Heat kernels on metric measure spaces with regular volume growth

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Abstract

In this survey we study heat kernel estimates of self-similar type on metric measure spaces with regular volume growth. One of the main results is the dichotomy phenomenon in such estimates: either they are sub-Gaussian like in the setting of diffusions on fractals, or they have a polynomial tail as the symmetric stable processes in $\mathbb{R}^n$. Despite the probabilistic motivation, all the statements and proofs are completely analytic.

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Contents

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1 Introduction

The amount of research on heat kernels in geometric settings has increased dramatically in the past decades. It is impossible to give in one article an overview of recent development of the subject. The diversity of the problems related to heat kernels is well reflected in recent collections [4], [36], [39].

Already in a first Analysis course, one sees a special role of the exponential function $t \mapsto e^t$. It is not surprising that a far reaching generalization of the exponential function - an operator semigroup $\{e^{-t\mathcal{L}}\}_{t\geq 0}$, where $\mathcal{L}$ is a positive definite operator, plays similarly an important role in Analysis, Geometry, Probability and other related fields. If operator $\mathcal{L}$ acts in a function space then frequently the action of the semigroup $e^{-t\mathcal{L}}$ is given by an integral operator. The kernel of this operator is called *the heat kernel* of $\mathcal{L}$. Needless to say that any knowledge of the heat kernel, for example, upper and/or lower estimates, can help in solving various problems related to the operator $\mathcal{L}$ (see, for example, [5], [16]). If in addition the operator $\mathcal{L}$ is Markovian, that is, generates a Markov process (for example, this is the case when $\mathcal{L}$ is a second order elliptic differential operator), then one can use information about the heat kernel to answer questions about the process itself (see, for example, [30] and references therein). A resolution of the Poincaré conjecture by G.Perel'man can be viewed as a spectacular example of application of heat kernels in Geometry.

In this survey we touch only one aspect of the study of heat kernels: what kind of two sided estimates of self-similar type are possible for a heat kernel in a metric measure space? This question will be stated more precisely below, after a series of examples, and is largely motivated by a recent progress in Analysis on fractal spaces (see, for example, [6], [40]). The results surveyed here were proved in [31], [32], [33]. The purpose of this survey is to present a self-contained, streamlined account of these results with complete proofs.

1.1 Heat kernel in $\mathbb{R}^n$

The classical Gauss-Weierstrass heat kernel is the following function

$$p_t(x,y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right),$$

where $x, y \in \mathbb{R}^n$ and $t > 0$. This function is a fundamental solution of the heat equation

$$\frac{\partial u}{\partial t} = \Delta u,$$

where $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator. Moreover, if f is a continuous bounded function on $\mathbb{R}^n$ then the function

$$u(t,x) = \int_{\mathbb{R}^n} p_t(x,y) f(y)\, dy$$

solves the Cauchy problem

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u \\ u(0,x) = f(x) \end{cases}.$$

In the modern terms, this also can be written in the form

$$u(t,\cdot) = \exp(-t\mathcal{L}) f\ ,$$

where $\mathcal{L}$ here is a self-adjoint extension of $-\Delta$ in $L^2(\mathbb{R}^n)$ and $\exp(-t\mathcal{L})$ is understood in the sense of the functional calculus of self-adjoint operators. This means that $p_t(x,y)$ is the integral kernel of the operator $\exp(-t\mathcal{L})$.

The function $p_t(x,y)$ has also a probabilistic meaning: it is the transition density of the Brownian motion $\{X_t\}_{t\geq 0}$ in $\mathbb{R}^n$. The graph of $p_t(x,0)$ as a function of x is shown on Fig. 1.

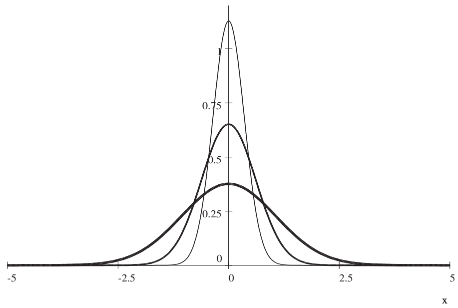

Figure 1: The Gauss-Weierstrass heat kernel at different values of t

The term $\frac{|x-y|^2}{t}$ determines the *space/time scaling*: if $|x-y|^2 \leq Ct$ then $p_t(x,y)$ is comparable with $p_t(y,y)$, that is, the probability density in the $C\sqrt{t}$-neighborhood of y is nearly constant.

1.2 Heat kernels on Riemannian manifolds

Let (M, g) be a connected Riemannian manifold, and Δ be the Laplace-Beltrami operator on M. Then the heat kernel $p_t(x, y)$ is can be defined as the integral kernel of the heat semigroup $\{\exp(-t\mathcal{L})\}_{t\geq 0}$ where $\mathcal{L}$ is the Dirichlet Laplace operator, that is, the minimal self-adjoint extension of $-\Delta$ in $L^2(M, \mu)$, and μ is the Riemannian volume. Alternatively, $p_t(x, y)$ is the minimal positive fundamental solution of the heat equation

$$\frac{\partial u}{\partial t} = \Delta u.$$

The function $p_t(x, y)$ can be used to define the Brownian motion $\{X_t\}_{t\geq 0}$ on M. Namely, $\{X_t\}_{t\geq 0}$ is a *diffusion process* (that is, a Markov process with continuous trajectories), such that

$$\mathbb{P}_x(X_t \in A) = \int_A p_t(x, y)\, d\mu(y)$$

for any Borel set $A \subset M$ (see Fig. 2).

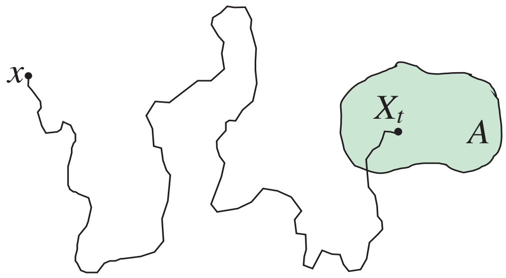

Figure 2: The Brownian motion X_t hits a set A

Let $d(x, y)$ be the geodesic distance on M. It turns out that the *Gaussian* type space/time scaling $\frac{d^2(x,y)}{t}$ appears in heat kernel estimates on general Riemannian manifolds:

1. (*Varadhan* [48]) For an arbitrary Riemannian manifold,

$$\log p_t(x, y) \sim -\frac{d^2(x, y)}{4t} \quad \text{as } t \to 0,$$

2. (*Li & Yau* [42]) If (M, g) is a complete manifold with $Ric \geq 0$ then

$$p_t(x, y) \asymp \frac{C}{V\left(x, \sqrt{t}\right)} \exp\left(-c\frac{d^2(x, y)}{t}\right),$$

   where $V(x, r) = \mu(B(x, r))$, $B(x, r)$ being a geodesic ball, c, C are positive constants, and the sign $\asymp$ means that both $\leq$ and $\geq$ take place, but possibly with different values of C and c.

3. (*Davies* [21]) For an arbitrary manifold M, for any two measurable sets $A, B \subset M$

$$\int_A \int_B p_t(x, y)\, d\mu(x)\, d\mu(y) \leq \sqrt{\mu(A)\, \mu(B)} \exp\left(-\frac{d^2(A, B)}{4t}\right)$$

4. ($Grigor'yan$ [26]) For an arbitrary manifold M, if for two points $x, y \in M$ and all $t > 0$

$$p_t(x,x) \le f_1(t) \text{ and } p_t(y,y) \le f_2(t)$$

where f_1 and f_2 functions with some regularity properties then, for all $t > 0$,

$$p_t(x,y) \le C\sqrt{f_1(Ct)\, f_2(Ct)} \exp\left(-c\frac{d^2(x,y)}{t}\right).$$

Technically, all these results depend upon the following property of the geodesic distance: $|\nabla d| \le 1$.

It is natural to ask the following question:

Are there settings where the space/time scaling is different from Gaussian?

1.3 Heat kernels of fractional powers of Laplacian

Easy examples can be constructed using another operator instead of the Laplacian. As above, let $\mathcal{L}$ be the Dirichlet Laplace operator on a Riemannian manifold M, and consider the evolution equation

$$\frac{\partial u}{\partial t} + \mathcal{L}^{\beta/2} u = 0,$$

where $\beta \in (0,2)$. The operator $\mathcal{L}^{\beta/2}$ is understood in the sense of the functional calculus in $L^2(M,\mu)$. Let $p_t(x,y)$ be now the heat kernel of $\mathcal{L}^{\beta/2}$, that is, the integral kernel of $\exp\left(-t\mathcal{L}^{\beta/2}\right)$.

The condition $\beta < 2$ leads to the fact that the semigroup $\exp\left(-t\mathcal{L}^{\beta/2}\right)$ is *Markovian*, which, in particular, means that $p_t(x,y) > 0$ (if $\beta > 2$ then $p_t(x,y)$ may be signed). Using the techniques of subordinators, developed in the theory of Markov processes, one obtains the following estimate for the heat kernel of $\mathcal{L}^{\beta/2}$ in $\mathbb{R}^n$ (see [13] or Section 4.3 below):

$$p_t(x,y) \asymp \frac{C}{t^{n/\beta}}\left(1 + \frac{|x-y|}{t^{1/\beta}}\right)^{-(n+\beta)} \asymp \frac{C}{t^{n/\beta}}\left(1 + \frac{|x-y|^\beta}{t}\right)^{-\frac{n+\beta}{\beta}}. \tag{1.1}$$

The heat kernel of $\sqrt{\mathcal{L}}$ in $\mathbb{R}^n$ (that is, the case $\beta = 1$) is known explicitly:

$$p_t(x,y) = \frac{c_n}{t^n}\left(1 + \frac{|x-y|^2}{t^2}\right)^{-\frac{n+1}{2}} = \frac{c_n t}{\left(t^2 + |x-y|^2\right)^{\frac{n+1}{2}}},$$

where $c_n = \Gamma\left(\frac{n+1}{2}\right)/\pi^{(n+1)/2}$ (which is nothing other but the Poisson kernel in the half-space $\mathbb{R}^{n+1}_+$, or the density of the Cauchy distribution in $\mathbb{R}^n$ with the parameter t).

As we see, the space/time scaling is given by the term $\frac{d^\beta(x,y)}{t}$ where $\beta < 2$. The heat kernel of the operator $\mathcal{L}^{\beta/2}$ coincides with the transition density of a *symmetric stable process of index* β, which belongs to the family of Levy processes. The trajectories of this process are discontinuous, thus allowing jumps. The heat kernel $p_t(x,y)$ of such process is nearly constant in a $Ct^{1/\beta}$-neighborhood of y. If t is large then

$$t^{1/\beta} \gg t^{1/2},$$

that is, this neighborhood is much larger than that for the diffusion process, which is not surprising because of the presence of jumps. The space/time scaling with $\beta < 2$ is called *super-Gaussian.*

1.4 Heat kernels on fractal spaces

A rich family of heat kernels for diffusion processes has come from Analysis on *fractals*. The notion of a fractal has appeared in Physics as a model for disordered media (cf. [43]). Mathematically, fractals are sets in $\mathbb{R}^n$ with certain self-similarity properties. On Fig. 3, the reader can see the first three steps of the construction procedure of a fractal set called *the Sierpinski gasket* (SG), which is similar to the construction of the Cantor set:

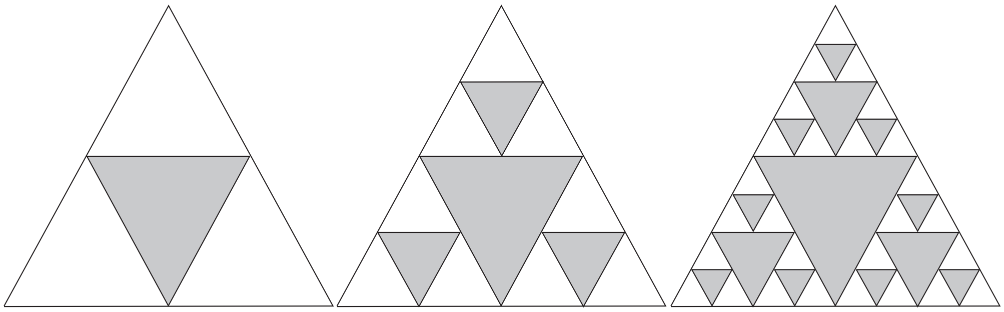

Figure 3: Construction of the Sierpinski gasket: one starts with a triangle as a closed subset of $\mathbb{R}^2$, then the open middle triangle is eliminated (shaded on the diagram), and similar procedures repeat for the remaining triangles, etc.

Hence, SG is a compact connected subset of $\mathbb{R}^2$. Using a similar procedure for expanding (see Fig. 4) one obtains the *unbounded* SG.

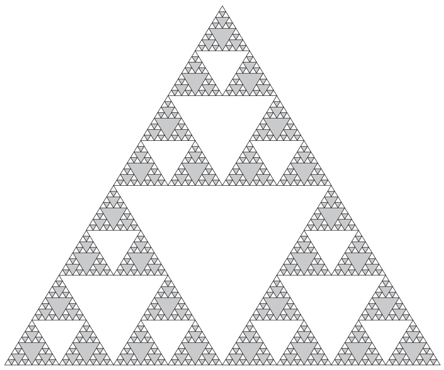

Figure 4: The unbounded SG is obtained from SG by merging the latter (at the left lower corner of the diagram) with two shifted copies and then by repeating this procedure at larger scales.

We refer the reader to [6], [23], [40] for the details of the construction of fractals.

Barlow and Perkins [12], Goldstein [25] and Kusuoka [41] have constructed by different methods a natural diffusion process on SG that has the same self-similarity as SG. Barlow and Perkins [12] considered random walks on the graph approximations of SG and showed that, with an appropriate scaling, the random walks converge to a diffusion process. Moreover, they proved that this process has a transition density $p_t(x, y)$ with respect to a proper Hausdorff measure μ of SG, and that p_t satisfies the following estimate:

$$p_t(x,y) \asymp \frac{C}{t^{\alpha/\beta}} \exp\left(-c\left(\frac{d^\beta(x,y)}{t}\right)^{\frac{1}{\beta-1}}\right), \tag{1.2}$$

where $d(x,y) = |x-y|$ and

$$\alpha = \dim_H SG = \frac{\log 3}{\log 2}, \qquad \beta = \frac{\log 5}{\log 2} > 2.$$

Similar estimates were proved by Barlow and Bass [8], [9], [10] for a large family of various fractals, and the parameters α and β in (1.2) are determined by the intrinsic properties of the fractal. In all cases, α is the Hausdorff dimension (which is also called the *fractal dimension*). The parameter β, that is called the *walk dimension*, is larger than 2 in all interesting examples.

In the case $\beta > 2$, the space/time scaling is called *sub-Gaussian;* the heat kernel $p_t(x,\cdot)$ is nearly constant in $Ct^{1/\beta}$-neighborhood of x and, for large t,

$$t^{1/\beta} \ll t^{1/2}.$$

The physicists call such a diffusion *anomalous* thus emphasizing that the estimates of the kind (1.2) are features of very specific singular spaces. However, surprisingly enough, (1.2) with $\beta > 2$ can occur on Riemannian manifolds, although for a restricted range of time t. Indeed, consider the *Sierpinski graph*, which is obtained similarly to the Sierpinski gasket, but using enlargement instead of shrinking (see Fig. 5).

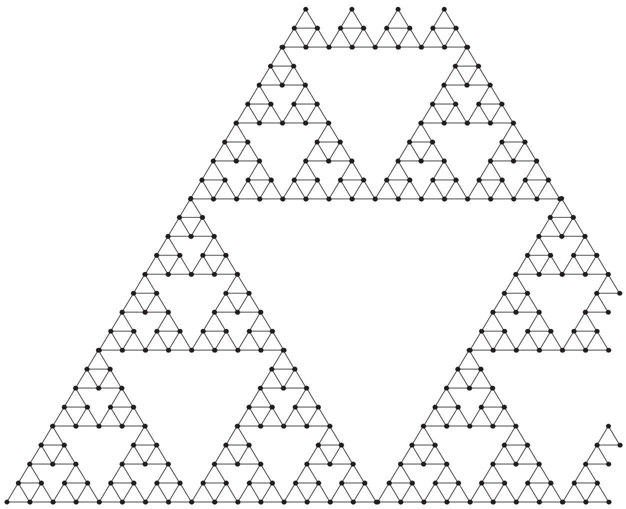

Figure 5: The Sierpinski graph

It is not difficult to believe that the heat kernel $p_t(x,y)$ of the simple random walk on this graph (where t is now an integer) admits the estimate (1.2) in the range

$$t \geq \max(1, d(x,y))$$

(see [35], [37]). Now make the Sierpinski graph into a manifold by blowing up the edges. Then the heat kernel $p_t(x,y)$ for the Laplace-Beltrami operator on this manifold also admit the sub-Gaussian estimate (1.2) for the above range of time. In the opposite case

$$t < \max(1, d(x,y))$$

the heat kernel satisfies the Gaussian estimate (cf. [11]).

1.5 Summary of examples

Observe now that in all the above examples, the heat kernel estimates can be unified in one formula as follows:

$$p_t(x,y) \asymp \frac{C}{t^{\alpha/\beta}} \Phi\left(c\frac{d(x,y)}{t^{1/\beta}}\right), \tag{1.3}$$

where α, β are positive parameters and $\Phi(s)$ is a positive decreasing function on $[0,+\infty)$. For example, for the Gauss-Weierstrass function we have $\alpha = n$, $\beta = 2$ and

$$\Phi(s) = \exp\left(-s^2\right).$$

For diffusions on fractals, we have $\beta > 2$ and

$$\Phi(s) = \exp\left(-s^{\frac{\beta}{\beta-1}}\right).$$

For the symmetric stable process of the index $\beta \in (0,2)$ in $\mathbb{R}^n$, we have

$$\Phi(s) = (1+s)^{-(\alpha+\beta)},$$

where $\alpha = n$.

In this survey we focus on the following question:

What values of the parameters α, β and what functions Φ can actually occur in the estimate (1.3)?

The answer (Theorem 6.7) will be given in the setting of metric measure spaces that will be described in the next section.

2 Abstract heat kernels

Let (M,d) be a locally compact separable metric space and μ be a Radon measure on M with full support. The triple (M,d,μ) will be called a *metric measure space.*

2.1 Basic definitions

Definition 2.1 *A family $\{p_t\}_{t>0}$ of measurable functions $p_t(x,y)$ on $M \times M$ is called a* heat kernel *if the following conditions are satisfied, for μ-almost all $x, y \in M$ and all $s, t > 0$:*

(i) Positivity: $p_t(x,y) \geq 0$.

(ii) The total mass inequality:

$$\int_M p_t(x,y)d\mu(y) \leq 1. \tag{2.1}$$

(iii) Symmetry: $p_t(x,y) = p_t(y,x)$.

(iv) The semigroup property:

$$p_{s+t}(x,y) = \int_M p_s(x,z)p_t(z,y)d\mu(z). \tag{2.2}$$

(v) *Approximation of identity: for any* $f \in L^2 := L^2(M,\mu)$,

$$\int_M p_t(x,y) f(y) d\mu(y) \xrightarrow{L^2} f(x) \quad as\ t \to 0+. \tag{2.3}$$

If in addition we have, for μ-a.a. $x \in M$ and all $t > 0$,

$$\int_M p_t(x,y) d\mu(y) = 1 \tag{2.4}$$

then the heat kernel p_t is called *stochastically complete* (or *conservative*).

Any heat kernel gives rise to the *heat semigroup* $\{P_t\}_{t \geq 0}$ where $P_0 = \mathrm{id}$ and P_t for $t > 0$ is the operator in L^2 defined by

$$P_t f(x) = \int_M p_t(x,y) f(y) d\mu(y). \tag{2.5}$$

It follows from $(i)-(ii)$ that the operator P_t is *Markovian*, that is, $f \geq 0$ implies $P_t f \geq 0$ and $f \leq 1$ implies $P_t f \leq 1$. It follows that P_t is a bounded operator in L^2 and, moreover, is a contraction, that is, $\|P_t\|_{L^2 \to L^2} \leq 1$. The symmetry property (iii) implies that the operator P_t is *symmetric* and, hence, self-adjoint. The semigroup property (iv) implies that $P_t P_s = P_{t+s}$, that is, the family $\{P_t\}_{t \geq 0}$ is a *semigroup.* It follows from (v) that

$$s\text{-}\lim_{t \to 0} P_t = \mathrm{id} = P_0$$

where s-lim stands for the *strong* limit. Hence, $\{P_t\}_{t \geq 0}$ is a strongly continuous, symmetric, Markovian semigroup in L^2. Conversely, if $\{P_t\}$ is such a semigroup and if it has the integral kernel $p_t(x,y)$ then the latter is a heat kernel in the sense of Definition 2.1.

Given a strongly continuous, symmetric, Markovian semigroup P_t in L^2, define the *infinitesimal generator* $\mathcal{L}$ of the semigroup by

$$\mathcal{L} f := \lim_{t \to 0} \frac{f - P_t f}{t}, \tag{2.6}$$

where the limit is understood in the L^2-norm. The *domain* $\mathrm{dom}(\mathcal{L})$ of the generator $\mathcal{L}$ is the space of functions $f \in L^2$ for which the limit in (2.6) exists. By the Hille–Yosida theorem, $\mathrm{dom}(\mathcal{L})$ is dense in L^2. Furthermore, $\mathcal{L}$ is a self-adjoint, positive definite operator, which immediately follows from the fact that the semigroup $\{P_t\}$ is self-adjoint and Markovian. Moreover, we have

$$P_t = \exp(-t\mathcal{L}), \tag{2.7}$$

where the right hand side is understood in the sense of spectral theory.

Heat kernels arise naturally from Markov processes. Let $\left(\{X_t\}_{t \geq 0}, \{\mathbb{P}_x\}_{x \in M}\right)$ be a reversible Markov process on M, and assume that it has the transition density $p_t(x,y)$, that is, a function such that, for all $x \in M$, $t > 0$, and all Borel sets $A \subset M$,

$$\mathbb{P}_x(X_t \in A) = \int_M p_t(x,y) d\mu(y).$$

Then $p_t(x,y)$ is a heat kernel in the sense of Definition 2.1. Furthermore, in this case all the properties $(i)-(iv)$ are satisfied for all $x, y \in M$ rather than for *almost* all.

All examples of heat kernels considered in Introduction, satisfy Definition 2.1. Now we can specify our main question as follows:

Let $p_t(x,y)$ be a heat kernel on a metric measure space (M,d,μ) and assume that it satisfies for all $t>0$ and μ-a.a. $x,y\in M$ the estimate

$$p_t(x,y) \asymp \frac{C}{t^{\alpha/\beta}}\Phi\left(c\frac{d(x,y)}{t^{1/\beta}}\right). \tag{2.8}$$

What values of the parameters α,β and what function Φ can actually occur in this estimate?

The answer will be given in Theorem 6.7. Before we embark on the study of this problem, let us show some simple examples of stochastically complete heat kernels that do not satisfy (2.8).

Example 2.2 *(A frozen heat kernel)* Let M be a countable set and let $\{x_k\}_{k=1}^{\infty}$ be the sequence of all distinct points from M. Let $\{\mu_k\}_{k=1}^{\infty}$ be a sequence of reals and define measure μ on M by $\mu(\{x_k\})=\mu_k$. Define a function $p_t(x,y)$ on $M\times M$ by

$$p_t(x,y)=\begin{cases}\frac{1}{\mu_k}, & x=y=x_k\\ 0, & \text{otherwise.}\end{cases}$$

We claim that $p_t(x,y)$ is a stochastically complete heat kernel. We call it frozen because it does not depend on time t. For example, let us check the approximation of identity: for any function $f\in L^2(M,\mu)$, we have

$$P_tf(x)=\int_M p_t(x,y)f(y)\,d\mu(y)=p_t(x,x)f(x)\mu(\{x\})=f(x),$$

whence the claim follows.

The Markov process associated with the frozen heat kernel is very simple: $X_t=X_0$ for all $t\geq 0$ so that it is a frozen diffusion.

Example 2.3 *(The heat kernel in $\mathbb{H}^3$)* The heat kernel of the Laplace-Beltrami operator on the 3-dimensional hyperbolic space $\mathbb{H}^3$ is given by the formula

$$p_t(x,y)=\frac{1}{(4\pi t)^{3/2}}\frac{r}{\sinh r}\exp\left(-\frac{r^2}{4t}-t\right), \tag{2.9}$$

where $r=d(x,y)$ is the geodesic distance between x,y. See [15], [20], [29].

Example 2.4 *(The Mehler heat kernel)* Let $M=\mathbb{R}$, measure μ be defined by

$$d\mu=e^{x^2}dx,$$

and the operator $\mathcal{L}$ be given by

$$\mathcal{L}=-e^{-x^2}\frac{d}{dx}\left(e^{x^2}\frac{d}{dx}\right)=-\frac{d^2}{dx^2}-2x\frac{d}{dx}.$$

Then the heat kernel of $\mathcal{L}$ is given by the formula

$$p_t(x,y)=\frac{1}{(2\pi\sinh 2t)^{1/2}}\exp\left(\frac{2xye^{-2t}-x^2-y^2}{1-e^{-4t}}-t\right) \tag{2.10}$$

Similarly, for the measure

$$d\mu = e^{-x^2}dx$$

and for the operator

$$\mathcal{L} = e^{x^2}\frac{d}{dx}\left(e^{-x^2}\frac{d}{dx}\right) = -\frac{d^2}{dx^2} + 2x\frac{d}{dx},$$

we have

$$p_t(x,y) = \frac{1}{(2\pi\sinh 2t)^{1/2}}\exp\left(\frac{2xye^{-2t} - \left(x^2+y^2\right)e^{-4t}}{1-e^{-4t}} + t\right).$$

See [19, p.181] and [29].

2.2 The Dirichlet form

Given a heat kernel $\{p_t\}$ on a metric measure space (M,d,μ), define for any $t>0$ a quadratic form $\mathcal{E}_t$ on L^2 by

$$\mathcal{E}_t[u] := \left(\frac{u-P_tu}{t}, u\right), \tag{2.11}$$

where $(\cdot,\cdot)$ is the inner product in L^2. An easy computation shows that $\mathcal{E}_t$ can be equivalently defined by

$$\begin{aligned}\mathcal{E}_t[u] &= \frac{1}{2t}\int_M\int_M |u(x)-u(y)|^2\, p_t(x,y)d\mu(y)d\mu(x) \\ &\quad+\frac{1}{t}\int_M (1-P_t1(x))\, u^2(x)d\mu(x).\end{aligned} \tag{2.12}$$

Indeed, by (2.5) we have

$$\begin{aligned}u(x)-P_tu(x) &= u(x)\,P_t1(x) - P_tu(x) + (1-P_t1(x))\,u(x) \\ &= \int_M (u(x)-u(y))\,p_t(x,y)d\mu(y) + (1-P_t1(x))\,u(x)\end{aligned}$$

whence by (2.11)

$$\begin{aligned}\mathcal{E}_t[u] &= \frac{1}{t}\int_M\int_M (u(x)-u(y))\,u(x)p_t(x,y)d\mu(y)d\mu(x) \\ &\quad+\frac{1}{t}\int_M (1-P_t1(x))\,u^2(x)d\mu(x).\end{aligned} \tag{2.13}$$

Interchanging the variables x and y in the first integral and using the symmetry of the heat kernel, we obtain also

$$\begin{aligned}\mathcal{E}_t[u] &= \frac{1}{t}\int_M\int_M (u(y)-u(x))\,u(y)p_t(x,y)d\mu(y)d\mu(x) \\ &\quad+\frac{1}{t}\int_M (1-P_t1(x))\,u^2(x)d\mu(x),\end{aligned} \tag{2.14}$$

and (2.12) follows by adding up (2.13) and (2.14).

Note that by (2.1) $P_t 1 \leq 1$ so that the second term in the right hand side of (2.12) is non-negative. If the heat kernel is stochastically complete, that is, $P_t 1 = 1$, then that term vanishes and we obtain

$$\mathcal{E}_t[u] = \frac{1}{2t} \int_M \int_M |u(x) - u(y)|^2 p_t(x,y) d\mu(y) d\mu(x). \tag{2.15}$$

In terms of the spectral resolution $\{E_\lambda\}$ of the generator $\mathcal{L}$, $\mathcal{E}_t$ can be expressed as follows

$$\mathcal{E}_t[u] = \int_0^\infty \frac{1 - e^{-t\lambda}}{t} d\|E_\lambda u\|_2^2,$$

which implies that $\mathcal{E}_t[u]$ is decreasing in t (indeed, this is an elementary exercise to show that the function $t \mapsto \frac{1-e^{-t\lambda}}{t}$ is decreasing).

Let us define a quadratic form $\mathcal{E}$ by

$$\mathcal{E}[u] := \lim_{t \to 0+} \mathcal{E}_t[u] = \int_0^\infty \lambda d\|E_\lambda u\|_2^2 \tag{2.16}$$

(where the limit may be $+\infty$ since $\mathcal{E}[u] \geq \mathcal{E}_t[u]$) and its *domain* $\mathcal{F} = \mathrm{dom}(\mathcal{E})$ by

$$\mathcal{F} := \{u \in L^2 : \mathcal{E}[u] < \infty\}.$$

It is clear from (2.12) and (2.16) that $\mathcal{E}_t$ and $\mathcal{E}$ are positive definite.

It is easy to see from (2.16) that $\mathcal{F} = \mathrm{dom}(\mathcal{L}^{1/2})$. It will be convenient for us to use the following notation:

$$\mathrm{dom}_{\mathcal{E}}(\mathcal{L}) := \mathcal{F} = \mathrm{dom}(\mathcal{L}^{1/2}). \tag{2.17}$$

The domain $\mathcal{F}$ is dense in L^2 because $\mathcal{F}$ contains $\mathrm{dom}(\mathcal{L})$. Indeed, if $u \in \mathrm{dom}(\mathcal{L})$ then using (2.6) and (2.11), we obtain

$$\mathcal{E}[u] = \lim_{t \to 0} \mathcal{E}_t[u] = (\mathcal{L}u, u) < \infty. \tag{2.18}$$

The quadratic form $\mathcal{E}[u]$ extends to a bilinear form $\mathcal{E}(u,v)$ by the polarization identity

$$\mathcal{E}(u,v) = \frac{1}{2}\left(\mathcal{E}[u+v] - \mathcal{E}[u-v]\right).$$

It follows from (2.18) that $\mathcal{E}(u,v) = (\mathcal{L}u, v)$ for all $u, v \in \mathrm{dom}(\mathcal{L})$.

The space $\mathcal{F}$ is naturally endowed with the inner product

$$\mathcal{E}_1(u,v) := (u,v) + \mathcal{E}(u,v). \tag{2.19}$$

It is possible to show that the form $\mathcal{E}$ is *closed*, that is, the space $\mathcal{F}$ is *Hilbert*.

The fact that P_t is Markovian implies that the form $\mathcal{E}$ satisfies the *Markov property*: if $u \in \mathcal{F}$ then $\widetilde{u} := \min(u_+, 1) \in \mathcal{F}$ and $\mathcal{E}[\widetilde{u}] \leq \mathcal{E}[u]$. Hence, $\mathcal{E}$ is a *Dirichlet form* (see [24]).

Definition 2.5 The form $(\mathcal{E}, \mathcal{F})$ is called *local* if $\mathcal{E}(u,v) = 0$ whenever the functions $u, v \in \mathcal{F}$ have compact disjoint supports. The form $(\mathcal{E}, \mathcal{F})$ is called *strongly local* if $\mathcal{E}(u,v) = 0$ whenever the functions $u, v \in \mathcal{F}$ have compact supports and $u \equiv \mathrm{const}$ in an open neighborhood of $\mathrm{supp}\, v$.

For example, if $p_t(x,y)$ is the heat kernel of the Laplace-Beltrami operator on a complete Riemannian manifold then

$$\mathcal{E}(u,v)=\int_M (\nabla u,\nabla v)\, d\mu$$

and $\mathcal{F}$ is the Sobolev space $W_2^1(M)$. Note that this Dirichlet form is strongly local because $u=\text{const}$ on $\operatorname{supp} v$ implies $\nabla u=0$ on $\operatorname{supp} u$ and, hence, $\mathcal{E}(u,v)=0$.

If $p_t(x,y)$ is the heat kernel of the symmetric stable process of index β in $\mathbb{R}^n$, that is, $\mathcal{L}=(-\Delta)^{\beta/2}$, then

$$\mathcal{E}(u,v)=c_{n,\beta}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+\beta}}dxdy,$$

and $\mathcal{F}$ is the Besov space $B_{2,2}^{\beta/2}(\mathbb{R}^n)$. This form is non-local.

Denote by $C_0(M)$ the space of continuous functions on M with compact supports, endowed with sup-norm.

Definition 2.6 The form $(\mathcal{E},\mathcal{F})$ is called *regular* if $\mathcal{F}\cap C_0(M)$ is dense both in $\mathcal{F}$ and in $C_0(M)$.

All the Dirichlet forms in the above examples are regular.

2.3 Identifying Φ in the non-local case

Fix two positive parameters α and β and a monotone decreasing function $\Phi:[0,+\infty)\to[0,+\infty)$.

Lemma 2.7 ([33]) *Assume that $\{p_t\}$ is a heat kernel on (M,d,μ) such that, for all $t>0$ and almost all $x,y\in M$,*

$$p_t(x,y)\leq\frac{1}{t^{\alpha/\beta}}\Phi\left(\frac{d(x,y)}{t^{1/\beta}}\right). \tag{2.20}$$

Then either the associated Dirichlet form $\mathcal{E}$ is local or

$$\Phi(s)\geq c(1+s)^{-(\alpha+\beta)} \tag{2.21}$$

for all $s>0$ and some $c>0$.

Proof. Consider the bilinear form $\mathcal{E}_t$ on $L^2(M,\mu)$, which is given by

$$\begin{aligned}\mathcal{E}_t(u,v) &= \frac{1}{2t}\int_M\int_M (u(x)-u(y))(v(x)-v(y))\,p_t(x,y)d\mu(y)d\mu(x) &(2.22)\\ &+\frac{1}{t}\int_M(1-P_t1(x))\,u(x)v(x)d\mu(x) &(2.23)\end{aligned}$$

(cf. (2.12)). Let $u,v\in L^2(M,\mu)$ be two non-negative functions with compact disjoint supports $A=\operatorname{supp} u$ and $B=\operatorname{supp} v$, and set

$$r=d(A,B)>0$$

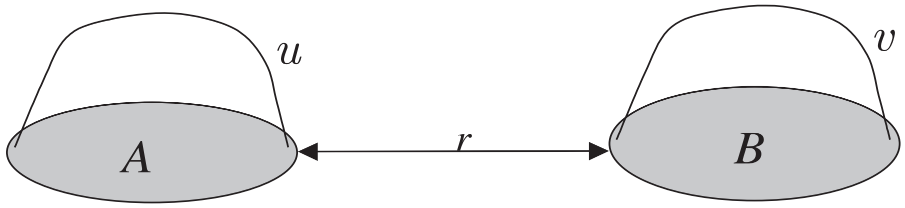

Figure 6: Functions u and v with disjoint supports

(see Fig. 6).

The integral (2.23) is clearly equal to 0. The integrand in (2.22) vanishes if either both x, y are outside A or both x, y are outside B. Hence, we can restrict the integration to the domain where one of the variables x, y is in A and the other is in B. Hence, we obtain, using the symmetry of the heat kernel,

$$\begin{aligned}\mathcal{E}_t(u,v) &= -\frac{1}{2t}\int_A\int_B u(x)v(y)\,p_t(x,y)d\mu(y)d\mu(x) \\ &\quad -\frac{1}{2t}\int_B\int_A u(y)v(x)\,p_t(x,y)d\mu(y)d\mu(x) \\ &= -\frac{1}{t}\int_A\int_B u(x)v(y)\,p_t(x,y)d\mu(y)d\mu(x). \end{aligned} \tag{2.24}$$

If $x \in A$ and $y \in B$ then $d(x,y) \geq r$. Therefore, for almost all $x \in A$ and $y \in B$,

$$p_t(x,y) \leq \frac{1}{t^{\alpha/\beta}}\Phi\left(\frac{r}{t^{1/\beta}}\right),$$

which together with (2.24) implies

$$|\mathcal{E}_t(u,v)| \leq \frac{1}{t^{1+\alpha/\beta}}\Phi\left(\frac{r}{t^{1/\beta}}\right)\|u\|_{L^1}\|v\|_{L^1} \tag{2.25}$$

(note that $\|u\|_{L^1} \leq \mu(A)^{1/2}\|u\|_{L^2} < \infty$ and the same holds for v). If (2.21) fails then there exists a sequence $\{s_k\} \to \infty$ such that

$$s_k^{\alpha+\beta}\Phi(s_k) \to 0 \quad \text{as } k \to \infty.$$

Define a sequence $\{t_k\}$ from the condition

$$s_k = \frac{r}{t_k^{1/\beta}}.$$

Then

$$s_k^{\alpha+\beta}\Phi(s_k) = \frac{r^{\alpha+\beta}}{t_k^{1+\alpha/\beta}}\Phi\left(\frac{r}{t_k^{1/\beta}}\right) \to 0 \quad \text{as } k \to \infty,$$

and (2.25) implies that

$$\mathcal{E}_{t_k}(u,v) \to 0 \quad \text{as } k \to \infty. \tag{2.26}$$

If in addition $u, v \in \mathcal{F}$ then, by (2.16) and (2.26),

$$\mathcal{E}(u, v) = \lim_{k \to \infty} \mathcal{E}_{t_k}(u, v) = 0,$$

whence the locality of $\mathcal{E}$ follows. ■

Lemma 2.8 ([33]) *Assume that $\{p_t\}$ is a heat kernel on (M, d, μ) such that, for all $t > 0$ and almost all $x, y \in M$,*

$$p_t(x, y) \geq \frac{1}{t^{\alpha/\beta}} \Phi\left(\frac{d(x, y)}{t^{1/\beta}}\right). \tag{2.27}$$

Then

$$\Phi(s) \leq C(1 + s)^{-(\alpha+\beta)} \tag{2.28}$$

for all $s > 0$ and some $C > 0$.

Proof. Let u be a non-constant function from $L^2(M, \mu)$. Choose a ball $Q \subset M$ where u is non-constant and let $a > b$ be two real values such that the sets

$$A = \{x \in Q : u(x) \geq a\} \text{ and } B = \{x \in Q : u(x) \leq b\}$$

have positive measures (see Fig. 7).

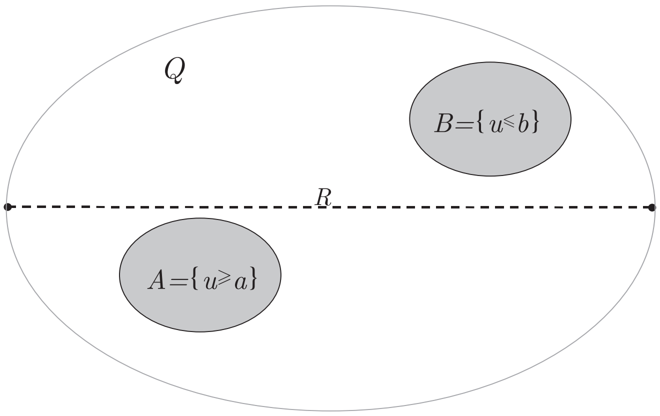

Figure 7: Sets A and B

If $R = \operatorname{diam} Q$ then, by (2.27), we have, for almost all $x, y \in Q$,

$$p_t(x, y) \geq \frac{1}{t^{\alpha/\beta}} \Phi\left(\frac{R}{t^{1/\beta}}\right),$$

whence by (2.12)

$$\begin{aligned} \mathcal{E}[u] &\geq \mathcal{E}_t[u] \geq \frac{1}{2t} \int_A \int_B (u(x) - u(y))^2 p_t(x, y) d\mu(y) d\mu(x) \\ &\geq (a - b)^2 \mu(A) \mu(B) \frac{1}{2t^{1+\alpha/\beta}} \Phi\left(\frac{R}{t^{1/\beta}}\right) \\ &= \frac{c'}{t^{1+\alpha/\beta}} \Phi\left(\frac{R}{t^{1/\beta}}\right), \end{aligned} \tag{2.29}$$

where $c' > 0$. If (2.28) fails then there exists a sequence $\{s_k\} \to \infty$ such that

$$s_k^{\alpha+\beta} \Phi(s_k) \to \infty \quad \text{as } k \to \infty. \tag{2.30}$$

Define a sequence $\{t_k\}$ from the condition

$$s_k = \frac{R}{t_k^{1/\beta}}.$$

Then

$$\frac{1}{t_k^{1+\alpha/\beta}} \Phi\left(\frac{R}{t_k^{1/\beta}}\right) = R^{-(\alpha+\beta)} s_k^{\alpha+\beta} \Phi(s_k) \to \infty \quad \text{as } k \to \infty,$$

and (2.29) yields $\mathcal{E}(u,u) = \infty$.

Hence, we have arrived at the conclusion that the domain of the form $\mathcal{E}$ contains only constants. Since $\mathcal{F}$ is dense in $L^2(M,\mu)$, it follows that $L^2(M,\mu)$ consists only of constants. Hence, there is a point $x \in M$ with a positive mass, that is, $\mu(\{x\}) > 0$. Then (2.1) implies that, for all $t > 0$,

$$p_t(x,x) \leq \frac{1}{\mu(\{x\})}. \tag{2.31}$$

On the other hand, (2.30) implies $\Phi(0) > 0$, whence by (2.27) $p_t(x,x) \to \infty$ as $t \to 0$, which contradicts (2.31). ■

Remark 2.9 *The last argument in the above proof can be stated as follows. If (2.27) holds with a function Φ such that $\Phi(0) > 0$, then $\mu(\{x\}) = 0$ for all $x \in M$. This simple observation will also be used below.*

Corollary 2.10 *Assume that the following estimate holds for all $t > 0$ and almost all $x, y \in M$:*

$$p_t(x,y) \asymp \frac{C}{t^{\alpha/\beta}} \Phi\left(c \frac{d(x,y)}{t^{1/\beta}}\right). \tag{2.32}$$

Then either the Dirichlet form $\mathcal{E}$ is local or

$$\Phi(s) \simeq (1+s)^{-(\alpha+\beta)}. \tag{2.33}$$

Proof. Indeed, if $\mathcal{E}$ is non-local then, by Lemmas 2.7 and 2.8, the function Φ must satisfy (2.21) and (2.28), whence (2.33) follows. ■

2.4 Volume of balls

Let

$$B(x,r) := \{y \in M : d(x,y) < r\}$$

be the metric ball in (M,d) of radius r centered at the point $x \in M$.

Theorem 2.11 ([32]) *Let p_t be a heat kernel on a metric measure space (M,d,μ). Let α, β be positive constants and Φ_1, Φ_2 be monotone decreasing functions from $[0,+\infty)$ to $[0,+\infty)$ such that $\Phi_1(s) > 0$ for some $s > 0$, and*

$$\int_0^\infty s^\alpha \Phi_2(s) \frac{ds}{s} < \infty. \tag{2.34}$$

(*a*) *If for μ-almost all $x, y \in M$ and all $t > 0$*

$$p_t(x,y) \geq \frac{1}{t^{\alpha/\beta}} \Phi_1 \left(\frac{d(x,y)}{t^{1/\beta}} \right) \tag{2.35}$$

then, for all $x \in M$ and $r > 0$,

$$\mu(B(x,r)) \leq C r^{\alpha}. \tag{2.36}$$

(*b*) *If $p_t(x,y)$ is stochastically complete and, for μ-almost all $x, y \in M$ and all $t > 0$,*

$$\frac{1}{t^{\alpha/\beta}} \Phi_1 \left(\frac{d(x,y)}{t^{1/\beta}} \right) \leq p_t(x,y) \leq \frac{1}{t^{\alpha/\beta}} \Phi_2 \left(\frac{d(x,y)}{t^{1/\beta}} \right) \tag{2.37}$$

then, for all $x \in M$ and $r > 0$,

$$\mu(B(x,r)) \simeq r^{\alpha}. \tag{2.38}$$

Proof. (*a*) Fix $r, t > 0$ and consider the following integral

$$\int_{B(x,r)} p_t(x,y)\, d\mu(y) := \int_M p_t(x,y)\, \mathbf{1}_{B(x,r)}(y)\, d\mu(y).$$

This definition is not entirely trivial as it requires the following justification. Indeed, the function $F(x,y) = p_t(x,y)\, \mathbf{1}_{B(x,r)}(y)$ is measurable jointly in x, y so that by Fubini's theorem the integral $\int_M F(x,y)\, d\mu(y)$ is well-defined for μ-a.a. $x \in M$ and is a measurable function of x.

For a fixed $t > 0$ (the value of $t = t(r)$ will be specified below), choose a pointwise version of $p_t(x,y)$ as a function of x, y. By Fubini's theorem, there is a subset $X \subset M$ of a full measure such, that, for any $x \in X$, the following is true:

1. the function $p_t(x,y)$ is measurable in y;
2. the following inequality is satisfied

$$\int_{B(x,r)} p_t(x,y) d\mu(y) \leq 1, \tag{2.39}$$

   which follows from (2.1);
3. the inequality (2.35) is satisfied for μ-a.a. $y \in M$.

It follows from (2.39) that, for any $x \in X$,

$$\mu(B(x,r)) \leq \left(\operatorname*{essinf}_{y \in B(x,r)} p_t(x,y) \right)^{-1}.$$

Choose $\varepsilon > 0$ so that $\Phi_1(\varepsilon) > 0$. Applying (2.35) and choosing t from the identity $r = \varepsilon t^{1/\beta}$ we obtain

$$\operatorname*{essinf}_{y \in B(x,r)} p_t(x,y) \geq \frac{1}{t^{\alpha/\beta}} \Phi_1 \left(\frac{r}{t^{1/\beta}} \right) = r^{-\alpha} \varepsilon^{\alpha} \Phi_1(\varepsilon),$$

whence (2.36) follows with $C = \left(\varepsilon^{\alpha}\Phi_1\left(\varepsilon\right)\right)^{-1}$. Finally, having proved (2.38) for all $x \in X$, we obtain (2.38) for all $x \in M$ because X is dense in M.

(b) We first show that the upper bound in (2.37) and (2.36) imply that, for μ-a.a. $x \in M$, $0 < t \leq \varepsilon r^{\beta}$,

$$\int_{M\setminus B(x,r)} p_t(x,y)d\mu(y) \leq \frac{1}{2}, \tag{2.40}$$

provided $\varepsilon > 0$ is sufficiently small (the measurability issues are handled in the same way as in part (a) so we skip the details). Setting $r_k = 2^k r$ and using the monotonicity of Φ_2 and (2.36) we obtain

$$\begin{aligned}
\int_{M\setminus B(x,r)} p_t(x,y)d\mu(y) &= \sum_{k=0}^{\infty}\int_{B(x,r_{k+1})\setminus B(x,r_k)} p_t\left(x,y\right)d\mu(y) \\
&\leq \sum_{k=0}^{\infty}\int_{B(x,r_{k+1})\setminus B(x,r_k)} t^{-\alpha/\beta}\Phi_2\left(\frac{r_k}{t^{1/\beta}}\right)d\mu(y) \\
&\leq \sum_{k=0}^{\infty} C r_{k+1}^{\alpha} t^{-\alpha/\beta}\Phi_2\left(\frac{r_k}{t^{1/\beta}}\right) \\
&= C'\sum_{k=0}^{\infty}\left(\frac{2^k r}{t^{1/\beta}}\right)^{\alpha}\Phi_2\left(\frac{2^k r}{t^{1/\beta}}\right) \\
&\leq C'\int_{\frac{1}{2}r/t^{1/\beta}}^{\infty} s^{\alpha}\Phi_2(s)\frac{ds}{s}.
\end{aligned} \tag{2.41}$$

Since by hypothesis (2.34) the integral in (2.41) is convergent, its value can be made arbitrarily small provided r^{β}/t is large enough, whence (2.40) follows.

From (2.4) (which is true by the stochastic completeness of the heat kernel) and (2.40), we conclude that the condition $0 < t \leq \varepsilon r^{\beta}$ implies

$$\int_{B(x,r)} p_t(x,y)d\mu(y) \geq \frac{1}{2}, \tag{2.42}$$

whence

$$\mu(B(x,r) \geq \frac{1}{2}\left(\operatorname*{esssup}_{y\in B(x,r)} p_t(x,y)\right)^{-1}.$$

Finally, choosing $t = \varepsilon r^{\beta}$ and using the upper bound

$$p_t(x,y) \leq t^{-\alpha/\beta}\Phi_2(0) = r^{-\alpha}\varepsilon^{-\alpha/\beta}\Phi_2\left(0\right),$$

we obtain

$$\mu\left(B\left(x,r\right)\right) \geq c r^{\alpha}, \tag{2.43}$$

where $c = \frac{1}{2}\varepsilon^{\alpha/\beta}\left(\Phi_2\left(0\right)\right)^{-1}$. Combining (2.36) and (2.43), we finish the proof. ■

Corollary 2.12 *Let a heat kernel $p_t\left(x,y\right)$ be stochastically complete and satisfy the two-sided estimate*

$$p_t\left(x,y\right) \asymp \frac{C}{t^{\alpha/\beta}}\Phi\left(c\frac{d\left(x,y\right)}{t^{1/\beta}}\right), \tag{2.44}$$

where Φ is a monotone decreasing function from $[0,+\infty)$ to $[0,+\infty)$. Then for all $x \in M$ and $r > 0$,

$$\mu(B(x,r)) \simeq r^{\alpha}. \tag{2.45}$$

Proof. Observe first that $\Phi(0) > 0$ because otherwise $p_t(x,y) = 0$ for almost all $x, y \in M$, which contradicts the stochastic completeness. Let us show that $\Phi(s) > 0$ for some $s > 0$. Indeed, assuming the contrary, we obtain that $p_t(x,y) = 0$ for μ-almost all $x \neq y$. The stochastic completeness of $p_t(x,y)$ then implies

$$\int_{\{x\}} p_t(x,y)\, d\mu(y) = 1 \tag{2.46}$$

for μ-a.a. $x \in M$, which implies that there is a point $x \in M$ with a positive mass, that is, $\mu(\{x\}) > 0$. However, by Remark 2.9, this is impossible in the presence of the lower bound in (2.44).

Hence, $\Phi(s) > 0$ for some $s > 0$. By Lemma 2.8, we obtain

$$\Phi(s) \leq C(1+s)^{-(\alpha+\beta)}. \tag{2.47}$$

In particular, Φ satisfies the condition (2.34) of Theorem 2.11. Hence, (2.45) follows from Theorem 2.11(b). ■

Lemma 2.13 *Assume that, for all $x \in M$ and $r \in (0, r_0)$*

$$\mu(B(x,r)) \simeq r^\alpha,$$

where $\alpha, r_0 > 0$. Then, for any non-empty open set $\Omega \subset M$,

$$\dim_H \Omega = \alpha.$$

Moreover, for all Borel sets $A \subset M$,

$$\mu(A) \simeq H^\alpha(A),$$

where H^α is the Hausdorff measure of the dimension α in M.

Proof. Recall that the Hausdorff measure H^s of a Borel subset $A \subset M$ is defined by

$$H^s(A) = \lim_{\varepsilon \to 0+} H^s_\varepsilon(A) \tag{2.48}$$

where

$$H^s_\varepsilon(A) = \inf\left\{\sum_i r_i^s : A \subset \bigcup_i B(x_i, r_i)\ ,\ x_i \in M, r_i < \varepsilon\right\}. \tag{2.49}$$

Note that $H^s_\varepsilon(A)$ increases as ε decreases so that $\lim_{\varepsilon\to 0}$ in (2.48) can be replaced by $\sup_{\varepsilon>0}$.

It follows from the definition that $H^s(A)$ decreases as s increases. This allows to defined the Hausdorff dimension of A by

$$\dim_H A = \sup\{s : H^s(A) > 0\}.$$

In fact, if there is α such that $0 < H^\alpha(A) < \infty$ then $\alpha = \dim_H A$. Indeed, it follows from (2.49) that if $s > \alpha$ then

$$H^s_\varepsilon(A) \leq \varepsilon^{s-\alpha} H^\alpha_\varepsilon(A) \leq \varepsilon^{s-\alpha} H^\alpha(A)$$

whence $H^s(A) = 0$.

Let Ω now be a bounded non-empty open subset of M and let us prove that $0 < H^{\alpha}(\Omega) < \infty$. In fact, it is suffices to prove that

$$H^{\alpha}(\Omega) \simeq \mu(\Omega)$$

(note that Ω contains a ball and is contained in a ball so that $0 < \mu(\Omega) < \infty$).

If $\{B(x_i, r_i)\}$ is any finite or countable sequence of balls covering Ω and $r_i < r_0$ then

$$\mu(\Omega) \leq \sum_i \mu(B(x_i, r_i)) \leq C \sum_i r_i^{\alpha}.$$

Taking inf over all such sequences with $r_i < \varepsilon$, where $\varepsilon < r_0$, we obtain

$$\mu(\Omega) \leq C H_{\varepsilon}^{\alpha}(\Omega) \leq C H^{\alpha}(\Omega).$$

To prove the opposite inequality, it suffices to show that, for any $\varepsilon \in (0, r_0)$ there is a covering of Ω by a sequence of ball $\{B(x_i, r_i)\}$ such that $r_i < \varepsilon$ and

$$\mu(\Omega) \geq c \sum_i r_i^{\alpha}$$

where the constant $c > 0$ is the same for all sets Ω and for all $\varepsilon \in (0, r_0)$. For any point $x \in \Omega$, there is $r_x \in (0, \varepsilon)$ such that the $B(x, r_x) \subset \Omega$. Using the ball covering argument (see, for example, [17, Lemma 2.6]), it is possible to select a finite or countable sequence $\{x_i\}$ of points from Ω such that the balls $B(x_i, r_i)$ cover Ω while the balls $B\left(x_i, \frac{1}{4} r_i\right)$ are disjoint (where $r_i = r_{x_i}$). Then

$$\mu(\Omega) \geq \sum_i \mu\left(B\left(x_i, \tfrac{1}{4} r_i\right)\right) \geq c \sum_i r_i^{\alpha},$$

which was to be proved.

Finally, let Ω be an arbitrary non-empty open set. Then it can be represented as the union of a sequence $\{\Omega_i\}_{i=1}^{\infty}$ of bounded open sets. Since $H^{\alpha}(\Omega) \geq H^{\alpha}(\Omega_i) > 0$ it follows that $\dim_H(\Omega) \geq \alpha$. On the other hand, if $s > \alpha$ then $H^{s}(\Omega_i) = 0$ which implies that $H^{s}(\Omega) = 0$ and, hence, $\dim_H(\Omega) \leq \alpha$. We conclude that $\dim_H(\Omega) = \alpha$, which finishes the proof.

The above argument yields also that the relation

$$H^{\alpha}(A) \simeq \mu(A) \tag{2.50}$$

holds for any bounded open set $A \subset M$. Since Borel sets in a metric space can be obtained from bounded open sets by applying the operation of a monotone limit, which preserves (2.50), it follows that (2.50) holds for all Borel sets. ■

Combining Theorem 2.11(b) or Corollary 2.12 with Lemma 2.13, we obtain the following statement.

Corollary 2.14 *Assume that either the hypotheses of* Theorem 2.11(b) *or those of* Corollary 2.12 *are satisfied. Then*

$$\alpha = \dim_H M \quad \textit{and} \quad \mu \simeq H^{\alpha}.$$

Consequently, the parameter α is the invariant of the metric space (M, d), and measure μ is determined (up to a factor $\simeq 1$) by the metric space (M, d) alone.

3 Besov spaces

3.1 Besov spaces in $\mathbb{R}^n$

Recall that the Sobolev space $W_p^1(\mathbb{R}^n)$, where $p \in [1, +\infty]$, consists of functions $u \in L^p(\mathbb{R}^n)$ such that $\frac{\partial u}{\partial x_i} \in L^p(\mathbb{R}^n)$ for all $i = 1, 2, ..., n$, where $\frac{\partial u}{\partial x_i}$ is the distributional derivative of u. It is known that if $p \in (1, +\infty]$ then a function $u \in L^p(\mathbb{R}^n)$ belongs to $W_p^1(\mathbb{R}^n)$ if and only if

$$\sup_{z \in \mathbb{R}^n \setminus \{0\}} \frac{\|u(x+z) - u(x)\|_p}{|z|} < \infty$$

(see [22, pp.277, 279]). Fix $p \in [1, +\infty]$, $\sigma \in (0, 1)$ and consider a more general *Besov-Nikol'skii* space $B_{p,\infty}^{\sigma}(\mathbb{R}^n)$ that consists of functions $u \in L^p(\mathbb{R}^n)$ such that

$$\sup_{z \in \mathbb{R}^n, 0<|z|\leq 1} \frac{\|u(x+z) - u(x)\|_p}{|z|^{\sigma}} < \infty, \tag{3.1}$$

and the norm in $B_{p,\infty}^{\sigma}$ is the sum of $\|u\|_p$ and the left hand side of (3.1).

A more general family $B_{p,q}^{\sigma}(\mathbb{R}^n)$ of *Besov* spaces is defined for any $1 \leq q \leq \infty$ but alongside the case $q = \infty$ considered above, we will need only the case $q = p < +\infty$. In this case, $u \in B_{p,p}^{\sigma}(\mathbb{R}^n)$ if $u \in L^p(\mathbb{R}^n)$ and

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x+z) - u(x)|^p}{|z|^{n+p\sigma}} dx\, dz < \infty, \tag{3.2}$$

with the obvious definition of the norm in $B_{p,p}^{\sigma}$. Here are some well known facts about Besov and Sobolev spaces, where we assume $1 < p < +\infty$ (see for example [3]).

1. $u \in B_{p,\infty}^{\sigma}(\mathbb{R}^n)$ if and only if $u \in L^p(\mathbb{R}^n)$ and
$$\sup_{0<r\leq 1} \frac{D_p(u,r)}{r^{n+p\sigma}} < \infty,$$
where
$$D_p(u,r) := \iint_{\{x,y\in\mathbb{R}^n : |x-y|<r\}} |u(y) - u(x)|^p\, dx\, dy = \int_{|z|<r} \|u(\cdot + z) - u\|_p^p\, d\mu(z).$$

2. $u \in B_{p,p}^{\sigma}(\mathbb{R}^n)$ if and only if $u \in L^p(\mathbb{R}^n)$ and
$$\int_0^{\infty} \frac{D_p(u,r)}{r^{n+p\sigma}} \frac{dr}{r} < \infty.$$
Indeed, assuming for simplicity that u is a smooth function with compact support, we obtain
$$\begin{aligned}
\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x+z) - u(x)|^p}{|z|^{n+p\sigma}} dx\, dz &= \int_{\mathbb{R}^n} \frac{\|u(\cdot + z) - u\|^p}{|z|^{n+p\sigma}} dz \\
&= \int_0^{\infty} \int_{z\in S_r} \frac{\|u(\cdot + z) - u\|^p}{|z|^{n+p\sigma}} dS_r dr \\
&= \int_0^{\infty} \frac{\partial_r D_p(u,r)}{r^{n+p\sigma}} dr \\
&= (n + p\sigma) \int_0^{\infty} \frac{D_p(u,r)}{r^{n+p\sigma}} \frac{dr}{r},
\end{aligned}$$

where $S_r = \{z \in \mathbb{R}^n : |z| = r\}$ and we have used that

$$\begin{aligned}
D_p(u,r) &= O\left(r^{n+p}\right) = o\left(r^{n+p\sigma}\right) \text{ as } r \to 0 \\
D_p(u,r) &= O(1) = o\left(r^{n+p\sigma}\right) \text{ as } r \to \infty.
\end{aligned}$$

3. For any $0 < \sigma < 1$ the following relations take place

$$\begin{array}{ccccc}
W_2^1(\mathbb{R}^n) & \subset & B_{2,2}^{\sigma}(\mathbb{R}^n) & \subset & B_{2,\infty}^{\sigma}(\mathbb{R}^n) \\
\| & & \| & & \\
\mathrm{dom}_{\mathcal{E}}(-\Delta) & \subset & \mathrm{dom}_{\mathcal{E}}(-\Delta)^{\sigma} & &
\end{array} \tag{3.3}$$

3.2 Besov spaces in a metric measure space

Fix $\alpha > 0$, $\sigma > 0$, $p \in [1, +\infty)$ and introduce the following functionals on $L^p = L^p(M, \mu)$:

$$D_p(u,r) = \iint_{\{x,y \in M : d(x,y) < r\}} |u(x) - u(y)|^p \, d\mu(x) d\mu(y), \tag{3.4}$$

and

$$N_{p,\infty}^{\alpha,\sigma}(u) = \sup_{0 < r \le 1} \frac{D_p(u,r)}{r^{\alpha + p\sigma}}. \tag{3.5}$$

Furthermore, for any $q \in [1, +\infty)$ set

$$N_{p,q}^{\alpha,\sigma}(u) = \left(\int_0^{\infty} \left(\frac{D_p(u,r)}{r^{\alpha+p\sigma}}\right)^{q/p} \frac{dr}{r}\right)^{p/q}. \tag{3.6}$$

We will need only particular case of (3.6) when $p = q$. In this case, we have

$$N_{p,p}^{\alpha,\sigma}(u) = \int_0^{\infty} \frac{D_p(u,r)}{r^{\alpha+p\sigma}} \frac{dr}{r}. \tag{3.7}$$

For all $1 \le p < +\infty$ and $1 \le q \le +\infty$ define the space

$$\Lambda_{p,q}^{\alpha,\sigma} = \left\{u \in L^p : N_{p,q}^{\alpha,\sigma}(u) < \infty\right\}$$

and the norm in this space by

$$\|u\|_{\Lambda_{p,q}^{\alpha,\sigma}}^p = \|u\|_p^p + N_{p,q}^{\alpha,\sigma}(u).$$

The space $\Lambda_{p,q}^{\alpha,\sigma}$ was denoted by $Lip(\sigma, p, q)$ in [38] and by $\Lambda_{\sigma}^{p,q}$ in [47]; the space $\Lambda_{p,\infty}^{\alpha,\sigma}$ was denoted by $W^{\sigma,p}$ in [32].

Comparing the equivalent definitions of the Besov spaces in $\mathbb{R}^n$ and $\Lambda_{p,q}^{\alpha,\sigma}$, we obtain the following identities:

$$\begin{aligned}
\Lambda_{p,q}^{n,\sigma}(\mathbb{R}^n) &= B_{p,q}^{\sigma}(\mathbb{R}^n), \ 0 < \sigma < 1, \\
\Lambda_{p,p}^{n,1}(\mathbb{R}^n) &= \{0\}, \\
\Lambda_{p,\infty}^{n,1}(\mathbb{R}^n) &= W_p^1(\mathbb{R}^n), \\
\Lambda_{p,q}^{n,\sigma}(\mathbb{R}^n) &= \{0\}, \ \sigma > 1.
\end{aligned}$$

The definitions of $\Lambda^{n,\sigma}_{p,q}$ and $B^{\sigma}_{p,q}$ match only for $\sigma < 1$. For $\sigma \geq 1$ the definition of $B^{\sigma}_{p,q}$ becomes more involved whereas the above definition of $\Lambda^{\alpha,\sigma}_{p,q}$ is valid for *all* $\sigma > 0$ even if the space $\Lambda^{\alpha,\sigma}_{p,q}$ degenerates to $\{0\}$ for sufficiently large σ. With some abuse of terminology, we refer to $\Lambda^{\alpha,\sigma}_{p,q}$ as a Besov space, too.

The fact that $D_p(u,r)$ is increasing in r implies that, for any $r > 0$,

$$\frac{D_p(u,r)}{r^{\alpha+p\sigma}} \leq 2^{\alpha+p\sigma}\int_r^{2r}\frac{D_p(u,\rho)}{\rho^{\alpha+p\sigma}}\frac{d\rho}{\rho}$$

whence $N^{\alpha,\sigma}_{p,\infty}(u) \leq CN^{\alpha,\sigma}_{p,p}(u)$ and

$$\Lambda^{\alpha,\sigma}_{p,p} \hookrightarrow \Lambda^{\alpha,\sigma}_{p,\infty}. \tag{3.8}$$

It is clear from (3.5) that $N^{\alpha,\sigma}_{p,\infty}(u)$ is monotone increasing in σ, which means that the space $\Lambda^{\alpha,\sigma}_{p,\infty}$ shrinks when σ increases. Let us show the same holds also for the spaces $\Lambda^{\alpha,\sigma}_{p,q}$ with $q < \infty$, provided the parameter α satisfies the following property:

$$\mu(B(x,r)) \leq Cr^{\alpha} \quad \text{for all } x \in M \text{ and } r > 0. \tag{3.9}$$

Indeed, using (3.9) we obtain, for any $u \in L^p$,

$$\begin{aligned} D_p(u,r) &\leq 2^{p-1}\iint\limits_{\{d(x,y)<r\}}\left(|u(x)|^p + |u(y)|^p\right)d\mu(x)\,d\mu(y) \\ &= 2^p\iint\limits_{\{d(x,y)<r\}}|u(y)|^p\,d\mu(x)\,d\mu(y) \\ &= 2^p\int_M |u(y)|^p\,\mu(B(y,r))\,d\mu(y) \\ &\leq Cr^{\alpha}\|u\|_p^p. \end{aligned}$$

Therefore, the integral (3.6) converges at ∞ for *all* $u \in L^p$ and $\sigma > 0$, and the condition $N^{\alpha,\sigma}_{p,q}(u) < \infty$ amounts to the convergence of the integral at 0. This implies the above claim that the space $\Lambda^{\alpha,\sigma}_{p,q}$ shrinks when σ increases.

3.3 Embedding of Besov spaces into Hölder spaces

Let us define a Hölder space $C^{\lambda} = C^{\lambda}(M,d,\mu)$ as follows: $u \in C^{\lambda}$ if[1]

$$\|u\|_{C^{\lambda}} := \|u\|_{\infty} + \operatorname*{esssup}_{\substack{x,\,y\,\in\,M \\ 0\,<\,d(x,\,y)\,\leq\,1/3}}\frac{|u(x)-u(y)|}{d(x,y)^{\lambda}} < \infty. \tag{3.10}$$

Theorem 3.1 ([32]) *Let (M,d,μ) satisfy*

$$\mu(B(x,r)) \simeq r^{\alpha} \tag{3.11}$$

for all $x \in M$ and $r > 0$. Then, for any $\sigma > \alpha/2$ and all $u \in L^2$,

$$\|u\|_{C^{\lambda}} \leq C\|u\|_{\Lambda^{\alpha,\sigma}_{2,\infty}}, \tag{3.12}$$

[1]The restriction $d(x,y) \leq 1/3$ in (3.10) is related to the restriction $r \leq 1$ in definition (3.5). If (M,d) satisfies the chain condition (see Definition 5.5 below) then the $1/3$ can be replaced by any other positive constant.

where

$$\lambda = \sigma - \alpha/2.$$

Consequently, we have the embedding

$$\Lambda_{2,\infty}^{\alpha,\sigma} \hookrightarrow C^{\lambda}.$$

Remark 3.2 From (3.8) it follows that also $\Lambda_{2,2}^{\alpha,\sigma} \hookrightarrow C^{\lambda}$.

Proof. For any $x \in M$ and $r > 0$, set

$$u_r(x) := \frac{1}{\mu(B(x,r))} \int_{B(x,r)} u(\xi) d\mu(\xi). \tag{3.13}$$

We claim that for any $u \in L^2$, any $0 < r \leq 1/3$, and all $x, y \in M$ such that $d(x,y) \leq r$, the following inequality holds:

$$|u_r(x) - u_r(y)| \leq C\, r^{\lambda} N_{2,\infty}^{\alpha,\sigma}(u)^{1/2}. \tag{3.14}$$

Indeed, setting $B_1 = B(x,r), B_2 = B(y,r)$, we have

$$u_r(x) = \frac{1}{\mu(B_1)} \int_{B_1} u(\xi) d\mu(\xi) = \frac{1}{\mu(B_1)\mu(B_2)} \int_{B_1} \int_{B_2} u(\xi) d\mu(\eta) d\mu(\xi),$$

and similarly

$$u_r(y) = \frac{1}{\mu(B_1)\mu(B_2)} \int_{B_1} \int_{B_2} u(\eta) d\mu(\eta) d\mu(\xi).$$

Applying the Cauchy-Schwarz inequality, (3.11) and (3.5), we obtain

$$\begin{aligned}
|u_r(x) - u_r(y)|^2 &= \left(\frac{1}{\mu(B_1)\mu(B_2)} \int_{B_1} \int_{B_2} (u(\xi) - u(\eta))\, d\mu(\eta) d\mu(\xi) \right)^2 \\
&\leq \frac{1}{\mu(B_1)\mu(B_2)} \int_{B_1} \int_{B_2} |u(\xi) - u(\eta)|^2\, d\mu(\eta) d\mu(\xi) \\
&\leq C\, r^{-2\alpha} \iint_{\{\xi,\eta \in M : d(\xi,\eta) < 3r\}} |u(\xi) - u(\eta)|^2\, d\mu(\eta) d\mu(\xi) \\
&= C r^{-2\alpha} D_2\,(u, 3r) \\
&\leq C\, r^{2\sigma - \alpha} N_{2,\infty}^{\alpha,\sigma}(u),
\end{aligned}$$

thus proving (3.14).

Similarly, one proves that, for any $0 < r \leq 1/3$ and $x \in M$,

$$|u_{2r}(x) - u_r(x)| \leq C\, r^{\lambda} N_{2,\infty}^{\alpha,\sigma}(u)^{1/2}. \tag{3.15}$$

By the Lebesgue theorem, we have

$$u_r\,(x) \to u\,(x) \text{ as } r \to 0, \text{ for } \mu\text{-a.a. } x \in M \tag{3.16}$$

(this theorem requires the doubling property of the measure, which is true by (3.11)). Setting $r_k = 2^{-k}r$ for any $k = 0, 1, 2, ...$ we obtain from (3.15) and (3.16), for μ-a.a. $x \in M$,

$$\begin{aligned} |u(x) - u_r(x)| &\leq \sum_{k=0}^{\infty} |u_{r_k}(x) - u_{r_{k+1}}(x)| \\ &\leq C \left(\sum_{k=0}^{\infty} r_k^{\lambda} \right) N_{2,\infty}^{\alpha,\sigma}(u)^{1/2} \\ &\leq C\, r^{\lambda} N_{2,\infty}^{\alpha,\sigma}(u)^{1/2}. \end{aligned} \tag{3.17}$$

Applying the Cauchy-Schwarz inequality

$$|u_r(x)| \leq C\, r^{-\alpha/2} \|u\|_2$$

and using (3.17) to some fixed value of r, say $r = 1/4$, we obtain

$$|u(x)| \leq |u(x) - u_r(x)| + |u_r(x)| \leq C \left(\|u\|_2 + N_{2,\infty}^{\alpha,\sigma}(u)^{1/2} \right),$$

whence

$$\|u\|_\infty \leq C\, \|u\|_{\Lambda_{2,\infty}^{\alpha,\sigma}}. \tag{3.18}$$

Using (3.14) and (3.17), we obtain, for μ-a.a. $x, y \in M$ such that $r := d(x, y) < 1/3$,

$$|u(x) - u(y)| \leq |u(x) - u_r(x)| + |u_r(x) - u_r(y)| + |u_r(y) - u(y)| \leq C\, r^{\lambda} N_{2,\infty}^{\alpha,\sigma}(u)^{1/2}.$$

Hence,

$$\frac{|u(x) - u(y)|}{d(x, y)^{\lambda}} \leq C\, N_{2,\infty}^{\alpha,\sigma}(u)^{1/2},$$

which together with (3.18) yields (3.12). ■

4 The energy domain

Recall that a heat kernel p_t on a metric measure space (M, d, μ) has the associated energy form $\mathcal{E}$ and the generator $\mathcal{L}$.

4.1 A local case

The following theorem identifies the domain $\mathcal{F}$ of the energy form in terms of Besov spaces.

Theorem 4.1 ([32]) *Let p_t be a heat kernel on (M, d, μ), α, β be positive constants, and Φ_1 and Φ_2 be monotone decreasing functions from $[0, +\infty)$ to $[0, +\infty)$.*

(a) If the heat kernel satisfies the lower bound

$$p_t\,(x, y) \geq \frac{1}{t^{\alpha/\beta}} \Phi_1 \left(\frac{d\,(x, y)}{t^{1/\beta}} \right)$$

where $\Phi_1\,(s) > 0$ for some $s > 0$, then, for any $u \in L^2\,(M, \mu)$,

$$\mathcal{E}\,[u] \geq c N_{2,\infty}^{\alpha,\beta/2}(u)$$

and, consequently, $\mathcal{F} \subset \Lambda_{2,\infty}^{\alpha,\beta/2}$.

(b) *If the heat kernel is stochastically complete and satisfies the upper bound*

$$p_t(x,y) \le \frac{1}{t^{\alpha/\beta}} \Phi_2 \left(\frac{d(x,y)}{t^{1/\beta}} \right)$$

where

$$\int^{\infty} s^{\alpha+\beta} \Phi_2(s) \frac{ds}{s} < \infty, \tag{4.1}$$

then, for any $u \in L^2(M,\mu)$,

$$\mathcal{E}[u] \le C N_{2,\infty}^{\alpha,\beta/2}(u)$$

and, consequently, $\mathcal{F} \supset \Lambda_{2,\infty}^{\alpha,\beta/2}$.

Corollary 4.2 *If the heat kernel is stochastically complete and satisfies the two-sided estimate* (2.37) *with functions* Φ_1 *and* Φ_2 *as in* Theorem 4.1 *then, for any* $u \in L^2(M,\mu)$,

$$\mathcal{E}[u] \simeq N_{2,\infty}^{\alpha,\beta/2}(u), \tag{4.2}$$

and, consequently, $\mathcal{F} = \Lambda_{2,\infty}^{\alpha,\beta/2}$.

The identity $\mathcal{F} = \Lambda_{2,\infty}^{\alpha,\beta/2}$ was obtained in different setting by Jonsson [38, Theorem 1], Pietruska-Paluba [44, Theorem 1], and [32, Theorem 4.2].

Let us show the sharpness of the condition (4.1). If $0 < \sigma < 1$ then the heat kernel of the operator $(-\Delta)^{\sigma}$ in $\mathbb{R}^n$ satisfies (2.37) with the function $\Phi_2(s) = C(1+s)^{-(\alpha+\beta)}$, where $\alpha = n$ and $\beta = 2\sigma$ (see Introduction or Lemma 4.4 below). For this function, the condition (4.1) breaks just on the borderline, and the identity $\mathcal{F} = \Lambda_{2,\infty}^{\alpha,\beta/2}$ is not valid either. Indeed, in $\mathbb{R}^n$ by (3.3) $\operatorname{dom}_{\mathcal{E}}(-\Delta)^{\sigma} = B_{2,2}^{\sigma}$ that is strictly smaller than $B_{2,\infty}^{\sigma} = \Lambda_{2,\infty}^{n,\beta/2}$. This case will be covered by Theorem 4.3 below.

As we will see in the proof below (cf. (4.5) and (4.7)), under hypothesis (4.1) we have in fact

$$\mathcal{E}[u] \simeq \limsup_{r\to 0} r^{-(\alpha+\beta)} \iint_{\{d(x,y)<r\}} |u(x)-u(y)|^2 d\mu(y) d\mu(x). \tag{4.3}$$

In particular, this implies that the energy form is strongly local, that is for all functions $u, v \in \mathcal{F}$ with compact supports, if $u \equiv \text{const}$ in an open neighborhood of the support of v then $\mathcal{E}(u,v) = 0$ (note that the locality of $\mathcal{E}$ follows also from Lemma 2.7). The operator $(-\Delta)^{\sigma}$ is not local for $0 < \sigma < 1$, and this explains why Theorem 4.1 does not apply to this operator.

Proof of Theorem 4.1. (a) For a function $u \in L^2$, we have by (3.4) and (3.5)

$$N_{2,\infty}^{\alpha,\beta/2}(u) = \sup_{0<r\le 1} \frac{D_2(u,r)}{r^{\alpha+\beta}},$$

where

$$D_2(u,r) = \iint_{\{d(x,y)<r\}} (u(x)-u(y))^2 \, d\mu(y) d\mu(x). \tag{4.4}$$

It suffices to prove that, for some $c > 0$ and for all $r > 0$,

$$\mathcal{E}[u] \ge c \frac{D_2(u,r)}{r^{\alpha+\beta}},$$

which would then imply

$$\mathcal{E}[u] \geq c \sup_{0<r<\infty} \frac{D_2(u,r)}{r^{\alpha+\beta}} \geq c N_{2,\infty}^{\alpha,\beta/2}(u). \tag{4.5}$$

Chose $\varepsilon > 0$ such that $\Phi_1(\varepsilon) > 0$. Let $r, t > 0$ be such that $r = \varepsilon t^{1/\beta}$. Using the lower bound of $p_t(x,y)$ and the monotonicity of Φ_1, we obtain from (2.12) that

$$\begin{aligned}
\mathcal{E}[u] &\geq \mathcal{E}_t[u] \geq \frac{1}{2t} \iint_{\{d(x,y)<r\}} (u(x)-u(y))^2 p_t(x,y) d\mu(y) d\mu(x) \\
&\geq \frac{1}{2} \frac{1}{t^{\alpha/\beta+1}} \Phi_1\left(\frac{r}{t^{1/\beta}}\right) \iint_{\{d(x,y)<r\}} (u(x)-u(y))^2 d\mu(y) d\mu(x) \\
&= \frac{1}{2} \frac{\varepsilon^{\alpha+\beta}}{r^{\alpha+\beta}} \Phi_1(\varepsilon) D_2(u,r),
\end{aligned}$$

which was to be proved.

(b) Let us prove that, for any $r > 0$,

$$\mathcal{E}[u] \leq C \sup_{0<\rho\leq r} \frac{D_2(u,\rho)}{\rho^{\alpha+\beta}}, \tag{4.6}$$

which would imply

$$\mathcal{E}[u] \leq C \limsup_{r\to 0+} \frac{D_2(u,r)}{r^{\alpha+\beta}} \leq C\, N_{2,\infty}^{\alpha,\beta/2}(u). \tag{4.7}$$

For any positive t, r, we have by (2.15)

$$\mathcal{E}_t[u] = \frac{1}{2t} \int_M \int_M (u(x)-u(y))^2 p_t(x,y) d\mu(y) d\mu(x) = A(t) + B(t) \tag{4.8}$$

where

$$\begin{aligned}
A(t) &= \frac{1}{2t} \int_M \int_{M\setminus B(x,r)} (u(x)-u(y))^2 p_t(x,y) d\mu(y) d\mu(x), &(4.9)\\
B(t) &= \frac{1}{2t} \int_M \int_{B(x,r)} (u(x)-u(y))^2 p_t(x,y) d\mu(y) d\mu(x). &(4.10)
\end{aligned}$$

To estimate $A(t)$ let us observe that by (2.41)

$$\begin{aligned}
\int_{M\setminus B(x,r)} p_t(x,y) d\mu(y) &\leq C \int_{\frac{1}{2} r t^{-1/\beta}}^{\infty} s^{\alpha} \Phi_2(s) \frac{ds}{s} \\
&\leq C \frac{t}{r^{\beta}} \int_{\frac{1}{2} r t^{-1/\beta}}^{\infty} s^{\alpha+\beta} \Phi_2(s) \frac{ds}{s}, &(4.11)
\end{aligned}$$

whence by (4.1)

$$\frac{1}{t} \int_{M\setminus B(x,r)} p_t(x,y) d\mu(y) = o(t) \quad \text{as } t \to 0+ \text{ uniformly in } x \in M. \tag{4.12}$$

Therefore, applying the elementary inequality $(a-b)^2 \leq 2(a^2+b^2)$ in (4.9) and using (4.12), we obtain that for $t \to 0$

$$\begin{aligned} A(t) &\leq \frac{1}{t} \iint_{\{x,y:d(x,y)\geq r\}} (u(x)^2+u(y)^2)p_t(x,y)d\mu(y)d\mu(x) \\ &= \frac{2}{t}\int_M u(x)^2 \left(\int_{M\setminus B(x,r)} p_t(x,y)d\mu(y)\right) d\mu(x) \\ &= o(1)\|u\|_2^2, \end{aligned}$$

whence

$$\lim_{t\to 0+} A(t) = 0. \tag{4.13}$$

The quantity $B(t)$ is estimated as follows using the upper bound of the heat kernel, (4.4), (4.1), and setting $r_k = 2^{-k}r$:

$$\begin{aligned} B(t) &= \frac{1}{2t}\sum_{k=0}^{\infty}\int_M \int_{B(x,r_k)\setminus B(x,r_{k+1})} (u(x)-u(y))^2 p_t(x,y)d\mu(y)d\mu(x) \\ &\leq \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{t^{1+\alpha/\beta}}\Phi_2\left(\frac{r_{k+1}}{t^{1/\beta}}\right)\int_M\int_{B(x,r_k)}(u(x)-u(y))^2 d\mu(y)d\mu(x) \\ &\leq C\sum_{k=0}^{\infty}\left(\frac{r_{k+1}}{t^{1/\beta}}\right)^{\alpha+\beta}\Phi_2\left(\frac{r_{k+1}}{t^{1/\beta}}\right)\frac{D_2(u,r_k)}{r_k^{\alpha+\beta}} \qquad (4.14) \\ &\leq C\sup_{0<\rho\leq r}\frac{D_2(u,\rho)}{\rho^{\alpha+\beta}}\int_0^{\infty} s^{\alpha+\beta}\Phi_2(s)\frac{ds}{s} \\ &\leq C\sup_{0<\rho\leq r}\frac{D_2(u,\rho)}{\rho^{\alpha+\beta}}. \qquad (4.15) \end{aligned}$$

Finally, (4.6) follows from (4.8), (4.13) and (4.15) by letting $t \to 0$. ■

The stochastic completeness in the hypotheses of Theorem 4.1(b) and in Corollary 4.2 is essential. Indeed, consider in $\mathbb{R}^n$, $n>2$, the operator $\mathcal{L} = -\Delta + c(x)$ where $c(x)$ is a non-negative continuous function in $\mathbb{R}^n$ such that

$$\sup_{x\in\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{c(y)}{|x-y|^{n-2}}dy < \infty. \tag{4.16}$$

It is possible to show (see [29]) that the heat kernel of $\mathcal{L}$ satisfies the Gaussian estimate

$$p_t(x,y) \asymp \frac{C}{t^{n/2}}\exp\left(-c\frac{|x-y|^2}{t}\right),$$

while the domain of the Dirichlet form is

$$\mathcal{F} = \left\{u \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n}\left(|\nabla u|^2 + c(x)u^2\right)dx < \infty\right\}.$$

Clearly, $\mathcal{F}$ is strictly smaller than

$$\Lambda_{2,\infty}^{n,1} = W_2^1 = \left\{u \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n}|\nabla u|^2 dx < \infty\right\},$$

provided function $c(x)$ is unbounded. It is easy to see that there exists an unbounded function $c(x)$ that satisfies (4.16). Indeed, let $c_0(x)$ be any non-zero non-negative function from $C_0^\infty(\mathbb{R}^n)$; any such function clearly satisfies (4.16). Choose a sequence $\{x_k\}_{k=1}^\infty$ in $\mathbb{R}^n$ such that $x_k \to \infty$ as $k \to \infty$ and set

$$c(x) = \sum_{k=1}^\infty k^\delta c_0(k(x - x_k)),$$

where $\delta \in (0,1)$. Then we have

$$\begin{aligned}
\sup_{x\in\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{c(y)}{|x-y|^{n-2}} dy &\leq \sum_{k=1}^\infty k^\delta \sup_{x\in\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{c_0(k(y-x_k))}{|x-y|^{n-2}} dy \\
&= \sum_{k=1}^\infty k^\delta \sup_{x\in\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{c_0(ky)}{|x-y|^{n-2}} dy \\
&= \sum_{k=1}^\infty k^\delta \sup_{x\in\mathbb{R}^n} k^{-n} \int_{\mathbb{R}^n} \frac{c_0(z)}{|x-z/k|^{n-2}} dz \\
&= \sum_{k=1}^\infty k^\delta \sup_{x\in\mathbb{R}^n} k^{-n} k^{n-2} \int_{\mathbb{R}^n} \frac{c_0(z)}{|x-z|^{n-2}} dz \\
&= \sum_{k=1}^\infty k^{\delta-2} \sup_{x\in\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{c_0(z)}{|x-z|^{n-2}} dz \\
&< \infty.
\end{aligned}$$

4.2 Non-local case

The following theorem identifies the domain $\mathcal{F}$ in the case of non-local form.

Theorem 4.3 ([27]) *Let p_t be a stochastically complete heat kernel on (M,d,μ) satisfying estimate* (2.37) *with functions Φ_1 and Φ_2 such that*

$$\Phi_1(s) \simeq s^{-(\alpha+\beta)} \quad \text{for } s>1 \quad \text{and} \quad \Phi_2(s) \leq Cs^{-(\alpha+\beta)} \quad \text{for } s>0. \tag{4.17}$$

Then, for any $u \in L^2(M,\mu)$,

$$\mathcal{E}[u] \simeq N_{2,2}^{\alpha,\beta/2}(u) \tag{4.18}$$

and, consequently, $\mathcal{F} = \Lambda_{2,2}^{\alpha,\beta/2}$.

The hypotheses (4.17) are satisfied provided the heat kernel admits the estimate

$$p_t(x,y) \simeq \frac{1}{t^{\alpha/\beta}} \left(1 + \frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)}. \tag{4.19}$$

In the next Section 4.3, a class of heat kernels satisfying (4.19) will be described.

Proof. The proof is similar to that of Theorem 4.1. Fix a decreasing geometric sequence $\{r_k\}_{k\in\mathbb{Z}}$ and observe that by (3.7)

$$N_{2,2}^{\alpha,\beta/2}(u) \simeq \sum_{k\in\mathbb{Z}} \frac{D_2(u,r_k)}{r_k^{\alpha+\beta}}.$$

Using (2.15) and the upper bounds in (2.37) and (4.17) we obtain

$$\begin{aligned}
2\mathcal{E}_t[u] &= \frac{1}{t}\sum_{k\in\mathbb{Z}}\int_M\int_{B(x,r_k)\setminus B(x,r_{k+1})}(u(x)-u(y))^2 p_t(x,y)d\mu(y)d\mu(x)\\
&\leq \sum_{k\in\mathbb{Z}}\frac{1}{t^{1+\alpha/\beta}}\Phi_2\left(\frac{r_{k+1}}{t^{1/\beta}}\right)\int_M\int_{B(x,r_k)\setminus B(x,r_{k+1})}(u(x)-u(y))^2 d\mu(y)d\mu(x)\\
&\leq \sum_{k\in\mathbb{Z}}\frac{1}{t^{1+\alpha/\beta}}\Phi_2\left(\frac{r_{k+1}}{t^{1/\beta}}\right)D_2(u,r_k)\\
&\leq C\sum_{k\in\mathbb{Z}}\frac{D_2(u,r_k)}{r_k^{\alpha+\beta}}\\
&\leq CN_{2,\infty}^{\alpha,\beta/2}(u),
\end{aligned}$$

whence

$$\mathcal{E}[u]=\lim_{t\to 0}\mathcal{E}_t[u]\leq CN_{2,2}^{\alpha,\beta/2}(u). \tag{4.20}$$

Similarly, using the lower bound in (2.37), we obtain

$$\begin{aligned}
2\mathcal{E}_t[u] &\geq \sum_{k\in\mathbb{Z}}\frac{1}{t^{1+\alpha/\beta}}\Phi_1\left(\frac{r_{k+1}}{t^{1/\beta}}\right)\int_M\int_{B(x,r_k)\setminus B(x,r_{k+1})}(u(x)-u(y))^2 d\mu(y)d\mu(x)\\
&= \sum_{k\in\mathbb{Z}}\frac{1}{t^{1+\alpha/\beta}}\Phi_1\left(\frac{r_{k+1}}{t^{1/\beta}}\right)(D_2(u,r_k)-D_2(u,r_{k+1}))\\
&= \sum_{k\in\mathbb{Z}}\frac{1}{t^{1+\alpha/\beta}}\left(\Phi_1\left(\frac{r_{k+1}}{t^{1/\beta}}\right)-\Phi_1\left(\frac{r_k}{t^{1/\beta}}\right)\right)D_2(u,r_k). \qquad (4.21)
\end{aligned}$$

The first part of the hypothesis (4.17) implies that there exists a large enough number a such that

$$\Phi_1\left(\frac{s}{a}\right)\geq 2\Phi_1(s)\quad \forall s>a.$$

Setting $r_k=a^{-k}$, we obtain from (4.21) and (4.17)

$$\mathcal{E}_t[u]\geq\frac{1}{2}\sum_{\{k:r_k>at^{1/\beta}\}}\frac{1}{t^{1+\alpha/\beta}}\Phi_1\left(\frac{r_k}{t^{1/\beta}}\right)D_2(u,r_k)\geq c\sum_{\{k:r_k>at^{1/\beta}\}}\frac{D_2(u,r_k)}{r_k^{\alpha+\beta}}.$$

Letting $t\to 0$, we conclude

$$\mathcal{E}[u]=\lim_{t\to 0}\mathcal{E}_t[u]\geq c\sum_{k\in\mathbb{Z}}\frac{D_2(u,r_k)}{r_k^{\alpha+\beta}}\simeq N_{2,2}^{\alpha,\beta/2}(u),$$

which together with (4.20) finishes the proof. ∎

4.3 Subordinated heat kernel

Let φ be a non-negative continuous function on $[0,+\infty)$ such that $\varphi(0)=0$, and let $\{\eta_t\}_{t>0}$ be a family of non-negative continuous functions on $(0,+\infty)$ such that for all $t>0$ and $\lambda\geq 0$

$$\exp(-t\varphi(\lambda))=\int_0^\infty\eta_t(s)e^{-s\lambda}ds. \tag{4.22}$$

Then, for any heat kernel p_t on a metric measure space (M, d, μ), the following expression

$$q_t(x, y) := \int_0^\infty \eta_t(s)\, p_s(x, y)\, ds \tag{4.23}$$

defines a new heat kernel $\{q_t\}_{t>0}$ on M, which is called a *subordinated heat kernel* to p_t (and η_t is called a *subordinator*). Indeed, applying (4.22) to the generator $\mathcal{L}$ of p_t we obtain

$$\exp(-t\varphi(\mathcal{L})) = \int_0^\infty \eta_t(s)\, P_s\, ds.$$

Comparing to (4.23) we see that q_t is the integral kernel of the semigroup $\left\{e^{-t\varphi(\mathcal{L})}\right\}_{t>0}$ generated by the operator $\varphi(\mathcal{L})$. Since $\left\{e^{-t\varphi(\mathcal{L})}\right\}_{t>0}$ is a self-adjoint strongly continuous contraction semigroup in L^2, the family $\{q_t\}_{t>0}$ satisfies the properties $(iii) - (v)$ of Definition 2.1. Let us show that q_t satisfies also (i) and (ii). Indeed, the positivity of q_t follows from $\eta_t \geq 0$, and the total mass inequality from

$$\int_M q_t(x, y)\, d\mu(y) = \int_0^\infty \eta_t(s) \left(\int_M p_s(x, y)\, d\mu(y) \right) ds \leq \int_0^\infty \eta_t(s)\, ds = 1, \tag{4.24}$$

where the last identity is obtained from (4.22) by taking $\lambda = 0$. Hence, $\{q_t\}_{t>0}$ is a heat kernel. It follows from (4.24) that q_t is stochastically complete if and only if p_t is stochastically complete.

For example, it follows from the definition of the gamma-function that, for all $t > 0$ and $\lambda \geq 0$,

$$\exp(-t \log(1 + \lambda)) = (1 + \lambda)^{-t} = \frac{1}{\Gamma(t)} \int_0^\infty s^{t-1} e^{-s(1+\lambda)} ds,$$

which takes the form (4.22) for $\varphi(\lambda) = \log(1 + \lambda)$ and $\eta_t(s) = \frac{s^{t-1}e^{-s}}{\Gamma(t)}$. Therefore, the operator $\log(\mathrm{I} + \mathcal{L})$, that generates the semigroup $\left\{(\mathrm{I} + \mathcal{L})^{-t}\right\}_{t\geq 0}$, has the heat kernel

$$q_t(x, y) = \frac{1}{\Gamma(t)} \int_0^\infty s^{t-1} e^{-s} p_s(x, y)\, ds.$$

It is well known that for any $\delta \in (0, 1)$ there exists a subordinator $\eta_t = \eta_t^{(\delta)}$ such that (4.22) takes place with $\varphi(\lambda) = \lambda^\delta$. In this case, (4.23) defines the heat kernel q_t of the operator $\mathcal{L}^\delta$.

For example, if $\delta = \frac{1}{2}$ then

$$\eta_t^{(1/2)}(s) = \frac{t}{\sqrt{4\pi s^3}} \exp\left(-\frac{t^2}{4s}\right).$$

For any $0 < \delta < 1$, the function $\eta_t^{(\delta)}(s)$ possesses the scaling property

$$\eta_t^{(\delta)}(s) = \frac{1}{t^{1/\delta}} \eta_1^{(\delta)}\left(\frac{s}{t^{1/\delta}}\right),$$

and satisfies the estimates

$$\eta_t^{(\delta)}(s) \leq C \frac{t}{s^{1+\delta}} \quad \forall s, t > 0, \tag{4.25}$$

$$\eta_t^{(\delta)}(s) \simeq \frac{t}{s^{1+\delta}} \quad \forall s \geq t^{1/\delta} > 0. \tag{4.26}$$

As $s \to 0+$, $\eta_1^{(\delta)}(s)$ goes to 0 exponentially fast so that for any $\gamma > 0$

$$\int_0^\infty s^{-\gamma} \eta_1^{(\delta)}(s)\, ds < \infty \tag{4.27}$$

(see [50] and [14]).

Lemma 4.4 ([27]) *Let a heat kernel p_t satisfy the estimate (2.37) where $\Phi_1(\xi) > 0$ for some $\xi > 0$ and Φ_2 satisfies the condition*

$$\int_0^\infty \xi^{\alpha+\beta'} \Phi_2(\xi) \frac{d\xi}{\xi} < \infty, \tag{4.28}$$

where $\beta' = \delta\beta$, $0 < \delta < 1$. Then the heat kernel $q_t(x, y)$ of operator $\mathcal{L}^\delta$ satisfies the estimate

$$q_t(x, y) \simeq \frac{1}{t^{\alpha/\beta'}} \left(1 + \frac{d(x, y)}{t^{1/\beta'}}\right)^{-(\alpha+\beta')} \simeq \min\left(t^{-\alpha/\beta'}, \frac{t}{d(x, y)^{\alpha+\beta'}}\right), \tag{4.29}$$

for μ-a.a. $x, y \in M$ and $t > 0$.

Proof. Chose $\varepsilon > 0$ such that $\Phi_1(\varepsilon) > 0$. Using (4.23), (2.37), (4.26) and setting $r = d(x, y)$, we obtain the lower bound in (4.29) as follows:

$$\begin{aligned} q_t(x, y) &\geq \int_{\max\left(t^{1/\delta}, (r/\varepsilon)^\beta\right)}^\infty \frac{1}{s^{\alpha/\beta}} \Phi_1\left(\frac{r}{s^{1/\beta}}\right) \eta_t^{(\delta)}(s)\, ds \\ &\geq c\Phi_1(\varepsilon) \int_{\max\left(t^{1/\delta}, (r/\varepsilon)^\beta\right)}^\infty \frac{1}{s^{\alpha/\beta}} \frac{t}{s^{1+\delta}} ds \\ &= ct \max\left(t^{1/\delta}, (r/\varepsilon)^\beta\right)^{-(\alpha/\beta+\delta)} \\ &\geq c' \min\left(t^{-\alpha/\beta'}, tr^{-(\alpha+\beta')}\right). \end{aligned}$$

Similarly, using (4.23), (2.37), (4.25), we obtain

$$\begin{aligned} q_t(x, y) &\leq \int_0^\infty \frac{1}{s^{\alpha/\beta}} \Phi_2\left(\frac{r}{s^{1/\beta}}\right) \eta_t^{(\delta)}(s)\, ds \\ &\leq C \int_0^\infty \frac{1}{s^{\alpha/\beta}} \Phi_2\left(\frac{r}{s^{1/\beta}}\right) \frac{t}{s^{1+\delta}} ds \\ &= C \frac{t}{r^{\alpha+\beta\delta}} \int_0^\infty \xi^{\alpha+\beta\delta} \Phi_2(\xi) \frac{d\xi}{\xi}. \end{aligned}$$

By (4.28) the above integral converges, whence

$$q_t(x, y) \leq C \frac{t}{r^{\alpha+\beta'}}. \tag{4.30}$$

On the other hand, using the upper bound $p_s(x, y) \leq Cs^{-\alpha/\beta}$ and the change $\tau = s/t^{1/\delta}$ we obtain

$$q_t(x, y) \leq C \int_0^\infty \frac{1}{s^{\alpha/\beta}} \eta_t^{(\delta)}(s)\, ds = t^{-\alpha/(\beta\delta)} \int_0^\infty \frac{1}{\tau^{\alpha/\beta}} \eta_1^{(\delta)}(\tau)\, d\tau \leq Ct^{-\alpha/\beta'}, \tag{4.31}$$

where the last inequality follows from (4.27). Combining (4.30) and (4.31) we obtain the upper bound in (4.29). ■

Corollary 4.5 ([27], [46]) *If $p_t(x,y)$ is a stochastically complete heat kernel that satisfies the hypotheses of* Lemma 4.4, *then*

$$\mathrm{dom}_{\mathcal{E}}(\mathcal{L}^{\delta}) = \Lambda_{2,2}^{\alpha,\beta'/2}. \tag{4.32}$$

Proof. By Lemma 4.4, the heat kernel q_t of the operator $\mathcal{L}^{\delta}$ satisfies the estimate

$$q_t(x,y) \simeq \frac{1}{t^{\alpha/\beta'}}\Phi\left(\frac{d(x,y)}{t^{1/\beta'}}\right),$$

where

$$\Phi(s) = (1+s)^{-(\alpha+\beta')}.$$

Since q_t is stochastically complete, Applying Theorem 4.3 to the heat kernel q_t and its generator $\mathcal{L}^{\delta}$ we obtain (4.32). ■

4.4 Bessel potential spaces

Let p_t be a heat kernel on a metric measure space (M,d,μ) and let $\mathcal{L}$ be its generator. Since $\mathcal{L}$ is positive definite, the operator $(\mathrm{I}+\mathcal{L})^{-s}$ is a bounded operator in L^2 for any $s \geq 0$. This operator is called the *Bessel potential.*

Fix $\beta > 0$, and for any $\sigma > 0$ define the *Bessel potential space* H^{σ} as the image of $(\mathrm{I}+\mathcal{L})^{-\sigma/\beta}$, that is

$$H^{\sigma} := (\mathrm{I}+\mathcal{L})^{-\sigma/\beta}(L^2) = \mathrm{dom}\,(\mathrm{I}+\mathcal{L})^{\sigma/\beta},$$

with the norm

$$\|u\|_{H^{\sigma}} := \|\,(\mathrm{I}+\mathcal{L})^{\sigma/\beta}\,u\|_2.$$

This definition of H^{σ} depends on the parameter β. A priori the value of β is arbitrary but normally β is taken the same as in (2.37) assuming that (2.37) holds. For example, for the Gauss-Weierstrass kernel in $\mathbb{R}^n$ we take $\beta = 2$. In this case $\mathcal{L} = -\Delta$ and it is easy to see that $H^{\sigma}(\mathbb{R}^n)$ consists of functions $u \in L^2(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} |\widetilde{u}(\xi)|^2\left(1+|\xi|^2\right)^{\sigma/2} d\xi < \infty,$$

where $\widetilde{u}$ is the Fourier transform of u. Of course, this coincides with the classical definition of the fractional Sobolev space $H^{\sigma}(\mathbb{R}^n)$.

The purpose of this section is to prove an embedding theorem for the space H^{σ} in a special case when $\alpha < \beta$.

Lemma 4.6 *For any $\sigma > 0$ we have $H^{\sigma} = \mathrm{dom}(\mathcal{L}^{\sigma/\beta})$.*

Proof. Indeed, let $\{E_{\lambda}\}_{\lambda\in\mathbb{R}}$ be the spectral resolution of the operator $\mathcal{L}$. Setting $s = \sigma/\beta$, we have

$$\begin{aligned} H^{\sigma} &= \mathrm{dom}(\mathrm{I}+\mathcal{L})^s = \left\{u \in L^2 : \int_0^{\infty}(1+\lambda)^{2s}\, d\|E_{\lambda}u\|^2 < \infty\right\} \\ &= \left\{u \in L^2 : \int_0^{\infty}\lambda^{2s}\, d\|E_{\lambda}u\|^2 < \infty\right\} \\ &= \mathrm{dom}(\mathcal{L}^s), \end{aligned}$$

which was to be proved. ■

Theorem 4.7 ([32]) *Assume that a heat kernel p_t is stochastically complete and satisfies the two sided estimates (2.37) where $\alpha < \beta$, function Φ_1 is such that $\Phi_1(s) > 0$ for some $s > 0$, and Φ_2 satisfies the condition*

$$\int_0^\infty s^{\alpha+\beta}\Phi_2(s)\frac{ds}{s} < \infty.$$

Then, for any $\sigma > \alpha/2$ we have

$$H^\sigma \hookrightarrow C^\lambda \quad \text{where } \lambda = \min(\sigma, \beta/2) - \alpha/2. \tag{4.33}$$

Proof. Note that $H^\sigma \hookrightarrow H^{\sigma'}$ wherever $\sigma \geq \sigma'$. Therefore, it suffices to prove the embedding (4.33) for $\sigma \leq \beta/2$, which will be assumed below.

By Corollary 4.2 we have $\mathrm{dom}_{\mathcal{E}}(\mathcal{L}) = \Lambda_{2,\infty}^{\alpha,\beta/2}$, and by Corollary 4.5 $\mathrm{dom}_{\mathcal{E}}(\mathcal{L}^\delta) = \Lambda_{2,2}^{\alpha,\delta\beta/2}$ provided $\delta \in (0,1)$. Using Lemma 4.6, we obtain that if $\sigma = \beta/2$ then

$$H^\sigma = \mathrm{dom}\left(\mathcal{L}^{1/2}\right) = \mathrm{dom}_{\mathcal{E}}(\mathcal{L}) = \Lambda_{2,\infty}^{\alpha,\beta/2} = \Lambda_{2,\infty}^{\alpha,\sigma},$$

and if $\sigma < \beta/2$ then

$$H^\sigma = \mathrm{dom}\left(\mathcal{L}^{\sigma/\beta}\right) = \mathrm{dom}_{\mathcal{E}}(\mathcal{L}^{2\sigma/\beta}) = \Lambda_{2,2}^{\alpha,\sigma}.$$

If $\sigma > \alpha/2$ then the both spaces $\Lambda_{2,\infty}^{\alpha,\sigma}$ and $\Lambda_{2,2}^{\alpha,\sigma}$ embed into C^λ (see Theorem 3.1 and Remark 3.2), which finishes the proof. ∎

For further results of this type see [18, Theorem 4.1], [27], [47, Theorem 3.13(a)], [49].

5 The walk dimension

5.1 Intrinsic characterization of the walk dimension

Definition 5.1 *Fix $\alpha > 0$ and set*

$$\beta^* := \sup\left\{\beta > 0 : \Lambda_{2,\infty}^{\alpha,\beta/2} \text{ is dense in } L^2(M,\mu)\right\}. \tag{5.1}$$

The number $\beta^* \in [0, +\infty]$ is called the critical exponent *of the family $\left\{\Lambda_{2,\infty}^{\alpha,\beta/2}\right\}_{\beta>0}$ of Besov spaces.*

Note that the value of β^* is an intrinsic property of the space (M, d, μ), which is defined independently of any heat kernel. For example, for $\mathbb{R}^n$ with $\alpha = n$ we have $\beta^* = 2$.

Theorem 5.2 ([32]) *Let p_t be a heat kernel on a metric measure space (M, d, μ).*

(*a*) *If the heat kernel satisfies the lower bound*

$$p_t(x,y) \geq \frac{1}{t^{\alpha/\beta}}\Phi_1\left(\frac{d(x,y)}{t^{1/\beta}}\right),$$

where $\Phi_1(s) > 0$ for some $s > 0$, then $\Lambda_{2,\infty}^{\alpha,\beta/2}$ is dense in $L^2(M,\mu)$; consequently, $\beta \leq \beta^*$.

(b) *If the heat kernel is stochastically complete and satisfies* (2.37), *where* Φ_1 *is as above and* Φ_2 *satisfies*

$$\int^{\infty} s^{\alpha+\beta+\varepsilon}\Phi_2(s)\frac{ds}{s}<\infty, \tag{5.2}$$

for some $\varepsilon>0$, *then* $\beta=\beta^*$.

If p_t is the heat kernel of the operator $(-\Delta)^{\beta/2}$ in $\mathbb{R}^n$, $0<\beta<2$, then by (1.1) it does not satisfy (5.2). In this case, the conclusion of Theorem 5.2(b) is not true either, because β is strictly smaller than $\beta^*=2$.

Proof. (a) By Theorem 4.1(a), we have the inclusion $\mathcal{F}\subset\Lambda_{2,\infty}^{\alpha,\beta/2}$. Since $\mathcal{F}$ is always dense in L^2, the conclusion follows.

(b) It suffices to prove that, for any $\beta'>\beta$, the space $\Lambda_{2,\infty}^{\alpha,\beta'/2}$ is not dense in L^2. It suffices to assume that $\beta'-\beta$ is sufficiently small so that the condition (5.2) holds with $\varepsilon=\beta'-\beta$.

Firstly, let us show that $u\in\Lambda_{2,\infty}^{\alpha,\beta'/2}$ implies $\mathcal{E}[u]=0$. We use again the decomposition $\mathcal{E}_t[u]=A(t)+B(t)$, where $A(t)$ and $B(t)$ are defined in (4.9) and (4.10) where we set $r=1$. Estimating $B(t)$ similarly to (4.14) but using $N_{2,\infty}^{\alpha,\beta'/2}$ instead of $N_{2,\infty}^{\alpha,\beta/2}$ and (5.2), we obtain

$$\begin{aligned} B(t) &\leq C\sum_{k=0}^{\infty}\frac{r_{k+1}^{\alpha+\beta'}}{t^{1+\alpha/\beta}}\Phi_2\left(\frac{r_{k+1}}{t^{1/\beta}}\right)\frac{D_2(u,r_k)}{r_k^{\alpha+\beta'}} \\ &= Ct^{\delta}\sum_{k=0}^{\infty}\left(\frac{r_{k+1}}{t^{1/\beta}}\right)^{\alpha+\beta'}\Phi_2\left(\frac{r_{k+1}}{t^{1/\beta}}\right)\frac{D_2(u,r_k)}{r_k^{\alpha+\beta'}} \\ &\leq Ct^{\delta}\int_0^{\infty}s^{\alpha+\beta'}\Phi_2(s)\frac{ds}{s}\left(\sup_{0<\rho\leq 1}\frac{D_2(u,\rho)}{\rho^{\alpha+\beta'}}\right) \\ &\leq Ct^{\delta}N_{2,\infty}^{\alpha,\beta'/2}(u), \end{aligned} \tag{5.3}$$

where δ is found from the identity

$$1+\frac{\alpha}{\beta}=\frac{\alpha+\beta'}{\beta}-\delta,$$

that is,

$$\delta=\frac{\beta'}{\beta}-1>0.$$

Putting together (4.8), (4.13), and (5.3), we obtain

$$\mathcal{E}_t[u]\leq A(t)+Ct^{\delta}N_{2,\infty}^{\alpha,\beta'/2}(u)\to 0 \quad \text{as } t\to 0,$$

whence

$$\mathcal{E}[u]=\lim_{t\to 0}\mathcal{E}_t[u]=0.$$

Since $\mathcal{E}_t[u]\leq\mathcal{E}[u]$, this implies back that $\mathcal{E}_t[u]\equiv 0$ for all $t>0$.

On the other hand, choose $s>0$ so that $\Phi_1(s)>0$. Then it follows from (2.15) and the lower bound in (2.37) that

$$\mathcal{E}_t[u]\geq\frac{1}{2t^{\alpha/\beta+1}}\Phi_1(s)\iint\limits_{\left\{d(x,y)\leq st^{1/\beta}\right\}}(u(x)-u(y))^2d\mu(y)d\mu(x),$$

which yields $u(x) = u(y)$ for μ-almost all x, y such that $d(x, y) \leq st^{1/\beta}$. Since t is arbitrary, we conclude that u is a constant function.

Hence, we have shown that the space $\Lambda_{2,\infty}^{\alpha,\beta'/2}$ consists of constants. However, as it was shown in the proof of Lemma 2.8, the constant functions are not dense in $L^2(M,\mu)$, which finishes the proof. ■

Corollary 5.3 *If the heat kernel p_t satisfies the hypotheses of* Theorem 5.2(*b*) *then the values of the parameters α and β are the invariants of the metric space (M, d) alone. Moreover, we have*

$$\mu \simeq H^{\alpha} \quad \text{and} \quad \mathcal{E} \simeq N_{2,\infty}^{\alpha,\beta/2}. \tag{5.4}$$

Consequently, both measure μ and the energy form $\mathcal{E}$ are determined (up to a factor $\simeq 1$) by the metric space (M, d) alone.

Proof. By Corollary 2.14, α is the Hausdorff dimension of M and, hence, is the invariants of (M, d). Furthermore, measure μ is comparable with the Hausdorff measure H^{α} so that measure μ in the definition of the Besov spaces can be replaced by H^{α}. Hence, the critical exponent β^* of the family of the Besov spaces is the invariant of the metric space (M, d) alone. By Theorem 5.2(*b*), we have $\beta = \beta^*$, which implies that β is determined by (M, d) as well.

The relations (5.4) follow from Corollaries 2.14 and 4.2. Finally, since measure μ in the definition of the seminorm $N_{2,\infty}^{\alpha,\beta/2}$ can be replaced by the Hausdorff measure H^{a}, the seminorm $N_{2,\infty}^{\alpha,\beta/2}$ is determined by the metric space structure alone, whence the same holds for the energy $\mathcal{E}$. ■

Assume that there are two heat kernels $p_t^{(i)}(x, y)$, $i = 1, 2$, on a metric measure spaces $\left(M, d, \mu^{(i)}\right)$ where the underlying metric space (M, d) is the same but measures $\mu^{(1)}$ and $\mu^{(2)}$ may be different. Let $\mathcal{E}^{(i)}$, $i = 1, 2$, be the corresponding energy forms. Assume that each heat kernel $p_t^{(i)}$ satisfies the hypotheses of Theorem 5.2(*b*) with parameters $\alpha^{(i)}$ and $\beta^{(i)}$, respectively. Then it follows from Corollary 5.3 that $\alpha^{(1)} = \alpha^{(2)}$, $\beta^{(1)} = \beta^{(2)}$, $\mu^{(1)} \simeq \mu^{(2)}$, and $\mathcal{E}^{(1)} \simeq \mathcal{E}^{(2)}$. Let us show an example of application of this result.

Example 5.4 Consider in $\mathbb{R}^n$ the Gauss-Weierstrass heat kernel

$$p_t(x, y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right)$$

and its generator $\mathcal{L} = -\Delta$ in $L^2(\mathbb{R}^n)$ with the Lebesgue measure. Then $\alpha = n$, $\beta = 2$, and

$$\mathcal{E}[u] = \int_{\mathbb{R}^n} |\nabla u|^2 \, dx.$$

Consider now another elliptic operator in $\mathbb{R}^n$:

$$\mathcal{L} = -\frac{1}{m(x)} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x) \frac{\partial}{\partial x_j}\right), \tag{5.5}$$

where $m(x)$ and $a_{ij}(x)$ are continuous functions, $m(x) > 0$ and the matrix $(a_{ij}(x))$ is positive definite. The operator $\mathcal{L}$ is symmetric with respect to measure

$$d\mu = m(x)\, dx,$$

and its energy form is

$$\mathcal{E}[u]=\int_{\mathbb{R}^n}a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}dx.$$

Let $d(x,y)=|x-y|$ and assume that the heat kernel $p_t(x,y)$ of $\mathcal{L}$ satisfies the conditions of Theorem 5.2(b). Then we conclude by Corollary 5.3 that α and β must be the same as for the Gauss-Weierstrass heat kernel, that is, $\alpha=n$ and $\beta=2$; moreover, measure μ must be comparable to the Lebesgue measure, which implies that $m\simeq 1$, and the energy form must admit the estimate

$$\mathcal{E}[u]\simeq\int_{\mathbb{R}^n}|\nabla u|^2\,dx,$$

which implies that the matrix $(a_{ij}(x))$ is uniformly elliptic. Hence, the operator $\mathcal{L}$ is uniformly elliptic. By Aronson's theorem (see [1], [45]), we obtain that the heat kernel of $\mathcal{L}$ satisfies the estimate

$$p_t(x,y)\simeq\frac{C}{t^{n/2}}\exp\left(-c\frac{|x-y|^2}{t}\right). \tag{5.6}$$

The conclusion is that the estimate (5.6) for the operator (5.5) holds if and only if $m\simeq 1$ and $(a_{ij}(x))$ is uniformly elliptic (assuming that $\mathcal{L}$ is stochastically complete). As far as we know, the necessity part of this statement is a new result.

5.2 Inequalities for the walk dimension

Definition 5.5 We say that a metric space (M,d) satisfies the *chain condition* if there exists a (large) constant C such that for any two points $x,y\in M$ and for any positive integer n there exists a sequence $\{x_i\}_{i=0}^n$ of points in M such that $x_0=x$, $x_n=y$, and

$$d(x_i,x_{i+1})\leq C\frac{d(x,y)}{n},\quad \text{for all } i=0,1,...,n-1. \tag{5.7}$$

The sequence $\{x_i\}_{i=0}^n$ is referred to as a chain connecting x and y.

For example, the chain condition is satisfied if (M,d) is a *length space*, that is if the distance $d(x,y)$ is defined as the infimum of the length of all continuous curves connecting x and y, with a proper definition of length. On the other hand, the chain condition is not satisfied if M is a locally finite graph, and d is the graph distance.

Recall that the critical exponent $\beta^*=\beta^*(M,d,\mu)$ of the family of Besov spaces $\Lambda_{2,\infty}^{\alpha,\sigma}$ was defined by (5.1).

Theorem 5.6 ([32]) *Let (M,d,μ) be a metric measure space.*

(a) *If $0<\mu(B(x,r))<\infty$ for all $x\in M$ and $r>0$, and*

$$\mu(B(x,r))\leq Cr^\alpha \tag{5.8}$$

for all $x\in M$ and $0<r\leq 1$ then

$$\beta^*\geq 2. \tag{5.9}$$

(b) *If the space* (M, d) *satisfies the chain condition and*

$$\mu(B(x,r)) \simeq r^\alpha \tag{5.10}$$

for all $x \in M$ *and* $0 < r \le 1$ *then*

$$\beta^* \le \alpha + 1. \tag{5.11}$$

Observe that the chain condition is essential for the inequality $\beta^* \le \alpha + 1$ to be true. Indeed, assume for a moment that the claim of Theorem 5.6(*b*) holds without the chain condition, and consider a new metric d' on M given by $d' = d^{1/\gamma}$ where $\gamma > 1$. Let us mark by a dash all notions related to the space (M, d', μ) as opposed to those of (M, d, μ). It is easy to see that $\alpha' = \alpha\gamma$ and $N'_{\sigma\gamma} = N_\sigma$; in particular, the latter implies $\beta^{*\prime} = \beta^*\gamma$. Hence, if Theorem 5.6 could be applied to the space (M, d', μ) it would yield $\beta^*\gamma \le \alpha\gamma + 1$ which implies $\beta^* \le \alpha$ because γ may be taken arbitrarily large. However, there are spaces with $\beta^* > \alpha$, for example SG.

Clearly, the metric d' does not satisfy the chain condition; indeed the inequality (5.7) implies

$$d'(x_i, x_{i+1}) \le C\frac{d'(x,y)}{n^{1/\gamma}}, \tag{5.12}$$

which is not good enough. Note that if in the inequality (5.7) we replace n by $n^{1/\gamma}$ then the proof below will give $\beta^* \le \alpha + \gamma$ instead of $\beta^* \le \alpha + 1$.

Corollary 5.7 *Let* p_t *be a stochastically complete heat kernel on a metric measure space* (M, d, μ) *that satisfies the estimate* (2.37) *where* $\Phi_1(s) > 0$ *for some* $s > 0$.

(a) *If for some* $\varepsilon > 0$

$$\int_0^\infty s^{\alpha+\beta+\varepsilon}\Phi_2(s)\frac{ds}{s} < \infty \tag{5.13}$$

then $\beta \ge 2$.

(b) *If* (M, d) *satisfies the chain condition and*

$$\int_0^\infty s^\alpha \Phi_2(s)\frac{ds}{s} < \infty \tag{5.14}$$

then $\beta \le \alpha + 1$.

(c) *If* (M, d) *satisfies the chain condition and for some* $\varepsilon > 0$

$$\int_0^\infty s^{2\alpha+1+\varepsilon}\Phi_2(s)\frac{ds}{s} < \infty \tag{5.15}$$

then

$$2 \le \beta \le \alpha + 1. \tag{5.16}$$

Note that by Lemma 2.7, the condition (5.15) can occur only for a local Dirichlet form $\mathcal{E}$. The set of couples (α, β) satisfying (5.16) is shaded on Fig. 8.

By [7], any couple of α, β satisfying (5.16) can be realized for the heat kernel estimate

$$p_t(x,y) \asymp \frac{C}{t^{\alpha/\beta}}\exp\left(-c\left(\frac{d^\beta(x,y)}{t}\right)^{\frac{1}{\beta-1}}\right), \tag{5.17}$$

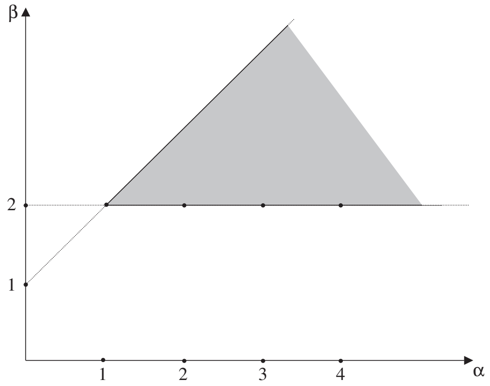

Figure 8: The set $2 \leq \beta \leq \alpha + 1$

with a local Dirichlet form.

In the case of a non-local form, we can only claim by Corollary 5.7(b) that

$$0 < \beta \leq \alpha + 1.$$

In fact, any couple α, β in the range $0 < \beta < \alpha + 1$ can be realized for the estimate

$$p_t(x, y) \simeq \frac{1}{t^{\alpha/\beta}} \left(1 + \frac{d(x, y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)}.$$

Indeed, if $\mathcal{L}$ is the generator of a diffusion with parameters α and β satisfying (5.17) then the operator $\mathcal{L}^\delta$, $\delta \in (0, 1)$, generates a jump process with the walk dimension $\beta' = \delta\beta$ and the same α (cf. Lemma 4.4). Clearly, β' can take any value from $(0, \alpha + 1)$. We do not know whether the walk dimension for a non-local form can be equal to $\alpha + 1$.

Proof of Corollary 5.7. (a) By Theorems 2.11(a) and 5.6(a) we have $\beta^* \geq 2$, and by Theorem 5.2(b) we have $\beta = \beta^*$, whence $\beta \geq 2$.

(b) By Theorems 2.11(b) and 5.6(b) we have $\beta^* \leq \alpha + 1$, and by Theorem 5.2(a) we have $\beta \leq \beta^*$, whence $\beta \leq \alpha + 1$.

(c) By part (b), we have $\beta \leq \alpha + 1$. Therefore, (5.15) implies (5.13), and by part (a) we obtain $\beta \geq 2$. ■

Corollary 5.8 *Let (M, d) satisfy the chain condition and p_t be a stochastically complete heat kernel on (M, d, μ) such that*

$$p_t(x, y) \asymp \frac{C}{t^{\alpha/\beta}} \Phi\left(c \frac{d(x, y)}{t^{1/\beta}}\right). \tag{5.18}$$

Then $\beta \leq \alpha + 1$.

Proof. As in the proof of Corollary 2.12, the estimate (5.18) implies that function Φ satisfies (5.14) and $\Phi(s) > 0$ for some $s > 0$, whence the claim follows by Corollary 5.7(b). ■

Proof of Theorem 5.6(a)**.** It suffices to show that $\Lambda_{2,\infty}^{\alpha,1}$ is dense in $L^2 = L^2(M,\mu)$. Let u be a Lipschitz function with a bounded support A. Let A_r be the closed r-neighborhood of A. If L is the Lipschitz constant of u then, for any $r \in (0,1]$,

$$\begin{aligned} D_2(u,r) &= \int_M \int_{B(x,r)} |u(x)-u(y)|^2 d\mu(y)d\mu(x) \\ &\leq \int_{A_r}\int_{B(x,r)} Lr^2 d\mu(y)d\mu(x) \\ &\leq Cr^{\alpha+2}\mu(A_r). \end{aligned}$$

It follows that

$$\sup_{0<r\leq 1} \frac{D_2(u,r)}{r^{\alpha+2}} < \infty,$$

whence we conclude that $u \in \Lambda_{2,\infty}^{\alpha,1}$.

Let now A be any bounded closed subset of M. For any positive integer n, consider the function on M

$$f_n(x) = (1 - nd(x,A))_+,$$

which is Lipschitz and is supported in $A_{1/n}$. Hence, $f_n \in \Lambda_{2,\infty}^{\alpha,1}$. Clearly, $f_n \to 1_A$ in L^2 as $n \to \infty$, whence it follows that $1_A \in \overline{\Lambda_{2,\infty}^{\alpha,1}}$, where the bar means closure in L^2. Since the linear combinations of the indicator functions of bounded closed sets form a dense subset in L^2, it follows that $\overline{\Lambda_{2,\infty}^{\alpha,1}} = L^2$, which was to be proved. ■

We precede the proof of Theorem 5.6(b) by a lemma.

Lemma 5.9 *Let $\{x_i\}_{i=0}^n$ be a sequence of points in a metric space (M,d) such that for some $\rho > 0$ we have $d(x_0,x_n) > 2\rho$ and*

$$d(x_i,x_{i+1}) < \rho \quad \textit{for all } i = 0,1,...,n-1. \tag{5.19}$$

Then there exists a subsequence $\{x_{i_k}\}_{k=0}^l$ such that

(a) $0 = i_0 < i_1 < ... < i_l = n$;

(b) $d(x_{i_k}, x_{i_{k+1}}) < 5\rho$ *for all* $k = 0,1,...,l-1$;

(c) $d(x_{i_k}, x_{i_j}) \geq 2\rho$ *for all distinct* $k,j = 0,1,...,l$.

The significance of conditions $(a),(b),(c)$ is that a sequence $\{x_{i_k}\}_{k=0}^l$ satisfying them gives rise to a chain of balls $B(x_{i_k},5\rho)$ connecting the points x_0 and x_n in a way that each ball contains the center of the next one whereas the balls $B(x_{i_k},\rho)$ are disjoint. This is similar to the classical ball covering argument, but additional difficulties arise from the necessity to maintain a proper order in the set of balls.

Proof. Let us say that a sequence of indices $\{i_k\}_{k=0}^l$ is *good* if the following conditions are satisfied:

(a') $0 = i_0 < i_1 < ... < i_l$;

(b') $d(x_{i_k}, x_{i_{k+1}}) < 3\rho$ for all $k = 0,1,...,l-1$;

(c') $d(x_{i_k}, x_{i_j}) \geq 2\rho$ for all distinct $k,j = 0,1,...,l$.

Note that a good sequence does not necessarily have $i_l = n$ as required in condition (a). We start with a good sequence that consists of a single index $i_0 = 0$, and will successively redefine it to increase at each step the value of i_l. A terminal good sequence will be used to construct a sequence satisfying $(a), (b), (c)$.

Assuming that $\{i_k\}_{k=0}^{l}$ is a good sequence, define the following set of indices

$$S := \{s : i_l < s \leq n \text{ and } d(x_s, x_{i_k}) \geq 2\rho \quad \text{for all } k \leq l\},$$

and consider two cases.

Set S is non-empty. In this case we will redefine $\{i_k\}$ to increase i_l. Let m be the minimal index in S. Therefore, $m-1$ is not in S, whence we have either $m-1 \leq i_l$ or

$$d(x_{m-1}, x_{i_k}) < 2\rho \quad \text{for some } k \leq l \tag{5.20}$$

(see Fig. 9). In the first case, we have in fact $m-1 = i_l$ so that (5.20) also holds (with $k = l$).

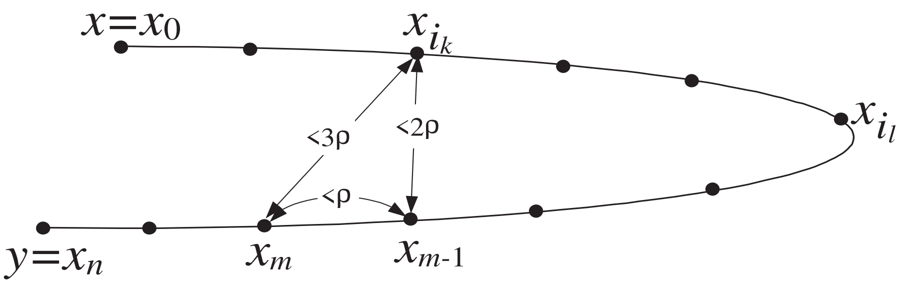

Figure 9: Illustration to the proof of Lemma 5.9

By (5.20) and (b') we obtain, for the same k as in (5.20),

$$d(x_m, x_{i_k}) \leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{i_k}) < 3\rho.$$

Now we modify the sequence $\{i_j\}$ as follows: keep $i_0, i_1, ..., i_k$ as before, forget the previously selected indices $i_{k+1}, ..., i_l$, and set $i_{k+1} := m$ and $l := k+1$.

Clearly, the new sequence $\{i_k\}_{k=0}^{l}$ is also good, and the value of i_l has increased (although l may have decreased). Therefore, this construction can be repeated only a finite number of times.

Set S is empty. In this case, we will use the existing good sequence to construct a sequence satisfying conditions (a), (b), (c). The set S can be empty for two reasons:

- either $i_l = n$
- or $i_l < n$ and for any index s such that $i_l < s \leq n$ there exists $k \leq l$ such that $d(x_s, x_{i_k}) < 2\rho$.

In the first case the sequence $\{i_k\}_{k=0}^{l}$ already satisfies (a), (b), (c), and the proof is finished. In the second case, choose the minimal $k \leq l$ such that $d(x_n, x_{i_k}) < 2\rho$ (see Fig. 10).

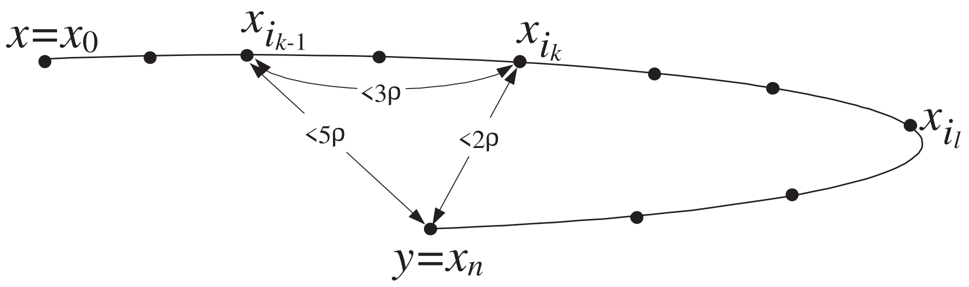

Figure 10: Illustration to the proof of Lemma 5.9

The hypothesis $d(x_n, x_0) \geq 2\rho$ implies $k \geq 1$, and we obtain from (b')

$$d(x_n, x_{i_{k-1}}) \leq d(x_n, x_{i_k}) + d(x_{i_k}, x_{i_{k-1}}) < 5\rho.$$

By the minimality of k, we have also $d(x_n, x_{i_j}) \geq 2\rho$ for all $j < k$. Hence, we define the final sequence $\{i_j\}$ as follows: keep $i_0, i_1, ..., i_{k-1}$ as before, forget $i_k, ..., i_l$, and set $i_k := n$ and $l := k$. Then this sequence satisfies $(a), (b), (c)$. ■

Let A be a subset of M of finite measure, that is $\mu(A) < \infty$. Then any function $u \in L^2$ is integrable on A, and let us set

$$u_A := \frac{1}{\mu(A)} \int_A u \, d\mu.$$

For any two measurable sets $A, B \subset M$ of finite measure, the following identity takes place

$$\begin{aligned} & \int_A \int_B |u(x) - u(y)|^2 \, d\mu(x) d\mu(y) \qquad (5.21) \\ = \; & \mu(A) \int_B |u - u_B|^2 \, d\mu + \mu(B) \int_A |u - u_A|^2 \, d\mu + \mu(A)\mu(B) \, |u_A - u_B|^2 , \end{aligned}$$

which is proved by a straightforward computation.

Proof of Theorem 5.6(b)**.** The hypothesis (5.10) implies that the space $L^2(M, \mu)$ is ∞-dimensional. The inequality $\beta^* \leq \alpha + 1$ will follow from (5.1) if we show that, for any $\sigma > \frac{\alpha+1}{2}$, the space $\Lambda_{2,\infty}^{\alpha,\sigma}$ contains only constants, that is, $N_{2,\infty}^{\alpha,\sigma}(u) < \infty$ implies $u \equiv \text{const}$. By definition (3.5) of $N_{2,\infty}^{\alpha,\sigma}$ and (5.10) we have, for any $0 < r \leq 1$,

$$N_{2,\infty}^{\alpha,\sigma}(u) \geq c r^{-2\sigma-\alpha} \iint\limits_{\{d(x,y)<r\}} |u(x) - u(y)|^2 d\mu(y) d\mu(x). \qquad (5.22)$$

Fix some $0 < r \leq 1$ and assume that we have a sequence of disjoint balls $\{B_k\}_{k=0}^{l}$ of the same radius $0 < \rho < 1$, such that for all $k = 0, 1, ..., l-1$

$$x \in B_k \quad \text{and} \quad y \in B_{k+1} \quad \Longrightarrow \quad d(x, y) < r. \qquad (5.23)$$

Then (5.22), (5.21), and (5.10) imply

$$\begin{aligned} N_{2,\infty}^{\alpha,\sigma}(u) \; & \geq \; c r^{-2\sigma-\alpha} \sum_{k=0}^{l-1} \int_{B_k} \int_{B_{k+1}} |u(x) - u(y)|^2 \, d\mu(y) d\mu(x) \\ & \geq \; c r^{-2\sigma-\alpha} \rho^{2\alpha} \sum_{k=0}^{l-1} \left| u_{B_k} - u_{B_{k+1}} \right|^2 . \qquad (5.24) \end{aligned}$$

By the chain condition, for any two distinct points $x, y \in M$ and for any positive integer n there exists a sequence of points $\{x_i\}_{i=0}^n$ such that $x_0 = x$, $x_n = y$, and

$$d(x_i, x_{i+1}) < C\frac{d(x,y)}{n} := \rho, \quad \text{for all } 0 \le i < n.$$

Assuming that n is large enough so that $d(x,y) > 2\rho$ and $\rho < 1/7$, we obtain by Lemma 5.9 that there exists a subsequence $\{x_{i_k}\}_{k=0}^l$ (where of course $l \le n$) such that $x_{i_0} = x$, $x_{i_l} = y$, the balls $\{B(x_{i_k}, \rho)\}$ are disjoint, and

$$d(x_{i_k}, x_{i_{k+1}}) < 5\rho, \tag{5.25}$$

for all $k = 0, 1, ..., l-1$.

Applying (5.24) to the balls $B_k := B(x_{i_k}, \rho)$ and setting $r = 7\rho < 1$ (which together with (5.25) ensures (5.23)) we obtain

$$\begin{aligned} N_{2,\infty}^{\alpha,\sigma}(u) &\ge c\rho^{-2\sigma+\alpha}\sum_{k=0}^{l-1}\left|u_{B_k} - u_{B_{k+1}}\right|^2 \\ &\ge c\rho^{-2\sigma+\alpha}\frac{1}{l}\left(\sum_{k=0}^{l-1}\left|u_{B_k} - u_{B_{k+1}}\right|\right)^2 \\ &\ge c\rho^{-2\sigma+\alpha}\frac{1}{n}\left|u_{B_0} - u_{B_l}\right|^2 \\ &\ge c\rho^{-2\sigma+\alpha+1}\frac{\left|u_{B(x,\rho)} - u_{B(y,\rho)}\right|^2}{d(x,y)}. \end{aligned} \tag{5.26}$$

By the Lebesgue theorem, we have, for μ-almost all $x \in M$,

$$\lim_{\rho\to 0} u_{B(x,\rho)} = u(x), \tag{5.27}$$

It follows from (5.26) and (5.27) as $n \to \infty$ (that is, as $\rho \to 0$) that, for μ-almost all $x, y \in M$,

$$\frac{|u(x) - u(y)|^2}{d(x,y)} \le C\, N_{2,\infty}^{\alpha,\sigma}(u)\lim_{\rho\to 0}\rho^{2\sigma-\alpha-1}.$$

Since $2\sigma > \alpha + 1$ and $N_{2,\infty}^{\alpha,\sigma}(u) < \infty$, the above limit is equal to 0 whence $u \equiv \text{const}$. ■

6 Two-sided estimates in the local case

6.1 The Dirichlet form in subsets

Let $(\mathcal{E}, \mathcal{F})$ be a Dirichlet form in a metric measure space (M, d, μ). Let $\mathcal{L}$ be the generator of $\mathcal{E}$ and consider the corresponding heat semigroup $P_t = e^{-t\mathcal{L}}$, $t \ge 0$. Denote by R_λ, $\lambda > 0$, the resolvent operator of $\mathcal{L}$, that is,

$$R_\lambda = (\mathcal{L} + \lambda \mathrm{I})^{-1}.$$

For any open subset $\Omega \subset M$, let $\mathcal{F}_0(\Omega)$ be the set of functions from $\mathcal{F}$ whose support is compact and is contained in Ω. Then define $\mathcal{F}(\Omega)$ as the closure of $\mathcal{F}_0(\Omega)$ in $\mathcal{F}$ with respect to the $\mathcal{E}_1$-norm. It follows that any function from $\mathcal{F}(\Omega)$ vanishes in Ω^c and, hence, can be identified as an element of $L^2(\Omega)$. If $\mathcal{F}(\Omega)$ is dense in $L^2(\Omega)$ then $(\mathcal{E}, \mathcal{F}(\Omega))$ is a Dirichlet form in $L^2(\Omega)$. In this case, denote by $\mathcal{L}^\Omega$, P_t^Ω, R_λ^Ω respectively the generator, the heat semigroup, and the resolvent of $(\mathcal{E}, \mathcal{F}(\Omega))$. If $f \in L^2(M)$ then set $P_t^\Omega f := P_t^\Omega(f|_\Omega)$ and $R_\lambda^\Omega f := R_\lambda^\Omega(f|_\Omega)$.

In general $\mathcal{F}(\Omega)$ need not be dense in $L^2(\Omega)$. However, if the Dirichlet form $(\mathcal{E}, \mathcal{F})$ is regular, that is, $C_0(M) \cap \mathcal{F}$ is dense both in $\mathcal{F}$ and $C_0(M)$ then $\mathcal{F}_0(\Omega)$ is obviously dense in $L^2(\Omega)$. In this case, $\mathcal{F}(\Omega)$ coincides with the closure of $\mathcal{F} \cap C_0(\Omega)$ in $\mathcal{F}$, and $(\mathcal{E}, \mathcal{F}(\Omega))$ is also a regular Dirichlet form (see [24, Corollary 2.3.1, p.95 and Theorem 4.4.2, p.154]).

6.2 Maximum principles

We cite here two lemmas referring to [31, Appendix] for their proofs.

We say that a function $w \in \mathcal{F}$ satisfies the inequality $\mathcal{L}w \le f$ weakly in an open set $\Omega \subset M$ where $f \in L^2(\Omega)$, if, for any non-negative function $\varphi \in \mathcal{F}(\Omega)$

$$\mathcal{E}(w, \varphi) \le (f, \varphi)_{L^2}.$$

Similar one defines the opposite inequality and equality. For example, if $f \in L^2(M)$ and $w = R_\lambda f$ then w satisfies the equation

$$\mathcal{L}w + \lambda w = f$$

weakly in any open set $\Omega \subset M$.

Lemma 6.1 *Let $(\mathcal{E}, \mathcal{F})$ be a local regular the Dirichlet form in (M, d, μ). Let $\Omega \subset M$ be a precompact open set and $\lambda > 0$. If a function $w \in \mathcal{F} \cap L^\infty(M)$ is such that $0 \le w \le 1$ in Ω and w satisfies weakly in Ω the inequality*

$$\mathcal{L}w + \lambda w \le 0, \tag{6.1}$$

then

$$w \le 1 - \lambda R_\lambda^\Omega \mathbf{1}_\Omega \text{ in } \Omega. \tag{6.2}$$

Lemma 6.2 *Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form in (M, d, μ). For any open subset $U \subset M$, for any compact set $K \subset U$, for any non-negative function $f \in L^2(M)$, for all $t > 0$, and for μ-almost all $x \in M$, we have*

$$0 \le P_t f(x) - P_t^U f(x) \le \sup_{s \in [0,t]} \operatorname*{essup}_{K^c} P_s f. \tag{6.3}$$

6.3 A tail estimate

If $B = B(x, r)$ then we use αB as a shorthand for $B(x, \alpha r)$.

Theorem 6.3 ([28], [31])*Assume that $(\mathcal{E}, \mathcal{F})$ is a regular conservative Dirichlet form in $L^2(M, \mu)$ and let all metric balls in M be precompact. Fix $\beta > 0$. The following conditions are equivalent:*

(i) For any $\varepsilon \in (0, 1)$ there exists $K > 0$ such that, for any $t > 0$ and any ball $B = B(x_0, r)$ with $r \ge K t^{1/\beta}$,

$$P_t \mathbf{1}_{B^c} \le \varepsilon \text{ a.e. in } \frac{1}{4}B. \tag{6.4}$$

(*ii*) *For any* $\varepsilon \in (0,1)$ *there exists* $K > 0$ *such that, for any* $t > 0$ *and any ball* $B = B(x_0, r)$ *with* $r \geq Kt^{1/\beta}$,

$$P_t^B \mathbf{1}_B \geq 1 - \varepsilon \text{ a.e. } in \ \frac{1}{4}B. \tag{6.5}$$

(*iii*) *For any* $\varepsilon \in (0,1)$ *there exists* $K > 0$ *such that, for any* $\lambda > 0$ *and any ball* $B = B(x_0, r)$ *with* $r \geq K\lambda^{-1/\beta}$,

$$\lambda R_\lambda^B \mathbf{1}_B \geq 1 - \varepsilon \text{ a.e. } in \ \frac{1}{4}B. \tag{6.6}$$

Remark 6.4 If the heat semigroup P_t possesses the heat kernel $p_t(x,y)$ then the condition (i) can be equivalently stated as follows: for any $\varepsilon \in (0,1)$ there exists $K > 0$ such that, for all $t > 0$, $r \geq Kt^{1/\beta}$, and almost all $x \in M$,

$$\int_{B(x,r)^c} p_t(x,y)\, d\mu(y) \leq \varepsilon. \tag{6.7}$$

Indeed, for any ball $B(x_0, r)$ and for almost all $x \in B(x_0, r/4)$ (or even $x \in B(x_0, r/2)$), we have

$$P_t \mathbf{1}_{B(x_0,r)^c}(x) = \int_{B(x_0,r)^c} p_t(x,y)\, d\mu(y) \leq \int_{B(x,r/2)^c} p_t(x,y)\, d\mu(y),$$

so that (6.7) implies (6.4) (with K being replaced by $2K$). Similarly, for almost all $x \in B(x_0, r/2)$,

$$\int_{B(x,r)^c} p_t(x,y)\, d\mu(y) \leq \int_{B(x_0,r/2)^c} p_t(x,y)\, d\mu(y) = P_t \mathbf{1}_{B(x_0,r/2)^c}(x),$$

so that (6.4) implies (6.7), for almost all $x \in B(x_0, r/8)$. Covering M by a countable family of balls of radius $r/8$, we obtain that (6.7) holds for almost all $x \in M$.

Proof of Theorem 6.3. $(i) \Rightarrow (ii)$. Applying the estimate (6.3) of Lemma 6.2 to function $f = \mathbf{1}_{\frac{1}{2}B}$, we obtain that

$$P_t^B \mathbf{1}_{\frac{1}{2}B}(x) \geq P_t \mathbf{1}_{\frac{1}{2}B}(x) - \sup_{s \in [0,t]} \operatorname*{essup}_{\left(\frac{3}{4}B\right)^c} P_s \mathbf{1}_{\frac{1}{2}B}, \tag{6.8}$$

for all $t > 0$ and a.e. $x \in M$. For any $x \in \frac{1}{4}B$, we have that $B(x, r/4) \subset \frac{1}{2}B$ (see Fig. 11). Using (6.4) and the identity $P_t 1 = 1$ a.e., we obtain, for any $x \in \frac{1}{4}B$,

$$P_t \mathbf{1}_{\frac{1}{2}B} = 1 - P_t \mathbf{1}_{\left(\frac{1}{2}B\right)^c} \geq 1 - P_t \mathbf{1}_{B(x,r/4)^c}.$$

Applying the hypothesis (i) for the ball $B(x, r/4)$, we obtain that

$$P_t \mathbf{1}_{B(x,r/4)^c} \leq \varepsilon \text{ a.e. in } B(x, r/16),$$

provided

$$\frac{r}{4} \geq Kt^{1/\beta} \tag{6.9}$$

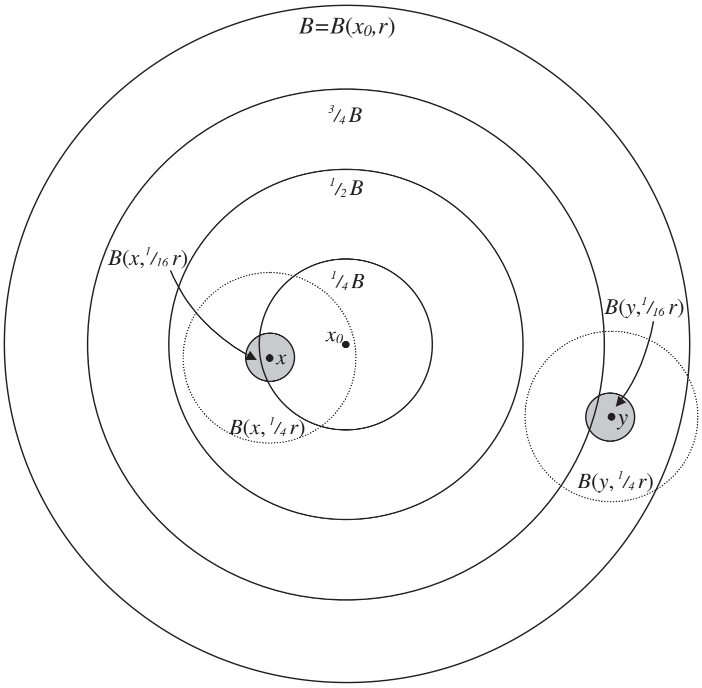

Figure 11: Illustration to the proof of $(i) \Rightarrow (ii)$

with sufficiently large K. It follows that, for any $x \in \frac{1}{4}B$,

$$P_t \mathbf{1}_{\frac{1}{2}B} \geq 1 - \varepsilon \text{ a.e. in } B\left(x, r/16\right),$$

whence

$$P_t \mathbf{1}_{\frac{1}{2}B} \geq 1 - \varepsilon \text{ a.e. in } \frac{1}{4}B. \tag{6.10}$$

On the other hand, for any $y \in \left(\frac{3}{4}B\right)^c$, we have $\frac{1}{2}B \subset B\left(y, r/4\right)^c$ (see Fig. 11)), whence

$$P_s \mathbf{1}_{\frac{1}{2}B} \leq P_s \mathbf{1}_{B(y,r/4)^c}.$$

Applying the hypothesis (i) for the ball $B\left(y, r/4\right)$ at time s, we obtain that if (6.9) holds for sufficiently large K then, for all $0 < s \leq t$,

$$P_s \mathbf{1}_{B(y,r/4)^c} \leq \varepsilon \text{ a.e. in } B\left(y, r/16\right).$$

It follows that, for any $y \in \left(\frac{3}{4}B\right)^c$,

$$P_s \mathbf{1}_{\frac{1}{2}B} \leq \varepsilon \text{ a.e. in } B\left(y, r/16\right),$$

whence

$$P_s \mathbf{1}_{\frac{1}{2}B} \leq \varepsilon \text{ a.e. in } \left(\frac{3}{4}B\right)^c. \tag{6.11}$$

Combining (6.8), (6.10) and (6.11), we obtain that, under the condition (6.9),

$$P_t^B \mathbf{1}_B \geq P_t^B \mathbf{1}_{\frac{1}{2}B} \geq 1 - 2\varepsilon \text{ a.e. in } \frac{1}{4}B, \tag{6.12}$$

which is equivalent to (6.5).

$(ii) \Rightarrow (iii)$. By (ii), we have (6.5) provided $t \le (r/K)^\beta$ whence

$$\lambda R_\lambda^B \mathbf{1}_B = \lambda \int_0^\infty e^{-\lambda t} P_t^B \mathbf{1}_B dt \ge \lambda \int_0^{(r/K)^\beta} e^{-\lambda t} P_t^B \mathbf{1}_B dt \ge (1-\varepsilon)\left(1 - e^{-\lambda (r/K)^\beta}\right),$$

which holds almost everywhere in $\frac{1}{4}B$. If

$$\lambda \left(\frac{r}{K}\right)^\beta \ge \log \frac{1}{\varepsilon} \tag{6.13}$$

then we obtain

$$\lambda R_\lambda^B \mathbf{1}_B \ge (1-\varepsilon)^2 \text{ a.e. in } \frac{1}{4}B,$$

which is equivalent to (6.6). The condition (6.13) is equivalent to

$$r \ge K \left(\frac{\log \frac{1}{\varepsilon}}{\lambda}\right)^{1/\beta}, \tag{6.14}$$

which can be assumed to be true by the hypothesis of (iii).

$(iii) \Rightarrow (i)$. Let us first show that, for all $t, \lambda > 0$,

$$P_t^B \mathbf{1}_B \ge 1 - e^{\lambda t}\left(1 - \lambda R_\lambda^B \mathbf{1}_B\right). \tag{6.15}$$

Indeed, using the facts that $P_s^B \mathbf{1}_B \le \mathbf{1}_B$ and

$$P_{s+t}^B \mathbf{1}_B = P_t^B \left(P_s^B \mathbf{1}_B\right) \le P_t^B \mathbf{1}_B,$$

we obtain that

$$\begin{aligned}
\lambda R_\lambda^B \mathbf{1}_B &= \lambda \int_0^\infty e^{-\lambda s} P_s^B \mathbf{1}_B \, ds \\
&= \lambda \int_0^t e^{-\lambda s} P_s^B \mathbf{1}_B \, ds + \lambda \int_t^\infty e^{-\lambda s} P_s^B \mathbf{1}_B \, ds \\
&\le \left(1 - e^{-\lambda t}\right) + \lambda \int_0^\infty e^{-\lambda (s+t)} P_{s+t}^B \mathbf{1}_B \, ds \\
&\le 1 - e^{-\lambda t} + e^{-\lambda t} P_t^B \mathbf{1}_B,
\end{aligned}$$

thus giving (6.15).

Given $\varepsilon \in (0,1)$, $t > 0$ and $r \ge K t^{1/\beta}$ (where K is defined by the hypothesis (iii)), choose λ from the condition $r = K \lambda^{-1/\beta}$. Then it follows from (6.6) and (6.15) that

$$P_t \mathbf{1}_B \ge P_t^B \mathbf{1}_B \ge 1 - \varepsilon e^{\lambda t} \text{ a.e. in } \frac{1}{4}B.$$

Using $P_t 1 \le 1$ and observing that

$$\lambda t \le \lambda \left(\frac{r}{K}\right)^\beta = 1,$$

we obtain

$$P_t \mathbf{1}_{B^c} = 1 - P_t \mathbf{1}_B \le \varepsilon e^{\lambda t} \le \varepsilon e \text{ a.e. in } \frac{1}{4}B,$$

which is equivalent to (6.4). ■

The following statement is an extension of Theorem 6.3 in the case of a local Dirichlet form.

Theorem 6.5 ([28], [31]) *Assume that all the hypotheses of Theorem 6.3 hold, and in addition that the Dirichlet form $(\mathcal{E},\mathcal{F})$ is local. Then each of the conditions (i), (ii), (iii) of Theorem 6.3 is equivalent to the following:*

(iv) *There are $c, C>0$ such that, for all $\lambda, t>0$ and any ball $B=B(x_0,r)$,*

$$\operatorname*{essup}_{\frac{1}{2}B} P_t \mathbf{1}_{B^c} \le C\exp\left(\lambda t - cr\lambda^{1/\beta}\right). \tag{6.16}$$

Remark 6.6 If in addition the heat semigroup P_t possesses the heat kernel $p_t(x,y)$ then the condition (iv) can be equivalently stated as follows: for all $\lambda, t, r>0$ and for almost all $x\in M$,

$$\int_{B(x,r)^c} p_t(x,y)\,d\mu(y) \le C\exp\left(\lambda t - cr\lambda^{1/\beta}\right) \tag{6.17}$$

(cf. Remark 6.4). If $\beta>1$ then taking $\lambda = \left(\frac{cr}{2t}\right)^{\frac{\beta}{\beta-1}}$, we obtain from (6.17)

$$\int_{B(x,r)^c} p_t(x,y)\,d\mu(y) \le C\exp\left(-c\left(\frac{r^\beta}{t}\right)^{\frac{1}{\beta-1}}\right).$$

Proof of Theorem 6.5. Let us first prove that $(iv)\Rightarrow(i)$. Assuming that $r\ge Kt^{1/\beta}$ and setting $\lambda=1/t$, we obtain

$$\lambda t - cr\lambda^{1/\beta} = 1-cK.$$

Hence, the right hand side of (6.16) can be made arbitrarily small, provided K is big enough, which yields (6.4).

Now we prove the main implication $(iii)\Rightarrow(iv)$. This proof is rather long and will be split into five steps.

Step 1. We claim that, for any $\varepsilon>0$, there exists $K>0$ with the following property: if a function $w\in\mathcal{F}\cap L^\infty(M)$ is such that $0\le w\le 1$ in a ball $B=B(x_0,r)$ and w satisfies weakly in B the equation

$$\mathcal{L}w+\lambda w=0,$$

where $\lambda>0$ and r are related by

$$r\ge K\lambda^{-1/\beta},$$

then

$$w\le\varepsilon \text{ a.e. in } \frac{1}{4}B.$$

Indeed, since the Dirichlet form $\mathcal{E}$ is local and the ball is precompact, we have by Lemma 6.1 that

$$w\le 1-\lambda R_\lambda^B \mathbf{1}_B \text{ a.e. in } B.$$

By (iii), we have

$$\lambda R_\lambda^B \mathbf{1}_B \ge 1-\varepsilon \text{ a.e. in } \frac{1}{4}B,$$

provided $r\ge K\lambda^{-1/\beta}$, where K is now defined by the condition (iii). Combining the above two lines, we finish the proof of the claim.

Step 2. Let us show that there exists $c > 0$ such that, for any ball $B = B(x_0, r)$ and any $\lambda > 0$,

$$\operatorname*{essup}_{B(x_0,\delta)} (\lambda R_\lambda 1_{B^c}) \leq \exp\left(-cr\lambda^{1/\beta} + 1\right), \tag{6.18}$$

where $\delta = \delta(\lambda) > 0$. Choose some $R > 4r$ and consider the functions

$$\phi = \mathbf{1}_{B(x_0,R)\setminus B(x_0,r)}$$

and

$$u = \lambda R_\lambda \phi. \tag{6.19}$$

It suffices to prove that

$$\operatorname*{essup}_{B(x_0,\delta)} u \leq \exp\left(-cr\lambda^{1/\beta} + 1\right), \tag{6.20}$$

and then let $R \to \infty$. Since $0 \leq \varphi \leq 1$ and $\phi \in L^2(M)$, we have $0 \leq u \leq 1$ on M, $u \in \operatorname{dom}\mathcal{L} \subset \mathcal{F}$, and u satisfies in M the equation

$$\mathcal{L}u + \lambda u = \lambda\phi. \tag{6.21}$$

It suffices to assume that

$$cr \geq \lambda^{-1/\beta}, \tag{6.22}$$

(where $c > 0$ is to be specified later) because otherwise (6.20) is trivially satisfied due to $u \leq 1$.

Let $n \geq 2$ be an integer to be determined later on. For any $1 \leq i \leq n$, set $r_i = \frac{ir}{n}$,

$$b_i = \operatorname*{essup}_{B(x_0,r_i)} u,$$

and, for $1 \leq i < n$,

$$w_i(x) = \frac{u(x)}{b_{i+1}}.$$

Clearly, $w_i \in \mathcal{F} \cap L^\infty(M)$. Since $\phi = 0$ in $B(x_0, r)$, it follows from (6.21) that

$$\mathcal{L}w_i + \lambda w_i = 0 \text{ weakly in } B(x_0, r).$$

By definition of b_{i+1}, we have $0 \leq w_i \leq 1$ in $B(x_0, r_{i+1})$. In particular, the same inequality holds in any ball $B(x, r_1)$ for any $x \in B(x_0, r_i)$ (see Fig. 12).

Therefore, by Step 1 with $\varepsilon = e^{-1}$, we have that

$$w_i \leq e^{-1} \text{ a.e. in } B\left(x, \frac{1}{4}r_1\right),$$

provided

$$r_1 \geq K\lambda^{-1/\beta}, \tag{6.23}$$

for an appropriate constant K. It follows that

$$\operatorname*{essup}_{B(x_0,r_i)} w_i \leq e^{-1},$$

that is,

$$b_i \leq e^{-1} b_{i+1}. \tag{6.24}$$

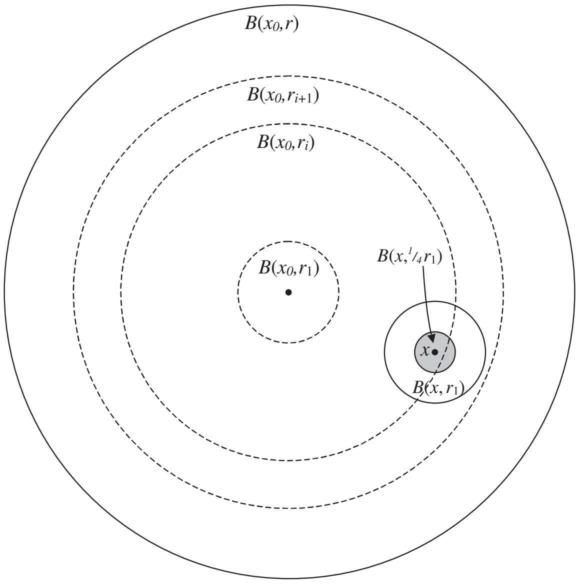

Figure 12: Balls $B(x_0, r_i)$ and $B(x_0, r_{i+1})$

Before we proceed further, let us make sure that the condition (6.23) is satisfied. Since $r_1 = r/n$, it is equivalent to

$$n \leq \frac{r\lambda^{1/\beta}}{K},$$

so that we can choose

$$n = \left[\frac{r\lambda^{1/\beta}}{K}\right].$$

Choosing in (6.22) $c = \frac{1}{2K}$, we obtain that $n \geq 2$. Note also that

$$n \geq 2cr\lambda^{1/\beta} - 1.$$

Now, iterating (6.24) and using the fact that $b_n \leq 1$, we obtain

$$b_1 \leq e^{-(n-1)} b_n \leq e^{-n/2} \leq \exp\left(-cr\lambda^{1/\beta} + 1\right).$$

Clearly, this implies (6.20), where δ can be anything $\leq r_1 = \frac{r}{n}$; for example, set $\delta = K\lambda^{-1/\beta}$.

Let us note that the iteration argument in this part of the proof is motivated by that in [34] for the setting of infinite graphs.

Step 3. Let us show that there is $K \geq 1$ such that for any ball $B = B(x_0, r)$ with

$$r \geq K\lambda^{-1/\beta}, \tag{6.25}$$

we have

$$\operatorname*{essinf}_{(2B)^c} \left(\lambda R_\lambda \mathbf{1}_{B^c}\right) \geq \frac{1}{2}. \tag{6.26}$$

Indeed, for any $x \in (2B)^c$, we have $B(x, r) \subset B^c$, whence by the condition (iii),

$$\lambda R_\lambda \mathbf{1}_{B^c} \geq \lambda R_\lambda \mathbf{1}_{B(x,r)} \geq \frac{1}{2} \text{ a.e. in } B\left(x, \frac{1}{4}r\right).$$

provided (6.25) is satisfied with an appropriate K. Hence, (6.26) follows.

Step 4. Let us show that, for any non-negative function $f \in L^{\infty}(M)$, the function $u = \lambda R_{\lambda} f$ satisfies the inequality

$$P_t u \le e^{\lambda t} u \ \text{ in } M. \tag{6.27}$$

for arbitrary $t, \lambda > 0$. Indeed, we have

$$\begin{aligned} P_t u &= \lambda \int_0^{\infty} e^{-\lambda s} P_{t+s} f \, ds \\ &= \lambda \int_t^{\infty} e^{-\lambda (s-t)} P_s f \, ds \\ &= e^{\lambda t} \lambda \int_t^{\infty} e^{-\lambda s} P_s f \, ds \le e^{\lambda t} u. \end{aligned}$$

Step 5. Finally, let us prove (6.16). Let c be the same as in (6.18) (Step 2), so that for any $\lambda > 0$ and for $u = \lambda R_{\lambda} 1_{B^c}$,

$$\operatorname*{essup}_{B(x_0,\delta)} u \le \exp\left(-cr\lambda^{1/\beta} + 1\right). \tag{6.28}$$

Let $\lambda > 0$ be such that (6.25) is satisfied. Then it follows from (6.26) that

$$u \ge \frac{1}{2} \mathbf{1}_{(2B)^c} \text{ in } M.$$

Applying P_t to the both sides of this inequality and using (6.27), we obtain

$$\frac{1}{2} P_t \mathbf{1}_{(2B)^c} \le P_t u \le e^{\lambda t} u, \tag{6.29}$$

which together with (6.28) yields

$$\operatorname*{essup}_{B(x_0,\delta)} P_t \mathbf{1}_{(2B)^c} \le C \exp\left(\lambda t - cr\lambda^{1/\beta}\right), \tag{6.30}$$

where $C = 2e$.

If λ is such that (6.25) fails, that is, $r < K\lambda^{-1/\beta}$ then (6.30) holds trivially with $C = e^{cK}$. Hence, (6.30) holds for all $\lambda > 0$, which is equivalent to (6.16). ∎

6.4 Identifying Φ in the local case

Now we can state and prove the main result.

Theorem 6.7 ([33]) *Assume that the metric space (M, d) satisfies the chain condition and all metric balls are precompact. Let $p_t(x, y)$ be a stochastically complete heat kernel in a metric measure space (M, d, μ). Assume that the associated Dirichlet form $\mathcal{E}$ is regular, and the following estimate holds with some $\alpha, \beta > 0$ and $\Phi : [0, +\infty) \to [0, +\infty)$:*

$$p_t(x, y) \asymp \frac{C}{t^{\alpha/\beta}} \Phi\left(c \frac{d(x, y)}{t^{1/\beta}}\right). \tag{6.31}$$

Then the following dichotomy holds:

- *either the Dirichlet form $\mathcal{E}$ is local, $2 \leq \beta \leq \alpha + 1$, and $\Phi(s) \asymp C \exp\left(-cs^{\frac{\beta}{\beta-1}}\right)$.*
- *or the Dirichlet form $\mathcal{E}$ is non-local, $\beta \leq \alpha + 1$, and $\Phi(s) \simeq (1+s)^{-(\alpha+\beta)}$.*

Proof. By Corollary 5.8, we have $\beta \leq \alpha + 1$. If the form $\mathcal{E}$ is non-local, then, by Corollary 2.10, Φ satisfies (2.33), which finishes the proof in this case.

Assume now that the form $\mathcal{E}$ is local. By Lemma 2.8, the lower bound in (6.31) implies that (2.28), that is,

$$\Phi(s) \leq C(1+s)^{-(\alpha+\beta)}. \tag{6.32}$$

By Corollary 2.11, we have for all balls

$$\mu(B(x,r)) \simeq r^{\alpha}. \tag{6.33}$$

Let us show that, for any $\varepsilon > 0$ there is $K > 0$ such that, for all $t > 0$, $r \geq Kt^{1/\beta}$ and almost all $x \in M$,

$$\int_{B(x,r)^c} p_t(x,y)\, d\mu(y) \leq \varepsilon. \tag{6.34}$$

Indeed, using (6.31) and (6.33), we obtain as in the proof of Theorem 2.11(b)

$$\int_{B(x,r)^c} p_t(x,y)\, d\mu(y) \leq C \int_{\frac{1}{2}r/t^{1/\beta}}^{\infty} s^{\alpha}\Phi(s)\frac{ds}{s}. \tag{6.35}$$

Since by (6.32) the integral in the right hand side of (6.35) converges, we see that the integral can be made smaller than ε provided $r/t^{1/\beta}$ is large enough, which was claimed.

The heat semigroup $\{P_t\}$ satisfies all the hypotheses of Theorems 6.3 and 6.5. Since the condition (i) of Theorem 6.3 is satisfied by (6.34), we conclude by Theorem 6.5 that, for all $t, r, \lambda > 0$ and for μ-a.a. $x \in M$,

$$\int_{B(x,r)^c} p_t(x,y)\, d\mu(y) \leq C \exp\left(\lambda t - cr\lambda^{1/\beta}\right). \tag{6.36}$$

If $\beta < 1$ then letting in (6.36) $\lambda \to \infty$, we obtain that the right hand side in (6.36) goes to 0. Letting then $r \to 0$, we obtain that, for almost all $x \in M$,

$$\int_{M\setminus\{x\}} p_t(x,y)\, d\mu(y) = 0.$$

Together with stochastic completeness, this implies that there is a point $x \in M$ of a positive mass, which is impossible by Remark 2.9. This contradiction proves that $\beta \geq 1$.

Setting in (6.36)

$$\lambda = \begin{cases} \left(\dfrac{cr}{2t}\right)^{\frac{\beta}{\beta-1}}, & \text{if } \beta > 1, \\ t^{-1}, & \text{if } \beta = 1 \end{cases}$$

we obtain that, for all positive r, t and almost all $x \in M$,

$$\int_{B(x,r)^c} p_t(x,y)\, d\mu(y) \leq \begin{cases} C \exp\left(-c\left(\frac{r^{\beta}}{t}\right)^{\frac{1}{\beta-1}}\right), & \text{if } \beta > 1 \\ C\exp\left(-c\frac{r}{t}\right), & \text{if } \beta = 1 \end{cases} \tag{6.37}$$

(where the constants c, C may be different from those of (6.36)).

By (2.2), we have, for all $t>0$, almost all $x, y \in M$, and $r := \frac{1}{2} d(x, y)$,

$$
\begin{aligned}
p_t(x, y) & = \int_M p_{\frac{t}{2}}(x, z)\, p_{\frac{t}{2}}(z, y)\, d\mu(z) \\
& \leq \left(\int_{B(x, r)^c}+\int_{B(y, r)^c}\right) p_{\frac{t}{2}}(x, z)\, p_{\frac{t}{2}}(z, y)\, d\mu(z) \\
& \leq \operatorname*{esssup}_{z \in M} p_{\frac{t}{2}}(z, y) \int_{B(x, r)^c} p_{\frac{t}{2}}(x, z)\, d\mu(z) \\
& \quad + \operatorname*{esssup}_{z \in M} p_{\frac{t}{2}}(x, z) \int_{B(y, r)^c} p_{\frac{t}{2}}(y, z)\, d\mu(z).
\end{aligned}
$$

Since by (6.31) $\operatorname{esssup} p_t \leq C t^{-\alpha / \beta}$, combining this with (6.37) we obtain, for almost all $x, y \in M$,

$$
p_t(x, y) \leq \begin{cases} \dfrac{C}{t^{\alpha / \beta}} \exp \left(-c\left(\dfrac{d^\beta(x, y)}{t}\right)^{\frac{1}{\beta-1}}\right), & \text{if } \beta>1 \\ \dfrac{C}{t^\alpha} \exp \left(-c \dfrac{r}{t}\right), & \text{if } \beta=1 \end{cases} \tag{6.38}
$$

Now we use Corollary 5.7(a). By (6.31) and (6.38), the heat kernel satisfies the two-sided estimates (2.37) with the following functions:

$$
\Phi_1(s) := C \Phi(c s)
$$

and

$$
\Phi_2(s) := \begin{cases} C \exp \left(-c s^{\frac{\beta}{\beta-1}}\right), & \text{if } \beta>1, \\ C \exp (-c s), & \text{if } \beta=1. \end{cases}
$$

As was mentioned above, the heat kernel cannot identically vanish off the diagonal, which implies that $\Phi_1(s)>0$ for some $s>0$. Function Φ_2 clearly satisfies the hypothesis (5.13) of Corollary 5.7(a). Hence, we conclude by Corollary 5.7(a) that $\beta \geq 2$.

If $\beta \geq 2$ (in fact, $\beta>1$ is enough), then the standard chaining argument using the chain condition (cf. [2], [32, Corollary 3.5]) shows that the lower bound in (6.31) implies the lower bounds

$$
p_t(x, y) \geq \frac{C}{t^{\alpha / \beta}} \exp \left(-c\left(\frac{d^\beta(x, y)}{t}\right)^{\frac{1}{\beta-1}}\right). \tag{6.39}
$$

Combining (6.38) and (6.39) with (6.31), we obtain

$$
\Phi(s) \asymp C \exp \left(-c s^{\frac{\beta}{\beta-1}}\right),
$$

which finishes the proof. ■

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