Prof. A. Grigoryan Analysis on Manifolds WS 2025/26

Blatt 11. Abgabe bis 16.01.2026

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Die mit **markierten Aufgaben sind zusätzlich und werden nicht korrigiert.

In all questions, $(M, \mathbf{g}, \mu)$ *is a weighted manifold,* $\Delta = \Delta_{\mathbf{g},\boldsymbol{\mu}}$*, and* Ω *is an open subset of* M*. The quantity* $\lambda_1(\Omega)$ *is defined in Exercise 67.*

66. Let R_α be the resolvent operator in Ω. Prove that, for any $\alpha > 0$ and $f \in L^2(\Omega)$, the function $u = \alpha R_\alpha f$ is a unique minimizer of the functional

$$E(v) := \|\nabla v\|^2_{\vec{L}^2} + \alpha \|v - f\|^2_{L^2}$$

in the domain $v \in W^1_0(\Omega)$.

Hint. Show that $E(u + \varphi) \geq E(u)$ for any $\varphi \in W^1_0(\Omega)$.

67. For any open subset $\Omega \subset M$, define $\lambda_1(\Omega)$ by

$$\lambda_1(\Omega) := \inf_{f \in \mathcal{D}(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla f|^2 \, d\mu}{\int_\Omega f^2 d\mu}. \tag{45}$$

Prove the following properties of $\lambda_1(\Omega)$.

(a) If $\Omega_1 \subset \Omega_2$ are two open sets then

$$\lambda_1(\Omega_1) \geq \lambda_1(\Omega_2).$$

(b) If $\{\Omega_k\}_{k=1}^\infty$ is an increasing sequence of open sets (that is, $\Omega_k \subset \Omega_{k+1}$) and $\Omega = \bigcup_k \Omega_k$ then

$$\lambda_1(\Omega) = \lim_{k \to \infty} \lambda_1(\Omega_k) = \inf_k \lambda_1(\Omega_k).$$

Remark. For any non-zero function $f \in \mathcal{D}(M)$, define its *Rayleigh quotient* by

$$\mathcal{R}(f) := \frac{\|\nabla f\|^2_{\vec{L}^2}}{\|f\|^2_{L^2}}.$$

Then (45) can be rewritten in the form

$$\lambda_1(\Omega) := \inf_{f \in \mathcal{D}(\Omega) \setminus \{0\}} \mathcal{R}(f).$$

68. Assume that $\lambda_1(\Omega) > 0$.

(a) Prove that the weak Dirichlet problem in Ω

$$\begin{cases} \Delta u = -f & \text{weakly in } \Omega, \\ u \in W^1_0(\Omega), \end{cases} \tag{46}$$

has exactly one solution u for any $f \in L^2(\Omega)$.

Hint. Set $[u, v] := (\nabla u, \nabla v)_{\vec{L}^2}$ for all $u, v \in W^1_0(\Omega)$ and prove that $[\cdot, \cdot]$ is an inner product in $W^1_0(\Omega)$. For that, use the hypothesis $\lambda_1(\Omega) > 0$.

(b) Prove that, for the solution u of (46),

$$\|u\|_{L^2} \le \lambda_1(\Omega)^{-1} \|f\|_{L^2} \tag{47}$$

and

$$\|\nabla u\|_{\vec{L}^2} \le \lambda_1(\Omega)^{-1/2} \|f\|_{L^2}. \tag{48}$$

69. Consider the following version of the weak Dirichlet problem in Ω: given a real constant α and functions $f \in L^2(\Omega)$, $g \in W^1(\Omega)$, find a function $u \in W^1(\Omega)$ such that

$$\begin{cases} \Delta u - \alpha u = -f & \text{weakly in } \Omega, \\ u - g \in W_0^1(\Omega). \end{cases} \tag{49}$$

Prove that if $\alpha > -\lambda_1(\Omega)$ then the problem (49) has exactly one solution.

70. * Let $f \in L^2(\Omega)$ and assume that u is a solution of the following weak Dirichlet problem:

$$\begin{cases} \Delta u = -f & \text{weakly in } \Omega, \\ u \in W_0^1(\Omega). \end{cases}$$

Prove that

$$\|u\|_{W^1}^2 \le c\left(\|u\|_{L^2}^2 + \|f\|_{L^2}^2\right), \tag{50}$$

where $c = \frac{1+\sqrt{2}}{2}$.

71. * (*Cheeger's inequality*) The *Cheeger constant* of Ω is defined by

$$h(\Omega) := \inf_{\varphi \in \mathcal{D}(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla \varphi| \, d\mu}{\int_\Omega |\varphi| \, d\mu}. \tag{51}$$

Prove that

$$\lambda_1(\Omega) \ge \frac{1}{4} h^2(\Omega).$$

Hint. Substitute in the right hand side of (51) φ^2 in place of φ and use the definition of $\lambda_1(\Omega)$.

72. ** Let d be the geodesic distance on a connected Riemannian manifold $(M, \mathbf{g})$. A function $f : M \to \mathbb{R}$ is called *Lipschitz* if there exists a constant L such that

$$|f(x) - f(y)| \le L d(x, y) \quad \text{for all } x, y \in M.$$

The number L is called the *Lipschitz constant* of f. Prove that if f is Lipschitz with the Lipschitz constant L then the weak gradient ∇f exists and

$$\|\nabla f\|_{\vec{L}^\infty} \le L. \tag{52}$$

Hint. In $\mathbb{R}^n$ this statement can be taken as known. In order to reduce the general case to that in $\mathbb{R}^n$, prove first the following claim: for any point $p \in M$ and for any $C > 1$, there exists a chart $U \ni p$ such that, for all $x \in U$ and $\xi \in T_x M$,

$$C^{-2}\left(\left(\xi^1\right)^2 + \ldots + \left(\xi^n\right)^2\right) \le g_{ij}(x)\,\xi^i \xi^j \le C^2\left(\left(\xi^1\right)^2 + \ldots + \left(\xi^n\right)^2\right).$$

This inequality was proved in lectures, however, with *some* constant C. Show that the constant C can be chosen arbitrarily close to 1.