# A long exact sequence of path homology of digraphs

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**Abstract** In this paper, we develop a long exact sequence for the path homology of digraphs, providing a useful tool for computing the path homology of digraphs. One application of this result is the proof of a conjecture proposed by S. Chowdhury, which was initially observed through extensive computational experiments. Another interesting application demonstrates that the path homology of $n$-dimensional grid-like digraphs is concentrated in dimension $\leq n-1$.

**Keywords** Digraph, path homology, Mayer-Vietoris sequence, grid-like digraph, directed cyclic network.

## 1 Introduction

The path homology theory of digraphs is based on a series of works by A. Grigor'yan, Y. Lin, Y. Muranov, and S.-T. Yau, providing a topological perspective for studying digraph invariants [4, 5, 6, 10, 7], which is referred to as GLMY theory. The path complex on a digraph is a key concept in this theory, understood as the collection of all paths on a digraph. In particular, if any sub sequence of a path on a digraph is also a path on the digraph, the path complex can reduce to the abstract simplicial complex. Therefore, the path homology of digraphs can be seen as a generalization of simplicial homology. The Mayer-Vietoris sequence for simplicial complexes plays an important role in computing homology groups. In this work, we attempt to develop a long exact sequence to assist in the computation of the path homology of digraphs.

The Mayer-Vietoris sequence for topological spaces asserts that for any covering $\{U_1, U_2\}$ of a topological space $X$, there is a long exact sequence:

$$\begin{aligned}\cdots \to H_n(U_1\cap U_2) \to H_n(U_1)\oplus H_n(U_2) \to H_n(X) \to H_{n-1}(U_1\cap U_2) \to \cdots\\ \cdots \to H_0(U_1)\oplus H_0(U_2) \to H_0(X) \to 0.\end{aligned}$$

However, establishing a Mayer-Vietoris sequence for the path homology of digraphs presents some inherent challenges. Let $G$ be a digraph, and let $G_1$ and $G_2$ be sub-digraphs of $G$ such that $G = G_1\cup G_2$. Then the path complex $P(G)$ of $G$ often contains many more paths than the union of

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the path complexes $P(G_1)\cup P(G_2)$. In fact, there are cases where the dimension of $P(G)$, defined as the length of the longest path, can exceed that of $P(G_1)\cup P(G_2)$. This discrepancy highlights the difficulties associated with applying the Mayer-Vietoris sequence to the path homology of digraphs.

In this work, we develop a long exact sequence for the path homology of digraphs, as detailed in Theorem 3.1. This sequence can aid in computing the path homology of digraphs. Specifically, consider digraphs $G_1$ and $G_2$, and let $G$ be a digraph containing $G_1$ and $G_2$ with parallel directed edges from $G_1$ to $G_2$. Theorem 3.2 asserts that there is a short exact sequence of path homology given by

$$0\to H_p(S)\to H_p(G_1)\oplus H_p(G_2)\to H_p(G)\to 0$$

for any $p\geq 2$. Here, $S$ denotes the induced sub-digraph of $G_1$ with vertex set consisting of the source points of the parallel edges.

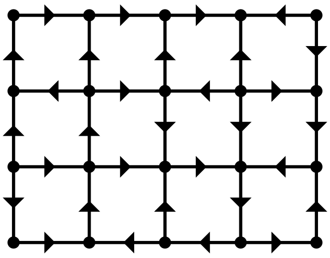
One application of our main theorems is the proof of a conjecture proposed by S. Chowdhury. In [2], S. Chowdhury observed through extensive computations that the (finite) temporal digraph representation of a directed cyclic network (DCN) has $\beta_p=0$ for $p>1$. In mathematical terms, this conjecture can be stated as follows:

**Theorem 1.1.** *Let $G$ be a finite simple digraph with a vertex set $V\subseteq\mathbb{Z}\times\mathbb{Z}$. The edges of $G$ are defined as follows:*

- *Horizontal edges: For any two vertices $(x,y)$ and $(x',y)$ in $V$ with $x<x'$, if there is no vertex $(x'',y)$ in $V$ such that $x<x''<x'$, then there is an edge $(x,y)\to(x',y)$.*
- *Vertical edges: For any vertex $(x,y)$ in $V$, there is at most one edge starting from $(x,y)$ to some vertex $(x,y')$ in $V$.*

*Then we have $H_p(G)=0$ for $p\geq 2$.*

Another interesting application of our main results demonstrates that any finite sub-digraph of an $n$-dimensional grid digraph has Betti numbers $\beta_p=0$ for all $p\geq n$. For example, any finite directed grid-like network has Betti numbers $\beta_p=0$ for $p\geq 2$. See Figure 1, where we report the Betti numbers associated with the path homology of all directed digraphs in the case of $2^{24}$ possibilities. The statistical result shows that the Betti numbers in dimension 2 are zero for all cases.



| $\beta_0$ | $\beta_1$ | $\beta_2$ | Number ($2^{24}$) |
|---|---|---|---|
| 1 | 0 | 0 | 64 |
| 1 | 1 | 0 | 1728 |
| 1 | 2 | 0 | 20736 |
| 1 | 3 | 0 | 145152 |
| 1 | 4 | 0 | 3919104 |
| 1 | 5 | 0 | 1959552 |
| 1 | 6 | 0 | 653184 |
| 1 | 7 | 0 | 5038848 |
| 1 | 8 | 0 | 3779136 |
| 1 | 9 | 0 | 1259712 |

Figure 1: The left is a grid-like digraph. The directed edges can be any vertical or horizontal arrows in different directions. The right is the number of digraphs of different cases.

In the next section, we provide a brief introduction to the path homology of digraphs. Section 3 presents the main results, and Section 4 contains the proofs of these key theorems.

## 2 Preliminaries

In this section, we will review some basic concepts and results related to GLMY theory that will be addressed in this paper. For more details, please refer to [**?** 8, 9]. To enhance the readability of this paper, some foundational knowledge of homological algebra is assumed, as outlined in [11]. From now on, $\mathbb{K}$ is the ground field.

**Path complex and path homology.** Let $V$ be a nonempty finite set. An *elementary $p$-path* on $V$ is a sequence $i_0 i_1 \cdots i_p$ for $i_0, i_1, \ldots, i_p \in V$, which is always denoted as $e_{i_0 i_1 \cdots i_p}$. Let $\Lambda_p(V)$ be the $\mathbb{K}$-linear space generated by all the elementary $p$-paths on $V$. An element in $\Lambda_p(V)$ is a *$p$-path*. Then we can obtain a chain complex $\Lambda_*(V)$ with the differential $\partial : \Lambda_*(V) \to \Lambda_{*-1}(V)$ given by

$$\partial e_{i_0} = 0 \text{ for any } i_0 \in V$$

and

$$\partial e_{i_0 i_1 \cdots i_p} = \sum_{t=0}^{p} (-1)^t e_{i_0 \cdots \widehat{i_t} \cdots i_p}, \quad p \geq 1,$$

where $\widehat{i_t}$ means omission of the index $i_t$.

Let $V$ be a nonempty finite set. A *path complex* over $V$ is defined as a collection $P$ of elementary paths on $V$, satisfying the condition that if $i_0 i_1 \cdots i_p \in P$, then $i_0 i_1 \cdots i_{p-1} \in P$ and $i_1 \cdots i_p \in P$ for any $p \geq 1$. Paths in $P$ are called *allowed*, while those not in $P$ are called *non-allowed*.

Let $P$ be a path complex on $V$. The path complex $P$ can be regarded as a graded set $\{P_n\}_{n\geq 0}$, where $P_n$ consists of elementary paths of length $n$ in $P$. Let $\mathcal{A}_n(P)$ be the $\mathbb{K}$-linear space generated by all the elementary paths in $P_n$. Then $\mathcal{A}_*(P) = \{\mathcal{A}_n(P)\}_{n\geq 0}$ is a graded linear space. Note that $\mathcal{A}_n(P)$ is a subspace of $\Lambda_n(P)$. Then the differential $\partial : \Lambda_*(V) \to \Lambda_{*-1}(V)$ restricts to a linear map

$$\partial : \mathcal{A}_*(P) \to \Lambda_{*-1}(V).$$

It is worth noting that $\partial \mathcal{A}_*(P)$ does not have to be a subspace of $\mathcal{A}_{*-1}(P)$. A direct example is the path complex $P = \{0, 1, 01, 12, 012\}$ over $V = \{0, 1, 2\}$. The element $\partial e_{012} = e_{01} - e_{02} + e_{12} \notin \mathcal{A}_1(P)$ since 02 is not an elementary path in $P$.

Let $\Omega_n(P) = \{x \in \mathcal{A}_n(P) \mid \partial x \in \mathcal{A}_{n-1}(P)\}$. An element in $\Omega_n(P)$ is called a $\partial$-invariant $n$-path. By construction, we have

$$\partial \Omega_n(P) \subseteq \Omega_{n-1}(P).$$

Then $\Omega_*(P)$ is a chain complex with the differential $\partial : \Omega_*(P) \to \Omega_{*-1}(P)$.

The *path homology* of $P$ is defined by

$$H_n(P) = H_n(\Omega_*), \quad n \geq 0.$$

**Path homology of digraphs.** A *directed graph* (digraph) $G$ is a pair $(V, E)$, where $V$ is a nonempty finite set and $E \subseteq V \otimes V$. An element $(v, w) \in E$ is called a directed edge, we also denote $v \to w$. If there is no edge $(v, w)$ in $E$, we denote $v \not\to w$.

A digraph is called *simply* if there are no loops or multiple edges. Let $G = (V, E)$ be a digraph. A digraph $G'$ is the *sub-digraph* of $G$ if its vertex set and edge set are subsets of those of $G$. A digraph $G' = (V', E')$ is an *induced sub-digraph* of $G$ if the edge set $E'$ is formed by all the edges in $G$ whose endpoints are in $V'$.

For a finite digraph $G = (V, E)$, the *path complex $P(G)$ associated with $G$* is constructed as follows: The elements in $P_n$ are elementary paths of the form $i_0 i_1 \cdots i_n$ such that $i_0, i_1, \ldots, i_n \in V$ and $(i_{t-1}, i_t) \in E$ for $1 \leq t \leq n$. These paths are called *allowed paths* on the digraph $G$. The *path homology of digraph $G$* is defined by

$$H_n(G) = H_n(P(G)), \quad n \geq 0.$$

The path homology of $G$ offers a new perspective on the topology of digraphs. Furthermore, this theory has already achieved significant success in practical applications [1, 3].

## 3 Main results

In this section, we present the main theorems. Our primary contribution is the formulation of a long exact sequence for computing the path homology of digraphs, which resembles the Mayer-Vietoris sequence for topological spaces. This result leads to some interesting findings when applied to grid-like digraphs.

A digraph $G$ has homology *concentrated in dimension $n$* if $H_p(G) = 0$ for any $p \geq n + 1$.

**Theorem 3.1.** *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be disjoint digraphs, and let $G$ be the union of $G_1$ and $G_2$ with a family $E$ of disjoint directed edges in $V_1 \times V_2$ or $V_2 \times V_1$. Let $S$ and $S'$ be the induced sub-digraphs of $G_1$, where the vertex set of $S$ consists of the source vertices of $E$ in $V_1$, and the vertex set of $S'$ consists of the target vertices of $E$ in $V_1$. Then, there is a long exact sequence of homology groups*

$$\cdots \to H_p(S) \oplus H_p(S') \to H_p(G_1) \oplus H_p(G_2) \to H_p(G) \to H_{p-1}(S) \oplus H_{p-1}(S') \to \cdots$$
$$\cdots \to H_2(G) \to H_1(S) \oplus H_1(S').$$

*Moreover, if any sub-digraph of $G_1$ has homology concentrated in dimension $\leq m - 1$ for some positive integer $m$, then $H_p(G) \cong H_p(G_2)$ for $p \geq m + 1$.*

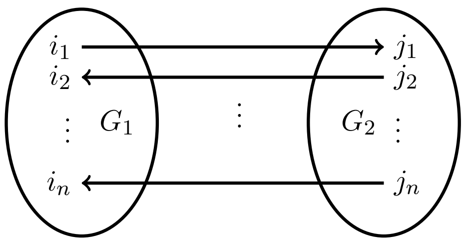



Figure 2: Illustration of the digraphs in Theorem 3.1.

**Theorem 3.2.** *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be disjoint digraphs, and let $G$ be the union of $G_1$ and $G_2$ with a family $E$ of disjoint directed edges in $V_1 \times V_2$. Let $S$ be the induced sub-digraph* *of $G_1$, where the vertex set of $S$ consists of the source vertices of $E$ in $V_1$. Then, there is a short exact sequence of homology groups*

$$0 \to H_p(S) \to H_p(G_1) \oplus H_p(G_2) \to H_p(G) \to 0$$

*for any $p \geq 2$.*

An *$n$-dimensional grid digraph* $G = (V, E)$ is a simple digraph where the vertex set is $V = \mathbb{Z}^n$. The edge set consists of directed edges of the form

$$(x_1, x_2, \ldots, x_n) \to (x_1, \ldots, x_k \pm 1, \ldots, x_n)$$

for each $(x_1, x_2, \ldots, x_n) \in \mathbb{Z}^n$ and $k = 1, \ldots, n$. As a corollary of Theorem 3.1, we have the following interesting result.

**Theorem 3.3.** *Any finite sub-digraph of an $n$-dimensional grid digraph has Betti numbers $\beta_p = 0$ for all $p \geq n$.*

*Proof.* We will prove the result by induction. The case for $n = 1$ is straightforward.

Assuming the theorem holds for $n \leq k$, we now consider the case for $n = k + 1$. Let us denote the digraph by $G$. The digraph $G$ can be viewed as a collection of layered sub-digraphs, where the vertex sets of these sub-digraphs lie in $\mathbb{R}^k$. Adjacent layers of sub-digraphs are connected by parallel directed edges.

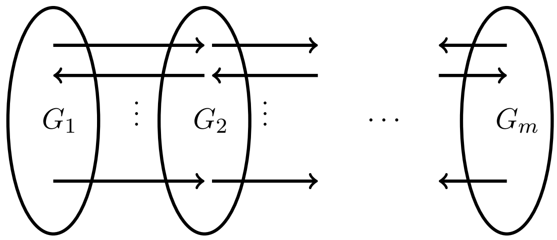



Figure 3: Illustration of the collection of layered sub-digraphs in Theorem 3.3.

Since the digraph $G$ is finite, we can partition $G$ into $m$ layers, as shown in Figure 3. Consider the first-layer digraph $G_1$ and the digraph $G_1'$ formed by all the remaining layers, which are connected by parallel directed edges. By the induction hypothesis, any subgraph of the digraph $G_1$ has homology concentrated in dimension $\leq k - 1$. Applying Theorem 3.1, we obtain the isomorphism

$$H_p(G) \cong H_p(G_1'), \quad p \geq k + 1.$$

Next, we partition the digraph $G_1'$ into the digraph $G_2$ and the digraph $G_2'$ formed by the remaining layers from the third layer. Reapplying Theorem 3.1, we obtain

$$H_p(G_1') \cong H_p(G_2'), \quad p \geq k + 1.$$

By repeating the above process, we ultimately obtain

$$H_p(G) \cong H_p(G_m), \quad p \geq k + 1.$$

By the induction hypothesis, $H_p(G_m) = 0$ for $p \geq k$. It follows that $H_p(G) = 0$ for $p \geq k + 1$.

Finally, by mathematical induction, the theorem is proved. □

**Example 3.1.** *Let $G = (\mathbb{Z}^n, S)$ be a Cayley digraph, where $S = \{e_1, e_2, \ldots, e_n\}$ is the standard basis of $\mathbb{Z}^n$. Then any finite sub-digraph of $G$ has Betti numbers $\beta_p = 0$ for $p \geq n$.*

For a directed $n$-cube, Theorem 3.3 demonstrates that its Betti numbers $\beta_p = 0$ for $p \geq n$. The following theorem extends this result by showing that $\beta_{n-1} = 0$ as well.

**Theorem 3.4.** *For $n \geq 3$, any directed $n$-cube has Betti numbers $\beta_p = 0$ for $p \geq n-1$. The same is true for any sub-digraph of a directed $n$-cube.*

*Proof.* Let $C^n$ be a directed $n$-cube, which is formed by connecting two $(n-1)$-cubes, $C_1^{n-1}$ and $C_2^{n-1}$, with $2^{n-1}$ disjoint directed edges. For the case when $n = 3$, see Figure 4.

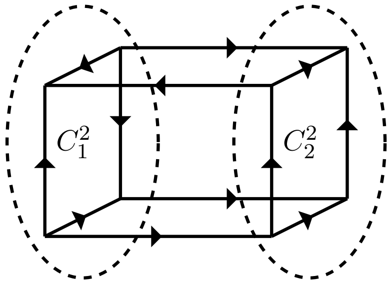



Figure 4: A directed 3-cube. The directed 3-cube can be viewed as two directed 2-cubes connected by four parallel directed edges.

When $n = 3$, by Theorem 3.1, we have

$$0 \to H_2(C^3) \to H_1(S) \oplus H_1(S'),$$

where $S$ and $S'$ are sub-digraphs of $C_1^2$ as defined in Theorem 3.1. Here, the term 0 on the far left of the above short exact sequence arises because

$$H_2(C_1^2) \oplus H_2(C_2^2) = 0.$$

If $H_1(S) \oplus H_1(S') = 0$, then we have $H_2(C^3) = 0$. If $H_1(S) \oplus H_1(S') \neq 0$, we assume $H_1(S) \neq 0$. It follows that $S' = \emptyset$. Moreover, the directed edges connecting $C_1^2$ and $C_2^2$ in $C_1^2 \times C_2^2$. This indicates that these four directed edges are oriented in the same direction.

Now, we place the directed 3-cube as a standard cube in the three-dimensional coordinate system. Suppose $H_2(C^3) \neq 0$. Along the $yOz$ plane, we can divide $C^3$ into two connecting 2-cubes. Since $H_2(C^3) \neq 0$, the four directed edges parallel to the $x$-axis are oriented in the same direction. Similarly, the four directed edges parallel to the $y$-axis in $C^3$ are oriented in the same direction, and the four directed edges parallel to the $z$-axis are also oriented in the same direction. This implies that $C^3 = I^3$, where $I$ is a directed interval path. Note that $H_2(I^3) = 0$. Therefore, we always have $H_2(C^3) = 0$.

For any proper sub-digraph $G$ of $C^3$, we can divide $G$ into two digraphs $G_1$ and $G_2$ connected by parallel directed edges. By Theorem 3.1, we have

$$0 \to H_2(G) \to H_1(S) \oplus H_1(S'),$$

where $S$ and $S'$ are sub-digraphs of $G_1$ as defined in Theorem 3.1. Since at least one of $G_1$ and $G_2$ is not a 2-cube, $S$ and $S'$ cannot be 2-cubes. This implies that $H_1(S) \oplus H_1(S') = 0$. Thus, we have $H_2(G) = 0$. Hence, each sub-digraph of $C^3$ has null homology in dimensions $\geq 2$.

For any 4-cube $C^4$, one can regard it as two 3-cube $C_1^3$ and $C_2^3$ connected by parallel directed edges. By Theorem 3.1, we have

$$0 \to H_p(C_1^3) \oplus H_p(C_2^3) \to H_p(C^4) \to 0$$

for $p \geq 3$. It follows that

$$H_p(C^4) \cong H_p(C_1^3) \oplus H_p(C_2^3) = 0$$

for $p \geq 3$. By induction, for any integer $n \geq 3$, we have $H_p(C^n) = 0$ for $p \geq n-1$. Similarly, by induction, any sub-digraph of $C^n$ has null homology in dimensions $\geq n-1$. □

**Example 3.2.** *Consider the directed* 4*-cube* $G = (V, E)$ *with* $V = \{1, 2, \ldots, 16\}$ *and*

$$\begin{aligned} E = \{&(1,2),(2,3),(3,4),(4,1),(5,6),(6,7),(7,8),(8,5),\\ &(1,5),(4,8),(2,6),(3,7),(9,10),(10,11),(11,12),(12,9),\\ &(13,14),(14,15),(15,16),(16,13),(9,13),(12,16),(10,14),(11,15),\\ &(1,9),(2,10),(3,11),(4,12),(13,5),(14,6),(15,7),(16,8)\}. \end{aligned}$$

*The Betti numbers of* $G$ *are given by* $\beta_0 = 1$*,* $\beta_1 = 2$*,* $\beta_2 = 1$*, and* $\beta_3 = 0$*. This indicates that for the case* $n = 4$*, Theorem 3.4 holds with* $p = 3$ *being the smallest integer such that* $\beta_p = 0$*.*

**Lemma 3.5.** *Let* $G = (V, E)$ *be a digraph. We denote* $d_{out}(i)$ *as the outdegree of vertex* $i$*, defined by*

$$d_{out}(i) = \sharp\{j \in V \mid (i,j) \in E\}.$$

*If* $d_{out}(i) \leq 1$ *for each* $i \in V$*, then* $\Omega_p(G) = 0$ *for* $p \geq 2$*.*

*Proof.* Suppose that $x \in \Omega_p(G)$ is a nonzero path. Choose an elementary summand $\lambda e_{i_0 i_1 \cdots i_p}$ of $x$ for some $\lambda \in \mathbb{K}$. Note that $e_{i_0 i_1 \cdots i_{p-2} i_p}$ is an elementary summand of $\partial e_{i_0 i_1 \cdots i_p}$. Since $d_{\mathrm{out}}(i_{p-2}) \leq 1$, we have $i_{p-2} \not\to i_p$. It follows that $e_{i_0 i_1 \cdots i_{p-2} i_p}$ is not allowed. To ensure that $\partial x \in \mathcal{A}_{p-1}(G)$, the elementary summand $e_{i_0 i_1 \cdots i_{p-2} i_p}$ of $\partial e_{i_0 i_1 \cdots i_p}$ must be annihilated by an elementary summand of $\partial e_{i_0 i_1 \cdots i_{p-2} j i_p}$ for some path $e_{i_0 i_1 \cdots i_{p-2} j i_p}$. However, $e_{i_0 i_1 \cdots i_{p-2} j i_p}$ is not an allowed path on $G$ since $d_{\mathrm{out}}(i_{p-2}) \leq 1$. This leads to a contradiction. Thus, we have $\Omega_p(G) = 0$. □

*Proof of Theorem 1.1.* Since the digraph is a finite, we can assume that it consists of a finite number of layers along the $x$-axis. We will prove the result by induction. Let $G_m$ denote the digraph with $m$ layers that satisfies the conditions of Theorem 1.1. For $m = 1$, by Lemma 3.5, we have $H_p(S) = 0$ for $p \geq 2$, where $S$ is any sub-digraph of $G_1$. Suppose the result holds for $m = k-1$, i.e., $H_p(G_{k-1}) = 0$ for any digraph $G_{k-1}$ and $p \geq 2$.

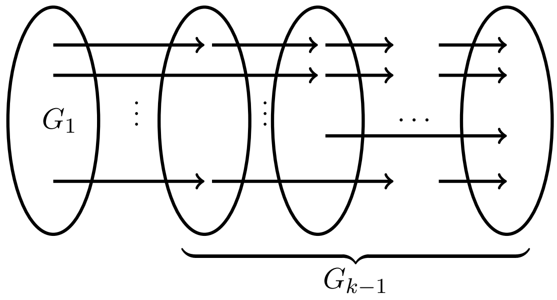



Figure 5: Illustration of the digraph $G_k$ with $k$ layers in Theorem 1.1.

By Theorem 3.2, we have a short exact sequence

$$0 \to H_p(S) \to H_p(G_1) \oplus H_p(G_{k-1}) \to H_p(G_k) \to 0$$

for any $p \geq 2$. Recall that $H_p(S) = H_p(G_1) = 0$ for $p \geq 2$. We have the isomorphism $H_2(G_{k-1}) \cong H_2(G_k)$. By the induction hypothesis, we have $H_p(G_k) = 0$ for $p \geq 2$. The desired result follows. □

# 4 Proofs of the main theorems

*The proof of Theorem 3.1.* We will divide it into the following four parts.

**Step** $(i)$. The construction $\Delta$.

Let $G_0$ be the digraph whose edge set is formed by the family $E$ of disjoint directed edges, and whose vertex set consists of the endpoints of $E$. For simplicity, we denote the condition that the directed edges in $G_0$ are disjoint by ($\sharp$). Let $G_1' = G_1 \cup G_0$ and $G_2' = G_2 \cup G_0$. Then we have $G = G_1' \cup G_2'$ and $G_0 = G_1' \cap G_2'$. There is a natural inclusion of the chain complexes of $\partial$-invariant paths:

$$\theta : \Omega_*(G_1') + \Omega_*(G_2') \hookrightarrow \Omega_*(G). \tag{1}$$

Let $\Gamma_*$ be the complement subspace of $\Omega_*(G_1') + \Omega_*(G_2')$ in $\Omega_*(G)$. Note that each elements in $\Omega_*(G)$ can be written as the sum of some elementary paths. Then, for any given $x \in \Gamma_p$, there exists an elementary summand $\lambda e_{i_0 i_1 \cdots i_p}$ of $x$ such that $(i_t, i_{t+1}) \in V_1 \times V_2$ or $(i_t, i_{t+1}) \in V_2 \times V_1$ some nonzero $\lambda \in \mathbb{K}$ and $1 \leq t \leq p-2$. We can assume without loss of generality that $(i_t, i_{t+1}) \in V_1 \times V_2$.

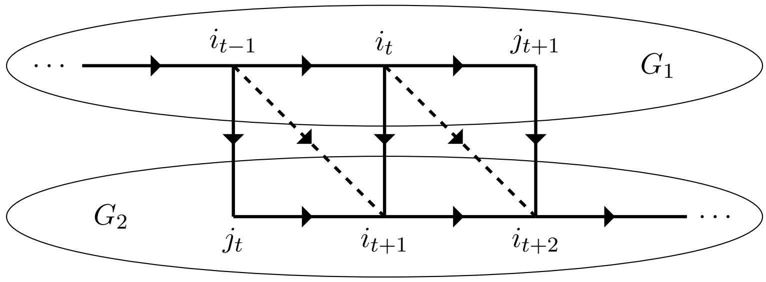



Figure 6: The path $e_{i_0 i_1 \cdots i_p}$ and the paths that can annihilate $e_{i_0 i_1 \cdots i_{t-1} i_{t+1} \cdots i_p}$ and $e_{i_0 i_1 \cdots i_t i_{t+2} \cdots i_p}$.

We write $x = x_1 + \lambda e_{i_0 i_1 \cdots i_p}$ for some $x_1 \in \Gamma_*$. By condition ($\sharp$), the elementary summand $e_{i_0 i_1 \cdots i_{t-1} i_{t+1} \cdots i_p}$ of $\partial e_{i_0 i_1 \cdots i_p}$ is not allowed on $G$. As $\partial x = \partial x_1 + \lambda \partial e_{i_0 i_1 \cdots i_p} \in \mathcal{A}_{p-1}(G)$ is allowed on $G$, the term $(-1)^{t+1}\lambda e_{i_0 i_1\cdots i_{t-1}i_{t+1}\cdots i_p}$ must be a summand of $\partial x_1$. Note that $e_{i_0 i_1\cdots i_{t-1}i_{t+1}\cdots i_p}$ can only be annihilated by a summand of $\partial e_{i_0 i_1\cdots i_{t-1}j_t i_{t+1}\cdots i_p}$ for some $j_t$. Therefore, $x_1$ must include the summand $-\lambda e_{i_0 i_1\cdots i_{t-1}j_t i_{t+1}\cdots i_p}$. Furthermore, the elementary path $e_{i_0 i_1\cdots i_{t-1}j_t i_{t+1}\cdots i_p}$ is allowed on $G$. If $j_t\in V_1$, then we would have $(j_t,i_{t+1})\in V_1\times V_2$ and $(i_t,i_{t+1})\in V_1\times V_2$, which contradicts condition ($\sharp$). Thus, $j_t$ must be in $V_2$. If $i_{t-1}\in V_2$, then we would have $(i_{t-1},i_t)\in V_2\times V_1$ and $(i_t,i_{t+1})\in V_1\times V_2$, which also contradicts condition ($\sharp$). Thus, $i_{t-1}$ must be in $V_1$. Consequently, $(i_{t-1},j_t)\in V_1\times V_2$. If $t-1\geq 1$, a similar argument shows that there is a summand $\lambda e_{i_0 i_1\cdots i_{t-2}j_{t-1}j_t i_{t+1}\cdots i_p}$ of $x$ for some $j_{t-1}$. Moreover, we have $(i_{t-2},j_{i-1})\in V_1\times V_2$. By induction, $x$ must include the summand $\sum\limits_{s=0}^{t-1}(-1)^{s+t}\lambda e_{i_0 i_1\cdots i_s j_{s+1}\cdots j_t i_{t+1}\cdots i_p}$. Here, $i_0,i_1,\ldots,i_t\in V_1$ and $j_1,j_2,\ldots,j_t\in V_2$, as described in Figures 6 and 7.

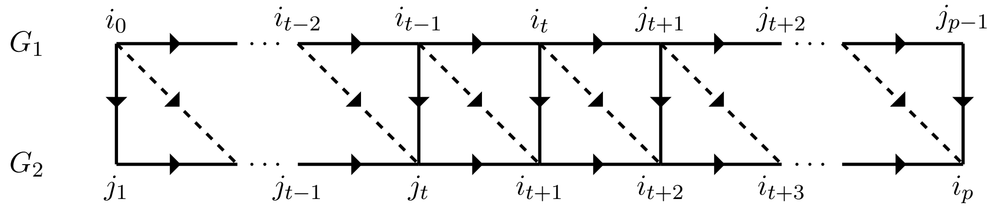



Figure 7: Illustration of all paths in $\Delta e_{i_0 i_1\cdots\cdots i_p}$.

Similarly, to ensure that the elementary summand $(-1)^{t+1}\lambda e_{i_0 i_1\cdots i_t i_{t+2}\cdots i_p}$ in $\lambda\partial e_{i_0 i_1\cdots i_p}$ can be annihilated, there must always be a summand $-\lambda e_{i_0 i_1\cdots i_t j_{t+1} i_{t+2}\cdots i_p}$ in $x_1$ for some $j_{t+1}$. Moreover, $e_{i_0 i_1\cdots i_t j_{t+1} i_{t+2}\cdots i_p}$ is an elementary path on $G$. If $j_{t+1}\in V_2$, we would have $(i_t,i_{t+1})\in V_1\times V_2$ and $(i_t,j_{t+1})\in V_1\times V_2$, which contradicts condition ($\sharp$). Thus, $j_{t+1}$ must be in $V_1$. A similar argument shows that $i_{t+2}$ must be in $V_2$. By induction, $x$ must include the summand $\sum\limits_{s=t+1}^{p-1}(-1)^{s+t}\lambda e_{i_0 i_1\cdots i_t j_{t+1}\cdots j_s i_{s+1}\cdots i_p}$, where $i_{t+1},\ldots,i_p\in V_2$ and $j_{t+1},\ldots,j_{p-1}\in V_1$.

For simplicity, we denote $k_0=i_0$, $k_1=i_1$, $\ldots$, $k_t=i_t$, $k_{t+1}=j_{t+1}$, $\ldots$, $k_{p-1}=j_{p-1}$, and $l_0=j_1$, $l_1=j_2$, $\ldots$, $l_{t-1}=j_t$, $l_t=i_{t+1}$, $\ldots$, $l_{p-1}=i_p$. Note that $k_s\in V_1$ and $l_s\in V_2$ for $s=0,1,\ldots,p-1$. Hence, $\sum\limits_{s=0}^{p-1}(-1)^s e_{k_0 k_1\cdots k_s l_s\cdots l_{p-1}}$ is a summand of $x$ up to a nonzero coefficient. By condition ($\sharp$), the construction of $\sum\limits_{s=0}^{p-1}(-1)^s e_{k_0 k_1\cdots k_s l_s\cdots l_{p-1}}$ from an elementary path $e_{i_0 i_1\cdots i_p}$ is unique. For convenience, we denote

$$\Delta e_{i_0 i_1\cdots i_p}=\sum_{s=0}^{p-1}(-1)^s e_{k_0 k_1\cdots k_s l_s\cdots l_{p-1}}.$$

Moreover, we have

$$\Delta e_{k_0 k_1\cdots k_s l_s\cdots l_{p-1}}=\Delta e_{i_0 i_1\cdots i_p},\quad 0\leq 1\leq p-1.$$

For the elementary path $e_{i_0 i_1\cdots\cdots i_p}$ on $G_1'$ or $G_2'$, we set $\Delta e_{i_0 i_1\cdots i_p}=0$. Then the construction $\Delta$ can extend to a linear map

$$\Delta:\Omega_*(G)\to\mathcal{A}_*(G),\quad x\mapsto\Delta x.$$

For any vertices $k_0, k_1, \dots, k_{p-1}$ and $l_0, l_1, \cdots, l_{p-1}$, let us denote

$$\begin{pmatrix} k_0k_1\cdots k_{p-1} \\ l_0l_1\cdots l_{p-1} \end{pmatrix}_G = \sum_{s=0}^{p-1}(-1)^s e_{k_0k_1\cdots k_sl_s\cdots l_{p-1}}.$$

If all the elementary paths $e_{k_0k_1\cdots k_sl_s\cdots l_{p-1}}$, $s = 0, 1, \dots, p-1$ are allowed on $G$, we have

$$\begin{pmatrix} k_0k_1\cdots k_{p-1} \\ l_0l_1\cdots l_{p-1} \end{pmatrix}_G = \Delta e_{k_0k_1\cdots k_sl_s\cdots l_{p-1}}, \quad 0 \le s \le p-1.$$

Let $\Delta\Lambda_*(G)$ be the $\mathbb{K}$-linear space generated by all elements of the form

$$\begin{pmatrix} k_0k_1\cdots k_{p-1} \\ l_0l_1\cdots l_{p-1} \end{pmatrix}_G$$

for any vertices $k_0, k_1, \dots, k_{p-1}$ and $l_0, l_1, \dots, l_{p-1}$. Let $\widetilde{\Omega}_* = \Omega_*(G) \cap \Delta\Lambda_*(G)$. In the next step, we will define a chain complex structure on $\widetilde{\Omega}_*$. In Step $(iii)$, we will prove that $\widetilde{\Omega}_* = \Gamma_*$ as $\mathbb{K}$-linear spaces.

**Step** $(ii)$. The chain complex $(\widetilde{\Omega}_*, \widetilde{\partial})$.

We will construct a differential on $(\widetilde{\Omega}_*, \widetilde{\partial})$. By definition, each element in $\widetilde{\Omega}_*$ is a linear combination of $\binom{k_0k_1\cdots k_{p-1}}{l_0l_1\cdots l_{p-1}}_G = \sum_{s=0}^{p-1}(-1)^s e_{k_0k_1\cdots k_sl_s\cdots l_{p-1}}$. The differential on $\binom{k_0k_1\cdots k_{p-1}}{l_0l_1\cdots l_{p-1}}_G$ is defined by

$$\widetilde{\partial}\begin{pmatrix} k_0k_1\cdots k_{p-1} \\ l_0l_1\cdots l_{p-1} \end{pmatrix}_G = \sum_{r=0}^{p-1}(-1)^{r+1}\begin{pmatrix} k_0\cdots \widehat{k_r}\cdots k_{p-1} \\ l_0\cdots \widehat{l_r}\cdots l_{p-1} \end{pmatrix}_G \in \Delta\Lambda_{p-1}(G), \quad p \ge 2$$

and $\widetilde{\partial} e_{k_0l_0} = 0$ for the case $p = 1$.

Indeed, a straightforward calculation shows that

$$\begin{aligned}
&\partial\begin{pmatrix} k_0k_1\cdots k_{p-1} \\ l_0l_1\cdots l_{p-1} \end{pmatrix}_G \\
=&\sum_{s=0}^{p-1}\sum_{r=0}^{s}(-1)^{s+r}e_{k_0k_1\cdots\widehat{k_r}\cdots k_sl_s\cdots l_{p-1}} + \sum_{s=0}^{p-1}\sum_{r=s+1}^{p}(-1)^{s+r}e_{k_0k_1\cdots k_sl_s\cdots\widehat{l_{r-1}}\cdots l_{p-1}} \\
=&\sum_{r=0}^{p-1}\sum_{s=r}^{p-1}(-1)^{s+r}e_{k_0k_1\cdots\widehat{k_r}\cdots k_sl_s\cdots l_{p-1}} + \sum_{r=1}^{p}\sum_{s=0}^{r-1}(-1)^{s+r}e_{k_0k_1\cdots k_sl_s\cdots\widehat{l_{r-1}}\cdots l_{p-1}} \\
=&\sum_{r=0}^{p-1}\sum_{s=r}^{p-1}(-1)^{s+r}e_{k_0k_1\cdots\widehat{k_r}\cdots k_sl_s\cdots l_{p-1}} + \sum_{r=0}^{p-1}\sum_{s=0}^{r}(-1)^{s+r+1}e_{k_0k_1\cdots k_sl_s\cdots\widehat{l_r}\cdots l_{p-1}} \\
=&\sum_{r=0}^{p-1}(-1)^{r+1}\left(\sum_{s=r}^{p-2}(-1)^s e_{k_0k_1\cdots\widehat{k_r}\cdots k_{s+1}l_{s+1}\cdots l_{p-1}} + \sum_{s=0}^{r-1}(-1)^s e_{k_0k_1\cdots k_sl_s\cdots\widehat{l_r}\cdots l_{p-1}}\right) \\
&+e_{l_0l_1\cdots l_{p-1}} - e_{k_0k_1\cdots k_{p-1}} \\
=&\left(\sum_{r=0}^{p-1}(-1)^{r+1}\begin{pmatrix} k_0\cdots \widehat{k_r}\cdots k_{p-1} \\ l_0\cdots \widehat{l_r}\cdots l_{p-1} \end{pmatrix}_G\right) + e_{l_0l_1\cdots l_{p-1}} - e_{k_0k_1\cdots k_{p-1}} \\
=&\widetilde{\partial}\begin{pmatrix} k_0k_1\cdots k_{p-1} \\ l_0l_1\cdots l_{p-1} \end{pmatrix}_G + e_{l_0l_1\cdots l_{p-1}} - e_{k_0k_1\cdots k_{p-1}}.
\end{aligned}$$

Here, $\widehat{i}$ denotes omission the index $i$. From a further calculation, we can obtain

$$\partial^2 \begin{pmatrix} k_0k_1\cdots k_{p-1} \\ l_0l_1\cdots l_{p-1} \end{pmatrix}_G = \widetilde{\partial}^2 \begin{pmatrix} k_0k_1\cdots k_{p-1} \\ l_0l_1\cdots l_{p-1} \end{pmatrix}_G + \Phi,$$

where

$$\Phi = \left(\sum_{r=0}^{p-1}(-1)^{r+1}(e_{l_0l_1\cdots\widehat{l_r}\cdots l_{p-1}} - e_{k_0k_1\cdots\widehat{k_r}\cdots k_{p-1}})\right) + \partial e_{l_0l_1\cdots l_{p-1}} - \partial e_{k_0k_1\cdots k_{p-1}} = 0.$$

This shows that $\widetilde{\partial}^2 = 0$.

On the other hand, any element in $\widetilde{\Omega}_p$ can be written as

$$x = \sum_{\gamma} \lambda_\gamma \gamma,$$

where $\gamma = \binom{k_0k_1\cdots k_{p-1}}{l_0l_1\cdots l_{p-1}}_G$ for some allowed path $e_{k_0k_1\cdots k_sl_s\cdots l_{p-1}}$ on $G$ for $s = 0, 1, \dots, p-1$. Since $\partial x \in \Omega_{p-1}(G)$ and $e_{l_0l_1\cdots l_{p-1}}, e_{k_0k_1\cdots k_{p-1}} \in \Omega_{p-1}(G)$, we have $\widetilde{\partial} x \in \Omega_{p-1}(G)$. It follows that $\widetilde{\partial} x \in \Omega_{p-1}(G) \cap \Delta\Lambda_{p-1}(G) = \widetilde{\Omega}_{p-1}(G)$. Hence, $\widetilde{\partial}$ is a differential on $\widetilde{\Omega}_*(G)$.

**Step** $(iii)$. $\widetilde{\Omega}_* = \Gamma_*$.

We define the $\mathbb{K}$-linear map $\varphi : \Omega_*(G) \to \widetilde{\Omega}_*$ on each elementary path as follows:

$$\varphi(e_{k_0k_1\cdots k_sl_s\cdots l_{p-1}}) = \frac{(-1)^s}{p} \Delta e_{k_0k_1\cdots k_sl_s\cdots l_{p-1}}.$$

We will show that $\varphi : \Omega_*(G) \to \widetilde{\Omega}_*$ is well-defined. Recall that

$$\Omega_*(G) = [\Omega_*(G_1') + \Omega_*(G_2')] \oplus \Gamma_*.$$

By construction, the map $\varphi$ is zero on $\Omega_*(G_1') + \Omega_*(G_2')$. For any summand $e_{k_0k_1\cdots k_sl_s\cdots l_{p-1}}$ in some element of $\Gamma_p$, it extends to a summand

$$\begin{pmatrix} k_0k_1\cdots k_{p-1} \\ l_0l_1\cdots l_{p-1} \end{pmatrix}_G = \sum_{s=0}^{p-1}(-1)^s e_{k_0k_1\cdots k_sl_s\cdots l_{p-1}} = \Delta e_{k_0k_1\cdots k_sl_s\cdots l_{p-1}},$$

where $e_{k_0k_1\cdots k_sl_s\cdots l_{p-1}}$ is an allowed path on $G$ for any $0 \leq s \leq p-1$. Thus, we have

$$\varphi\left(\begin{pmatrix} k_0k_1\cdots k_{p-1} \\ l_0l_1\cdots l_{p-1} \end{pmatrix}_G\right) = \sum_{s=0}^{p-1}(-1)^s \varphi\left(e_{k_0k_1\cdots k_sl_s\cdots l_{p-1}}\right) = \begin{pmatrix} k_0k_1\cdots k_{p-1} \\ l_0l_1\cdots l_{p-1} \end{pmatrix}_G.$$

Here, we use the fact that

$$\Delta e_{k_0k_1\cdots k_sl_s\cdots l_{p-1}} = \begin{pmatrix} k_0k_1\cdots k_{p-1} \\ l_0l_1\cdots l_{p-1} \end{pmatrix}_G$$

for $s = 0, 1, \dots, p-1$. Hence, $\varphi = \mathrm{id}$ on $\Gamma_*$.

By definition, $\varphi$ is a surjection. By a direct calculation, we have

$$\widetilde{\partial}\varphi\left(\begin{pmatrix} k_0k_1\cdots k_{p-1} \\ l_0l_1\cdots l_{p-1} \end{pmatrix}_G\right) = \widetilde{\partial}\begin{pmatrix} k_0k_1\cdots k_{p-1} \\ l_0l_1\cdots l_{p-1} \end{pmatrix}_G = \sum_{r=0}^{p-1}(-1)^{r+1}\begin{pmatrix} k_0\cdots\widehat{k_r}\cdots k_{p-1} \\ l_0\cdots\widehat{l_r}\cdots l_{p-1} \end{pmatrix}_G.$$

On the other hand, we obtain

$$\varphi\left(\partial\begin{pmatrix}k_0k_1\cdots k_{p-1}\\ l_0l_1\cdots l_{p-1}\end{pmatrix}_G\right)=\sum_{r=0}^{p-1}(-1)^{r+1}\begin{pmatrix}k_0\cdots \widehat{k_r}\cdots k_{p-1}\\ l_0\cdots \widehat{l_r}\cdots l_{p-1}\end{pmatrix}_G.$$

It follows that $\varphi\partial=\widetilde{\partial}\varphi$ on $\Gamma_*$. Since $\varphi=0$ on $\Omega_*(G_1')+\Omega_*(G_2')$, we have $\varphi\partial=\widetilde{\partial}\varphi$ on $\widetilde{\Omega}_*$. Thus, $\varphi$ is a morphism of chain complexes.

Recall the inclusion map $\theta:\Omega_*(G_1')+\Omega_*(G_2')\hookrightarrow\Omega_*(G)$ as defined in Eq. (1). It is evident that $\varphi\theta=0$. We will prove that $\ker\varphi\subseteq \mathrm{im}\theta$. The morphism $\varphi$ can be expressed as

$$\varphi:[\Omega_*(G_1')+\Omega_*(G_2')]\oplus\Gamma_*\to\widetilde{\Omega}_*.$$

Suppose $\varphi(x_1+x_2)=0$ for $x_1\in\Omega_*(G_1')+\Omega_*(G_2')$ and $x_2\in\Gamma_*$. We then have $\varphi(x_1)=0$ and $\varphi(x_2)=x_2$. It follows that $x_2=0$, which implies $\ker\varphi=\Omega_*(G_1')+\Omega_*(G_2')$. Thus, we obtain a short exact sequence of chain complexes:

$$0\to\Omega_*(G_1')+\Omega_*(G_2')\xrightarrow{\theta}\Omega_*(G)\xrightarrow{\varphi}\widetilde{\Omega}_*\to 0. \tag{2}$$

Hence, $\widetilde{\Omega}_*=\Gamma_*$ as $\mathbb{K}$-linear spaces.

**Step** $(iv)$. The main result.

Let $S=(V_S,E_S)$ be the induced sub-digraph of $G_1$ with vertex set consisting of the source vertices of directed edges in $V_1\times V_2$. The edge set $E_S$ includes all directed edges between the vertices in $V_S$ within $G_1$. Similarly, let $S'=(V_{S'},E_{S'})$ be the induced sub-digraph of $G_1$ with vertex set consisting of the target vertices of directed edges in $V_1\times V_2$.

If $S\cap S'\neq\emptyset$, then there exists an $i\in S\cap S'$. By construction, there is an edge $(i,j)$ in $E_S$ and an edge $(j,k)$ in $E_{S'}$. This contradicts condition ($\sharp$). Thus, we have

$$S\cap S'=\emptyset.$$

It follows that $\Omega_*(S\cup S')=\Omega_*(S)\oplus\Omega_*(S')$. Let $\Omega_*(S)[1]$ be the chain complex with $\Omega_p(S)[1]=\Omega_{p-1}(S)$. Consider the $\mathbb{K}$-linear map

$$\phi:\widetilde{\Omega}_*\to\Omega_*(S)[1]\oplus\Omega_*(S')[1]$$

given by

$$\phi\left(\begin{pmatrix}k_0k_1\cdots k_{p-1}\\ l_0l_1\cdots l_{p-1}\end{pmatrix}_G\right)=\begin{cases}(-1)^{p-1}e_{k_0k_1\cdots k_{p-1}}, & \text{if } k_0,k_1,\dots,k_{p-1}\in V_1 \text{ and } l_0,l_1,\dots,l_{p-1}\in V_2;\\ (-1)^{p-1}e_{l_0l_1\cdots l_{p-1}}, & \text{if } k_0,k_1,\dots,k_{p-1}\in V_2 \text{ and } l_0,l_1,\dots,l_{p-1}\in V_1.\end{cases}$$

For the case where $k_0,k_1,\dots,k_{p-1}\in V_1$ and $l_0,l_1,\dots,l_{p-1}\in V_2$, a straightforward calculation yields:

$$\begin{aligned}\phi\left(\widetilde{\partial}\begin{pmatrix}k_0k_1\cdots k_{p-1}\\ l_0l_1\cdots l_{p-1}\end{pmatrix}_G\right)&=\phi\left(\sum_{r=0}^{p-1}(-1)^{r+1}\begin{pmatrix}k_0\cdots \widehat{k_r}\cdots k_{p-1}\\ l_0\cdots \widehat{l_r}\cdots l_{p-1}\end{pmatrix}_G\right)\\ &=\sum_{r=0}^{p-1}(-1)^{r+p-1}e_{k_0k_1\cdots\widehat{k_r}\cdots k_{p-1}}\\ &=(-1)^{p-1}\partial e_{k_0k_1\cdots k_{p-1}}\\ &=\partial\phi\left(\begin{pmatrix}k_0k_1\cdots k_{p-1}\\ l_0l_1\cdots l_{p-1}\end{pmatrix}_G\right).\end{aligned}$$

For the case where $k_0, k_1, \dots, k_{p-1} \in V_2$ and $l_0, l_1, \dots, l_{p-1} \in V_1$, the calculation is similar. Thus, $\phi$ is a morphism of chain complexes. It can be directly verified that $\phi$ is a bijection for $p \geq 1$. Consequently, we have

$$H_p(\widetilde{\Omega}_*) \cong H_{p-1}(S) \oplus H_{p-1}(S'), \quad p \geq 2. \tag{3}$$

Note that $\Omega_*(G_1') \cap \Omega_*(G_2') = \Omega_*(G_0)$. We have a short exact sequence

$$0 \to \Omega_*(G_0) \xrightarrow{\rho} \Omega_*(G_1') \oplus \Omega_*(G_2') \xrightarrow{\pi} \Omega_*(G_1') + \Omega_*(G_2') \to 0$$

where $\rho(\sigma) = (\sigma, -\sigma)$ and $\pi(\sigma, \tau) = \sigma + \tau$. This short exact sequence induces a long exact sequence of homology groups:

$$\cdots \to H_p(G_0) \xrightarrow{\rho^*} H_p(G_1') \oplus H_p(G_2') \xrightarrow{\pi^*} H_p(\Omega_*(G_1') + \Omega_*(G_2')) \to H_{p-1}(G_0) \to \cdots.$$

Since $H_p(G_0) = 0$ for $p \geq 1$, we obtain

$$H_p(\Omega_*(G_1') + \Omega_*(G_2')) \cong H_p(G_1') \oplus H_p(G_2'), \quad p \geq 2. \tag{4}$$

By the short exact sequence (2), we have a long exact sequence of homology groups:

$$\cdots \to H_p(\Omega_*(G_1') + \Omega_*(G_2')) \xrightarrow{\theta^*} H_p(G) \xrightarrow{\varphi^*} H_p(\widetilde{\Omega}_*) \to H_{p-1}(\Omega_*(G_1') + \Omega_*(G_2')) \to \cdots.$$

Combining with the isomorphisms (3) and (4), we obtain a long exact sequence of homology groups

$$\begin{aligned} \cdots \to H_p(S) \oplus H_p(S') \xrightarrow{\delta} H_p(G_1') \oplus H_p(G_2') \xrightarrow{\overline{\theta}^*} H_p(G) \xrightarrow{\overline{\varphi}^*} H_{p-1}(S) \oplus H_{p-1}(S') \xrightarrow{\delta} \cdots \\ \cdots \to H_2(G) \xrightarrow{\overline{\varphi}^*} H_1(S) \oplus H_1(S') \xrightarrow{\delta} H_1(\Omega_*(G_1') + \Omega_*(G_2')) \xrightarrow{\overline{\theta}^*} H_1(G). \end{aligned}$$

By [5, Theorem 5.1], we have the isomorphism

$$H_p(G_1') \oplus H_p(G_2') \cong H_p(G_1) \oplus H_p(G_2).$$

This leads to the desired long exact sequence

$$\begin{aligned} \cdots \to H_p(S) \oplus H_p(S') \xrightarrow{\delta} H_p(G_1) \oplus H_p(G_2) \xrightarrow{\overline{\theta}^*} H_p(G) \xrightarrow{\overline{\varphi}^*} H_{p-1}(S) \oplus H_{p-1}(S') \xrightarrow{\delta} \cdots \\ \cdots \to H_2(G) \xrightarrow{\overline{\varphi}^*} H_1(S) \oplus H_1(S') \xrightarrow{\delta} H_1(\Omega_*(G_1') + \Omega_*(G_2')) \xrightarrow{\overline{\theta}^*} H_1(G). \end{aligned} \tag{5}$$

Now, we will describe the morphisms in the long exact sequence. For a directed edge $(k, l) \in V_1 \times V_2$ or $(l, k) \in V_2 \times V_1$, let $Tk = l$. For each elementary path $e_{k_0 k_1 \cdots k_p}$ with directed edges $(k_0, l_0), (k_1, l_1), \dots, (k_p, l_p) \in V_1 \times V_2$ or $(l_0, k_0), (l_1, k_1), \dots, (l_p, k_p) \in V_2 \times V_1$, we define the $\mathbb{K}$-linear map $T : \Omega_*(S) \to \Omega_*(G_2)$ by

$$Te_{k_0 k_1 \cdots k_p} = e_{l_0 l_1 \cdots l_p}.$$

Consider the case where $(k_0, l_0), (k_1, l_1), \dots, (k_p, l_p) \in V_1 \times V_2$. The other case is similar. Given a cycle

$$x = \sum_{e_{k_0 k_1 \cdots k_p}} \lambda_{e_{k_0 k_1 \cdots k_p}} e_{k_0 k_1 \cdots k_p} \in \Omega_p(S),$$

we have

$$\phi^{-1}(x) = \sum_{e_{k_0k_1\cdots k_p}} \lambda_{e_{k_0k_1\cdots k_p}} \Delta e_{k_0k_1\cdots Tk_p} = \Delta z,$$

where $z = \sum_{e_{k_0k_1\cdots k_p}} \lambda_{e_{k_0k_1\cdots k_p}} e_{k_0k_1\cdots Tk_p}$. Note that the preimage of $\varphi$ at $\Delta z$ in $\Omega_*(G)$ is $\Delta z$. The map $\delta$ is defined by

$$\delta[x] = [\partial \Delta z].$$

By definition, we have

$$\partial \Delta z = \widetilde{\partial} \Delta z + Tx - x \in \Omega_*(G_1') + \Omega_*(G_2'),$$

and $Tx - x \in \Omega_*(G_1') + \Omega_*(G_2')$. Thus, we obtain $\widetilde{\partial}\Delta z = 0$ in $\Omega_*(G_1') + \Omega_*(G_2')$. It follows that

$$\delta[x] = [\partial \Delta z] = [\widetilde{\partial} \Delta z + Tx - x] = [Tx - x].$$

Hence, $\delta$ is given by $\delta([x] + [x']) = [Tx' - x] + [Tx - x'] \in H_p(G_1' + G_2')$ for cycles $x \in \Omega_p(S)$ and $x' \in \Omega_p(S')$. Finally, $\overline{\theta}^*([\sum e_{k_0k_1\cdots k_p}]) = [\Delta \sum e_{k_0k_1\cdots k_pTk_p}]$ for $\sum e_{k_0k_1\cdots k_p} \in S_1$ or $\sum e_{k_0k_1\cdots k_p} \in S_2$, and $\overline{\varphi}^* = H(\phi \circ \varphi)$.

Moreover, if each sub-digraph of $G_1$ has null homology for $p \geq k$, we obtain a short exact sequence

$$0 \to H_p(G_2) \to H_p(G) \to 0, \quad p \geq k + 1.$$

This completes the proof. □

*The proof of Theorem 3.2.* Applying to Eq. (5) in the proof of Theorem 3.1, we have a long exact sequence

$$\cdots \to H_p(S) \xrightarrow{\delta} H_p(G_1) \oplus H_p(G_2) \xrightarrow{\overline{\theta}^*} H_p(G) \xrightarrow{\overline{\varphi}^*} H_{p-1}(S) \xrightarrow{\delta} \cdots$$
$$\cdots \to H_2(G) \xrightarrow{\overline{\varphi}^*} H_1(S) \xrightarrow{\delta} H_1(\Omega_*(G_1') + \Omega_*(G_2')) \xrightarrow{\overline{\theta}^*} H_1(G).$$

We follow the notation from the proof of Theorem 3.1. For $p \geq 2$, recall that the map $\delta : H_p(S) \to H_p(G_1) \oplus H_p(G_2)$ is given by $\delta([x]) = [Tx] - [x]$. Since $x \in \Omega_p(G_1)$ and $Tx \in \Omega_p(G_2)$, we have that $\delta([x]) = 0$ implies $[x] = 0$. Therefore, $\delta$ is injective. For the case when $p = 1$, the map $\delta : H_1(S) \to H_1(\Omega_*(G_1') + \Omega_*(G_2'))$ is also injective by a similar verification. Thus, we have a short exact sequence

$$0 \to H_p(S) \to H_p(G_1) \oplus H_p(G_2) \to H_p(G) \to 0$$

for any $p \geq 2$. □

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