Heat kernels and function theory on metric measure spaces

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1 Introduction

The classical heat kernel in $\mathbb{R}^n$ is the fundamental solution to the heat equation, which is given by the following formula

$$p_t(x,y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right). \tag{1.1}$$

It is worth observing that the *Gaussian term* $\exp\left(-\frac{|x-y|^2}{4t}\right)$ does not depend in n, whereas the other term $(4\pi t)^{-n/2}$ reflects the dependence of the heat kernel on the underlying space via its dimension.

The notion of heat kernel extends to any Riemannian manifold M. In this case, the heat kernel $p_t(x,y)$ is the minimal positive fundamental solution to the heat equation $\frac{\partial u}{\partial t} = \Delta u$ where Δ is the Laplace-Beltrami operator on M, and it always exists (see [11], [13], [16]). Under certain assumptions about M, the heat kernel can be estimated similarly to (1.1). For example, if M is geodesically complete and has non-negative Ricci curvature then by a theorem of Li and Yau [25]

$$p_t(x,y) \asymp \frac{C}{V\left(x,\sqrt{t}\right)} \exp\left(-\frac{d^2(x,y)}{Ct}\right).$$

Here $V(x,r)$ is the volume of the geodesic ball $B(x,r)$ of radius r centered at x, and the sign $\asymp$ means that both inequality signs $\leq$ and $\geq$ can be used instead but the positive constant C may be different in upper and lower bounds. Again, most information about the geometry of M sits in the volume term, whereas the Gaussian term $\exp\left(-\frac{d^2(x,y)}{Ct}\right)$ seems to be more robust.

The recent development of analysis on fractals, notably by M.Barlow and R.Bass (see [5] and references therein), has shed new light on the nature of the Gaussian term. Without going into the definition of fractals sets[1], let us give an example. The most well studied fractal set is the *Sierpinski gasket* SG which is obtain from an equilateral triangle by removing its middle triangle, then removing the middle triangles from the remaining triangles, etc. (see Fig. 1 where three iterations are shown and the removed triangles are blank).

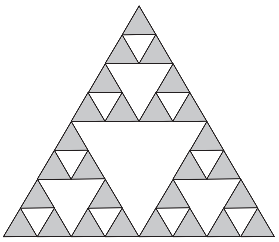

Figure 1:

We regard SG as a *metric measure energy* space. A metric d on SG is defined as the restriction of the Euclidean metric. A measure μ on SG is the Hausdorff measure

[1]For a detailed account of fractals see the article of M.Barlow [3] in the same volume.

H_α where $\alpha := \dim_H \mathrm{SG}$ is called the *fractal dimension* of SG (it is possible to show that $\alpha = \log_2 3$). Defining an energy form is highly non-trivial. By an energy form we mean an analogue of the Dirichlet form

$$\mathcal{E}[f] = \int_M |\nabla f|^2 \, d\mathrm{vol}$$

which is defined on any Riemannian manifold M. One first defines a discrete analogue of this form on graphs approximating SG, and then takes a certain scaling limit. Barlow and Perkins [8] have proved that the resulting functional $\mathcal{E}[f]$ is a local regular Dirichlet form in $L^2(\mathrm{SG}, \mu)$. Furthermore, they have shown that the associate diffusion process X_t has a transition density $p_t(x,y)$ which is a continuous function on $x, y \in \mathrm{SG}$ and $t > 0$ and satisfies the following estimate

$$p_t(x,y) \asymp \frac{C}{t^{\alpha/\beta}} \exp\left(-\left(\frac{d^\beta(x,y)}{Ct}\right)^{\frac{1}{\beta-1}}\right) \tag{1.2}$$

provided $0 < t < 1$ (the restriction $t < 1$ can be removed if one considers an unbounded version of SG). Here α is the same as above - the Hausdorff dimension of SG, whereas β is a new parameter called the *walk dimension.* It is determined in terms of the process X_t as follows: the mean exit time from a ball of radius r is of the order r^β. For SG it is known that $\beta = \log_2 5$. Barlow and Bass [5] extended the above construction and the estimate (1.2) to the large family of fractals including Sierpinski carpet and its higher dimensional analogues (see also [2]). Moreover, it turns out that estimates similar to (1.2) are valid on *fractal-like graphs* (see [6], [19], [20], [22], [30]). An example of such a graph – the *graphical Sierpinski gasket* – is shown on Fig. 2. Furthermore, one easily makes a smooth Riemannian manifold out

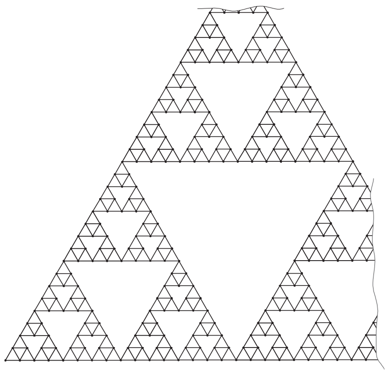

Figure 2:

of this graph by replacing the edges by tubes, and the heat kernel on this manifold also satisfies (1.2) provided $t \geq \max(d(x,y), 1)$ where d is the geodesic distance.

In all the cases the nature of the parameters α and β in (1.2) is of great interest. Assume that a *metric measure space* (M, d, μ) admits a heat kernel satisfying (1.2) with some $\alpha > 0$ and $\beta > 1$ (a precise definition of a heat kernel will be given in the next section – to some extent, specifying a heat kernel is equivalent to choosing an appropriate energy form on M). Although a priori the parameters α and β are defined through the heat kernel, a posteriori they happen to be invariants of the underlying metric measure structure *alone*. The parameter α turns out to be the *volume growth exponent* of M (see Theorem 3.1) which also implies that α is the Hausdorff dimension of M. The parameter β is characterized intrinsically as the *critical exponent* of a family of function spaces on M. This approach for characterizing of the walk dimension originated by A.Jonsson [23] in the setting of SG, and was later used by K.Pietruska-Pałuba [26] and A.Stós [27] for so called d-sets in $\mathbb{R}^n$ supporting a fractional diffusion. In full generality this result was proved in [18].

The structure of the paper and the main results are as follows. In Section 2 we define a heat kernel on a metric measure space and related notions. The main result of Section 3 is Theorem 3.1 characterizing the parameter α from (1.2) as the volume growth exponent. In Section 4 we define the energy form associated with the heat kernel and prove that the domain of the energy form embeds compactly into L^2 (Theorem 4.1). In Section 5 we introduce a family of function spaces on M generalizing Besov spaces, and prove Theorems 5.1 and 5.2 characterizing the domain of the energy form in terms of the family of Besov spaces. In Section 6 we prove Theorem 6.2 giving an intrinsic characterization of the parameter β from (1.2) in terms of Besov spaces. In Section 7 we prove Theorem 7.2 saying that under mild assumptions about the underlying metric space, the parameters α and β are related by the inequalities

$$2 \leq \beta \leq \alpha + 1$$

(see also Corollary 7.3(iii)). In Section 8 we prove an embedding of Besov spaces into Hölder spaces C^λ (Theorem 8.1). In Section 9 we introduce *Bessel potential spaces* generalizing fractional Sobolev spaces in $\mathbb{R}^n$, and prove the embedding of these spaces into L^q and C^λ (Theorem 9.2(i) and (ii), respectively).

Most results surveyed here are taken from [18] (these are Theorems 3.1, 4.1, 5.1, 6.2, 7.2, 8.1). Theorem 5.2 is new although it is largely motivated by a result of A.Stós [27]. Theorem 9.2(ii) is also new giving a partial answer to a conjecture of R.Strichartz [28].

In this survey we do not touch the methods of obtaining estimates like (1.2), and for that we refer the reader to [7], [12], [19], [20], [21], [24].

Notation. For non-negative functions f and g, we write $f \simeq g$ if there is a constant $C > 0$ such that $C^{-1}g \leq f \leq Cg$ in the domain of the functions f, g or in a specified range of the arguments. Letters c and C are normally used to denote unimportant positive constants, whose values may change at each occurrence.

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2 Definition of a heat kernel and examples

Let (M, μ) be *a measure space*; that is, μ is a measure on a σ-algebra of subsets of a non-empty set M. For any $q \in [1, +\infty]$, set $L^q = L^q(M, \mu)$ and

$$\|u\|_q := \|u\|_{L^q(M,\mu)}.$$

All functions in L^q are considered to be real-valued.

Definition 2.1 *A family $\{p_t\}_{t>0}$ of $\mu \otimes \mu$-measurable functions $p_t(x,y)$ on $M \times M$ is called a* heat kernel *if the following conditions are satisfied, for μ-almost all $x, y \in M$ and all $s, t > 0$:*

(i) Positivity: $p_t(x,y) \geq 0$.

(ii) Stochastic completeness:

$$\int_M p_t(x,y)d\mu(y) \equiv 1. \tag{2.1}$$

(iii) Symmetry: $p_t(x,y) = p_t(y,x)$.

(iv) Semigroup property:

$$p_{s+t}(x,y) = \int_M p_s(x,z)p_t(z,y)d\mu(z). \tag{2.2}$$

(v) Approximation of identity: for any $u \in L^2$

$$\int_M p_t(x,y)u(y)d\mu(y) \xrightarrow{L^2} u(x) \quad \text{as } t \to 0+. \tag{2.3}$$

For example, the Gauss-Weierstrass kernel (1.1) in $\mathbb{R}^n$ satisfies this definition. Alternatively, the function (1.1) can be viewed as the density of the normal distribution in $\mathbb{R}^n$ with the mean y and the variance $2t$. Another elementary example is the Poisson kernel in $\mathbb{R}^n$

$$p_t(x,y) = \frac{C_n}{t^n} \frac{1}{\left(1 + \frac{|x-y|^2}{t^2}\right)^{\frac{n+1}{2}}}, \tag{2.4}$$

where $C_n = \Gamma\left(\frac{n+1}{2}\right)/\pi^{(n+1)/2}$, which can be viewed as the Cauchy distribution in $\mathbb{R}^n$ with parameters y and t.

On any Riemannian manifold M, the heat kernel associated with the Laplace-Beltrami operator satisfies all properties of Definition 2.1 (with respect to the Riemannian volume μ) except for the stochastic completeness; however, the latter too can be obtained under certain mild hypotheses about M (see [15] and [17]).

Under additional assumptions about the space (M, μ) a heat kernel p_t gives rise to a Markov process $\{X_t\}$ for which p_t is the *transition density.* This means that for any measurable set $A \subset M$ and for all $x \in M$, $t > 0$,

$$\mathbb{P}_x \{X_t \in A\} = \int_M p_t(x,y)d\mu(y).$$

For example, the process associated with the Gauss-Weierstrass kernel is the standard Brownian motion in $\mathbb{R}^n$, and process associated with the Poisson kernel is a certain jump process in $\mathbb{R}^n$.

Any heat kernel gives rise to the *heat semigroup* $\{P_t\}_{t>0}$ where P_t is the operator in L^2 defined by

$$P_t u(x) = \int_M p_t(x,y)u(y)d\mu(y). \tag{2.5}$$

Indeed, we obtain by the Cauchy-Schwarz inequality and (2.1)

$$\begin{aligned}
\|P_t u\|_2^2 &= \int_M \left(\int_M p_t(x,y)u(y)d\mu(y) \right)^2 d\mu(x) \\
&\le \int_M \left(\int_M p_t(x,y)d\mu(y) \int_M p_t(x,y)u(y)^2 d\mu(y) \right) d\mu(x) \\
&= \int_M \int_M p_t(x,y)u(y)^2 d\mu(x)d\mu(y) \\
&= \|u\|_2^2,
\end{aligned}$$

which implies that P_t is a bounded operator in L^2 and $\|P_t\|_{2\to 2} \le 1$. The symmetry of the heat kernel implies that P_t is a self-adjoint operator.

The semigroup identity (2.2) implies that $P_t P_s = P_{t+s}$, that is, the family $\{P_t\}_{t>0}$ is a *semigroup.* Finally, it follows from (2.3) that

$$s\text{-}\lim_{t\to 0} P_t = \mathrm{I},$$

where I is the identity operator in L^2 and s-lim stands for *strong* limit. Hence, $\{P_t\}_{t>0}$ is a strongly continuous, self-adjoint, contraction semigroup in L^2.

Define the *infinitesimal generator* $\mathcal{L}$ of a semigroup $\{P_t\}_{t>0}$ by

$$\mathcal{L}u := \lim_{t\to 0} \frac{u - P_t u}{t}, \tag{2.6}$$

where the limit is understood in the L^2-norm. The *domain* $\mathrm{dom}(\mathcal{L})$ of the generator $\mathcal{L}$ is the space of functions $u \in L^2$ for which the limit in (2.6) exists. By the Hille–Yosida theorem, $\mathrm{dom}(\mathcal{L})$ is dense in L^2. Furthermore, $\mathcal{L}$ is a self-adjoint, positive definite[2] operator, which immediately follows from the fact that the semigroup $\{P_t\}$ is self-adjoint and contractive. Moreover, we have

$$P_t = \exp(-t\mathcal{L}), \tag{2.7}$$

where the right hand side is understood in the sense of spectral theory.

For example, the generator of the Gauss-Weierstrass kernel in $\mathbb{R}^n$ is $-\Delta$ where $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ is the classical Laplace operator with a properly defined domain in

[2]Normally the generator of a semigroup $\{P_t\}$ is defined as the limit

$$\lim_{t\to 0} \frac{P_t u - u}{t}.$$

Our choice of the sign in (2.6) makes the generator positive definite.

$L^2(\mathbb{R}^n)$. The generator of the Poisson kernel is $(-\Delta)^{1/2}$. It is well known that for any $0 < m < 2$ the operator $(-\Delta)^{m/2}$ is the generator of a heat kernel associated with the *symmetric stable process* of index m, which belongs to the family of Levy processes.

Any positive definite self-adjoint operator $\mathcal{L}$ in L^2 determines by (2.7) a semigroup P_t satisfying the above properties. It is not always the case that the semigroup $\{P_t\}$ defined by (2.7) possesses an integral kernel. However, when it does, the integral kernel satisfies all the properties of Definition 2.1 except for positivity and stochastic completeness; to ensure the latter properties, some additional assumptions about $\mathcal{L}$ are needed (see [14]).

Definition 2.2 *We say that a triple (M, d, μ) is a* metric measure space *if (M, d) is a non-empty metric space and μ is a Borel measure on M.*

In the sequel, we will always work in a metric measure space (M, d, μ). Let p_t be a heat kernel on such a space, and consider the following estimates for p_t, which in general may be true or not:

$$\frac{1}{t^{\alpha/\beta}}\Phi_1\left(\frac{d(x,y)}{t^{1/\beta}}\right) \le p_t(x,y) \le \frac{1}{t^{\alpha/\beta}}\Phi_2\left(\frac{d(x,y)}{t^{1/\beta}}\right), \tag{2.8}$$

for μ-almost all $x, y \in M$ and all $t \in (0, \infty)$. Here α, β are positive constants, and Φ_1 and Φ_2 are a priori non-negative monotone decreasing functions on $[0, +\infty)$. For example, the Gauss-Weierstrass kernel (1.1) satisfies (2.8) with $\alpha = n$, $\beta = 2$, and

$$\Phi_1(s) = \Phi_2(s) = \frac{1}{(4\pi)^{n/2}} \exp\left(-\frac{s^2}{4}\right).$$

The Poisson heat kernel (2.4) satisfies (2.8) with $\alpha = n$, $\beta = 1$, and

$$\Phi_1(s) = \Phi_2(s) = \frac{C_n}{(1+s^2)^{\frac{n+1}{2}}}.$$

For any $0 < m < 2$ the heat kernel of the operator $(-\Delta)^{m/2}$ in $\mathbb{R}^n$ satisfies the following estimate

$$p_t(x,y) \simeq \frac{1}{t^{n/m}} \frac{1}{\left(1 + \frac{|x-y|}{t^{1/m}}\right)^{n+m}} \tag{2.9}$$

(see [9] or Lemma 5.4 below). Hence, the estimate (2.8) is satisfied with $\alpha = n$, $\beta = m$, and with functions Φ_1 and Φ_2 of the form

$$\Phi(s) = \frac{C}{(1+s)^{\alpha+\beta}}, \tag{2.10}$$

where the constant C may be different for Φ_1 and Φ_2.

The estimate (1.2) for heat kernels on fractals mentioned in Introduction is a particular case of (2.8) with the same α and β and with functions Φ_1, Φ_2 of the form

$$\Phi(s) = C \exp\left(-Cs^{\frac{\beta}{\beta-1}}\right).$$

Definition 2.3 *We say that a heat kernel p_t on a metric measure space (M, d, μ) satisfies hypothesis $\mathcal{H}(\theta)$ (where $\theta > 0$) if p_t satisfies the estimate (2.8) with some positive parameters α, β and non-negative decreasing functions Φ_1 and Φ_2 such that $\Phi_1(1) > 0$ and*

$$\int^{\infty} s^{\theta} \Phi_2(s) \frac{ds}{s} < \infty. \tag{2.11}$$

Note that hypothesis $\mathcal{H}(\theta)$ gets stronger with increasing of θ.

For example, every function of the form

$$\Phi_2(s) = \exp\left(-Cs^{\gamma}\right) \tag{2.12}$$

satisfies (2.11) for any θ provided $C > 0$ and $\gamma > 0$, and the function

$$\Phi_2(s) = \frac{C}{(1+s)^{\gamma}}$$

satisfies (2.11) for $\theta < \gamma$. In particular, the Gauss-Weierstrass kernel satisfies $\mathcal{H}(\theta)$ for all θ whereas the heat kernel of the operator $(-\Delta)^{m/2}$ in $\mathbb{R}^n$ satisfies $\mathcal{H}(\theta)$ for $\theta < n + m$.

3 Volume of balls

Let $B(x, r) := \{y \in M : d(x, y) < r\}$ be the ball in M of radius r centered at the point $x \in M$.

Theorem 3.1 *If a heat kernel p_t on a metric measure space (M, d, μ) satisfies hypothesis $\mathcal{H}(\alpha)$ then for all $x \in M$ and $r > 0$*

$$C^{-1} r^{\alpha} \leq \mu(B(x, r)) \leq C r^{\alpha}. \tag{3.1}$$

Remark 3.2 *Note that in all examples considered in Section 2, hypothesis $\mathcal{H}(\alpha)$ is satisfied. Let us mention also that the estimate (3.1) can be true only for a single value of α which is called the* exponent of the volume growth *and is determined by intrinsic properties of the space (M, d, μ). Hence, under hypothesis $\mathcal{H}(\alpha)$ the value of the parameter α in (2.8) is an invariant of the underlying space.*

Proof. Fix $x \in M$ and prove first the upper bound

$$\mu(B(x, r)) \leq C r^{\alpha}, \tag{3.2}$$

for all $r > 0$. Indeed, for any $t > 0$, we have

$$\int_{B(x,r)} p_t(x, y) d\mu(y) \leq \int_M p_t(x, y) d\mu(y) = 1 \tag{3.3}$$

whence

$$\mu(B(x, r)) \leq \left(\inf_{y \in B(x,r)} p_t(x, y) \right)^{-1}.$$

Applying the lower bound in (2.8) and taking $t = r^\beta$ we obtain

$$\inf_{y\in B(x,r)} p_t(x,y) \geq \frac{1}{t^{\alpha/\beta}}\Phi_1\left(\frac{r}{t^{1/\beta}}\right) = c\,r^{-\alpha},$$

where $c = \Phi_1(1) > 0$, whence (3.2) follows.

Let us prove the opposite inequality

$$\mu(B(x,r)) \geq c\,r^{\alpha}. \tag{3.4}$$

We first show that the upper bound in (2.8) and (3.2) imply the following inequality

$$\int_{M\setminus B(x,r)} p_t(x,y)d\mu(y) \leq \frac{1}{2} \quad \text{for all } t \leq \varepsilon r^\beta, \tag{3.5}$$

provided $\varepsilon > 0$ is sufficiently small. Setting $r_k = 2^k r$ and using the monotonicity of Φ_2 and (3.2) we obtain

$$\begin{aligned}
\int_{M\setminus B(x,r)} p_t(x,y)d\mu(y) &\leq \int_{M\setminus B(x,r)} t^{-\alpha/\beta}\Phi_2\left(\frac{d(x,y)}{t^{1/\beta}}\right)d\mu(y)\\
&\leq \sum_{k=0}^{\infty}\int_{B(x,r_{k+1})\setminus B(x,r_k)} t^{-\alpha/\beta}\Phi_2\left(\frac{r_k}{t^{1/\beta}}\right)d\mu(y)\\
&\leq C\sum_{k=0}^{\infty} r_k^\alpha t^{-\alpha/\beta}\Phi_2\left(\frac{r_k}{t^{1/\beta}}\right)\\
&= C\sum_{k=0}^{\infty}\left(\frac{2^k r}{t^{1/\beta}}\right)^\alpha \Phi_2\left(\frac{2^k r}{t^{1/\beta}}\right)\\
&\leq C\int_{\frac{1}{2}r/t^{1/\beta}}^{\infty} s^\alpha \Phi_2(s)\frac{ds}{s}.
\end{aligned} \tag{3.6}$$

Since by (2.11) the integral in (3.6) is convergent, its value can be made arbitrarily small provided r^β/t is large enough, whence (3.5) follows.

From (2.1) and (3.5), we conclude that for such r and t

$$\int_{B(x,r)} p_t(x,y)d\mu(y) \geq \frac{1}{2}, \tag{3.7}$$

whence

$$\mu(B(x,r) \geq \frac{1}{2}\left(\sup_{y\in B(x,r)} p_t(x,y)\right)^{-1}.$$

Finally, choosing $t := \varepsilon r^\beta$ and using the upper bound

$$p_t(x,y) \leq t^{-\alpha/\beta}\Phi_2(0) = Cr^{-\alpha},$$

we obtain (3.4). ■

Corollary 3.3 *If a metric measure space (M,d,μ) admits a heat kernel p_t satisfying* (2.8) *then $\mu(M) = \infty$. If in addition $\Phi_1(1) > 0$ then* $\operatorname{diam} M = \infty$.

Proof. Let us show that the upper bound in (2.8) implies $\mu(M) = \infty$. Indeed, fix a point $x_0 \in M$ and observe that the family of functions $u_t(x) = p_t(x, x_0)$ satisfies the following two conditions: $\|u_t\|_1 = 1$ and

$$\|u_t\|_\infty \le Ct^{-\alpha/\beta} \to 0 \quad \text{as } t \to \infty.$$

Hence, we obtain

$$\mu(M) \ge \frac{\|u_t\|_1}{\|u_t\|_\infty} \to \infty \quad \text{as } t \to \infty,$$

that is $\mu(M) = \infty$.

On the other hand, the first part of the proof of Theorem 3.1, based on the hypothesis $\Phi_1(1) > 0$, says that the measure of any ball is finite. Hence, M is not contained in any ball, that is $\operatorname{diam} M = \infty$. ■

4 Energy form

Given a heat kernel $\{p_t\}$ on a measure space (M, μ), define for any $t > 0$ a quadratic form $\mathcal{E}_t$ on L^2 by

$$\mathcal{E}_t [u] := \left(\frac{u - P_t u}{t}, u \right), \tag{4.1}$$

where $(\cdot, \cdot)$ is the inner product in L^2. An easy computation shows that $\mathcal{E}_t$ can be equivalently defined by

$$\mathcal{E}_t [u] = \frac{1}{2t} \int_M \int_M |u(x) - u(y)|^2 \, p_t(x, y) d\mu(y) d\mu(x). \tag{4.2}$$

Indeed, by (2.1) and (2.5) we have

$$u(x) - P_t u(x) = \int_M (u(x) - u(y)) \, p_t(x, y) d\mu(y)$$

whence by (4.1)

$$\mathcal{E}_t [u] = \frac{1}{t} \int_M \int_M (u(x) - u(y)) \, u(x) p_t(x, y) d\mu(y) d\mu(x). \tag{4.3}$$

Interchanging the variables x and y and using the symmetry of the heat kernel, we obtain also

$$\mathcal{E}_t [u] = \frac{1}{t} \int_M \int_M (u(y) - u(x)) \, u(y) p_t(x, y) d\mu(y) d\mu(x), \tag{4.4}$$

and (4.2) follows by adding up (4.3) and (4.4).

In terms of the spectral resolution $\{E_\lambda\}$ of the generator $\mathcal{L}$, $\mathcal{E}_t$ can be expressed as follows

$$\mathcal{E}_t [u] = \int_0^\infty \frac{1 - e^{-t\lambda}}{t} d\|E_\lambda u\|_2^2,$$

which implies that $\mathcal{E}_t [u]$ is decreasing in t (indeed, this is an elementary exercise to show that the function $t \mapsto \frac{1-e^{-t\lambda}}{t}$ is decreasing).

Let us define a quadratic form $\mathcal{E}$ by

$$\mathcal{E}[u] := \lim_{t\to 0+} \mathcal{E}_t [u] = \int_0^\infty \lambda \, d\|E_\lambda u\|_2^2 \tag{4.5}$$

(where the limit may be $+\infty$ since $\mathcal{E}[u] \geq \mathcal{E}_t[u]$) and its *domain* $\mathcal{D}(\mathcal{E})$ by

$$\mathcal{D}(\mathcal{E}) : = \{u \in L^2 : \mathcal{E}[u] < \infty\}.$$

It is clear from (4.2) and (4.5) that $\mathcal{E}_t$ and $\mathcal{E}$ are positive definite.

It is easy to see from (4.5) that $\mathcal{D}(\mathcal{E}) = \mathrm{dom}(\mathcal{L}^{1/2})$. It will be convenient for us to use the following notation:

$$\mathrm{dom}_{\mathcal{E}}(\mathcal{L}) := \mathcal{D}(\mathcal{E}) = \mathrm{dom}(\mathcal{L}^{1/2}). \tag{4.6}$$

The domain $\mathcal{D}(\mathcal{E})$ is dense in L^2 because $\mathcal{D}(\mathcal{E})$ contains $\mathrm{dom}(\mathcal{L})$. Indeed, if $u \in \mathrm{dom}(\mathcal{L})$ then using (2.6) and (4.1), we obtain

$$\mathcal{E}[u] = \lim_{t\to 0} \mathcal{E}_t[u] = (\mathcal{L}u, u) < \infty. \tag{4.7}$$

The quadratic form $\mathcal{E}[u]$ extends to a bilinear form $\mathcal{E}(u,v)$ by the polarization identity

$$\mathcal{E}(u,v) = \frac{1}{2}\left(\mathcal{E}[u+v] - \mathcal{E}[u-v]\right).$$

It follows from (4.7) that $\mathcal{E}(u,v) = (\mathcal{L}u, v)$ for all $u, v \in \mathrm{dom}(\mathcal{L})$. The space $\mathcal{D}(\mathcal{E})$ is naturally endowed with the inner product

$$[u,v] := (u,v) + \mathcal{E}(u,v). \tag{4.8}$$

It is possible to show that the form $\mathcal{E}$ is *closed*, that is the space $\mathcal{D}(\mathcal{E})$ is *Hilbert.*

It is easy to see from (2.5) and the definition of a heat kernel that the semigroup $\{P_t\}$ is *Markovian*, that is $0 \leq u \leq 1$ implies $0 \leq P_t u \leq 1$. This implies that the form $\mathcal{E}$ satisfies the Markov property, that is $u \in \mathcal{D}(\mathcal{E})$ implies $v := \min(u_+, 1) \in \mathcal{D}(\mathcal{E})$ and $\mathcal{E}[v] \leq \mathcal{E}[u]$. Hence, $\mathcal{E}$ is a *Dirichlet form.*

In the next statement, we demonstrate how the heat kernel estimate (2.8) allows to prove a compact embedding theorem.

Theorem 4.1 *Let (M, d, μ) be a metric measure space, and p_t be a heat kernel in M satisfying* (2.8)*. Then for any bounded sequence $\{u_k\} \subset \mathcal{D}(\mathcal{E})$ there exists a subsequence $\{u_{k_i}\}$ that converges to a function $u \in L^2(M, \mu)$ in the following sense:*

$$\|u_{k_i} - u\|_{L^2(B,\mu)} \to 0,$$

for any set $B \subset M$ of finite measure.

Remark 4.2 The estimate (2.8) without further assumptions on Φ_1 and Φ_2 is equivalent to the upper bound

$$\mu\text{-}\operatorname*{ess\,sup}_{x,y\in M} p_t(x,y) \leq C t^{-\alpha/\beta}, \quad \text{for all } t > 0. \tag{4.9}$$

Proof. Let $\{u_k\}$ be a bounded sequence in $\mathcal{D}(\mathcal{E})$. Since $\{u_k\}$ is also bounded in L^2, there exists a subsequence, still denoted by $\{u_k\}$, such that $\{u_k\}$ weakly converges to some function $u \in L^2$. Let us show that in fact $\{u_k\}$ converges to u in $L^2(B) = L^2(B, \mu)$ for any set $B \subset M$ of finite measure.

For any $t > 0$, we have by the triangle inequality

$$\|u_k - u\|_{L^2(B)} \leq \|u_k - P_t u_k\|_{L^2(M)} + \|P_t u_k - P_t u\|_{L^2(B)} + \|P_t u - u\|_{L^2(M)}. \quad (4.10)$$

For any function $v \in L^2$ we have

$$\begin{aligned}
\|v - P_t v\|_2^2 &= \int_M \left(\int_M (v(x) - v(y)) p_t(x,y) d\mu(y) \right)^2 d\mu(x) \\
&\leq \int_M \left\{ \int_M p_t(x,y) d\mu(y) \int_M |v(x) - v(y)|^2 p_t(x,y) d\mu(y) \right\} d\mu(x) \\
&= 2t\, \mathcal{E}_t[v] \\
&\leq 2t\, \mathcal{E}[v].
\end{aligned}$$

Since $\mathcal{E}[u_k]$ is uniformly bounded in k by the hypothesis, we obtain that for all k and $t > 0$

$$\|u_k - P_t u_k\|_2 \leq C\sqrt{t}. \quad (4.11)$$

Since $\{u_k\}$ converges to u weakly in L^2 and $p_t(x, \cdot) \in L^2$, we see that for μ-almost all $x \in M$

$$P_t u_k(x) = \int_M p_t(x,y) u_k(y) d\mu(y) \longrightarrow \int_M p_t(x,y) u(y) d\mu(y) = P_t u(x).$$

Using the definition (2.5) of the semigroup P_t, the Cauchy-Schwarz inequality, and (2.1) we obtain that, for any $v \in L^2$,

$$\begin{aligned}
|P_t v(x)| &\leq \int_M p_t(x,y) |v(y)|\, d\mu(y) \\
&\leq \left\{ \int_M p_t(x,y) v(y)^2 d\mu(y) \right\}^{1/2} \left\{ \int_M p_t(x,y) d\mu(y) \right\}^{1/2} \\
&\leq C\, t^{-\frac{\alpha}{2\beta}} \|v\|_2. \quad (4.12)
\end{aligned}$$

Hence, we have by (4.12)

$$\|P_t u_k\|_\infty \leq C t^{-\frac{\alpha}{2\beta}} \|u_k\|_2$$

so that the sequence $\{P_t u_k\}$ is uniformly bounded in k for any $t > 0$. Since $\{P_t u_k\}$ converges to $P_t u$ almost everywhere, the dominated convergence theorem yields

$$P_t u_k \longrightarrow P_t u \quad \text{in} \quad L^2(B), \quad (4.13)$$

because $\mu(B) < \infty$. Hence, we obtain from (4.10), (4.11), and (4.13) that for any $t > 0$

$$\limsup_{k \to \infty} \|u_k - u\|_{L^2(B)} \leq C\sqrt{t} + \|P_t u - u\|_{L^2(M)}.$$

Since $P_t u \to u$ in $L^2(M)$ as $t \to 0$, we finish the proof by letting $t \to 0$. ■

Corollary 4.3 *Let (M, d, μ) be a metric measure space, and p_t be a heat kernel in M satisfying* (2.8) *with $\Phi_1(1) > 0$. Then for any bounded sequence $\{u_k\} \subset \mathcal{D}(\mathcal{E})$ there exists a subsequence $\{u_{k_i}\}$ that converges to a function $u \in L^2(M, \mu)$ almost everywhere.*

Proof. By the first part of the proof of Theorem 3.1, the hypothesis $\Phi_1(1) > 0$ implies finiteness of the measure of any ball. Fix a point $x \in M$ and consider the sequence of balls $B_N = B(x, N)$, where $N = 1, 2, \ldots$. By Theorem 4.1 we can assume that the sequence $\{u_k\}$ converges to $u \in L^2(M)$ in the norm of $L^2(B_N)$ for any N. Therefore, there exists a subsequence that converges almost everywhere in B_1. From this sequence, let us select a subsequence that converges to u almost everywhere in B_2, and so on. Using the diagonal process, we obtain a subsequence that converges to u almost everywhere in M. ∎

5 Besov spaces and energy domain

The purpose of this section is to give an alternative characterization of $\mathrm{dom}_{\mathcal{E}}(\mathcal{L})$ in terms of *Besov spaces*, which are defined *independently* of the heat kernel.

5.1 Besov spaces in $\mathbb{R}^n$

Recall that the Sobolev space $W_2^1(\mathbb{R}^n)$ consists of functions $u \in L^2(\mathbb{R}^n)$ such that $\frac{\partial u}{\partial x_i} \in L^2(\mathbb{R}^n)$ for all $i = 1, 2, \ldots, n$. It is known that a function $u \in L^2(\mathbb{R}^n)$ belongs to $W_2^1(\mathbb{R}^n)$ if and only if

$$\sup_{z \in \mathbb{R}^n \setminus \{0\}} \frac{\|u(x+z) - u(x)\|_2}{|z|} < \infty.$$

Fix $1 \le p < \infty$, $0 < \sigma < 1$, and consider a more general *Besov-Nikol'skii* space $B^{\sigma}_{p,\infty}(\mathbb{R}^n)$ that consists of functions $u \in L^p(\mathbb{R}^n)$ such that

$$\sup_{z \in \mathbb{R}^n, 0<|z|\le 1} \frac{\|u(x+z) - u(x)\|_p}{|z|^{\sigma}} < \infty, \tag{5.1}$$

and the norm in $B^{\sigma}_{p,\infty}$ is the sum of $\|u\|_p$ and the left hand side of (5.1).

A more general family $B^{\sigma}_{p,q}$ of *Besov* spaces is defined for any $1 \le q \le \infty$ but alongside the case $q = \infty$ considered above, we will need only the case $q = p$. By definition, $u \in B^{\sigma}_{p,p}$ if $u \in L^p$ and

$$\iint\limits_{\mathbb{R}^n\,\mathbb{R}^n} \frac{|u(y) - u(x)|^p}{|y - x|^{n+p\sigma}} dx\, dy < \infty, \tag{5.2}$$

with the obvious definition of a norm in $B^{\sigma}_{p,p}$. Here are some well known facts about Besov and Sobolev spaces (see for example [1]).

1. $u \in B^{\sigma}_{p,\infty}(\mathbb{R}^n)$ if and only if $u \in L^p(\mathbb{R}^n)$ and there is a constant C such that for all $0 < r \le 1$

$$D_p(u, r) := \iint\limits_{\{x,y \in \mathbb{R}^n : |x-y|<r\}} |u(y) - u(x)|^p \, dx\, dy \le C r^{n+p\sigma}. \tag{5.3}$$

2. $u \in B^{\sigma}_{p,p}(\mathbb{R}^n)$ if and only if $u \in L^p(\mathbb{R}^n)$ and

$$\int_0^{\infty} \frac{D_p(u,r)}{r^{n+p\sigma}} \frac{dr}{r} < \infty.$$

3. For any $0 < \sigma < 1$ the following relations take place

$$\begin{array}{ccccc} W^1_2(\mathbb{R}^n) & \subset & B^{\sigma}_{2,2}(\mathbb{R}^n) & \subset & B^{\sigma}_{2,\infty}(\mathbb{R}^n) \\ \| & & \| & & \\ \mathrm{dom}_{\mathcal{E}}(-\Delta) & \subset & \mathrm{dom}_{\mathcal{E}}(-\Delta)^{\sigma} & & \end{array} \tag{5.4}$$

5.2 Besov spaces in a metric measure space

Fix $\alpha > 0$, and for any $\sigma > 0$ introduce the following functionals on $L^2(M,\mu)$:

$$D(u,r) := \iint_{\{x,y \in M : d(x,y) < r\}} |u(x) - u(y)|^2 \, d\mu(x) d\mu(y), \tag{5.5}$$

$$N_{\sigma,\infty}(u) := \sup_{0 < r \leq 1} \frac{D(u,r)}{r^{\alpha+2\sigma}} \tag{5.6}$$

and

$$N_{\sigma,2}(u) := \int_0^{\infty} \frac{D(u,r)}{r^{\alpha+2\sigma}} \frac{dr}{r}. \tag{5.7}$$

More generally, for any $q \in [1,+\infty)$ one can set

$$N_{\sigma,q}(u) := \left(\int_0^{\infty} \left(\frac{D(u,r)}{r^{\alpha+2\sigma}} \right)^{q/2} \frac{dr}{r} \right)^{2/q}. \tag{5.8}$$

For any $q \in [1,+\infty]$ define the space

$$\Lambda^{\sigma}_{2,q} := \left\{ u \in L^2 : N_{\sigma,q}(u) < \infty \right\}$$

and the norm in this space by

$$\|u\|^2_{\Lambda^{\sigma}_{2,q}} := \|u\|^2_2 + N_{\sigma,q}(u).$$

An obvious modification of the above formulas allows to introduce the space $\Lambda^{\sigma}_{p,q}$ for any $p \in [1,+\infty)$ (the space $\Lambda^{\sigma}_{p,q}$ was denoted by $Lip(\sigma,p,q)$ in [23] and by $\Lambda^{p,q}_{\sigma}$ in [28]; the space $\Lambda^{\sigma}_{2,\infty}$ was denoted by $W^{\sigma,2}$ in [18]).

For example, we have

$$\begin{array}{ll} \Lambda^{\sigma}_{2,q}(\mathbb{R}^n) = B^{\sigma}_{2,q}(\mathbb{R}^n) & \text{if } 0 < \sigma < 1, \\ \Lambda^{\sigma}_{2,q}(\mathbb{R}^n) = \{0\}, & \text{if } \sigma > 1, \\ \Lambda^{1}_{2,\infty}(\mathbb{R}^n) = W^1_2(\mathbb{R}^n), & \\ \Lambda^{1}_{2,2}(\mathbb{R}^n) = \{0\}. & \end{array}$$

The definitions of $\Lambda^{\sigma}_{2,q}$ and $B^{\sigma}_{2,q}$ match only for $\sigma < 1$. For $\sigma \geq 1$ the definition of $B^{\sigma}_{2,q}$ becomes more involved whereas the above definition of $\Lambda^{\sigma}_{2,q}$ is valid for *all*

$\sigma > 0$ even if the space $\Lambda^{\sigma}_{2,q}$ degenerates to $\{0\}$ for sufficiently large σ. With some abuse of terminology, we refer to $\Lambda^{\sigma}_{p,q}$ also as Besov spaces.

The fact that $D(u,r)$ is increasing in r implies that for any $r > 0$

$$\frac{D(u,r)}{r^{\alpha+2\sigma}} \leq 2^{\alpha+2\sigma} \int_r^{2r} \frac{D(u,\rho)}{\rho^{\alpha+2\sigma}} \frac{d\rho}{\rho}$$

whence $N_{\sigma,\infty}(u) \leq C N_{\sigma,2}(u)$ and

$$\Lambda^{\sigma}_{2,2} \hookrightarrow \Lambda^{\sigma}_{2,\infty}. \tag{5.9}$$

The above definition of the spaces $\Lambda^{\sigma}_{2,q}$ depends on the choice of α. A priori α is any number but we will use the above definition in the presence of the following condition

$$\mu(B(x,r)) \simeq r^{\alpha} \quad \text{for all } x \in M \text{ and } r > 0, \tag{5.10}$$

and α in (5.6)-(5.8) will always be the same as in (5.10). In particular, (5.10) implies that for any $u \in L^2$

$$\begin{aligned} D(u,r) &\leq 2 \iint_{\{d(x,y)<r\}} \left(|u(x)|^2 + |u(y)|^2\right) d\mu(x)\, d\mu(y) \\ &= 4 \iint_{\{d(x,y)<r\}} |u(y)|^2\, d\mu(x)\, d\mu(y) \\ &= 4 \int_M |u(y)|^2 \mu(B(y,r))\, d\mu(y) \\ &\leq C r^{\alpha} \|u\|_2. \end{aligned}$$

Therefore, the integrals in (5.7) and (5.8) converge at ∞ for *all* $u \in L^2$ and $\sigma > 0$, and the point of the condition $N_{\sigma,q}(u) < \infty$ is the convergence of the integral at 0. Consequently, we see that the space $\Lambda^{\sigma}_{2,q}$ decreases when σ increases.

5.3 Identification of energy domains and Besov spaces

Recall that a heat kernel p_t on a metric measure space (M,d,μ) has the associated energy form $\mathcal{E}$ and the generator $\mathcal{L}$. The following theorem identifies the domain $\mathcal{D}(\mathcal{E})$ of the energy form in terms of Besov spaces.

Theorem 5.1 *Let p_t be a heat kernel on (M,d,μ) satisfying hypothesis $\mathcal{H}(\alpha+\beta)$. Then*

$$\mathrm{dom}_{\mathcal{E}}(\mathcal{L}) := \mathcal{D}(\mathcal{E}) = \Lambda^{\beta/2}_{2,\infty}$$

and for any $u \in \mathcal{D}(\mathcal{E})$

$$\mathcal{E}[u] \simeq N_{\beta/2,\infty}(u). \tag{5.11}$$

The result of Theorem 5.1 was first obtained by Jonsson [23, Theorem 1] for SG and then was extended to more general fractal diffusions by Pietruska-Pałuba [26, Theorem 1]. In the present form it was proved in [18, Theorem 4.2].

Recall that hypothesis $\mathcal{H}(\alpha+\beta)$ means that p_t satisfies (2.8) with functions Φ_1 and Φ_2 such that $\Phi_1(1)>0$ and

$$\int^{\infty} s^{\alpha+\beta}\Phi_2(s)\frac{ds}{s}<\infty. \tag{5.12}$$

Let us show the sharpness of the condition (5.12). As it was mentioned above if $0<\sigma<1$ then the heat kernel of the operator $(-\Delta)^{\sigma}$ in $\mathbb{R}^n$ satisfies (2.8) with the function $\Phi_2(s)=\frac{C}{1+s^{\alpha+\beta}}$, where $\alpha=n$ and $\beta=2\sigma$ (see also Lemma 5.4 below). For this function, the condition (5.12) breaks just on the borderline, and the conclusion of Theorem 5.1 is not valid either. Indeed, in $\mathbb{R}^n$ by (5.4) $\mathrm{dom}_{\mathcal{E}}(-\Delta)^{\sigma}=B^{\sigma}_{2,2}$ that is strictly smaller than $B^{\sigma}_{2,\infty}=\Lambda^{\beta/2}_{2,\infty}$. This case will be covered by Theorem 5.2 below.

As we will see in the proof below (cf. (5.15) and (5.17)), under hypothesis $\mathcal{H}(\alpha+\beta)$ we have in fact

$$\mathcal{E}[u]\simeq \limsup_{r\to 0} r^{-\alpha-\beta}\iint\limits_{\{d(x,y)<r\}}|u(x)-u(y)|^2d\mu(y)d\mu(x). \tag{5.13}$$

In particular, this implies that the energy form is *strongly local*, that is for all functions $u,v\in\mathcal{D}(\mathcal{E})$ with compact supports, if $u\equiv\mathrm{const}$ in an open neighborhood of the support of v then $\mathcal{E}(u,v)=0$. The operator $(-\Delta)^{\sigma}$ is not local for $0<\sigma<1$, and this explains why Theorem 5.1 does not apply to this operator.

Proof of Theorem 5.1. Since the expressions $\mathcal{E}[u]$ and $N_{\beta/2,\infty}(u)$ are defined (with possibility of infinite values) for all $u\in L^2$, it suffices to show that (5.11) holds for all $u\in L^2$. Fix a function $u\in L^2$ and recall that by (5.6) and (5.5)

$$N_{\beta/2,\infty}(u)=\sup_{0<r\le 1}\frac{D(u,r)}{r^{\alpha+\beta}}$$

where

$$D(u,r)=\int_M\int_{B(x,r)}|u(x)-u(y)|^2d\mu(y)d\mu(x). \tag{5.14}$$

We first prove that for some $c>0$ and for all $r>0$

$$\mathcal{E}[u]\ge c\frac{D(u,r)}{r^{\alpha+\beta}}$$

which would imply

$$\mathcal{E}[u]\ge c\sup_{r>0}\frac{D(u,r)}{r^{\alpha+\beta}}\ge cN_{\beta/2,\infty}(u). \tag{5.15}$$

Using the lower bound in (2.8) and the monotonicity of Φ_1, we obtain from (4.2) that for any $r>0$ and $t=r^{\beta}$,

$$\begin{aligned}
\mathcal{E}[u] &\ge \frac{1}{2t}\int_M\int_{B(x,r)}(u(x)-u(y))^2p_t(x,y)d\mu(y)d\mu(x)\\
&\ge \frac{1}{2}\left(\frac{1}{t}\right)^{\alpha/\beta+1}\Phi_1\left(\frac{r}{t^{1/\beta}}\right)\int_M\int_{B(x,r)}(u(x)-u(y))^2d\mu(y)d\mu(x)\\
&= \frac{1}{2}\frac{1}{r^{\alpha+\beta}}\Phi_1(1)\int_M\int_{B(x,r)}(u(x)-u(y))^2d\mu(y)d\mu(x)\\
&\ge c\frac{D(u,r)}{r^{\alpha+\beta}},
\end{aligned}$$

which was to be proved.

Let us now prove that for any $r>0$

$$\mathcal{E}[u] \le C \sup_{0<\rho\le r} \frac{D(u,\rho)}{\rho^{\alpha+\beta}}, \tag{5.16}$$

which would imply

$$\mathcal{E}[u] \le C \limsup_{r\to 0+} \frac{D(u,r)}{r^{\alpha+\beta}} \le C\, N_{\beta/2,\infty}(u). \tag{5.17}$$

For any $t>0$ and $r>0$ we have

$$\mathcal{E}_t[u] = \frac{1}{2t}\int_M\int_M (u(x)-u(y))^2 p_t(x,y)d\mu(y)d\mu(x) = A(t)+B(t) \tag{5.18}$$

where

$$A(t) = \frac{1}{2t}\int_M\int_{M\setminus B(x,r)} (u(x)-u(y))^2 p_t(x,y)d\mu(y)d\mu(x), \tag{5.19}$$

$$B(t) = \frac{1}{2t}\int_M\int_{B(x,r)} (u(x)-u(y))^2 p_t(x,y)d\mu(y)d\mu(x). \tag{5.20}$$

To estimate $A(t)$ let us observe that by (3.6)

$$\int_{M\setminus B(x,r)} p_t(x,y)d\mu(y) \le C\int_{\frac{1}{2}rt^{-1/\beta}}^{\infty} s^{\alpha}\Phi_2(s)\frac{ds}{s} \le C\frac{t}{r^{\beta}}\int_{\frac{1}{2}rt^{-1/\beta}}^{\infty} s^{\alpha+\beta}\Phi_2(s)\frac{ds}{s}, \tag{5.21}$$

whence by (5.12)

$$\frac{1}{t}\int_{M\setminus B(x,r)} p_t(x,y)d\mu(y) = o(t) \quad \text{as } t\to 0+ \text{ uniformly in } x\in M. \tag{5.22}$$

Therefore, applying the elementary inequality $(a-b)^2 \le 2(a^2+b^2)$ in (5.19) and using (5.22), we obtain that for small enough $t>0$

$$\begin{aligned} A(t) &\le \frac{1}{t}\iint_{\{x,y:d(x,y)<r\}} (u(x)^2+u(y)^2)p_t(x,y)d\mu(y)d\mu(x) \\ &= \frac{2}{t}\int_M u(x)^2\left(\int_{M\setminus B(x,r)} p_t(x,y)d\mu(y)\right)d\mu(x) \\ &= o(1)\|u\|_2^2, \end{aligned}$$

whence

$$\lim_{t\to 0+} A(t) = 0. \tag{5.23}$$

The quantity $B(t)$ is estimated as follows using (2.8), (5.14), and (5.12), setting $r_k = 2^{-k}r$:

$$
\begin{aligned}
B(t) &= \frac{1}{2t}\sum_{k=0}^{\infty}\int_M\int_{B(x,r_k)\setminus B(x,r_{k+1})}(u(x)-u(y))^2 p_t(x,y)d\mu(y)d\mu(x) \\
&\le \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{t^{1+\alpha/\beta}}\Phi_2\left(\frac{r_{k+1}}{t^{1/\beta}}\right)\int_M\int_{B(x,r_k)}(u(x)-u(y))^2 d\mu(y)d\mu(x) \\
&\le C\sum_{k=0}^{\infty}\left(\frac{r_{k+1}}{t^{1/\beta}}\right)^{\alpha+\beta}\Phi_2\left(\frac{r_{k+1}}{t^{1/\beta}}\right)\frac{D(u,r_k)}{r_k^{\alpha+\beta}} \qquad (5.24) \\
&\le C\sup_{0<\rho\le r}\frac{D(u,\rho)}{\rho^{\alpha+\beta}}\int_0^{\infty}s^{\alpha+\beta}\Phi_2(s)\frac{ds}{s} \\
&\le C\sup_{0<\rho\le r}\frac{D(u,\rho)}{\rho^{\alpha+\beta}}. \qquad (5.25)
\end{aligned}
$$

Finally, (5.16) follows from (5.18), (5.23) and (5.25) by letting $t\to 0$. ■

Theorem 5.2 *Let p_t be a heat kernel on (M,d,μ) satisfying estimate* (2.8) *with functions Φ_1 and Φ_2 such that*

$$\Phi_1(s)\simeq s^{-(\alpha+\beta)} \quad \text{for } s>1 \quad \text{and} \quad \Phi_2(s)\le Cs^{-(\alpha+\beta)} \quad \text{for } s>0. \qquad (5.26)$$

Then

$$\mathrm{dom}_{\mathcal{E}}(\mathcal{L}) = \Lambda_{2,2}^{\beta/2},$$

and for any $u\in\mathcal{D}(\mathcal{E})$

$$\mathcal{E}[u]\simeq N_{\beta/2,2}(u). \qquad (5.27)$$

Remark 5.3 Condition (5.26) implies that $\mathcal{H}(\alpha+\beta)$ is not satisfied while $\mathcal{H}(\alpha+\beta-\varepsilon)$ is satisfied for any $\varepsilon>0$. In this case, neither the hypotheses nor the conclusions of Theorem 5.1 are satisfied.

Proof. The proof is similar to that of Theorem 5.1. It suffices to show that (5.27) holds for any $u\in L^2$. Fix a decreasing geometric sequence $\{r_k\}_{k\in\mathbb{Z}}$ and observe that by (5.7)

$$N_{\beta/2,2}(u)\simeq\sum_{k\in\mathbb{Z}}\frac{D(u,r_k)}{r_k^{\alpha+\beta}}.$$

Using (4.2) and the upper bounds in (2.8) and (5.26) we obtain

$$
\begin{aligned}
2\mathcal{E}_t[u] &= \frac{1}{t}\sum_{k\in\mathbb{Z}}\int_M\int_{B(x,r_k)\setminus B(x,r_{k+1})}(u(x)-u(y))^2 p_t(x,y)d\mu(y)d\mu(x) \\
&\le \sum_{k\in\mathbb{Z}}\frac{1}{t^{1+\alpha/\beta}}\Phi_2\left(\frac{r_{k+1}}{t^{1/\beta}}\right)\int_M\int_{B(x,r_k)\setminus B(x,r_{k+1})}(u(x)-u(y))^2 d\mu(y)d\mu(x) \\
&\le \sum_{k\in\mathbb{Z}}\frac{1}{t^{1+\alpha/\beta}}\Phi_2\left(\frac{r_{k+1}}{t^{1/\beta}}\right)D(u,r_k) \\
&\le C\sum_{k\in\mathbb{Z}}\frac{D(u,r_k)}{r_k^{\alpha+\beta}} \\
&\le CN_{\beta/2,2}(u),
\end{aligned}
$$

whence

$$\mathcal{E}\left[u\right]=\lim_{t\to 0}\mathcal{E}_t\left[u\right]\le CN_{\beta/2,2}\left(u\right). \tag{5.28}$$

Similarly, using the lower bound in (2.8), we obtain

$$\begin{aligned}
2\mathcal{E}_t\left[u\right] &\ge \sum_{k\in\mathbb{Z}}\frac{1}{t^{1+\alpha/\beta}}\Phi_1\left(\frac{r_{k+1}}{t^{1/\beta}}\right)\int_M\int_{B(x,r_k)\setminus B(x,r_{k+1})}(u(x)-u(y))^2d\mu(y)d\mu(x)\\
&= \sum_{k\in\mathbb{Z}}\frac{1}{t^{1+\alpha/\beta}}\Phi_1\left(\frac{r_{k+1}}{t^{1/\beta}}\right)\left(D\left(u,r_k\right)-D\left(u,r_{k+1}\right)\right)\\
&= \sum_{k\in\mathbb{Z}}\frac{1}{t^{1+\alpha/\beta}}\left(\Phi_1\left(\frac{r_{k+1}}{t^{1/\beta}}\right)-\Phi_1\left(\frac{r_k}{t^{1/\beta}}\right)\right)D\left(u,r_k\right).
\end{aligned} \tag{5.29}$$

The first part of the hypothesis (5.26) implies that there exists a large enough number a such that

$$\Phi_1\left(\frac{s}{a}\right)\ge 2\Phi_1\left(s\right)\quad\forall s>a.$$

Setting $r_k=a^{-k}$, we obtain from (5.29) and (5.26)

$$\mathcal{E}_t\left[u\right]\ge\frac{1}{2}\sum_{\left\{k:r_k>at^{1/\beta}\right\}}\frac{1}{t^{1+\alpha/\beta}}\Phi_1\left(\frac{r_k}{t^{1/\beta}}\right)D\left(u,r_k\right)\ge c\sum_{\left\{k:r_k>at^{1/\beta}\right\}}\frac{D\left(u,r_k\right)}{r_k^{\alpha+\beta}}.$$

Letting $t\to 0$, we conclude

$$\mathcal{E}\left[u\right]=\lim_{t\to 0}\mathcal{E}_t[u]\ge c\sum_{k\in\mathbb{Z}}\frac{D\left(u,r_k\right)}{r_k^{\alpha+\beta}}\simeq N_{\beta/2,2}\left(u\right),$$

which together with (5.28) finishes the proof. ■

5.4 Subordinated heat kernel

Let φ be a non-negative continuous function on $[0,+\infty)$ such that $\varphi\left(0\right)=0$, and let $\left\{\eta_t\right\}_{t>0}$ be a family of non-negative continuous functions on $(0,+\infty)$ such that for all $t>0$ and $\lambda\ge 0$

$$\exp\left(-t\varphi\left(\lambda\right)\right)=\int_0^\infty\eta_t\left(s\right)e^{-s\lambda}ds. \tag{5.30}$$

Then, for any heat kernel p_t on a metric measure space (M,d,μ), the following expression

$$q_t\left(x,y\right):=\int_0^\infty\eta_t\left(s\right)p_s\left(x,y\right)ds \tag{5.31}$$

defines a new heat kernel $\left\{q_t\right\}_{t>0}$ on M, which is called a *subordinated heat kernel* to p_t (and η_t is called a *subordinator*). Indeed, applying (5.30) to the generator $\mathcal{L}$ of p_t we obtain

$$\exp\left(-t\varphi\left(\mathcal{L}\right)\right)=\int_0^\infty\eta_t\left(s\right)P_s\,ds.$$

Comparing to (5.31) we see that q_t is the integral kernel of the semigroup $\left\{e^{-t\varphi(\mathcal{L})}\right\}_{t>0}$ generated by the operator $\varphi(\mathcal{L})$. Since $\left\{e^{-t\varphi(\mathcal{L})}\right\}_{t>0}$ is a self-adjoint strongly continuous contraction semigroup in L^2, the family $\{q_t\}_{t>0}$ satisfies the properties 3, 4, 5 of Definition 2.1. The positivity of q_t follows from $\eta_t \geq 0$, and the stochastic completeness of q_t follows from that of p_t and

$$\int_M q_t(x,y)\,d\mu(y) = \int_0^\infty \eta_t(s)\left(\int_M p_s(x,y)\,d\mu(y)\right)ds = \int_0^\infty \eta_t(s)\,ds = 1,$$

where the last equality is obtained from (5.30) by taking $\lambda = 0$. Hence, $\{q_t\}_{t>0}$ is a heat kernel.

For example, it follows from the definition of the gamma-function that for all $t > 0$ and $\lambda \geq 0$

$$\exp(-t\log(1+\lambda)) = (1+\lambda)^{-t} = \frac{1}{\Gamma(t)}\int_0^\infty s^{t-1}e^{-s(1+\lambda)}ds,$$

which takes the form (5.30) for $\varphi(\lambda) = \log(1+\lambda)$ and $\eta_t(s) = \frac{s^{t-1}e^{-s}}{\Gamma(t)}$. Therefore, the operator $\log(1+\mathcal{L})$ that generates the semigroup $\left\{(\mathrm{I}+\mathcal{L})^{-t}\right\}_{t>0}$ has the heat kernel

$$q_t(x,y) = \frac{1}{\Gamma(t)}\int_0^\infty s^{t-1}e^{-s}p_s(x,y)\,ds.$$

It is well known that for any $\delta \in (0,1)$ there exists a subordinator $\eta_t = \eta_t^{(\delta)}$ such that (5.30) takes place with $\varphi(\lambda) = \lambda^\delta$. In this case, (5.31) defines the heat kernel q_t of the operator $\mathcal{L}^\delta$. For example, if $\delta = \frac{1}{2}$ then

$$\eta_t^{(1/2)}(s) = \frac{t}{\sqrt{4\pi s^3}}\exp\left(-\frac{t^2}{4s}\right).$$

For any $0 < \delta < 1$, the function $\eta_t^{(\delta)}(s)$ possesses the scaling property

$$\eta_t^{(\delta)}(s) = \frac{1}{t^{1/\delta}}\eta_1^{(\delta)}\left(\frac{s}{t^{1/\delta}}\right),$$

and satisfies the estimates

$$\eta_t^{(\delta)}(s) \leq C\frac{t}{s^{1+\delta}} \quad \forall s,t>0, \tag{5.32}$$

$$\eta_t^{(\delta)}(s) \simeq \frac{t}{s^{1+\delta}} \quad \forall s \geq t^{1/\delta} > 0. \tag{5.33}$$

As $s \to 0+$, $\eta_1^{(\delta)}(s)$ goes to 0 exponentially fast so that for any $\gamma > 0$

$$\int_0^\infty s^{-\gamma}\eta_1^{(\delta)}(s)\,ds < \infty \tag{5.34}$$

(see [31] and [10]).

Lemma 5.4 *If a heat kernel p_t satisfies hypothesis $\mathcal{H}(\alpha+\beta')$ where $\beta'=\delta\beta$, $0<\delta<1$, then the heat kernel $q_t(x,y)$ of operator $\mathcal{L}^\delta$ satisfies the estimate*

$$q_t(x,y)\simeq\frac{1}{t^{\alpha/\beta'}}\frac{1}{\left(1+\frac{d(x,y)}{t^{1/\beta'}}\right)^{\alpha+\beta'}}\simeq\min\left(t^{-\alpha/\beta'},\frac{t}{d(x,y)^{\alpha+\beta'}}\right), \tag{5.35}$$

for all $x,y\in M$ and $t>0$.

Proof. Setting $r=d(x,y)$ and using (5.31), (2.8), (5.32) we obtain

$$\begin{aligned} q_t(x,y) &\leq \int_0^\infty\frac{1}{s^{\alpha/\beta}}\Phi_2\left(\frac{r}{s^{1/\beta}}\right)\eta_t^{(\delta)}(s)\,ds\\ &\leq C\int_0^\infty\frac{1}{s^{\alpha/\beta}}\Phi_2\left(\frac{r}{s^{1/\beta}}\right)\frac{t}{s^{1+\delta}}ds\\ &= C\frac{t}{r^{\alpha+\beta\delta}}\int_0^\infty\Phi_2(\xi)\,\xi^{\alpha+\beta\delta}\frac{d\xi}{\xi}.\end{aligned}$$

By $\mathcal{H}(\alpha+\beta')$ the above integral converges, whence

$$q_t(x,y)\leq C\frac{t}{r^{\alpha+\beta'}}. \tag{5.36}$$

On the other hand, using the upper bound $p_s(x,y)\leq Cs^{-\alpha/\beta}$ and the change $\tau=s/t^{1/\delta}$ we obtain

$$q_t(x,y)\leq C\int_0^\infty\frac{1}{s^{\alpha/\beta}}\eta_t^{(\delta)}(s)\,ds=t^{-\alpha/(\beta\delta)}\int_0^\infty\frac{1}{\tau^{\alpha/\beta}}\eta_1^{(\delta)}(\tau)\,d\tau\leq Ct^{-\alpha/\beta'} \tag{5.37}$$

where the last inequality follows from (5.34). Combining (5.36) and (5.37) we obtain the upper bound in (5.35).

Finally, (5.31), (2.8), (5.33) imply the lower bound in (5.35) as follows:

$$\begin{aligned} q_t(x,y) &\geq \int_{\max\left(t^{1/\delta},r^\beta\right)}^\infty\frac{1}{s^{\alpha/\beta}}\Phi_1\left(\frac{r}{s^{1/\beta}}\right)\eta_t^{(\delta)}(s)\,ds\\ &\geq c\Phi_1(1)\int_{\max\left(t^{1/\delta},r^\beta\right)}^\infty\frac{1}{s^{\alpha/\beta}}\frac{t}{s^{1+\delta}}ds\\ &= ct\max\left(t^{1/\delta},r^\beta\right)^{-\alpha/\beta-\delta}\\ &= c\min\left(t^{-\alpha/\beta'},tr^{-\alpha-\beta'}\right).\end{aligned}$$

■

Corollary 5.5 *If a heat kernel p_t satisfies hypothesis $\mathcal{H}(\alpha+\beta')$ where $\beta'=\delta\beta$, $0<\delta<1$, then*

$$\mathrm{dom}_{\mathcal{E}}(\mathcal{L}^\delta)=\Lambda_{2,2}^{\beta'/2}. \tag{5.38}$$

Remark 5.6 *This result was essentially proved by A.Stós [27].*

Proof. By Lemma 5.4, the heat kernel q_t of the operator $\mathcal{L}^\delta$ satisfies the estimate (2.8)

$$q_t(x,y) \simeq \frac{1}{t^{\alpha/\beta'}} \Phi\left(\frac{d(x,y)}{t^{1/\beta'}}\right)$$

where

$$\Phi(s) = \frac{1}{(1+s)^{\alpha+\beta'}}.$$

Applying Theorem 5.2 to the heat kernel q_t and its generator $\mathcal{L}^\delta$ we obtain (5.38). ■

6 Intrinsic characterization of walk dimension

Definition 6.1 *Let us set*

$$\beta^* := 2\sup\left\{\sigma : \dim \Lambda^\sigma_{2,\infty} = \infty\right\} \tag{6.1}$$

and refer to β^* as the critical exponent *of the family $\left\{\Lambda^\sigma_{2,\infty}\right\}_{\sigma>0}$ of Besov spaces.*

Note that the value of β^* is an intrinsic property of the space (M,d,μ), which is defined independently of any heat kernel. For example, for $\mathbb{R}^n$ we have $\beta^* = 2$.

Theorem 6.2 *Let p_t be a heat kernel on a metric measure space (M,d,μ).*

(i) If p_t satisfies hypothesis $\mathcal{H}(\alpha)$ then $\dim \Lambda^{\beta/2}_{2,\infty} = \infty$. Consequently, $\beta \le \beta^*$.

(ii) If p_t satisfies hypothesis $\mathcal{H}(\alpha+\beta+\varepsilon)$ for some $\varepsilon > 0$ then $\Lambda^\sigma_{2,\infty} = \{0\}$ for any $\sigma > \beta/2$. Consequently, $\beta = \beta^*$.

If p_t is the heat kernel of the operator $(-\Delta)^{\beta/2}$ in $\mathbb{R}^n$, $0 < \beta < 2$, then by (2.9) it satisfies $\mathcal{H}(\alpha)$ (with $\alpha = n$) but not $\mathcal{H}(\alpha+\beta+\varepsilon)$. In this case the conclusion of Theorem 6.2(ii) is not true, because β can actually be smaller than $\beta^* = 2$.

Proof of Theorem 6.2(i)**.** As one can see from the proof of Theorem 5.1, the inclusion $\mathcal{D}(\mathcal{E}) \subset \Lambda^{\beta/2}_{2,\infty}$ requires only the lower estimates in (2.8) and (3.1) (and the opposite inclusion requires only the upper estimates in (2.8) and (3.1)). By Theorem 3.1 hypothesis $\mathcal{H}(\alpha)$ implies (3.1), and by the above remark we obtain $\mathcal{D}(\mathcal{E}) \subset \Lambda^{\beta/2}_{2,\infty}$. On the other hand, $\mathcal{D}(\mathcal{E})$ is always dense in L^2, whereas it follows from Corollary 3.3 that $\dim L^2 = \infty$. Therefore, $\dim \Lambda^{\beta/2}_{2,\infty} = \infty$, whence $\beta \le \beta^*$. ■

Proof of Theorem 6.2(ii)**.** Let us first explain why $\beta = \beta^*$ follows from the first claim. Indeed, $\Lambda^\sigma_{2,\infty} = \{0\}$ implies that $\sigma \ge \beta^*/2$, and since this is true for any $\sigma > \beta/2$, we obtain $\beta \ge \beta^*$. Since the opposite inequality holds by part (i), we conclude $\beta = \beta^*$.

Let us prove that $\Lambda^\sigma_{2,\infty} = \{0\}$ for any $\sigma > \beta/2$. It suffices to assume that $\sigma - \beta/2$ is positive but sufficiently small. Namely, we can assume that $\varepsilon := 2\sigma - \beta$ is so small that hypothesis $\mathcal{H}(\alpha+\beta+\varepsilon)$ holds. Recall that this hypothesis means that the estimate (2.8) holds with functions Φ_1 and Φ_2 such that $\Phi_1(1) > 0$ and

$$\int^\infty s^{\alpha+\beta+\varepsilon}\Phi_2(s)\frac{ds}{s} < \infty. \tag{6.2}$$

Let us show that for any function $u \in \Lambda^{\sigma}_{2,\infty}$ we have $\mathcal{E}[u] = 0$. We use again the decomposition $\mathcal{E}_t[u] = A(t) + B(t)$, where $A(t)$ and $B(t)$ are defined in (5.19) and (5.20) with $r = 1$. Estimating $B(t)$ similarly to (5.24) but using $N_{\sigma,\infty}$ instead of $N_{\beta/2,\infty}$ and setting $r_k = 2^{-k}$ we obtain

$$\begin{aligned} t^{-\varepsilon/\beta} B(t) &\le C \sum_{k=0}^{\infty} \left(\frac{r_{k+1}}{t^{1/\beta}}\right)^{\alpha+\beta+\varepsilon} \Phi_2\left(\frac{r_{k+1}}{t^{1/\beta}}\right) \frac{D(u, r_k)}{r_k^{\alpha+\beta+\varepsilon}} \\ &\le C \sup_{0<\rho\le 1} \frac{D(u,\rho)}{\rho^{\alpha+2\sigma}} \int_0^{\infty} s^{\alpha+\beta+\varepsilon} \Phi_2(s) \frac{ds}{s} \\ &\le C N_{\sigma,\infty}(u). \end{aligned}$$

Together with (5.18) and (5.23) this yields

$$\mathcal{E}_t[u] \le A(t) + C t^{\varepsilon/\beta} N_{\sigma,\infty}(u) \to 0 \quad \text{as } t \to 0,$$

whence

$$\mathcal{E}[u] = \lim_{t\to 0} \mathcal{E}_t[u] = 0.$$

Since $\mathcal{E}_t[u] \le \mathcal{E}[u]$, this implies back $\mathcal{E}_t[u] \equiv 0$ for all $t > 0$. On the other hand, it follows from (4.2) and the lower bound in (2.8) that

$$\mathcal{E}_t[u] \ge \frac{1}{2t^{\alpha/\beta+1}} \Phi_1(1) \iint_{\{d(x,y)\le t^{1/\beta}\}} (u(x) - u(y))^2 d\mu(y) d\mu(x),$$

which yields $u(x) = u(y)$ for μ-almost all x, y such that $d(x,y) \le t^{1/\beta}$. Since t is arbitrary, we conclude that u is a constant function. Since $\mu(M) = \infty$ (see Corollary 3.3), we obtain $u \equiv 0$. ■

Corollary 6.3 *If a heat kernel p_t satisfies $\mathcal{H}(\alpha + \beta + \varepsilon)$ then the values of the parameters α and β in (2.8) are invariants of the metric measure space (M, d, μ).*

Proof. By Theorem 3.1, $\mu(B(x,r))$ satisfies (3.1), which uniquely determines α as the exponent of the volume growth of (M, d, μ). By Theorem 6.2(ii), β is uniquely determined as the critical exponent of the family of Besov spaces of (M, d, μ). ■

7 Inequalities for walk dimension

Definition 7.1 *We say that a metric space (M, d) satisfies the* chain condition *if there exists a (large) constant C such that for any two points $x, y \in M$ and for any positive integer n there exists a sequence $\{x_i\}_{i=0}^n$ of points in M such that $x_0 = x$, $x_n = y$, and*

$$d(x_i, x_{i+1}) \le C \frac{d(x,y)}{n}, \quad \textit{for all } i = 0, 1, ..., n-1. \tag{7.1}$$

The sequence $\{x_i\}_{i=0}^n$ is referred to as a chain connecting x and y.

For example, the chain condition is satisfied if (M,d) is a *length space*, that is if the distance $d(x,y)$ is defined as the infimum of the length of all continuous curves connecting x and y, with a proper definition of length. On the other hand, the chain condition is not satisfied if M is a locally finite graph, and d is the graph distance.

Recall that the critical exponent $\beta^* = \beta^*(M,d,\mu)$ of the family of Besov spaces $\Lambda^{\sigma}_{2,\infty}$ was defined by (6.1).

Theorem 7.2 *Let (M,d,μ) be a metric measure space.*

(i) If $0 < \mu(B(x,r)) < \infty$ for all $x \in M$ and $r > 0$, and

$$\mu(B(x,r)) \le Cr^{\alpha} \tag{7.2}$$

for all $x \in M$ and $0 < r \le 1$ then

$$\beta^* \ge 2. \tag{7.3}$$

(ii) If the space (M,d) satisfies the chain condition and

$$\mu(B(x,r)) \simeq r^{\alpha} \tag{7.4}$$

for all $x \in M$ and $0 < r \le 1$ then

$$\beta^* \le \alpha + 1. \tag{7.5}$$

Observe that the chain condition is essential for the inequality $\beta^* \le \alpha + 1$ to be true. Indeed, assume for a moment that the claim of Theorem 7.2(ii) holds without the chain condition, and consider a new metric d' on M given by $d' = d^{1/\gamma}$ where $\gamma > 1$. Let us mark by a dash all notions related to the space (M,d',μ) as opposed to those of (M,d,μ). It is easy to see that $\alpha' = \alpha\gamma$ and $N'_{\sigma\gamma} = N_{\sigma}$; in particular, the latter implies $\beta^{*\prime} = \beta^*\gamma$. Hence, if Theorem 7.2 could be applied to the space (M,d',μ) it would yield $\beta^*\gamma \le \alpha\gamma + 1$ which implies $\beta^* \le \alpha$ because γ may be taken arbitrarily large. However, there are spaces with $\beta^* > \alpha$, for example SG.

Clearly, the metric d' does not satisfy the chain condition; indeed the inequality (7.1) implies

$$d'(x_i, x_{i+1}) \le C\frac{d'(x,y)}{n^{1/\gamma}}, \tag{7.6}$$

which is not good enough. Note that if in the inequality (7.1) we replace n by $n^{1/\gamma}$ then the proof below will give $\beta^* \le \alpha + \gamma$ instead of $\beta^* \le \alpha + 1$.

Corollary 7.3 *Let p_t be a heat kernel on a metric measure space (M,d,μ).*

(i) If p_t satisfies hypothesis $\mathcal{H}(\alpha + \beta + \varepsilon)$ for some $\varepsilon > 0$ then $\beta \ge 2$.

(ii) If p_t satisfies hypothesis $\mathcal{H}(\alpha)$ and (M,d) satisfies the chain condition then $\beta \le \alpha + 1$.

(iii) If p_t satisfies hypothesis $\mathcal{H}(2\alpha + 1 + \varepsilon)$ for some $\varepsilon > 0$ and (M,d) satisfies the chain condition then

$$2 \le \beta \le \alpha + 1. \tag{7.7}$$

M.Barlow [4] has proved that if α and β satisfy (7.7) then there exists a graph such that the transition probability for the random walk on this graph satisfies (1.2). There is no doubt that similar examples can be constructed in the setting of metric measure spaces satisfying the chain condition.

Proof of Corollary 7.3. (i) By Theorems 3.1 and 7.2(i) we have $\beta^* \geq 2$, and by Theorem 6.2(ii) we have $\beta = \beta^*$, whence $\beta \geq 2$.

(ii) By Theorems 3.1 and 7.2(ii) we have $\beta^* \leq \alpha + 1$, and by Theorem 6.2(i) we have $\beta \leq \beta^*$, whence $\beta \leq \alpha + 1$.

(iii) Clearly, $\mathcal{H}(2\alpha + 1 + \varepsilon)$ implies $\mathcal{H}(\alpha)$, and by part (ii) we obtain $\beta \leq \alpha + 1$. Therefore, $\mathcal{H}(2\alpha + 1 + \varepsilon)$ implies $\mathcal{H}(\alpha + \beta + \varepsilon)$, and by part (i) we obtain $\beta \geq 2$. ■

Proof of Theorem 7.2(i)**.** Let us show that $\dim \Lambda^1_{2,\infty} = \infty$, which would imply by definition (6.1) of β^* that $\beta^* \geq 2$. Fix a ball $B(x_0, R)$ in M of a positive radius R, and let $u(x)$ be the tent function of this ball, that is

$$u(x) = (R - d(x, x_0))_+ . \tag{7.8}$$

Let us show that $u \in \Lambda^1_{2,\infty}$. Indeed, by (5.6), it suffices to prove that for some constant C and for all $0 < r < 1$

$$D(u,r) = \int_M \int_{B(x,r)} |u(x) - u(y)|^2 d\mu(y) d\mu(x) \leq C r^{2+\alpha}. \tag{7.9}$$

Since the function u vanishes outside the ball $B(x_0, R)$ and $r \leq 1$, the exterior integration in (7.9) can be reduced to $B(x_0, R+1)$. Using the obvious inequality

$$|u(x) - u(y)| \leq d(x,y) \leq r,$$

and (7.2) we obtain

$$D(u,r) \leq C \int_{B(x_0,R+1)} r^2 \mu\left(B\left(x,r\right)\right) d\mu(x) = C r^{2+\alpha},$$

whence (7.9) follows. Note also that $\|u\|_2 \neq 0$ which follows from $\mu(B(x_0, R)) > 0$.

Observe that M contains infinitely many points. Indeed, by (7.2) the measure of any single point is 0. Since any ball has positive measure, it has uncountable many points. Let $\{x_i\}_{i=1}^\infty$ be a sequence of distinct points in M. Fix a positive integer n and choose $R > 0$ small enough so that all balls $B(x_i, R)$, $i = 1, 2, ..., n$, are disjoint. The tent functions of these balls are linearly independent, which implies $\dim \Lambda^1_{2,\infty} \geq n$. Since this is true for any n, we conclude $\dim \Lambda^1_{2,\infty} = \infty$. ■

We precede the proof of the second part of Theorem 7.2 by a lemma.

Lemma 7.4 *Let $\{x_i\}_{i=0}^n$ be a sequence of points in a metric space (M, d) such that for some $\rho > 0$ we have $d(x_0, x_n) > 2\rho$ and*

$$d(x_i, x_{i+1}) < \rho \quad \textit{for all } i = 0, 1, ..., n-1. \tag{7.10}$$

Then there exists a subsequence $\{x_{i_k}\}_{k=0}^l$ such that

(a) $0 = i_0 < i_1 < ... < i_l = n;$

(b) $d(x_{i_k}, x_{i_{k+1}}) < 5\rho$ *for all* $k = 0, 1, ..., l-1$*;*

(c) $d(x_{i_k}, x_{i_j}) \geq 2\rho$ *for all distinct* $k, j = 0, 1, ..., l$.

The significance of conditions $(a), (b), (c)$ is that a sequence $\{x_{i_k}\}_{k=0}^{l}$ satisfying them gives rise to a chain of balls $B(x_{i_k}, 5\rho)$ connecting the points x_0 and x_n in a way that each ball contains the center of the next one whereas the balls $B(x_{i_k}, \rho)$ are disjoint. This is similar to the classical ball covering argument, but additional difficulties arise from the necessity to maintain a proper order in the set of balls.

Proof. Let us say that a sequence of indices $\{i_k\}_{k=0}^{l}$ is *good* if the following conditions are satisfied:

(a') $0 = i_0 < i_1 < ... < i_l$;

(b') $d(x_{i_k}, x_{i_{k+1}}) < 3\rho$ for all $k = 0, 1, ..., l-1$;

(c') $d(x_{i_k}, x_{i_j}) \geq 2\rho$ for all distinct $k, j = 0, 1, ..., l$.

Note that a good sequence does not necessarily have $i_l = n$ as required in condition (a). We start with a good sequence that consists of a single index $i_0 = 0$, and will successively redefine it to increase at each step the value of i_l. A terminal good sequence will be used to construct a sequence satisfying $(a), (b), (c)$.

Assuming that $\{i_k\}_{k=0}^{l}$ is a good sequence, define the following set of indices

$$S := \{s : i_l < s \leq n \text{ and } d(x_s, x_{i_k}) \geq 2\rho \quad \text{for all } k \leq l\},$$

and consider two cases.

Set S is non-empty. In this case we will redefine $\{i_k\}$ to increase i_l. Let m be the minimal index in S. Therefore, $m-1$ is not in S, whence we have either $m-1 \leq i_l$ or

$$d(x_{m-1}, x_{i_k}) < 2\rho \quad \text{for some } k \leq l \tag{7.11}$$

(see Fig. 3). In the first case, we have in fact $m - 1 = i_l$ so that (7.11) also holds (with $k = l$).

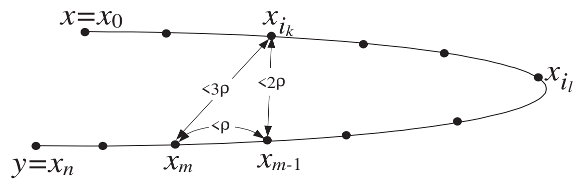

Figure 3:

By (7.11) and (b') we obtain, for the same k as in (7.11),

$$d(x_m, x_{i_k}) \leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{i_k}) < 3\rho.$$

Now we modify the sequence $\{i_j\}$ as follows: keep $i_0, i_1, ..., i_k$ as before, forget the previously selected indices $i_{k+1}, ..., i_l$, and set $i_{k+1} := m$ and $l := k + 1$.

Clearly, the new sequence $\{i_k\}_{k=0}^{l}$ is also good, and the value of i_l has increased (although l may have decreased). Therefore, this construction can be repeated only a finite number of times.

Set S is empty. In this case, we will use the existing good sequence to construct a sequence satisfying conditions (a), (b), (c). The set S can be empty for two reasons:

- either $i_l = n$
- or $i_l < n$ and for any index s such that $i_l < s \leq n$ there exists $k \leq l$ such that $d(x_s, x_{i_k}) < 2\rho$.

In the first case the sequence $\{i_k\}_{k=0}^{l}$ already satisfies (a), (b), (c), and the proof is finished. In the second case, choose the minimal $k \leq l$ such that $d(x_n, x_{i_k}) < 2\rho$ (see Fig. 4).

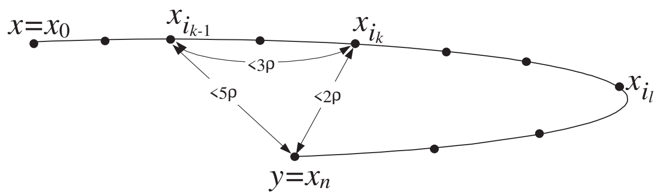

Figure 4:

The hypothesis $d(x_n, x_0) \geq 2\rho$ implies $k \geq 1$, and we obtain from (b')

$$d(x_n, x_{i_{k-1}}) \leq d(x_n, x_{i_k}) + d(x_{i_k}, x_{i_{k-1}}) < 5\rho.$$

By the minimality of k, we have also $d(x_n, x_{i_j}) \geq 2\rho$ for all $j < k$. Hence, we define the final sequence $\{i_j\}$ as follows: keep $i_0, i_1, ..., i_{k-1}$ as before, forget $i_k, ..., i_l$, and set $i_k := n$ and $l := k$. Then this sequence satisfies (a), (b), (c). ■

Let A be a subset of M of finite measure, that is $\mu(A) < \infty$. Then any function $u \in L^2$ is integrable on A, and let us set

$$u_A := \frac{1}{\mu(A)} \int_A u \, d\mu.$$

For any two measurable sets $A, B \subset M$ of finite measure, the following identity takes place

$$\begin{aligned} & \int_A \int_B |u(x) - u(y)|^2 \, d\mu(x) d\mu(y) & (7.12) \\ = \; & \mu(A) \int_B |u - u_B|^2 \, d\mu + \mu(B) \int_A |u - u_A|^2 \, d\mu + \mu(A)\mu(B) \, |u_A - u_B|^2 , \end{aligned}$$

which is proved by a straightforward computation.

Proof of Theorem 7.2(ii). The inequality $\beta^* \leq \alpha + 1$ will follow from (6.1) if we show that for any $\sigma > \frac{\alpha+1}{2}$ the space $\Lambda^{\sigma}_{2,\infty}$ is trivial, that is $N_{\sigma,\infty}(u) < \infty$ implies $u \equiv 0$. By definition (5.6) of $N_{\sigma,\infty}$ and (7.4) we have, for any $0 < r \leq 1$,

$$N_{\sigma,\infty}(u) \geq c r^{-2\sigma-\alpha} \iint_{\{d(x,y)<r\}} |u(x) - u(y)|^2 d\mu(y) d\mu(x). \tag{7.13}$$

Fix some $0 < r \leq 1$ and assume that we have a sequence of disjoint balls $\{B_k\}_{k=0}^{l}$ of the same radius $0 < \rho < 1$, such that for all $k = 0, 1, ..., l-1$

$$x \in B_k \quad \text{and} \quad y \in B_{k+1} \quad \Longrightarrow \quad d(x,y) < r. \tag{7.14}$$

Then (7.13), (7.12), and (7.4) imply

$$\begin{aligned} N_{\sigma,\infty}(u) &\geq cr^{-2\sigma-\alpha} \sum_{k=0}^{l-1} \int_{B_k} \int_{B_{k+1}} |u(x) - u(y)|^2 \, d\mu(y) d\mu(x) \\ &\geq cr^{-2\sigma-\alpha} \rho^{2\alpha} \sum_{k=0}^{l-1} \left| u_{B_k} - u_{B_{k+1}} \right|^2. \end{aligned} \tag{7.15}$$

By the chain condition, for any two distinct points $x, y \in M$ and for any positive integer n there exists a sequence of points $\{x_i\}_{i=0}^{n}$ such that $x_0 = x$, $x_n = y$, and

$$d(x_i, x_{i+1}) < C\frac{d(x,y)}{n} := \rho, \quad \text{for all } 0 \leq i < n.$$

Assuming that n is large enough so that $d(x,y) > 2\rho$ and $\rho < 1/7$, we obtain by Lemma 7.4 that there exists a subsequence $\{x_{i_k}\}_{k=0}^{l}$ (where of course $l \leq n$) such that $x_{i_0} = x$, $x_{i_l} = y$, the balls $\{B(x_{i_k}, \rho)\}$ are disjoint, and

$$d(x_{i_k}, x_{i_{k+1}}) < 5\rho, \tag{7.16}$$

for all $k = 0, 1, ..., l-1$.

Applying (7.15) to the balls $B_k := B(x_{i_k}, \rho)$ and setting $r = 7\rho < 1$ (which together with (7.16) ensures (7.14)) we obtain

$$\begin{aligned} N_{\sigma,\infty}(u) &\geq c\rho^{-2\sigma+\alpha} \sum_{k=0}^{l-1} \left| u_{B_k} - u_{B_{k+1}} \right|^2 \\ &\geq c\rho^{-2\sigma+\alpha} \frac{1}{l} \left(\sum_{k=0}^{l-1} \left| u_{B_k} - u_{B_{k+1}} \right| \right)^2 \\ &\geq c\rho^{-2\sigma+\alpha} \frac{1}{n} \left| u_{B_0} - u_{B_l} \right|^2 \\ &\geq c\rho^{-2\sigma+\alpha+1} \frac{\left| u_{B(x,\rho)} - u_{B(y,\rho)} \right|^2}{d(x,y)}. \end{aligned} \tag{7.17}$$

The property (7.4) of measure of balls implies that for μ-almost all $x \in M$ one has

$$\lim_{\rho \to 0} u_{B(x,\rho)} = u(x), \tag{7.18}$$

and any point x satisfying (7.18) is called a *Lebesgue point* of u. Assuming that x and y are Lebesgue points of u, using $N_{\sigma,\infty}(u) < \infty$ we obtain from (7.17) as $n \to \infty$ (that is, as $\rho \to 0$)

$$\frac{|u(x) - u(y)|^2}{d(x,y)} \leq C \, N_{\sigma,\infty}(u) \lim_{\rho \to 0} \rho^{2\sigma-\alpha-1}. \tag{7.19}$$

Since $2\sigma > \alpha + 1$, the above limit is zero whence $u \equiv \text{const}$. Finally, $\mu(M) = \infty$ implies $u \equiv 0$. ∎

8 Embedding of Besov spaces into Hölder spaces

In addition to the spaces L^p and $\Lambda^\sigma_{2,q}$ defined above, let us define a Hölder space $C^\lambda = C^\lambda(M, d, \mu)$ as follows: $u \in C^\lambda$ if

$$\|u\|_{C^\lambda} := \|u\|_\infty + \mathop{\mu\text{-ess}\sup}_{\substack{x,\, y \,\in\, M \\ 0 \,<\, d(x,y) \,\le\, 1/3}} \frac{|u(x) - u(y)|}{d(x,y)^\lambda} < \infty.$$

The restriction $d(x, y) \le 1/3$ here is related to the restriction $r \le 1$ in definition (5.6). If (M, d) satisfies the chain condition then the $1/3$ can be replaced by any other positive constant.

Theorem 8.1 *Let* (M, d, μ) *satisfy* (5.10). *Then for any* $\sigma > \alpha/2$

$$\Lambda^\sigma_{2,\infty} \hookrightarrow C^\lambda \quad \textit{where} \quad \lambda = \sigma - \alpha/2.$$

That is, for any $u \in L^2$ *we have*

$$\|u\|_{C^\lambda} \le C \|u\|_{\Lambda^\sigma_{2,\infty}}. \tag{8.1}$$

Remark 8.2 From (5.9) it follows that also $\Lambda^\sigma_{2,2} \hookrightarrow C^\lambda$, which will be used in the proof of Theorem 9.2(ii).

Proof. For any $x \in M$ and $r > 0$, set

$$u_r(x) := \frac{1}{\mu(B(x,r))} \int_{B(x,r)} u(\xi) d\mu(\xi). \tag{8.2}$$

We claim that for any $u \in L^2$, any $0 < r \le 1/3$, and all $x, y \in M$ such that $d(x, y) \le r$, the following inequality holds:

$$|u_r(x) - u_r(y)| \le C\, r^\lambda N_{\sigma,\infty}(u)^{1/2}. \tag{8.3}$$

Indeed, setting $B_1 = B(x, r), B_2 = B(y, r)$, we have

$$u_r(x) = \frac{1}{\mu(B_1)} \int_{B_1} u(\xi) d\mu(\xi) = \frac{1}{\mu(B_1)\mu(B_2)} \int_{B_1}\int_{B_2} u(\xi) d\mu(\eta) d\mu(\xi),$$

and similarly

$$u_r(y) = \frac{1}{\mu(B_1)\mu(B_2)} \int_{B_1}\int_{B_2} u(\eta) d\mu(\eta) d\mu(\xi).$$

Applying the Cauchy-Schwarz inequality, (3.1) and (5.6), we obtain

$$\begin{aligned}
|u_r(x) - u_r(y)|^2 &= \left\{ \frac{1}{\mu(B_1)\mu(B_2)} \int_{B_1}\int_{B_2} (u(\xi) - u(\eta))\, d\mu(\eta) d\mu(\xi) \right\}^2 \\
&\le \frac{1}{\mu(B_1)\mu(B_2)} \int_{B_1}\int_{B_2} |u(\xi) - u(\eta)|^2\, d\mu(\eta) d\mu(\xi) \\
&\le C\, r^{-2\alpha} \int_M \int_{B(\eta,3r)} |u(\xi) - u(\eta)|^2\, d\mu(\eta) d\mu(\xi) \\
&\le C\, r^{-\alpha+2\sigma} N_{\sigma,\infty}(u),
\end{aligned}$$

thus proving (8.3).

Similarly, one proves that for any $0 < r \leq 1/3$ and $x \in M$

$$|u_{2r}(x) - u_r(x)| \leq C\, r^{\lambda} N_{\sigma,\infty}(u)^{1/2}. \tag{8.4}$$

Let x be a Lebesgue point of u. Setting $r_k = 2^{-k}r$ for any $k = 0, 1, 2, \ldots$ we obtain from (8.4)

$$\begin{aligned} |u(x) - u_r(x)| &\leq \sum_{k=0}^{\infty} |u_{r_k}(x) - u_{r_{k+1}}(x)| \\ &\leq C \left(\sum_{k=0}^{\infty} r_k^{\lambda} \right) N_{\sigma,\infty}(u)^{1/2} \\ &\leq C\, r^{\lambda} N_{\sigma,\infty}(u)^{1/2}. \end{aligned} \tag{8.5}$$

Applying the Cauchy-Schwarz inequality

$$|u_r(x)| \leq C\, r^{-\alpha/2} \|u\|_2$$

and using (8.5) to some fixed value of r, say $r = 1/4$, we obtain

$$|u(x)| \leq |u(x) - u_r(x)| + |u_r(x)| \leq C \left(\|u\|_2 + N_{\sigma,\infty}(u)^{1/2} \right),$$

whence

$$\|u\|_{\infty} \leq C\, \|u\|_{\Lambda^{\sigma}_{2,\infty}}. \tag{8.6}$$

If y is another Lebesgue point of u such that $r := d(x, y) < 1/3$ then we obtain from (8.3), (8.5), and a similar inequality for y

$$|u(x) - u(y)| \leq |u(x) - u_r(x)| + |u_r(x) - u_r(y)| + |u_r(y) - u(y)| \leq C\, r^{\lambda} N_{\sigma,\infty}(u)^{1/2}.$$

Hence,

$$\frac{|u(x) - u(y)|}{d(x, y)^{\lambda}} \leq C\, N_{\sigma,\infty}(u)^{1/2},$$

which together with (8.6) yields (8.1). ■

9 Bessel potential spaces

Let p_t be a heat kernel on a metric measure space (M, d, μ) and let $\mathcal{L}$ be its generator. Since $\mathcal{L}$ is positive definite, the operator $(\mathrm{I} + \mathcal{L})^{-s}$ is a bounded operator in L^2 for any $s > 0$ (cf. Section 5.4). This operator is called a *Bessel potential.* Fix $\beta > 0$, and for any $\sigma > 0$ define the *Bessel potential space* H^{σ} as the image of $(\mathrm{I} + \mathcal{L})^{-\sigma/\beta}$, that is $H^{\sigma} := (\mathrm{I} + \mathcal{L})^{-\sigma/\beta}(L^2)$, with the norm

$$\|u\|_{H^{\sigma}} := \|\, (\mathrm{I} + \mathcal{L})^{\sigma/\beta}\, u\|_2.$$

This definition of H^{σ} depends on the parameter β. A priori the value of β is arbitrary but in fact we take for β the corresponding exponent in (2.8) assuming that (2.8) holds. For example, for the Gauss-Weierstrass kernel in $\mathbb{R}^n$ we take $\beta = 2$. In this case $\mathcal{L} = -\Delta$ and it is easy to see that $H^{\sigma}(\mathbb{R}^n)$ consists of functions $u \in L^2(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} |\widetilde{u}(\xi)|^2 \left(1 + |\xi|^2\right)^{\sigma/2} d\xi < \infty,$$

where $\widetilde{u}$ is the Fourier transform of u. Of course, this is the classical definition of the fractional Sobolev space $H^{\sigma}(\mathbb{R}^n)$.

The purpose of this section is to prove embedding theorems for the space H^{σ} in the abstract setting, which generalize the classical Sobolev embedding theorems in $\mathbb{R}^n$. We start with a lemma.

Lemma 9.1 *For any $0 < \sigma \leq \beta/2$ we have $H^{\sigma} = \mathrm{dom}_{\mathcal{E}}(\mathcal{L}^{2\sigma/\beta})$.*

Proof. Indeed, let $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ be the spectral resolution of the operator $\mathcal{L}$. Setting $s = \sigma/\beta$, we have

$$\begin{aligned} H^{\sigma} &= \mathrm{dom}(\mathrm{I} + \mathcal{L})^s = \left\{ u \in L^2 : \int_0^\infty (1+\lambda)^{2s}\, d\|E_\lambda u\|^2 < \infty \right\} \\ &= \left\{ u \in L^2 : \int_0^\infty \lambda^{2s}\, d\|E_\lambda u\|^2 < \infty \right\} \\ &= \mathrm{dom}_{\mathcal{E}}(\mathcal{L}^{2s}), \end{aligned}$$

which was to be proved. ■

Theorem 9.2 (Embedding of H^{σ})

(*i*) *If a heat kernel p_t satisfies* (2.8) *and $0 < \sigma < \alpha/2$ then*

$$H^{\sigma} \hookrightarrow L^q \quad \text{where } q = \frac{2\alpha}{\alpha - 2\sigma}. \tag{9.1}$$

(*ii*) *If a heat kernel p_t satisfies $\mathcal{H}(\alpha+\beta)$ and $\alpha/2 < \sigma \leq \beta/2$ then*

$$H^{\sigma} \hookrightarrow C^{\lambda} \quad \text{where } \lambda = \sigma - \alpha/2. \tag{9.2}$$

Remark 9.3 It is curious that under the chain condition the Hölder exponent λ in (9.2) does not exceed 1/2, which follows from Corollary 7.3(*ii*).

Proof. (*i*) This is essentially a result of Varopoulos. Indeed, [29, Theorem II.2.7] says that if for all $t > 0$ and some $\nu > 0$

$$\|P_t\|_{1\to\infty} \leq C t^{-\nu} \tag{9.3}$$

then for any $0 < s < \nu/2$ the operator $(\mathrm{I} + \mathcal{L})^{-s}$ is bounded from L^2 to L^q where $q = \frac{2\nu}{\nu - 2s}$. In our case (9.3) with $\nu = \alpha/\beta$ follows from (4.12). Applying the above result with $s = \sigma/\beta$ we obtain (9.1).

An alternative proof is given in [28, Theorem 3.11] under a stronger assumption that the heat kernel satisfies the upper bound in (1.2).

(*ii*) Using Lemma 9.1, Theorem 5.1, Corollary 5.5, Theorem 8.1, and Remark 8.2 we obtain

$$H^{\sigma} = \mathrm{dom}_{\mathcal{E}}(\mathcal{L}^{2\sigma/\beta}) = \begin{cases} \Lambda^{\sigma}_{2,\infty}, & \sigma = \beta/2, \\ \Lambda^{\sigma}_{2,2}, & \sigma < \beta/2, \end{cases} \hookrightarrow C^{\lambda},$$

which was to be proved. ■

For SG the claim of Theorem 9.2(*ii*) was proved by Strichartz [28, Theorem 3.13(a)]. He has conjectured a certain scale of embedding theorems for Sobolev spaces on fractals. Indeed, for any $1 < p < \infty$ consider the space

$$H^{p,\sigma} := (\mathrm{I} + \mathcal{L})^{-\sigma/\beta} (L^p) .$$

Then Conjecture 3.14 from [28] adapted to our setting says that if the heat kernel satisfies (1.2) then for all $\sigma > \alpha/p$ the space $H^{p,\sigma}$ embeds into a certain Hölder-Zygmund space, where the latter coincides with $C^{\sigma-\alpha/p}$ provided $\sigma \leq \beta/p$. Hence, our embedding (9.2) proves a particular case of this conjecture when $p = 2$ and $\alpha/2 < \sigma \leq \beta/2$. Note also that the identity $H^\sigma = \Lambda^\sigma_{2,2}$ (where $\alpha/2 < \sigma < \beta/2$) used in the above proof was also stated in [28] as an open question.

A result closely related to Theorem 9.2(*ii*) was obtained by Coulhon [12, Theorem 4.1]. Namely, he has proved that, under the chain condition, the estimate (1.2) with $\alpha < \beta$ is equivalent to the conjunction of the following three conditions: (*i*) the upper bound in (1.2); (*ii*) the volume estimate (5.10); (*iii*) the embedding $H^{p,\sigma} \hookrightarrow C^{\sigma-\alpha/p}$ for all $p > 1$ and $\sigma > \alpha/p$ provided $\sigma - \alpha/p$ is sufficiently small. Hence, Coulhon's result proves the above conjecture for all $p > 1$ but for a smaller range of σ.

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