_2 < ... < i_j} n_{i_1} n_{i_2} ... n_{i_j}.$$

   For example, $a_1 = n_1 + ... + n_m$ (the number of vertices),

   $$a_2 = \left(n_1 n_2 + n_1 n_3 + ... + n_1 n_m\right) + \left(n_2 n_3 + n_2 n_4 + ... + n_2 n_m\right) + ... + n_{m-1} n_m$$

   (the number of edges),

   $$a_2 = \left(n_1 n_2 n_3 + ... + n_1 n_2 n_m\right) + ... + n_{m-2} n_{m-1} n_m$$

   (the number of triangles), etc, and

   $$a_m = n_1 ... n_m$$

   is the number of subgraphs K_m. By Viète's formulas, the coefficients of the polynomial of z, that has the roots $n_1, ..., n_m$, are exactly $\left(-1\right)^i a_i$, which proves the claim.

8. A subset S of a group G is called generating if any element $x \in G$ can be represented in the form

   $$x = s_1 * s_2 * ... * s_n$$

   for some positive integer n and with some $s_k \in S$. Prove that if S is a symmetric generating subset of G then the Cayley graph (G, S) is connected. (A graph (V, E) is called connected if, for any two vertices $x, y \in V$, there is an edge path in (V, E) connecting x and y.)

Solution. For any two vertices $x, y \in G$, we need to construct a path connecting x and y in (G, S). The group element $x^{-1} * y$ can be represented in the form

$$x^{-1} * y = s_1 * s_2 * ... * s_n$$

for some positive integer n and $s_k \in S$. It follows that

$$y = x * s_1 * s_2 * ... * s_n.$$

Consider a sequence $\{x_k\}_{k=0}^{n}$ defined by

$$\begin{aligned} x_0 &= x \\ x_1 &= x * s_1 \\ x_2 &= x * s_1 * s_2 \\ & ... \\ x_n &= x * s_1 * ... * s_n. \end{aligned}$$

Clearly, $x_{k+1} = x_k * s_{k+1}$ whence $x_k^{-1} * x_{k+1} = s_{k+1} \in S$ so that $x_k \sim x_{k+1}$. Since $x_0 = x$ and $x_n = y$, we have obtained a path connecting x and y, which means that the graph (G, S) is connected.

9. A graph (V, E) is called regular if $\deg(x)$ is the same for all $x \in V$. The following graphs are obviously regular: all Cayley graphs, cycles C_n, complete graphs K_n, complete multipartite graphs $K_{n,...,n}$, and their products.

   (a) List all connected regular graphs with at most 6 vertices. Show that every such graph is a Cayley graph. Show that every such graph belongs to one of the families

   $$C_n,\ K_n,\ K_n \Box K_m,\ K_{n,..,n}. \tag{4}$$

   (b) Give an example of a connected regular graph with 7 vertices that is non-Cayley and that does not belong to any of the families (4).

Solution. (a) All regular graphs with ≤ 6 vertices are as follows:

- $\mathbb{Z}_2 = K_2$,
- $\mathbb{Z}_3 = K_3$,
- $(\mathbb{Z}_4, \{\pm 1\}) = C_4 = K_{2,2} = K_2 \Box K_2$,
- $(\mathbb{Z}_4, \{\pm 1, 2\}) = K_4$ (tetrahedron),
- $(\mathbb{Z}_5, \{\pm 1\}) = C_5$,
- $\{\mathbb{Z}_5, \{\pm 1, \pm 2\}\} = K_5$,
- $(\mathbb{Z}_6, \{\pm 1\}) = C_6$,
- $(\mathbb{Z}_6, \{\pm 1, \pm 2\}) = K_{2,2,2}$ (octahedron),
- $(\mathbb{Z}_6, \{\pm 1, \pm 2, 3\}) = K_6$,
- $(\mathbb{Z}_3 \times \mathbb{Z}_2, \{(0,1), (\pm 1, 1)\}) = K_{3,3}$,
- $(\mathbb{Z}_3 \times \mathbb{Z}_2, \{(0,1), (\pm 1, 0)\}) = K_3 \Box K_2$.

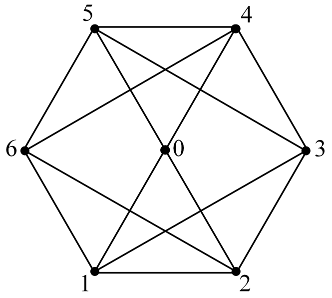


Figure 2: A 4-regular graph with 7 vertices

The last graph can also being seen as $(D_3, \{r, s\})$ where D_3 is a dihedral group, that is, the group of symmetries of an equilateral triangle, and r refers to a rotation, s – to a mirror symmetry.

(b) Fig. 2 contains a 4-regular graph with 7 vertices, that is not a Cayley graph.

Its edges form three cycles $1 \sim 2 \sim 3 \sim 4 \sim 5 \sim 6 \sim 1$, $0 \sim 1 \sim 3 \sim 5 \sim 0$ and $0 \sim 2 \sim 6 \sim 4 \sim 0$. This graph is not Cayley because there are three triangles with the vertex 1, namely, 126, $123, 102$ while there are only two triangles with the vertex 0: 054 and 012. Clearly, this graph is neither C_n, nor K_n, nor $K_{n,\dots,n}$ (the latter because 7 is a prime).

10. Let P, Q be Markov kernels on a finite or countable set V. Consider a function $P \circ Q$ on $V \times V$ that is defined by

$$(P \circ Q)(x, y) = \sum_{z \in V} P(x, z) Q(z, y).$$

   (a) Prove that $P \circ Q$ is a Markov kernel.

   (b) Prove that if R is a Markov kernel then

$$(P \circ Q) \circ R = P \circ (Q \circ R).$$

   (c) Prove that if P and Q are reversible with respect to the same function $\mu(x)$ and $P \circ Q = Q \circ P$ then $P \circ Q$ is also reversible with respect to $\mu(x)$.

Solution. (a) Clearly, $P \circ Q(x, y) \geq 0$. We need to verify that

$$\sum_{y \in V} P \circ Q(x, y) = 1.$$

Indeed, we have

$$\begin{aligned}
\sum_{y \in V} P \circ Q(x, y) &= \sum_{y \in V} \sum_{z \in V} P(x, z) Q(z, y) \\
&= \sum_{z \in V} \sum_{y \in V} P(x, z) Q(z, y) \\
&= \sum_{z \in V} \left(P(x, z) \sum_{y \in V} Q(z, y) \right) \\
&= \sum_{z \in V} P(x, z) \\
&= 1.
\end{aligned}$$

(b) We have

$$\begin{aligned}(P\circ Q)\circ R(x,y) &= \sum_{y\in V}(P\circ Q)(x,z)\,R(z,y)\\ &= \sum_{y\in V}\left(\sum_{u\in V}P(x,u)\,Q(u,z)\right)R(z,y)\\ &= \sum_{y\in V}\sum_{u\in V}P(x,u)\,Q(u,z)\,R(z,y)\\ &= \sum_{u\in V}P(x,u)\sum_{y\in V}Q(u,z)\,R(z,y)\\ &= \sum_{u\in V}P(x,u)\,(Q\circ R)(u,y)\\ &= P\circ(Q\circ R)(x,y)\,.\end{aligned}$$

(c) If

$$\mu(x)\,P(x,y)=\mu(y)\,P(y,x)$$

and the same is true for Q then

$$\begin{aligned}\mu(x)\,(P\circ Q)(x,y) &= \sum_{z\in V}\mu(x)\,P(x,z)\,Q(z,y)\\ &= \sum_{z\in V}\mu(z)\,P(z,x)\,Q(z,y)\\ &= \sum_{z\in V}\mu(z)\,Q(z,y)\,P(z,x)\\ &= \sum_{z\in V}\mu(y)\,Q(y,z)\,P(z,x)\\ &= \mu(y)\sum_{z\in V}Q(y,z)\,P(z,x)\\ &= \mu(y)\,(Q\circ P)(y,x)\\ &= \mu(y)\,(P\circ Q)(y,x)\,,\end{aligned}$$

where in the last line we have used that $P\circ Q=Q\circ P$.

11. Let (V,E) be a finite graph without loops and let μ be a simple weight on (V,E). Prove that $\operatorname{trace}\Delta_\mu=-|V|$.

Solution. The Laplace operator has the form

$$\Delta_\mu f(x)=\frac{1}{\mu(x)}\sum_{y\in V}f(y)\,\mu_{xy}-f(x)\,,$$

and it acts in the space $\mathcal{F}$ of all functions on V. Denote the vertices of V by $1,2,...,m$, where $m=|V|$. Then every function f on V is identifies with the sequence $\{f_1,...,f_m\}$ where $f_k=f(k)$. With this identification, the space $\mathcal{F}$ is linear isomorphic to $\mathbb{R}^m$. With this notation, we have

$$(\Delta_\mu f)_i=\frac{1}{\mu_i}\sum_{j=1}^m\mu_{ij}f_j-f_i$$

so that the matrix A of the operator Δ_μ in the canonical basis of $\mathbb{R}^n$ is given by

$$A_{ij} = \begin{cases} -1, & i = j \\ \frac{\mu_{ij}}{\mu_i}, & i \neq j \end{cases}$$

where we have used that $\mu_{ii} = 0$. Hence,

$$\operatorname{trace} \Delta_\mu = \operatorname{trace} A = -m.$$

12. Let Γ be a simple graph with $m \geq 3$ vertices and n edges.

    (a) Prove that if $n \geq \left\lfloor \frac{m^2}{4} \right\rfloor + 1$ then $N(\Gamma, K_3) \geq 1$.

    (b) For any $m \geq 3$, give and example of a graph Γ with $n = \left\lfloor \frac{m^2}{4} \right\rfloor$ edges such that $N(\Gamma, K_3) = 0$.

    (c) Prove that if $n \geq \left\lfloor \frac{m^2}{4} \right\rfloor + 1$ then, in fact, $N(\Gamma, K_3) \geq \left\lfloor \frac{m}{2} \right\rfloor$.

    Solution. (a) is a theorem of Mantel, (c) is its improvement.

13. Consider the *Petersen graph* as on Fig 3.

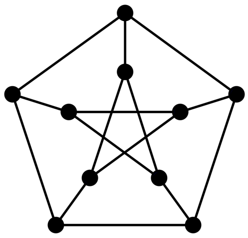


Figure 3: Petersen graph

Prove that this graph is not a Cayley graph.

Solution. Assume that the Petersen graph is a Cayley graph of a group G with the edge generating set S. Observe first that the Petersen graph does not contain any 4-cycle, which implies that any two distinct elements $a, b \in S$ do not commute. Indeed, if $ab = ba$ then we obtain a 4-cycle

$$e \sim a \sim ab \sim b \sim e.$$

Since the graph has 10 vertices, we have $|G| = 10$ so that the degree of each group element is 2, 5 or 10. If G contains an element of degree 10 then G is cyclic and, hence, commutative, which is not possible. Hence, G may contain only elements of degree 5 and 2.

Since the graph is 3-regular, for the set S there are two possibilities: either it consists of one element of degree 2 and one element of degree 5, or it consists of 3 elements of degree 2 each.

Consider the first case and denote by a an element of S of degree 5 and by b an element of degree 2. Then we have a chain

$$e \sim b \sim ba. \tag{5}$$

On the other hand, by inspection of the graph, we see that any chain of two edges can be continued by three more edges to be closed to a 5-cycle. In other words, the vertex ba can be obtained from e by a chain of three edges:

$$e \sim x \sim y \sim ba, \tag{6}$$

where x and y are not equal to e, b, ba. Logically there are the following possibilities for the latter chain:

$$\begin{aligned} e &\sim a \sim a^2 \sim ba & (7)\\ e &\sim a \sim ab \sim ba & (8)\\ e &\sim a^{-1} \sim a^{-2} \sim ba & (9)\\ e &\sim a^{-1} \sim a^{-1}b \sim ba. & (10) \end{aligned}$$

In the case (7), $a^2 \sim ba$ is only possible if

$$a^2 b = ba. \tag{11}$$

In the case (8), $ab \sim ba$ is only possible if $aba^{-1} = ba$ that is,

$$ab = ba^2. \tag{12}$$

In the case (9), $a^{-2} \sim ba$ is only possible if

$$a^3 b = ba. \tag{13}$$

In the case (10), $a^{-1}b \sim ba$ is only possible if $a^{-1}ba^{-1} = ba$, that is,

$$a^{-1}b = ba^2.$$

By renaming a^{-1} into a, we obtain

$$ab = ba^3. \tag{14}$$

Let us bring to contradiction (11) and (13). Rewrite each of them in the form

$$ba = a^r b,$$

where $r = 2$ or 3. Observe first that

$$abab = a\,(ba)\,b = a\,(a^r b)\,b = a^{r+1}.$$

Next, we have

$$ababab = (abab)\,ab = \left(a^{r+1}\right)ab = a^{r+2}b \tag{15}$$

and

$$\begin{aligned} ababab &= ab\,(abab) = aba^{r+1} = a\,(ba)\,a^r = a\,(a^r b)\,a^r \\ &= a^{r+1}ba^r = a^{r+1}\,(ba)\,a^{r-1} = a^{r+1}\,(a^r b)\,a^{r-1} \\ &= a^{2r+1}ba^{r-1} = a^{2r+1}\,(ba)\,a^{r-2} = a^{2r+1}\,(a^r b)\,a^{r-2} \\ &= a^{3r+1}ba^{r-2} = \ldots = a^{(r+1)r+1}b. \end{aligned}$$

Comparing with (15), we obtain that

$$a^{r+2} = a^{r^2+r+1}$$

that is,

$$\begin{aligned} r^2 + r + 1 &= r + 2 \bmod 5 \\ r^2 &= 1 \bmod 5, \end{aligned}$$

which is not the case for $r = 2, 3$. Similarly, one brings to contradiction (12) and (14).

Consider now the second case when S consists of three elements a, b, c each of the degree 2. Considering again the chains (5) and (6), we see that there are only the following two possibilities for (6):

$$\begin{aligned} e &\sim a \sim ac \sim ba \\ e &\sim c \sim cb \sim ba. \end{aligned}$$

The first case is only possible if

$$ba = acb \tag{16}$$

and the second case – if

$$ba = cbc \tag{17}$$

(the case $ba = cbb = c$ is not possible since we obtain a 3-cycle $e \sim b \sim c \sim e$). In the case (37), we have

$$abab = a\,(ba)\,b = a\,(acb)\,b = c.$$

It follows that

$$ababab = ab\,(abab) = abc$$

and

$$ababab = (abab)\,ab = cab,$$

that is,

$$abc = cab.$$

Hence, we obtain the following 6-cycle:

$$e \sim a \sim ab \sim abc = cab \sim ca \sim c \sim e,$$

which is impossible on the Petersen graph. In this case (17), we have similarly

$$bcbcbc = bcb\,(cbc) = bcbba = bca$$

and

$$bcbcbc = b\,(cbc)\,bc = bbabc = abc,$$

so that

$$abc = bca.$$

Hence, we obtain a 6-cycle

$$e \sim a \sim ab \sim abc = bca \sim bc \sim b \sim e,$$

which finishes the proof.

14. Let A_4 be the group of even permutations of the sequence $\{1, 2, 3, 4\}$. Consider the cyclic permutations $a = (234)$ and $b = (123)$ as well as the product of two transpositions $c = (12)\,(34)$.
    (a) Verify that $a^3 = b^3 = c^2 = \mathrm{id}$ and $ba = a^2b^2 = c$.
    (b) Prove that A_4 is isomorphic to the group of rotations of a regular tetrahedron in $\mathbb{R}^3$.
    (c) Consider a symmetric set $S = \{a, a^{-1}, b, b^{-1}, c\}$ and prove that the Cayley graph (A_4, S) is isomorphic to the *icosahedron* graph on Fig. 4.

Solution. (*a*) and (*b*) are straightforward. Solution of (*c*) is contained on Fig. 5.

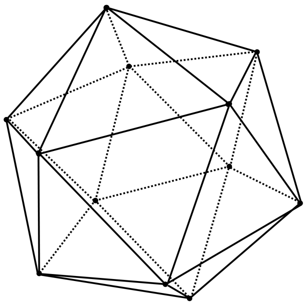


Figure 4: Icosahedron

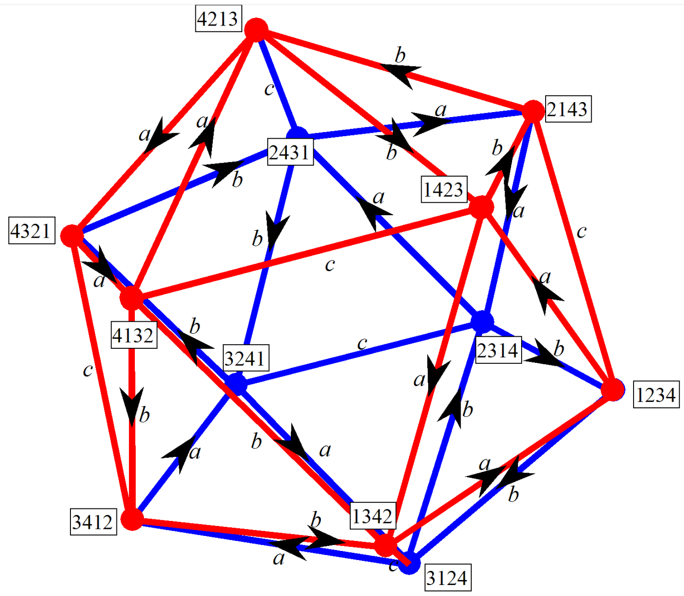


Figure 5: Icosahedron as the Cayley graph (A_4, S)

15. Prove the following identities for arbitrary functions f, g on a weighted graph (V, μ):

    (a) $\nabla_{xy}(fg) = (\nabla_{xy} f) g + (\nabla_{xy} g) f + (\nabla_{xy} f)(\nabla_{xy} g)$.

    (b) $\Delta_\mu (fg) = (\Delta_\mu f) g + (\Delta_\mu g) f + \frac{1}{\mu(x)} \sum_{y \sim x} (\nabla_{xy} f)(\nabla_{xy} g) \mu_{xy}$.

    Solution. (a) We have

$$\begin{aligned}
\nabla_{xy}(fg) &= f(y) g(y) - f(x) g(x) \\
&= (f(y) - f(x)) g(y) + f(x) g(y) - f(x) g(x) \\
&= (\nabla_{xy} f)(g(y) - g(x)) + (\nabla_{xy} f) g(x) + (\nabla_{xy} g) f(x) \\
&= (\nabla_{xy} f)(\nabla_{xy} g) + (\nabla_{xy} f) g(x) + (\nabla_{xy} g) f(x)
\end{aligned}$$

    (b) Using the Markov kernel $P(x, y) = \frac{\mu_{xy}}{\mu(x)}$, we obtain

$$\begin{aligned}
& \Delta_\mu (fg) - (\Delta_\mu f) g - (\Delta_\mu g) f \\
= & \sum_y P(x, y) f(y) g(y) - f(x) g(x)
\end{aligned}$$

$$
\begin{aligned}
& -\left(\sum_y P(x,y) f(y) - f(x)\right) g(x) \\
& -\left(\sum_y P(x,y) g(y) - g(x)\right) f(x) \\
= & \sum_y P(x,y)\left[f(y) g(y) - f(y) g(x) - g(y) f(x)\right] + f(x) g(x) \\
= & \sum_y P(x,y)\left[f(y) g(y) - f(y) g(x) - g(y) f(x) + f(x) g(x)\right] \\
= & \sum_y P(x,y)(f(y) - f(x))(g(y) - g(x)),
\end{aligned}
$$

which finishes the proof.

16. Consider the equation $\Delta_\mu u = f$ on a finite connected weighted graph (V, μ). Here f is a given function whereas u is an unknown function.

    (a) Prove that if one solution u exists then all other solutions are $u + \text{const}$.

    (b) Prove that if a solution u exists then

    $$\sum_{x \in V} f(x) \mu(x) = 0. \tag{18}$$

    (c) Prove that if (18) is satisfied the a solution u exists.

Solution. (a) If u_1 is another solution then $\Delta_\mu (u_1 - u) = 0$. It follows that $u_1 - u = \text{const}$, whence $u_1 = u + \text{const}$.

(b) Using the Green formula

$$\sum_{x \in V} \Delta_\mu u(x) v(x) \mu(x) = -\frac{1}{2} \sum_{x,y \in V} (\nabla_{xy} u)(\nabla_{xy} v) \mu_{xy}$$

with $v \equiv 1$, we obtain

$$
\begin{aligned}
\sum_{x \in V} f(x) \mu(x) &= \sum_{x \in V} \Delta_\mu u(x) v(x) \mu(x) \\
&= -\frac{1}{2} \sum_{x,y \in V} (u(x) - u(y))(v(x) - v(y)) \mu_{xy} \\
&= 0.
\end{aligned}
$$

(c) The operator $\Delta_\mu : \mathcal{F} \to \mathcal{F}$ has the one-dimensional kernel $\{\text{const}\}$. By the rank-nullity theorem, we have

$$\dim \ker \Delta_\mu + \dim \operatorname{image} \Delta_\mu = \dim \mathcal{F}$$

whence

$$\dim \operatorname{image} \Delta_\mu = \dim \mathcal{F} - 1.$$

The image of Δ_μ is contained in the space of functions $f \in \mathcal{F}$ satisfying (18). The latter space has also dimension $\dim \mathcal{F} - 1$, whence we obtain the identity of these two spaces. In other words, $\Delta_\mu u = f$ has a solution u if and only if f satisfies (18), which was to be proved.

17. Let $\{X_n\}$ be a simple random walk on $\mathbb{Z}$, and set

$$v_n(x) = \mathbb{P}_0(X_n = x).$$

(a) Prove that

$$v_n(x) = \begin{cases} \frac{1}{2^n}\binom{n}{\frac{x+n}{2}}, & x \equiv n \mod 2 \\ 0, & \text{otherwise,} \end{cases} \tag{19}$$

where $\binom{n}{m}$ is the binomial coefficient that is defined by

$$\binom{n}{m} = \begin{cases} \frac{n!}{m!(n-m)!} & \text{if } 0 \le m \le n, \\ 0, & \text{otherwise.} \end{cases}$$

(b) Prove that, for even n,

$$v_n(0) \sim \sqrt{\frac{2}{\pi n}} \text{ as } n \to \infty.$$

Hint: Use the Stirling formula $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$ as $n \to \infty$.

Solution. (a) Induction in n. Since $v_0(x) = 1$ for $x = 0$ and $v_0(x) = 0$ for $x \neq 0$, we see that (19) is satisfied for $n = 0$. To make the inductive step from n to $n+1$, let us use the forward equation

$$\begin{aligned} v_{n+1}(x) &= \sum_y \frac{1}{\mu(y)} v_n(y)\,\mu_{xy} \\ &= \frac{1}{2}\left(v_n(x-1) + v_n(x+1)\right). \end{aligned}$$

If $x \equiv n+1 \mod 2$ then $x \pm 1 \equiv n \mod 2$ whence

$$\begin{aligned} v_{n+1}(x) &= \frac{1}{2}\left(v_n(x-1) + v_n(x+1)\right) \\ &= \frac{1}{2^{n+1}}\left(\binom{n}{\frac{x-1+n}{2}} + \binom{n}{\frac{x+1+n}{2}}\right) \\ &= \frac{1}{2^{n+1}}\binom{n+1}{\frac{x+n+1}{2}}, \end{aligned}$$

which finishes the proof. We have used here the following property of binomial coefficients:

$$\binom{n}{m-1} + \binom{n}{m} = \binom{n+1}{m}.$$

If $x \not\equiv n+1 \mod 2$ then $v_n(x-1) = v_n(x+1) = 0$ and, hence, $v_{n+1}(x) = 0$, which was to be proved.

(b) If $x = 0$ and n is even, then we obtain from (19) that

$$\begin{aligned} v_n(0) &= \frac{1}{2^n}\binom{n}{n/2} \\ &= \frac{1}{2^n}\frac{n!}{((n/2)!)^2} \\ &\sim \frac{1}{2^n}\frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}{\sqrt{2\pi\frac{n}{2}}^2\left(\frac{n}{2e}\right)^n} \\ &= \sqrt{\frac{2}{\pi n}}. \end{aligned}$$

18. Let $\mathcal{F}$ be the space of real-valued functions on a finite weighted graph (V, μ), endowed by the inner product

$$(f, g) = \sum f(x)\, g(x)\, \mu(x).$$

Set $\|f\| = \sqrt{(f, f)}$. Let P be the Markov operator of (V, μ) and $\mathcal{L}$ be the positive definite Laplace operator of (V, μ).

   (a) Prove that, for any $f \in \mathcal{F}$, $(Pf)^2 \leq P\left(f^2\right)$

   (b) Prove that, for any $f \in \mathcal{F}$, $\|Pf\| \leq \|f\|$.

   (c) Use (b) to show that $\operatorname{spec} P \in [-1, 1]$.

   (d) Conclude that $\operatorname{spec} \mathcal{L} \subset [0, 2]$.

Remark: This gives an alternative proof of the fact that all the eigenvalues of $\mathcal{L}$ are contained in $[0, 2]$.

Solution. (a) Using Cauchy-Schwarz inequality, we obtain, for any $x \in V$,

$$\begin{aligned}
Pf(x)^2 &= \left(\sum_y P(x, y)\, f(y)\right)^2 \\
&= \left(\sum_y P(x, y)^{1/2}\, P(x, y)^{1/2}\, f(y)\right)^2 \\
&\leq \left(\sum_y P(x, y)\right)\left(\sum_y P(x, y)\, f^2(y)\right) \\
&= \sum_y P(x, y)\, f^2(y) \\
&= \left(Pf^2\right)(x).
\end{aligned}$$

(b) It follows from (a) that

$$\begin{aligned}
\|Pf\|^2 &= \sum_x Pf(x)^2\, \mu(x) \\
&\leq \sum_x \sum_y P(x, y)\, f^2(y)\, \mu(x) \\
&= \sum_y \sum_x P(y, x)\, f^2(y)\, \mu(y) \\
&= \sum_y f^2(y)\, \mu(y) = \|f\|^2,
\end{aligned}$$

where we have used the reversibility of $P(x, y)$ and that $\sum_x P(y, x) = 1$.

(c) Since P is a symmetric operator in $\mathcal{F}$, its spectrum is real. If α is an eigenvalue of P with eigenfunction f, then $Pf = \alpha f$ whence $\|Pf\| = |a|\, \|f\|$. Since $\|Pf\| \leq \|f\|$, it follows that $|\alpha| \leq$ so that $\operatorname{spec} P \subset [-1, 1]$. Since $\mathcal{L} = \operatorname{id} - P$, it follows that

$$\operatorname{spec} \mathcal{L} = 1 - \operatorname{spec} P \subset [0, 2].$$

19. Prove that if (X, E_1) and (Y, E_2) are two connected graph then the graph $(V, E) = (X, E_1) \square (Y, E_2)$ is also connected.

    Solution. Indeed, if we need to connect by a path the vertices (x, y) and (x', y') in (V, E) then first connect x to x' in E_1 by a path $x = x_0 \sim x_1 \sim ... \sim x_n = x'$ and y to y' in E_2 by a path $y = y_0 \sim y_1 \sim ... \sim y_m = y'$ and form the path

    $$(x, y) = (x_0, y) \sim (x_1, y) \sim ... \sim (x', y) = (x', y_0) \sim (x', y_1) \sim ... (x', y'),$$

    that connects (x, y) and (x', y').

20. Let (X, E_1) and (Y, E_2) be two finite connected graphs with more than one vertex. Prove that their product $(X, E_1) \square (Y, E_2)$ is bipartite if and only if both (X, E_1) and (Y, E_2) are bipartite.

    Solution. Set $(V, E) = (X, E_1) \square (Y, E_2)$. Let a be a simple weight on (X, E_1) and b be a simple weight on (Y, E_2). Consider the product of weighted graphs

    $$(V, \mu) = (X, a) \square_{p,q} (Y, b)$$

    with some fixed parameters $p, q > 0$ (for example, can take $p = q = 1$). Note that the set of edges E on V is compatible with the weight μ.

    Let $\{\alpha_k\}$ be the sequence of the eigenvalues of the Markov operator on (X, a) and $\{\beta_l\}$ be the sequence of the eigenvalues of the Markov operator on (Y, b), both with multiplicities. Then the eigenvalues of the Markov operator on the product graph (V, μ) are $\frac{p\alpha_k + q\beta_l}{p+q}$. We use the fact that a graph is bipartite if and only if its Markov operator has eigenvalue -1. If (X, a) and (Y, b) are bipartite then for some k, l we have $\alpha_k = -1$ and $\beta_l = -1$ whence

    $$\frac{p\alpha_k + q\beta_l}{p + q} = -1, \tag{20}$$

    so that (V, μ) is also bipartite. Conversely, if (V, μ) is bipartite then (20) holds for some k, l. However, we have $\alpha_k \geq -1$ and $\beta_l \geq -1$ so that (20) is only possible when $\alpha_k = \beta_l = -1$. Hence, both (X, a) and (Y, b) are bipartite.

    Another solution can be obtained using Exercise 3

21. Let P be the Markov kernel of a locally finite weighted graph (V, μ) and E be the corresponding set of edges.

    (a) Fix a positive integer n, and consider two vertices $x, y \in V$. Prove that $P_n(x, y) > 0$ if and only if there is a path of length n in the graph (V, E) that connects x and y.

    (b) Define a new set of edges E_n on V as follows: $(x, y) \in E_n$ if $P_n(x, y) > 0$. Prove that if (V, E) bipartite then (V, E_2) is disconnected.

    (c) Let (V, E) be finite, connected and non-bipartite. Prove that (V, E_n) is complete for some n.

    Solution. (a) Iterating the identity

    $$P_n(x, y) = \sum_{x_1 \in V} P(x, x_1) P_{n-1}(x_1, y)$$

    we obtain

    $$P_n(x, y) = \sum_{x_1, x_2 \in V} P(x, x_1) P(x_1, x_2) P_{n-2}(x_2, y)$$
    $$...$$

$$= \sum_{x_1,...,x_{n-1}\in V} P(x,x_1)P(x_1,x_2)...P(x_{n-1},y).$$

Note that the term $P(x,x_1)P(x_1,x_2)...P(x_{n-1},y)$ is strictly positive if and only if

$$x_0 := x \sim x_1 \sim x_2 \sim ... \sim x_{n-1} \sim y =: x_n$$

that is, if the sequence $\{x_k\}_{k=0}^n$ is a path connecting x and y. Hence, if such a path exists then $P_n(x,y) > 0$. If such path does not exist then all the terms in the above sum vanish, and $P_n(x,y) = 0$.

(b) Let V^+ and V^- be the bipartition of V, so that $x \sim y$ implies that x and y are contained in different sets V^+, V^-. Assuming that $x \in V^+$ and $y \in V^-$, we have

$$\begin{aligned} P_2(x,y) &= \sum_{z\in V} P(x,z)P(z,y) \\ &= \sum_{z\in V^+} P(x,z)P(z,y) + \sum_{z\in V^-} P(x,z)P(z,y) \\ &= 0, \end{aligned}$$

because in the first sum $P(x,z) = 0$ while in the second some $P(z,y) = 0$. Hence, there is no edges in (V,E_2) between vertices in V^+ and V^-, and (V,E_2) is disconnected.

(c) We have, for all $x_0, x \in V$ and $n \geq 1$,

$$\left| P_n(x_0,x) - \frac{\mu(x)}{\mu(V)} \right| \leq \rho^n \sqrt{\frac{\mu(x)}{\mu(x_0)}},$$

where $\rho \in (-1,1)$. Since the functions $\frac{\mu(x)}{\mu(V)}$ and $\frac{\mu(x)}{\mu(x_0)}$ are uniformly bounded between positive constants and $\rho^n \to 0$ as $n \to \infty$, we obtain that $P_n(x_0,x) > 0$ for large enough n and for all $x_0, x \in V$. By (a) it follows that $x_0 \sim x$ in (V,E_n), which was to be proved.

22. Let (V,μ) be a bipartite finite connected weighted graph. Let V^+, V^- be a bipartition of V.

    (a) For any function f on V, consider the function $\widetilde{f}$ on V that takes two values as follows:

    $$\widetilde{f}(x) = \frac{2}{\mu(V)} \left\{ \begin{array}{ll} \sum_{y\in V^+} f(y)\mu(y), & x \in V^+, \\ \sum_{y\in V^-} f(y)\mu(y), & x \in V^-. \end{array} \right.$$

    Prove that if n is even and $n \to \infty$ then $P^n f(x) \to \widetilde{f}(x)$ for all $x \in V$.

    (b) Consider the distribution $v_n(x) = \mathbb{P}_{x_0}(X_n = x)$ of the random walk on (V,μ). Prove that if $x_0 \in V^+$ and n is even then as $n \to \infty$

    $$v_n(x) \to \left\{ \begin{array}{ll} \frac{2\mu(x)}{\mu(V)}, & x \in V^+, \\ 0, & x \in V^-. \end{array} \right.$$

Solution. (a) Let $|V| = N$ and $\{v_0, ..., v_{N-1}\}$ be an orthonormal basis in the space $\mathcal{F}$ of functions on V that consists of eigenfunctions of P, with eigenvalues

$$1 = \alpha_0 > \alpha_1 \geq ... \geq \alpha_{N-2} > \alpha_{N-1} = -1.$$

Note that the eigenvalues $\alpha_0 = 1$ and $\alpha_{N-1} = -1$ are simple, and their eigenfunctions are as follows: $v_0 \equiv c_0$ for some constant c and

$$v_{N-1}(x) = \left\{ \begin{array}{ll} c, & x \in V^+ \\ -c, & x \in V^- \end{array} \right.$$

also for a constant c. Using the normalization conditions $\|v_0\| = 1 = \|v_{N-1}\|$, we obtain that

$$c_0 = c = \frac{1}{\sqrt{\mu(V)}}.$$

Given a function f on V, represent it in the form

$$f = \sum_{k=0}^{N-1} a_k v_k$$

where $a_k = (f, v_k)$. Then

$$\begin{aligned} P^n f &= \sum_{k=0}^{N-1} \alpha_k^n a_k v_k \\ &= a_0 v_0 + (-1)^n a_{N-1} v_{N-1} + \sum_{k=1}^{N-2} \alpha_k^n a_k v_k. \end{aligned}$$

Set

$$\rho = \max_{1 \le k \le N-2} |\alpha_k|$$

and observe that $\rho < 1$. It follows that

$$\|P^n f - a_0 v_0 - (-1)^n a_{N-1} v_{N-1}\|^2 \le \rho^{2n} \sum_{k=1}^{N-2} a_k^2 \le \rho^{2n} \|f\|^2.$$

Assuming that n is even and letting $n \to \infty$, we obtain that

$$P^n f \to a_0 v_0 + a_{N-1} v_{N-1}.$$

Next,

$$a_0 v_0 = (f, v_0)\, v_0 = c_0^2 \sum_{y \in V} f(y)\, \mu(y) = \frac{1}{\mu(V)} \sum_{y \in V} f(y)\, \mu(y)$$

and, for $x \in V^+$,

$$a_{N-1} v_{N-1}(x) = (f, v_{N-1})\, c = \frac{1}{\mu(V)} \left(\sum_{y \in V^+} f(y)\, \mu(y) - \sum_{y \in V^-} f(y)\, \mu(y) \right)$$

while for $x \in V^-$

$$a_{N-1} v_{N-1}(x) = -(f, v_{N-1})\, c = \frac{1}{\mu(V)} \left(-\sum_{y \in V^+} f(y)\, \mu(y) + \sum_{y \in V^-} f(y)\, \mu(y) \right).$$

Hence, if $x \in V^+$ then

$$a_0 v_0 + a_{N-1} v_{N-1}(x) = \frac{2}{\mu(V)} \sum_{y \in V^+} f(y)\, \mu(y) = \widetilde{f}(x)$$

and if $x \in V^-$ then

$$a_0 v_0 + a_{N-1} v_{N-1}(x) = \frac{2}{\mu(V)} \sum_{y \in V^-} f(y)\, \mu(y) = \widetilde{f}(x).$$

Finally, we obtain $P^n f \to \widetilde{f}$ as $n \to \infty$ and n is even, which was to be proved. Moreover, we have proved that

$$\left\| P_n f - \widetilde{f} \right\| \leq \rho^n \|f\|$$

Remark. Note that always $\mu(V^+) = \mu(V^-) = \frac{1}{2}\mu(V)$ because

$$\mu\left(V^+\right) = \sum_{x \in V^+} \mu(x) = \sum_{x \in V^+} \sum_{y \in V^-} \mu_{xy} = \sum_{y \in V^-} \sum_{x \in V^+} \mu_{yx} = \mu\left(V^-\right).$$

Note also that on bipartite graphs the sequence of eigenvalues $\{\alpha_k\}$ is symmetric at 0, that is,

$$\alpha_{N-1-k} = -\alpha_k.$$

In particular, this means $\alpha_{N-2} = -\alpha_1$ and, hence,

$$\rho = \max_{1 \leq k \leq N-2} |\alpha_k| = \alpha_1 = 1 - \lambda_1,$$

where λ_1 is the first positive eigenvalue of the Laplace operator.

(b) Apply (a) with $f = \mathbf{1}_{\{x_0\}}$. Using that

$$\widetilde{f}(x) = \begin{cases} \frac{2\mu(x_0)}{\mu(V)}, & x \in V^+ \\ 0, & x \in V^- \end{cases}$$

and $u_n(x) := P^n f = \frac{v_n(x)\mu(x_0)}{\mu(x)}$, we obtain

$$\frac{v_n(x)\,\mu(x_0)}{\mu(x)} \to \widetilde{f}(x),$$

whence

$$v_n(x) \to \begin{cases} \frac{2\mu(x)}{\mu(V)}, & x \in V^+ \\ 0, & x \in V^- \end{cases}$$

which was to be proved. In fact, we have

$$\sum_{x \in V} \left(v_n(x) \frac{\mu(x_0)}{\mu(x)} - \widetilde{f}(x) \right)^2 \mu(x) \leq \rho^{2n} \mu(x_0)$$

whence for all $x \in V^+$

$$\left| v_n(x) - \frac{2\mu(x)}{\mu(V)} \right| \leq \rho^n \sqrt{\frac{\mu(x)}{\mu(x_0)}}$$

and for all $x \in V^-$

$$|v_n(x)| \leq \rho^n \sqrt{\frac{\mu(x)}{\mu(x_0)}}.$$

23. Prove that the positive definite Laplace operator $\mathcal{L}$ on a complete bipartite graph $K_{n,m}$ (where $n+m > 2$) with a simple weight has the following eigenvalues: $0, 1, 2$. What are their multiplicities?

Solution. The value 0 is always a simple eigenvalue, and since $K_{n,m}$ is bipartite, the value $2-0=2$ is also a simple eigenvalue. Let λ be any other eigenvalue and f be its eigenfunction, so that

$\mathcal{L}f = \lambda f$, which is equivalent to $Pf = \alpha f$ where $P = \mathrm{id} - \mathcal{L}$ is the Markov operator and $\alpha = 1 - \lambda \in (-1, 1)$. For any $x \in V$, we have

$$\alpha f(x) = \sum_{y \in V} P(x, y) f(y).$$

Let the partition of $K_{n,m}$ be V^+ and V^-, where $|V^+| = n$ and $|V^-| = m$. Then, for $x \in V^+$,

$$\alpha f(x) = \sum_{y \in V^-} P(x, y) f(y). \tag{21}$$

Since $P(x, y) = \frac{1}{\deg(x)} = \frac{1}{n}$, we see that the right hand side is independent of x. If $\alpha \neq 0$ then we obtain that $f = c_+$ on V^+ for some constant c_+ and similarly $f = c_-$ on V^-. Then (21) implies $\alpha c_+ = c_-$ and similarly $\alpha c_- = c_+$, whence we conclude that $\alpha^2 = 1$ and $\alpha = \pm 1$. However, we are considering only eigenvalues $\alpha \in (-1, 1)$ which means that we must have $\alpha = 0$ and, hence, $\lambda = 1$. Since the total sum of multiplicities of all eigenvalues must be $|V| = n + m$ and the eigenvalues $\lambda = 0$ and $\lambda = 2$ have multiplicities 1, it follows that the multiplicity of $\lambda = 1$ is $n + m - 2$.

Remark. An interesting particular case of $K_{n,m}$ is a "star" graph $K_{n,1}$.

24. Fix integers $m, n \geq 2$. Prove that the positive definite Laplace operator $\mathcal{L}$ on a complete m-partite graph $K_{\underbrace{n, n, \dots, n}_{m}}$ (cf. Exercises 6 and 9) with simple weight has the following eigenvalues: $0, 1, \frac{m}{m-1}$. What are their multiplicities?

Observing that $K_{2,2,2}$ is isomorphic to the *octahedron* (Fig. 6), evaluate the eigenvalues of the Laplacian on the octahedron.

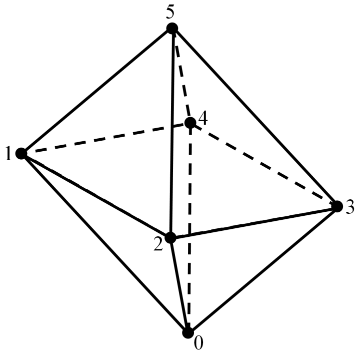


Figure 6: Octahedron

Solution. The value 0 is always a simple eigenvalue of $\mathcal{L}$. Let $\lambda > 0$ be any other eigenvalue and f be its eigenfunction, so that $\mathcal{L}f = \lambda f$. Let $V_1, .., V_m$ be an m-partition of $V = K_{\underbrace{n, n, \dots, n}_{m}}$. For any $x \in V_k$, we have

$$(1 - \lambda) f(x) = \sum_{y \notin V_k} P(x, y) f(y) = \frac{1}{\deg(x)} \sum_{y \notin V_k} f(y). \tag{22}$$

Since $\deg(x) = (m - 1) n = \text{const}$ on V_k, it follows that the right hand side is a constant function with respect to $x \in V_k$. It follows that either $\lambda = 1$ or $f(x) = \text{const}$ on V_k. Consider first the case $\lambda = 1$. Then the above equation is equivalent to

$$\sum_{y \notin V_k} f(y) = 0.$$

Since this has to be true for any $k = 1, ..., m$, we obtain m equations that define all eigenfunctions of the eigenvalue $\lambda = 1$. Hence, the multiplicity of $\lambda = 1$ is

$$|V| - m = (n-1)\,m.$$

Let now $\lambda \neq 1$ so that function f must be a constant on each V_k; let $f = c_k$ on V_k. Consider now the sets $V_1, ..., V_m$ as vertices in a new graph $\mathcal{V}$. Then function f gives rise to a function F on $\mathcal{V}$ as follows: $F(V_k) = c_k$. Rewrite (22) as follows:

$$(1-\lambda)\,F(V_k) = \frac{1}{(m-1)\,n}\sum_{j\neq k} F(V_j)\,n = \frac{1}{(m-1)}\sum_{j\neq k} F(V_j)\,.$$

It follows that F is the eigenfunction of the Laplace operator on $\mathcal{V}$ considered as a complete graph with simple weight. The complete graph K_m has the eigenvalues 0 and $\frac{m}{m-1}$ the latter being with multiplicity $m-1$. Hence, we obtain that $\lambda = \frac{m}{m-1}$ is also an eigenvalue for V with the same multiplicity. So, the answer is: $\lambda = 0$ is simple, $\lambda = 1$ has multiplicity $(n-1)\,m$, $\lambda = \frac{m}{m-1}$ has multiplicity $m-1$.

Note that the octahedron on Fig. 6 is isomorphic to the graph $K_{2,2,2}$ with the 3-partition $\{1,3\}, \{2,4\}, \{0,5\}$. Hence, the eigenvalues of $\mathcal{L}$ on the octahedron are 0 (simple), 1 (triple) and $\frac{3}{2}$ (double).

Alternatively, one could try to compute these eigenvalues by a brute force. Indeed, enumerating the vertices of the octahedron as on Fig. 6, we obtain that $\mathcal{L}$ is given by a 6×6 matrix

$$\begin{pmatrix} 1 & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & 0 \\ -\frac{1}{4} & 1 & -\frac{1}{4} & 0 & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & 1 & -\frac{1}{4} & 0 & -\frac{1}{4} \\ -\frac{1}{4} & 0 & -\frac{1}{4} & 1 & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & 0 & -\frac{1}{4} & 1 & -\frac{1}{4} \\ 0 & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & 1 \end{pmatrix}.$$

The eigenvalues of this matrix can be computed using an appropriate software, which gives, of course, the same result as above.

25. (*The Dirichlet principle*) Let Ω be a finite set of vertices on a connected weighted graph (V, μ) such that Ω^c is non-empty. Consider the Dirichlet problem

$$\begin{cases} \Delta_\mu u(x) = 0 & \text{for all } x \in \Omega, \\ u(x) = g(x) & \text{for all } x \in \Omega^c, \end{cases} \tag{23}$$

where g is a given function on Ω^c.

(a) Prove that a solution u of the Dirichlet problem (23) has the smallest value of the *Dirichlet integral*

$$D(u) = \frac{1}{2}\sum_{x,y\in\overline{\Omega}} (\nabla_{xy}u)^2\,\mu_{xy},$$

among all other functions u that satisfy the same boundary condition $u = g$ in Ω^c. Here $\overline{\Omega}$ is the union of Ω with all its neighbors.

(b) Prove that if u minimizes the Dirichlet integral among all functions with the boundary condition $u = g$ in Ω^c then u is solves (23).

(c) Prove that there exists a function u that minimizes the Dirichlet integral among all functions with the boundary condition $u = g$ in Ω^c. *Remark:* This provides an alternative proof of the existence of solution of (23).

Solution. (a) It suffices to prove that

$$D(u+v) \geq D(u)$$

for any v that vanishes in Ω^c. Note that

$$\begin{aligned} D(u+v) &= \frac{1}{2}\sum_{x,y\in\overline{\Omega}} \left(\nabla_{xy}(u+v)\right)^2 \mu_{xy} \\ &= \frac{1}{2}\sum_{x,y\in\overline{\Omega}} (\nabla_{xy}u)^2 \mu_{xy} + \sum_{x,y\in\overline{\Omega}} (\nabla_{xy}u)(\nabla_{xy}v)\mu_{xy} + \frac{1}{2}\sum_{x,y\in\overline{\Omega}} (\nabla_{xy}v)^2 \mu_{xy} \end{aligned}$$

whence

$$D(u+v) - D(u) \geq \sum_{x,y\in\overline{\Omega}} (\nabla_{xy}u)(\nabla_{xy}v)\mu_{xy}.$$

By the Green formula, we have

$$\sum_{x,y\in\overline{\Omega}} (\nabla_{xy}u)(\nabla_{xy}v)\mu_{xy} = -2\sum_{x\in\overline{\Omega}} \Delta_\mu u(x)\, v(x)\, \mu(x) + 2\sum_{x\in\overline{\Omega}}\sum_{y\in\overline{\Omega}^c} (\nabla_{xy}u)\, v(x)\, \mu_{xy}.$$

If $x \in \Omega$ then $\Delta_\mu u(x) = 0$. If $x \notin \Omega$ then $v(x) = 0$ so that $\Delta_\mu u(x)\, v(x) \equiv 0$. In the last sum, one can restrict summation to neighboring x and y. Since y is outside $\overline{\Omega}$, every $x \sim y$ must be outside Ω, whence $v(x) = 0$. Hence, the both sums on the right hand side vanishes, and we obtain

$$D(u+v) - D(u) \geq 0.$$

(b) By the above argument, we have always

$$D(u+v) - D(u) = -2\sum_{x\in\overline{\Omega}} \Delta_\mu u(x)\, v(x)\, \mu(x) + \frac{1}{2}\sum_{x,y\in\overline{\Omega}} (\nabla_{xy}v)^2 \mu_{xy}.$$

Replacing v by tv

$$D(u+tv) - D(u) = -2t\sum_{x\in\overline{\Omega}} \Delta_\mu u(x)\, v(x)\, \mu(x) + \frac{t^2}{2}\sum_{x,y\in\overline{\Omega}} (\nabla_{xy}v)^2 \mu_{xy}.$$

If $\Delta_\mu u(x_0) > 0$ at some point $x_0 \in \Omega$, then consider a function $v = \mathbf{1}_{\{x_0\}}$ so that

$$\sum_{x\in\overline{\Omega}} \Delta_\mu u(x)\, v(x)\, \mu(x) = \Delta_\mu u(x_0)\, v(x_0)\, \mu(x_0) > 0.$$

Taking t to be small enough and positive, we obtain that

$$D(u+tv) - D(u) < 0$$

which contradicts the minimality of $D(u)$. In the same way, if $\Delta_\mu u(x_0) < 0$ then use $v = -\mathbf{1}_{\{x_0\}}$. Hence, $\Delta_\mu u(x) = 0$ for all $x \in \Omega$.

(c) The set of functions u on $\overline{\Omega}$ forms a finite dimensional vector space $\mathcal{F}_{\overline{\Omega}}$ over $\mathbb{R}$, and $D(u)$ can be considered as a continuous function on $\mathcal{F}_{\overline{\Omega}}$. Set

$$D_{\min} = \inf\left\{D(u) : u \in \mathcal{F}_{\overline{\Omega}},\ u = g \text{ on } \partial\Omega\right\},$$

where $\partial\Omega = \overline{\Omega} \setminus \Omega$, and let $\{u_n\}$ be a sequence of functions from $\mathcal{F}_{\overline{\Omega}}$ such that $D(u_n) \to D_{\min}$ as $n \to \infty$. Let us show that the sequence $\{u_n\}$ is bounded[1]. If this is known already, then by Bolzano-Weierstrass theorem $\{u_n\}$ has a convergent subsequence, say $u_{n_k} \to u$ as $k \to \infty$. Then $D(u_{n_k}) \to D(u)$ whence it follows that $D(u) = D_{\min}$ and u is a minimizer.

We are left to show that the sequence $\{u_n\}$ is bounded, that is, the sequence of the norms $\{\|u_n\|\}$ is bounded. We can choose any norm $\|\cdot\|$ here. Consider the following norm in $\mathcal{F}_{\overline{\Omega}}$:

$$\|u\| = \sqrt{D(u)} + \max_{y \in \partial\Omega} |u(y)|.$$

Indeed, $\max_{y\in\partial\Omega} |u(y)|$ and $\sqrt{D(u)}$ are seminorms, so that their sum is also a seminorm. If $\|u\| = 0$ then $u(y) = 0$ for all $y \in \partial\Omega^c$, and $D(u) = 0$. Any vertex $x \in \Omega$ can be connected to some vertex $y \in \partial\Omega$ by a path, say, $\{x_k\}_{k=0}^n$, that is contained in $\overline{\Omega}$. For this path we have

$$D(u) \geq \sum_{k=0}^{N-1} (u(x_{k+1}) - u(x_k))^2 \mu_{x_k x_{k+1}},$$

which implies that $u(x_{k+1}) = u(x_k)$ for all k, whence $u(x) = u(y) = 0$ for all $x \in \Omega$., that is, $u = 0$ as an element of $\mathcal{F}_{\overline{\Omega}}$. This proves that $\|\cdot\|$ is indeed a norm.

Finally, since the sequence $\{D(u_n)\}$ is bounded and $\max_{\partial\Omega} |u_n| = \max_{\partial\Omega} |g|$ is independent of n and, hence, also is bounded, we obtain that the sequence $\{\|u_n\|\}$ is bounded.

26. Let (V, μ) be a finite connected weighted graph without loops, and let $\lambda_0 = 0 < \lambda_1 \leq ... \leq \lambda_{N-1}$ be the eigenvalues of the Laplace operator $\mathcal{L}$ on (V, μ). Assume that, for some positive integer k, there are $k+1$ functions $f_1, f_2, ..f_{k+1}$ on V such that:

    (i) their supports $A_i = \{x \in V : f_i(x) \neq 0\}$ are disjoint and not connected, that is, if $x \in A_i$ and $y \in A_j$ with $i \neq j$ then $x \neq y$ and $x \not\sim y$.

    (ii) $\mathcal{R}(f_i) \leq a$ for some real a and for all $i = 1, 2, ..., k+1$.

    Prove that $\lambda_k \leq a$.

    Solution. Let $\mathcal{F}$ be the space of all real-valued functions on V. Denote by F the subspace $\mathcal{F}$ that consist of all linear combinations of functions $f_1, ..., f_{k+1}$. Observe first that $(i) + (ii)$ imply that $\mathcal{R}(f) \leq a$ also for any $f \in F$, which follows from $f_i f_j \equiv 0$ and $(\mathcal{L} f_i) f_j = 0$ for all $i \neq j$. Consider the subspace E of $\mathcal{F}$ spanned by all eigenfunctions of the eigenvalues λ_i with $i \geq k$. Then $\dim E = N - k$ and $\dim F = k + 1$ so that E and F must intersect on a non-zero vector f. Since $f \in F$, we have $\mathcal{R}(f) \leq a$. However, if we assume that $\lambda_k > a$, then $f \in E$ implies $\mathcal{R}(f) > a$. This contradiction shows that $\lambda_k \leq a$.

27. Let D be the diameter of the graph (V, μ), that is,

    $$D = \max_{x,y \in V} d(x, y).$$

    Prove that, for any $k \leq [D/2]$, we have $\lambda_k \leq 1$.

    Hint: Use Exercise 26.

    Solution. Choose $x, y \in V$ so that $d(x, y) = D$ and let $\{x_k\}_{k=0}^D$ be a path of length D that connects x and y. Note that if $|k - m| > 1$ then $x_k \neq x_m$ and $x_k \not\sim x_m$ because otherwise by skipping some vertices we could obtain a shorter path between x, y. Hence, choosing the vertices x_0, x_2, x_4 etc on the above path, we obtain a sequence of $[\frac{D}{2}] + 1$ distinct vertices that are not connected each

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[1] In any finite dimensional space (in particular, in $\mathcal{F}_\Omega$) all norms are equivalent. Hence, when speaking about boundedness or convergence of sequences in $\mathcal{F}_\Omega$, one can use any norm.

to other by an edge. Denoting the chosen vertices by z_i, $i = 0, 1, ..., [\frac{D}{2}]$, consider the functions $f_i = \mathbf{1}_{\{z_i\}}$ so that we have $[\frac{D}{2}] + 1$ functions f_i with $\mathcal{R}(f_i) \leq 1$. Obviously, the supports of the functions f_i, being $\{z_i\}$, are disjoint and not connected. Using Exercise 26, we conclude that $\lambda_k \leq 1$ whenever $k \leq [\frac{D}{2}]$.

28. Evaluate the eigenvalues and eigenfunctions of the Markov operator on a *path graph* (V, E) with simple weight, that is, $V = \{0, 1, ..., N-1\}$ and the edges are defined by

$$0 \sim 1 \sim ... \sim N-1.$$

Solution. This graph is always bipartite so that -1 and 1 are simple eigenvalues. Let $\alpha \in (-1, 1)$ be an eigenvalue with an eigenfunction f. Then f satisfies the equation $Pf = \alpha f$, where

$$Pf(k) = \begin{cases} \frac{1}{2}\left(f(k-1) + f(k+1)\right) & 1 \leq k \leq N-2 \\ f(1), & k = 0 \\ f(N-2), & k = N-1 \end{cases}$$

Hence, the equation $Pf = \alpha f$ becomes:

$$\begin{aligned} f(k+1) - 2\alpha f(k) + f(k-1) &= 0 \text{ for } k = 1, ..., N-2 && (24) \\ f(1) &= \alpha f(0) && (25) \\ f(N-2) &= \alpha f(N-1) && (26) \end{aligned}$$

The difference equation (24) has in $\mathbb{Z}$ solutions $f(k) = C_1 \cos k\theta + C_2 \sin k\theta$ where $\theta \in (0, \pi)$ is determined by $\cos\theta = \alpha$. Choose C_1 and C_2 to satisfy (25) and (26):

$$\begin{aligned} C_1 \cos\theta + C_2 \sin\theta &= \alpha C_1 \\ C_1 \cos(N-2)\theta + C_2 \sin(N-2)\theta &= \alpha\left(C_1 \cos(N-1)\theta + C_2 \sin(N-1)\theta\right) \end{aligned}$$

Since $\cos\theta = \alpha$ and $\sin\theta \neq 0$, the first equation gives $C_2 = 0$. Hence, the second equation becomes

$$\cos(N-2)\theta = \cos\theta \cos(N-1)\theta.$$

Since $\cos(N-2)\theta = \cos\theta\cos(N-1)\theta + \sin\theta\sin(N-1)\theta$, we obtain the equivalent equation

$$\sin(N-1)\theta = 0$$

whence $\theta = \frac{\pi l}{N-1}$ with $l = 1, 2, ..., N-2$. Therefore, we obtain $N-2$ eigenvalues

$$\alpha_l = \cos\frac{\pi l}{N-1} \tag{27}$$

where $l = 1, ..., N-2$. The previously known eigenvalues ± 1 can also be written in this form for $l = 0$ and $l = N-1$, respectively, Hence, (27) gives a full set of N eigenvalues where $l = 0, 1, ..., N-1$, all the eigenvalues being simple.

29. Let (V, μ) be a finite connected weighted graph with $N > 1$ vertices. Let P be the Markov operator on (V, μ), $\{v_k\}_{k=0}^{N-1}$ be an orthonormal basis of eigenfunctions of P with eigenvalues α_k, where $1 = \alpha_0 > \alpha_1 \geq \alpha_2 \geq ... \geq \alpha_{N-1}$. Fix a point $x_0 \in V$ and set $f = \mathbf{1}_{\{x_0\}}$.

    (a) Assume that there is a constant c such that $|v_k(x)| \leq c$ for all $x \in V$ and $k = 1, ..., N-1$. Prove that

    $$\left| P^n f(x) - \frac{\mu(x_0)}{\mu(V)} \right| \leq c^2 \mu(x_0) \sum_{k=1}^{n} |\alpha_k|^n$$

    for all $x \in V$ and positive integers n.

(b) Prove that if (V, μ) is a cycle graph $C_N = \mathbb{Z}_N$ with an odd N and with a simple weight μ then

$$\left|P^n f(x) - \frac{1}{N}\right| \le \frac{4}{N} \frac{1}{e^{\frac{4n}{N^2}} - 1}$$

for all $x \in V$ and positive integers n. Conclude that the mixing time T admits the estimate $T \simeq N^2$.

Hint: Use the explicit eigenvalues and eigenfunctions of $\mathbb{Z}_N$ and the inequality $0 \le \cos z \le e^{-\frac{z^2}{2}}$ for $z \in (0, \pi/2)$.

Solution. (a) We have the identity

$$P^n f(x) - \overline{f} = \sum_{k=1}^{N-1} \alpha_k^n a_k v_k(x)$$

where $\overline{f} = \frac{\mu(x_0)}{\mu(V)}$ and $a_k = (f, v_k) = v_k(x_0)\mu(x_0)$. Since $|v_k| \le c$ and, hence, $|a_k| = |v_k(x_0)\mu(x_0)| \le c\mu(x_0)$, we obtain

$$\left|P^n f(x) - \overline{f}\right| \le c^2 \mu(x_0) \sum_{k=1}^{N-1} |\alpha_k|^n .$$

(b) For a cycle graph C_N with odd N, we have have $\alpha_l = \cos\frac{2\pi l}{N}$ where $l = 1, ..., \frac{N-1}{2}$, and each eigenvalue is double. Moreover, the eigenfunctions are $c_l \cos\frac{2\pi l}{N}x$ and $c_l' \sin\frac{2\pi l}{N}x$, where the constant c_l, c_l' are determined by

$$c_l = \left\|\cos\frac{2\pi l}{N}x\right\|^{-1} \text{ and } c_l' = \left\|\sin\frac{2\pi l}{N}x\right\|^{-1}.$$

We have

$$\begin{aligned}\left\|\cos\frac{2\pi l}{N}x\right\|^2 &= \frac{1}{N}\sum_{x=0}^{N-1}\cos^2\frac{2\pi l}{N}x = \frac{1}{N}\sum_{x=0}^{N-1}\sin^2\frac{2\pi l}{N}x \\ &= \frac{1}{2N}\sum_{x=0}^{N-1}\left(\cos^2\frac{2\pi l}{N}x + \sin^2\frac{2\pi l}{N}x\right) = \frac{1}{2}\end{aligned}$$

whence $c_l = \sqrt{2} = c_l'$. It follows that all the eigenfunctions are bounded by the constant $\sqrt{2}$. Using part (a), we obtain

$$\left|P^n f(x) - \overline{f}\right| \le \left(\sqrt{2}\right)^2 \frac{2}{N}\sum_{l=1}^{\frac{N-1}{2}} |\alpha_l|^n = \frac{4}{N}\sum_{l=1}^{\frac{N-1}{2}}\left|\cos\frac{2\pi l}{N}\right|^n . \tag{28}$$

We claim that the right hand side here is equal to

$$\frac{4}{N}\sum_{k=1}^{\frac{N-1}{2}}\cos^n\frac{\pi k}{N} \tag{29}$$

(note that $\frac{\pi k}{N} \in (0, \pi/2)$ and $\cos\frac{\pi k}{N} > 0$). Indeed, if in (28) $2l \in \left[1, \frac{N-1}{2}\right]$ then set $k = 2l$. If $2l \in \left[\frac{N+1}{2}, N-1\right]$ then set $k = N - 2l \in \left[1, \frac{N-1}{2}\right]$. In the first case, k runs over all even numbers in the range $[1, \frac{N-1}{2}]$, and in the second case, k runs over all odd numbers in the same range. In the both cases, we have $\cos\frac{2\pi l}{N} = \pm\cos\frac{\pi k}{N}$, whence (29) follows.

Use the following inequalities: if $z \in (0, \pi/2)$ then

$$0 \leq \cos z \leq e^{-z^2/2}.$$

It follows that if $\theta \in (0, 1/2)$ then

$$0 \leq \cos \pi\theta \leq e^{-\frac{\pi^2}{2}\theta^2} < e^{-4\theta^2}.$$

Therefore,

$$\begin{aligned} \left|P^n f(x) - \overline{f}\right| &\leq \frac{4}{N} \sum_{k=1}^{\frac{N-1}{2}} e^{-n\frac{4k^2}{N^2}} \\ &\leq \frac{4}{N} \sum_{k=1}^{\infty} e^{-\frac{4n}{N^2}s} \\ &= \frac{4}{N} \frac{e^{-\frac{4n}{N^2}}}{1 - e^{-\frac{4n}{N^2}}} = \frac{4}{N} \frac{1}{e^{\frac{4n}{N^2}} - 1}. \end{aligned}$$

Finally, $\overline{f} = \frac{1}{N}$, whence

$$\left|P^n f(x) - \frac{1}{N}\right| \leq \frac{4}{N} \frac{1}{e^{\frac{4n}{N^2}} - 1}.$$

If $n \geq N^2$ then the right hand side $\leq \frac{4}{N}\frac{1}{e^4 - 1} < \frac{1}{10N}$ so that the error of approximation $P^n f(x) \approx \frac{1}{N}$ becomes significantly smaller than $\frac{1}{N}$, so that indeed the mixing time is of the order N^2.

30. Let (V, μ) be a finite connected weighted graph with $N > 1$ vertices. Let P be the Markov operator on (V, μ), $\{v_k\}_{k=0}^{N-1}$ be an orthonormal basis of eigenfunctions of P with eigenvalues α_k, where $1 = \alpha_0 > \alpha_1 \geq \alpha_2 \geq ... \geq \alpha_{N-1}$. Fix a point $x_0 \in V$ and set $f = \mathbf{1}_{\{x_0\}}$.

    (a) Prove that for any positive integer n,

    $$\left\|P^n f - \frac{\mu(x_0)}{\mu(V)}\right\|^2 = \sum_{k=1}^{N-1} \alpha_k^{2n} v_k^2(x_0)\, \mu^2(x_0). \tag{30}$$

    (b) Assume in addition that (V, μ) is *vertex transitive*, that is, for any two vertices $x, y \in V$, there is an graph isomorphism $\varphi : V \to V$, that is, a bijection that preserves weight μ, such that $\varphi(x) = \varphi(y)$. Prove that

    $$\sum_{x \in V} \left(P^n f(x) - \frac{1}{N}\right)^2 = \frac{1}{N} \sum_{k=1}^{N-1} \alpha_k^{2n}.$$

Solution. (a) We use the identity

$$\left\|P^n f - \overline{f}\right\|^2 = \sum_{k=1}^{N-1} \alpha_k^{2n} a_k^2,$$

where $a_k = (f, v_k)$ and $\overline{f} = \frac{\mu(x_0)}{\mu(V)}$. We have

$$a_k = \sum_{x \in V} f(x)\, v_k(x)\, \mu(x) = v_k(x_0)\, \mu(x_0),$$

which implies (30).

(b) We have

$$\mu(\varphi(x)) = \sum_y \mu_{\varphi(x)y} = \sum_z \mu_{\varphi(x)\varphi(z)} = \sum_z \mu_{xz} = \mu(x).$$

Since for all x, y there is an isomorphism φ such that $\varphi(x) = y$, we obtain that $\mu(x) = \text{const}$ and, in particular, $\frac{\mu(x)}{\mu(V)} = \frac{1}{N}$ for all x. Next, we have

$$P(\varphi(x), \varphi(y)) = \frac{1}{\mu(\varphi(x))}\mu_{\varphi(x)\varphi(y)} = \frac{\mu_{xy}}{\mu(x)} = P(x, y).$$

It follows that

$$\begin{aligned} (Pf)(\varphi(x)) &= \sum_y P(\varphi(x), y) f(y) = \sum_z P(\varphi(x), \varphi(z)) f(\varphi(z)) \\ &= \sum_z P(x, z) f \circ \varphi(z) = P(f \circ \varphi)(x), \end{aligned}$$

that is, $(Pf) \circ \varphi = P(f \circ \varphi)$. By induction, the same is true for P^n. Consider the function

$$\begin{aligned} F_n(x_0) &= \sum_{x \in V} \left(P^n f(x) - \frac{1}{N}\right)^2 \\ &= \frac{1}{\mu(x_0)} \sum_{x \in V} \left(P^n f(x) - \frac{\mu(x_0)}{\mu(V)}\right)^2 \mu(x) \\ &= \frac{1}{\mu(x_0)} \left\| P^n f - \frac{\mu(x_0)}{\mu(V)} \right\|^2 \\ &= \sum_{k=1}^{N-1} \alpha_k^{2n} v_k^2(x_0) \mu(x_0), \end{aligned}$$

where we have used (30). We have

$$\begin{aligned} F_n(x_0) &= \sum_{x \in V} \left(P^n f(\varphi(x)) - \frac{1}{N}\right)^2 \\ &= \sum_{x \in V} \left(P^n (f \circ \varphi)(x) - \frac{1}{N}\right)^2 \\ &= F_n(\varphi(x_0)). \end{aligned}$$

Since there is φ that maps x_0 to any other point, it follows that $F_n = \text{const}$. Therefore,

$$\begin{aligned} F_n(x_0) &= \frac{1}{N} \sum_{x_0 \in V} F_n(x_0) \\ &= \frac{1}{N} \sum_{x_0 \in V} \sum_{k=1}^{N-1} \alpha_k^{2n} v_k^2(x_0) \mu(x_0) \\ &= \frac{1}{N} \sum_{k=1}^{N-1} \alpha_k^{2n} (v_k, v_k) \\ &= \frac{1}{N} \sum_{k=1}^{N-1} \alpha_k^{2n}. \end{aligned}$$

31. Let (V,E) be a finite connected k-regular graph. Let $a,b\in V$ be two distinct vertices of V such that $x\sim a$ implies $x\sim b$. Prove that the following function on V

$$f(x)=\begin{cases}1, & x=a\\ -1, & x=b\\ 0, & \text{otherwise}\end{cases}$$

is an eigenfunction of the Laplace operator $\mathcal{L}$ on (V,E) (with a simple weight). What is its eigenvalue?

Solution. We have

$$\mathcal{L}f(x)=f(x)-\frac{1}{k}\sum_{y\sim x}f(y).$$

Assume first $x\neq a$ and $x\neq b$ so that $f(x)=0$. If among vertices $y\sim x$ there is a then there is also b so that

$$\sum_{y\sim x}f(y)=0. \tag{31}$$

If among $y\sim x$ there is neither a nor b then (31) holds again. Hence, in the both cases $\mathcal{L}f(x)=0$. Consider next two cases:

Case 1. $a\sim b$. If $x=a$ then among $y\sim a$ there is $y=b$ whence

$$\sum_{y\sim x}f(y)=-1$$

and $\mathcal{L}f(a)=1+\frac{1}{k}=\frac{k+1}{k}$. Similarly, if $x=b$ then $\mathcal{L}f(b)=-\frac{k+1}{k}$. Hence, we obtain

$$\mathcal{L}f(x)=\frac{k+1}{k}f(x)$$

for all $x\in V$.

Case 2. $a\not\sim b$. Then we obtain that $\mathcal{L}f(a)=1$ and $\mathcal{L}f(b)=-1$ so that $\mathcal{L}f=f$.

Hence, in the both case, f is an eigenfunction, and the eigenvalue is either $\frac{k+1}{k}$ or 1.

32. Let (V,μ) be a finite connected weighted graph and i be a weight preserving involution of (V,μ), that is, a non-identical mapping $i:V\to V$ such that $i^2=\mathrm{id}$ and $\mu_{i(x)i(y)}=\mu_{xy}$ for all $x,y\in V$.

   (a) Prove that there exists a non-constant eigenfunction $f(x)$ of the Laplace operator $\mathcal{L}$ on (V,μ) such that $f\circ i=-f$.

   (b) Prove that if there exist vertices $x_1,x_2\in V$ such that the four vertices $x_1,x_2,i(x_1),i(x_2)$ are all distinct then there exists a non-constant eigenfunction $f(x)$ of the Laplace operator $\mathcal{L}$ on (V,μ) such that $f\circ i=f$.

Solution. It follows from the hypotheses that

$$\mu(i(x))=\sum_y\mu_{i(x),y}=\sum_z\mu_{i(x)i(z)}=\sum_z\mu_{xz}=\mu(x),$$

whence

$$P(i(x),i(y))=P(x,y).$$

Therefore,

$$P(f\circ i)(x) = \sum_y P(x,y)(f\circ i)(y)=\sum_y P(x,y)f(i(y))=\sum_z P(x,i(z))f(z)$$

$$= \sum_z P(i(x), z) f(x) = (Pf)(i(x)).$$

(a) Let J be a subspace of $\mathcal{F}$ that consists of all functions f with the property $f \circ i = -f$. Since i is not identical, the space J is non-trivial. Indeed, if $i(x_0) \neq x_0$ for some $x_0 \in V$ then define

$$f(x) = \begin{cases} 1, x = x_0 \\ -1, x = i(x_0) \\ 0, \text{otherwise} \end{cases}$$

so that $f \in J$. For any $f \in J$ we have

$$Pf(i(x)) = P(f \circ i)(x) = -Pf(x),$$

that is, $(Pf) \circ i = -Pf$ and $Pf \in J$. Hence, J is an invariant space of P and, hence, P has an eigenfunction in J, and it will be also an eigenfunction of $\mathcal{L}$. This eigenfunction is non-constant because $1 \notin J$.

(b) Let K be a subspace of $\mathcal{F}$ that consists of functions f with the properties $f \circ i = f$ and $f \perp 1$ (that is, $(f, 1) = 0$). As above, we have, for any $f \in K$,

$$Pf(i(x)) = P(f \circ i)(x) = Pf(x),$$

and

$$(Pf, 1) = (f - \mathcal{L}f, 1) = (f, 1) - (\mathcal{L}f, 1) = 0.$$

Hence, $Pf \in K$ and K is an invariant space for P. The space K contains a non-zero function, because the function

$$f(x) = \begin{cases} \frac{1}{\mu(x_1)}, \text{if } x = x_1 \text{ or } x = i(x_1) \\ -\frac{1}{\mu(x_2)}, \text{ if } x = x_2 \text{ or } x = i(x_2) \\ 0, \text{ otherwise} \end{cases}$$

belongs to K. Hence, K contains an eigenfunction of P, which is also an eigenfunction of $\mathcal{L}$.

33. Evaluate the eigenvalues of the Laplace operator of the graph on Fig. 7 with a simple weight.

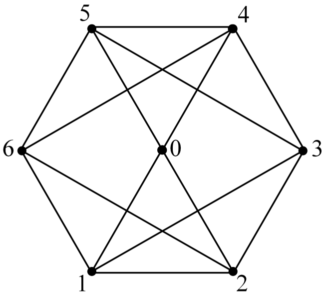

Figure 7: A 4-regular graph with 7 vertices

Hint: Use Exercises 31 and 32 to build various eigenfunctions of the Laplace operator.

Solution. Note that the graph in question is regular with degree $k = 4$. A function $f_0 \equiv 1$ is always an eigenfunction with eigenvalue 0. A pair $0, 3$ satisfies the condition of Exercise 31 so that we obtain eigenfunction f_1 defined by

$$f_1(x) = \begin{cases} 1, x = 0 \\ -1, x = 3 \\ 0, \text{otherwise,} \end{cases}$$

and similarly we obtain an eigenfunction f_2 from the pair $0, 6$. Both f_1 and f_2 have eigenvalues 1 because $0 \not\sim 3$ and $0 \not\sim 6$.

Also, pairs $1, 2$ and $4, 5$ give in the same way eigenfunctions f_3, f_4 with eigenvalues $\frac{k+1}{k} = \frac{5}{4}$ because $1 \sim 2$ and $4 \sim 5$.

To use Exercise 32, define involution $i : V \to V$ as permutation $(15)(24)$ (that is, i swaps 1 with 5 and 2 with 4, leaving the points $0, 3, 6$ fixed). Then $\mathcal{L}$ has an eigenfunction f with the property $f \circ i = -f$, which means that $f(0) = f(3) = f(6) = 0$, $\alpha := f(1) = -f(5)$ and $\beta := f(2) = -f(4)$. We only have to determine, for which values α and β this function is an eigenfunction. We have

$$\begin{aligned}
\mathcal{L}f(1) &= \alpha - \frac{1}{4}\left(f(0) + f(2) + f(3) + f(6)\right) = \alpha - \frac{1}{4}\beta \\
\mathcal{L}f(2) &= \beta - \frac{1}{4}\left(f(0) + f(1) + f(3) + f(6)\right) = \beta - \frac{1}{4}\alpha
\end{aligned}$$

then similar identities holds for $x = 4$ and 5, while for $x = 0, 3, 6$ we have $\mathcal{L}f(x) = 0 = f(x)$. To have $\mathcal{L}f(x) = \lambda f(x)$ for all x, it suffices to have it for $x = 1$ and $x = 2$, which amounts to

$$\begin{aligned}
\alpha - \frac{1}{4}\beta &= \lambda\alpha \\
\beta - \frac{1}{4}\alpha &= \lambda\beta
\end{aligned}$$

that is, $\alpha(1-\lambda) = \frac{1}{4}\beta$ and $(1-\lambda)\beta = \frac{1}{4}\alpha$, which yields $(1-\lambda)^2 = \frac{1}{4^2}$ and, hence, $(1-\lambda) = \pm\frac{1}{4}$. In the case $1 - \lambda = -\frac{1}{4}$ we obtain $\lambda = \frac{5}{4}$ that is the same eigenvalue as in the previous argument. In the case $1 - \lambda = \frac{1}{4}$, we obtain a new eigenvalue $\lambda = \frac{3}{4}$ with eigenfunction f_5 that is defined as above with $\alpha = \beta = 1$.

Finally, having found six eigenvalues 0, 1 (double), $\frac{5}{4}$ (double) and $\frac{3}{4}$, we obtain the last seventh eigenvalue λ from the condition that $\operatorname{trace}\mathcal{L} = 7$, that is,

$$\lambda = 7 - (0 + 1 \cdot 2 + 5/4 \cdot 2 + 3/4) = \frac{7}{4}.$$

Obviously, this eigenvalue has to be simple. Note that the eigenfunction f_6 of $\lambda = \frac{7}{4}$ has to satisfy $f_6 \circ i = f_6$ but we do not need to compute f_6.

Alternatively, one can find all the eigenvalues of $\mathcal{L}$ by a brute force computation of the roots of the characteristic polynomial of $\mathcal{L}$ (that is a 7-th order polynomial) using software packages like Maple. The matrix of the Laplace operator is

$$\begin{pmatrix}
1 & -\frac{1}{4} & -\frac{1}{4} & 0 & -\frac{1}{4} & -\frac{1}{4} & 0 \\
-\frac{1}{4} & 1 & -\frac{1}{4} & -\frac{1}{4} & 0 & 0 & -\frac{1}{4} \\
-\frac{1}{4} & -\frac{1}{4} & 1 & -\frac{1}{4} & 0 & 0 & -\frac{1}{4} \\
0 & -\frac{1}{4} & -\frac{1}{4} & 1 & -\frac{1}{4} & -\frac{1}{4} & 0 \\
-\frac{1}{4} & 0 & 0 & -\frac{1}{4} & 1 & -\frac{1}{4} & -\frac{1}{4} \\
-\frac{1}{4} & 0 & 0 & -\frac{1}{4} & -\frac{1}{4} & 1 & -\frac{1}{4} \\
0 & -\frac{1}{4} & -\frac{1}{4} & 0 & -\frac{1}{4} & -\frac{1}{4} & 1
\end{pmatrix}$$

whence one obtains the same eigenvalues as above.

34. Evaluate the eigenvalues of the Laplace operator on the Petersen graph from Exercise 13.

    Solution. Let us enumerate the vertices of the Petersen graph as on Fig. 8.

    To use Exercise 32, define involution $i : V \to V$ as permutation

$$i = (13)(04)(68)(59),$$

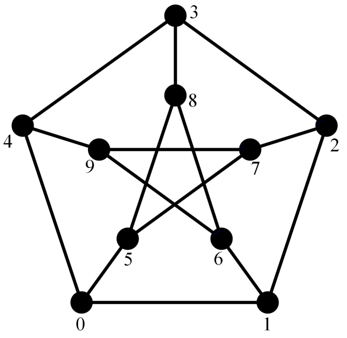

Figure 8: The Petersen graph

thus living the points 2 and 7 fixed. By Exercise 32, there exists an eigenfunction f such that $f\circ i=-f$. It follows that

$$\begin{gathered} f(2)=f(7)=0 \\ \alpha:=f(1)=-f(3) \\ \beta:=f(0)=-f(4) \\ \gamma:=f(6)=-f(8) \\ \delta:=f(5)=-f(9), \end{gathered} \tag{32}$$

where $\alpha,\beta,\gamma,\delta$ are to be determined. The equation $\mathcal{L}f(x)=\lambda f(x)$ at $x=1,0,6,5$ yields

$$\begin{aligned} \lambda\alpha &= \lambda f(1)=\mathcal{L}f(1)=f(1)-\frac{1}{3}(f(0)+f(2)+f(6))=\alpha-\frac{1}{3}(\beta+\gamma) \\ \lambda\beta &= \lambda f(0)=\mathcal{L}f(0)=f(0)-\frac{1}{3}(f(1)+f(4)+f(5))=\beta-\frac{1}{3}(\alpha-\beta+\delta) \\ \lambda\gamma &= \lambda f(6)=\mathcal{L}f(6)=f(6)-\frac{1}{3}(f(1)+f(8)+f(9))=\gamma-\frac{1}{3}(\alpha-\gamma-\delta) \\ \lambda\delta &= \lambda f(5)=\mathcal{L}f(5)=f(5)-\frac{1}{3}(f(0)+f(7)+f(8))=\delta-\frac{1}{3}(\beta-\gamma). \end{aligned}$$

Hence, we obtain

$$\left\{\begin{array}{l} (\lambda-1)\alpha+\frac{1}{3}\beta+\frac{1}{3}\gamma=0 \\ \frac{1}{3}\alpha+\left(\lambda-\frac{4}{3}\right)\beta+\frac{1}{3}\delta=0 \\ \frac{1}{3}\alpha+\left(\lambda-\frac{4}{3}\right)\gamma-\frac{1}{3}\delta=0 \\ \frac{1}{3}\beta-\frac{1}{3}\gamma+(\lambda-1)\delta=0 \end{array}\right. \tag{33}$$

Since the determinant of this linear system is

$$\begin{aligned} \det\begin{pmatrix} \lambda-1 & \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & \lambda-\frac{4}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & 0 & \lambda-\frac{4}{3} & -\frac{1}{3} \\ 0 & \frac{1}{3} & -\frac{1}{3} & \lambda-1 \end{pmatrix} &= \lambda^4-\frac{14}{3}\lambda^3+\frac{23}{3}\lambda^2-\frac{140}{27}\lambda+\frac{100}{81} \\ &= \frac{1}{81}(3\lambda-2)^2(3\lambda-5)^2, \end{aligned}$$

we obtain the double eigenvalues $\lambda=\frac{2}{3}$ and $\lambda=\frac{5}{3}$.

Considering another involution

$$j=(03)(58)(67)(12)$$

with fixed points 4 and 9, we obtain the same eigenvalues, but with different eigenfunctions. Indeed, if an eigenfunction f of one of the eigenvalues λ satisfies both $f\circ i=-f$ and $f\circ j=-f$ then

$$f\circ i\circ j=f.$$

On the other hand, $i\circ j$ acts as follows: $i\circ j:1\mapsto 2\to 3\to 4\to 0\to 1$, which implies that

$$f(0)=f(1)=f(2)=f(3)=f(4).$$

Comparing with (32), we conclude that $\alpha=\beta=0$, and substituting into (33) yields also $\gamma=\delta=0$. Hence, the involutions i and j give different eigenfunctions, which implies that each of the eigenvalues $\frac{2}{3}$ and $\frac{5}{3}$ has already 4 independent eigenfunctions. Since $\lambda=0$ is also an eigenvalue, we have obtained thus 9 independent eigenfunctions. The remaining 10^{th} eigenvalue can be found from $\operatorname{trace}\mathcal{L}=10$, which yields the value

$$10-\left(0+\frac{2}{3}\cdot 4+\frac{5}{3}\cdot 4\right)=\frac{2}{3}.$$

Hence, the eigenvalues of the Petersen graphs are 0 (simple), $\frac{2}{3}$ (multiplicity 5) and $\frac{5}{3}$ (multiplicity 4).

The same result can be obtain by computing directly the eigenvalues of the matrix of $\mathcal{L}$ which is

$$\begin{pmatrix}
1 & -\frac{1}{3} & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 & 0\\
-\frac{1}{3} & 1 & -\frac{1}{3} & 0 & 0 & 0 & -\frac{1}{3} & 0 & 0 & 0\\
0 & -\frac{1}{3} & 1 & -\frac{1}{3} & 0 & 0 & 0 & -\frac{1}{3} & 0 & 0\\
0 & 0 & -\frac{1}{3} & 1 & -\frac{1}{3} & 0 & 0 & 0 & -\frac{1}{3} & 0\\
-\frac{1}{3} & 0 & 0 & -\frac{1}{3} & 1 & 0 & 0 & 0 & 0 & -\frac{1}{3}\\
-\frac{1}{3} & 0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{3} & -\frac{1}{3} & 0\\
0 & -\frac{1}{3} & 0 & 0 & 0 & 0 & 1 & 0 & -\frac{1}{3} & -\frac{1}{3}\\
0 & 0 & -\frac{1}{3} & 0 & 0 & -\frac{1}{3} & 0 & 1 & 0 & -\frac{1}{3}\\
0 & 0 & 0 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & 1 & 0\\
0 & 0 & 0 & 0 & -\frac{1}{3} & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & 1
\end{pmatrix}$$

35. Let (V,μ) be a finite connected weighted graph. Prove that if the diameter of the graph is $D\geq 1$ then there exist at least $D+1$ distinct eigenvalues of the Laplace operator.

    Solution. It suffices to prove the same for the eigenvalues of the Markov operator P. Let f be a non-negative function on V. Set $A=\operatorname{supp}f=\{x\in V: f(x)\neq 0\}$ and $B=\operatorname{supp}Pf$.

    Claim. The set $B\setminus A$ consists exactly of those points $x\in V\setminus A$ that are connected to A.

    Indeed, if x is not connected to A then

    $$Pf(x)=\sum_{y\sim x}P(x,y)f(y)=\sum_{y\notin A,y\sim x}P(x,y)f(y)=0$$

    while if x is connected to A then

    $$Pf(x)=\sum_{y\sim x}P(x,y)f(y)=\sum_{y\in A,y\sim x}P(x,y)f(y)>0.$$

    Let x_0,y_0 be two points such that $d(x_0,y_0)=D$. Start with function $f=\mathbf{1}_{\{x_0\}}$ and consider the sequence $f_k=P^kf$. Also, set

    $$B_k=\{x\in V: d(x,x_0)\leq k\}\quad\text{and}\quad S_k=\{x\in V: d(x,x_0)=k\}.$$

    Let us prove by induction in k that

    $$S_k\subset\operatorname{supp}P^kf\subset B_k. \tag{34}$$

Indeed, the inductive basis for $k=0$ is trivial as $S_0 = B_0 = \{x_0\}$. If (34) holds for some $k < D$ then $\operatorname{supp} P^{k+1}P \subset B_{k+1}$ because $V \setminus B_{k+1}$ is not connected to B_k. If $x \in S_{k+1}$ then x is connected to S_k and is outside B_k. Hence, x belongs to $V \setminus \operatorname{supp} P^k f$ and x is connected to $\operatorname{supp} P^k f$, which implies by the above Claim that $x \in \operatorname{supp} P^{k+1} f$.

For any $k \leq D$, the set S_k is non-empty. This implies that the functions

$$f, Pf, ..., \ P^D f$$

are linearly independent. Indeed, if for some constants c_k

$$\sum_{k=0}^{D} c_k P^k f = 0$$

then choose maximal j with $c_j \neq 0$. Then $\operatorname{supp} P^j f$ contains S_j while $\operatorname{supp} P^k f \subset B_{k-1}$ for all $k < j$. This means that, for any $x \in S_j$,

$$\sum_{k=0}^{D} c_k P^k f(x) = c_j P^j f(x) \neq 0.$$

Finally, let us show that P has only simple eigenvalues. Let P have distinct eigenvalues say $\lambda_1, ..., \lambda_m$. Consider the polynomial

$$F(\lambda) = (\lambda - \lambda_1) \ldots (\lambda - \lambda_m)$$

of degree m. Then $F(P) = 0$ (which is obvious in the eigenfunction basis) which means that the operators $\operatorname{id}, P, P^2, ..., P^m$ are linearly dependent. However, we have shown above that this is not the case if $m \leq D$. Therefore, $m \geq D+1$, which was to be proved.

36. Let (V, μ) be a finite connected weighted graph. Prove that, for any subset $\Omega \subset V$,

$$\mu(\partial\Omega) \geq \lambda_1 \frac{\mu(\Omega)\mu(\Omega^c)}{\mu(V)},$$

where λ_1 is the smallest positive eigenvalue of the Laplace operator on (V, μ).

Solution. If Ω or Ω^c is empty then the inequality is trivial. Otherwise, rewrite it in the form

$$\lambda_1 \leq \frac{\mu(\partial\Omega)\mu(V)}{\mu(\Omega)\mu(\Omega^c)}.$$

Hence, it suffices to construct a non-zero function $f \perp 1$ such that

$$\mathcal{R}(f) \leq \frac{\mu(\partial\Omega)\mu(V)}{\mu(\Omega)\mu(\Omega^c)}.$$

Consider the following function

$$f(x) = \begin{cases} 1, & x \in \Omega \\ -a, & x \notin \Omega^c \end{cases}$$

where a is chosen so that $f \perp 1$, that is, $\mu(\Omega) = a\mu(\Omega^c)$ whence

$$a = \frac{\mu(\Omega)}{\mu(\Omega^c)}.$$

We have

$$(f, f) = \sum_{x \in V} f(x)^2 \mu(x) = \mu(\Omega) + a^2 \mu(\Omega^c) = (1+a)\mu(\Omega)$$

and

$$
\begin{aligned}
(\mathcal{L}f, f) &= \frac{1}{2}\sum_{x,y} (f(x) - f(y))^2 \mu_{xy} \\
&= \sum_{x\in\Omega, y\in\Omega^c} (f(x) - f(y))^2 \mu_{xy} \\
&= (1+a)^2 \sum_{x\in\Omega, y\in\Omega^c} \mu_{xy} \\
&= (1+a)^2 \mu(\partial\Omega).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\mathcal{R}(f) &\leq \frac{(1+a)^2 \mu(\partial\Omega)}{(1+a)\mu(\Omega)} \\
&= (1+a)\frac{\mu(\partial\Omega)}{\mu(\Omega)} \\
&= \left(1 + \frac{\mu(\Omega)}{\mu(\Omega^c)}\right)\frac{\mu(\partial\Omega)}{\mu(\Omega)} \\
&= \frac{\mu(\partial\Omega)\mu(V)}{\mu(\Omega)\mu(\Omega^c)},
\end{aligned}
$$

which was to be proved.

37. Let (V, μ) be a finite connected weighted graph with $N > 1$ vertices. Fix a positive integer r and define the *expansion factor* F_r of the graph by

$$
F_r = \inf_{X\subset V} \frac{\mu(X_r \setminus X)\mu(V)}{\mu(X)\mu(X^c)},
$$

where $X_r = \{x \in V : d(x, X) \leq r\}$. Prove that

$$
F_r \geq 1 - \left(\frac{\lambda_{N-1} - \lambda_1}{\lambda_{N-1} + \lambda_1}\right)^{2r},
$$

where λ_1 and λ_{N-1} are the eigenvalues of the Laplace operator on (V, μ).

Solution. Assume without loss of generality that $\mu(V) = 1$ and set $a = \mu(X)$ and $b = \mu(X_r)$. Then we have

$$
b \geq \frac{1}{1 + \frac{1-a}{a}\delta^{2r}} = \frac{a}{a + (1-a)\delta^{2r}},
$$

where $\delta = \frac{\lambda_{N-1}-\lambda_1}{\lambda_{N-1}+\lambda_1}$. It follows that

$$
\mu(X_1 \setminus X) = b - a \geq \frac{a - a^2 - (a - a^2)\delta^{2r}}{a + (1-a)\delta^{2r}}
$$

whence

$$
\begin{aligned}
\frac{\mu(X_r \setminus X)\mu(V)}{\mu(X)\mu(X^c)} &\geq \frac{1}{a(1-a)}\frac{a - a^2 - (a - a^2)\delta^{2r}}{a + (1-a)\delta^{2r}} \\
&= \frac{1 - \delta^{2r}}{a + (1-a)\delta^{2r}} \\
&\geq 1 - \delta^{2r},
\end{aligned}
$$

which proves the claim.

38. Let (V, E) be a simple connected graph of diameter $D \geq 2$, and let μ be a simple weight on (V, E). Set

$$k = \max_{x \in V} \deg(x).$$

(a) Fix $x_0 \in V$ and define for any non-negative integer r a set E_r of edges as follows:

$$E_r = \{xy \in E : \ d(x, x_0) = r, \ d(y, x_0) = r + 1\}.$$

Prove that

$$|E_R| \leq (k-1)^{R-r} |E_r|$$

for all non-negative integers $r \leq R$.

(b) Fix a positive integer R and consider the following function on V:

$$f(x) = \begin{cases} 1, & x = x_0 \\ e^{-\alpha(r-1)}, & \text{if } r := d(x, x_0) \in [1, R] \\ 0, & \text{if } r > R, \end{cases}$$

where $a = \frac{1}{2} \ln(k-1)$. Prove that

$$\mathcal{R}(f) \leq 1 - \frac{2\sqrt{k-1}}{k}\left(1 - \frac{1}{R}\right) - \frac{1}{kR}.$$

(c) Let $\{\lambda_m\}_{m=0}^{N-1}$ be an increasing sequence of the eigenvalues of the Laplace operator $\mathcal{L}$ on (V, μ) counted with multiplicities, where $N = |V|$. Prove that if $4m \leq D$ then

$$\lambda_m \leq 1 - \frac{2\sqrt{k-1}}{k}\left(1 - \frac{1}{R}\right) - \frac{1}{kR}$$

where

$$R = \left[\frac{D}{2m}\right] - 1.$$

Remark: If R is large then the main part of this estimate is given by the term $1 - 2\frac{\sqrt{k-1}}{k}$. There are k-regular graphs with arbitrarily large diameters and with

$$\lambda_1 \geq 1 - 2\frac{\sqrt{k-1}}{k}.$$

Such graphs are called *Ramanujan graphs*.

Solution. (a) Introduce the following notation, for any $r = 0, 1, \ldots$:

$$S_r = \{x \in X : d(x_0, x) = r\}.$$

Then we have

$$E_r = \{xy \in E : x \in S_r \text{ and } y \in S_{r+1}\}.$$

Note that $|S_{r+1}| \leq |E_r|$ and each vertex $x \in S_{r+1}$ gives $\leq k-1$ edges to S_{r+2}. Hence,

$$|E_{r+1}| \leq (k-1)|S_{r+1}| \leq (k-1)|E_r|.$$

Then by induction we obtain

$$|E_R| \leq (k-1)^{R-r} |E_r|,$$

for all $r < R$. Note that the connectedness of (V, E) and $D \geq 2$ imply that $k \geq 2$.

(b) Observe that

$$f(x) = \begin{cases} 1, & x \in S_0 \\ e^{-\alpha(r-1)}, & x \in S_r \text{ with } r \in [1, R] \\ 0, & x \in S_r \text{ with } r > R. \end{cases}$$

Also, we have, for any $r \geq 1$,

$$\mu(S_r) = \sum_{x \in S_r} \deg(x) \geq |E_{r-1}| + |E_r|$$

because $\sum_{x \in S_r} \deg(x)$ is the number of all edges coming out of S_r, and this set of edges contains E_{r-1} and E_r. It follows

$$\begin{aligned} (f, f) &= \sum_{x \in V} f(x)^2 \mu(x) \\ &= \sum_{r=0}^{R} \sum_{x \in S_r} f(x)^2 \mu(x) \\ &= \mu(x_0) + \sum_{r=1}^{R} e^{-2a(r-1)} \mu(S_r) \\ &> \sum_{r=1}^{R} e^{-2a(r-1)} \left(|E_{r-1}| + |E_r|\right) \\ &\geq \sum_{r=1}^{R} e^{-2a(r-1)} \frac{k}{k-1} |E_r| \end{aligned}$$

where in the last line we have used that $|E_{r-1}| \geq \frac{1}{k-1} |E_r|$.

To estimate $(\mathcal{L}f, f) = \frac{1}{2} \sum_{x \sim y} (f(x) - f(y))^2$ observe first that if $x \sim y$ and $x \in S_r$ then y belongs to one of the sets S_{r-1}, S_r, S_{r+1}. If $y \in S_r$ then $f(x) - f(y) = 0$ so that such edges (x, y) can be excluded from the summation. If $y \in S_{r-1}$ then swap the notation x and y and rename $r-1$ to r. Hence, we can restrict summation to $x \in S_r$ and $y \in S_{r+1}$. If $r = 0$ or $r > R$ then again $f(x) - f(y) = 0$ so that the range of r can be restricted to $[1, R]$. Hence, we have

$$\begin{aligned} (\mathcal{L}f, f) &= \sum_{r=1}^{R} \sum_{x \in S_r, y \in S_{r+1}, x \sim y} (f(x) - f(y))^2 \\ &= \sum_{r=1}^{R-1} \left(e^{-a(r-1)} - e^{-ar}\right)^2 |E_r| + e^{-2a(R-1)} |E_R| \\ &= \sum_{r=1}^{R} \left(e^{-a(r-1)} - e^{-ar}\right)^2 |E_r| + \left[e^{-2a(R-1)} - \left(e^{-a(R-1)} - e^{-aR}\right)^2\right] |E_R|. \end{aligned}$$

Noticing that

$$\left(e^{-a(r-1)} - e^{-ar}\right)^2 = \left(1 - e^{-a}\right)^2 e^{-2a(r-1)}$$

and

$$\begin{aligned} e^{-2a(R-1)} - \left(e^{-a(R-1)} - e^{-aR}\right)^2 &= 2e^{-a(2R-1)} - e^{-2aR} \\ &= \left[2e^{-a} - e^{-2a}\right] e^{-2a(R-1)} \end{aligned}$$

we obtain

$$(\mathcal{L}f,f)=\sum_{r=1}^{R}\left(1-e^{-a}\right)^2 e^{-2a(r-1)}\left|E_r\right|+\left[2e^{-a}-e^{-2a}\right]e^{-2a(R-1)}\left|E_R\right|. \tag{35}$$

Comparing with the above lower bound of (f,f) we obtain

$$\sum_{r=1}^{R}\left(1-e^{-a}\right)^2 e^{-2a(r-1)}\left|E_r\right|\le\left(1-e^{-a}\right)^2\frac{k-1}{k}(f,f).$$

To estimate the second term in (35) we use part (a) and the identity $2a=\ln(k-1)$ which yields

$$\frac{e^{-2a(R-1)}\left|E_R\right|}{e^{-2a(r-1)}\left|E_r\right|}=(k-1)^{-(R-r)}\frac{\left|E_R\right|}{\left|E_r\right|}\le 1.$$

Hence,

$$e^{-2a(R-1)}\left|E_R\right|\le\frac{1}{R}\sum_{r=1}^{R}e^{-2a(r-1)}\left|E_r\right|\le\frac{1}{R}\frac{k-1}{k}(f,f).$$

Combining the above estimates, we obtain

$$\begin{aligned}
\mathcal{R}(f) &= \frac{(\mathcal{L}f,f)}{(f,f)}\le\left(1-e^{-a}\right)^2\frac{k-1}{k}+\left[2e^{-a}-e^{-2a}\right]\frac{1}{R}\frac{k-1}{k}\\
&= \left(1-\frac{1}{\sqrt{k-1}}\right)^2\frac{k-1}{k}+\left(\frac{2}{\sqrt{k-1}}-\frac{1}{k-1}\right)\frac{1}{R}\frac{k-1}{k}\\
&= \left(1+\frac{1}{k-1}-\frac{2}{\sqrt{k-1}}\right)\frac{k-1}{k}+\frac{1}{R}\frac{2\sqrt{k-1}-1}{k}.\\
&= 1-2\frac{\sqrt{k-1}}{k}+\frac{1}{R}\frac{2\sqrt{k-1}-1}{k}\\
&= 1-\frac{2\sqrt{k-1}}{k}\left(1-\frac{1}{R}\right)-\frac{1}{kR}.
\end{aligned}$$

(c) Fix x_0,y_0 such that $d(x_0,y_0)=D$, and let $\{z_i\}_{i=0}^{D}$ be a shortest path connecting x_0 and y_0. To obtain an upper bound for λ_m, we need to construct $m+1$ functions $f_0,...,f_m$ with disjoint, not connected supports with controlled $\mathcal{R}(f_i)$. Let $R=\left[\frac{D}{2m}\right]-1$. Then choose a sequence $\{x_i\}_{i=0}^{m}$ of vertices as follows:

$$x_i=z_{(2R+2)i}.$$

Since $(2R+2)m\le D$, we can indeed choose such a sequence. Construct f_i as above using x_i instead of x_0. Then $\operatorname{supp} f_i$ is located in R-neighborhood of x_i, and since $d(x_i,x_{i+1})=2R+2$, we see that $\operatorname{supp} f_i$ and $\operatorname{supp} f_{i+1}$ are disjoint and not connected. By part (b), we have

$$\mathcal{R}(f_i)\le 1-\frac{2\sqrt{k-1}}{k}\left(1-\frac{1}{R}\right)-\frac{1}{kR},$$

for all $i=0,1,...,m-1$ whence the required estimate for λ_m follows.

39. Fix a positive integer N and consider the following subset Ω of $\mathbb{Z}^2$:

$$\Omega=\{(j,0):j=1,2,...,N\}.$$

Evaluate all the eigenvalues and eigenfunctions of the Dirichlet Laplace operator $\mathcal{L}_\Omega$.

Solution. It will be more convenient to evaluate the eigenvalues and eigenfunctions of $P_\Omega = \operatorname{id} - \mathcal{L}_\Omega$. Let f be a function on Ω that is extended to $\mathbb{Z}^2$ by setting $f = 0$ outside Ω. Since $\deg(x) = 4$ for any $x \in \mathbb{Z}^2$, we have

$$Pf(k) = \frac{1}{4}(f(k-1) + f(k+1)),$$

for any integer $k \in \Omega$ (for simplicity, we denote the elements of Ω by k rather than $(k, 0)$). Since $P_\Omega f = Pf$ on Ω, the equation $P_\Omega f = \alpha f$ for the eigenfunctions of P_Ω becomes

$$\frac{1}{4}(f(k-1) + f(k+1)) = \alpha f(k), \quad k = 1, ..., N, \tag{36}$$

with additional condition that $f(k) = 0$ if $k \le 0$ or $k \ge N+1$. Let us solve first (36) for all $k \in \mathbb{Z}$ and then select solutions that satisfy the conditions

$$f(0) = f(N+1) = 0. \tag{37}$$

Those f will be eigenfunctions of P_Ω.

Search solutions to (36) in the form $f(k) = r^k$ where r is to be found. Substituting into (36), we obtain the equation for r:

$$r^2 - 4\alpha r + 1 = 0.$$

So solve this quadratic equation, we first assume that $|\alpha| < 1/2$ so that the roots are imaginary:

$$r = 2\alpha \pm i\sqrt{1 - 4\alpha^2}$$

(as we will see, already imaginary roots give us all eigenfunctions so that we do not need to consider the case $|\alpha| \ge 1/2$). Since $|r| = 1$, we can represent r in the form $r = e^{\pm i\theta}$ where θ is determined by two conditions

$$\cos\theta = 2\alpha \qquad \text{and} \quad \sin\theta = \sqrt{1 - 4\alpha^2}. \tag{38}$$

Assuming in addition $\theta \in (0, \pi)$ we see that (38) define a bijection between the values of $\theta \in (0, \pi)$ and $\alpha \in (-1/2, 1/2)$. Hence, we obtain two complex solutions of (36) in $\mathbb{Z}$:

$$\begin{aligned} f_1(k) &= e^{i\theta k} = \cos k\theta + i \sin k\theta \\ f_2(k) &= e^{-i\theta k} = \cos k\theta - i \sin k\theta \end{aligned}$$

which yields the following family of real solutions:

$$f(k) = C_1 \cos k\theta + C_2 \sin k\theta$$

where C_1 and C_2 are arbitrary real constants. This solution satisfies (37) provided $C_1 = 0$ and $\sin(N+1)\theta = 0$. The latter is equivalent to

$$(N+1)\theta = \pi l \quad \text{for } l \in \mathbb{Z},$$

that is,

$$\theta = \frac{\pi l}{N+1}.$$

The restriction $\theta \in (0, \pi)$ is equivalent to $1 \le l \le N$. Hence, we obtain N values for θ:

$$\theta_l = \frac{\pi l}{N+1}, \quad l = 1, ..., N$$

and N corresponding solutions

$$f_l(k) = \sin k\theta_l,$$

which are the eigenfunctions of P_Ω with the eigenvalues

$$\alpha_l = \frac{1}{2}\cos\theta_l.$$

Since $\dim \mathcal{F}_\Omega = N$ and we have found N independent eigenfunctions of P_Ω, there is no other eigenfunction. The eigenfunctions of $\mathcal{L}_\Omega$ are also f_l, the eigenvalues of $\mathcal{L}_\Omega$ are

$$\lambda_l = 1 - \alpha_l = 1 - \frac{1}{2}\cos\frac{\pi l}{N+1}.$$

40. Let (V, E) be a connected locally finite infinite graph without loops. A finite or infinite sequence $\{x_k\}$ of vertices on V is called a *geodesic* if $d(x_k, x_n) = |k-n|$ for all indices k, n. Prove that there is an infinite geodesic starting at any given vertex $x \in V$.

    Solution. Observe that every two distinct points $x, y \in V$ can be connected be a shortest path of length $d(x,y)$, and this path is necessarily a geodesic. Since the graph is infinite and connected, there are hence the geodesics coming out of x of arbitrarily large length. Consider the family G_x of all geodesic coming out of x. We need to prove the existence of an infinite geodesic in G_x. Assume that G_x contains only finite geodesics. Since the length of them is unbounded, the number of geodesics in G_x is infinite. It follows that there is an edge xx_1 that belongs to an infinite number of geodesics from G_x; denote this family of geodesics by G_{xx_1}. Then there is an edge x_1x_2 that belong to infinite number of geodesics from G_{xx_1}, denote this family $G_{xx_1x_2}$, etc. In the end, we construct an infinite path $x, x_1, x_2, \ldots$ so that for any index n, we have $x \sim x_1 \sim \ldots \sim x_n$ and the family of geodesics $G_{xx_1\ldots x_n}$ that starts from the path $x \sim x_1 \sim \ldots \sim x_n$, is infinite. Of course, the path $x \sim x_1 \sim \ldots \sim x_n$ itself is a geodesic. It follows that the infinite sequence $\{x_k\}$ is also a geodesic, which finishes the proof.

In all the remaining questions, (V, μ) is an infinite, locally finite, connected weighted graph, P is the corresponding Markov kernel, $p_n(x,y)$ is the heat kernel, and Ω is a finite non-empty subset of V.

41. Prove that if

    $$\frac{(\mathcal{L}_\Omega f, f)}{(f,f)} = \lambda_1(\Omega)$$

    for some non-zero function $f \in \mathcal{F}_\Omega$ then f is an eigenfunction of $\mathcal{L}_\Omega$ with the eigenvalue $\lambda_1(\Omega)$.

    Solution. Let $\{\varphi_k\}_{k=1}^N$ be an orthonormal basis in $\mathcal{F}_\Omega$ that consists of the eigenfunctions of $\mathcal{L}_\Omega$. Let the eigenvalue of φ_k be $\lambda_k = \lambda_k(\Omega)$. Any function $f \in \mathcal{F}_\Omega$ can be expanded in the basis $\{\varphi_k\}$ as $f = \sum_k a_k\varphi_k$ with some coefficients a_k. Then $\mathcal{L}_\Omega f = \sum_k a_k\lambda_k\varphi_k$ and

    $$\frac{(\mathcal{L}_\Omega f, f)}{(f,f)} = \frac{\sum_k \lambda_k a_k^2}{\sum_k a_k^2}.$$

    Since $\frac{(\mathcal{L}_\Omega f, f)}{(f,f)} = \lambda$, it follows that

    $$\begin{aligned} \sum_k \lambda_k a_k^2 &= \lambda_1 \sum_k a_k^2, \\ \sum_{k=1}^N (\lambda_k - \lambda_1) a_k^2 &= 0. \end{aligned}$$

    It follows that if $\lambda_k > \lambda_1$ then $a_k = 0$. Hence, f is a linear combination of some eigenfunctions φ_k all of them having the eigenvalue λ_1. Hence, f is also an eigenfunction with the eigenvalue λ_1.

42. Let the weight μ be simple and (V, E) be m-regular. Let Ω be a finite non-empty subset of V such that every vertex of Ω has at most k neighbors in Ω where $2 \leq k < m$ (for example, if Ω is a path or a cycle then $k = 2$). Prove that

$$h(\Omega) \geq \frac{m-k}{m} \quad \text{and} \quad \lambda_1(\Omega) \geq \frac{1}{2}\left(\frac{m-k}{m}\right)^2.$$

Solution. Let U be any non-empty subset of Ω. Any vertex $x \in U$ has at most two neighbors in U. Hence, it has at least $m - k > 0$ neighbors outside U. Hence, the total number of edges that has one end in U and the other end outside U is at least $(m-k)|U|$. It follows that $\mu(\partial U) \geq (m-k)|U|$. Since $\mu(U) = m|U|$, we obtain that

$$\frac{\mu(\partial U)}{\mu(U)} \geq \frac{m-k}{m},$$

whence $h(\Omega) \geq \frac{m-k}{m}$. By the Cheeger inequality, $\lambda_1(\Omega) \geq \frac{1}{2}h^2 = \frac{1}{2}\left(\frac{m-k}{m}\right)^2$.

43. For any positive integer r set $\Omega_r = U_r(\Omega)$. Prove that

$$\lambda_1(\Omega_r) \leq \frac{\mu(\Omega_{r+1})}{r^2\mu(\Omega)}.$$

Solution. Consider the following function on V

$$f(x) = (r - d(x, \Omega))_+ .$$

If $x \notin \Omega_r$ then $d(x, \Omega) > r$ so that $f(x) = 0$. Hence, f can be considered as a function on Ω_r that is extended by 0 to Ω_r^c. Let us estimate $\mathcal{R}(f)$. Since $f|_\Omega = r$, we have

$$(f, f) = \sum_{x \in V} f^2(x)\mu(x) \geq \sum_{x \in \Omega} f^2(x)\mu(x) \geq r^2\mu(\Omega).$$

For arbitrary vertices $x, y \in V$, that are connected by an edge, we have $|f(x) - f(y)| \leq 1$. Indeed, using the obvious inequality $|a_+ - b_+| \leq |a - b|$, we obtain

$$|f(x) - f(y)| \leq |d(x, \Omega) - d(y, \Omega)| \leq d(x, y) = 1.$$

Hence,

$$\begin{aligned}
(\mathcal{L}_\Omega f, f) &= \frac{1}{2}\sum_{x,y \in V}(f(x) - f(y))^2\mu_{xy} \\
&= \frac{1}{2}\sum_{x,y \in \Omega_{r+1}}(f(x) - f(y))^2\mu_{xy} \\
&\leq \frac{1}{2}\sum_{x,y \in \Omega_{r+1}}\mu_{xy} \\
&\leq \frac{1}{2}\mu(\Omega_{r+1}).
\end{aligned}$$

It follows that

$$\mathcal{R}(f) = \frac{(\mathcal{L}_\Omega f, f)}{(f, f)} \leq \frac{1}{2r^2}\frac{\mu(\Omega_{r+1})}{\mu(\Omega)}.$$

whence the claim follows.

44. Let $B_r = \{x \in \mathbb{Z}^m : d(x,0) \le r\}$ be the ball of radius r in $\mathbb{Z}^m$. Prove that

$$\lambda_1(B_r) \le \frac{C}{r^2}$$

and

$$\lambda_1(B_r) \le C' \mu(B_r)^{-2/m},$$

where C and C' are constants depending only on m.

Solution. The number of vertices in a cube

$$Q_n = \left\{x \in \mathbb{Z}^m : \max_{1\le k\le m} |x_k| \le n\right\}$$

is equal to $(2n+1)^m \simeq n^m$. Note that

$$B_r = \left\{x \in \mathbb{Z}^m : \sum_{k=1}^{m} |x_k| \le r.\right\}$$

Since the ball B_r can be squeezed between two cubes

$$Q_{r/m} \le B_r \le Q_r,$$

it follows that $\mu(B_r) \simeq r^m$. By Exercise 43, we have

$$\lambda_1(B_r) \le \frac{1}{2r^2}\frac{\mu(B_{2r+1})}{\mu(B_r)} \le \text{const}\,\frac{1}{r^2}\frac{(2r+1)^m}{r^m} \le \frac{C}{r^2}.$$

Finally, since $r^2 \simeq \mu(B_r)^{2/m}$, we obtained the second claim.

45. Consider the Dirichlet problem $\mathcal{L}_\Omega u = f$ where f and u are functions from $\mathcal{F}_\Omega$. Prove the following inequalities:

(a) $\|u\| \ge \frac{1}{2}\|f\|$.

(b) $\|u\| \le \frac{1}{\lambda_1(\Omega)}\|f\|$.

(c) Let $\Omega_1 = U_1(\Omega)$. Then

$$\frac{1}{2}\sum_{x,y\in\Omega_1} (\nabla_{xy} u)^2 \mu_{xy} \le \frac{1}{\lambda_1(\Omega)}\|f\|^2.$$

Solution. (a) We have

$$\|f\| = \|\mathcal{L}_\Omega u\| \le \|\mathcal{L}_\Omega\|\,\|u\| = \lambda_N(\Omega)\|u\| \le 2\|u\|,$$

because λ_N is the maximal eigenvalue of $\mathcal{L}_\Omega$ and it is bounded by 2.

(b) Similarly, one obtains from $u = \mathcal{L}_\Omega^{-1} f$ that

$$\|u\| = \left\|\mathcal{L}_\Omega^{-1} f\right\| \le \left\|\mathcal{L}_\Omega^{-1}\right\|\|f\| = \frac{1}{\lambda_1(\Omega)}\|f\|$$

because the eigenvalues of $\mathcal{L}_\Omega^{-1}$ are $\frac{1}{\lambda_1(\Omega)}, \ldots, \frac{1}{\lambda_N(\Omega)}$, so that the maximal eigenvalue is $\frac{1}{\lambda_1(\Omega)}$.

(c) From $\mathcal{L}_\Omega u = f$ we obtain

$$(\mathcal{L}_\Omega u, u) = (f, u) \le \|f\|\,\|u\| \le \frac{1}{\lambda_1(\Omega)}\|f\|^2.$$

On the other hand, by the Green formula, we have

$$(\mathcal{L}_\Omega u, u) = \frac{1}{2}\sum_{x,y\in\Omega_1} (\nabla_{xy}u)^2 \mu_{xy},$$

whence the claim follows.

46. Set

$$c := \inf_{x,y\in V: x\sim y} \mu_{xy}.$$

Prove that, for any function $f \in \mathcal{F}_\Omega$ and for any point $x_0 \in \Omega$, the following inequality holds:

$$(\mathcal{L}_\Omega f, f) \geq \frac{c}{d(x_0, \Omega^c)} f(x_0)^2.$$

Hint: Consider a path $\{x_k\}_{k=0}^n$ connecting x_0 to the nearest point from Ω^c and use the sum $\sum_{k=1}^n (f(x_{k-1}) - f(x_k))^2 \mu_{x_{k-1}x_k}$.

Solution. Let $\{x_k\}_{k=0}^n$ be the shortest path connecting x_0 to a vertex outside Ω so that $n = d(x_0, \Omega^c)$, $x_n \in \Omega^c$, and $x_{n-1} \in \Omega$ (see Fig. 9). It follows that $x_n \in \Omega_1 := U_1(\Omega)$.

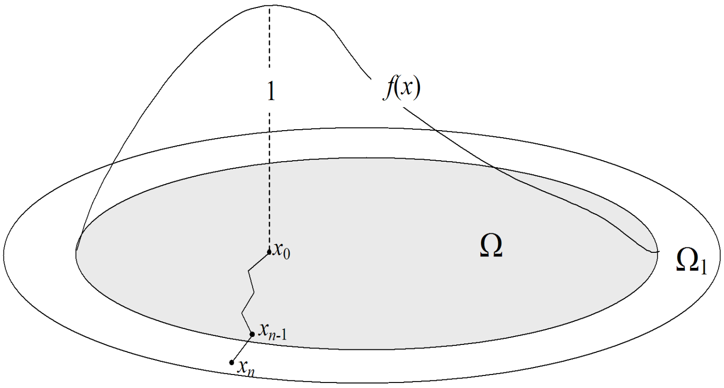

Figure 9: Path $\{x_k\}_{k=0}^n$

Using the Green formula for the operator $\mathcal{L}_\Omega$, we obtain

$$\begin{aligned}(\mathcal{L}_\Omega f, f) &= \frac{1}{2}\sum_{x,y\in\Omega_1} (\nabla_{xy}f)^2 \mu_{xy} \\ &\geq \sum_{k=1}^n (f(x_{k-1}) - f(x_k))^2 \mu_{x_{k-1}x_k} \\ &\geq \frac{c}{n}\left(\sum_{k=1}^n |f(x_{k-1}) - f(x_k)|\right)^2 \\ &\geq \frac{c}{n} = \frac{c}{d(x_0, \Omega^c)},\end{aligned}$$

where we have used that $\mu_{xy} \geq c$, the Cauchy-Schwarz inequality and the inequality

$$\sum_{k=1}^n |f(x_{k-1}) - f(x_k)| \geq |f(x_0) - f(x_n)| = 1.$$

47. Let Ω be connected and f be an eigenfunction of $\mathcal{L}_\Omega$ with the eigenvalue $\lambda_1(\Omega)$.

   (a) Prove that if $f \geq 0$ then $f > 0$ in Ω.

   (b) Prove that f_+ and f_- are also the eigenfunctions of $\lambda_1(\Omega)$, provided they do not vanish identically. Here $f_+ = \max(f, 0)$ and $f_- = -\min(f, 0)$ so that $f = f_+ - f_-$. *Hint*: Use Exercise 41.

   (c) Prove that either $f > 0$ in Ω or $f < 0$ in Ω. *Hint*: Assume the contrary and use (b).

   (d) Prove that $\lambda_1(\Omega)$ is a simple eigenvalue. *Hint*: Assuming that there exist two linearly independent eigenfunctions, consider their linear combination that vanishes at some vertex, and use (c).

Solution. Denote for simplicity $\lambda = \lambda_1(\Omega)$ so that $\mathcal{L}_\Omega f = \lambda f$ and

$$P_\Omega f = (1 - \lambda) f. \tag{39}$$

(a) If $f(x) = 0$ at some $x \in \Omega$ then (39) gives at x that

$$\sum_{y \sim x} P(x, y) f(y) = 0,$$

which implies that $f(y) = 0$ for all $y \sim x$. Arguing as in Exercise **??**, we conclude that $f \equiv 0$, which contradicts the definition of an eigenfunction.

(b) By Exercise 41, it suffices to prove that $\mathcal{R}(f_+) = \mathcal{R}(f_-) = \lambda$ where

$$\mathcal{R}(g) := \frac{(\mathcal{L}_\Omega g, g)}{\|g\|^2}.$$

Since λ is the smallest eigenvalue, we have a priori that $\mathcal{R}(g) \geq \lambda$ for all non-zero functions $g \in \mathcal{F}_\Omega$, in particular, for $g = f_+$ and $g = f_-$, while $\mathcal{R}(f) = \lambda$. Assume that $\mathcal{R}(f_+) > \lambda$ or $\mathcal{R}(f_-) > \lambda$ and show that it implies that $\mathcal{R}(f) > \lambda$, which will be a contradiction. Using $f = f_+ - f_-$, we obtain

$$\begin{aligned} \mathcal{R}(f) &= \frac{(\mathcal{L}_\Omega (f_+ - f_-), f_+ - f_-)}{\|f_+ - f_-\|^2} \\ &= \frac{(\mathcal{L}_\Omega f_+, f_+) + (\mathcal{L}_\Omega f_-, f_-) - (\mathcal{L}_\Omega f_+, f_-) - (\mathcal{L}_\Omega f_-, f_+)}{\|f_+\|^2 + \|f_-\|^2 - 2(f_+, f_-)} \end{aligned} \tag{40}$$

Note that if $f_+(x) > 0$ then $f(x) > 0$ and, hence, $f_-(x) = 0$. Therefore, $f_+ f_- \equiv 0$ and $(f_+, f_-) = 0$. Let us show that

$$(\mathcal{L}_\Omega f_+, f_-) \leq 0 \tag{41}$$

and in the same way $(\mathcal{L}_\Omega f_-, f_+) \leq 0$. Indeed, by the Green formula we have

$$\begin{aligned} (\mathcal{L}_\Omega f_+, f_-) &= \sum_{x \in \Omega} \mathcal{L} f_+(x) f_-(x) \mu(x) \\ &= \frac{1}{2} \sum_{x, y \in \Omega} (\nabla_{xy} f_+)(\nabla_{xy} f_-) \mu_{xy} - \sum_{x \in \Omega, y \in \Omega^c} (\nabla_{xy} f_+) f_-(x) \mu_{xy}. \end{aligned}$$

The summand in the first sum is equal to

$$\begin{aligned} (\nabla_{xy} f_+)(\nabla_{xy} f_-) &= (f_+(y) - f_+(x))(f_-(y) - f_-(x)) \\ &= f_+(y) f_-(y) + f_+(x) f_-(x) - f_+(x) f_-(y) - f_+(y) f_-(x) \end{aligned}$$

$$\begin{aligned} &= -f_+(x) f_-(y) - f_+(y) f_-(x) \\ &\le 0. \end{aligned}$$

Since in the second sum $y \in \Omega^c$, we have $f(y) = 0$ and

$$(\nabla_{xy} f_+) f_-(x) = -f_+(x) f_-(x) = 0.$$

Hence, (41) follows. Using that

$$(\mathcal{L}_\Omega f_+, f_+) \ge \lambda \|f_+\|^2, \quad (\mathcal{L}_\Omega f_-, f_-) \ge \lambda \|f_-\|^2$$

and that one of these inequalities is *strict*, we obtain from (40) and (41) that

$$\mathcal{R}(f) > \frac{\lambda \|f_+\|^2 + \lambda \|f_-\|^2}{\|f_+\|^2 + \|f_-\|^2} = \lambda,$$

which finishes the proof.

(c) Assume that f takes both positive and negative values so that f_+ and f_- are not identical zeros. By (b), both f_+ and f_- are the eigenfunctions with the eigenvalue λ. Since both f_+ and f_- are non-negative, by (a) they cannot take value 0. However, $f_+ \equiv 0$ on $\operatorname{supp} f_-$ which is a contradiction. Therefore, either $f \ge 0$ in Ω or $f \le 0$ in Ω. By (a), we conclude that either $f > 0$ in Ω or $f < 0$ in Ω.

(d) Let f_1 and f_2 be two linearly independent eigenfunctions of λ. By (c), we can assume that both f_1 and f_2 are strictly positive (if $f_i < 0$ then replace f_i by $-f_i$). Fix a point $x \in \Omega$ and choose positive constants C_1, C_2 so that

$$C_1 f_1(x) = C_2 f_2(x).$$

Then the function $f = C_1 f_1 - C_2 f_2$ also satisfies the equation $\mathcal{L}_\Omega f = \lambda f$ and $f(x) = 0$. By (c), f cannot be an eigenfunction, which means that $f \equiv 0$. This implies that f_1 and f_2 are linearly dependant, which means that the eigenvalue λ is simple.

48. Let f be a function on V with a finite support. Set

$$u_n(x) = P^n f(x).$$

(a) Prove that $\sup_x |u_n(x)|$ is a decreasing function of n.
(b) Prove that $\|u_n\|$ is a decreasing function of n.
(c) Prove that the heat kernel $p_{2n}(x,x)$ is a decreasing function of n, for any fixed vertex x.

Solution. (a) We have $u_{n+1} = P^{n+1} f = P(P^n f) = P u_n$, whence

$$|u_{n+1}(x)| = \left| \sum_y P(x,y) u_n(y) \right| \le \sum_y P(x,y) |u_n(y)| \le \sup |u_n| \sum_y P(x,y) = \sup |u_n|$$

whence $\sup |u_{n+1}| \le \sup |u_n|$.

(b) Since $\|P\| \le 1$, we have $\|u_{n+1}\| = \|P u_n\| \le \|P\| \|u_n\| = \|u\|$.

(c) For any fixed $x \in V$, we have

$$p_n(x,y) = \sum_{z \in V} p_n(y,z) \mathbf{1}_{\{x\}}(z) = \sum_{z \in V} p_n(y,z) \frac{1_{\{x\}}(z)}{\mu(x)} \mu(z) = P_n f(y)$$

where $f = \frac{1}{\mu(x)} 1_{\{x\}}$. It follows that,

$$p_{2n}(x,x) = \sum_{y \in V} p_n(x,y)^2 \mu(y) = \|P_n f\|^2.$$

Since $\|P_n f\|$ is decreasing, it follows that $p_{2n}(x,x)$ is also decreasing.

49. Assume that $\mu(x) \geq 1$ for all $x \in V$ and that the heat kernel on (V, μ) satisfies

$$p_n(x, x) \leq C n^{-\alpha}$$

for all $x \in V$ and all positive integers n, where $C, \alpha > 0$. Prove that, for any $0 < \varepsilon < \alpha$, for all $x, y \in V$ and all positive integers n,

$$p_n(x, y) \leq \frac{C'}{n^{\alpha-\varepsilon}} \exp\left(-c \frac{d^2(x, y)}{n}\right)$$

where $C', c > 0$.

Hint: Prove first that $p_n(x, y) \leq \operatorname{const} n^{-\alpha}$ for all x, y and n, and then combine this estimate with the estimate of Carne-Varopoulos.

Solution. Since

$$p_{2k}(x, y) \leq \sqrt{p_{2k}(x, x)\, p_{2k}(y, y)},$$

it follows that

$$p_{2k}(x, y) \leq C(2k)^{-\alpha}.$$

For $n = 2k + 1$ we have

$$p_{2k+1}(x, y) = \sum_z p_{2k}(x, z)\, p(z, y)\, \mu(z) \leq C(2k)^{-\alpha} \sum_z p(z, y)\, \mu(z) = C(2k)^{-\alpha} \leq C 2^\alpha (2k+1)^{-\alpha}.$$

Combining the above two estimates together, we obtain that

$$p_n(x, y) \leq 2^\alpha C n^{-\alpha}$$

for all x, y and n. On the other hand, by Carne-Varopoulos estimate,

$$p_n(x, y) \leq \frac{2}{\sqrt{\mu(x)\, \mu(y)}} \exp\left(-\frac{d^2(x, y)}{2n}\right) \leq 2 \exp\left(-\frac{d^2(x, y)}{2n}\right).$$

Combining these two estimates, we obtain

$$\begin{aligned} p_n(x, y) &= p_n(x, y)^{1-\varepsilon/\alpha}\, p_n(x, y)^{\varepsilon/\alpha} \\ &\leq \left(2^\alpha C n^{-\alpha}\right)^{1-\varepsilon/a} 2^{\varepsilon/\alpha} \exp\left(-\frac{\varepsilon d^2(x, y)}{2\alpha n}\right) \\ &\leq C 2^{\alpha+1} n^{-(\alpha-\varepsilon)} \exp\left(-c \frac{d^2(x, y)}{n}\right) \end{aligned}$$

where $c = \varepsilon/2\alpha$, which finishes the proof.

50. Prove that if (V, μ) is a Cayley graph with the exponential volume growth then the heat kernel of (V, μ) admits the following estimate

$$p_n(x, y) \leq C \exp\left(-\frac{d^2(x, y)}{4n} - c n^{1/3}\right),$$

for all $x, y \in V$, positive integers n, and some constants $C, c > 0$.

Solution. For Cayley graphs with exponential volume growth, we have

$$p_n(x, y) \leq C_1 \exp\left(-c_1 n^{1/3}\right).$$

By Carne-Varopoulos estimate, we have also

$$p_n(x, y) \leq 2 \exp\left(-\frac{d^2(x, y)}{2n}\right).$$

Multiplying these inequalities and taking square root, we obtain the claim.

51. Assume that there exists a constant $p_0 > 0$ such that

$$P(x,y) \geq p_0 \text{ for all } x \sim y.$$

(a) Prove that $\deg(x) \leq 1/p_0$ for any vertex $x \in V$.

(b) Prove that, for all $x, y \in V$,

$$\mu(x) \geq p_0^{d(x,y)} \mu(y)$$

(c) Prove that any ball $B(x,r)$ contains at most C^r vertices where $C = C(p_0)$.

(d) Prove that, for any finite set $A \subset V$ and for any positive integer r,

$$\mu(U_r(A)) \leq K^r \mu(A)$$

where $K = K(p_0)$.

Solution. (a) We have

$$\sum_{y \sim x} P(x,y) = 1.$$

Since $P(x,y) \geq p_0$, it follows that

$$p_0 \deg(x) \geq 1,$$

whence the claim follows.

(b) Let $x \sim y$. Since $P(x,y) = \frac{\mu_{xy}}{\mu(x)}$ and $\mu_{xy} \leq \mu(y)$, the hypothesis $P(x,y) \geq p_0$ implies $p_0\mu(x) \leq \mu(y)$. Similarly, $p_0\mu(y) \leq \mu(x)$. Iterating the last inequality, we obtain, for arbitrary x and y,

$$p_0^{d(x,y)} \mu(y) \leq \mu(x). \tag{42}$$

(c) It follows from (a) that, for any non-negative integer r,

$$|B(x,r+1)| - |B(x,r)| \leq \sum_{y \in B(x,r)} \deg(y) \leq \frac{1}{p_0} |B(x,r)|,$$

where $|A|$ denotes the number of vertices in a set A. Hence,

$$|B(x,r+1)| \leq C |B(x,r)|$$

where $C = 1 + \frac{1}{p_0}$, which implies by induction that

$$|B(x,r)| \leq C^r.$$

(d) By (42), any point $y \in B(x,r)$ has measure $\leq p_0^{-r}\mu(x)$, whence we obtain

$$\mu(B(x,r)) = \sum_{y \in B(x,r)} \mu(y) \leq p_0^{-r} \mu(x) |B(x,r)| \leq C^{2r} \mu(x).$$

Since

$$U_r(A) = \bigcup_{x \in A} B(x,r),$$

we obtain

$$\mu(U_r(A)) \leq \sum_{x \in A} \mu(B(x,r)) \leq \sum_{x \in A} C^{2r} \mu(x) = C^{2r} \mu(A).$$

Setting $K = C^2$, we finish the proof.

52. Assume that $\mu(x) \geq 1$ for all $x \in V$,

$$\mu(B_r(x)) \leq C r^\alpha$$

for all $x \in V$ and positive integers r, and that

$$p_n(x,y) \leq \frac{C}{n^{\alpha/2}} \exp\left(-c\frac{d^2(x,y)}{n}\right)$$

for all $x, y \in V$ and positive integers n, where $C, c, \alpha > 0$ (for example, all these hypotheses hold on $\mathbb{Z}^m$ with $m = \alpha$ or, more generally, on any Cayley graph of polynomial volume growth). Prove that, for all $x \in V$ and for all positive even integers n,

$$p_n(x,x) \geq \frac{c'}{n^{\alpha/2}},$$

with some constant $c' > 0$.

Solution. Let us use the inequality

$$\sum_{y \in B(x,r)^c} \exp\left(-\frac{d^2(x,y)}{2n}\right) \mu(y) \leq C' \exp\left(-\frac{r^2}{2n}\right) r^\alpha.$$

It follows that

$$\sum_{y \in B(x,r)^c} p_n(x,y)\,\mu(y) \leq \frac{C''}{n^{\alpha/2}} \exp\left(-\frac{r^2}{2n}\right) r^\alpha.$$

Choose $r = \sqrt{2Kn}$ with some constant K. Then

$$\sum_{y \in B(x,r)^c} p_n(x,y)\,\mu(y) \leq \frac{C''}{n^{a/2}} \exp(-K)\,(2Kn)^{\alpha/2} = C''' \exp(-K)\, K^{\alpha/2}.$$

If K is large enough then the right hand side is $< \frac{1}{2}$, so that

$$\sum_{y \in B(x,r)^c} p_n(x,y)\,\mu(y) \leq \frac{1}{2}.$$

It follows that

$$p_{2n}(x,x) \geq \frac{1/4}{\mu(B(x,r))},$$

which implies $p_{2n}(x,x) \geq c' n^{-\alpha/2}$, which was to be proved.

53. Let μ^*_{xy} be a weight on V that is associated with the Markov kernel $P_2(x,y)$, that is,

$$\mu^*_{xy} = P_2(x,y)\,\mu(x).$$

Then (V, μ^*) is a weighted graph. Let us mark by $*$ all quantities related to (V, μ^*) as opposed to those of (V, μ).

(a) Prove that for all $x \in V$, $\mu^*(x) = \mu(x)$.

(b) Prove that, for any finite subset $\Omega \subset V$,

$$P^*_\Omega = (P_U)^2\Big|_\Omega,$$

where $U = U_1(\Omega)$.

(c) Prove that

$$\lambda_1^*(\Omega) \geq \lambda_1(U).$$

Solution. *(a)* By definition,

$$\mu^*(x) = \sum_y \mu^*_{xy} = \sum_y P_2(x,y)\,\mu(x) = \mu(x)\sum_y P_2(x,y) = \mu(x).$$

(b) By definition, the kernel $P_\Omega(x,y)$ is obtained from $P(x,y)$ by setting it to be 0 outside Ω. In operator terms, P_Ω is obtained by restricting P to the class $\mathcal{F}_\Omega$ of functions on Ω, that are regarded as 0 outside Ω. We have

$$\begin{aligned}
P^*_\Omega(x,y) &= P^*(x,y)\,\mathbf{1}_\Omega(x)\,\mathbf{1}_\Omega(y) \\
&= P_2(x,y)\,\mathbf{1}_\Omega(x)\,\mathbf{1}_\Omega(y) \\
&= \sum_{z\in V} P(x,z)\,P(z,y)\,\mathbf{1}_\Omega(x)\,\mathbf{1}_\Omega(y) \\
&= \sum_{z\in U} P(x,z)\,P(z,y)\,\mathbf{1}_\Omega(x)\,\mathbf{1}_\Omega(y) \\
&= (P_U)^2(x,y)\,\mathbf{1}_\Omega(x)\,\mathbf{1}_\Omega(y)
\end{aligned}$$

Hence, restricting operator $(P_U)^2$ to $\mathcal{F}_\Omega$, we obtain P^*_Ω. Note that it is not true in general that $P^*_\Omega = (P_\Omega)^2$.

(c) Denote by $\alpha_{\max}(A)$ the maximal eigenvalue of an operator A, and by $\alpha_{\min}(A)$ - the minimal eigenvalue. Then it suffices to prove that

$$\alpha_{\max}(P^*_\Omega) \leq \alpha_{\max}(P_U),$$

Since restricting an operator to a subspace can only diminish its maximal eigenvalue, we have by *(b)*

$$\alpha_{\max}(P^*_\Omega) = \alpha\left((P_U)^2|_\Omega\right) \leq \alpha_{\max}\left((P_U)^2\right).$$

It remains to prove that

$$\alpha_{\max}\left(P_U^2\right) \leq \alpha_{\max}(P_U).$$

Observe that that

$$\alpha_{\max}\left(P_U^2\right) = \max\left\{\alpha^2_{\max}(P_U),\, \alpha^2_{\min}(P_U)\right\}$$

so that it suffices to verify the two inequalities

$$\begin{aligned}
\alpha^2_{\max}(P_U) &\leq \alpha_{\max}(P_U) \\
\alpha^2_{\min}(P_U) &\leq \alpha_{\max}(P_U).
\end{aligned}$$

Indeed, they follow from $\alpha_{\max}(P_U) \in [0,1]$ and $\alpha_{\min}(P_U) \in [-\alpha_{\max}(P_U), \alpha_{\max}(P_U)]$ respectively.