Blatt 12. Abgabe bis 23.01.2026

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Die mit **markierten Aufgaben sind zusätzlich und werden nicht korrigiert.

In all questions, $(M, \mathbf{g}, \mu)$ is a weighted manifold, Ω is a precompact open subset of M, $\{\lambda_k\}_{k=1}^{\infty}$ is the sequence of the Dirichlet eigenvalues of $\Delta = \Delta_{\mathbf{g},\mu}$ in Ω in the increasing order, and $\{v_k\}$ is the sequence of the corresponding eigenfunctions that forms an orthonormal basis in $L^2(\Omega)$.

73. Recall that any function $u \in L^2(\Omega)$ admits an eigenfunction expansion

$$u = \sum_{k=1}^{\infty} a_k v_k \ , \tag{53}$$

where $a_k \in \mathbb{R}$ and the series converges in $L^2(\Omega)$.

(a) Prove that if $u \in W_0^1(\Omega)$ then the series (53) converges also in $W^1(\Omega)$ and

$$\|u\|_{W^1}^2 = \sum_{k=1}^{\infty} (\lambda_k + 1)\, a_k^2. \tag{54}$$

Hint. Use the Parseval identity in $L^2(\Omega)$ and in $W_0^1(\Omega)$.

(b) Prove that if $u \in W_0^1(\Omega)$ and $\Delta u \in L^2(\Omega)$ then

$$\Delta u = -\sum_{k=1}^{\infty} \lambda_k a_k v_k, \tag{55}$$

where the series converges in $L^2(\Omega)$.

74. *(Variational property of the bottom eigenvalue)* For any non-zero function $u \in W^1(\Omega)$, consider its *Rayleigh quotient*:

$$\mathcal{R}(u) := \frac{\|\nabla u\|_{\vec{L}^2}^2}{\|u\|_{L^2}^2}.$$

Prove that the bottom Dirichlet eigenvalue $\lambda_1(\Omega)$ satisfies the following identity:

$$\lambda_1(\Omega) = \min_{u \in W_0^1(\Omega)\setminus\{0\}} \mathcal{R}(u) = \inf_{u \in \mathcal{D}(\Omega)\setminus\{0\}} \mathcal{R}(u). \tag{56}$$

Hint. Use Exercise 73.

Remark. The notation $\lambda_1(\Omega)$ for any open set Ω was defined in Exercise 67 as $\lambda_1(\Omega) = \inf_{u \in \mathcal{D}(\Omega)\setminus\{0\}} \mathcal{R}(u)$. The identity (56) shows that, for precompact Ω, this notation is consistent with the notation $\lambda_1(\Omega)$ for the bottom Dirichlet eigenvalue.

75. Let a function $f \in L^2(\Omega)$ have an eigenfunction expansion

$$f = \sum_{k=1}^{\infty} a_k v_k.$$

(a) Prove that, for any $\alpha > 0$,

$$R_\alpha f = \sum_{k=1}^{\infty} \frac{1}{\alpha + \lambda_k} a_k v_k. \tag{57}$$

Hint. Use Exercise 73.

(b) Using (57), prove the following *resolvent identity* for all $\alpha, \beta > 0$:

$$R_\alpha - R_\beta = (\beta - \alpha) R_\alpha R_\beta. \tag{58}$$

76. * Let $f \in L^2(\Omega)$.

(a) Prove that $\alpha R_a f \xrightarrow{L^2} f$ as $\alpha \to +\infty$.
Hint. Use Exercise 75.

(b) Prove that if in addition $f \in W_0^1(\Omega)$ and $\Delta f \in L^2(\Omega)$ then, for all $\alpha > 0$,

$$\|\alpha R_\alpha f - f\|_{L^2} \leq \frac{1}{\alpha} \|\Delta f\|_{L^2}.$$

Hint. Use Exercise 73.

77. Prove that there exists a positive constant c_n such that, for any ball B_R of radius R in $\mathbb{R}^n$,

$$\lambda_1(B_R) = \frac{c_n}{R^2}.$$

Hint. Using the variational property (56) of Exercise 74, prove first that

$$\lambda_1(B_R) = R^{-2} \lambda_1(B_1).$$

You can assume without loss of generality that the ball B_R is centered at the origin of $\mathbb{R}^n$.

Remark. Letting $R \to \infty$ and using Exercise 67 we conclude that $\lambda_1(\mathbb{R}^n) = 0$.

78. ** Prove that, for any geodesic ball $B_R = B(x_0, R)$ on an arbitrary connected weighted manifold M,

$$\lambda_1(B_R) \leq \frac{4}{R^2} \frac{\mu(B_R)}{\mu(B_{R/2})}.$$

79. ** Let M be connected. Fix a point $x_0 \in M$ and consider the function $\rho(x) = d(x, x_0)$.

(a) Prove that $\rho \in W^1_{loc}(M)$ and $\|\nabla \rho\|_{\vec{L}^\infty} \leq 1$.
Hint. Use Exercise 72.

(b) Assume ρ has in Ω the weak Laplacian $\Delta\rho$, and that $\Delta\rho$ satisfies in Ω the inequality $\Delta\rho \geq a$, where a is a positive constant. Prove that

$$\lambda_1(\Omega) \geq \frac{a^2}{4}.$$

Hint. First prove that $h(\Omega) \geq a$ where $h(\Omega)$ is the Cheeger constant from Exercise 71, and then use the Cheeger inequality.

80. ** Prove that

$$\lambda_1(\mathbb{H}^n) \geq \frac{(n-1)^2}{4}.$$

Hint. Use Exercise 79 and the polar coordinates in $\mathbb{H}^n$.

81. ** Prove that

$$\lambda_1(\mathbb{H}^n) = \frac{(n-1)^2}{4}.$$

Hint. It suffices to find for any $\varepsilon > 0$ a function $f \in W_0^1(\mathbb{H}^n)$ such that

$$\mathcal{R}(f) \leq \frac{(n-1)^2}{4} + \varepsilon.$$

Look for f in the form $f(x) = e^{-cr}$ where r is the polar radius. You may use without proof the fact that if $f \in W^1(M)$ and $f(x) \to 0$ as $x \to \infty$ then $f \in W_0^1(M)$.