The Heat Kernel on Hyperbolic Space

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1 Introduction and the main result

The purpose of this note is to provide a new proof for the explicit formulas of the heat kernel on hyperbolic space. By definition, the hyperbolic space $\mathbb{H}^n$ is a (unique) simply connected complete n-dimensional Riemannian manifold with a constant negative sectional curvature -1.

Let Δ denote the Laplacian on a Riemannian manifold X. The heat kernel on X is a function $p(x, y, t)$ on $X \times X \times (0, \infty)$ which is the minimal positive fundamental solution to the heat equation

$$\frac{\partial v}{\partial t} = \Delta v.$$

In other words, the Cauchy problem

$$\begin{cases} \frac{\partial v}{\partial t} = \Delta v, \\ v|_{t=0} = v_0(x), \end{cases} \tag{1.1}$$

has a solution

$$v(x, t) = \int_X p(x, y, t)\, v_0(y)\, dy, \tag{1.2}$$

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provided that v_0 is a bounded continuous function. If, in addition, $v_0 \geq 0$, then (1.2) defines the minimal positive solution to (1.1) (see [4] for details).

If X is the Euclidean space $\mathbb{R}^n$ then the heat kernel is given by the classical formula

$$p(x,y,t) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{\rho^2}{4t}\right), \tag{1.3}$$

where $\rho = |x-y|$. Due to homogeneity of the hyperbolic space, the heat kernel on $\mathbb{H}^n$ also depends only on t and ρ (where $\rho = \operatorname{dist}(x,y)$ is now the geodesic distance on $\mathbb{H}^n$). Let us denote the heat kernel on $\mathbb{H}^n$ by $p_n(\rho,t)$. Then we have the following.

Theorem 1.1 *The heat kernel $p_n(\rho,t)$ on the hyperbolic space $\mathbb{H}^n$ is given by the following formulas.*

If $n = 2m+1$, then

$$p_n(\rho,t) = \frac{(-1)^m}{2^m \pi^m} \frac{1}{(4\pi t)^{\frac{1}{2}}} \left(\frac{1}{\sinh\rho} \frac{\partial}{\partial\rho}\right)^m e^{-m^2 t - \frac{\rho^2}{4t}}. \tag{1.4}$$

If $n = 2m+2$, then

$$p_n(\rho,t) = \frac{(-1)^m}{2^{m+\frac{5}{2}} \pi^{m+\frac{3}{2}}} t^{-\frac{3}{2}} e^{-\frac{(2m+1)^2}{4}t} \left(\frac{1}{\sinh\rho} \frac{\partial}{\partial\rho}\right)^m \int_\rho^\infty \frac{s e^{-\frac{s^2}{4t}}}{(\cosh s - \cosh\rho)^{\frac{1}{2}}} ds. \tag{1.5}$$

In particular, if $n = 1$, then (1.4) coincides with the one-dimensional Euclidean heat kernel (1.3). If $n = 3$, then (1.4) becomes

$$p_3(\rho,t) = \frac{1}{(4\pi t)^{\frac{3}{2}}} \frac{\rho}{\sinh\rho} e^{-t - \frac{\rho^2}{4t}}, \tag{1.6}$$

whereas (1.5) yields, for $n = 2$,

$$p_2(\rho,t) = \frac{\sqrt{2}}{(4\pi t)^{\frac{3}{2}}} e^{-\frac{1}{4}t} \int_\rho^\infty \frac{s e^{-\frac{s^2}{4t}}}{(\cosh s - \cosh\rho)^{\frac{1}{2}}} ds. \tag{1.7}$$

The formulas (1.4) and (1.5) are not new. The heat kernel in dimension two (formula (1.7)) was found by McKean [6] (see also [1, pp. 242-246]). The three-dimensional case (formula (1.6)) was proved in [3, p. 396]. For $n > 3$, the formulas (1.4) and (1.5) can be obtained inductively by using the recurrence relation

$$p_{n+2}(\rho,t) = -\frac{\exp(-nt)}{2\pi \sinh\rho} \frac{\partial}{\partial\rho} p_n(\rho,t). \tag{1.8}$$

The identity (1.8) is attributed in [3, p. 396] to Millson (unpublished). Its proof can be found in [2, Theorem 2.1].

In this note, we provide an independent proof of (1.4) and (1.5) by using a completely different approach, based on the following two ingredients:

(1) the relation between the heat kernel and the wave kernel which follows from the spectral theory;

(2) the explicit formula for the wave kernel on symmetric spaces which is found in [5].

These are enough to derive (1.4) and (1.5) directly, without using (1.8).

2 Proof of the main theorem

Crucial for the proof is the following relation between the heat equation and the wave equation. Let L denote an elliptic operator on a manifold X, and let us consider the Cauchy problem for the wave equation in $X \times (-\infty, \infty)$,

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = Lu, \\ u|_{t=0} = u_0(x), \\ \frac{\partial u}{\partial t}\Big|_{t=0} = 0, \end{cases} \tag{2.1}$$

for a given $u_0 \in C(X)$. Assuming that the operator L is initially defined on $C_0^\infty(X)$ and has a self-adjoint non-positive definite extension in $\mathcal{L}^2(X)$ (which will also be denoted by L), the solution to (2.1) can be represented as

$$u(x,t) = \cos\left(t\sqrt{-L}\right) u_0(x).$$

Similarly, for the Cauchy problem for the heat equation in $X \times (0, \infty)$

$$\begin{cases} \frac{\partial v}{\partial t} = Lv, \\ v|_{t=0} = v_0(x), \end{cases} \tag{2.2}$$

one has

$$v(x,t) = \exp\left(tL\right) v_o(x), \quad t > 0.$$

On the other hand, we have the following Fourier transform identity (where $t > 0$):

$$e^{-t\lambda^2} = \frac{1}{(4\pi t)^{\frac{1}{2}}} \int_{-\infty}^{\infty} e^{-\frac{s^2}{4t}} e^{is\lambda}\, ds = \frac{1}{(4\pi t)^{\frac{1}{2}}} \int_{-\infty}^{\infty} e^{-\frac{s^2}{4t}} \cos\left(s\lambda\right)\, ds,$$

where λ is either a real number or a self-adjoint operator. Put $\lambda = \sqrt{-L}$; then

$$e^{tL} = \frac{1}{(4\pi t)^{\frac{1}{2}}} \int_{-\infty}^{\infty} e^{-\frac{s^2}{4t}} \cos\left(s\sqrt{-L}\right) ds,$$

whence we have the following proposition.

Proposition 2.1 *Let L be as above, and let $v_0 = u_0$ be a bounded continuous function on X. Then the solutions $u(x,t)$ of (2.1) and $v(x,t)$ of (2.2) are related as follows:*

$$v(x,t) = \frac{1}{(4\pi t)^{\frac{1}{2}}} \int_{-\infty}^{\infty} e^{-\frac{s^2}{4t}} u(x,s)\, ds. \tag{2.3}$$

Let us describe the next ingredient of the proof: the explicit formula for $u(x,s)$ on $\mathbb{H}^n$. Let us denote by $S_r(x)$ the geodesic sphere on $\mathbb{H}^n$ with centre $x \in \mathbb{H}^n$ and radius r. It is known that the area of $S_r(x)$ is equal to $A(r) = \Omega_n \sinh^{n-1} r$, where

$$\Omega_n := \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \tag{2.4}$$

is the area of the unit sphere in $\mathbb{R}^n$. For any number $r > 0$, denote by M^r the averaging operator on $\mathbb{H}^n$:

$$(M^r f)(x) := \frac{1}{A(r)} \int_{S_r(x)} f(y)\, d\omega(y), \tag{2.5}$$

where $d\omega$ is the area element of $S_r(x)$. For convenience, let us denote

$$\left(N^r_{m,k} f\right)(x) := \left(\frac{\partial}{\partial \cosh r}\right)^m \left(M^r f(x) \sinh^k s\right), \tag{2.6}$$

where k and m are non-negative integers.

The next assertion follows from the general mean value theorem for symmetric spaces and can be found in [5, Chapter 2].

Proposition 2.2 *Let $u(x,s)$ be the solution of the Cauchy problem on $\mathbb{H}^n \times (-\infty, \infty)$*

$$\begin{cases} Lu = 0, \\ u|_{t=0} = u_0, \\ \frac{\partial u}{\partial t}\Big|_{t=0} = u_1, \end{cases} \tag{2.7}$$

where

$$L = \Delta + \left(\frac{n-1}{2}\right)^2 \tag{2.8}$$

and u_0 and u_1 are continuous initial functions. Then for any $x \in \mathbb{H}^n$ and $s > 0$, we have the following.

If $n \geq 3$ is odd, then

$$u(x,s) = c_n \left(\frac{\partial}{\partial s} N^s_{\frac{n-3}{2},n-2}\, u_0 + N^s_{\frac{n-3}{2},n-2}\, u_1\right) \tag{2.9}$$

where

$$c_n = \frac{\Omega_n}{2(n-3)!!\Omega_{n-1}} = \frac{1}{(n-2)!!}.$$

(We use the notation $k!! := 1 \cdot 3 \cdot 5 \cdot \ldots \cdot k$ if k is odd, $k!! = 2 \cdot 4 \cdot 6 \cdot \ldots \cdot k$ if k is even and $k!! = 1$ if $k \leq 0$.)

If $n \geq 2$ is even, then the following equation holds:

$$\frac{1}{2} \int_0^s \frac{u(x,r) + u(x,-r)}{(\cosh s - \cosh r)^{\frac{1}{2}}} dr = c_n N^s_{\frac{n-2}{2},n-2} u_0 \tag{2.10}$$

where $c_n = \frac{\Omega_n}{\sqrt{2}(n-3)!!\Omega_{n-1}} = \frac{\pi}{2(n-2)!!}$.

Proposition 2.2 simplifies for the case $u_1 = 0$, as follows.

Corollary 2.3 *Assume, under the hypotheses of Proposition 2.2, that* $u_1 = 0$. *Then, for any* $x \in \mathbb{H}^n$ *and* $s > 0$, *we have the following.*

If $n \geq 3$ *is odd, then*

$$u(x,s) = c_n \frac{\partial}{\partial s} N^s_{\frac{n-3}{2},n-2} u_0(x). \tag{2.11}$$

If $n \geq 2$ *is even, then*

$$u(x,s) = \frac{c_n}{\pi} \frac{\partial}{\partial s} \int_0^s \frac{\sinh \rho}{(\cosh s - \cosh \rho)^{\frac{1}{2}}} N^\rho_{\frac{n-2}{2},n-2} u_0(x)\, d\rho \tag{2.12}$$

Proof. Since (2.11) is obvious from (2.9), let us concentrate on (2.12). Observe that $u_1 = 0$ implies that the solution $u(x,t)$ is an even function in t. Therefore (2.10) acquires the form

$$\int_0^s K(s,r) u(x,r)\, dr = c_n N^s_{\frac{n-2}{2},n-2} u_0(x), \tag{2.13}$$

where we denote

$$K(s,r) := \frac{1}{(\cosh s - \cosh r)^{\frac{1}{2}}}, \quad s \neq r.$$

By multiplying (2.13) by $K(t,s) \sinh s$ and integrating in s, we have, for any $t > 0$,

$$\int_0^t K(t,s) \sinh s \int_0^s u(x,r) K(s,r)\, dr\, ds = c_n \int_0^t K(t,s) \sinh s N^s_{\frac{n-2}{2},n-2} u_0(x)\, ds\,. \tag{2.14}$$

The left-hand side is computed by changing the order of integration:

$$\int_0^t dr \int_r^t u(x,r) K(t,s) K(s,r) \sinh s\, ds = \pi \int_0^t u(x,r)\, dr, \tag{2.15}$$

where we have used the identity

$$\int_r^t K(t,s) K(s,r) \sinh s\, ds = \int_a^b \frac{d\xi}{\sqrt{(b-\xi)(\xi-a)}} = \pi\,.$$

By (2.14) and (2.15),

$$\pi \int_0^t u(x,r)\, dr = c_n \int_0^t K(t,s) \sinh s N^s_{\frac{n-2}{2},n-2} u_0(x)\, ds$$

whence we obtain (2.12) by differentiating in t. ■

Our next goal will be to extend (2.11) and (2.12) to the case $s < 0$. Fix x, and denote, for any $\rho > 0$,

$$U(\rho) := \int_{S_\rho(x)} u_0(y)\, d\omega(y) = A(\rho)\, M^\rho u_0(x), \tag{2.16}$$

and extend $U(\rho)$ to non-positive ρ by $U(0) = 0$ and $U(\rho) = U(|\rho|)$.

Lemma 2.4 *We have $U(r) \in C^{n-1}(-\infty, \infty)$. Moreover, $U^{(l)}(0) = 0$ for any $l = 0, 1, ..., n-1$.*

Proof. The function u_0 is infinitely smooth, and so is $U(\rho)$ for $\rho \neq 0$. To handle the case $\rho = 0$, let us rewrite (2.16) in the polar coordinates $y = (\rho, \theta)$ centred at x:

$$U(\rho) = \int_{\mathbb{S}^n} u_0(\rho, \theta)\, A(\rho)\, d\theta.$$

The trouble is that the distance function $\rho(y) := \operatorname{dist}(x, y)$ is not smooth at $y = x$. However, its square ρ^2 is infinitely smooth in y, which implies that the function u_0 is C^∞ in ρ^2. Since

$$\frac{\partial}{\partial \rho} = 2\rho \frac{\partial}{\partial (\rho^2)},$$

it is not difficult to see by induction that $\left(\frac{\partial}{\partial \rho}\right)^k$ is a sum of the terms proportional to

$$\rho^{2j-k} \left(\frac{\partial}{\partial\left(\rho^2\right)}\right)^j,$$

where $j = 1, 2, ..., k$.

Obviously, the function $\left(\frac{\partial}{\partial \rho}\right)^m A$ has, at $\rho = 0$, a zero of order ρ^{n-1-m}, that is, it can be represented as

$$\left(\frac{\partial}{\partial \rho}\right)^m A = \rho^{n-1-m} A_m(\rho),$$

where $A_m(\rho)$ is a continuous function of ρ. When differentiating l times the product $u_0(\rho, \theta) A(\rho)$ in ρ, we have the sum of the terms proportional to

$$\left(\frac{\partial}{\partial \rho}\right)^k u_0 \cdot \left(\frac{\partial}{\partial \rho}\right)^{l-k} A,$$

which splits further, to the sum of terms as

$$\rho^{2j-k} \left(\frac{\partial}{\partial\left(\rho^2\right)}\right)^j u_0 \cdot \rho^{n-1-l+k} A_{l-k}(\rho) = \rho^{2j+(n-1-l)} \left(\frac{\partial}{\partial\left(\rho^2\right)}\right)^j u_0 \cdot A_{l-k}(\rho).$$

If $l \leq n-1$, then the latter function is continuous up to $\rho = 0$ and vanishes at $\rho = 0$ whence we obtain $U \in C^{n-1}$ and $U(0) = U'(0) = U''(0) = ... = U^{(n-1)}(0) = 0$.

Corollary 2.5 *We have the following, under the hypotheses of Corollary 2.3, for all $x \in \mathbb{H}^n$ and $s \in (-\infty, \infty)$.*

If $n \geq 3$ is odd, then

$$u(x, s) = \frac{c_n}{\Omega_n} \frac{\partial}{\partial s} \left(\frac{1}{\sinh s} \frac{\partial}{\partial s}\right)^{\frac{n-3}{2}} \left[\frac{1}{\sinh s} U(s)\right]. \tag{2.17}$$

If $n \geq 2$ is even, then

$$u(x,s) = \frac{c_n}{\pi\Omega_n}\frac{\partial}{\partial s}\int_0^s K(s,\rho)\sinh\rho\left(\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right)^{\frac{n-2}{2}}\left(\frac{1}{\sinh\rho}U(\rho)\right)d\rho\,. \tag{2.18}$$

Proof. Indeed, (2.17) coincides with (2.11) if $s > 0$. Since both sides of (2.17) are even functions in s, (2.17) holds also for $s < 0$. The case $s = 0$ follows by continuity. Each derivation $\frac{\partial}{\partial s}$ of $U(s)$, or division by $\sinh s$, reduces smoothness at 0 by at most 1. After $n-1$ such operations as in (2.17), we still have a continuous function on the right-hand side of (2.17). In particular, this argument shows that the right-hand side of (2.17) has meaning for $s = 0$.

The equation (2.18) follows in the same way from (2.12).

Proof of Theorem 1.1. It is clear from (2.8) that

$$e^{t\Delta} = \exp\left(-\left(\frac{n-1}{2}\right)^2 t\right)e^{tL},$$

so the kernels of the semigroups $e^{t\Delta}$ and e^{tL} are related by the same equation. Therefore it will suffice to find the kernel of the semigroup e^{tL}. Note that the operator L is non-positive definite, for the top of the spectrum of Δ in $\mathcal{L}^2(\mathbb{H}^n)$ is equal to $-(\frac{n-1}{2})^2$.

We shall use the notation $u(x,t)$ and $v(x,t)$ for the solutions to the initial problems (2.1) and (2.2), respectively, for the operator (2.8). The initial function $u_0 = v_0$ is supposed to be in $C_0^\infty(\mathbb{H}^n)$.

Case I: $n = 2m+1$.

We have, by (2.3) and (2.17),

$$v(x,t) = \frac{c_n}{(4\pi t)^{\frac{1}{2}}}\int_{-\infty}^{\infty} e^{-\frac{s^2}{4t}}\frac{\partial}{\partial s}\left(\frac{1}{\sinh s}\frac{\partial}{\partial s}\right)^{\frac{n-3}{2}}\left[\frac{1}{\sinh s}U(s)\right]ds\,, \tag{2.19}$$

whence we obtain, by integration by parts,

$$\begin{aligned}
v(x,t) &= \frac{c_n\Omega_n^{-1}}{(4\pi t)^{\frac{1}{2}}}\int_{-\infty}^{\infty} e^{-\frac{s^2}{4t}}\frac{\partial}{\partial s}\left(\frac{1}{\sinh s}\frac{\partial}{\partial s}\right)^{m-1}\left[\frac{1}{\sinh s}U(s)\right]ds \\
&= \frac{(-1)^m c_n\Omega_n^{-1}}{(4\pi t)^{\frac{1}{2}}}\int_{-\infty}^{\infty}\left\{\left(\frac{1}{\sinh s}\frac{\partial}{\partial s}\right)^m e^{-\frac{s^2}{4t}}\right\}U(s)\,ds \\
&= \frac{2(-1)^m c_n\Omega_n^{-1}}{(4\pi t)^{\frac{1}{2}}}\int_0^{\infty}\left\{\left(\frac{1}{\sinh s}\frac{\partial}{\partial s}\right)^m e^{-\frac{s^2}{4t}}\right\}U(s)\,ds \\
&= \frac{2(-1)^m c_n\Omega_n^{-1}}{(4\pi t)^{\frac{1}{2}}}\int_{\mathbb{H}^n}\left(\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right)^m e^{-\frac{\rho^2}{4t}}u_0(y)\,dy,
\end{aligned}$$

where $\rho = \operatorname{dist}(x,y)$.

Observing that

$$2c_n\Omega_n^{-1} = \frac{1}{(n-3)!!\Omega_{n-1}} = \frac{1}{2^m\pi^m},$$

we conclude that the operator e^{tL} is an integral operator with kernel

$$\frac{(-1)^m}{2^m\pi^m}\frac{1}{(4\pi t)^{\frac{1}{2}}}\left(\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right)^m e^{-\frac{\rho^2}{4t}}.$$

Therefore the heat kernel $p_n(\rho,t)$ of $e^{t\Delta}$ is

$$p_n(\rho,t)=\frac{(-1)^m}{2^m\pi^m}\frac{1}{(4\pi t)^{\frac{1}{2}}}\left(\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right)^m e^{-m^2t-\frac{\rho^2}{4t}},$$

which was to be proved.

Case II: $n=2m+2$.

We have, by (2.3) and (2.18), by changing the order of the integrals and by integration by parts,

$$\begin{aligned}
&v(x,t)\\
=&\ \frac{c_n'}{\sqrt{t}}\int_{-\infty}^{\infty}e^{-\frac{s^2}{4t}}\left[\frac{\partial}{\partial s}\int_0^s K(s,\rho)\sinh\rho\left(\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right)^m\left(\frac{1}{\sinh\rho}U(\rho)\right)d\rho\right]ds\\
=&\ \frac{c_n'}{\sqrt{t}}\int_{-\infty}^{\infty}\frac{2s}{4t}e^{-\frac{s^2}{4t}}\left[\int_0^s K(s,\rho)\sinh\rho\left(\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right)^m\left(\frac{1}{\sinh\rho}U(\rho)\right)d\rho\right]ds\\
=&\ \frac{c_n'}{t^{\frac{3}{2}}}\int_0^{\infty}\left(\int_\rho^\infty se^{-\frac{s^2}{4t}}K(s,\rho)ds\right)\sinh\rho\left(\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right)^m\left(\frac{1}{\sinh\rho}U(\rho)\right)d\rho\\
=&\ \frac{(-1)^m c_n'}{t^{\frac{3}{2}}}\int_0^{\infty}\left(\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right)^m\left(\int_\rho^\infty se^{-\frac{s^2}{4t}}K(s,\rho)ds\right)U(\rho)\,d\rho\\
=&\ \frac{(-1)^m c_n'}{t^{\frac{3}{2}}}\int_{\mathbb{H}^n}\left(\left(\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right)^m\int_\rho^\infty se^{-\frac{s^2}{4t}}K(s,\rho)ds\right)u_0(y)\,dy,
\end{aligned}$$

where $c_n'=\frac{c_n}{2\pi^{\frac{3}{2}}\Omega_n}=\frac{1}{2^{m+\frac{5}{2}}\pi^{m+\frac{3}{2}}}$.

This shows that the kernel of the integral operator e^{tL} is

$$\frac{(-1)^m}{2^{m+\frac{5}{2}}\pi^{m+\frac{3}{2}}}t^{-\frac{3}{2}}\left(\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right)^m\int_\rho^\infty\frac{se^{-\frac{s^2}{4t}}}{(\cosh s-\cosh\rho)^{\frac{1}{2}}}ds,$$

whence the heat kernel is

$$p_{2m+2}(x,y,t)=\frac{(-1)^m}{2^{m+\frac{5}{2}}\pi^{m+\frac{3}{2}}}e^{-t\frac{(2m+1)^2}{4}}t^{-\frac{3}{2}}\left(\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right)^m\int_\rho^\infty\frac{se^{-\frac{s^2}{4t}}}{(\cosh s-\cosh\rho)^{\frac{1}{2}}}ds.$$

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