LOWER ESTIMATES FOR PERTURBED DIRICHLET SOLUTIONS

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1. Introduction

Let X be a Riemannian manifold (for example, an open subset of $\mathbb{R}^n$) and Δ be the Laplace operator associated with the Riemannian structure. Alongside with the Laplace equation on X,

$$\Delta u = 0, \tag{1.1}$$

let us consider the *perturbed* equation

$$\Delta u - u\mu = 0 \tag{1.2}$$

where μ is, in general, a signed Radon measure on X. In particular, μ may have a density with respect to the Riemannian measure μ_0 in which case μ in (1.2) will be identified with its density. In general, we understand (1.2) in the sense of distributions. In particular, a solution u should be in $\mathcal{L}^1_{loc}(X, \mu_0) \cap \mathcal{L}^1_{loc}(X, |\mu|)$ so that both terms Δu and $u\mu$ are distributions.

The results of this note are new even if μ is a smooth function in $X = \mathbb{R}^d$. In this case (1.2) can be understood in the classical sense and is an elliptic Schrödinger equation. The question we address here is as follows:

How to compare solutions to the Dirichlet problem for the equation (1.2) with solutions to the Dirichlet problem for the equation (1.1)?

Let V denote a precompact open subset of X which is regular (that is, every point of the boundary ∂V is regular with respect to the Dirichlet problem for the Laplace equation; in particular, this is the case when $\partial V \in C^1$). Denote by $G_V(x, y)$ the Green function of the Dirichlet problem for the Laplace equation in V. Denote also by K_V^μ the integral operator on functions in V which acts by

$$K_V^\mu h = \int_V G_V(\cdot, y)h(y)d\mu(y). \tag{1.3}$$

Let f be a continuous function on ∂V and consider the following two Dirichlet problems in V:

$$\begin{cases} \Delta h = 0, \\ h|_{\partial V} = f \end{cases} \tag{1.4}$$

and

$$\begin{cases} \Delta u - u\mu = 0, \\ u|_{\partial V} = f\ . \end{cases} \tag{1.5}$$

The main result of this note is the following lower bound for u via h. Suppose that $f \geq 0$ and $f \not\equiv 0$. Then

$$\frac{u}{h} \geq \exp\left(-\frac{K_V^\mu h}{h}\right) \quad \text{in } V, \tag{1.6}$$

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assuming that μ is an arbitrary signed (local) Kato measure.

The latter means that, for any precompact regular open set $U \subset X$, the function

$$K_U^{|\mu|}1 = \int_U G_U(\cdot, y) d|\mu|(y)$$

is finite and continuous on U. In particular, this ensures that the expression $K_V^\mu h$ in (1.6) does make sense. Any locally bounded measurable function on X is a density of a (local) Kato measure because the singularity of the Green kernel is summable like in $\mathbb{R}^n$.

If $\mu \geq 0$ then inequality (1.6) is known and was proved in [5, p.558] (for such μ, we obviously have also $u \leq h$). However, the method of [5] does not work for a signed measure μ. Here we give an entirely different proof which works for any μ and which is based on the Feynman-Kac formula.

Let us emphasize the general nature of inequality (1.6). Although it is a *pointwise* inequality, its validity does not depend on any particular property of the underlying space X. Moreover, it holds in a much more general setting of harmonic spaces.

So Section 2 will be devoted to a short discussion of perturbations of harmonic spaces. In particular, we shall recall that we always can find an associated Hunt process such that perturbed solutions are given by a Feynman-Kac formula. A reader who accepts the Feynman-Kac formula may skip that section.

The inequality (1.6) will be proved in Section 3 (Theorem 3.1) by an application of Jensen's inequality.

Let us finally note that we might get rid of the continuity of the potentials defining the perturbation. This could be achieved either by studying a more general perturbation from the very beginning (see e.g. [3, Section 2.2]) or, having established Theorem 3.1 below, by using a limit procedure to extend the validity of the inequality.

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2. Harmonic spaces and Feynman-Kac formula

Let $(X, \mathcal{H})$ be a $\mathcal{P}$-harmonic Bauer space and let $\mathcal{M}^+(\mathcal{H})$ denote the convex cone of all sections of continuous real potentials. For a short introduction of these notions and related definitions and properties the reader is referred to [3, Section 7].

We denote by $\mathcal{U}_c$ the set of all precompact open subsets of X. By definition of a Bauer space, for any set $V \in \mathcal{U}_c$ there is a harmonic operator H_V which maps any function $f \in C(\partial V)$ to a function $H_V f \in \mathcal{H}(V)$, and if $f \geq 0$ then also $H_V f \geq 0$. There is a rich enough family of regular sets in $\mathcal{U}_c$ for which "the Dirichlet problem is solvable" that is $H_V f$ is continuous in $\overline{V}$ and is equal to f on ∂V.

Let $\mathcal{M}(\mathcal{H})$ be the vector space generated by $\mathcal{M}^+(\mathcal{H})$ (see [2, p.105]). Using the ordering (called *specific order*) induced by $\mathcal{M}^+(\mathcal{H})$ the space $\mathcal{M}(\mathcal{H})$ is a Riesz space. In particular, each $M \in \mathcal{M}(\mathcal{H}))$ has a unique decomposition $M = M^+ - M^-$ such that $M^+, M^- \in \mathcal{M}^+(\mathcal{H})$ and the specific infimum of M^+ and M^- is 0. Then $|M| = M^+ + M^-$ is the specific supremum of M and $-M$.

In the case when X is a Riemannian manifold and $\mathcal{H}$ is the sheaf of harmonic functions on X (and in many other cases, too), any $M \in \mathcal{M}(\mathcal{H})$ can be identified with a measure: The corresponding (local) Kato measure μ is the unique signed measure μ on X such that

$$M_V = \int_V G_V(\cdot, y)\, \mu(dy)$$

for all $V \in \mathcal{U}_c$. If $\mu = \mu^+ - \mu^-$ is the decomposition of μ into its positive part μ^+ and its negative part μ^-, then μ^+ corresponds to M^+, μ^- corresponds to M^-, and $|\mu| = \mu^+ + \mu^-$ corresponds to $|M|$.

In the following assume that $M \in \mathcal{M}(\mathcal{H})$. Then we define

$$K_V^M = K_V^{M^+} - K_V^{M^-} \qquad (V \in \mathcal{U}_c)$$

(where $K_V^{M^\pm}$ is the potential kernel associated with $M_V^\pm$) and, for every open subset U of X,

$$\mathcal{H}^M(U) = \{u \in \mathcal{C}(U) : u + K_V^M u \in \mathcal{H}(V) \text{ for every } V \in \mathcal{U}_c \text{ with } \overline{V} \subset U\}.$$

From [2] we quote the

Theorem 2.1. *$(X, \mathcal{H}^M)$ is a Bauer space.*

Let $\mathcal{U}^M(\mathcal{H})$ denote the set of all $V \in \mathcal{U}_c$ such that the operator $I + K_V^M$ on $B_b(V)$ is invertible and $(I + K_V^M)^{-1}s \geq 0$ for every $s \in \mathcal{S}_b^+(V)$. If $M \in \mathcal{M}^+(\mathcal{H})$ then $\mathcal{U}^M(\mathcal{H}) = \mathcal{U}_c$. In the general case, we have $\sup M_V^-(X) < 1$ if V is sufficiently small and then trivially

$$(I + K_V^M)^{-1} = \sum_{n=0}^{\infty} \left[(I + K_V^{M^+})^{-1} K_V^{M^-}\right]^n (I + K_V^{M^+})^{-1} \tag{2.1}$$

showing that $V \in \mathcal{U}^M(\mathcal{H})$. In fact, (2.1) holds for every $V \in \mathcal{U}^M(\mathcal{H})$ and the boundedness of the kernel on the right side characterizes those $V \in \mathcal{U}_c$ which are contained in $\mathcal{U}^M(\mathcal{H})$ (see [4, p.136]). For $V \in \mathcal{U}^M(\mathcal{H})$ we define

$$H_V^M = (I + K_V^M)^{-1} H_V. \tag{2.2}$$

Then for every $f \in \mathcal{C}(X)$, $H_V^M f$ is the unique function $h \in \mathcal{H}_b^M(V)$ such that $\lim_{n\to\infty} h(x_n) = f(z)$ for every regular sequence (x_n) in V converging to a point $z \in \partial V$. In particular, every regular set V in X which is sufficiently small, is M-regular, i.e., regular with respect to $\mathcal{H}^M$, and the corresponding harmonic kernel is H_V^M (see [4, p.140]).

Remark: Using perturbation of Bauer spaces which are not necessarily $\mathcal{P}$-harmonic we may get back to the original space $(X, \mathcal{H})$ by a perturbation of $(X, \mathcal{H}^M)$ (see [2, p.109]):

$$(X, \mathcal{H}) = (X, (\mathcal{H}^M)^N)$$

where

$$N_V = -(I + K_V^M)^{-1} M_V \qquad (V \in \mathcal{U}^M(\mathcal{H})).$$

So in the context of harmonic spaces the relation between $(X, \mathcal{H})$ and $(X, \mathcal{H}^M)$ is completely symmetric!

Let us assume in the following that the function 1 is superharmonic. Then Feynman-Kac integrals can be used. The key is the following result (see [2]):

Theorem 2.2. *Given $M \in \mathcal{M}(\mathcal{H})$, there exists a Hunt process*

$$\mathcal{X} = (\Omega, \mathbb{M}, \mathbb{M}_t, X_t, \theta_t, \mathbb{P}_x)$$

on X having the following properties:

(i) *For every $V \in \mathcal{U}_c$, every $x \in V$ and every Borel set B in X,*

$$H_V(x, B) = \mathbb{P}_x[X_{\tau_V} \in B]$$

where τ_V denotes the first exit time from V and $H_V(x, B)$ denotes the kernel of the operator H_V.

(ii) *The potential kernel W of $\mathcal{X}$ is proper (i.e., $W1_L = \int_0^\infty P_t 1_L\,dt$ is finite for every compact subset L of X) and there exists a locally bounded function $g \in \mathcal{B}(X)$ such that, for every $V \in \mathcal{U}_c$,*

$$M_V = W(g1_V) - H_V W(g1_V).$$

If X is a Riemannian manifold and if μ has a locally bounded density, then the Brownian motion on X is such a process. Let us briefly sketch how we can find such a process (even with bounded potential kernel) in our general situation.

We already noted in [3, Section 7.2] that we have an injection

$$j : \mathcal{P}(X) \cap \mathcal{C}(X) \to \mathcal{M}^+(\mathcal{H})$$

given by

$$(j(p))_V = p - H_V p \qquad (V \in \mathcal{U}_c).$$

Now fix $M \in \mathcal{M}(\mathcal{H})$ and an exhaustion (U_n) of X. Then, for each $n \in \mathbb{N}$, $p_n := K^{M^+}_{U_{n+1}} 1_{U_n}$ is a continuous real potential on U_{n+1} which is harmonic on $U_{n+1} \setminus \overline{U}_n$. Hence there exists a unique continuous real potential q_n on X such that q_n is harmonic on $X \setminus \overline{U}_n$ and $p_n - q_n$ is harmonic on U_{n+1}. Then $\sup q_n(X) = \sup q_n(\overline{U}_n) < \infty$ and it is easily seen that

$$j(q_n) = 1_{U_n} M^+.$$

Similarly, there exist $q_n' \in \mathcal{P}(X) \cap \mathcal{C}_b(X)$, $n \in \mathbb{N}$, such that

$$j(q_n') = 1_{U_n} M^-.$$

Define

$$q = \sum_{n=1}^\infty \alpha_n q_n, \qquad q' = \sum_{n=1}^\infty \alpha_n q_n', \qquad \varphi = \sum_{n=1}^\infty \alpha_n 1_{U_n}$$

where

$$\alpha_n = \frac{1}{2^n \sup(q_n + q_n' + 1)(X)}.$$

Then $q, q' \in \mathcal{P}(X) \cap \mathcal{C}_b(X)$, $\varphi \le 1$, $\inf \varphi(U_n) > 0$ for every $n \in \mathbb{N}$, and

$$j(q) = \varphi M^+, \qquad j(q') = \varphi M^-.$$

Let $q_0 \in \mathcal{P}(X) \cap \mathcal{C}_b(X)$ be a strict potential and take

$$p := q_0 + q + q'.$$

Then p is a strict potential in $\mathcal{C}_b(X)$, hence by [1] there exists a Hunt process $\mathcal{X} = (\Omega, \mathbb{M}, \mathbb{M}_t, X_t, \theta_t, \mathbb{P}_x)$ on X such that (i) holds and the potential kernel $W = \int_0^\infty P_t\,dt$ of $\mathcal{X}$ satisfies

$$W1 = p.$$

Moreover, by [1], there exist $\psi, \psi' \in \mathcal{B}_b^+(X)$ (less than 1) such that

$$W\psi = q, \qquad W\psi' = q'.$$

Defining

$$g := \frac{\psi - \psi'}{\varphi}$$

we then have

$$M = gj(p),$$

i.e., (ii) holds. Moreover,

$$M^+ = g^+ j(p), \qquad M^- = g^- j(p)$$

and, for every $V \in \mathcal{U}_c$, $x \in V$, and every $f \in \mathcal{B}_b(X)$,

$$\begin{aligned} K_V^{M^\pm} f(x) &= W(fg^\pm 1_V)(x) - H_V W(fg^\pm 1_V)(x) \\ &= \mathbb{E}_x \left(\int_o^{\tau_V} (fg^\pm)(X_t)\, dt \right), \end{aligned}$$

$$K_V^M f(x) = \mathbb{E}_x \left(\int_0^{\tau_V} (fg)(X_t)\, dt \right). \tag{2.3}$$

Proceeding as in [2, p.125-127] this finally leads to the following result:

Theorem 2.3. *Let $V \in \mathcal{U}_c$ such that $H_V 1 > 0$. Then the following statements are equivalent:*

(i) $V \in \mathcal{U}^M(\mathcal{H})$.
(ii) *The function $x \mapsto \mathbb{E}_x(\exp\left(-\int_0^{\tau_V} g(X_t)\, dt\right) 1_{\{\tau_V<\infty\}})$ is locally bounded on V.*
(iii) *For every $f \in \mathcal{C}(\partial V)$, there exists a unique function $h \in \mathcal{H}_b^M(V)$ such that $\lim_{n\to\infty} h(x_n) = f(z)$ for every regular sequence (x_n) converging to a point $z \in \partial V$, and $h \geq 0$ if $f \geq 0$.*

In this case, the function h is given by

$$h(x) = \mathbb{E}_x \left(\exp\left(- \int_0^{\tau_V} g(X_t)\, dt \right) f(X_{\tau_V}) \right). \tag{2.4}$$

For later purpose we finally note an easy consequence of (2.3):

Proposition 2.4. *For every $V \in \mathcal{V}_c$ and every $f \in \mathcal{C}(X)$, the function $h = H_V f$ satisfies*

$$K_V^M h(x) = \mathbb{E}_x \left(f(X_{\tau_V}) \int_0^{\tau_V} g(X_t)\, dt \right) \qquad (x \in V).$$

Proof. Fix $x \in V$ and let $\tau = \tau_V$. Since $t + \tau \circ \theta_t = \tau$ on $\{t < \tau\}$, we have

$$X_\tau \circ \theta_t = X_\tau \ \text{ on } \{t < \tau\},$$

hence, by the (weak) Markov property,

$$\begin{aligned} K_V^M h(x) &= \mathbb{E}_x \left(\int_0^\tau g(X_t) h(X_t) dt \right) \\ &= \int_0^\infty \mathbb{E}_x \left(1_{\{t<\tau\}} g(X_t) \mathbb{E}_{X_t}(f \circ X_\tau) \right) dt \\ &= \int_0^\infty \mathbb{E}_x \left(1_{\{t<\tau\}} g(X_t) f \circ X_\tau \circ \theta_t \right) dt \\ &= \int_0^\tau \mathbb{E}_x \left(1_{\{t<\tau\}} g(X_t) f \circ X_\tau \right) dt = \mathbb{E}_x \left(f(X_\tau) \int_0^\tau g(X_t) dt \right), \end{aligned}$$

which was to be proved. ∎

3. The lower estimate

We are now ready to formulate our general lower estimate of perturbed Dirichlet solutions:

Theorem 3.1. *Let $M \in \mathcal{M}(\mathcal{H})$ and $V \in \mathcal{U}^M(\mathcal{H})$ such that $H_V 1 > 0$. Given $f \in C_b^+(X)$ let us denote $h := H_V f$ and $u := H_V^M f$. Then*

$$\frac{u}{h} \geq \exp\left\{-\frac{K_V^M h}{h}\right\} \qquad \text{on } \{h > 0\}. \tag{3.1}$$

Remark: For the case $M \in \mathcal{M}^+(\mathcal{H})$, this inequality was proved in [5, Proposition 1.9]. Being based on (2.4) our proof below is completely different and goes through regardless of the sign of M. We were inspired by [6, Proposition 2.5] to use the Feynman-Kac formula to get our estimate.

Proof. Let $\mathcal{X}$ be a Hunt process on X having the properties (i) and (ii) of Theorem 2.3, and let τ be the first exit time from V. Then, by Theorem 2.3,

$$h(x) = \mathbb{E}_x\left\{f(X_\tau)\right\}, \qquad u(x) = \mathbb{E}_x\left\{\exp\left(-\int_0^\tau g(X_t)dt\right) f(X_\tau)\right\}.$$

Let us introduce random variables

$$\xi = f(X_\tau), \qquad \eta = \int_0^\tau g(X_t)dt$$

so that

$$h(x) = \mathbb{E}_x(\xi), \qquad u(x) = \mathbb{E}_x\left\{e^{-\eta}\xi\right\}.$$

Using Jensen's inequality (see the following Lemma 3.2) we obtain that

$$\frac{u(x)}{h(x)} \geq \exp\left(-\frac{\mathbb{E}_x(\xi\eta)}{h(x)}\right). \tag{3.2}$$

It remains to observe that, by Proposition 2.4,

$$\mathbb{E}_x(\xi\eta) = \mathbb{E}_x\left(f(X_\tau)\int_0^\tau g(X_t)dt\right) = K_V^M h(x).$$

∎

Finally, let us show how to obtain (3.2).

Lemma 3.2. *For every $x \in V$ and for all real random variables ξ, η such that $\xi \geq 0$, $\mathbb{E}_x(\xi) > 0$, we have*

$$\mathbb{E}_x\left(e^{-\eta}\xi\right) \geq \mathbb{E}_x(\xi)\exp\left(-\frac{\mathbb{E}_x(\xi\eta)}{\mathbb{E}_x(\xi)}\right) \tag{3.3}$$

Proof. Consider the probability measure

$$\mathbb{Q} = \frac{\xi}{\mathbb{E}_x(\xi)}\mathbb{P}_x$$

on $(\Omega, \mathbb{M})$. By Jensen's inequality, we have

$$\frac{\mathbb{E}_x\left(e^{-\eta}\xi\right)}{\mathbb{E}_x(\xi)} = \int e^{-\eta}d\mathbb{Q} \geq \exp\left(-\int \eta d\mathbb{Q}\right) = \exp\left(-\frac{\int \eta\xi d\mathbb{P}_x}{\mathbb{E}_x(\xi)}\right) = \exp\left(-\frac{\mathbb{E}_x(\xi\eta)}{\mathbb{E}_x(\xi)}\right),$$

which was to be proved. ∎

Let us consider some particular cases of Theorem 3.1.

Corollary 3.3. *Let X be a Riemannian manifold and μ be a signed (local) Kato measure on X. Let V be a precompact open regular subset of X such that $\overline{V} \neq X$. Assume that $u \in C(\overline{V})$ solves in V the equation*

$$\Delta u - u\mu = 0, \tag{3.4}$$

and $h \in C(\overline{V})$ is a positive harmonic function in V such that $h|_{\partial V} = u|_{\partial V}$. Then, for any $x \in V$,

$$\frac{u(x)}{h(x)} \geq \exp\left(-\frac{\int_V G_V(x,y)h(y)d\mu(y)}{h(x)}\right). \tag{3.5}$$

Proof. Let $\mathcal{H}$ be the sheaf of harmonic functions on X. If the manifold X is non-parabolic, i.e., admits a global Green function $G(x,y)$ then $(X,\mathcal{H})$ is a $\mathcal{P}$-harmonic space. If X is parabolic then we will use the hypothesis $\overline{V} \neq X$ which excludes the situation when X is compact and V is dense in X. It is possible to prove that the Dirichlet Laplace operator in a non-dense precompact open subset U of X has a positive bottom of the spectrum, which implies the finiteness of the Green function G_U. Since V is not dense in X, there is a precompact open neighborhood U of $\overline{V}$ which is not dense in X either. Let us rename U by X so that X is now non-parabolic.

For any precompact open set $V \subset X$, we define the potential M_V on V by

$$M_V = \int_V G_V(\cdot, y)d\mu(y).$$

The perturbation M is the family of all potentials $\{M_V\}_{V\in\mathcal{U}_c}$. Then $\mathcal{H}^M$ is the sheaf of M-harmonic functions, i.e., the functions satisfying the Schrödinger equation (3.4), and the potential kernel K_V^M is defined by (1.3). Hence, (3.5) follows by Theorem 3.1. ∎

Let U be a bounded region in $\mathbb{R}^n$, which lies in the half-space $\{x \in \mathbb{R}^n : x_1 > 0\}$ and has a part of the boundary on the hyperplane $\{x \in \mathbb{R}^n : x_1 = 0\}$. Denote $\Gamma_0 = \partial U \cap \{x_1 = 0\}$ and $\Gamma_+ = \partial U \cap \{x_1 > 0\}$ and consider the following mixed boundary value problem in U

$$\begin{cases} \Delta u = 0 & \text{in } U\,, \\ u = f & \text{on } \Gamma_+\,, \\ \frac{\partial u}{\partial x_1} - qu = 0 & \text{on } \Gamma_0\,, \end{cases} \tag{3.6}$$

where q is a function on Γ_0. Denote by U^* the domain obtained from U by reflection at $\{x_1 = 0\}$ and let V be the set of all interior points of $\overline{U \cup U^*}$. Also, extend evenly the boundary function f to ∂V. Then we have the following lower bound for $u(x)$.

Corollary 3.4. *If V is regular and if $f \in C^+(\partial V)$ and $q \in C(\Gamma_0)$ then, for any $x \in U$,*

$$\frac{u(x)}{h(x)} \geq \exp\left(-\frac{2\int_{\Gamma_0} G_V(x,y)q(y)h(y)d\sigma_{\Gamma_0}(y)}{h(x)}\right), \tag{3.7}$$

where h solves the Dirichlet problem in V

$$\begin{cases} \Delta h = 0, \\ h|_{\partial V} = f, \end{cases}$$

and σ_{Γ_0} is the $(n-1)$-dimensional Lebesgue measure supported by Γ_0.

Proof. Extend evenly the function u to V. It is possible to prove that u solves the following boundary value problem in V (cf. [3, Section 6.6])

$$\begin{cases} \Delta u - u\,\mu = 0 \\ u|_{\partial V} = f\,, \end{cases} \tag{3.8}$$

where

$$\mu := 2q\sigma_{\Gamma_0}.$$

Since q is continuous on Γ_0 and σ_{Γ_0} is a Kato measure, we see that μ is also a Kato measure. Hence, (3.7) follows by Corollary 3.3. ∎

Observe that the estimate (3.7) gives a non-trivial result even if $f \equiv 1$, in which case $h \equiv 1$ and

$$u(x) \geq \exp\left(-2\int_{\Gamma_0} G_V(x,y)q(y)d\sigma_{\Gamma_0}(y)\right). \tag{3.9}$$

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